RATIONAL DECOMPOSITION OF MODULAR FORMS

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1. Introduction

The space S_{2k} of modular cusp forms of even weight $2k$ for the full modular group $\Gamma = PSL_2(\mathbb{Z})$ is endowed with several rational structures. Besides the subspace S_{2k}^0 of forms with rational Fourier coefficients, there are the spaces S_{2k}^+ and S_{2k}^- defined by Kohnen and Zagier [KZ84], of forms with rational even, and respectively odd, periods. In this paper we prove explicit formulas decomposing an arbitrary $f \in S_{2k}$ in terms of forms belonging to the three rational structures. Using the Shimura correspondance, we also prove a similar decomposition for forms of half integral weight $k + 1/2$ when k is even.

To state the results, let $w = 2k - 2$, and for $0 \le n \le w$ define the n^{th} period of $f \in S_{2k}$ by:

$$r_n(f) = \int_0^{i\infty} f(z) z^n dz.$$

Let $R_n \in S_{2k}$ be the cusp form defined by requiring that the Petersson product $(f, R_n) = r_n(f)/i^{n+1}$, for all $f \in S_{2k}$. The forms R_n with n even (resp. odd) span the space S_{2k}^- (resp. S_{2k}^+) of forms f with $r_j(f)/i^{j+1}$ rational for j odd (resp. even).

Theorem 1.1. *For all $f \in S_{2k}$ we have the decompositions:*

$$\begin{aligned} f &= (-1)^{k-1} 3^{-1} 2^{-w} \sum_{\substack{n=2 \\ n \text{ even}}}^{w-2} \binom{w}{n} s_n^-(f) (-1)^{n/2} R_n \\ &= i(-1)^{k-1} 3^{-1} 2^{-w} \sum_{\substack{j=1 \\ j \text{ odd}}}^{w-1} \binom{w}{j} s_j^+(f) (-1)^{(j+1)/2} R_j, \end{aligned}$$

where $s_n^-(f), s_j^+(f)$ are the following linear combinations of odd, respectively even periods of f:

$$s_n^-(f) = \sum_{\substack{j=1 \\ j \text{ odd}}}^{n} \binom{n}{j} r_j(f), \qquad s_j^+(f) = \sum_{\substack{n=0 \\ n \text{ even}}}^{j} \binom{j}{n} r_n(f).$$

Key words and phrases. Forms with rational periods; Shimura correspondence; Rankin-Cohen brackets.

The theorem can be seen as providing explicit inverses to the Eichler-Shimura maps which attach to a cusp form its even or odd period polynomial. In Section 4, we restate it in terms of the natural pairing on period polynomials induced by the cup product in parabolic cohomology. It turns out that the formula in the theorem refines (and implies) a formula due to Haberland expressing the Petersson product of two cusp forms in terms of their periods. It also implies the invariance of a modified period polynomial pairing, with respect to the Hecke action. The proof of Theorem 1.1–given in Section 3–follows immediately from the Eichler-Shimura relations, recalled in Section 2, and the explicit computation by Kohnen and Zagier of the periods of R_n.

In Section 5, Theorem 1.1 is combined with work of Kohnen and Zagier to prove a formula for $r_n(f)f$ in terms of forms with rational Fourier coefficients, when f is a Hecke cusp form (by Hecke form we always mean an eigenform of the Hecke operators, normalized so that the Fourier coefficient of q is 1). The forms with rational coefficients that appear are Rankin-Cohen brackets of Eisenstein series; in the simplest case $n = 0$, these forms are products of Eisenstein series, and the formula in Theorem 5.1 becomes:

$$r_0(f)f = \frac{2}{3} \sum_{\substack{j=1 \\ j \text{ odd}}}^{w-1} \binom{w}{j} s_j^+(f) \left[G_{j+1}G_{\tilde{j}+1} + \frac{\delta_{j,1} + \delta_{j,w-1}}{4w\pi i} G'_w + \right. \\ \left. + \frac{kB_{j+1}B_{\tilde{j}+1}}{B_{2k}(j+1)(\tilde{j}+1)} G_{2k} \right] \tag{1.1}$$

where $\tilde{j} = w - j$ and

$$G_l(z) = -\frac{B_l}{2l} + \sum_{n\geq 1} \sigma_{l-1}(n)q^n, \quad q := e^{2\pi i z}$$

is the Eisenstein series of weight l for $l \geq 2$ even (G_2 is not a modular form, and the term involving the derivative (with respect to z) G'_w is needed to complete G_2G_w to a modular form of weight $2k$). A similar formula for $r_0(f)f$ was proved by Manin [Ma73] (the Coefficients theorem).

The last result involves the Shimura correspondence between forms of integral and half integral weight. Let $S_{k+1/2}$ denote the Kohnen space of cusp forms of weight $k+1/2$ for the group $\Gamma_0(4)$. They are characterized by the condition that their Fourier coefficients $c(n)$ vanish if $(-1)^k n \equiv 2, 3 \pmod 4$. The space $S_{k+1/2}$ admits a basis of eigenforms for Hecke operators of square index, and the Shimura correspondence provides a bijection between these eigenforms and Hecke cusp forms in S_{2k}.

Assume now that k is even. If $g \in S_{k+1/2}$ is a Hecke eigenform, and $f \in S_{2k}$ the corresponding Shimura lift, then we have the following explicit formula for g in terms of the odd periods of f and certain "indefinite theta series" θ_n (the terminology is explained in Section 6):

$$\frac{c(1)(f,f)}{(g,g)(-1)^{k/2}2^k}\, g = \frac{2}{3} \sum_{\substack{n=2 \\ n \text{ even}}}^{w-2} s_n^-(f)\theta_n \tag{1.2}$$

where $c(1)$ is the first Fourier coefficient of g, and

$$\theta_n = -\theta_{w-n} = -\frac{1}{(w/2-n/2)!}[\theta(\tau), G_{k-n}(4\tau)]_{n/2} \quad (0<n<k,\ n \text{ even}). \tag{1.3}$$

Here $\theta(z)=\sum_{n\in\mathbb{Z}} q^{n^2}$, and the definition of the Rankin-Cohen bracket is recalled in Section 5.

