SIMION STOILOW
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DOCTORAL THESIS

Contributions to the homogenization of the composite heterogeneous media

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Contents

COMPENDIUM

Abstract

In the last four decades, many studies have been dedicated to the theme of my thesis: mathematical modelling of the macroscopic behaviour of microscopically heterogeneous materials by the homogenization theory methods. Macroscopic scale analysis of such materials was initiated by Rayleigh, Maxwell, Einstein and continued by J.L. Lions, E. Sanchez-Palencia, H. I. Ene, L. Tartar, D. Ciorănescu, U. Hornung. Having important implications on the setting of transport and transmission problems in composite materials, they received attention not only from the mathematical community, but also from other scientific communities, including engineering, materials science and physics.

Keywords: Homogenization, heat conduction, first-order jump interface, two-scale convergence, fractured porous media, Stokes flow, Beavers-Joseph interface, partially fissured media, diffusion problem.

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1 Mathematical background

The first part of Chapter 1 is conceived as an introduction to the basic notions and results that are used throughout this work. We refer the reader for instance to Ciorănescu and Donato [11], Lukkassen, Nguetseng and Wall [25], Allaire [3], Poliševschi [33] and Girault and Raviart [14].

In the second part of this chapter, two classical homogenization problems are reconsidered. The first one is the asymptotic behaviour of an elastic structure, containing $\varepsilon-$periodically distributed fissures. Following [36] (Ch.6), we consider the boundary value problem for a fissured elastic body subject to one-side constraint without friction, that is, the two lips of the fissure may be open but they cannot overlap and if the fissure is closed at a certain point then the forces act in the normal direction. We determine the variational formulation of this problem and prove that it is well posed. Then, the homogenized system is obtained by the formal method of double-scale asymptotic expansions. Finally, we consider the Dirichlet problem for $\varepsilon-$periodically layered materials. Their foundations were laid down by Murat and Tartar in [28]. Here, following [11](Ch.5, Sec.5.4), we present the formulae of the effective coefficients for the $\varepsilon-$periodically layered materials, one of the main achievements of the homogenization theory, besides the Darcy's law in fluid mechanics and the modelling of composite materials in elasticity.

2 Heat transfer model for a two-component media with first-order jump interfaces

In the Chapter 2 we study the asymptotic behaviour ($\varepsilon \to 0$) of the temperature governed by the heat transfer problem in the ε-periodic structure introduced by [33], which is a realistic periodic structure composed of two connected components separated by an interface on which the heat flux is continuous and the temperature subjects to a first-order jump condition, namely the flow of heat is proportional to the jump of the temperature field(see [9] for a physical justification of the model). The influence of interfacial resistances on the macroscopic conductivity has been investigated analytically and experimentally in [5] (and the references there). Until that time, thermal barrier resistances were not explicitly introduced, but the interphase heat transfer was supposed to be proportional to the temperature difference (see H.S. Carslaw and J.C. Jaeger [9] for a physical justification of the model). Also, macroscopic heat transfer in periodic composite materials using asymptotic developments (see [7] and [36]) was studied in [4] with the classical boundary conditions between the constituents (e.g. continuity of the temperature and normal flux). Therefore, J.L. Auriault and H. Ene (see [5]) presented the case of periodic composite media, made up of two connected solids with conductivities of the same order of magnitude, separated by a thermal barrier resistance. In their analysis, the macroscopic description of heat transfer has been shown to strongly depend on the relative value of the barrier resistance with respect to the resistance of the components. In that paper, using a method of double-scale asymptotic developments, five different macroscopic models are obtained, for different values of paramater r: the first there types were one-temperature field models whereas the last two were two-temperature filed models. A rigorous treatment of a special case of heat conduction in the presence of interfacial barrier, without any connectivity assumptions, was done by R. Lipton in [22]. The problem of heat conduction in a composite material containing two components, one connected and the other disconnected, separated by a contact surface, was treated by P. Donato and S. Monsurro in [12]. For different values of r ($r > 1$ and $-1 < r \leq 1$), the paper presented the macroscopic equations, using Tartar's method of oscillating test function (see [38]).

Now, returning to our problem, we set the reference conductor (where the conductivity is of unity order with respect to ε) in the ambient component, the only one which is reaching the boundary of the domain (in this way, we avoid the $H^1_{loc}(\Omega)$ extension constructed in [1]). The second component contains the core material of the structure, where the conductivity is set of $\varepsilon^{2\beta}$-order, with $\beta \in [0, 1]$. Let us remark here that for $\beta > 1$ the temperature becomes singular with respect to ε. On the interface between the reference conductor and the core material we set

ε^r to be the order of the transmission coefficient in the jump condition. A counterexample of [18] shows that the temperature cannot be asymptotically finite for $r > 1$ (unless one reduces the heat source inside the inclusions); furthermore, we restrain to $r \in (-1, 1]$. An important property of our structure is the existence of a bounded extension operator, similar to that introduced in [10] for the case of isolated inclusions. Also, for the a priori estimates of the temperature on each structures of the regarded domain, we use certain inequalities obtained by D. Poliševski in [33]. In order to derive the macroscopic laws and the effective coefficients in all regular cases we apply the two-scale convergence technique of the periodic homogenization theory (see [3], [30] and [11]). In the present framework, it turns out that there are exactly six distinct cases, given by $\beta = 0$, $\beta \in (0, 1)$ or $\beta = 1$ and $r = 1$ or $r \in (-1, 1)$. We determine in each case the specific local-periodic problems. The solutions of these specific problems define the effective coefficients which allow the identification of the homogenized systems which uniquely define the asymptotic behaviour of the temperature. We have to mention that besides heat conduction there are many other phenomena which lead to asymptotic problems similar to the one studied here; for instance, the pressure distribution in a partially fractured porous medium, the dispersion of a concentration of solute in a domain with highly different diffusivities or the diffusion of a dissolved chemical in a fluid flowing through a porous medium with highly different permeabilities. Thus, in such different frameworks, this problem has already been treated when the core material is composed of isolated inclusions for $\beta = 0$ and $r = 0$ in [22] and for $\beta = 0$ and various values of r, especially $r = 1$, which corresponds to the case when the transmission coefficient balance the total measure of the interface, in [5], [31], [8], [18], [27] and [12]. For our geometry, only the case $\beta = 0$ and $r = 1$ have already been studied in [13].

The results of Chapter 2 can also be found in the paper [34], which has been accepted for publication.

