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HABILITATION THESIS

Semisimple Hopf algebras and fusion categories

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To Anca and Maria for their patience and love

Abstract

In this thesis we present our recent works on the structure of semisimple Hopf algebras and their representation theory. The subjects we analyze are grouped in three main thematic parts which are detailed below.

The first one, contained in Chapters 1 and 2 is concerned with the study of normal Hopf subalgebras of semisimple Hopf algebras. The second one is devoted to the study of representations of semisimple Hopf algebras. It contains Chapters 3 and 4. The third part is the study of the group actions on fusion categories an it contains Chapters 5 and 6.

We resume now the main subjects that we address in the three parts of this thesis.

Part I. Normal Hopf subalgebras and kernels of representations

In Chapter 1 the notion of a Hopf kernel of a representation of a semisimple Hopf algebra is introduced in this part. Similar properties to the kernel of a group representation are proved in some special cases. In particular, every normal Hopf subalgebra of a semisimple Hopf algebra H is the kernel of a representation of H. The maximal normal Hopf subalgebras of H are also described. A well known result of Brauer in group representation theory is generalized in the context of representations of a semisimple Hopf algebra. It will be shown that this results holds only for those characters which are central in H^*.

In Chapter 2 we propose a different notion of the kernel of a representation of a semisimple Hopf algebra. We define left and right kernels of representations of Hopf algebras. In the case of group algebras, left and right kernels coincide and they are the usual kernels of representations. In the general case we show that these kernels coincide with the categorical left and right Hopf kernels of morphisms of Hopf algebras defined in [2]. Brauer's theorem for kernels over group algebras is generalized to any characters of semisimple Hopf algebras. Recently it was proven in [10] that Hopf subalgebras are normal if and only if they are depth two subalgebras. In this chapter we extend this result to coideal subalgebras. Moreover, we show that in this situation, depth two and normality in the sense defined by Rieffel in [105] also coincide. Note that normal subalgebras as defined by Rieffel were recently revised in [25, Section 4]. In the same chapter it is also shown any coideal subalgebra of a semisimple Hopf algebra is also semisimple.

Chapter 1 of this part is based on the paper [15] and Chapter 2 is based on [17].

Part II. Representation theory of semisimple Hopf algebras. The second part of this thesis contains two chapters. The first chapter is devoted to the study of double cosets for semisimple Hopf algebras. The second chapter studies Clifford theory for semisimple Hopf algebras.

In Chapter 3 we construct double cosets relative to two Hopf subalgebras of a given semisimple Hopf algebra. The results in this Chapter are contained in [14] and generalizes the results from [95]. Using Frobenius-Perron theory for nonnegative Hopf algebras the results from [95] are generalized and proved in a simpler manner in this chapter. Applying a dual version of these double cosets we obtain new results concerning character theory of normal Hopf subalgebras. Applying these double cosets for the dual Hopf algebra we study the restriction functor from a semisimple Hopf algebra to a normal Hopf subalgebra. We define a notion of conjugate module similar to the one for modules over normal subgroups of a group. Some results from group theory hold in this more general setting. In particular we show that the induced module restricted back to the original normal Hopf subalgebra has as irreducible constituents the constituents of all the conjugate modules.

In Chapter 4 we study an analogue of initial's Clifford approach for groups. We consider an extension of semisimple Hopf algebras A/B where B is a normal Hopf subalgebra of A and let M be an irreducible B-module. The conjugate B-modules of M are defined as in the Chapter 3 and the stabilizer Z of M is a Hopf subalgebra of A containing B. We say that the Clifford correspondence holds for M if induction from Z to A provides a bijection between the sets of isomorphism classes of irreducible Z (respectively A)-modules that contain M as a B-submodule.

It is shown that the Clifford correspondence holds for M if and only if Z is a stabilizer in the sense proposed in [120]. A necessary and sufficient condition for this to happen is given in Proposition 4.2. Our approach uses the character theory for Hopf algebras and normal Hopf subalgebras. If the extension

$$\Bbbk \longrightarrow B \xrightarrow{i} A \xrightarrow{\pi} H \longrightarrow \Bbbk$$

is cocentral then we prove that this condition is alawys satisfied (see Corollary 6.5.1).

Part III. Fusion categories: Group actions on fusion categories

In Chapter 5 we determine the fusion rules of the equivariantization of a fusion category $\mathcal{C}$ under the action of a finite group G in terms of the fusion rules of $\mathcal{C}$ and some group-theoretical data associated to the group action. As an application we obtain a formula for the fusion rules in an equivariantization of a pointed fusion category in terms of only group-theoretical data. This entails a description of the fusion rules in any braided group-theoretical fusion category. Chapter 5 is based on a work in collaboration with S. Natale. It mainly contains the results from [26].

In Chapter 6 a general Mackey type decomposition for representations of semisimple Hopf algebras is investigated. We show that such a decomposition occurs in the case that the module is induced from an arbitrary Hopf subalgebra and it is restricted back to a group subalgebra. Some other examples when such a decomposition occurs are also constructed. They arise from gradings on the category of corepresentations of a semisimple Hopf algebra and provide new examples of Green functors in the literature. The results of this chapter are included in [22].

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Rezumat

In această teză vom prezenta câteva din rezultatele noastre recente privind structura algebrelor Hopf semisimple şi teoria lor de reprezentare. Subiectele pe care le vom analiza sunt cuprinse în trei părti care sunt detaliate mai jos.

Prima parte, conţinută în Capitolele 1 şi 2, tratează subalgebrele Hopf normale ale unei algebre Hopf semisimple. Cea de a doua parte este destinată studiului reprezentă rilor algebrelor Hopf semisimple şi este conţiunta în Capitolele 3 şi 4. A treia parte conţine Capitolele 5 şi 6 şi studiaza acţiunea grupurilor finite supra categoriilor de fuziune.

Mai jos vom face un rezumat al subiectelor adresate în cele trei părţi ale tezei.

Partea I -a. Subalgebre Hopf normale şi nuclee de reprezentări.

În Capitolul 1 este introdusă noţiunea de nucleu Hopf al unei reprezentări a unei algebre Hopf semisimple. În cazuri speciale sunt demonstrate proprietaţi similare cu nucleul unei reprezentări a unui grup finit. În particular, este arătat că orice subalgebră Hopf normală a unei algebre Hopf semisimple H este nucleul unei reprezentări a lui H. Subalgebrele Hopf normale maximale ale lui H sunt de asemenea descrise. Un rezultat bine cunoscut în teoria reprezentărilor grupurilor este Teorema Brauer privind caracterele fidele. Acest rezultat este generalizat în contextul algebrelor Hopf semisimple. Se va arăta că acest rezultat are loc doar pentru acele caractere care sunt centrale în algebra duală H^*.

În Capitolul 2 este propusă o noţiune diferită de nucleu pentru o reprezentare a unei algebre Hopf semisimple. Vom defini nuclee la stânga şi la dreapta pentru reprezentări de algebre Hopf arbitrare, nu neapărat semisimple. În cazul algebrelor grupale, nucleele la stânga şi la dreapta coincid şi sunt egale cu nucleele obişnuite de reprezentări. În cazul general, vom arăta că aceste nuclee coincid cu nucleele de morfisme în categoria de algebre Hopf (la stânga şi la dreapta), aşa cum au fost definite în [2]. Teorema Brauer mai sus menţionată, pentru caractere fidele de algebre grupale finite, este generalizată în acest caz la orice caracter al unei algebre Hopf semisimple. Recent, a fost demonstrat în [10] că subalgebrele Hopf sunt normale, dacă şi numai dacă acestea sunt subalgebre de adancime doi. În acest capitol vom extinde acest rezultat la subalgebrele coideal ale unei algebre Hopf arbitrare. Mai mult decât atât, vom arăta că în această situaţie, adâncimea 2 şi noţiunea de normalitate, în sensul definit de Rieffel în [105], de asemenea coincid. De remarcat este faptul că pentru subalgebrele Hopf normale, aşa cum sunt definite de Rieffel în [105],

acest rezultat a fost recent demonstrat în [25, Sectiunea 4]. De asemenea, în acest capitol este arătat că orice subalgebră coideal al unei algebre Hopf semisimple este de asemenea semisimplă.

Capitolul 1 este parte a articolului [15]. Capitolul 2 este continut in articolul [17].

Partea II-a. Reprezentări de algebre Hopf semisimple.

Cea de a doua parte a acestei teze conţine două capitole. Primul capitol este destinat studiului coseturilor duble pentru algebrele Hopf semisimple. Cel de al doilea prezintă o teorie de tip Clifford pentru reprezentări de algebre Hopf semisimple

În Capitolul 3 construim spaţiul coseturilor duble relativ la două subalgebre Hopf ale unei algebre Hopf semisimple. Rezultatele acestui capitol sunt conţinute în [14] şi generalizează rezultate din [95]. Folosind teoria Frobenius-Perron pentru matrici cu intrări nenegative rezultatele din [95] sunt generalizate şi demonstrate intr-o manieră mai simplă în acest capitol. Aplicând o versiune duală a acestor rezultate obţinem noi informaţii cu privire la teoria caracterelor subalgebrelor Hopf normale. Mai exact, aplicând aceste coseturi duble pentru algebra Hopf duală studiem functorul inducţie şi restricţie de la o algebră Hopf semisimplă la o subalgebră Hopf normală. Definim de asemenea o noţiune de modul conjugat similar celei din teoria grupurilor. Anumite rezultate din teoria reprezentărilor de subgrupuri normale au loc de asemenea în acest context mai general. In particular este arătat că modulul indus şi restricţionat înapoi la subalgebra Hopf normală are ca şi constituenţi simpli exact constitutenţii tuturor modulelor conjugate.

In Capitlolul 4 studiem un analog al teoriei Clifford pentru grupuri finite urmând abordarea initială a acestei teorii definită de Clifford. Astfel s-au adus contribuţii noi la inţelegerea reprezentărilor algebrelor Hopf semisimple. Considerăm extensii A/B de algebre Hopf semisimple unde B este o subalgebră normală a lui A şi M un modul ireductibil al lui B. Modulele conjugate ale lui M sunt definite în Capitolul 3 şi stabilizatorul lui M este o subalgebră Hopf a lui A care conţine B. Spunem că teoria Clifford funcţionează pentru M dacă functorul inducţie produce o bijecţie intre clasele de izomorfism ale Z-modulelor (respectiv A-modulelor) simple care contin pe M ca B-modul.

Este arătat că teoria Clifford funcţionează pentru M dacă şi numai dacă Z este stabilizator în sensul definit în [120]. O condiţie necesară şi suficientă ca aceasta să se întâmple este dată în Propoziţia 4.2. Abordarea noastră foloseşte teoria caracterelor pentru algebre Hopf semisimple şi subalgebre Hopf normale. Dacă extinderea

$$\Bbbk \longrightarrow B \xrightarrow{i} A \xrightarrow{\pi} H \longrightarrow \Bbbk$$

este cocentrală demonstrăm că această condiţie este automat satisfacută (vezi Corolarul 6.5.1).

Partea III-a. Grupuri finite: Acţiuni de grupuri finite pe categorii de fuziune

În Capitolul 5 determinăm regulile de fuziune ale unei categorii echivariantizate $\mathcal{C}^G$ sub acţiunea unui grup finit G pe o categorie de fuziune $\mathcal{C}$. Acestea sunt date în funcţie de regulile de fuziune ale lui $\mathcal{C}$ şi anumite date grup teoretice asociate acţiunii lui G. Ca aplicaţie obţinem o formulă pentru regulile de fuziune ale unei echivariantizări a unei categorii de fuziune punctate numai în funcţie de date grup-teoretice. Aceasta în schimb permite descrierea regulilor de fuziune pentru orice categorie braided grup-teoreticală. Capitolul 5 se bazează pe articolul [26], scris în colaborare cu S. Natale.

În Capitolul 6 o descompunere generală de tip Mackey pentru reprezentările algebrelor Hopf semisimple este investigată. Arătăm că o astfel de descompunere are loc în cazul când modulul este indus de la o subalgebră Hopf arbitrară şi restricţionat la o subalgebră grupală. Alte exemple când o astfel de descompunere are loc sunt de asemenea construite. Ele sunt date de graduări pe categoria de coreprezentări a algebrei Hopf semisimple şi produc noi exemple de functori Green. Rezultatele acestui capitol sunt conţinute în [22].

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Contents

Part I

Normal Hopf subalgebras and kernels of representations

Chapter 1

Normal Hopf Subalgebras of Semisimple Hopf Algebras

This part is devoted to the study of the notion of kernel of a representation of a semisimple Hopf algebra. Similar properties to the kernel of a group representation are proved. The results of the first chapter of this part are contained in author's paper [15].

Let G be a finite group and $\mathrm{X} : G \to \mathrm{End}_{\Bbbk}(M)$ be a finite dimensional complex representation of G which affords the character χ. The kernel of the representation M is defined as $\ker\, \chi = \{g \in G|\ \chi(g) = \chi(1)\}$ and it is the set of all group elements $g \in G$ which act as identity on M. (For example, see [63].) Every normal subgroup N of G is the kernel of a character, namely the character of the regular representation of G/N. If $\mathrm{Z} = \{g \in G|\ \ |\chi(g)| = \chi(1)\}$ then Z is called the *center* of the character χ and it is the set of group elements of G which act as a unit scalar on M. The properties of Z and $\ker\, \chi$ are described in [Lemma 2.27, [63]] which shows that $\mathrm{Z}/\ker\, \chi$ is a cyclic subgroup of the center of $G/\ker\, \chi$.

If M is a representation of a finite dimensional semisimple Hopf algebra A and $\chi \in C(A)$ is its associated character then $\ker\, \chi \subset A$ is defined as the set of all irreducible A^*-characters $d \in A$ such that d acts as the scalar $\epsilon(d)$ on M. We prove that $\ker\, \chi = \{d \in \mathrm{Irr}(A^*)|\ \chi(d) = \epsilon(d)\chi(1)\}$. Similarly, the set of all irreducible A^*-characters $d \in A$ that act as a scalar of absolute value $\epsilon(d)$ on M is characterized as $\mathrm{z}_{\chi} = \{d \in \mathrm{Irr}(A^*)|\ \ |\chi(d)| = \epsilon(d)\chi(1)\}$.

The organization of this chapter is as follows. Section 1.2 presents the definition and the main properties of the *kernel* of a character χ and its center z_{χ}. It is shown that these sets of characters are closed under multiplication and duality operation " * ". Thus they generate Hopf subalgebras of A denoted by A_{χ} and Z_{χ}, respectively. We say that a Hopf subalgebra K of A is the kernel of a representation if $K = A_{\chi}$ for a certain character χ of A.

Section 1.3 studies the relationship between normal Hopf subalgebras and the Hopf algebras generated by kernels. It is shown that any normal Hopf subalgebra is the kernel of a character which is central in A^*.

In Section 1.4 the converse of the above statement is proven. More precisely, it is shown that for a representation M of H affording a character χ which is central in A^* the Hopf subalgebra A_χ is normal in A. Therefore this implies that a Hopf subalgebra is normal if and only if it is the kernel of a character which is central in the dual Hopf algebra. Under the same assumption on χ it is shown that the irreducible representations of $A//A_\chi := A/AA_\chi^+$ are precisely the irreducible representations of A which are constituents of some tensor power of M. Using a basis description given in [124] for the algebra generated by the characters which are central in A^* we describe a finite collection of normal Hopf subalgebras of A which are the maximal normal Hopf subalgebras of A (under inclusion). We show that any other normal Hopf subalgebra is an intersection of some of these Hopf algebras. Two other results that hold for group representation are presented in this section.

1.1 Notations

For a vector space V we denote by $|V|$ the dimension $\dim_{\Bbbk} V$. The comultiplication, counit and antipode of a Hopf algebra are denoted by Δ, ϵ and S, respectively. We use Sweedler's notation $\Delta(x) = x_1 \otimes x_2$ for all $x \in A$ with the sum symbol dropped. All the other notations are those used in [82]. All considered modules are left modules.

Throughout this section A will denote a semisimple Hopf algebra over the algebraically closed field $\Bbbk$ of characteristic zero. It follows that A is also cosemisimple [73]. Recall that a Hopf algebra is called cosemisimple if it is cosemisimple as a coalgebra. Moreover, a coalgebra is called cosemisimple if the category of finite dimensional right (left) comodules is completely reducible.