This formula allows for easy computation of all the coefficients of the half integral weight form g, once the odd periods of its Shimura lift f are known. The right-hand side of this identity appears in disguise in a formula in [KZ84] (on p.237), expressing the coefficients of g in terms of the odd periods of f and certain arithmetic functions related to the Fourier coefficients of θ_n. It is this identity that provided the motivation for Theorem 1.1.

The present article can be seen as an addition to the program initiated in [KZ84], of studying modular forms through their periods rather than their Fourier expansions. An interesting feature of this approach is that formulas for Fourier coefficients of Hecke eigenforms of integral and half integral weight are obtained without directly using the action of Hecke operators. Instead, the main tool is a generalization of an identity of Rankin to Rankin-Cohen brackets, obtained by Zagier [Za77].

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2. Eichler-Shimura-Manin theory

In this section we review the Eichler-Shimura-Manin theory, in the form needed for the proof of Theorem 1.1. We also give a short proof, based on Theorem 1.1, of the extra relation satisfied by the even periods of all cusp forms, found by Kohnen and Zagier.

For $f\in S_{2k}$, the period polynomial $r(f)$ is defined by:

$$r(f)(x) = \int_0^{i\infty} f(z)(x-z)^w dz = \sum_{n=0}^{w}(-1)^n \binom{w}{n} r_{w-n}(f)x^n.$$

The group $PGL_2(\mathbb{Z})$ acts on the space V_w of polynomials of degree $\le w$ by

$$(P|\gamma)(x) = (cx+d)^w P\left(\frac{ax+b}{cx+d}\right), \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

It is easy to see that $(r(f)|\gamma)(x) = \int_{\gamma^{-1}0}^{\gamma^{-1}\infty} f(z)(x-z)^w dz$, and taking for γ the two generators $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and $U = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ of Γ it follows that the period polynomials $r(f)$ belong to the subspace W_w of V_w defined by the relations:

$$W_w = \{P \in V_w : P + P|S = 0, P + P|U + P|U^2 = 0\}.$$

This subspace is invariant under the action of $\eta = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, which acts on polynomials by $P|\eta(x) = P(-x)$; therefore we have a decomposition $W_w = W_w^+ \oplus W_w^-$, corresponding to the decomposition of a polynomial $P = P^+ + P^-$ into its even and odd parts.

Writing the relations defining $W_w^\pm$ in terms of coefficients yields the Eichler-Shimura relations satisfied by the periods of f. For later use, it is convenient to state them in terms of the sums, defined for $0 \le n \le w$:

$$s_n^-(f) = \sum_{\substack{j=1 \\ j \text{ odd}}}^{n} \binom{n}{j} r_j(f), \quad s_n^+(f) = \sum_{\substack{j=0 \\ j \text{ even}}}^{n} \binom{n}{j} r_j(f).$$

For n an integer, we denote by $\widetilde{n} = w - n$. Then for $0 \le n \le w$ we have [Ma73],[La76], [KZ84]:

(ES.odd) $\qquad s_n^-(f) + s_{\widetilde{n}}^-(f) = 0, \quad n \text{ even}; \qquad s_n^-(f) + s_{\widetilde{n}}^-(f) = r_n(f), \quad n \text{ odd}$

(ES.even) $\qquad s_n^+(f) - s_{\widetilde{n}}^+(f) = 0, \quad n \text{ odd}; \qquad s_n^+(f) - s_{\widetilde{n}}^+(f) = r_n(f), \quad n \text{ even}$

Note that the relations above imply that $r_n(f) = (-1)^{n+1} r_{\widetilde{n}}(f)$.

In Appendix A, we also prove the relations satisfied by periods of Eisenstein series, which can also be written compactly in terms of Bernoulli number identities involving sums similar to s_n^- and s_n^+.

Remark 2.1. The periods of $f = \sum_{n\ge1} a_n q^n$ are related to the critical values of its (analytically continued) L-function $L(f,s) = \sum_{n\ge1} a_n n^{-s}$ by

$$r_n(f) = \frac{n! i^{n+1}}{(2\pi)^{n+1}} L(f, n+1).$$

If f is a Hecke cusp form, the periods $r_n(f)$ are nonzero if $n \ne k-1$. Indeed $L(f,s)$ is given by an absolutely convergent Euler product for $\Re(s) > k + 1/2$, and the Euler factors do not vanish at $s = n+1 > k$.

The Eichler-Shimura relations consist of all but one of the linear relations satisfied by the periods of all $f \in S_{2k}$. Consider the maps

$$r^{\pm} : S_{2k} \to W_w^{\pm}$$

taking f to the even $r^+(f)$ or odd $r^-(f)$ parts of its period polynomial. The Eichler-Shimura theorem states that the map r^- is an isomorphism, while r^+ is an isomorphism onto a codimension 1 subspace of W_w^+, which does not contain the polynomial $p_0(x) = x^w - 1$.

The extra linear relation satisfied by the even periods of all cusp forms was found by Kohnen and Zagier, and it immediately follows from Theorem 1.1. Indeed, taking the 0th period of the second identity, one obtains (after using the computation of $r_0(R_n)$ in [KZ84] and rewriting the result using the Bernoulli number identity (3.2) applied to $N = w + 1$):

$$\sum_{\substack{j=1 \\ j \text{ odd}}}^{w-1} \binom{w}{j} s_j^+(f) r_j(G_{2k}) + \frac{B_{2k}}{2k} \frac{s_{w+1}^+(f)}{w+1} = 0, \tag{2.1}$$

which is the Kohnen-Zagier relation, as stated in [CZ93, p.91] (s_{w+1}^+ is defined in the same way as s_n^+ for $0 \le n \le w$). The periods $r_j(G_{2k})$ of Eisenstein series are defined in Appendix A.