2.1 The heat conduction problem

Let Ω be an open connected bounded set in $\mathbb{R}^N$ $(N \geq 3)$, locally located on one side of the boundary $\partial\Omega$, a Lipschitz manifold composed of a finite number of connected components. Ω consists of two connected components disposed $\varepsilon-$periodic, with $\varepsilon \in (0, 1)$. For convenience, the periodicity will be described using the cube $Y = (0, 1)^N$, as follows:

Let Y_a be a Lipschitz open connected subset of the unit cube $Y = (0, 1)^N$. We assume that $Y_b = Y \setminus \overline{Y}_a$ has a locally Lipschitz boundary and that the intersections of ∂Y_b with ∂Y are reproduced identically on the opposite faces of the cube, denoted for every $i \in \{1, 2, \dots, N\}$ by

$$\Sigma^{+i} = \{y \in \partial Y : \ y_i = 1\} \text{ and } \Sigma^{-i} = \{y \in \partial Y : \ y_i = 0\}, \tag{2.1}$$

with the property that

$$\overline{Y}_b \cap \Sigma^{\pm i} \subset\subset \Sigma^{\pm i}, \quad \forall i \in \{1, 2, ..., N\}. \tag{2.2}$$

We assume that repeating Y by periodicity, the reunion of all the $\overline{Y}_a$ parts is a connected domain in $\mathbb{R}^N$ with a locally C^2 boundary; we denote it by $\mathbb{R}^N_a$ and further $\mathbb{R}^N_b = \mathbb{R}^N \setminus \mathbb{R}^N_a$. Obviously, the origin of the coordinate system can be set such that there exists $R > 0$ with the property $B(0, R) \subseteq \mathbb{R}^N_a$.

Now, we define the two components of Ω with the following sets of indices:

For any $\varepsilon \in (0, 1)$ we denote

$$\mathbb{Z}_\varepsilon = \{k \in \mathbb{Z}^N : \ \varepsilon k + \varepsilon Y \subseteq \Omega\}, \tag{2.3}$$

$$I_\varepsilon = \{k \in \mathbb{Z}_\varepsilon : \ \varepsilon k \pm \varepsilon e_i + \varepsilon Y \subseteq \Omega, \ \forall i \in \{1, ..., N\}\}, \tag{2.4}$$

where e_i are the unit vectors of the canonical basis in $\mathbb{R}^N$.

The core component of our structure is defined by

$$\Omega_{\varepsilon b} = \operatorname{int} \left(\bigcup_{k \in I_\varepsilon} (\varepsilon k + \varepsilon \overline{Y}_b) \right) \tag{2.5}$$

and the reference conductor by

$$\Omega_{\varepsilon a} = \Omega \setminus \overline{\Omega}_{\varepsilon b}. \tag{2.6}$$

The interface between the two components is denoted by

$$\Gamma_\varepsilon = \partial\Omega_{\varepsilon a} \cap \partial\Omega_{\varepsilon b} = \partial\Omega_{\varepsilon b}. \tag{2.7}$$

Finally, let us remark that all the boundaries are at least locally Lipschitz, $\Omega_{\varepsilon a}$ is connected and $\Omega_{\varepsilon b}$ can be, in particular, connected too.

We introduce the Hilbert space

$$H_\varepsilon = \left\{ v \in L^2(\Omega) : v\Big|_{\Omega_{\varepsilon a}} \in H^1(\Omega_{\varepsilon a}), \ v\Big|_{\Omega_{\varepsilon b}} \in H^1(\Omega_{\varepsilon b}), \ v = 0 \text{ on } \partial\Omega \right\} \tag{2.8}$$

endowed with the scalar product

$$(u,v)_{H_\varepsilon} = \int_{\Omega_{\varepsilon a}} \nabla u \nabla v + \varepsilon^2 \int_{\Omega_{\varepsilon b}} \nabla u \nabla v + \varepsilon \int_{\Gamma_\varepsilon} [u][v], \tag{2.9}$$

where $[u] = \gamma_{\varepsilon b} u - \gamma_{\varepsilon a} u$ and $\gamma_{\varepsilon a} u$, $\gamma_{\varepsilon b} u$ are the traces of u on Γ_ε defined in $H^1(\Omega_{\varepsilon a})$ and $H^1(\Omega_{\varepsilon b})$, respectively.

From now on, let us denote $\Gamma := \partial Y_a \cap \partial Y_b$. Obviously,

$$\bigcup_{k \in \mathbb{Z}_\varepsilon} (\varepsilon k + \varepsilon \Gamma) \subseteq \Gamma_\varepsilon \tag{2.10}$$

and if ν is the normal on Γ(exterior to Y_a) and $x \in (\varepsilon k + \varepsilon \Gamma)$ for some $k \in \mathbb{Z}_\varepsilon$ then

$$\nu^\varepsilon(x) = \nu\left(\left\{\frac{x}{\varepsilon}\right\}\right) \tag{2.11}$$

where $\left\{\frac{x}{\varepsilon}\right\}$ is formed by the fractional parts of the components of $\varepsilon^{-1}x$.

For any $\varepsilon \in (0,1)$ we introduce the transmission factor $h^\varepsilon(x) = h(x/\varepsilon)$ and the symmetric conductivities $a^\varepsilon_{ij}(x) = a_{ij}(x/\varepsilon)$ and $b^\varepsilon_{ij}(x) = b_{ij}(x/\varepsilon)$, where h, a_{ij} and b_{ij} belong to $L^\infty_{per}(Y)$ and have the property that there exists $\delta > 0$ such that

$$h \geq \delta, \quad \text{a.e.} \quad \text{on } Y, \tag{2.12}$$

$$a_{ij}\xi_j\xi_i \geq \delta\xi_i\xi_i \quad \text{and} \quad b_{ij}\xi_j\xi_i \geq \delta\xi_i\xi_i, \quad \forall \xi \in \mathbb{R}^N, \quad \text{a.e.} \quad \text{on } Y. \tag{2.13}$$

Considering that $\beta \in [0,1]$, $r \in (-1;1]$ and $f \in L^2(\Omega)$ are also given, we look for the temperature u^ε which satisfies the heat conduction equations

$$-\frac{\partial}{\partial x_i}\left(a^\varepsilon_{ij}\frac{\partial u^\varepsilon}{\partial x_j}\right) = f \quad \text{in } \Omega_{\varepsilon a}, \tag{2.14}$$

$$-\varepsilon^{2\beta}\frac{\partial}{\partial x_i}\left(b^\varepsilon_{ij}\frac{\partial u^\varepsilon}{\partial x_j}\right) = f \quad \text{in } \Omega_{\varepsilon b}, \tag{2.15}$$

with the following transmission and boundary conditions

$$a^\varepsilon_{ij}\frac{\partial u^\varepsilon}{\partial x_j}\nu^\varepsilon_i = \varepsilon^{2\beta} b^\varepsilon_{ij}\frac{\partial u^\varepsilon}{\partial x_j}\nu^\varepsilon_i = \varepsilon^r h^\varepsilon\left(\gamma_{\varepsilon b}u^\varepsilon - \gamma_{\varepsilon a}u^\varepsilon\right) \text{ on } \Gamma_\varepsilon, \tag{2.16}$$

$$u^\varepsilon = 0 \text{ on } \partial\Omega. \tag{2.17}$$

The variational formulation of the problem (2.14)-(2.17) is the following:

To find $u^\varepsilon \in H_\varepsilon$ such that

$$a_\varepsilon(u^\varepsilon, v) := \int_{\Omega_{\varepsilon a}} a^\varepsilon_{ij}\frac{\partial u^\varepsilon}{\partial x_j}\frac{\partial v}{\partial x_i} + \varepsilon^{2\beta}\int_{\Omega_{\varepsilon b}} b^\varepsilon_{ij}\frac{\partial u^\varepsilon}{\partial x_j}\frac{\partial v}{\partial x_i} + \varepsilon^r \int_{\Gamma_\varepsilon} h^\varepsilon [u^\varepsilon][v] = \int_\Omega f v, \quad \forall v \in H_\varepsilon. \tag{2.18}$$

Theorem 1. *By using the Lax-Milgram Theorem in an appropriate way, we prove that, for any $\varepsilon \in (0,1)$ there exists a unique $u^\varepsilon \in H_\varepsilon$, solution of the problem (2.18).*

2.2 A priori estimates of the temperature

Hereafter, for any $u \in H^1(\Omega_{\varepsilon\alpha})$, $\alpha \in \{a,b\}$, we use the notations

$$\widehat{u}^\varepsilon_\alpha = \begin{cases} u & \text{in } \Omega_{\varepsilon\alpha}, \\ 0 & \text{in } \Omega - \Omega_{\varepsilon\alpha}, \end{cases} \qquad \widehat{\nabla u}^\varepsilon_\alpha = \begin{cases} \nabla u & \text{in } \Omega_{\varepsilon\alpha}, \\ 0 & \text{in } \Omega - \Omega_{\varepsilon\alpha}. \end{cases} \tag{2.19}$$

From the a priori estimates of u^ε, solution of (2.18), for any $\beta \in [0,1]$ and $r \in (-1,1]$, we obtain the main compactness result:

Theorem 2. *For every $\beta \in [0,1]$ and $r \in (-1,1]$ there exists $u_a \in H^1_0(\Omega)$, $\eta_a \in L^2\Big(\Omega; \widetilde{H}^1_{per}(Y_a)\Big)$ and $u_b \in L^2(\Omega, L^2_{per}(Y_b))$ such that the following convergences hold on some subsequence*

$$\widehat{u}^\varepsilon_a \xrightarrow{2s} \chi_a u_a, \qquad \widehat{u}^\varepsilon_b \xrightarrow{2s} \chi_b u_b, \tag{2.20}$$

$$\widehat{\nabla u}^\varepsilon_a \xrightarrow{2s} \chi_a\left(\nabla_x u_a + \nabla_y \eta_a(\cdot, y)\right), \tag{2.21}$$

where $\chi_\alpha : L^2(\Omega \times Y_\alpha) \to L^2(\Omega \times Y)$, $\alpha \in \{a,b\}$, denotes the straight prolongation with zero; sometimes it can be identified with the characteristic value of Y_α.

When $\beta = 0$ we find that u_b is independent of y, with $u_b \in H^1(\Omega)$. Moreover, there exists $\eta_b \in L^2\left(\Omega; \widetilde{H}^1_{per}(Y_b)\right)$ such that it holds

$$\widehat{\nabla u_b}^{\varepsilon} \xrightarrow{2s} \chi_b \left(\nabla_x u_b + \nabla_y \eta_b(\cdot, y)\right). \tag{2.22}$$

When $\beta \in (0,1)$ we find that u_b is independent of y, with $u_b \in L^2(\Omega)$.

When $\beta = 1$ it holds

$$\varepsilon \widehat{\nabla u_b}^{\varepsilon} \xrightarrow{2s} \chi_b \nabla_y u_b. \tag{2.23}$$

Now, for any $k \in \{1, 2, ..., N\}$, we define $\eta_{ak} \in \widetilde{H}^1_{per}(Y_a)$ as the unique solution of the local-periodic problem

$$-\frac{\partial}{\partial y_i}\left(a_{ij}\frac{\partial\left(\eta_{ak} + y_k\right)}{\partial y_j}\right) = 0 \quad \text{in } Y_a, \qquad a_{ij}\frac{\partial\left(\eta_{ak} + y_k\right)}{\partial y_j}\nu_i = 0 \quad \text{on } \Gamma. \tag{2.24}$$

The effective conductivity A is given by the classical formula

$$A_{ij} = \int_{Y_a} a_{ij} + a_{ik}\frac{\partial \eta_{aj}}{\partial y_k}\, dy, \qquad \forall i, j \in \{1, 2, ..., N\}. \tag{2.25}$$

Remark 3. The homogenized tensor A is symmetric and positively defined.

Remark 4. Similarly to (2.24), for any $k \in \{1, 2, ..., N\}$, we consider the local-periodic problem associated to b_{ij} in Y_b; its solution is denoted by $\eta_{bk} \in \widetilde{H}^1_{per}(Y_b)$. Correspondingly, we define the effective conductivity B_{ij} like in (2.25).