The set of irreducible characters of A is denoted by $\mathrm{Irr}(A)$. The Grothendieck group $\mathcal{G}(A)$ of the category of finite dimensional left A-modules is a ring under the tensor product of modules. Then the character ring $C(A) := \mathcal{G}(A) \otimes_{\mathbb{Z}} \Bbbk$ is a semisimple subalgebra of A^* [125, 64] and it has a basis given by the characters of the irreducible A-modules. Also $C(A) = \mathrm{Cocom}(A^*)$, the space of cocommutative elements of A^*. By duality, the character ring of A^* is a semisimple subalgebra of A and under this identification it follows that $C(A^*) = \mathrm{Cocom}(A)$. If M is a finite dimensional A-module with character χ then M^* is also an A-module with character $\chi^* = \chi \circ S$. This induces an involution $^* : C(A) \to C(A)$ on $C(A)$.

1.1.1 The subcoalgebra associated to a comodule

Let W be any right A-comodule. Since A is finite dimensional it follows that W is a left A^*-module via the module structure $f.w = f(w_1)w_0$, where $\rho(w) = w_0 \otimes w_1$ is the given right A-comodule structure of W. Then one can associate to W the coefficient subcoalgebra denoted by C_W [72]. Recall that C_W is the minimal subcoalgebra C of A with the property that $\rho(W) \subset W \otimes C$. Moreover, it can be shown that $C_W = (\mathrm{Ann}_{A^*}(W))^{\perp}$ and C_W is called the subcoalgebra of A associated to the right A-comodule W. If W is a simple right A-comodule (or equivalently W is an irreducible left A^*-module) then the associated subcoalgebra C_W is a co-matrix coalgebra. More precisely, if $\dim W = q$ then $\dim C_W = q^2$ and it has a a $\Bbbk$-linear basis given by x_{ij} with $1 \leq i, j \leq q$. The coalgebra structure of C_W is then given by $\Delta(x_{ij}) = \sum_l x_{il} \otimes x_{lj}$ for all $1 \leq i, j \leq q$. Then $W \cong \mathbb{C} < x_{1i} | \; 1 \leq i \leq q >$ as right A-comodules where $\rho(x_{1i}) = \Delta(x_{1i}) = \sum_{l=1}^{q} x_{1l} \otimes x_{li}$ for all $1 \leq i \leq q$. Moreover the irreducible character $d \in C(A^*)$ of W is given by formula $d = \sum_{i=1}^{q} x_{ii}$. Then $\epsilon(d) = q$ and the simple subcoalgebra $C_{_W}$ is also denoted by C_d. As already mentioned, it is easy to check that W is an irreducible A^*-module if and only if C_W is a simple subcoalgebra of A. This establishes a canonical bijection between the set $\mathrm{Irr}(A^*)$ of simple right A^*-comodules and the set of simple subcoalgebras of A.

Recall also that if M and N are two right A-comodules then $M \otimes N$ is also a comodule with $\rho(m \otimes n) = m_0 \otimes n_0 \otimes m_1 n_1$. The associated coalgebra of $M \otimes N$ is is CD where C and D are the associated subcoalgebras of M and N respectively (see [95]).

For any two subcoalgebras C and D of A we denote by CD the subcoalgebra of A generated as a $\Bbbk$-vector space by all elements of the type cd with $c \in C$ and $d \in D$.

Remark 1.1.1. *For a simple subcoalgebra $C \subset A$ we denote by M_C the simple A-comodule associated to C. Following [96], if C and D are simple subcoalgebras of a semisimple Hopf algebra A then the simple comodules entering in the decomposition of $M_C \otimes M_D$ are in bijection with the set of all simple subcoalgebras of the product subcoalgebra CD of A. Moreover, this bijection is given by $W \mapsto C_W$ for any simple subcomodule W of $M_C \otimes M_D$.*

Subsets closed under multiplication and duality

Recall from [96] that a subset $X \subset \mathrm{Irr}(A^*)$ is closed under multiplication if for every two elements $c, d \in X$ in the decomposition of the product $cd = \sum_{e \in \mathrm{Irr}(A^*)} m_{c,d}^{e} e$ one has $e \in X$ whenever $m_e \neq 0$. Also a subset $X \subset \mathrm{Irr}(A^*)$ is closed under " $*$ " if $x^* \in X$ for all $x \in X$.

Following [96] any subset $X \subset \mathrm{Irr}(A^*)$ closed under multiplication generates a subbialgebra $A(X)$ of A defined by

$$A(X) := \oplus_{x \in X} C_x.$$

Moreover if the set X is also closed under " $*$ " then $A(X)$ is a Hopf subalgebra of A.

Remark 1.1.2. *Since in our case A is finite dimensional, it is well known that any subbialgebra of A is also a Hopf subalgebra. Therefore in this case any set X of irreducible characters closed under product is also closed under "$*$".*

Integrals in Hopf algebras

We use the notation $\Lambda_H \in H$ for the idempotent integral of H ($\epsilon(\Lambda_H) = 1$) and $t_H \in H^*$ for the idempotent integral of H^* ($t_H(1) = 1$). From [72, Proposition 4.1] it follows that the regular character of H is given by the formula

$$|H|t_{H} = \sum_{\chi \in \mathrm{Irr}(H)} \chi(1)\chi. \tag{1.1.1}$$

The dual formula is

$$|H|\Lambda_{H} = \sum_{d \in \mathrm{Irr}(H^*)} \epsilon(d)d. \tag{1.1.2}$$

One also has $t_{H}(\Lambda_{H}) = \frac{1}{|H|}$, see [73].

Central primitive idempotents in Hopf algebras

Let μ be any irreducible character of H and $\xi_{\mu} \in \mathrm{Z}(H)$ be the central primitive idempotent associated to it. Then $\nu(\xi_\mu) = \delta_{\mu,\,\nu}\mu(1)$ for any other irreducible character ν of H and $\{\xi_{\mu}\}_{\mu \in \mathrm{Irr}(H)}$ is the complete set of central orthogonal idempotents of H. Dually, since H^* is semisimple to any irreducible character $d \in C(H^*)$ one has an associated central primitive idempotent $\xi_d \in \mathrm{Z}(H^*)$. As before one can view $d \in H^{**} = H$ and the above relation becomes $\xi_d(d') = \delta_{d,\,d'}\epsilon(d)$ for any other irreducible character $d' \in C(H^*)$. Also $\{\xi_{d}\}_{d \in \mathrm{Irr}(H^*)}$ is the complete set of central orthogonal idempotents of H^*.

Normal Hopf subalgebras

If K is a Hopf subalgebra of H then K is a semisimple and cosemisimple Hopf algebra [82]. A Hopf subalgebra K of H is called normal if $h_1xS(h_2) \in K$ and $S(h_1)xh_2 \in K$ for all $x \in K$ and $h \in H$. If H is a semisimple Hopf algebra as above then $S^2 = \mathrm{Id}$ (see [73]) and K is normal in H if and only if $h_1xS(h_2) \in K$ for all $x \in K$ and $h \in H$. If $K^+ = \mathrm{Ker}(\epsilon) \cap K$ then K is normal Hopf subalgebra of A if and only if $AK^+ = K^+A$. In this situation $A//K := A/AK^+$ is a quotient Hopf algebra of A via the canonical map $\pi : A \to A//K$ (see [82, Lemma 3.4.2]). In our settings K is normal in A if and only if Λ_{K} is central in A (see [78, Lemma 1], on page 1932).

Remark 1.1.1. Suppose that K is a normal Hopf subalgebra of A and let $L = A//K$ be the quotient Hopf algebra of A via $\pi : A \to L$. Then $\pi^* : L^* \to A^*$ is an injective Hopf algebra map and $\pi^*(C(L)) \subseteq C(A)$. It follows that $\pi^*(L^*)$ is normal in A^* and it is easy to see that $(A^*//L^*)^* \cong K$. The representations of $L = A//K$ are those representations M of A such that each $x \in K$ acts as $\epsilon(x)Id_M$ on M. If χ is the character of M as L-module then $\pi^*(\chi) \in C(A)$ is the character of M as A-module and with the notations from the next section $A_{\pi^*(\chi)} \supset K$.

1.2 Kernels of representations and their properties

Throughout this chapter H will denote a semisimple Hopf algebra. Recall that the exponent of H is the smallest positive number $m > 0$ such that $h^{[m]} = \epsilon(h)1$ for all $h \in H$. The generalized power $h^{[m]}$ is defined by $h^{[m]} = \sum_{(h)} h_1 h_2 ... h_m$. The exponent of a finite dimensional semisimple Hopf algebra is always finite and divides the third power of the dimension of H, [40]. For the rest of this chapter we work over the field $\mathbb{C}$ of complex numbers.

Proposition 1.1. *Let H be a finite dimensional semisimple Hopf algebra and M be a representation of H affording the character $\chi \in C(H)$. If W is an irreducible representation of H^* affording the character $d \in C(H^*)$ then the following hold:*

1. $|\chi(d)| \leq \chi(1)\epsilon(d)$
2. *Equality holds if and only if d acts as $\alpha\epsilon(d)Id_M$ on M for a root of unity $\alpha \in \mathbb{C}$.*

Proof. 1. W is a right $H-$comodule and one can define a map T which is similar to the one defined in the [69, Paragraph 3.1]:

$$\begin{array}{llll} \mathrm{T}: & M \otimes W & \longrightarrow & M \otimes W \\ & m \otimes w & \longmapsto & \sum w_1 m \otimes w_0. \end{array}$$

It can be checked that $\mathrm{T}^p(m \otimes w) = \sum w_1^{[p]} m \otimes w_0$ for all $p \geq 0$. Thus, if $m = \exp(H)$ then $\mathrm{T}^m = \mathrm{Id}_{M \otimes W}$. Therefore T is a semisimple operator and all its eigenvalues are roots of unity. It follows that $\mathrm{tr}(\mathrm{T})$ is the sum of all these eigenvalues and in consequence $|\mathrm{tr}(\mathrm{T})| \leq \dim_{\mathbb{C}}(M \otimes W) = \chi(1)\epsilon(d)$. It is easy to see that $\mathrm{tr}(\mathrm{T}) = \chi(d)$. Indeed using the above remark one can suppose that $W = \mathbb{C} < x_{1i} \mid 1 \leq i \leq q >$ where $C_W = \mathbb{C} < x_{ij} | \ 1 \leq i, j \leq q >$ is the coalgebra associated to W. Then the formula for T becomes $\mathrm{T}(m \otimes x_{1i}) = \sum_{j=1}^{\epsilon(d)} x_{ji} m \otimes x_{1j}$ which shows that $\mathrm{tr}(\mathrm{T}) = \sum_{i=1}^{\epsilon(d)} \chi(x_{ii}) = \chi(d)$.

2. Equality holds if and only if $\mathrm{T} = \alpha \mathrm{Id}_{M \otimes W}$ for some root of unity α. The above expression for T implies that in this case $x_{ij} m = \delta_{i,j} \alpha m$ for any $1 \leq i, j \leq \epsilon(d)$. Therefore $dm = \alpha\epsilon(d)m$ for any $m \in M$. The converse is immediate.

□

Let M be a representation of H which affords the character χ. Define $\ker \chi$ as the set of all irreducible characters $d \in \mathrm{Irr}(H^*)$ which act as the scalar $\epsilon(d)$ on M. The previous proposition implies that $\ker \chi = \{d \in \mathrm{Irr}(H^*) | \ \chi(d) = \epsilon(d)\chi(1)\}$. Similarly let z_χ be the the set of all irreducible characters $d \in \mathrm{Irr}(H^*)$ which act as a scalar $\alpha\epsilon(d)$ on M, where α is a root of unity. Then from the same proposition it follows $\mathrm{z}_\chi = \{d \in \mathrm{Irr}(H^*) | \ \ |\chi(d)| = \epsilon(d)\chi(1)\}$. Clearly $\ker \chi \subset \mathrm{z}_\chi$.

Remark 1.2.1. *1. The proof of Proposition 1.1 implies that for a representation M of H affording a character $\chi \in C(H)$ and an irreducible character $d \in \mathrm{Irr}(H^*)$ the following assertions are equivalent:*

1) $d \in \ker \chi$. *2)* $\chi(d) = \epsilon(d)\chi(1)$.
3) $\chi(x_{ij}) = \delta_{ij}\chi(1)$ *for all* i, j. *4)* $dm = \epsilon(d)m$ *for all* $m \in M$.
5) $x_{ij}m = \delta_{ij}m$ *for all* i, j *and* $m \in M$.

2. *Similarly one has the following equivalences:*

1)$d \in \mathrm{z}_\chi$. 2) $|\chi(d)| = \epsilon(d)\chi(1)$.
3) There is a root of unity $\alpha \in \mathbb{C}$ *such that* $\chi(x_{ij}) = \alpha\delta_{ij}\chi(1)$ *for all* i, j.
4) There is a root of unity $\alpha \in \mathbb{C}$ *such that* $dm = \alpha\epsilon(d)m$ *for all* $m \in M$.
5) There is a root of unity $\alpha \in \mathbb{C}$ *such that* $x_{ij}m = \alpha\delta_{ij}m$ *for all* i, j *and* $m \in M$.

3. *Let* $\mathrm{Irr}(H) = \{\chi_0, \cdots, \chi_s\}$ *be the set of all irreducible* H*-characters and* M *be a representation of* H *which affords the character* χ. *If* $\chi = \sum_{i=0}^{s} m_i\chi_i$ *where* $m_i \in \mathbb{Z}_{\geq 0}$ *then* $\ker \chi = \cap_{m_i \neq 0} \ker \chi_i \subset \mathrm{z}_\chi \subset \cap_{m_i \neq 0} \mathrm{z}_{\chi_i}$.

A subset $X \subset \mathrm{Irr}(H^*)$ is closed under multiplication if for every $\chi, \mu \in X$ in the decomposition of $\chi\mu = \sum_{\gamma \in \mathrm{Irr}(H^*)} m_\gamma \gamma$ one has $\gamma \in X$ if $m_\gamma \neq 0$. A subset $X \subset \mathrm{Irr}(H^*)$ is closed under " * " if $x^* \in X$ for all $x \in X$.

Proposition 1.2. *Let* H *be a finite dimensional semisimple Hopf algebra and* M *a representation of* H *affording the character* $\chi \in C(H)$. *Then the subsets* $\ker \chi$ *and* z_χ *of* $\mathrm{Irr}(H^*)$ *are closed under multiplication and " * ".*

Proof. Proposition 1.1 implies that $\chi(d) = \epsilon(d)\chi(1)$ if and only if d acts as $\epsilon(d)\mathrm{Id}_M$ on M. Therefore if $d \in \ker \chi$ then $d^* = S(d) \in \ker \chi$ since $\chi(d^*) = \overline{\chi(d)}$ [95]. Let $d, d' \in \ker \chi$. Then dd' acts as $\epsilon(dd')Id_M$ on M since d acts as $\epsilon(d)Id_M$ and d' acts as $\epsilon(d')Id_M$ on M. Write $dd' = \sum_{i=1}^{q} m_i d_i$ where d_i are irreducible characters of H^* and $m_i \neq 0$ for all $1 \leq i \leq q$. Then $\chi(dd') = \sum_{i=1}^{q} m_i\chi(d_i)$ and

$$\chi(1)\epsilon(dd') = |\chi(dd')| \leq \sum_{i=1}^{q} m_i|\chi(d_i)| \leq \chi(1)\sum_{i=1}^{q} m_i\epsilon(d_i) = \chi(1)\epsilon(dd').$$

It follows by Proposition 1.1 that $\chi(d_i) = \chi(1)\epsilon(d_i)$ and therefore $d_i \in \ker \chi$ for all $1 \leq i \leq q$. The proof for z_χ is similar. □

Remark 1.2.2. *1. For later use let us notice that* $\ker \chi \subset \ker \chi^n$ *for all* $n \geq 0$. *Indeed if* $d \in \mathrm{Irr}(H^*)$ *is an element of* $\ker \chi$ *then one has a simple subcoalgebra* C_d *associated to* d *and* $d = \sum_{i=1}^{\epsilon(d)} x_{ii}$. *Item 1 of Remark 1.2.1 implies that* $\chi(x_{ij}) = \chi(1)\delta_{ij}$. *Thus*

$$\chi^n(d) = \sum_{i=1}^{\epsilon(d)} \sum_{i_1,\cdots,i_{n-1}=1}^{\epsilon(d)} \chi(x_{ii_1})\chi(x_{i_1i_2})\cdots\chi(x_{i_{n-1}i}) = \chi(1)^n\epsilon(d).$$

Similarly it can be shown that $\mathrm{z}_\chi \subset \mathrm{z}_{\chi^n}$ *for all* $n \geq 0$.