3. Proof of the main theorem

In this section we prove Theorem 1.1. It is enough to prove the first identity; the second follows from the first by expressing $(f, g) = \overline{(g, f)}$ using Corollary 4.1 in the next section.

The proof is based on Kohnen and Zagier's computation of the periods of R_n, which we state here only in the case of interest to us.

Proposition 3.1. [KZ84, Thm. 1] *If m is odd and n is even, $0 < m, n < w$, then:*

$$(-1)^k 2^{-w} w! r_m[(-1)^{n/2} R_n] = n!\widetilde{m}!\beta_{n-m} - \widetilde{n}!\widetilde{m}!\beta_{\widetilde{n}-m} + n!m!\beta_{m-\widetilde{n}} - \widetilde{n}!m!\beta_{m-n}$$

where $\widetilde{n} = w - n$, and β_{j-1} equals $B_j/j!$ if $j \ge 0$ is even, and zero otherwise.

We have to show that:

$$f = (-1)^{k-1} 3^{-1} 2^{-w} \sum_{\substack{n=0 \\ n \text{ even}}}^{w} \binom{w}{n} s_n^-(f)(-1)^{n/2} R_n. \tag{3.1}$$

Note that $s_0^-(f) = s_w^-(f) = 0$, but it will be convenient to have these terms as well. By the Eichler-Shimura isomorphism, it is enough to check that for m odd, $1 \le m \le w-1$, the m^{th} period of the right-hand side equals $r_m(f)$.

Let $f^{\#}\in S_{2k}$ denote the right-hand side of (3.1). By Proposition 3.1, we write:

$$r_m(f^{\#})=3^{-1}\sum_{\substack{n=0\\ n \text{ even}}}^{w} s_n^-(f)\left[\frac{\widetilde{m}!}{n!}\beta_{\widetilde{n}-m}-\frac{\widetilde{m}!}{\widetilde{n}!}\beta_{n-m}+\frac{m!}{n!}\beta_{m-n}-\frac{m!}{\widetilde{n}!}\beta_{m-\widetilde{n}}\right].$$

Since $s_{\widetilde{n}}^-(f)=-s_n^-(f)$ for n even, the first two terms inside the brackets give the same contribution to the sum, as do the last two terms. We can write therefore

$$r_m(f^{\#})=\frac{2}{3}(S_1+S_2)$$

where $S_1=\sum_{\substack{n=0\\ n \text{ even}}}^{w} s_n^-(f)\frac{\widetilde{m}!}{n!}\beta_{\widetilde{n}-m}$ and S_2 is defined like S_1 with m replaced by $\widetilde{m}$. Expanding s_n^- and interchanging the order of summation yields:

$$\begin{aligned} S_1&=\sum_{\substack{j=1\\ j \text{ odd}}}^{w}\sum_{\substack{n=j\\ n \text{ even}}}^{w}\frac{\widetilde{m}!}{j!(n-j)!}r_j(f)\beta_{\widetilde{n}-m}\\ &=\sum_{\substack{j=1\\ j \text{ odd}}}^{\widetilde{m}}\sum_{\substack{n=j\\ n \text{ even}}}^{\widetilde{m}+1}\binom{\widetilde{m}}{j}\frac{r_j(f)}{\widetilde{m}-j+1}\binom{\widetilde{m}-j+1}{\widetilde{m}-n+1}B_{\widetilde{m}-n+1}. \end{aligned}$$

The sum over n can be computed using the Bernoulli number identity

$$\sum_{\substack{n=0\\ n \text{ even}}}^{N}\binom{N}{n}B_n=\frac{N}{2}+B_N+\delta_{1,N} \tag{3.2}$$

(for $N=\widetilde{m}-j+1$). We obtain $S_1=\frac{1}{2}(s_{\widetilde{m}}^-(f)+r_m(f))$. Replacing m by $\widetilde{m}$ yields S_2, and using (ES.odd) we obtain $r_m(f^{\#})=r_m(f)$. $\square$

4. Bilinear form on period polynomials

In this section we show that Theorem 1.1 refines a formula of Haberland expressing the Petersson inner product of two cusp forms in terms of their periods. It is also equivalent to the Hecke invariance of a natural pairing on period polynomials induced from cup product on the parabolic cohomology group $H^1_{\text{par}}(\Gamma,V_k)$.

To explain the connection, it is convenient to restate Theorem 1.1 in terms of the pairing on the space V_w of polynomials of degree $\le w$ given by:

$$\left\langle\sum_{n=0}^{w}a_nx^n,\sum_{n=0}^{w}b_nx^n\right\rangle=\sum_{n=0}^{w}(-1)^n\binom{w}{n}^{-1}a_nb_{w-n}.$$

This pairing is symmetric, and $PGL_2(\mathbb{Z})$ invariant:

$$< P|\gamma, Q|\gamma >=< P,Q >,\quad \text{for } \gamma\in PGL_2(\mathbb{Z}), P,Q\in V_w.$$

Taking the Petersson product of the identities in Theorem 1.1 with an arbitrary cusp form g and rewriting the result in terms of this pairing yields:

Corollary 4.1. *If $f, g \in S_{2k}$ then:*

$$\begin{aligned} c_k(f,g) &= i < r(f), \overline{r^+}(g)|(T - T^{-1}) > \\ &= i < r(f), \overline{r^-}(g)|(T - T^{-1}) >, \end{aligned} \tag{4.1}$$

where $c_k = (-1)^k \cdot 3 \cdot 2^{2k-1}$.