Next, we introduce the functions w_0 and w_1, which are the only solutions in $H^1_{per}(Y_b)$ of the following two problems:

$$-\frac{\partial}{\partial y_i}\left(b_{ij}\frac{\partial w_0}{\partial y_j}\right) = 1 \quad \text{in } Y_b, \qquad\qquad w_0 = 0 \quad \text{on } \Gamma \tag{2.26}$$

$$-\frac{\partial}{\partial y_i}\left(b_{ij}\frac{\partial w_1}{\partial y_j}\right) = 1 \text{ in } Y_b, \qquad -b_{ij}\frac{\partial w_1}{\partial y_j}\nu_i + h w_1 = 0 \text{ on } \Gamma. \tag{2.27}$$

Due to the existence of the first-order jump interface Γ_ε, there are two effective coefficients describing the microscopic transfer:

$$\widetilde{h} = \int_\Gamma h(y) ds, \tag{2.28}$$

$$\widetilde{w_1 h} = \int_\Gamma w_1(y) h(y) ds. \tag{2.29}$$

2.3 The homogenization process for $\beta = 0$ and $r = 1$

Using density arguments it follows that

$$((u_a, u_b), (\eta_a, \eta_b)) \in V_1 := \left[H^1_0(\Omega) \times H^1(\Omega)\right] \times \left[L^2(\Omega, \widetilde{H}^1_{per}(Y_\alpha))\right]$$

$\alpha \in \{a, b\}$, is solution of the problem: To find $((u_a, u_b), (\eta_a, \eta_b)) \in V_1$ satisfying

$$\sum_{\alpha \in \{a,b\}} \int_{\Omega \times Y_\alpha} \alpha_{ij}\left(\frac{\partial u_\alpha}{\partial x_j} + \frac{\partial \eta_\alpha}{\partial y_j}\right)\left(\frac{\partial \Phi_\alpha}{\partial x_i} + \frac{\partial \varphi_\alpha}{\partial y_i}\right) + \widetilde{h}\int_\Omega (u_b - u_a)(\Phi_b - \Phi_a) =$$

$$= \int_{\Omega \times Y} \left(\chi_a \Phi_a + \chi_b \Phi_b\right) f, \quad \forall\, ((\Phi_a, \Phi_b), (\varphi_a, \varphi_b)) \in V_1. \tag{2.30}$$

Theorem 5. *If u^ε is the solution of (2.18) then*

$$u^\varepsilon \xrightarrow{2s} \chi_a u_a + \chi_b u_b \tag{2.31}$$

where $(u_a, u_b) \in H^1_0(\Omega) \times H^1(\Omega)$ is the unique solution of

$$\int_\Omega A_{ij}\frac{\partial u_a}{\partial x_j}\frac{\partial \Phi_a}{\partial x_i} + \int_\Omega B_{ij}\frac{\partial u_b}{\partial x_j}\frac{\partial \Phi_b}{\partial x_i} + \widetilde{h}\int_\Omega (u_b - u_a)(\Phi_b - \Phi_a) =$$

$$= \int_\Omega \left(|Y_a|\Phi_a + |Y_b|\Phi_b\right) f, \quad \forall\, (\Phi_a, \Phi_b) \in H^1_0(\Omega) \times H^1(\Omega). \tag{2.32}$$

2.4 The homogenization process for $\beta = 0$ and $r \in (-1, 1)$

Due to the a priori estimates specific for this case, the compactness result is completed by:

Lemma 6. *There exists $u_b \in H^1(\Omega)$ and $\eta_b \in L^2\left(\Omega; \widetilde{H}^1_{per}(Y_b)\right)$ such that:*

$$\widehat{\nabla u_b^\varepsilon} \xrightarrow{2s} \chi_b \left(\nabla_x u_b + \nabla_y \eta_b(\cdot, y)\right). \tag{2.33}$$

Moreover, there exists $u \in H^1_0(\Omega)$ such that

$$u_a = u_b = u \text{ in } \Omega. \tag{2.34}$$

Using density arguments it follows that

$$(u, \eta_a, \eta_b) \in V_2 := H^1_0(\Omega) \times L^2(\Omega, \widetilde{H}^1_{per}(Y_a)) \times L^2(\Omega, \widetilde{H}^1_{per}(Y_b))$$

is solution of the problem: To find $(u, \eta_a, \eta_b) \in V_2$ satisfying

$$\sum_{\alpha \in \{a,b\}} \int_{\Omega \times Y_\alpha} \alpha_{ij} \left(\frac{\partial u}{\partial x_j} + \frac{\partial \eta_\alpha}{\partial y_j}\right)\left(\frac{\partial \Phi}{\partial x_i} + \frac{\partial \varphi_\alpha}{\partial y_i}\right) dxdy = \int_\Omega f\Phi \, dx, \ \ \forall (\Phi, \varphi_a, \varphi_b) \in V_2. \tag{2.35}$$

In this case the homogenization process is summarized by:

Theorem 7. *If u^ε is the solution of the problem (2.18) then*

$$u^\varepsilon \xrightarrow{2s} u \tag{2.36}$$

where $u \in H^1_0(\Omega)$ is the unique solution of the homogenized problem

$$\int_\Omega (A + B) \nabla u \nabla \Phi = \int_\Omega f\Phi, \ \ \forall \Phi \in H^1_0(\Omega), \tag{2.37}$$

and A, B are the effective positive matrices defined by (2.25).

2.5 The homogenization process for $\beta \in (0, 1)$ and $r = 1$

Using density arguments it follows that

$$(u_a, u_b, \eta_a) \in V_3 := H^1_0(\Omega) \times L^2(\Omega) \times L^2(\Omega, \widetilde{H}^1_{per}(Y_a))$$

is solution of the problem: To find $(u_a, u_b, \eta_a) \in V_3$ satisfying

$$\int_{\Omega \times Y_a} a_{ij} \left(\frac{\partial u_a}{\partial x_j} + \frac{\partial \eta_a}{\partial y_j}\right)\left(\frac{\partial \Phi_a}{\partial x_i} + \frac{\partial \varphi_a}{\partial y_i}\right) + \widetilde{h} \int_\Omega (u_b - u_a)(\Phi_b - \Phi_a) =$$

$$= \int_{\Omega \times Y} (\chi_a \Phi_a + \chi_b \Phi_b) f, \quad \forall (\Phi_a, \Phi_b, \varphi_a) \in V_3. \tag{2.38}$$

Theorem 8. *If u^ε is the solution of the problem (2.18) then*

$$u^\varepsilon \xrightarrow{2s} u + \frac{|Y_b|}{\widetilde{h}} \chi_b f, \tag{2.39}$$

where $u \in H^1_0(\Omega)$ is the unique solution of the Dirichlet problem

$$\int_\Omega A \nabla u \nabla \Phi = \int_\Omega f\Phi, \ \ \forall \Phi \in H^1_0(\Omega). \tag{2.40}$$

2.6 The homogenization process for $\beta \in (0, 1)$ and $r \in (-1, 1)$

Here it is the preliminary result specific to this case:

Lemma 9. *There exists $u \in H^1_0(\Omega)$ such that*

$$u_a = u_b = u \text{ in } \Omega. \tag{2.41}$$

Moreover, for any $\Phi \in \mathcal{D}(\Omega)$ and $\varphi_a \in \mathcal{D}(\Omega; C^\infty_{per}(Y_a))$, it holds

$$\int_{\Omega \times Y_a} a_{ij} \left(\frac{\partial u}{\partial x_j} + \frac{\partial \eta_a}{\partial y_j}\right)\left(\frac{\partial \Phi}{\partial x_i} + \frac{\partial \varphi_a}{\partial y_i}\right) dxdy = \int_\Omega f\Phi \, dx. \tag{2.42}$$