2. *If $X \subset \mathrm{Irr}(H^*)$ is closed under multiplication and " * " then it generates a Hopf subalgebra of H denoted by H_X [92]. One has $H_X = \oplus_{d \in X} C_d$. Using this, since the sets* $\ker \chi$ *and* z_χ *are closed under multiplication and " * " they generate Hopf subalgebras of H denoted by H_χ and Z_χ, respectively.*

3. *The proof of Proposition 1.1 implies that $\chi \downarrow_{H_\chi} = \chi(1)\epsilon_{H_\chi}$ where $\chi \downarrow_{H_\chi}$ is the restriction of χ to the subalgebra H_χ and ϵ_{H_χ} is the character of the trivial module over the Hopf algebra H_χ.*

4. *Suppose that M and N are two H-modules affording the characters χ and μ. If M is a submodule of N then it can be easily seen that* $\ker \mu \subset \ker \chi$ *and consequently $H_\mu \subset H_\chi$.*

1.3 Normal Hopf subalgebras

We say that a Hopf subalgebra K of H is the kernel of a character if $K = H_\chi$ for some character $\chi \in C(H)$. The following is the main result of this section.

Theorem 1.3.1. *Let H be a finite dimensional semisimple Hopf algebra. Any normal Hopf subalgebra K of H is the kernel of a character which is central in H^*. More precisely, with the above notations one has:*

$$K = H_{|L|\pi^*(t_L)}.$$

Proof. Let $L = H//K$. Then L is a semisimple and cosemisimple Hopf algebra [82]. The above remark shows that the representations of L are exactly those representations M of H such that $H_{\chi_M} \supset K$ where χ_M is the H-character of M. Let $\pi : H \to L$ be the natural projection and $\pi^* : L^* \to H^*$ be its dual map. Then π^* is an injective Hopf algebra map and L^* can be identified with a Hopf subalgebra of H^*. Therefore, if $t_L \in L^*$ is the idempotent integral of L then $|L|t_L$ is the regular character of L and $H_{|L|\pi^*(t_L)} \supset K$. Since $\pi^*(L^*)$ is a normal Hopf subalgebra of H^* it follows that $\pi^*(t_L)$ is a central element of H^*. We have to show that $H_{|L|\pi^*(t_L)} = K$ and the proof will be complete.

With the above notations, since $\pi^*(t_L)$ is a central idempotent of H^* one can write it as a sum of central primitive orthogonal idempotents:

$$\pi^*(t_L) = \sum_{d \in X} \xi_d$$

where X is a subset of $\mathrm{Irr}(H^*)$. It follows that for any $d \in \mathrm{Irr}(H^*)$ one has that $\pi^*(t_L)(d) = \epsilon(d)$ if $d \in X$ and $\pi^*(t_L)(d) = 0$ otherwise which shows that $X = \ker |L|\pi^*(t_L)$. Since $H_{|L|\pi^*(t_L)} \supset K$ one has $\pi^*(t_L)(d) = \epsilon(d)$ for all $d \in \mathrm{Irr}(K^*)$ and thus $\mathrm{Irr}(K^*) \subset X$. Let

$\Lambda_H \in H$ and $\Lambda_L \in L$ be the idempotent integrals of H and L. Since π is a surjective Hopf algebra map one has $\pi(\Lambda_H) = \Lambda_L$. Then

$$\pi^*(t_L)(\Lambda_H) = t_L(\pi(\Lambda_H)) = t_L(\Lambda_L) = \frac{1}{|L|}.$$

On the other hand, since $\Lambda_H = \frac{1}{|H|}\sum_{d\in \mathrm{Irr}(H^*)} \epsilon(d)d$ it follows that

$$\pi^*(t_L)(\Lambda_H) = \frac{1}{|H|}\sum_{d\in X} \epsilon(d)^2$$

which implies that $\sum_{d\in X}\epsilon(d)^2 = \frac{|H|}{|L|} = |K|$. Since $\sum_{d\in \mathrm{Irr}(K^*)}\epsilon(d)^2 = |K|$ and $\mathrm{Irr}(K^*) \subset X$ we conclude that $\mathrm{Irr}(K^*) = X$ and $H_{|L|\pi^*(t_L)} = K$. □

Let $\mathbb{C}$ be the trivial K-module via the augmentation map ϵ_K. Denote by $\epsilon\uparrow_K^H := \epsilon_K \uparrow_K^H$ the character of the induced module $H \otimes_K \mathbb{C}$.

Corollary 1.3.2. *Let K be a Hopf subalgebra of H. Then K is normal in H if and only if $H_{\epsilon\uparrow_K^H} = K$.*

Proof. Suppose K is a normal Hopf subalgebra of H. With the notations from the above theorem, since $\epsilon\uparrow_K^H = |L|\pi^*(t_L)$ and $H_{|L|\pi^*(t_L)} = K$, it follows $H_{\epsilon\uparrow_K^H} = K$. Conversely, suppose that $H_{\epsilon\uparrow_K^H} = K$. Then using the third item of Remark 6.1.3 it follows $\epsilon\uparrow_K^H\downarrow_K^H = \frac{|H|}{|K|}\epsilon_K$. Using Frobenius reciprocity this implies that for any irreducible character χ of H we have that the value of $m(\chi\downarrow_K, \epsilon_K) = m(\chi, \epsilon\uparrow_K^H)$ is either $\chi(1)$ if χ is a constituent of $\epsilon\uparrow_K^H$ or 0 otherwise. But if Λ_K is the idempotent integral of K then $m(\chi\downarrow_K, \epsilon_K) = \chi(\Lambda_K)$. Thus $\chi(\Lambda_K)$ is either zero or $\chi(1)$ for any irreducible character χ of H. This implies that Λ_K is a central idempotent of H and therefore K is a normal Hopf subalgebra of H by [78] (see also [93, Proposition 1.7.2]). □

1.4 Central characters

Let H be finite dimensional semisimple Hopf algebra over $\mathbb{C}$. Consider the central subalgebra of H defined by $\hat{\mathrm{Z}}(H) = \mathrm{Z}(H)\bigcap C(H^*)$. It is the algebra of H^*-characters which are central in H. Let $\hat{\mathrm{Z}}(H^*) := \mathrm{Z}(H^*)\bigcap C(H)$ be the dual concept, the subalgebra of H-characters which are central in H^*.

Let $\phi : H^* \to H$ given by $f \mapsto f \rightharpoonup \Lambda_H$ where $f \rightharpoonup \Lambda_H = f(S(\Lambda_{H_1}))\Lambda_{H_2}$. Then ϕ is an isomorphism of vector spaces [82].

Remark 1.4.1. *With the notations from the previous section, it can be checked that $\phi(\xi_d) = \frac{\epsilon(d)}{|H|}d^*$ and $\phi^{-1}(\xi_\chi) = \chi(1)\chi$ for all $d \in \mathrm{Irr}(H^*)$ and $\chi \in \mathrm{Irr}(H)$ (see for example [82]).*

We use the following description of $\hat{\mathrm{Z}}(H^*)$ and $\hat{\mathrm{Z}}(H)$ which is given in [124]. Since $\phi(C(H)) = \mathrm{Z}(H)$ and $\phi(\mathrm{Z}(H^*)) = C(H^*)$ it follows that the restriction

$$\phi|_{\hat{\mathrm{Z}}(H^*)} : \hat{\mathrm{Z}}(H^*) \to \hat{\mathrm{Z}}(H)$$

is an isomorphism of vector spaces.

Since $\hat{\mathrm{Z}}(H^*)$ is a commutative semisimple algebra it has a vector space basis given by its primitive idempotents. Since $\hat{\mathrm{Z}}(H^*)$ is a subalgebra of $\mathrm{Z}(H^*)$ each primitive idempotent of $\hat{\mathrm{Z}}(H^*)$ is a sum of primitive idempotents of $\mathrm{Z}(H^*)$. But the primitive idempotents of $\mathrm{Z}(H^*)$ are of the form ξ_d where $d \in \mathrm{Irr}(H^*)$. Thus, there is a partition $\{\mathcal{Y}_j\}_{j\in J}$ of the set of irreducible characters of H^* such that the elements $(e_j)_{j\in J}$ given by

$$e_j = \sum_{d\in\mathcal{Y}_j} \xi_d$$

form a basis for $\hat{\mathrm{Z}}(H^*)$. Note that $e_j(d) = \epsilon(d)$ if $d \in \mathcal{Y}_j$ and $e_j(d) = 0$ if $d \notin \mathcal{Y}_j$. Since $\phi(\hat{\mathrm{Z}}(H^*)) = \hat{\mathrm{Z}}(H)$ it follows that $\widehat{e_j} := |H|\phi(e_j)$ is a $\Bbbk$-linear basis for $\hat{\mathrm{Z}}(H)$. Using the first formula from Remark 1.4.1 one has

$$\widehat{e_j} = \sum_{d\in\mathcal{Y}_j} \epsilon(d)d^*.$$

Remark 1.4.2. *1. By duality, the set of irreducible characters of H can be partitioned into a finite collection of subsets $\{\mathcal{X}_i\}_{i\in I}$ such that the elements $(f_i)_{i\in I}$ given by*

$$f_i = \sum_{\chi\in\mathcal{X}_i} \chi(1)\chi$$

form a $\mathbb{C}$-basis for $\hat{\mathrm{Z}}(H^*)$. Then the elements $\phi(f_i) = \sum_{\chi\in\mathcal{X}_i} \xi_\chi$ are the central orthogonal primitive idempotents of $\hat{\mathrm{Z}}(H)$ and therefore they form a linear basis for this space. Clearly $|I| = |J|$.

*2. Let M be a representation of a semisimple Hopf algebra H. Consider the set $\mathcal{X}$ of all simple representations of H which are a direct summand in some tensor power $M^{\otimes n}$. Then $\mathcal{X}$ is closed under tensor product and " * " and it generates a Hopf algebra L which is a quotient of H (see [103] or [104]). Note that if $\mathcal{X} \subset \mathrm{Irr}(H)$ is closed under multiplication and " * " then using the dual version of item 2 of Remark 6.1.3 it follows that $\mathcal{X}$ generates a Hopf subalgebra $H^*_{\mathcal{X}}$ of H^*. It follows that $L = (H^*_{\mathcal{X}})^*$ (see also [97, Proposition 3.11]). If M has character $\chi \in H^*$ then the character $\pi^*(t_L) \in C(H)$ can be expressed as a polynomial in χ with rational coefficients (see [95, Corollary 19]).*

Next proposition represents a generalization of Brauer's theorem from group representations to Hopf algebras representations with central characters.

Proposition 1.3. *Suppose χ is a character of H which is central in H^*. Then H_χ is a normal Hopf subalgebra of H and the simple representations of $H//H_\chi$ are the simple constituents of all the powers of χ.*

Proof. Since $\chi \in \hat{\mathrm{Z}}(H^*)$ with the above notations one has $\chi = \sum_{j\in J} \alpha_j e_j$, where $\alpha_j \in \mathbb{C}$. It follows that $\chi(d) = \alpha_j \epsilon(d)$ if $d \in \mathcal{Y}_j$. Therefore if $d \in \mathcal{Y}_j$ then $d \in \ker\chi$ if and only if $\alpha_j = \chi(1)$. This implies $\ker\chi$ is the union of all the sets $\mathcal{Y}_j$ such that $\alpha_j = \chi(1)$. Using formula (1.1.2) the integral $|H_\chi|\Lambda_{H_\chi}$ can be written as

$$|H_\chi|\Lambda_{H_\chi} = \sum_{d\in \ker\chi} \epsilon(d)d = \sum_{\{j \mid \alpha_j=\chi(1)\}} \sum_{d\in\mathcal{Y}_j} \epsilon(d)d$$

and therefore

$$|H_\chi|\Lambda_{H_\chi} = |H_\chi|S(\Lambda_{H_\chi}) = \sum_{\{j \mid \alpha_j=\chi(1)\}} \sum_{d\in\mathcal{Y}_j} \epsilon(d)d^* = \sum_{\{j \mid \alpha_j=\chi(1)\}} \widehat{e}_j.$$

Then Λ_{H_χ} is central in H since each $\widehat{e}_j$ is central in H. As above this implies that H_χ is normal in H.

Let V be an H-module with character χ and $I = \bigcap_{m\geq 0} \mathrm{Ann}(V^{\otimes m})$. If L is the quotient Hopf algebra of H generated by the constituents of all the powers of χ then from [104, 103] one has that $L = H/I$. Note that $I \supset HH^+_{|L|\pi^*(t_L)}$. Using item 2 of Remark 1.4.2 one has that $\pi^*(t_L)$ is a polynomial in χ with rational coefficients. Since χ is central in H^* it follows that $\pi^*(t_L)$ is a central element of H^* and thus L^* is a normal Hopf subalgebra of H^*. Using (1.1.1) (for $L^* \hookrightarrow H^*$) it follows that $H//(H^*//L^*)^* = L$. Then if $K = (H^*//L^*)^*$ one has $H//K = L$. Theorem 4.2.2 implies that $H_{|L|\pi^*(t_L)} = K$ thus $H//H_{|L|\pi^*(t_L)} = H//K = L$. But $L = H/I$ and since $I \supset HH^+_{|L|\pi^*(t_L)}$ it follows that $HH^+_{|L|\pi^*(t_L)} = I$. It is easy to see that $HH^+_\chi \subset I$ since the elements of H_χ act as ϵ on each tensor power of V (see item 1 of Remark 6.1.3).

On the other hand $|L|t_L$ is the regular character of L. Then $\ker\chi \supset \ker|L|\pi^*(t_L)$ since χ is a constituent of $|L|\pi^*(t_L)$. Thus $I \supset HH^+_\chi \supset HH^+_{|L|\pi^*(t_L)}$. Since $HH^+_{|L|\pi^*(t_L)} = I$ it follows $HH^+_\chi = I$ and thus $H//H_\chi = L$. □

Theorem 4.2.2 and the previous proposition imply the following corollary:

Corollary 1.4.1. *A Hopf subalgebra of H is normal if and only if it is the kernel of a character χ which is central in H^*.*

Let $H_i := H_{f_i}$, see Remark 1.4.2 for the definition of f_i. From Proposition 1.3 it follows that H_i is a normal Hopf subalgebra of H. If K is any other normal Hopf subalgebra of H then Theorem 4.2.2 implies that $K = H_\chi$ for some central character χ. Following [124] one has $\chi = \sum_{i\in I'} m_i f_i$ for some rational positive numbers m_i and some subset $I' \subset I$. Then $\ker\chi = \cap_{i\in I'} \ker f_i$ which implies that $H_\chi = \bigcap_{i\in I'} H_i$. Thus any normal Hopf subalgebra is an intersection of some of these Hopf algebras H_i.

Remark 1.4.3. *If K and L are normal Hopf subalgebras of H then it is easy to see that $KL = LK$ is a normal Hopf subalgebra of H that contains both K and L.*

Let L be any Hopf subalgebra of H. We define $core(L)$ to be the biggest Hopf subalgebra of L which is normal in H. Based on Remark 1.4.3 clearly $\text{core}(L)$ exists and it is unique. If A is a Hopf subalgebra of H then there is an isomorphism of H-modules $H/HA^+ \cong H\otimes_A \mathbb{C}$ given by $\bar{h} \mapsto h \otimes_A 1$. Thus if $A \subset B \subset H$ are Hopf subalgebras of H then $\epsilon \uparrow_B^H$ is a constituent of $\epsilon \uparrow_A^H$ since there is a surjective H-module map $H/HA^+ \to H/HB^+$.

Remark 1.4.4. *Suppose χ an μ are two characters of two representations M and N of H such that μ is central in H^* and χ is an irreducible character which is a constituent of μ. Using item 1 of Remark 1.4.2 it follows that $\chi \in \mathcal{X}_{i_0}$ for some $i_0 \in I$. Since μ is central in H^* it follows that μ is a linear combination with nonnegative rational coefficients of the elements f_i. Since χ is a constituent of μ it follows that f_{i_0} is also a constituent of an integral multiple of μ. Thus* $\ker \mu \subset \ker f_{i_0}$.

Theorem 1.4.2. *If χ is an irreducible character of H such that $\chi \in \mathcal{X}_i$ for some $i \in I$ then* $\text{core}(H_\chi) = H_{f_i}$.