The second identity follows from the first, applied to $\overline{(g, f)}$. Indeed, the modified pairing

$$\{P, Q\} := < P, Q|(T - T^{-1}) >, \quad P, Q \in V_w$$

is antisymmetric and anti-invariant under the action of η: $\{P|\eta, Q|\eta\} = -\{P, Q\}$, therefore $\{P, Q\}$ vanishes if P, Q are both even or both odd.

Adding the two identities in formula (4.1), we obtain Haberland's formula (see the version given in [KZ84, p.243], which is equivalent to the version given here):

$$2c_k(f, g) = i < r(f), \overline{r}(g)|(T - T^{-1}) > . \tag{4.2}$$

Therefore the widely used formula of Haberland is made of two simpler identities. Notice that in Haberland's identity both odd and even periods of f and g appear, while in the right hand sides of Corollary 4.1 only odd periods of f and even period of g appear (or viceversa).

An application of Corollary 4.1 to a basis of Hecke eigenforms implies that the pairing $\{\cdot, \cdot\}$ is Hecke invariant when restricted to $W_w^+ \times W_w^-$:

$$\{r^+(f)|\widetilde{T}_n, \overline{r^-(g)}\} = \{r^+(f), \overline{r^-(g)}|\widetilde{T}_n\}.$$

Here the action of Hecke operators on period polynomials is defined such that $r(f)|\widetilde{T}_n = r(f|T_n)$ (see [Za90] for a concrete definition and further properties of $\widetilde{T}_n$). The Hecke invariance property is also stated without proof in [GKZ, p.96].

The Hecke invariance property also points to a way of generalizing Theorem 1.1 and Corollary 4.1 to other congruence subgroups. This approach is taken in an upcoming joint work with V. Pasol.

5. Modular forms with rational coefficients

In this section we show that the main theorem implies a decomposition of Hecke cusp forms in terms of forms with rational Fourier coefficients. These forms are given by Rankin-Cohen brackets of Eisenstein series, and we give a brief review of their properties, following [KZ84, Sec. 1.4].

The transition between forms with rational periods and forms with rational coefficients is realized by the linear functionals $\rho_m : S_{2k} \to S_{2k}$ introduced in [KZ84], defined by

$$\rho_m(f) = \frac{r_m(f)}{i^{m+1}} f, \quad f \text{ Hecke form.}$$

By linearity, the functional ρ_m acts on an arbitrary $g \in S_{2k}$ by

$$\rho_m(g) = i^{-m-1} \sum_{l=1}^{\infty} r_m(g|T_l) q^l$$

and since the Hecke operators T_l preserve S_{2k}^+ and S_{2k}^-, it follows that ρ_m maps $S_{2k}^{\pm}$ (with $(-1)^m = \pm 1$) to the space S_{2k}^0 of cusp forms with rational Fourier coefficients.

It follows that if m, n have opposite parity the form

$$X_{m,n} := \rho_m(R_n)$$

has rational Fourier coefficients. By decomposing R_n with respect to a basis of Hecke cusp forms, one can also characterize $X_{m,n}$ as the unique cusp form such that:

$$(f, X_{m,n}) = \frac{r_m(f) r_n(f)}{i^{m+n+2}}, \quad \text{for all } f \text{ Hecke cusp forms.} \tag{5.1}$$

This is an identity of Rankin-Selberg type for $m = 0$, and in that case $X_{0,n}$ is essentially a product of Eisenstein series $G_{n+1} G_{\tilde{n}+1}$ (up to a constant, and a correcting factor if $n = 1$ or $n = w - 1$). In general, Zagier showed that the previous identity is satisfied by the Rankin-Cohen bracket of two Eisenstein series [Za77]. We refer to [KZ84, p.214-215] for the definition of Rankin-Cohen brackets, and here we only recall, to fix notations, that if f, g are modular forms of weights a, b for any subgroup of $SL_2(\mathbb{R})$, their Rankin-Cohen bracket of index $m \geq 0$ is a modular form of weight $a + b + 2m$ given by:

$$[f, g]_m = (2\pi i)^{-m} \sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} \frac{\Gamma(a+m)\Gamma(b+m)}{\Gamma(a+i)\Gamma(b+m-i)} f^{(i)} g^{(m-i)}. \tag{5.2}$$

It is a cusp form if $m > 0$, and it can also be defined if one or both of f, g equal the Eisenstein series G_2 [KZ84].

We can now identify the modular form $X_{m,n} \in S_{2k}$ determined by (5.1). If $0 \leq m < n \leq k-1$ have opposite parity, then the generalization of the Rankin-Selberg method due to Zagier yields [KZ84, p.215]:

$$X_{m,n} = (-1)^{k+(n-m+1)/2} 2^{w+1} \frac{(w-m)!}{w!} \cdot$$
$$\cdot \left[[G_{\tilde{n}-m+1}, G_{n-m+1}]_m + \delta_{m,0} \frac{k}{B_{2k}} \frac{B_{n+1} B_{\tilde{n}+1}}{(n+1)(\tilde{n}+1)} G_{2k} \right].$$

For other values $m, n \in [0, w]$ of opposite parity, the form $X_{m,n}$ is determined by the relations:

$$X_{m,n} = X_{n,m} = (-1)^k X_{m,w-n},$$

which follow from (5.1)

Applying the maps ρ_m to the identities in Theorem 1.1 we obtain:

Theorem 5.1. *If $f \in S_{2k}$ is a Hecke form and $0 \le m \le w$, one has the following decomposition of $r_m(f)f$ in terms of forms with rational coefficients:*

$$r_m(f)f = 3^{-1}2^{-w} \sum_{\substack{n=0 \\ n \not\equiv m \pmod 2}}^{w} \binom{w}{n} s_n^{\pm}(f)(-1)^{k+(n-m+1)/2} X_{m,n},$$

where the sign of $s_n^{\pm}(f)$ is chosen according as $(-1)^m = \pm 1$.