By density arguments we remark that

$$(u, \eta_a) \in V_4 := H_0^1(\Omega) \times L^2(\Omega, \widetilde{H}^1_{per}(Y_a))$$

is solution of the problem: To find $(u, \eta_a) \in V_4$ satisfying

$$\int_{\Omega\times Y_a} a_{ij} \left(\frac{\partial u}{\partial x_j} + \frac{\partial \eta_a}{\partial y_j}\right)\left(\frac{\partial \Phi}{\partial x_i} + \frac{\partial \varphi_a}{\partial y_i}\right) dxdy = \int_\Omega f\Phi dx, \;\; \forall\, (\Phi, \varphi_a) \in V_4. \tag{2.43}$$

Theorem 10. *If u^ε is the solution of the problem (2.18) then,*

$$u^\varepsilon \xrightarrow{2s} u, \tag{2.44}$$

where $u \in H_0^1(\Omega)$ is the unique solution of (2.40).

2.7 The homogenization process for $\beta = 1$ and $r = 1$

Using density arguments it follows that

$$(u_a, u_b, \eta_a) \in V_5 := H_0^1(\Omega) \times L^2(\Omega; H^1_{per}(Y_b)) \times L^2(\Omega, \widetilde{H}^1_{per}(Y_a))$$

is solution of the problem: To find $(u_a, u_b, \eta_a) \in V_5$ satisfying

$$\int_{\Omega\times Y_a} a_{ij} \left(\frac{\partial u_a}{\partial x_j} + \frac{\partial \eta_a}{\partial y_j}\right)\left(\frac{\partial \Phi}{\partial x_i} + \frac{\partial \varphi_a}{\partial y_i}\right) + \int_{\Omega\times Y_b} b_{ij} \frac{\partial u_b}{\partial y_j}\frac{\partial \varphi_b}{\partial y_i} +$$

$$+ \int_{\Omega\times\Gamma} h\,(u_b - u_a)\,(\varphi_b - \Phi) = \int_{\Omega\times Y_a} f\Phi + \int_{\Omega\times Y_b} f\varphi_b, \;\; \forall\, (\Phi, \varphi_b, \varphi_a) \in V_5. \tag{2.45}$$

Theorem 11. *If u^ε is the solution of (2.18) then*

$$u^\varepsilon \xrightarrow{2s} \left(\mid Y_a \mid + \widetilde{w_1 h}\right) u + w_1 \chi_b f \tag{2.46}$$

where $u \in H_0^1(\Omega)$ is the unique solution of the homogenized problem (2.40).

2.8 The homogenization process for $\beta = 1$ and $r \in (-1, 1)$

Lemma 12. *The limits u_a and u_b satisfy:*

$$u_a = u_b \;\; on \;\; \Omega \times \Gamma. \tag{2.47}$$

Moreover, for any $\Phi \in \mathcal{D}(\Omega)$ and $\varphi_\alpha \in \mathcal{D}(\Omega; C^\infty_{per}(Y_\alpha))$, $\alpha \in \{a, b\}$ such that

$$\varphi_b(x, y) = \Phi(x), \;\; \forall (x, y) \in \Omega \times \Gamma, \tag{2.48}$$

we have:

$$\int_{\Omega\times Y_a} a_{ij} \left(\frac{\partial u_a}{\partial x_j} + \frac{\partial \eta_a}{\partial y_j}\right)\left(\frac{\partial \Phi}{\partial x_i} + \frac{\partial \varphi_a}{\partial y_i}\right) + \int_{\Omega\times Y_b} b_{ij} \frac{\partial u_b}{\partial y_j}\frac{\partial \varphi_b}{\partial y_i} = \int_{\Omega\times Y_a} f\Phi + \int_{\Omega\times Y_b} f\varphi_b. \tag{2.49}$$

Using density arguments it follows that

$$((u_a, u_b), \eta_a) \in V_6 := V \times L^2(\Omega; \widetilde{H}^1_{per}(Y_a)),$$

$$V := \left\{(\Phi, \varphi) \in H_0^1(\Omega) \times L^2(\Omega; H^1_{per}(Y_b)), \;\; \varphi = \Phi \text{ on } \Omega \times \Gamma\right\}. \tag{2.50}$$

is solution of the problem: To find $((u_a, u_b), \eta_a) \in V_6$ satisfying

$$\int_{\Omega\times Y_a} a_{ij} \left(\frac{\partial u_a}{\partial x_j} + \frac{\partial \eta_a}{\partial y_j}\right)\left(\frac{\partial \Phi}{\partial x_i} + \frac{\partial \varphi_a}{\partial y_i}\right) + \int_{\Omega\times Y_b} b_{ij} \frac{\partial u_b}{\partial y_j}\frac{\partial \varphi_b}{\partial y_i} =$$

$$= \int_{\Omega\times Y_a} f\Phi + \int_{\Omega\times Y_b} f\varphi_b, \quad \forall\, ((\Phi, \varphi_b), \varphi_a) \in V \times L^2(\Omega; \widetilde{H}^1_{per}(Y_a)). \tag{2.51}$$

Theorem 13. *If u^ε is the solution of the problem (2.18) then,*

$$u^\varepsilon \xrightarrow{2s} |Y_a| u + w_0 \chi_b f, \tag{2.52}$$

where $u \in H_0^1(\Omega)$ is the unique solution of (2.40).

3 The effective permeability of fractured porous media subject to the Beavers-Joseph contact law

In Chapter 3 we study the asymptotic behaviour of an incompressible Stokes fluid flow contained in an $\varepsilon-$periodic distribution of fractures perturbating a porous medium where the Darcy's law is valid and the coupling is modeled by the Beavers-Joseph interface condition. The first region represents the system of fractures, which is connected and where the flow is governed by the Stokes system. The second region, which is also connected, stands for the system of porous blocks, which have a certain permeability and where the flow is governed by Darcy's law. These two flows are coupled on the interface by the Saffman's variant [35] of the Beavers-Joseph condition ([6] and [24]), which was confirmed by [19] as the limit of a homogenization process. Besides the continuity of the normal component of the velocity, it imposes the proportionality of the tangential velocity with the tangential component of the viscous stress on the fluid-side of the interface. We prove the existence and uniqueness of the solution of this model in our ε-periodic framework.