Proof. Let $K = \text{core}(H_\chi)$. Since χ is a constituent of f_i by item 4 of Remark 6.1.3 one has that $H_{f_i} \subset H_\chi$. The normality of H_{f_i} implies that $H_{f_i} \subset K$. By the proof of Corollary 2.5 $\mu := \epsilon \uparrow_K^H$ is the H-character of $H//K = H/K^+H$ and is central in H^*. Since $\chi \downarrow_{H_\chi} = \chi(1)\epsilon_{H_\chi}$ and $K \subset H_\chi$ it follows that $\chi \downarrow_K = \chi(1)\epsilon_K$. By Frobenius reciprocity one has that χ is a constituent of the character μ. Using Remark 1.4.4 it follows that $\ker f_i \supset \ker \mu$ and $H_{f_i} \supset H_\mu = K$. □

Remark 1.4.5. *Item 3 of the Remark 6.1.3 implies that χ is a constituent of $\epsilon \uparrow_{H_\chi}^H$ and therefore $H_\chi \supseteq H_{\epsilon\uparrow_{H_\chi}^H}$. Let $H_1 = H_\chi$ and*

$$H_{s+1} = H_{\epsilon\uparrow_{H_s}^H} \quad \textit{for } \ s \geq 1.$$

The above argument implies that $H_s \supseteq H_{s+1}$. Since H is finite dimensional we conclude that there is $l \geq 1$ such that $H_l = H_{l+1} = \cdots = H_{l+n} = \cdots$. Corollary 1.3.2 gives that H_l is a normal Hopf subalgebra of H. We claim that $\text{core}(H_\chi) = H_l$. Indeed, for any normal Hopf subalgebra K of H with $K \subset H_\chi \subset H$ we have that $\epsilon \uparrow_{H_\chi}^H$ is a constituent of $\epsilon \uparrow_K^H$ and then using Corollary 1.3.2 it follows that $K = H_{\epsilon\uparrow_K^H} \subseteq H_{\epsilon\uparrow_{H_\chi}^H} = H_2$. Inductively, it can be shown that $K \subset H_s$ for any $s \geq 1$, which implies that $\text{core}(H_\chi) = H_l$.

Proposition 1.4. *Let H be a semisimple Hopf algebra. Then*

$$\bigcap_{\chi \in \text{Irr}(H)} \mathrm{z}_\chi = \bar{G}(H)$$

where $\bar{G}(H)$ is the set of all central grouplike elements of H.

Proof. Any central grouplike element g of H acts as a scalar on each simple H-module. Since $g^{\exp(H)} = 1$ it follows that this scalar is a root of unity and then

$$\bar{G}(H) \subset \bigcap_{\chi \in Irr(H)} \mathrm{z}_{\chi}.$$

Let $d \in \bigcap_{\chi \in \mathrm{Irr}(H)} \mathrm{z}_{\chi}$. If C_d is the simple subcoalgebra of H associated to d (see subsection 1.1.1) then $d = \sum_{i=1}^{\epsilon(d)} x_{ii}$. Item 2 of Remark 1.2.1 implies that x_{ij} acts as $\delta_{i,j}\alpha_{\chi} Id_{M_{\chi}}$ on M_{χ} where α_{χ} is a root of unity. For $i \neq j$, it follows that x_{ij} acts as zero on each irreducible representation of H. Therefore $x_{ij} = 0$ for all $i \neq j$ and d is a grouplike element of H. Since d acts as a scalar on each irreducible representation of H we have $d \in \mathrm{Z}(H)$ and therefore $d \in \bar{G}(H)$. □

The next theorem is the generalization of the fact that $\mathrm{Z}/\ker \chi$ is a cyclic subgroup of $G/\ker \chi$ for any character of the finite group G.

Theorem 1.4.3. *Let M be a representation of H such that its character χ is central in H^*. Then Z_{χ} is a normal Hopf subalgebra of H and $\mathrm{Z}_{\chi}//H_{\chi}$ is the group algebra of a cyclic subgroup of $\mathbb{C}\bar{G}(H//H_{\chi})$.*

Proof. Since $\chi \in \hat{\mathrm{Z}}(H^*)$ one can write $\chi = \sum_{j \in J} \alpha_j e_j$ with $\alpha_j \in \mathbb{C}$. A similar argument to the one in Proposition 1.3 shows that

$$|\mathrm{Z}_{\chi}|\Lambda_{\mathrm{Z}_{\chi}} = |\mathrm{Z}_{\chi}|S(\Lambda_{\mathrm{Z}_{\chi}}) = \sum_{\{j \mid |\alpha_j| = \chi(1)\}} \sum_{d \in \mathcal{Y}_j} \epsilon(d) d^* = \sum_{\{j \mid |\alpha_j| = \chi(1)\}} \widehat{e_j}.$$

Therefore $\Lambda_{\mathrm{Z}_{\chi}}$ is central in H and Z_{χ} is normal Hopf subalgebra of H. Let $\pi : H \to H//H_{\chi}$ be the canonical projection. Since H is a free Z_{χ}-module there is also an injective Hopf algebra map $i : \mathrm{Z}_{\chi}//H_{\chi} \to H//H_{\chi}$ such that $i(\bar{z}) = \pi(z)$ for all $z \in \mathrm{Z}_{\chi}$. Proposition 1.3 implies that the irreducible representations of $H//H_{\chi}$ are precisely the irreducible constituents of tensor powers of χ. From item 1 of Remark 6.1.3 it follows that $\mathrm{Z}_{\chi} \subset \mathrm{Z}_{\chi^l}$ for any nonnegative integer l. Let $d \in \mathrm{z}_{\chi}$ and $C_d =< x_{ij} >$ be the coalgebra associated to d as in subsection 1.1.1. Item 2 of Remark 1.2.1 implies that for $i \neq j$ the element x_{ij} acts as zero on any tensor power of χ and therefore its image under π is zero. Since π is a coalgebra map one has

$$\Delta(\pi(x_{ii})) = \sum_{j=1}^{\epsilon(d)} \pi(x_{ij}) \otimes \pi(x_{ji}) = \pi(x_{ii}) \otimes \pi(x_{ii}).$$

Thus $\pi(x_{ii})$ is a grouplike element of $H//H_{\chi}$. Since $\pi(x_{ii})$ acts as a scalar on each irreducible representation of $H//H_{\chi}$ it follows that $\pi(x_{ii})$ is a central grouplike element of $H//H_{\chi}$. This proves that the image under i of $\mathrm{Z}_{\chi}//H_{\chi}$ is inside $\mathbb{C}\bar{G}(H//H_{\chi})$. The grouplike elements that act as a scalar on the representation M of $H//H_{\chi}$ form a cyclic group by [69, Theorem 5.4] and the proof is finished. □

Remark 1.4.6. *If $\chi \in \hat{\mathrm{Z}}(H^*)$ is an irreducible character of H then Proposition 1.3 together with [69, Theorem 5.4] imply that $\bar{G}(H//H_\chi)$ is a cyclic group of order equal to the index of the character χ.*

We remark that recently, based on the papers [52, 53], Natale and Galindo provided examples of noncentral charactesr χ with non-normal Hopf kernels. These characters are one dimensional characters of the simple Hopf algebras described as twists of group algebras of non-simple groups in [53]. Note that these Hopf algebras also appeared in [31].

Chapter 2

Left and right kernels of representations

The notion of kernel of a representation of a group is a very important notion in studying the representations of groups (see [63]). For example, a classical result of Brauer in group theory states that over an algebraically closed field $\Bbbk$ of characteristic zero if χ is a faithful character of G then any other irreducible character of G is a direct summand in some tensor power of χ.

In [15] the author introduced a similar notion for the kernel of representations of any semisimple Hopf algebra. The notion uses the fact that the exponent of a semisimple Hopf algebra is finite [40]. In this chapter we will extend the notion of kernel to arbitrary Hopf algebras, not necessarily semisimple. The aforementioned Brauer's theorem for groups was extended in [15] for semisimple Hopf algebras A and for those representations of A whose characters are central in A^*. With the help of this new notion of kernel, we extend this theorem to an arbitrary representation whose character is not necessarily central in the dual Hopf algebra, see Theorem 2.3.1. We also prove a version of Brauer's theorem for arbitrary Hopf algebras, giving a new insight to the main results from [103] and [104].

Note that the group algebra $\Bbbk G$ is cocommutative. Lack of cocommutativity suggests the introduction of left and right kernels of modules. It will also be shown that these left and right kernels coincide with the left and respectively right categorical kernels of morphisms of Hopf algebras that has been introduced in [2] and extensively studied in [1]. This coincidence proven in Theorem 2.1.6 suggests also how to generalize the notion of kernel of modules to the non semisimple case.

Recently it was proven in [10] that Hopf subalgebras are normal if and only if they are depth two subalgebras. We extend this result to coideal subalgebras. Moreover we show that in this situation, depth two and normality in the sense defined by Rieffel in [105] also coincide. Normal subalgebras as defined by Rieffel were recently revised in [25, Section 4].

This chapter is organized as follows. The second section introduces the notion of left and right kernels of modules and proves their coincidence with the categorical kernels introduced in [2]. A general version of Brauer's theorem is also stated in this section. The third section is concerned with depth two coideal subalgebras of a Hopf algebra. We prove a left coideal subalgebra is right depth two subalgebra if and only if it is a left normal coideal subalgebra. Section 2.3 considers coideal subalgebras of semisimple Hopf algebra. An extension of Brauer's theorem in this case is also considered.

The Hopf algebra notations from [82] are used in this part but we drop the sigma symbol from Sweedler's notation of comultiplication. Recall a left coideal subalgebra S of A is a subalgebra of A with $\Delta(S) \subset A \otimes S$. We say that a left coideal subalgebra S of A is left normal if and only if it is invariant under the left the adjoint action of A, i.e $a_1 x S(a_2) \in S$ for all $a \in A$ and $x \in S$. Such a coideal subalgebra will be called a left normal coideal subalgebra. Also it follows from [109] that if A is finite dimensional then A is free as left or right S-module. In this Chapter we work over an arbitrary base field $\Bbbk$, unless otherwise specified.

2.1 Left and right kernels

Let A be an arbitrary Hopf algebra over a field $\Bbbk$ and M be an A-module. We say that an element $a \in A$ acts trivially on M if and only $am = \epsilon(a)m$ for all $a \in A$. If S is a subalgebra of A we say that S acts trivially on M if each element of S acts trivially on M.

If M is an A-module then define

$$\mathcal{A}_{\mathbf{M}} = \{a \in A \mid am = \epsilon(a)m \text{ for all } m \in M\} \tag{2.1.1}$$

It is easy to verify that $\mathcal{S}_M$ is a subalgebra of A, thus the largest subalgebra of A which acts trivially on M. If A is semisimple and M a simple A-module then $\dim_{\Bbbk}(\mathcal{S}_M) = \dim_{\Bbbk}(A) - \dim_{\Bbbk}(M)^2$.

2.1.1 Definition of left kernels

Let M be an A-module and $\mathrm{L}_M : A \otimes M \to A \otimes M$ be the linear operator given by $a \otimes m \mapsto a_1 \otimes a_2 m$. Let LKer_M be the largest subspace B of A such that $\mathrm{L}_M|_{B \otimes M} = \mathrm{id}$. Then LKer_M is called the left kernel of M. Thus

$$\mathrm{LKer}_M = \{a \in A \mid a_1 \otimes a_2 m = a \otimes m, \text{ for all } m \in M\} \tag{2.1.1}$$

If A is a semisimple Hopf algebra and M a module with character χ we also write LKer_χ instead of LKer_M.

Remark 2.1.1. *If $A = \Bbbk G$ and M a G-module then LKer_M is the group algebra $\Bbbk[\mathrm{Ker}_G(M)]$ of the usual kernel of M.*

Proposition 2.1.1. *Let A be a Hopf algebra and M be an A-module. Then LKer_M is a left normal coideal subalgebra of A.*

Proof. Applying $\epsilon\otimes\mathrm{Id}$ to the defining relation (2.1.1) of $\mathrm{LKer}_{_M}$ it follows that $\mathrm{LKer}_{_M}\subset\mathcal{S}_{_M}$. It is easy to verify that $\mathrm{LKer}_{_M}$ is an algebra.

Next we will show that $\Delta(\mathrm{LKer}_{_M})\subset A\otimes\mathrm{LKer}_{_M}$. Let $a\in\mathrm{LKer}_{_M}$ and $\Delta(a)=\sum_{i=1}^n h_i\otimes x_i$ where h_i is a basis of A. It will be shown that $x_i\in\mathrm{LKer}_{_M}$ for all $1\leq i\leq n$. Indeed $\Delta^2(a)=\sum_{i=1}^n h_i\otimes\Delta(x_i)$. Since $a_1\otimes a_2m=a\otimes m$ it follows that $a_1\otimes a_2\otimes a_3m=a_1\otimes a_2\otimes m$ for all $m\in M$. Thus $\sum_{i=1}^n h_i\otimes(x_i)_1\otimes(x_i)_2m=\sum_{i=1}^n h_i\otimes x_i\otimes m$ which implies that $(x_i)_1\otimes(x_i)_2m=x_i\otimes m$ for all $m\in M$ and $1\leq i\leq n$.

Now we will show invariance under the left adjoint action of A. Let $h\in A$ and $a\in\mathrm{LKer}_{_M}$. Then $h_1aS(h_2)\in\mathrm{LKer}_{_M}$ since

$$\begin{aligned}(h_1aS(h_2))_1\otimes(h_1aS(h_2))_2m &= h_1a_1S(h_4)\otimes h_2a_2S(h_3)m\\ &= h_1aS(h_4)\otimes h_2S(h_3)m\\ &= h_1aS(h_2)\otimes m\end{aligned}$$

□

Proposition 2.1.2. *Let A be a Hopf algebra and*

$$0\longrightarrow N\xrightarrow{\ i\ }M\xrightarrow{\ p\ }P\longrightarrow 0$$

be a short exact sequence of modules. Then

$$\mathrm{LKer}_{_M}\subseteq\mathrm{LKer}_{_N}\cap\mathrm{LKer}_{_P}.$$

Proof. Clearly $\mathrm{LKer}_{_M}\subset\mathrm{LKer}_{_N}$. Next we will show $\mathrm{LKer}_{_M}\subset\mathrm{LKer}_{_P}$. Indeed if $a\in\mathrm{LKer}_{_M}$ then $a_1\otimes a_2m=a\otimes m$ for all $m\in M$ and applying $\mathrm{id}\otimes p$ it follows that $a\in\mathrm{LKer}_{_P}$ since p is surjective. □

Recall that the exponent of A is the smallest positive number $m>0$ such that $a^{[m]}=\epsilon(a)1$ for all $a\in A$. The generalized power $a^{[m]}$ is defined by $a^{[m]}=a_1a_2...a_m$.