By linearity, the identity in the theorem holds for any $f \in S_{2k}$ if we replace the left-hand side by:

$$\sum_{l \ge 1} r_m(f|T_l) q^l \in S_{2k}.$$

Therefore the theorem can be interpreted as giving explicit formulas for the periods of $f|T_l$ in terms of the periods of f, divisor functions, and Bernoulli numbers. A formula for the action of Hecke operators on period polynomials has been previously obtained by Zagier [Za90, Theorem 2] by a different method.

For $m = 0$, this is the identity quoted in the introduction. A similar identity was proved by Manin in [Ma73] (the Coefficients theorem), expressing the coefficients of $r_0(f)(G_{2k} - f)$, for f a Hecke cusp form, in terms of the even periods of f and certain arithmetic functions. Here the arithmetic functions are explicitly identified in terms of coefficients of modular forms.

As a consequence of the Coefficients theorem, Manin shows that when S_{2k} is one dimensional (that is in weights 12, 16, 18, 20, 22, 26) the coefficients a_n of the unique Hecke form f satisfy the Ramanujan-type congruences:

$$a_n \equiv \sigma_{2k-1}(n) \pmod{N_{2k}}$$

where N_{2k} is the gcd of the numerators of $r_n(f)/r_0(f)$, $2 \le n \le w-2$, n even. Numerically:

$$N_{12} = 691; \quad N_{16} = 3617; \quad N_{18} = 43867;$$

$$N_{20} = 283 \cdot 617; \quad N_{22} = 131 \cdot 593; \quad N_{26} = 657931.$$

In all six cases, N_{2k} is also a divisor of the numerator of B_{2k}.

The congruences also follow from our formula, after we modify it using the Kohnen-Zagier relation among even periods. Using the linear combination F_m of products $G_{j+1}G_{\tilde{j}+1}$ defined by (A.2), the identity from the introduction becomes, after using equation (2.1) to simplify the coefficient of G_{2k}:

$$r_0(f)f = \frac{2}{3}\left[-\frac{s_{w+1}^{+}}{w+1} G_{2k} - \sum_{\substack{n=2 \\ n \text{ even}}}^{w} \binom{w}{n} r_n(f) F_n \right].$$

Now we collect all the terms containing $r_0(f)$ and $r_w(f) = -r_0(f)$ on the right-hand side, using the identity

$$F_w = \frac{w+3}{2(w+1)} G_{2k},$$

which is the case $m = w$ of the Eisenstein series identity (A.3). We finally obtain

$$r_0(f)(G_{2k} - f) = \frac{2}{3} \sum_{\substack{n=2 \\ n \text{ even}}}^{w-2} \frac{1}{n+1} \binom{w}{n} r_n(f)[(n+1)F_n - G_{2k}].$$

Assuming S_{2k} is one dimensional, the Ramanujan-type congruences now follow as in Manin's proof, from the fact that the numbers N_{2k} above divide the numerators of all the ratios $r_n(f)/r_0(f)$, $2 \le n \le w-2$ (n even). Indeed it is easy to check that the denominators of the coefficients of the forms F_n and G_{2k} are coprime with N_{2k}, as they involve only fractions of type B_m/m, for $m \le 2k$.

6. Modular forms of half integral weight

In this section we assume $k \ge 2$ is even. To prove the decomposition of half integral forms from the introduction, we consider an explicit version of the Shimura lift $\mathcal{S}_1 : S_{k+1/2} \to S_{2k}$, defined as follows:

$$\mathcal{S}_1\Big[\sum_{n\ge1} c(n) q^n\Big] = \sum_{n\ge1}\Big[\sum_{d|n} d^{k-1} c(n^2/d^2)\Big] q^n. \tag{6.1}$$

This map commutes with Hecke operators, therefore if $g \in S_{k+1/2}$ and $f \in S_{2k}$ are eigenforms in Shimura correspondence (f normalized) we have

$$\mathcal{S}_1(g) = c(1) f.$$

It follows that the adjoint of $\mathcal{S}_1$ with respect to the Petersson inner product on the two spaces is the linear map $\mathcal{S}_1^* : S_{2k} \to S_{k+1/2}$ defined by

$$\mathcal{S}_1^*(f) = \frac{\overline{c(1)}(f,f)}{(g,g)} g, \quad f \text{ Hecke cusp form} \tag{6.2}$$

with $c(1)$ the first Fourier coefficient of g.

Kohnen and Zagier have computed the image of R_n, n even, under the map $\mathcal{S}_1^*$. By using Rankin's method for Rankin-Cohen brackets due to Zagier [Za77], it is shown in [KZ84, Sec.2.1] that the form of half integral weight $\mathcal{S}_1^*(R_n)$, $0 \le n \le w$ even, is a multiple of a Rankin-Cohen bracket of the weight 1/2 theta series and an Eisenstein series. We use this result in the following form:

$$\mathcal{S}_1^*((-1)^{n/2} R_n) = -(-1)^{k/2} 2^{3k-1} \binom{w}{n}^{-1} \theta_n \quad (0 < n < w,\ n \text{ even}), \tag{6.3}$$

where $\theta_n = -\theta_{w-n}$ is given by (1.3). Since there is a typo in the normalization of formula (6.3) in [KZ84, p.219], in Appendix B we give a different proof. Namely, we show that the Shimura lift $\mathcal{S}_1$ maps the Rankin-Cohen bracket appearing in the definition of θ_n to another Rankin-Cohen bracket of weight $2k$.

Applying the map $\mathcal{S}_1^*$ to the identity of Theorem 1 proves the following explicit construction of the adjoint Shimura lift in terms of periods. Formula (1.2) given in the introduction follows immediately from the theorem and (6.2).

Theorem 6.1. *For all $f \in S_{2k}$ (k even) we have:*

$$(-1)^{k/2} 2^{-k} \mathcal{S}_1^*(f) = \frac{2}{3} \sum_{\substack{n=2 \\ n \text{ even}}}^{w} s_n^-(f)\theta_n.$$

The formula is normalized such that the first Fourier coefficient on both sides is the central period $r_{k-1}(f)$.