Proper rescaling of the fractures shows that the Beavers-Joseph condition influences the asymptotic behaviour of the system as long as the permeability coefficients obey one of two alternatives. As one of them was studied in [15], in Chapter 3 we consider the case where permeability is of the same order of magnitude as ε^2. Under this assumption, we find the asymptotic behaviour of the fractured porous medium using arguments of the homogenization theory. The present framework may be seen as a further development of the periodic homogenization in percolation. It does not deal with the formal method of asymptotic expansions [20], [21], [36] but rather extends the first rigorous proof based on the construction of a pressure extension due to [37] and followed by contextual variants [2], [23], [26], with the restriction that, unlike previous works [2], [17], [37] relying on specific constructions, the velocity and pressure of the fluid have natural bounded extensions in the porous medium. Homogenization of phenomena in fractured media were studied later in [32], [2] and [33], when the connectedness of both regions could be assumed. Obviously, ε-periodicity allows the use of the homogenization theory procedure. It is initiated in Section 3.2 thanks to a priori estimates and compacity arguments of the two-scale convergence theory, identifying the limit by constructing special test functions (see [3], [25] and [30]). We find the homogenized problem verified by the two-scale limit of the couple velocity-pressure. It is well-posed and can be decoupled. The asymptotic pressure is purely macroscopic, unlike the velocity field which still depends on the microscopic variable.

The results of Chapter 3 can also be found in the paper [16], which has been accepted for publication.

3.1 The flow through the ε-periodic structure

Let Ω be an open connected bounded set in $\mathbb{R}^N (N \geq 2)$, locally located on one side of the boundary $\partial\Omega$, a Lipschitz manifold composed of a finite number of connected components.

Let Y_f be a Lipschitz open connected subset of the unit cube $Y = (0,1)^N$, such that the intersections of ∂Y_f with ∂Y are reproduced identically on the opposite faces of the cube and $0 \notin \overline{Y}_f$. The outward normal on ∂Y_f is denoted by ν. Repeating Y by periodicity, we assume that the reunion of all the $\bar{Y}_f$ parts, denoted by $\mathbb{R}^N_f$, is a connected domain in $\mathbb{R}^N$ with a boundary of class $\mathcal{C}^2$. Defining $Y_s = Y \setminus \overline{Y}_f$, we assume also that the reunion of all the $\bar{Y}_s$ parts is a connected domain in $\mathbb{R}^N$.

For any $\varepsilon \in (0,1)$ we denote

$$Z_\varepsilon = \{k \in \mathbb{Z}^N,\ \varepsilon k + \varepsilon Y \subseteq \Omega\} \tag{3.1}$$

$$I_\varepsilon = \{k \in Z_\varepsilon, \varepsilon k \pm \varepsilon e_i + \varepsilon Y \subseteq \Omega, \forall i \in \overline{1,N}\} \tag{3.2}$$

where e_i are the unit vectors of the canonical basis in $\mathbb{R}^N$.

Finally, we define the system of fractures by

$$\Omega_{\varepsilon f} = \operatorname{int}\left(\bigcup_{k \in I_\varepsilon} (\varepsilon k + \varepsilon \bar{Y}_f)\right) \tag{3.3}$$

and the porous matrix of our structure by $\Omega_{\varepsilon s} = \Omega \backslash \bar{\Omega}_{\varepsilon f}$. The interface between the porous blocks and the fluid is denoted by $\Gamma_\varepsilon = \partial\Omega_{\varepsilon f}$. Its normal is:

$$\nu^\varepsilon(x) = \nu\left(\frac{x}{\varepsilon}\right), \quad x \in \Gamma_\varepsilon \tag{3.4}$$

where ν has been periodically extended to $\mathbb{R}^N$. Let us remark that $\Omega_{\varepsilon s}$ and $\Omega_{\varepsilon f}$ are connected and that the fracture ratio of this structure is given by

$$m = |Y_f| \in]0,1[, \quad \text{as} \quad \frac{|\Omega_{\varepsilon f}|}{|\Omega|} \to m \quad \text{when} \quad \varepsilon \to 0. \tag{3.5}$$

To the previous structure we associate a model of fluid flow through a fractured porous medium by assuming that there is a filtration flow in $\Omega_{\varepsilon s}$ obeying the Darcy's law and that there is a viscous flow in $\Omega_{\varepsilon f}$ governed by the Stokes system. These two flows are coupled by a Saffman's variant [35] of the Beavers-Joseph condition [6], [24]. This system is completed by an impermeability condition on $\partial\Omega$:

$$\mathrm{div} v^{\varepsilon s} = 0 \quad \text{in} \quad \Omega_{\varepsilon s} \tag{3.6}$$

$$\mu_\varepsilon v^{\varepsilon s} = K^\varepsilon (g^\varepsilon - \nabla p^{\varepsilon s}) \quad \text{in} \quad \Omega_{\varepsilon s}, \tag{3.7}$$

$$\mathrm{div} v^{\varepsilon f} = 0 \quad \text{in} \quad \Omega_{\varepsilon f}, \tag{3.8}$$

$$\sigma_{ij}^\varepsilon = -p^{\varepsilon f}\delta_{ij} + 2\mu_\varepsilon e_{ij}(v^{\varepsilon f}) \quad \text{in} \quad \Omega_{\varepsilon f} \tag{3.9}$$

$$-\frac{\partial}{\partial x_j}\sigma_{ij}^\varepsilon = g_i^\varepsilon \quad \text{in} \quad \Omega_{\varepsilon f} \tag{3.10}$$

$$v^{\varepsilon s}\cdot \nu^\varepsilon = v^{\varepsilon f}\cdot\nu^\varepsilon \quad \text{on} \quad \Gamma_\varepsilon, \tag{3.11}$$

$$-p^{\varepsilon s}\nu_i^\varepsilon - \sigma_{ij}^\varepsilon\nu_j^\varepsilon = \alpha_\varepsilon\mu_\varepsilon\beta_\varepsilon^{-1}(v_i^{\varepsilon f} - (v^{\varepsilon f}\cdot\nu^\varepsilon)\nu_i^\varepsilon) \quad \text{on} \quad \Gamma_\varepsilon, \tag{3.12}$$

$$v^{\varepsilon s}\cdot n = 0 \quad \text{on} \quad \partial\Omega, \quad n \text{ the outward normal on } \Omega, \tag{3.13}$$

where $v^{\varepsilon s}$, $v^{\varepsilon f}$ and $p^{\varepsilon s}$, $p^{\varepsilon f}$ stand for the corresponding velocities and pressures, $\mu^\varepsilon > 0$ is the viscosity of the fluid, $\alpha_\varepsilon \in L^\infty(\Omega)$ is the positive non-dimensional Beavers-Joseph number, $g^\varepsilon \in L^2(\Omega)^N$ is the exterior force and $e(v)$ denotes the symmetric tensor of the velocity gradient defined by $e_{ij}(v) = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$.