Remark 2.1.2. *Let A be a Hopf algebra and M be an A-module. Then it is easy to check that $L_M^s(a\otimes m)=a_1\otimes a_2^{[s]}m$ for all $s\geq 1$. Thus if A is of a finite exponent m then $L_M^m=\mathrm{id}$ on $A\otimes M$.*

Theorem 2.1.3. *Let A be a Hopf algebra of finite exponent over an algebraically closed field $\Bbbk$ and let*

$$0\longrightarrow N\xrightarrow{\ i\ }M\xrightarrow{\ p\ }P\longrightarrow 0$$

be a short exact sequence of finite dimensional modules. Then

$$\mathrm{LKer}_{_M}=\mathrm{LKer}_{_N}\cap\mathrm{LKer}_{_P}.$$

Proof. Let $L := \mathrm{LKer}_{N} \cap \mathrm{LKer}_{P}$ and consider the exact sequence

$$0 \longrightarrow L \otimes N \xrightarrow{1\otimes i} L \otimes M \xrightarrow{1\otimes p} L \otimes P \longrightarrow 0\cdot$$

If $l \in L$ and $m \in M$ then

$$(1 \otimes p)(L_{M}(l \otimes m) - l \otimes m) = l_1 \otimes p(l_2 m) - l \otimes p(m) = l_1 \otimes l_2 p(m) - l \otimes p(m) = 0$$

since $l \in \mathrm{LKer}_{P}$. Thus $L_{M}(l\otimes m)-l\otimes m \in L\otimes N$. It follows that $L_{N}(L_{M}(l\otimes m)-l\otimes m)) = L_{M}(l \otimes m) - l \otimes m$. This also implies that $L^2_M(l \otimes m) - L_M(l \otimes m) = L_{M}(l \otimes m) - l \otimes m$. Thus $(L_{M} - \mathrm{id})^2 = 0$ on $L \otimes M$. On the other hand since A has finite exponent one has $L^m_{M} = \mathrm{id}$ on $L \otimes M$ by the previous remark. Since $\Bbbk$ is algebraically closed it follows that $L_{M} = \mathrm{id}$ on $L \otimes M$, i.e $L \subset \mathrm{LKer}_{M}$. □

Remark 2.1.3. *Among the Hopf algebras with finite exponent we recall semisimple algebras and group algebras of finite groups over a field of characteristic diving the order of the group. The fact that the exponent of a finite dimensional semisimple Hopf algebra is finite was proven in [40]. In the same paper it is also shown that in this case the exponent divides the third power of the dimension of A.*

Remark 2.1.4. *Let S_{M} the inverse operator of L_{M}. Then S_{M} is given by $S_{M}(a \otimes m) = a_1 \otimes S(a_2)m$ for all $a \in A$ and $m \in M$. It follows that $S_{M}|_{\mathrm{LKer}_M \otimes A} = \mathrm{id}$ and thus one also has that:*

$$\mathrm{LKer}_{M} = \{a \in A|\ a_1 \otimes S(a_2)m = a \otimes m, \ \ \textit{for all} \ \ m \in M\}.$$

Proposition 2.1.4. *Let A be a Hopf algebra and M be an A-module. Then LKer_{M} is the largest coideal subalgebra of A that acts trivially on M.*

Proof. Proposition 2.1.1 implies that LKer_{M} acts trivially on M. On the other hand if $\Delta(S) \subset A \otimes S$ and S acts trivially on M then $s_1 \otimes s_2 m = s_1 \otimes \epsilon(s_2)m = s \otimes m$, which shows that $S \subset \mathrm{LKer}_{M}$. □

Lemma 2.1.1. *Let A be a Hopf algebra with bijective antipode and M be a finite dimensional module over A. Then $\mathrm{LKer}_{M} = \mathrm{LKer}_{M^*}$.*

Proof. For the operator $L_{M^*} : A\otimes M^* \to A\otimes M^*$ one has that $\mathrm{LKer}_{M^*} = \{a \in A|\ a_1\otimes a_2 f = a \otimes f\}$ for all $f \in M^*$. This is equivalent to $a_1 f(S(a_2)m) = af(m)$ for all $m \in M$ and $f \in M^*$. Therefore $a \in \mathrm{LKer}_{M^*}$ if and only if $a_1 \otimes S(a_2)m = a \otimes m$ for all $m \in M$ which implies by Remark 2.1.4 that $\mathrm{LKer}_{M^*} = \mathrm{LKer}_{M}$. □

2.1.2 Definition of right kernel

Let M be an A-module and $\mathrm{R}_{M} : A \otimes M \to A \otimes M$ given by $a \otimes m \mapsto a_2 \otimes a_1 m$. Let RKer_{M} be the largest subspace B of A such that $\mathrm{R}_{M}|_{B\otimes M} = \mathrm{id}$. Thus

$$\mathrm{RKer}_{M} = \{a \in A | \; a_2 \otimes a_1 m = a \otimes m, \;\text{ for all }\; m \in M\} \tag{2.1.1}$$

Suppose that A has a bijective antipode S with inverse S^{-1}. Then the inverse operator U_{M} of R_{M} is given by $U_{M}(a \otimes m) = a_2 \otimes S^{-1}(a_1)m$ for all $a \in A$ and $m \in M$. It follows that $U_{M}|_{\mathrm{RKer}_{M} \otimes A} = \mathrm{id}$ and thus one has also that

$$\mathrm{RKer}_{M} = \{a \in A | \; a_2 \otimes S^{-1}(a_1)m = a \otimes m, \;\text{ for all }\; m \in M\} \tag{2.1.2}$$

It is also easy to check that $\Delta(\mathrm{RKer}_{M}) \subset \mathrm{RKer}_{M} \otimes A$.

Remark 2.1.5. *1)Suppose that the Hopf algebra A has a bijective antipode and let M be an A-module. Then* $\mathrm{RKer}_{M} = S(\mathrm{LKer}_{M})$. *Indeed if* $a \in \mathrm{LKer}_{M}$ *then*

$$S(a)_2 \otimes S(a)_1 m \;=\; S(a_1) \otimes S(a_2)m = S(a) \otimes m$$

which shows that $S(\mathrm{LKer}_{M}) \subset \mathrm{RKer}_{M}$. *Similarly it can be checked that if* $a \in \mathrm{RKer}_{M}$ *then* $S^{-1}a \in \mathrm{LKer}_{M}$. *Thus* $\mathrm{RKer}_{M} = S(\mathrm{LKer}_{M})$.
2)Applying $\epsilon \otimes \mathrm{Id}$ *to the relation* (2.1.1) *it also follows that* $\mathrm{RKer}_{M} \subset \mathcal{S}_{M}$.

2.1.3 Description as categorical kernels

Lemma 2.1.2. *If M and N are two A-modules then* $\mathrm{LKer}_{M} \cap \mathrm{LKer}_{N} \subset \mathrm{LKer}_{M \otimes N}$. *In particular* $\mathrm{LKer}_{M} \subset \mathrm{LKer}_{M^{\otimes n}}$ *for all* $n \geq 1$.

Proof. Suppose $m \in M$, $n \in M$ and $a \in \mathrm{LKer}_{M} \cap \mathrm{LKer}_{N}$. Since $a_1 \otimes a_2 n = a \otimes n$ applying $\Delta \otimes \mathrm{Id}$ one has $a_1 \otimes a_2 \otimes a_3 n = a_1 \otimes a_2 \otimes n$. Thus $a_1 \otimes a_2 m \otimes a_3 n = a_1 \otimes a_2 m \otimes n = a_1 \otimes m \otimes n$. □

Definition 2.1.1. *Let A be a Hopf algebra and M be an A-module. We define the Hopf kernel A_{M} of M as the largest sub-bialgebra of A contained in $\mathcal{S}_{M}$.*

If A is finite dimensional then A_{M} is also the largest Hopf subalgebra of A. It is easy to see that in the case of a semisimple Hopf algebra this notion of kernel coincides with the kernel A_{M} of the module M introduced in [15].

Lemma 2.1.3. *Let A be a Hopf algebra and M be an A-module. Then:*
1) The kernel A_{M} is the largest subcoalgebra in $\mathcal{S}_{M}$.
2) The kernel A_{M} is the largest subcoalgebra in LKer_{M}.
3) The kernel A_{M} is the largest subcoalgebra in RKer_{M}.

Proof. 1) If C is a subcoalgebra of A contained in $\mathcal{S}_{M}$ then $< C >= \oplus_{n \geq 0} C^n$ is a sub-bialgebra of A. Thus one has $< C > \subset A_M$.

2)Clearly A_{M} is a subcoalgebra of $\mathrm{LKer}_{M} \subset \mathcal{S}_{M}$. On the other hand any subcoalgebra of $\mathcal{S}_{M}$ is by definition included in LKer_{M}. Thus the largest subcoalgebra of $\mathcal{S}_{M}$ is A_{M} and coincides with the largest subcoalgebra of LKer_{M}.

3)The proof of 3) is similar to that of 2). □

Proposition 2.1.5. *Let A be a Hopf algebra and M be an A-module. Then*

$$A_{M} = \{a \in A \mid a_1 \otimes a_2 m \otimes a_3 = a_1 \otimes m \otimes a_2 \ \ \textit{for all} \ \ m \in M\}$$

Proof. Let $A'_{M} = \{a \in A \mid a_1 \otimes a_2 m \otimes a_3 = a_1 \otimes m \otimes a_2 \ \text{ for all } \ m \in M\}$. Clearly $A_{M} \subset A'_{M}$, since A_{M} is a coalgebra. On the other hand it easy to check that A'_{M} is a sub-bialgebra of A contained inside $\mathcal{S}_{\mathbf{M}}$. Maximality of the kernel A_{M} implies the other inclusion. □

Let $\pi : A \to B$ a Hopf map. Recall that the Hopf kernel of π was defined in [2] as:

$$\mathrm{HKer}(\pi) = \{a \in A \mid a_1 \otimes \pi(a_2) \otimes a_3 = a_1 \otimes \pi(1) \otimes a_2\} \tag{2.1.1}$$

Also the left and right kernels of π are defined as $\mathrm{LKer}(\pi) = A^{co\,\pi}$ and $\mathrm{RKer}(\pi) = {}^{co\,\pi}A$.

Now, let M be an A-module and $I_{M} = \cap_{n\geq 0}\mathrm{Ann}_A(M^{\otimes\, n})$. Then I_{M} is a Hopf ideal [103] that will be called the Hopf ideal generated by M for the rest of this chapter. Also let $\pi_M : A \to A/I_{M}$ be the canonical projection.

Theorem 2.1.6. *Suppose that M is a finite dimensional module over a finite dimensional Hopf algebra A. Let $I_{M} = \cap_{n\geq 0}\mathrm{Ann}_A(M^{\otimes\, n})$ and $\pi : A \to A/I_{M}$ be the canonical projection. Then*

1) $A^{co\,\pi} = \mathrm{LKer}_{M}$ *and* ${}^{co\,\pi}A = \mathrm{RKer}_{M}$.

2) $\mathrm{Hker}(\pi) = A_M$.

Proof. 1)If $a \in A^{co\,\pi}$ then $a_1 \otimes a_2 m = a_1 \otimes \pi(a_2)m = a \otimes \pi(1)m = a \otimes m$, therefore $a \in \mathrm{LKer}_{M}$. Conversely suppose that $a \in \mathrm{LKer}_{M}$ and let $\Delta(a) = \sum_{i=1}^{s} a_i \otimes x_i$ with a_i a $\Bbbk$-basis of A. Then $x_i \in \mathrm{LKer}_M$ and the Lemma 2.1.2 implies that $x_i \in \mathrm{LKer}_{M^{\otimes\, n}}$ for all $n \geq 1$. Therefore $x_i - \epsilon(x_i)1 \in I_{M}$ which implies that $\pi(x_i) = \epsilon(x_i)1$. Thus $A^{co\,\pi} = \mathrm{LKer}_{M}$ and applying the antipode S it follows that ${}^{co\,\pi}A = \mathrm{RKer}_{M}$.

2)It is easy to see that $A_M \subset \mathrm{HKer}(\pi)$. On the other hand since $\mathrm{HKer}(\pi)$ acts trivially on M it follows that $\mathrm{HKer}(\pi) \subset A_M$ by maximality of A_{M}. □

The following Corollary can be regarded as a generalization of Brauer's theorem for groups.

Corollary 2.1.7. *Suppose that M is a finite dimensional module over a finite dimensional Hopf algebra A. Then*

$$\cap_{n\geq 0}\mathrm{Ann}_A(M^{\otimes\, n}) = \omega(\mathrm{LKer}_{M})A.$$

For a coideal subalgebra S of A denote by ϵ_{S} the character of the left trivial S-module. Then ϵ_{S} is the restriction of the counit ϵ to S.

Remark 2.1.6. *One has that $A \otimes_S \Bbbk$, the trivial left S-module induced to A, is isomorphic to A/AS^+ via the map $a \otimes_S 1 \mapsto \bar{a}$.*

Next Lemma is the first item of [110, Theorem 1.1].

Lemma 2.1.4. *Suppose that the antipode S of the Hopf algebra A is bijective. Let S be a coideal subalgebra of A and $\pi : A \to A/AS^+$ the canonical coalgebra projection. If A is left S-faithfully flat then $A^{co\,\pi} = S$.*

Remark 2.1.7. *From [110, Lemma 4.2] it follows that S is normal whenever $AS^+ \subset S^+A$.*

Proposition 2.1.8. *Let L be a normal left coideal subalgebra of a Hopf algebra A with bijective antipode S. Then*

$$\mathrm{LKer}_{\epsilon_L \uparrow_L^A} = L \tag{2.1.2}$$

Proof. We show first that $L \subset \mathrm{LKer}_{\epsilon_L \uparrow_L^A}$. Indeed, for all $l \in L$ and $a \in A$ one has

$$l_1 \otimes l_2(Sa \otimes_L 1) = l_1 \otimes Sa_1 \otimes_L (a_2 l_2 Sa_3)1) = l \otimes (Sa \otimes_L 1)$$

since L is normal. Since the antipode S is bijective it follows that $L \subset \mathrm{LKer}_{\epsilon_L \uparrow_L^A}$.

Let $\pi : A \to A//L$ be the canonical projection. It follows that $A^{co\,\pi} = L$ by Lemma 2.1.4. Suppose now that $a \in \mathrm{LKer}_M$ with $M = \Bbbk_L \uparrow_L^A \cong A/AL^+$. Then $a \otimes m = a_1 \otimes a_2 m = a_1 \otimes \pi(a_2)m$ for all $m \in M$. In particular for $m = \bar{1}$ one has $a_1 \otimes \pi(a_2) = a \otimes \pi(1)$. Therefore $a \in A^{co\,\pi} = L$. □

Corollary 2.1.9. *Suppose that L is a normal coideal subalgebra of A and $M := \Bbbk \uparrow_L^A$ is the trivial L-module induced up to A. Let I_M be the Hopf ideal generated by M in A and $\pi : A \to A/I_M$ be the canonical projection. Then $A^{co\pi} = L$, i.e $A/I_M = A//L$.*

Proof. By Proposition 2.1.8 it follows that $\mathrm{LKer}_M = L$. On the other hand Corollary 2.1.7 implies that $A^{co\pi} = \mathrm{LKer}_M$. □

Next we will give two examples of left (right) kernels and kernels.

Example 2.1.1. *1) Let A be a Hopf algebra and consider the left adjoint action of A on itself. Then $L := \mathrm{LKer}(A)$ is the largest central coideal subalgebra of A. In this situation the kernel A_A of the adjoint action coincides with the largest central sub-bialgebra of A and it is in fact a Hopf subalgebra called, the Hopf center of A in [1].*

2) Let A be a finite dimensional Hopf algebra and

$$L := \cap_{S \in \mathrm{Irr}(A)} \mathrm{LKer}(S).$$

Then $A//L$ is the largest semisimple Hopf algebra quotient of A. Moreover A has Chevalley property if and only if the Jacobson radical $\mathrm{rad}(A)$ equals $\omega(L)A$.

Remark 2.1.8. *Let $\pi : A \to H$ be a Hopf algebra map with A and H finite dimensional Hopf algebras. Then H can be regarded as A-module via π and let $L := \mathrm{LKer}_A(H)$. It is to see that in this case $A//L$ is isomorphic with the Hopf image of π as defined in [5]. Thus π is inner faithful if and only if L is trivial.*

2.1.4 Core of a coideal subalgebra

Let S be a left coideal subalgebra of A. If L and K are left normal coideal subalgebras of A contained in S then it is easy to see that LK is also a left normal coideal subalgebra contained in S. Thus one can define $L := \mathrm{core}(S)$ as the largest left normal coideal subalgebra of A contained in S.

The next Proposition gives a description of the core of a coideal subalgebra. It generalizes Theorem 3.7 and Remark 3.8 from [15].

Proposition 2.1.10. *Suppose that S is a coideal subalgebra of A and let $L := \mathrm{LKer}_{\epsilon_S \uparrow_S^A}$. Then* $\mathrm{core}(S) = L$.

Proof. First we show that $L \subset S$. Since $\Bbbk \uparrow_S^A = A/AS^+$ one has that

$$L = \{l \in A \mid l_1 \otimes \pi(l_2 a) = l \otimes \pi(a) \text{ for all } a \in A\} \tag{2.1.1}$$

Thus for $a = 1$ one gets that $L \subset A^{co\,\pi} = S$. Now suppose that K is any left normal coideal subalgebra of A contained in S. Then one has a canonical projection of A modules $A/AK^+ \to A/AS^+$ and Proposition 2.1.2 implies $K = \mathrm{LKer}_{A/AK^+} \subset L$. □

Corollary 2.1.11. *Let S be a coideal subalgebra of A. Then S is normal if and only if* $\epsilon_S \uparrow_S^A \downarrow_S^A = \frac{|A|}{|S|} \epsilon_S$

Proof. If S normal the statement follows from Proposition 2.1.8. The converse follows from previous Proposition and Proposition 2.1.4. □

2.1.5 On two endofunctors on $A - \mathrm{mod}$ and respectively $S - \mathrm{mod}$

Let A be a Hopf algebra. Then A is a right comodule subalgebra of A with the usual multiplication and comultiplication given by Δ.