To emphasize the explicit nature of the formula in the theorem, we give the Fourier coefficients $e_{k,n}(D)$ of θ_n. Denoting by $d_{k,n}(a,b,c)$ the coefficient of x^n in $(ax^2+bx+c)^{k-1}$, the computation of the Fourier coefficients of the Rankin-Cohen bracket (1.3) gives

$$\begin{aligned} e_{k,n}(D) = \sum_{\substack{|b|\le\sqrt{D} \\ b\equiv D \mod 2}} d_{k,n}\left(\frac{D-b^2}{4}, b, -1\right)\sigma_{k-1-n}\left(\frac{D-b^2}{4}\right) - \\ -D^{k/2}/k, \ [\text{if } D \text{ is square and } n=k-2\ ]. \end{aligned} \tag{6.4}$$

where the second line has to be taken into account only if the conditions inside the bracket are met. In this formula, for D a square we define $\sigma_{N-1}(0) = -\frac{B_N}{2N}$ (the constant term of the Eisenstein series G_N).

We point out that θ_n can be interpreted as an indefinite theta series. When viewed as homogeneous polynomials in a,b,c, the functions $d_{k,n}(a,b,c)$ for $0 \le n \le w$ form a basis for the space of spherical polynomials attached to the indefinite quadratic form b^2-4ac. One can easily check from the formulas for $e_{k,n}(D)$ that

$$\theta_n(z) = \sum_{\substack{a,b,c\in\mathbb{Z} \\ a>0>c}} d_{k,n}(a,b,c)q^{b^2-4ac} + (\text{correction for } b^2-4ac \text{ a square}),$$

where the correction term is explicit, and can be viewed as a contribution due to the boundary of the cone determined by $a>0>c$. This formula is analogous to the usual construction of theta series with spherical polynomials attached to positive definite quadratic forms, with the difference that the usual summation over a lattice has to be truncated in order to make the series converge. For a general theory of indefinite theta series that includes the present example as a special case, see [Zw02].

Remark 6.2. Using the map $\mathcal{S}_1$, we obtain a simple formula for the coefficient $a(p)$ of a Hecke form $f \in S_{2k}$, when p is prime. Assuming $r_{k-1}(f) \neq 0$ we have

$$a(p) = p^{k-1} + \frac{4}{3} \sum_{\substack{n=2 \\ n \text{ even}}}^{k-2} \frac{s_n^-(f)}{r_{k-1}(f)} e_{k,n}(p^2).$$

Remark 6.3. According to a conjecture of Zagier, the central period $r_{k-1}(f)$ is nonzero for all Hecke forms f (recall that k is even, and that $r_{k-1}(f)$ is proportional to the central value $L(f,k)$). Zagier's conjecture is equivalent to the statement that $\mathcal{S}_1^*$ (or $\mathcal{S}_1$) is an isomorphism. Theorem 6.1 and relation (6.2) show that a failure of Zagier's conjecture would imply an *explicit* linear relation among the Rankin-Cohen brackets θ_n.

The nonvanishing of $L(f,k)$ is implied by the conjecture that the Hecke cusp forms in S_{2k} form a single Galois orbit. For $k \leq 1000$, the latter statement was checked in [CFW], by checking Maeda's conjecture on the irreducibility of the characteristic polynomial of the Hecke operator T_2 .

7. Numerical examples

We have checked numerically the identities in Theorems 5.1 and 6.1 (and implicitly in Theorem 1.1), when the spaces S_{2k} or $S_{k+1/2}$ have small dimension (≤ 3). The Fourier coefficients of Hecke forms in S_{2k} are available in the computer algebra systems MAGMA or SAGE, as well as on the webpage of William Stein.

For the convenience of the reader who would like to reproduce the computations, we give the values of s_n^-/r_{k-1}, n even, in the one dimensional cases $k = 6, 8, 10$, and in the two dimensional case $k = 12$. Since $s_{w-n}^- = -s_n^-$, we restrict to the range $2 \leq n \leq k-2$, n even.

k	s_2^-/r_{k-1}	s_4^-/r_{k-1}	s_6^-/r_{k-1}	s_8^-/r_{k-1}	s_{10}^-/r_{k-1}
6	$24/5$	$23/5$			
8	$-936/35$	$-1382/35$	$-652/35$		
10	408	$4181/6$	$27835/42$	273	
12	$\frac{77a-893568}{55}$	$\frac{11729a-136025472}{4620}$	$\frac{1131a-13094944}{385}$	$\frac{4643a-53630808}{1980}$	$\frac{1034a-11920026}{1155}$

When $k = 12$, we denote by a a fixed root of the characteristic polynomial $x^2 - 1080x - 20468736$ of T_2 acting on S_{24}. It is the periods of the Hecke form $q + aq^2 + \ldots$ that are listed for $k = 12$ (the other Hecke form in S_{24} is conjugate to this one, and has conjugate period ratios).

When S_{2k} is one dimensional, the period ratios are given in [Ma73]. In general, we computed the period ratios by the method described in the last section of [Za91], using the coefficients of the Rankin-Cohen brackets $X_{m,n}$, and the known coefficients of Hecke forms $f \in S_{2k}$. The latter form a single Galois orbit for $k \leq 1000$, so only one check is needed for each weight.

Appendix A

Here we prove the relations satisfied by periods of Eisenstein series, which yield interesting Bernoulli number identities. The method used is sketched in [Sk93] and [Za93], but the resulting identities are simpler to state in terms of the sums s_n^-, s_n^+ already introduced in Section 2.