Finally, the positive tensor of permeability is defined by:

$$K^\varepsilon(x) = \beta_\varepsilon^2 K\left(\frac{x}{\varepsilon}\right), \tag{3.14}$$

where $K \in L^\infty(Y)^{N\times N}$ and $\beta_\varepsilon > 0$ stands for the magnitude of $(TrK^\varepsilon)^{1/2}$ with respect to $\varepsilon \to 0$. As usual, we use the notations:

$$H_0(\mathrm{div},\Omega) = \{v \in H(\mathrm{div},\Omega), \quad v\cdot\nu = 0 \quad \text{on} \quad \partial\Omega\} \tag{3.15}$$

$$L_0^2(\Omega) = \{p \in L^2(\Omega), \quad \int_\Omega p = 0\} \tag{3.16}$$

$$V_0(\mathrm{div},\Omega) = \{v \in H_0(\mathrm{div},\Omega), \quad \mathrm{div} v = 0 \quad \text{in} \quad \Omega\} \tag{3.17}$$

Next, we define

$$H_\varepsilon = \{v \in H_0(\mathrm{div},\Omega), \quad v \in H^1(\Omega_{\varepsilon f})^N\}, \tag{3.18}$$

the Hilbert space endowed with the scalar product

$$(u,v)_{H_\varepsilon} = \int_{\Omega_{\varepsilon s}} uv + \int_{\Omega_{\varepsilon s}} \mathrm{div} u \, \mathrm{div} v + \varepsilon^2\int_{\Omega_{\varepsilon f}} e(u)e(v) + \varepsilon\int_{\Gamma_\varepsilon}(\gamma^\varepsilon u - (\gamma_\nu^\varepsilon u)\nu^\varepsilon)\gamma^\varepsilon v \tag{3.19}$$

where γ^ε and γ_ν^ε denote respectively the trace and the normal trace operators on Γ_ε with respect to $\Omega_{\varepsilon f}$. Its corresponding subspace of divergence free velocities is

$$V_\varepsilon = \{v \in V_0(\mathrm{div},\Omega), \quad v \in H^1(\Omega_{\varepsilon f})^N\}. \tag{3.20}$$

A straightforward consequence, via the corresponding Korn's inequality, is

Lemma 14. *There exists some constant $C > 0$, independent of ε, such that*

$$|u|_{L^2(\Omega_{\varepsilon f})} + \varepsilon|\nabla u|_{L^2(\Omega_{\varepsilon f})} \le C|u|_{H_\varepsilon}, \quad \forall u \in H_\varepsilon. \tag{3.21}$$

Denoting

$$A^\varepsilon = (K^\varepsilon)^{-1} \tag{3.22}$$

and using the positivity of K^ε, we can assume without loss of generality that

$$\exists a_0 > 0 \quad \text{such that} \quad A_{ij}^\varepsilon(\cdot)\,\xi_i\xi_j \ge a_0|\xi|^2, \quad \forall\xi\in\mathbb{R}^N, \quad \text{a.e. in } \Omega. \tag{3.23}$$

Rescaling the velocity by

$$u^\varepsilon = \left\{\begin{array}{ll} u^{\varepsilon s} & \text{in} \quad \Omega_{\varepsilon s} \\ u^{\varepsilon f} & \text{in} \quad \Omega_{\varepsilon f}\end{array}\right. = \frac{\mu_\varepsilon}{\beta_\varepsilon^2}\left\{\begin{array}{ll} v^{\varepsilon s} & \text{in} \quad \Omega_{\varepsilon s} \\ v^{\varepsilon f} & \text{in} \quad \Omega_{\varepsilon f}\end{array}\right. \tag{3.24}$$

then, for any $u, v \in H_\varepsilon$ and $q \in L^2_0(\Omega)$, we define

$$a_\varepsilon(u,v) = \int_{\Omega_{\varepsilon s}} A^\varepsilon uv + 2\beta_\varepsilon^2 \int_{\Omega_{\varepsilon f}} e(u)e(v) + \beta_\varepsilon \int_{\Gamma_\varepsilon} \alpha_\varepsilon(\gamma^\varepsilon u - (\gamma_\nu^\varepsilon u)\nu^\varepsilon)\gamma^\varepsilon v \tag{3.25}$$

$$b_\varepsilon(q,v) = -\int_\Omega q \,\mathrm{div} v. \tag{3.26}$$

We see that if the pair $(u^\varepsilon, p^\varepsilon)$ is a smooth solution of the problem (3.6)–(3.13), then it is also a solution of the following problem: To find $(u^\varepsilon, p^\varepsilon) \in H_\varepsilon \times L^2_0(\Omega)$ such that

$$a_\varepsilon(u^\varepsilon, v) + b_\varepsilon(p^\varepsilon, v) = \int_\Omega g^\varepsilon v, \quad \forall v \in H_\varepsilon \tag{3.27}$$

$$b_\varepsilon(q, u^\varepsilon) = 0, \quad \forall q \in L^2_0(\Omega) \tag{3.28}$$

Theorem 15. *There exists a unique pair $(u^\varepsilon, p^\varepsilon) \in H_\varepsilon \times L^2_0(\Omega)$ solution of (3.27)–(3.28).*

3.2 The homogenization process

In this section for any function φ defined on $\Omega \times Y$ we shall use the notations

$$\varphi^h = \varphi|_{\Omega\times Y_h}, \quad \tilde{\varphi}^h = \frac{1}{|Y_h|}\int_{Y_h} \varphi(\cdot, y)dy, \quad h \in \{s, f\}, \tag{3.29}$$

$$\tilde{\varphi} = \int_Y \varphi(\cdot, y)dy, \quad \text{that is} \quad \tilde{\varphi} = (1-m)\tilde{\varphi}^s + m\tilde{\varphi}^f. \tag{3.30}$$

Applying the V_ε-ellipticity of a_ε, from the a priori estimates we find that $\exists u \in L^2(\Omega \times Y)^N$ such that, on some subsequence

$$u^\varepsilon \overset{2}{\rightharpoonup} u \tag{3.31}$$

$$u^\varepsilon \rightharpoonup \int_Y u(\cdot, y)dy \in V_0(\mathrm{div}, \Omega) \quad \text{weakly in} \quad L^2(\Omega)^N. \tag{3.32}$$

Denoting $\chi_{\varepsilon f}(x) = \chi_f\left(\frac{x}{\varepsilon}\right)$ and $\chi_{\varepsilon s}(x) = \chi_s\left(\frac{x}{\varepsilon}\right)$, where χ_f and χ_s are the characteristic functions of Y_f and Y_s in Y, we see that $(\chi_{\varepsilon s}u^\varepsilon)_\varepsilon$, $(\chi_{\varepsilon f}u^\varepsilon)_\varepsilon$ and $\left(\varepsilon\chi_{\varepsilon f}\frac{\partial u^\varepsilon}{\partial x_i}\right)_\varepsilon$ are bounded in $(L^2(\Omega))^N$, $\forall i \in \{1, 2, \cdots, N\}$.