Let S be a left A-comodule subalgebra of A, i.e a left coideal subalgebra of A. Then for any A-module M and any S-module V the comodule structure $\rho : S \to A \otimes S$ defines via pullback, an S-module structure on $M \otimes V$. Denote this module structure by $M \odot V$.

This makes the category of S-modules a left module category over the tensor category A-modules.

Proposition 2.1.12. *Let $S \subset A$ be a right A-comodule subalgebra of A. Then $M \otimes V \uparrow_S^A \cong (M \downarrow_S^A \odot V) \uparrow_S^A$ for any S-module V and any A-module M.*

Proof. The map $T : A \otimes_S (M \odot V) \to M \otimes (A \otimes_S V)$ given by $a \otimes_S (m \otimes v) \mapsto a_1 \otimes a_1 m \otimes (a_2 \otimes_S v)$ is a well defined map and a morphism of A-modules with inverse given by $m \otimes (a \otimes_S v) \mapsto a_2 \otimes_S (S^{-1}(a_1) m \otimes v)$. □

Corollary 2.1.13. *Let $S \subset A$ be a right A-comodule subalgebra of A. Then*

$$M \otimes \epsilon_S \uparrow_S^A = M \downarrow_S^A \uparrow_S^A \tag{2.1.1}$$

for any A-module S.

Proof. Put $V = \Bbbk$, the trivial S-module in the above relation. Note also that $M \odot \Bbbk = M \downarrow_S^A$. $\square$

Define the endofunctors:

$$\mathcal{T} : S - \text{mod} \to S - \text{mod} \quad \text{given by} \quad \mathcal{T}(V) = V \uparrow_S^A \downarrow_S^A \tag{2.1.2}$$

for any S-module V. Also define

$$\mathcal{V} : A - \text{mod} \to A - \text{mod} \quad \text{given by} \quad \mathcal{V}(M) = M \downarrow_S^A \uparrow_S^A \tag{2.1.3}$$

for any A-module M.

The following Lemma is straightforward. It shows that the restriction functor res : $A - \text{mod} \to S - \text{mod}$ is a morphism of $A - \text{mod}$ categories.

Lemma 2.1.1. *Let S be a coideal subalgebra of A and M, N be two A-modules. Then*

$$(M \otimes N) \downarrow_S^A = M \odot N \downarrow_S^A$$

Then one has the following relations:

Lemma 2.1.5. *For any S-module V and any A-module M it follows that:*

1. $$\mathcal{V}^n(M) = M \otimes (\epsilon_S \uparrow_S^A)^n$$

2. $$\mathcal{T}^{n+1}(V) = V \uparrow_S^A \odot \mathcal{T}^n(\epsilon_S)$$

for all $n \geq 1$.

Proof. The first statement follows from Corollary 2.1.13. The second statement is easily proven by induction on n. $\square$

In the next Proposition we need the Frobenius-Perron theory on Grothendieck rings of hopf algebras developed in [46].

Proposition 2.1.14. *Let A be a finite dimensional Hopf algebra and $L := \mathrm{LKer}_M$ be the kernel of a finite dimensional A-module. Then $\epsilon_L \uparrow_L^A$ is the regular character of $A//L$.*

Proof. For any left $A//L$-module one has that $N \downarrow_L^A = |N|\epsilon_L$. Applying previous lemma it follows that $N \otimes \epsilon_L \uparrow_L^A = |N|\epsilon_L \uparrow_L^A$. Thus $\epsilon_L \uparrow_L^A$ is the common Frobenius-Perron eigenvector of the operators of left multiplication by all $A//L$-modules, thus the regular character of $A//L$ (see [46]). $\square$

2.2 Depth two coideal subalgebras

In this section we work over a commutative ring R instead of the field $\Bbbk$. Recall that an extension of R-algebras $B \subset A$ is called right (left) depth two if the module $A \otimes_B A$ is a direct summand in A^n in the category ${}_B\mathrm{Mod}_A$ (respectively ${}_A\mathrm{Mod}_B$) for an arbitrary $n \geq 1$.

In this section we will show that a left coideal subalgebra of a Hopf algebra A is right depth two if and only if it is left normal, i.e closed under the left adjoint action. For this reason in this section we have to work with right modules instead of left. Our treatment is very similar to the one used in [10] for depth two Hopf subalgebras.

2.2.1 Depth two coideal subalgebras

Proposition 2.2.1. *Let S be a right coideal subalgebra of a Hopf algebra A.*

1)Then the map

$$\beta : A \otimes_S A \to A \otimes A/S^+A \quad \textit{given by} \quad a \otimes_S b \mapsto ab_1 \otimes \overline{b_2}$$

is a well defined morphism of (A, S)-bimodules. The (A, S)-bimodule actions are defined as $c(a \otimes_S b)x = ca \otimes bx$ on $A \otimes_S A$ and $c(a \otimes \bar{b})x = cax \otimes b$ for $a, b, c \in A$ and $x \in S$.

2) If $AS^+ \subset S^+A$ then the above map β is an isomorphism.

Proof. The first item follows by direct computation. For the second item consider

$$\gamma : A \otimes A/S^+A \to A \otimes_S A \quad \text{given by} \quad a \otimes_S \bar{b} \mapsto ab_1 \otimes_S Sb_2$$

It is not hard to check that if $AS^+ \subset S^+A$ then γ is well defined. Moreover γ is an inverse for β. □

For the rest of this section let $\bar{A} := A/AS^+A$ and $\pi : A \to \bar{A}$ be the canonical projection. Recall that the extension A/S is $\bar{A}$ -right Hopf Galois [82] if $A^{co\,\pi} = S$ and the canonical map β is bijective.

Proposition 2.2.2. *If A is left or right faithfully flat over S and $AS^+ \subset SA^+$ then A/S is a right $\bar{A}$-Galois extension. In particular S is a left normal coideal subalgebra.*

Proof. By Proposition 2.2.1 one knows that β is bijective. If A is left faithfully flat over S the second theorem of [117, Section 13.1] implies that S equals the equalizer of i_1 and i_2 where $i_1 : A \to A \otimes_S A \;\; a \mapsto a \otimes_S 1$ and $i_2 : A \to A \otimes_S A \;\; a \mapsto 1 \otimes_S a$. Since β is bijective this coincides with the equalizer of $i_1 \circ \beta$ and $i_2 \circ \beta$. But this last equalizer is exactly $A^{co\,\pi}$. Thus S is also closed under the left adjoint action.

If A is right faithfully flat over S a "right version" of the same second theorem from [117, Section 13.1] would also imply that S is the equalizer i_1 and i_2. □

Theorem 2.2.3. *If $AS^+ \subset SA^+$ and A is finitely generated projective as left S-module then S is of right depth two inside A.*

Proof. Using [10, Lemma 2.7] it follows that $\bar{A}|R^n$ in Mod_R and therefore $A \otimes_S A \cong A \otimes \bar{A}|A \otimes R^n \cong A^n$ in ${}_A\mathrm{Mod}_S$. □

Proposition 2.2.4. *Let S be a left coideal subalgebra of A such that A is faithfully flat S-module. If S has right depth two inside A then $AS^+ \subset S^+A$.*

Proof. If S has right depth two inside A then $\Bbbk \otimes_A (A \otimes_S A)$ is a direct summand in $(\Bbbk \otimes_A A)^n$ in Mod_S. Note that $\Bbbk \otimes_S A$ is the trivial S-modules and therefore $\Bbbk \otimes_S A$ divides $\Bbbk^n$ in S-mod. Since S^+ annihilates the S-module $\Bbbk$ this implies that the two sided ideal generated AS^+A annihilates the A-module $\Bbbk \otimes_S A$. Previous remark implies that $AS^+A = S^+A$ and in particular $AS^+ \subset S^+A$. □

Corollary 2.2.5. *Let S be a coideal subalgebra of A such that A is faithfully flat over S. Then S is a right depth two subalgebra of A if and only if S is normal, i.e closed under the left adjoint action.*

2.2.2 Rieffel's normality for coideal subalgebras

Let $B \subset A$ an extension of finite dimensional $\Bbbk$-algebras. An ideal J of B is called A-invariant if $AJ = JA$. Following [105] the extension A/B is called normal if for every maximal two sided ideal I of A the ideal $B \cap I$ is A-invariant.

If S is closed under adjoint action and A has bijective antipode then it easy to verify that S is normal in Rieffel's sense. Indeed, if I is a two-sided ideal of A and $x \in S \cap I$ then $ax = a_1xS(a_2)a_3 \in (I \cap S)A$. Also $xSa = a_3Sa_2xSa_1 \in A(I \cap S)$. Conversely, if a coideal subalgebra S of a semisimple Hopf algebra A is normal in Rieffel's sense then $AS^+ = S^+A$ since $S^+ = A^+ \cap S$ is a maximal two sided ideal of S. Then Proposition 2.2.2 implies that S is closed under the left adjoint action. Thus normality, depth two and Rieffel's normality coincide for coideal subalgebras of a Hopf algebra with bijective antipode.

Remark 2.2.1. *If A is semisimple it will be shown in the next section that S is also semisimple. Note that in this case left depth two coincides with right depth two by [25, Theorem 4.6].*

This remark brings up the question whether left and right depth two coincide on coideal subalgebras. Further one can ask whether the extension A/S is a quasi-Frobenius extension, at least in the finite dimensional case. Results from [109] shows that S is a Frobenius algebra if S is finite dimensional.

2.3 The semisimple case

Throughout of this section we assume that the Hopf algebra A is semisimple. Let S be a left coideal subalgebra of A.

Since A is free over S [109] there is a decomposition $A = S \oplus R$ as left S-modules. Consider $\Lambda_{_A} = x + r$ the decomposition of the idempotent integral $\Lambda_{_A}$ in the above direct sum. Then clearly $sx = \epsilon(s)x$ for all $s \in S$ and $\epsilon(x) = 1$. Similarly since A is free as right S there is a decomposition $A = S \oplus R'$ as left S-module. Consider $\Lambda_{_A} = y + r'$ the decomposition of the idempotent integral $\Lambda_{_A}$ in the above direct sum. Then clearly $ys = \epsilon(s)x$ for all $s \in S$ and $\epsilon(y) = 1$. Thus $x = yx = y$.

Lemma 2.3.1. *Let S be a left coideal subalgebra of a semisimple Hopf algebra A. Then S is also semisimple.*

Proof. We will use a Maschke type argument for the category of S-modules. Suppose that W is an S- submodule of V and $f : V \to W$ a $\Bbbk$-linear projection. Then it easy to check $\tilde{f} : V \to W$ given by

$$\tilde{f}(v) = \sum Sx_1 f(x_2 v)$$

is an S-projection. Indeed one has that $\tilde{f}(sv) = \sum Sx_1 f(x_2 sv) = s_1 S(s_2) Sx_1 f(x_2 s_3 v) = s\tilde{f}(v)$ and $\tilde{f}(w) = Sx_1 x_2 w = w$. □

Thus the element x from above is the central idempotent corresponding to the trivial module $\epsilon_{_S}$ of S.

2.3.1 Rieffel's equivalence relation for a coideal subalgebra

Let $S \subseteq A$ be a left coideal subalgebra of A. Let V and W two S-modules. that We say that $V \sim W$ if and only if there is a simple A-module M such that V and W are both constituents of $M \downarrow_{S}^{A}$. The relation $\sim$ is reflexive and symmetric but not transitive in general. Its transitive closure is denoted by $\approx$. Thus we say that $V \approx W$ if and only if there is $m \geq 1$ and a sequence $V_{i_0}, V_{i_1}, \cdots, V_{i_{m-1}}, V_{i_m}$ of simple S-modules such that $V = V_{i_0} \sim V_{i_1} \sim V_{i_2} \sim \cdots \sim V_{i_{m-1}} \sim V_{i_m} = W$. As explained in [25] it follows that $V \approx W$ if and only if there is $n \geq 0$ such that V is a constituent to $\mathcal{T}^n(W)$. This is equivalent to W to be a constituent of $\mathcal{T}^n(V)$.

We denote the above equivalence relation by d_{S}^{A}. This equivalence relation is considered in [105] in the context of any extension of semisimple Hopf algebras.

Similarly one can define an equivalence relation u_{B}^{A} on the set of irreducible A-modules. We say that $M \sim N$ for two simple A-modules M and N if and only if their restriction to S have a common constituent. Then u_{S}^{A} is the transitive closure of $\sim$. Similarly M and N are equivalent if and only if there is $n \geq 0$ such that M is a constituent to $\mathcal{V}^n(N)$. This is equivalent to N to be a constituent of $\mathcal{V}^n(M)$.

2.3.2 Tensor powers of a character

The following Theorem can be viewed as a generalization of Brauer's theorem for groups:

Theorem 2.3.1. *Let A be a semisimple Hopf algebra and M be an A-module with character χ. If $L := \mathrm{LKer}_{M}$ then the irreducible modules of $A//L$ are precisely all the irreducible constituents the tensor powers $M^{\otimes n}$ with $n \geq 0$.*

Proof. One has that $A//L$ is a semisimple Hopf algebra. On the other hand since L is left normal Corollary 2.1.11 implies that the simple A-submodules of $\Bbbk \uparrow_L^A$ and the simple A -submodules of $A//L$ coincide. Then description of the Hopf ideal I_{M} given in Corollary 2.1.7 implies the conclusion. □

Next Corollary follows from [15, Proposition 3.3].

Corollary 2.3.2. *Let A be a semisimple Hopf algebra and M an A module with character χ. If χ is central in A^* then* $\mathrm{Lker}_{\chi} = \mathrm{Rker}_{\chi} = A_{\chi}$.

Proof. Let $L := \mathrm{Lker}_{\chi}$. Then from the previous Theorem and [15, Proposition 3.3] the quotient Hopf algebras $A//L$ and $A//A_{\chi}$ have the same irreducible representations, namely the irreducible representations of all tensor powers of M. It follows that $\dim_{\Bbbk} L = \dim_{\Bbbk} A_{\chi}$. Since $A_{\chi} \subset L$ then one has $A_{\chi} = L$. Then $A_{\chi} = \mathrm{Rker}_{\chi}$ by applying the antipode S. □

In view of the last corollary the next theorem can be viewed as an extension of [25, Corollary 6.5] from representations of group algebras.

Theorem 2.3.3. *Let A be a semisimple Hopf algebra and S be a coideal subalgebra of A. Let $L := \mathrm{core}(S)$. Then the equivalence relations u_{S}^{A} and u_{L}^{A} coincide.*

Proof. From Proposition 2.1.1 it follows that $L := \mathrm{LKer}_{\epsilon_S \uparrow_S^A}$. Two A-modules M and N are equivalent if and only if N is a constituent of $\mathcal{V}^n(M)$ for some $n \geq 1$. From the formula of $\mathcal{V}^n(M)$ it follows that M and N are equivalent if and only if N is a constituent of $M \otimes (\epsilon_S \uparrow_S^A)^n$ for some $n \geq 1$.

But from Theorem 2.3.1 it follows that $\epsilon_L \uparrow_L^A$ has as constituents all the irreducible constituents of all tensor powers $(\epsilon_S \uparrow_S^A)^n$ with $n \geq 0$. Thus two A-modules M and N are equivalent relative to S if and only if they are equivalent relative to L. □

2.3.3 The integral element of S

Let $x_{S} := x$ denote the element from the beginning of this section.

In this situation one has $AS^+ = A(1 - x_{S})$. Indeed if $s \in S^+$ then $s = sx_{S} + s(1 - x_{S}) = \epsilon(s)x_{S} + s(1 - x_{S}) = s(1 - x_{S})$.

Proposition 2.3.4. *Let A be a semisimple Hopf algebra and S be a coideal subalgebra of A. Then the following statements are equivalent:*

1)S is normal in A

2)ϵ_S by itself from an equivalence class of d_S^A.

3)The element x_S is central in A.

Proof. Corollary 2.1.11 implies the equivalence between the first two items. [25, Proposition 3.1] shows the equivalence between the last two items. By results from [110] it follows that S is normal if and only if $AS^+ = S^+A$. □

Note that the equivalence between first and third item generalizes [78, Proposition 1].

The depth of Hopf subalgebras and coideal subalgebras of a finite dimensional Hopf algebra was subsequently studied by others autors in [9, 34, 51, 65, 66, 114, 115].

Part II

Representation theory of semisimple Hopf algebras

Chapter 3

Coset decomposition for semisimple Hopf algebras

In this chapter we introduce a notion of double coset for semisimple finite dimensional Hopf algebras, similar to the one for groups. This is achieved by considering an equivalence relation on the set of irreducible characters of the dual Hopf algebra. The equivalence relation that we define generalizes the equivalence relation introduced in [92]. Using Frobenius-Perron theory for nonnegative Hopf algebras the results from [92] are generalized and proved in a simpler manner.