To define periods of Eisenstein series, we use the connection with special values of L-functions mentioned in Remark 2.1. Since $L(G_{2k}, s) = \zeta(s)\zeta(s-2k+1)$, one easily finds:

$$r_j(G_{2k}) = \frac{1}{2}\frac{B_{j+1}}{j+1}\frac{B_{\widetilde{j}+1}}{\widetilde{j}+1}, \quad \text{for } 0 < j < w,\ j \text{ odd };$$

$$r_n(G_{2k}) = 0 \text{ if } 0 < n < w,\ n \text{ even}; \quad r_0(G_{2k}) = -r_w(G_{2k}) = -\frac{w!}{(2\pi i)^{w+1}}\zeta(w+1).$$

For Eisenstein series, only the sums $s_n^-(G_{2k})$ are interesting, and the 'Eichler-Shimura relations' for Eisenstein series are, for $0 \le n \le w$:

(A.1)

$$s_n^-(G_{2k}) + s_{\widetilde{n}}^-(G_{2k}) = (1-\delta_{n,0}-\delta_{n,w})r_n(G_{2k}) - \frac{B_{2k}}{4k}\left(\frac{(-1)^n n!\widetilde{n}!}{(w+1)!} + \frac{2k}{(n+1)(\widetilde{n}+1)}\right).$$

This is the content of the Proposition in Sec. 2 of [Za91]; another proof is given in [Kr87, p. 69] (the formula has a typo there, but it is corrected in the proof).

Following an idea sketched in [Sk93] and [Za93], we show that (A.1) extends to an identity relating products of Eisenstein series $G_{j+1}G_{\widetilde{j}+1}$ and G_{2k}. This Eisenstein series identity, which we will use again in Section 5, follows from the Eichler-Shimura relations for cusp forms via an identity of Rankin. Set

$$H_j = G_{j+1}G_{\widetilde{j}+1} + \frac{\delta_{j,1}+\delta_{\widetilde{j},1}}{4w\pi i}G_w', \quad 1 \le j \le w-1,\ j \text{ odd},$$

and let also $H_e = 0$ for e even. These products appear in the formula for $\widetilde{r_0(f)f}$ from the introduction, where the presence of the second term when $j = 1$ or $\widetilde{j} = 1$ is explained. Rankin's identity, also proved in [KZ84], states that for $f \in S_{2k}$ a Hecke form:

$$(f, H_j) = c_w r_w(f) r_j(f), \quad 0 < j < w,\ j \text{ odd},$$

with c_w a nonzero constant depending only on w. As $r_w(f) \neq 0$ for f a Hecke form (see Remark 2.1), from this and the Eichler-Shimura relations (ES.odd) we deduce that $F_m + F_{\widetilde{m}} - H_m$ $(0 \le m \le w)$ is orthogonal to all Hecke cusp forms, where we denote by F_m the linear combination:

$$F_m = \sum_{\substack{j=1 \\ j \text{ odd}}}^{m} \binom{m}{j} H_j. \tag{A.2}$$

Therefore the modular form $F_m + F_{\widetilde{m}} - H_m$ is a multiple of the Eisenstein series G_{2k}. The exact multiple can be found by looking at the coefficient of q on both sides, and using two Bernoulli number identities that we prove below. Hence we have the following Eisenstein series identity:

$$F_m + F_{\widetilde{m}} - H_m = \left(\frac{k}{(m+1)(\widetilde{m}+1)} + (-1)^m \frac{m!\widetilde{m}!}{2(w+1)!} \right) G_{2k}, \tag{A.3}$$

for $0 \le m \le w$. The constant term of this identity gives relation (A.1), therefore the identity is equivalent–via Rankin's identity–to the Eichler-Shimura relations for odd periods of cusp forms *and* G_{2k}.

Coming back to the equality of the coefficients of q on both sides of (A.3), it reduces to two Bernoulli number identities (the first is identity (3.2)):

$$\sum_{\substack{n=0 \\ n \text{ even}}}^{N} \binom{N}{n} B_n = \frac{N}{2} + B_N + \delta_{1,N}$$

(recall $B_N = 0$ if $N > 1$ is odd and $B_1 = -1/2$) and

$$\sum_{\substack{j=1 \\ j \text{ odd}}}^{m} \binom{m}{j} \frac{B_{\widetilde{j}+1}}{\widetilde{j}+1} + \sum_{\substack{j=1 \\ j \text{ odd}}}^{\widetilde{m}} \binom{\widetilde{m}}{j} \frac{B_{\widetilde{j}+1}}{\widetilde{j}+1} = \frac{(-1)^{m+1} m!\widetilde{m}!}{(w+1)!} + \frac{1}{2}(\delta_{0,m} + \delta_{w,m})$$

for $0 \le m \le w$ ($w \ge 2$ even). Both identities follow from the well-known property $B_n(1-x) = (-1)^n B_n(x)$ of the Bernoulli polynomials $B_n(x) = \sum_{i=0}^{n} \binom{n}{i} B_i x^{n-i}$. The first identity is obtained by setting $x = 1$, while to prove the second we write it as:

$$\sum_{j=m}^{w} \binom{w+1}{j+1} \binom{j}{m} B_{j+1} + \sum_{j=\widetilde{m}}^{w} \binom{w+1}{j+1} \binom{j}{\widetilde{m}} B_{j+1} = (-1)^{m+1}$$

where we included the terms involving B_1 to get rid of the delta functions. Denoting by $C_w(m)$ the left-hand side, we easily sum:

$$\sum_{m=0}^{w} C_w(m) X^m Y^{w-m} = (X+Y)^w \left[B_{w+1}\left(\frac{X}{X+Y}\right) + B_{w+1}\left(\frac{Y}{X+Y}\right) \right] - \frac{X^{w+1} + Y^{w+1}}{X+Y}.$$

Since w is even, the above property of Bernoulli polynomials implies that the expression inside the bracket vanishes. The coefficient of $X^m Y^{w-m}$ in the remaining part is $(-1)^{m+1}$, thus proving the claim.