Lemma 16. *Denoting*

$$\tilde{H}^1_{\mathrm{per}}(Y_f) = \{\varphi \in H^1_{\mathrm{loc}}(\mathbb{R}^N_f), \quad \varphi \quad \textit{is } Y\textit{-periodic}, \quad \int_{Y_f} \varphi = 0\},$$

then $u^f \in L^2(\Omega, (\tilde{H}^1_{\mathrm{per}}(Y_f))^N)$ satisfies

$$\varepsilon\chi_{\varepsilon f}\nabla u_i^\varepsilon \overset{2}{\rightharpoonup} \chi_f \nabla_y u_i^f. \tag{3.33}$$

Moreover,

$$\mathrm{div}\widetilde{u} = 0 \quad \textit{in} \quad \Omega, \tag{3.34}$$

$$\gamma_n \widetilde{u} = 0 \quad \textit{on} \quad \partial\Omega, \tag{3.35}$$

$$\mathrm{div}_y u = 0 \quad \textit{in} \quad \Omega \times Y. \tag{3.36}$$

Proposition 17. *There exists a constant $C > 0$ independent of ε such that*

$$|p^\varepsilon|_{L^2(\Omega)} + |\nabla p^\varepsilon|_{L^2(\Omega_{\varepsilon s})} \le C. \tag{3.37}$$

Lemma 18. *There exists $p \in L^2_0(\Omega \times Y)$ with $p^s = \tilde{p}^s \in H^1(\Omega)$ and $p^f = \tilde{p}^f \in L^2(\Omega)$ such that, up to some subsequence, we have:*

$$p^\varepsilon \overset{2}{\rightharpoonup} p. \tag{3.38}$$

Moreover: $p^s = p^f = p$ and thus $p \in \tilde{H}^1(\Omega)$.

3.3 The homogenized problem

Consider the Hilbert space:

$$X = \{u \in L^2(\Omega \times Y),\ u^f \in L^2(\Omega, \tilde{H}^1_{\text{per}}(Y_f)),\ (\tilde{u}^s, \tilde{u}^f) \in H_0(\text{div}, \Omega)^2,\ \text{div}_y u = 0\}$$

endowed with the scalar product:

$$(u, v)_X = \int_{\Omega \times Y_s} u\, v + \int_\Omega \text{div}\tilde{u}\, \text{div}\tilde{v} + \int_{\Omega \times Y_f} e_y(u) e_y(v) + \int_{\Omega \times \Gamma} \alpha(y)(\gamma u \gamma v - \gamma_\nu u \gamma_\nu v)$$

and set:

$$X_0 = \{u \in X,\ \text{div}\tilde{u} = 0\}, \quad M = L^2_0(\Omega).$$

We can present our first homogenization result:

Proposition 19. *The limit problem reads: Find $(v, q) \in X \times M$ such that*

$$a(v, \varphi) + b(q, \varphi) = \int_\Omega g\tilde{\varphi}, \quad \forall \varphi \in X \tag{3.39}$$

$$b(\pi, v) = 0, \quad \forall \pi \in M \tag{3.40}$$

where we set

$$a(v, \varphi) = \int_{\Omega \times Y_s} A v \varphi + 2\beta \int_{\Omega \times Y_f} e_y(v) e_y(\varphi) + \int_{\Omega \times \Gamma} \alpha(y)(\gamma v - (\gamma_\nu v)\nu)\gamma\varphi \tag{3.41}$$

$$b(\pi, v) = -\int_\Omega \pi \text{div}\tilde{v}. \tag{3.42}$$

Proposition 20. *Using the "inf-sup" condition we see that the problem (3.39)–(3.40) is well-posed.*

Proposition 21. *The problem (3.39)–(3.40) equivalently reads: Find $u \in X_0$ such that*

$$a(u, \varphi) = \int_\Omega (g - \nabla p)\tilde{\varphi}, \quad \forall \varphi \in X_0. \tag{3.43}$$

Remark 22. In this case, we can decouple the problem (3.43) as follows: an homogenized problem and a local one. For this purpose, we define

$$W = \{w \in L^2(Y), \quad w^f \in \tilde{H}^1_{\text{per}}(Y_f), \quad \text{div}_y w = 0 \quad \text{in} \quad Y\}$$

and we find that the solution of the homogenized problem reads:

$$u(x, y) = w^i(y)\left(g_i(x) - \frac{\partial p}{\partial x_i}\right), \tag{3.44}$$

where for every $i \in \{1, \cdots, N\}$, $w^i \in W$ is the solution of the local problem:

$$\int_{Y_s} A w^i \psi + 2\beta \int_{Y_f} e_y(w^i) e_y(\psi) + \int_\Gamma \alpha(\gamma w^i - (\gamma_\nu w^i)\nu)\psi = \int_Y \psi e_i, \quad \forall \psi \in W \tag{3.45}$$

and e_i are the unit vectors of the canonical basis in $\mathbb{R}^N$.

Now, we set

$$K_{ji} = \int_Y w^i_j. \tag{3.46}$$

Proposition 23. *The effective permeability tensor K is symmetric and positively defined.*

From (3.44) we get the Darcy's law

$$\tilde{u} = K(g - \nabla p), \tag{3.47}$$

where $\tilde{u} \in H_0(\text{div}, \Omega)$ and $p \in \tilde{H}^1(\Omega)$ is the unique solution of the following boundary value problem:

$$\text{div}(K\nabla p) = \text{div}(Kg) \quad \text{in} \quad \Omega, \tag{3.48}$$

$$K\nabla p \cdot n = Kg \cdot n \quad \text{on} \quad \partial\Omega, \tag{3.49}$$

where obviously $\text{div}(Kg) \in H^{-1}(\Omega)$.

Proposition 24. *The problem (3.48)–(3.49) is well-posed.*

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