The chapter is organized as follows. In the first Section we recall some basic results for finite dimensional semisimple Hopf algebras that we need in the other sections.

In section 3.2 the equivalence relation on the set of irreducible characters of the dual Hopf algebra is introduced and the coset decomposition it is proven. Using this coset decomposition in the next section we prove a result concerning the restriction of a module to a normal Hopf subalgebra. A formula for the induction from a normal Hopf subalgebra is also described using Frobenius reciprocity. A formula equivalent to the Mackey decomposition formula for groups is given in the situation of a unique double coset.

Section 3.4 considers one of the above equivalence relations but for the dual Hopf algebra. In the situation of normal Hopf subalgebras this relation can be written in terms of the restriction of the characters to normal Hopf subalgebras. Some results similar to those in group theory are proved.

The next sections studies the restriction functor from a semisimple Hopf algebra to a normal Hopf subalgebra. We define a notion of conjugate module similar to the one for modules over normal subgroups of a group. Some results from group theory hold in this more general setting. In particular we show that the induced module restricted back to the original normal Hopf subalgebra has as irreducible constituents the constituents of all the conjugate modules. Note that results of this chapter can be applied in the program of classification of semisimple Hopf algebras of low dimension, see for example [39, 45, 55, 88, 89, 91].

3.1 Notations

Throughout this chapter, H denotes a semisimple Hopf algebra over $\mathbb{C}$. It follows that H is also cosemisimple [73]. If K is a Hopf subalgebra of H then K is also semisimple and cosemisimple Hopf algebra [82]. For a (finite dimensional) semisimple Hopf algebra H we use as above the notation $\Lambda_H \in H$ for the integral of H with $\epsilon(\Lambda_H) = |H|$ and $t_H \in H^*$ for the integral of H^* with $t_H(1) = |H|$.

All algebras and coalgebras in this chapter are defined over the complex numbers $\mathbb{C}$. As before, for a vector space V over $\mathbb{C}$, by $|V|$ is denoted the dimension $\dim_{\mathbb{C}}V$.

3.2 Double coset formula for cosemisimple Hopf algebras

In this section let H be a semisimple finite dimensional Hopf algebra as before and K and L be two Hopf subalgebras. Then H can be decomposed as sum of $K-L$ bimodules which are free both as K-modules and L-modules and are analogues of double cosets in group theory. To the end we give an application in the situation of a unique double coset.

There is a bilinear form $m : C(H^*) \otimes C(H^*) \to \Bbbk$ defined as follows (see [92]). If M and N are two H-comodules with characters c and d then $m(c,\ d)$ is defined as $\dim_{\Bbbk}\mathrm{Hom}^H(M,\ N)$. The following properties of m (see [92, Theorem 10]) will be used later:

$$m(x,\ yz) = m(y^*,\ zx^*) = m(z^*,\ x^*y) \text{ and}$$

$$m(x, y) = m(y,\ x) = m(y^*,\ x^*)$$

for all $x, y, z \in C(H^*)$.

Let H be a finite dimensional cosemisimple Hopf algebra and K, L be two Hopf subalgebras of H. We define an equivalence relation $r^H_{K,\ L}$ on the set of simple subcoalgebras of H as following: $C \sim D$ if $C \subset$ KDL.

Since the set of simple subcoalgebras is in bijection with $\mathrm{Irr}(H^*)$ the above relation in terms of H^*-characters becomes the following: $c \sim d$ if $m(c\ , \Lambda_{_K} d\Lambda_{_L}) > 0$ where $\Lambda_{_K}$ and $\Lambda_{_L}$ are the integrals of K and L with $\epsilon(\Lambda_{_K}) = |K|$ and $\epsilon(\Lambda_{_L}) = |L|$ and $c,\ d \in \mathrm{Irr}(H^*)$.

It is easy to see that $\sim$ is an equivalence relation. Clearly $c \sim c$ for any $c \in \mathrm{Irr}(H^*)$ since both $\Lambda_{_K}$ and $\Lambda_{_L}$ contain the trivial character.

Using the above properties of the bilinear form m , one can see that if $c \sim d$ then $m(d,\ \Lambda_{_K} c\Lambda_{_L}) = m(\Lambda^*_{_K},\ c\Lambda_{_L} d^*) = m(c^*,\ \Lambda_{_L} d^* \Lambda_{_K}) = m(c,\ \Lambda^*_{_K} d\Lambda^*_{_L}) = m(c,\ \Lambda_{_K} d\Lambda_{_L})$ since $\Lambda^*_{_K} = \Lambda_{_K}$ and $\Lambda^*_{_L} = \Lambda_{_L}$. Thus $d \sim c$.

The transitivity can be easier seen that holds in terms of simple subcoalgebras. Suppose that $c \sim d$ and $d \sim e$ and $c,\ d$, and e are three irreducible characters associated to the simple subcoalgebras $C,\ D$ and E respectively. Then $C \subset KDL$ and $D \subset KEL$. The last relation implies that $KDL \subset K^2EL^2 = KEL$. Thus $C \subset KEL$ and $c \sim e$.

If $\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_l$ are the equivalence classes of $r^H_{K,\,L}$ on $\mathrm{Irr}(H^*)$ then let

$$a_i = \sum_{d \in \mathcal{C}_i} \epsilon(d)d \tag{3.2.1}$$

for $1 \leq i \leq l$.

For any character $d \in C(H^*)$ let L_d and R_d be the left and right multiplication with d on $C(H^*)$.

Proposition 3.1. *With the above notations it follows that a_i are eigenvectors of the operator $T = L_{\Lambda_K} \circ R_{\Lambda_L}$ on $C(H^*)$ corresponding to the eigenvalue $|K||L|$.*

Proof. Definition of $r^H_{K,\,L}$ implies that $d \sim d'$ if and only if $m(d',\ T(d)) > 0$. It follows that $T(a_i)$ has all the irreducible constituents in $\mathcal{C}_i$ for all $1 \leq i \leq l$. Since $\Lambda_H = \sum_{i=1}^{l} a_i$ the formula $T(\Lambda_H) = |H||K|\Lambda_H$ gives that $T(a_i) = |K||L|a_i$ for all $1 \leq i \leq l$. □

In the sequel, we use the Frobenius-Perron theorem for matrices with nonnegative entries (see [54]). If A is such a matrix then A has a positive eigenvalue λ which has the biggest absolute value among all the other eigenvalues of A. The eigenspace corresponding to λ has a unique vector with all entries positive. λ is called the principal value of A and the corresponding positive vector is called the principal vector of A. Also the eigenspace of A corresponding to λ is called the principal eigenspace of A.

The following result is also needed:

Proposition 3.2.1. *([54], Proposition 5.) Let A be a matrix with nonnegative entries such that A and A^t have the same principal eigenvalue and the same principal vector. Then after a permutation of the rows and the same permutation of the columns A can be decomposed in diagonal blocks $A = A_1,\ A_2,\ \cdots,\ A_l$ with each block an indecomposable matrix.*

Recall from [54] that a matrix $A \in M_n(\mathbb{C})$ is called decomposable if the set $I = \{1,\ 2,\ \cdots .\ n\}$ can be written as a disjoint union $I = J_1 \cup J_2$ such that $a_{uv} = 0$ whenever $u \in J_1$ and $v \in J_2$. Otherwise the matrix A is called indecomposable.

Theorem 3.2.2. *Let H be a finite dimensional semisimple Hopf algebra and K, L be two Hopf subalgebras of H. Consider the linear operator $T = L_{\Lambda_K} \circ R_{\Lambda_L}$ on the character ring $C(H^*)$ and $[T]$ the matrix associated to T with respect to the standard basis of $C(H^*)$ given by the irreducible characters of H^*.*

1. *The principal eigenvalue of $[T]$ is $|K||L|$.*
2. *The eigenspace corresponding to the eigenvalue $|K||L|$ has $(a_i)_{1 \leq i \leq l}$ as $\mathbb{C}$-linear basis, were a_i are defined in* (3.2.1).

Proof. 1. Let λ be the biggest eigenvalue of T and v the principal eigenvector corresponding to λ. Then $\Lambda_K v \Lambda_L = \lambda v$. Applying ϵ on both sides of this relation it follows that $|K||L|\epsilon(v) = \lambda\epsilon(v)$. But $\epsilon(v) > 0$ since v has positive entries. It follows that $\lambda = |K||L|$.

2. It is easy to see that the transpose of the matrix $[T]$ is also $[T]$. To check that let $x_1, \cdots, x_s$ be the basis of $C(H^*)$ given by the irreducible characters of H^* and suppose that $T(x_i) = \sum_{j=1}^{s} t_{ij}x_j$. Thus $t_{ij} = m(x_j, \Lambda_K x_i \Lambda_L)$ and $t_{ji} = m(x_i, \Lambda_K x_j \Lambda_L) = m(\Lambda_K^*, x_j \Lambda_L x_i^*) = m(x_j^*, \Lambda_L x_i^* \Lambda_K) = m(x_j, \Lambda_K^* x_i \Lambda_L^*) = t_{ij}$ since $\Lambda_K^* = S(\Lambda_K) = \Lambda_K$ and also $\Lambda_L^* = \Lambda_L$.

   Proposition 3.2.1 implies that after a permutation of the rows and the same permutation of the columns the matrix $[T]$ decomposes in diagonal blocks $A = \{A_1, A_2, \cdots, A_s\}$ with each block an indecomposable matrix. This decomposition of $[T]$ in diagonal blocks gives a partition $\mathrm{Irr}(H^*) = \cup_{i=1}^{s} \mathcal{A}_i$ on the set of irreducible characters of H^*, where each $\mathcal{A}_i$ corresponds to the rows (or columns) indexing the block A_i. The eigenspace of $[T]$ corresponding to the eigenvalue λ is the sum of the eigenspaces of the diagonal blocks $A_1, A_2, \cdots A_l$ corresponding to the same value. Since each A_i is an indecomposable matrix it follows that the eigenspace of A_i corresponding to λ is one dimensional (see [54]). If $b_j = \sum_{d \in \mathcal{A}_j} \epsilon(d)d$ then as in the proof of Proposition 3.1 it can be seen that b_j is eigenvector of T corresponding to the eigenvalue $\lambda = |K||L|$. Thus b_j is the unique eigenvector of A_j corresponding to the eigenvalue $|K||L|$. Therefore each a_i is a linear combination of these vectors. But if $d \in \mathcal{A}_i$ and $d' \in \mathcal{A}_j$ with $i \neq j$ then $m(d', T(d)) = 0$ and the definition of $r^H_{K,\,L}$ implies that $d \nsim d'$. This means that a_i is a scalar multiple of some b_j and this defines a bijective correspondence between the diagonal blocks and the equivalence classes of the relation $r^H_{K,\,L}$. Thus the eigenspace corresponding to the principal eigenvalue $|K||L|$ has a $\mathbb{C}$- basis given by a_i with $1 \leq i \leq l$.

□

Corollary 3.2.3. *Let H be a finite dimensional semisimple Hopf algebra and K, L be two Hopf subalgebras of H. Then H can be decomposed as*

$$H = \bigoplus_{i=1}^{l} B_i \tag{3.2.2}$$

where each B_i is a (K, L)- bimodule free as both left K-module and right L-module.

Proof. Consider as above the equivalence relation $r^H_{K,\,L}$ relative to the Hopf subalgebras K and L. For each equivalence class $\mathcal{C}_i$ let $B_i = \bigoplus_{C \in \mathcal{C}_i} C$. Then $KB_iL = B_i$ from the definition of the equivalence relation. It follows that $B_i = KB_iL \in {}_K\mathcal{M}^H_L$ which implies that B_i is free as left K-module and right L-module [96]. □

The bimodules B_i from the above corollary will be called a double coset for H with respect to K and L.

Corollary 3.2.4. *With the above notations, if $d \in \mathcal{C}_i$ then*

$$\frac{\Lambda_K}{|K|} d \frac{\Lambda_L}{|L|} = \epsilon(d) \frac{a_i}{\epsilon(a_i)} \tag{3.2.3}$$

Proof. One has that $\Lambda_K d\Lambda_L$ is an eigenvector of $T = L_{\Lambda_K} \circ R_{\Lambda_L}$ with the maximal eigenvalue $|K||L|$. From Theorem 3.2.2 it follows that $\Lambda_K d\Lambda_L$ is a linear combination of the elements a_j. But $\Lambda_K d\Lambda_L$ cannot contain any a_j with $j \neq i$ because all the irreducible characters entering in the decomposition of the product are in $\mathcal{C}_i$. Thus $\Lambda_K d\Lambda_L$ is a scalar multiple of a_i and formula (3.2.3) follows. □

Remark 3.2.1. *Let C_1 and C_2 be two subcoalgebras of H and $K = \sum_{n\geq 0} C_1^n$ and $L = \sum_{n\geq 0} C_2^n$ be the two Hopf subalgebras of H generated by them [95]. The above equivalence relation $r^H_{K,\,L}$ can be written in terms of characters as follows: $c \sim d$ if $m(c,\ c_1^n d c_2^m) > 0$ for some natural numbers $m, n \geq 0$.*

Remark 3.2.2. *Setting $C_1 = \Bbbk$ in the above remark Theorem 3.2.2 gives [92, Theorem 7]. The above equivalence relation is denoted by $r^H_{\Bbbk,\,L}$ and can be written as $c \sim d$ if and only if $m(c,\ dc_2^m) > 0$ for some natural number $m \geq 0$. The equivalence class corresponding to the simple coalgebra $\Bbbk 1$ consists of the simple subcoalgebras of all the powers C_2^m for $m \geq 0$ that is all the simple subcoalgebras of L. Without loss of generality we may assume that this equivalence class is $\mathcal{C}_1$. It follows that $a_1 = \Lambda_L$ and*

$$\frac{d}{\epsilon(d)}\frac{\Lambda_L}{|L|} = \frac{a_i}{\epsilon(a_i)} \tag{3.2.4}$$

for any irreducible character $d \in \mathcal{C}_i$.

Let K be a Hopf subalgebra of H and $s = |H|/|K|$. Then H is free as left K-module [96]. If $\{a_i\}_{i=1,\,s}$ is a basis of H as left K-module then $H = Ka_1 \oplus Ka_2 \cdots \oplus Ka_s$ as left K-modules. Consider the operator L_{Λ_K} given by left multiplication with Λ_K on H. The eigenspace corresponding to the eigenvalue $|K|$ has a basis given by $\Lambda_K a_i$, thus it has dimension s. If we restrict the operator L_{Λ_K} on $C(H^*) \subset H$ then Theorem 3.2.2 implies that the number of equivalence classes of $r^H_{\Bbbk,\,K}$ is equal to the dimension of the eigenspace of L_{Λ_K} corresponding to the eigenvalue $|K|$. Thus the number of the equivalence classes of $r^H_{\Bbbk,\,K}$ is always less or equal then the index of K in H.

Example 3.2.1. *Let $H = \Bbbk Q_8 \#^{\alpha} \Bbbk C_2$ be the 16-dimensional Hopf algebra described in [68]. Then $G(H^*) =< x > \times < y > \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathrm{Irr}(H^*)$ is given by the four one dimensional characters $1,\ x,\ y,\ xy$ and 3 two dimensional characters denoted by $d_1,\ d_2,\ d_3$. The algebra structure of $C(H^*)$ is given by:*

$$x.d_1 = d_3 = d_1.x,\ x.d_2 = d_2 = d_2.x,\ \ x.d_3 = d_1 = d_3.x$$

$$y.d_i = d_i = d_i.y\ \text{ for all }\ i = 1, 3$$

$$d_1^2 = d_3^2 = x + xy + d_2,\ \ d_2^2 = 1 + x + y + xy,\ \ d_1 d_2 = d_2 d_1 = d_1 + d_3$$

$$d_1 d_3 = d_3 d_1 = 1 + y + d_2$$

Consider $K = \Bbbk < x >$ as Hopf subalgebra of H and the equivalence relation $r^H_{\Bbbk,\, K}$ on $\mathrm{Irr}(H^*)$. *The equivalence classes are given by $\{1,\ x\}$, $\{y,\ xy\}$, $\{d_2\}$ and $\{d_1,\ d_3\}$ and the number of them is strictly less than the index of K and H. If $C_2 \subset H$ is the coalgebra associated to d_2 then the third equivalence class gives in the decomposition* (3.2.2) *the free K-module $C_2K = C_2$ whose rank is strictly less then the dimension of C_2.*

3.3 More on coset decomposition

Let H be a semisimple Hopf algebra and A be a Hopf subalgebra. Define $H//A = H/HA^+$ and let $\pi : H \to H//A$ be the natural module projection. Since HA^+ is a coideal of H it follows that $H//A$ is a coalgebra and π is also a coalgebra map.