Remark A.1. Another version of the Eisenstein series identities (A.3) was proved in [Pa06], using a different method.

Appendix B

In the appendix we assume that k is even. As mentioned in the introduction, it is shown in [KZ84] that

$$\mathcal{S}_1^*(R_n) = c_{k,n}[\theta(\tau), G_{k-n}(4\tau)]_{n/2}, \quad 2 \le n \le k-2, n \text{ even},$$

where $\theta(z) = \sum_{n\in\mathbb{Z}} q^{n^2}$ is the weight 1/2 theta series. We check here that the constant is

$$c_{k,n} = (-1)^{k/2+n/2} 2^{3k-1} \binom{w}{n}^{-1} (w/2-n/2)!^{-1}$$

thus checking, via (1.3), the constant in formula (6.3).

The following proposition could also be extracted from the computations in [Co76] (see Lemma 4.2 there). Since the proof is simpler in the present situation, we give it below.

Proposition B.1. *If* $2 \leq n \leq k-2$ *is even, then*

$$\frac{1}{(w/2-n/2)!}\mathcal{S}_1\left([\theta(\tau), G_{k-n}(4\tau)]_{n/2}\right) = \frac{1}{n!}[G_{k-n}, G_{k-n}]_n.$$

Before giving the proof, we show how the proposition implies the formula for $c_{k,n}$. By acting with $\mathcal{S}_1 \circ \mathcal{S}_1^*$ on Hecke cusp forms, and using the identity [KZ81]

$$\frac{c(1)^2(f,f)}{(g,g)(-1)^{k/2}2^k} = r_{k-1}(f),$$

it follows that

$$\mathcal{S}_1 \circ \mathcal{S}_1^* = 2^k \rho_{k-1},$$

where the map ρ_{k-1} has been defined in Section 5. Applying this identity to R_n for $2 \leq n \leq k-2$, n even, and taking into account formula (5.1) for $\rho_{k-1}(R_n) = X_{n,k-1}$, and the Proposition, we find the value $c_{k,n}$ above. Note that this argument proves the formula for $\mathcal{S}_1^*(R_n)$ without assuming it to be proportional to the Rankin-Cohen bracket, in case that $\mathcal{S}_1^*$ is an isomorphism (see Remark 6.3).

From the proof it will be clear that the Proposition holds if G_{k-n} is replaced by any Hecke form of full level and weight $k-n$.

Proof of Proposition. We use the definition of $\mathcal{S}_1$ in (6.1) to show that the Fourier coefficients of the left-hand side match those of the right-hand side. The prototype of this calculation is already present in [KZ81, p.186], where it is shown that

$$\mathcal{S}_1[\theta(\tau)G_k(4\tau)] = G_k^2,$$

after extending the map $\mathcal{S}_1$ to the whole of $M_{k+1/2}$. This identity can be interpreted as the case $n=0$ of the proposition.

First we assume that $n \neq k-2$, so that the Rankin-Cohen bracket is given by equation (5.2). For $m \geq 1$, the mth Fourier coefficient $C_R(m)$ of the right-hand side of the formula to prove is:

$$C_R(m) = \sum_{m_1+m_2=m} \sum_{i=0}^{n} (-1)^{n-i} \binom{k-1}{i}\binom{k-1}{n-i} m_1^i m_2^{n-i} \sigma_{k-1-n}(m_1)\sigma_{k-1-n}(m_2),$$

where m_1, m_2 take nonnegative values, and $\sigma_{k-1-n}(0)$ has been defined following equation (6.4).

The Dth Fourier coefficient of $[\theta, G_{k-n}(4\cdot)]_{n/2}/(w/2-n/2)!$ is equal to $-e_{k,n}(D)$, given in equation (6.4). By the definition (6.1) of $\mathcal{S}_1$, the mth Fourier coefficient $C_L(m)$ of the left-hand side of the formula to prove is given by:

$$C_L(m) = -\sum_{d|m} d^{k-1} \sum_{\substack{|b|\leq m/d \\ b\equiv m/d \mod 2}} d_{k,n}\left(\frac{m^2-b^2d^2}{4d^2}, b, -1\right) \sigma_{k-1-n}\left(\frac{m^2-b^2d^2}{4d^2}\right)$$

We replace the sum over b by making the change of variables:

$$m_1 = \frac{m-bd}{2}, \quad m_2 = \frac{m+bd}{2}.$$

Using that $d|m$ if and only if $d|(m_1, m_2)$, and that

$$d_{k,n}(a/d^2, b/d, c) = d^{-n} d_{k,n}(a,b,c)$$

we have:

$$C_L(m) = -\sum_{m_1+m_2=m} d_{k,n}(m_1m_2, m_2-m_1, -1) \sum_{d|(m_1,m_2)} d^{k-1-n} \sigma_{k-1-n}\left(\frac{m_1m_2}{d^2}\right).$$

The inner sum equals $\sigma_{k-1-n}(m_1)\sigma_{k-1-n}(m_2)$, by the multiplicativity of the divisor function σ_{k-1-n}. The equality of $C_L(m)$ and $C_R(m)$ is now a consequence of the combinatorial identity:

$$d_{k,n}(m_1m_2, m_2-m_1, -1) = \sum_{i=0}^{n} (-1)^{i+1} \binom{k-1}{i}\binom{k-1}{n-i} m_1^i m_2^{n-i},$$

which follows from the definition of $d_{k,n}(m_1m_2, m_2-m_1, -1)$ as the coefficient of x^n in

$$[m_1m_2x^2 + (m_2-m_1)x - 1]^{k-1} = (m_1x+1)^{k-1}(m_2x-1)^{k-1}.$$

This finishes the proof of the proposition if $n \neq k-2$. For $n = k-2$, it is easy to check that the correction terms needed to define the Rankin-Cohen brackets involving G_2 match on the two sides of the identity in the proposition. □

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