Let $\Bbbk$ be the trivial A-module via the counit ϵ. It can be checked that $H//A \cong H \otimes_A \Bbbk$ as H-modules via the map $\hat{h} \mapsto h \otimes_A 1$. Thus $\dim_{\Bbbk} H//A = \mathrm{rank}_A H$.

If L and K are Hopf subalgebras of H define $LK//K := LK/LK^+$. LK is a right free K-module since $LK \in \mathcal{M}^H_K$. A similar argument to the one above shows that $LK//K \cong LK \otimes_K \Bbbk$ as left L-modules where $\Bbbk$ is the trivial K-module. Thus $\dim_{\Bbbk} LK//K = \mathrm{rank}_K LK$. It can be checked that LK^+ is a coideal in LK and therefore $LK//K$ has a natural coalgebra structure.

Theorem 3.3.1. *Let H be a semisimple Hopf algebra and K, L be two Hopf subalgebras of H. Then $L//L \cap K \cong LK//K$ as coalgebras and left L-modules.*

Proof. Define the map $\phi : L \to LK//K$ by $\phi(l) = \hat{l}$. Then ϕ is the composition of $L \hookrightarrow LK \to LK//K$ and is a coalgebra map as well as a morphism of left L-modules. Moreover ϕ is surjective since $\hat{lk} = \epsilon(k)\hat{l}$ for all $l \in L$ and $k \in K$. Clearly $L(L \cap K)^+ \subset \ker(\phi)$ and thus ϕ induces a surjective map $\phi : L//L \cap K \to LK//K$.

Next it will be shown that

$$\frac{|L|}{|L \cap K|} = \frac{|KL|}{|K|}$$

which implies that ϕ is bijective since both spaces have the same dimension. Consider on $\mathrm{Irr}(H^*)$ the equivalence relation introduced above and corresponding to the linear operator $L_{\Lambda_L} \circ R_{\Lambda_K}$. Assume without loss of generality that $\mathcal{C}_1$ is the equivalence class of the character 1 and put $d = 1$ the trivial character, in formula (3.2.3). Thus $\frac{\Lambda_L}{|L|}\frac{\Lambda_K}{|K|} = \frac{a_1}{\epsilon(a_1)}$. But from the definition of $\sim$ it follows that a_1 is formed by the characters of the coalgebra LK. On the other hand $\Lambda_L = \sum_{d \in Irr(L^*)} \epsilon(d)d$ and $\Lambda_K = \sum_{d \in Irr(K^*)} \epsilon(d)d$ (see [72]). Equality 3.2 follows counting the multiplicity of the irreducible character 1 in $\Lambda_K \Lambda_L$. Using [92, Theorem 10] we know that $m(1,\ dd') > 0$ if and only if $d' = d^*$ in which case $m(1,\ dd') = 1$. Then $m(1,\ \frac{\Lambda_L}{|L|}\frac{\Lambda_K}{|K|}) = \frac{1}{|L||K|}\sum_{d \in \mathrm{Irr}(L \cap K)} \epsilon(d)^2 = \frac{|L \cap K|}{|L||K|}$ and $m(1,\ \frac{a_1}{\epsilon(a_1)}) = \frac{1}{\epsilon(a_1)} = \frac{1}{|LK|}$. ☐

Corollary 3.3.2. *If K and L are Hopf subalgebras of H then* $\mathrm{rank}_K LK = \mathrm{rank}_{L \cap K} L$.

Proposition 3.2. *Let H be a finite dimensional cosemisimple Hopf algebra and K, L be two Hopf subalgebras of H such that $KL = LK$. If M is a K-module then*

$$M \uparrow_K^{LK} \downarrow_L \cong (M \downarrow_{L\cap K}) \uparrow^L$$

as left L-modules.

Proof. For any K-module M one has

$$M \uparrow^{LK} \downarrow_L = LK \otimes_K M$$

while

$$(M \downarrow_{L\cap K}) \uparrow^L = L \otimes_{L\cap K} M.$$

The previous Corollary implies that $\mathrm{rank}_K LK = \mathrm{rank}_{L\cap K} L$ thus both modules above have the same dimension.

Define the map $\phi : L \otimes_{L\cap K} M \to LK \otimes_K M$ by $\phi(l \otimes_{L\cap K} m) = l \otimes_K m$ which is the composition of $L \otimes_{L\cap K} M \hookrightarrow LK \otimes_{L\cap K} M \to LK \otimes_K M$. Clearly ϕ is a surjective homomorphism of L-modules. Equality of dimensions implies that ϕ is an isomorphism. □

If $LK = H$ then the previous theorem is the generalization of Mackey's theorem decomposition for groups in the situation of a unique double coset.

3.4 A dual relation

Let K be a normal Hopf subalgebra of H and $L = H//K$. Then the natural projection $\pi : H \to L$ is a surjective Hopf map and then $\pi^* : L^* \to H^*$ is an injective Hopf map. We identify L^* with its image $\pi^*(L^*)$ in H^*. This is a normal Hopf subalgebra of H^*. In this section we will study the equivalence relation $r^{H^*}_{L^*,\,\Bbbk}$ on $\mathrm{Irr}(H^{**}) = \mathrm{Irr}(H)$.

The following result was proven in [15].

Proposition 3.3. *Let K be a normal Hopf subalgebra of a finite dimensional semisimple Hopf algebra H and $L = H//K$. If $t_{_L} \in L^*$ is the integral on L with $t_L(1) = |L|$ then $\epsilon_{_K} \uparrow_K^H = t_L$ and $t_L \downarrow_K^H = \frac{|H|}{|K|}\epsilon_{_K}$.*

Proposition 3.4. *Let K be a normal Hopf subalgebra of a semisimple Hopf algebra H and $L = H//K$. Consider the equivalence relation $r^{H^*}_{L^*,\,\Bbbk}$ on $\mathrm{Irr}(H)$. Then $\chi \sim \mu$ if and only if their restrictions to K have a common constituent.*

Proof. The equivalence relation $r^{H^*}_{L^*,\,\Bbbk}$ on $\mathrm{Irr}(H)$ becomes the following: $\chi \sim \mu$ if and only if $m_{_H}(\chi,\ t_{_L}\mu) > 0$. On the other hand, applying the previous Proposition it follows that:

$$\begin{aligned} m_{_H}(\chi,\ t_{_L}\mu) &= m_{_H}(t^*_{_L},\ \mu\chi^*) = m_{_H}(t_{_L},\ \mu\chi^*) \\ &= = m_{_H}(\epsilon \uparrow_{_K}^H,\ \mu\chi^*) = m_{_K}(\epsilon_{_K},\ (\mu\chi^*) \downarrow_{_K}) \\ &= m_{_K}(\epsilon_{_K},\ \mu \downarrow_{_K} \chi^* \downarrow_{_K}) = m_{_K}(\chi \downarrow_{_K},\ \mu \downarrow_{_K}) \end{aligned}$$

Thus $\chi \sim \mu$ if and only if their restriction to K have a common constituent. □

Theorem 3.4.1. *Let K be a normal Hopf subalgebra of a semisimple Hopf algebra H and $L = H//K$. Consider the equivalence relation $r^{H^*}_{L^*,\,\Bbbk}$ on $\mathrm{Irr}(H)$. Then $\chi \sim \mu$ if and only if $\frac{\chi\downarrow_K}{\chi(1)} = \frac{\mu\downarrow_K}{\mu(1)}$.*

Proof. Let $\mathcal{C}_1, \mathcal{C}_2, \cdots \mathcal{C}_l$ be the equivalence classes of $r^{H^*}_{L^*,\,\Bbbk}$ on $\mathrm{Irr}(H)$ and let

$$a_i = \sum_{\chi \in \mathcal{C}_i} \chi(1)\chi \tag{3.4.1}$$

for $1 \le i \le l$. If $\mathcal{C}_1$ is the equivalence class of the trivial character ϵ then the definition of $r^{H^*}_{L^*,\,\Bbbk}$ implies that $a_1 = t_L$. Formula from Remark 3.2.2 becomes

$$\frac{t_L}{|L|}\frac{\chi}{\chi(1)} = \frac{a_i}{a_i(1)}$$

for any irreducible character $\chi \in \mathcal{C}_i$.

Restriction to K of the above relation combined with Proposition 4.1 gives:

$$\frac{\chi\downarrow_K}{\chi(1)} = \frac{a_i\downarrow_K}{a_i(1)} \tag{3.4.2}$$

Thus if $\chi \sim \mu$ then $\frac{\chi\downarrow_K}{\chi(1)} = \frac{\mu\downarrow_K}{\mu(1)}$. □

3.4.1 Formulae for restriction and induction

The previous theorem implies that the restriction of two irreducible H-characters to K either have the same common constituents or they have no common constituents. Let t_H be the integral on H with $t_H(1) = |H|$. One has that $t_H = \sum_{i=1}^{l} a_i$ as t_H is the regular character of H. Since H is free as K-module [96] it follows that the restriction of t_H to K is the regular character of K multiplied by $|H|/|K|$. Thus $t_H\downarrow_K = |H|/|K|t_K$. But $t_K = \sum_{\alpha \in \mathrm{Irr}(K)} \alpha(1)\alpha$ and Theorem 4.3 implies that the set of the irreducible characters of K can be partitioned in disjoint subsets $\mathcal{A}_i$ with $1 \le i \le l$ such that

$$a_i\downarrow_K = \frac{|H|}{|K|}\sum_{\alpha \in \mathcal{A}_i} \alpha(1)\alpha.$$

Then if $\chi \in \mathcal{C}_i$ formula (3.4.2) implies that

$$\chi\downarrow_K = \frac{\chi(1)}{a_i(1)}\frac{|H|}{|K|}\sum_{\alpha \in \mathcal{A}_i} \alpha(1)\alpha.$$

Let $|\mathcal{A}_i| = \sum_{\alpha \in \mathcal{A}_i} \alpha^2(1)$. Evaluating at 1 the above equality one gets $a_i(1) = \frac{|H|}{|K|}|\mathcal{A}_i|$. By Frobenius reciprocity the above restriction formula implies that if $\alpha \in \mathcal{A}_i$ then

$$\alpha\uparrow_K^H = \frac{\alpha(1)}{a_i(1)}\frac{|H|}{|K|}\sum_{\chi \in \mathcal{C}_i} \chi(1)\chi \tag{3.4.1}$$

3.5 Restriction of modules to normal Hopf subalgebras

Let G be a finite group and H a normal subgroup of G. If M is an irreducible H-module then

$$M \uparrow_H^G \downarrow_H^G = \oplus_{i=1}^{s} {}^{g_i}M$$

where ${}^gM$ is a conjugate module of M and $\{g_i\}_{i=1,\, s}$ is a set of representatives for the left cosets of H in G. For $g \in G$ the H-module ${}^gM$ has the same underlying vector space as M and the multiplication with $h \in H$ is given by $h.n = (g^{-1}hg)n$ for all $n \in N$. It is easy to see that $gN \cong g'N$ if $gN = g'N$.

Let K be a normal Hopf subalgebra of H and M be an irreducible K-module. In this section we will define the notion of a conjugate module to M similar to group situation. If $d \in \mathrm{Irr}(H^*)$ we define a conjugate module ${}^dM$. The left cosets of K in H correspond to the equivalence classes of $r^H_{K,\, \Bbbk}$. We will show that if $d,\ d'$ are two irreducible characters in the same equivalence class of $r^H_{K,\, \Bbbk}$ then the modules ${}^dM$ and ${}^{d'}M$ have the same constituents. We will show that the irreducible constituents of $M \uparrow_K^H \downarrow_K^H$ and $\oplus_{d \in \mathrm{Irr}(H^*)} {}^dM$ are the same.

Remark 3.5.1. *Since K is a normal Hopf subalgebra it follows that Λ_K is a central element of H (see [78]) and by their definition $r^H_{K,\, \Bbbk} = r^H_{\Bbbk,\, K}$. Thus the left cosets are the same with the right cosets in this situation.*

Let K be a normal Hopf subalgebra of H and M be a K-module. If W is an H^*-module then $W \otimes M$ becomes a K-module with

$$k(w \otimes m) = w_0 \otimes (S(w_1)kw_2)m \tag{3.5.1}$$

In order to check that $W \otimes M$ is a K-module one has that

$$\begin{aligned} k.(k'(w \otimes m)) &= k(w_0 \otimes (S(w_1)k'w_2)m) \\ &= w_0 \otimes (S(w_1)kw_2)((S(w_3)k'w_4)m) \\ &= w_0 \otimes (S(w_1)kk'w_2)m \\ &= (kk')(w \otimes m) \end{aligned}$$

for all $k, k' \in K$, $w \in W$ and $m \in M$.

It can be checked that if $W \cong W'$ as H^*-modules then $W \otimes M \cong W' \otimes M$. Thus for any irreducible character $d \in \mathrm{Irr}(H^*)$ associated to a simple H-comodule W one can define the K-module ${}^dM \cong W \otimes M$.

Proposition 3.5.1. *Let K be a normal Hopf subalgebra of H and M be an irreducible K-module with character $\alpha \in C(K)$. Suppose that W is a simple H^*-module with character $d \in \mathrm{Irr}(H^*)$. Then the character α_d of the K-module ${}^dM$ is given by the following formula:*

$$\alpha_d(x) = \alpha(Sd_1xd_2)$$

for all $x \in K$.

Proof. Indeed one may suppose that $W \cong \Bbbk < x_{1i} \mid 1 \leq i \leq q >$ where $C_d = \Bbbk < x_{ij} \mid 1 \leq i, j \leq q >$ is the coalgebra associated to W and $q = \epsilon(d) = |W|$. Then formula (4.2.1) becomes $k(x_{1i} \otimes w) = \sum_{j,\, l=1}^{q} x_{1j} \otimes (S(x_{jl})kx_{lj})m$. Since $d = \sum_{i=1}^{q} x_{ii}$ one gets the the formula for the character α_d. □

For any $d \in \mathrm{Irr}(H^*)$ define the linear operator $c_d : C(K) \to C(K)$ which on the basis given by the irreducible characters is given by $c_d(\alpha) = \alpha_d$ for all $\alpha \in \mathrm{Irr}(K)$.

Remark 3.5.2. *From the above formula it can be directly checked that* $^{dd'}\alpha =^d (^{d'}\alpha)$ *for all* $d,\ d' \in \mathrm{Irr}(H^*)$ *and* $\alpha \in C(K)$. *This shows that* $C(K)$ *is a left* $C(H^*)$*-module. Also one can verify that* $^d(\alpha^*) = (^d\alpha)^*$.

Proposition 3.5.2. *Let* K *be a normal Hopf subalgebra of* H *and* M *be an irreducible* K*-module with character* $\alpha \in C(K)$. *If* $d,\ d' \in \mathrm{Irr}(H^*)$ *lie in the same coset of* $r^H_{K,\,\Bbbk}$ *then* ^dM *and* $^{d'}M$ *have the same irreducible constituents. Moreover* $\frac{\alpha_d}{\epsilon(d)} = \frac{\alpha_{d'}}{\epsilon(d')}$.

Proof. Consider the equivalence relation $r^H_{\Bbbk,\,K}$ from section 2 and $H = \oplus_{i=1}^{s} B_i$ the decomposition from Corollary 2.5. Let $\mathcal{B}_1, \cdots, \mathcal{B}_s$ be the equivalence classes and let b_i be defined as in subsection 3.4. Then formula (3.2.4) becomes

$$\frac{d}{\epsilon(d)}\frac{\Lambda_K}{|K|} = \frac{b_i}{\epsilon(b_i)} \tag{3.5.2}$$

where Λ_K is the integral in K with $\epsilon(\Lambda_K) = |K|$ and $d \in B_i$. Thus

$$\begin{aligned}
\alpha_{b_i}(x) &= \alpha(S(b_i)_1 x (b_i)_2) = \\
&= \frac{\epsilon(b_i)}{\epsilon(d)|K|}\alpha(S(d\Lambda_K)_1)x(d\Lambda_K)_2) = \\
&= \frac{\epsilon(b_i)}{\epsilon(d)|K|}\alpha(S((\Lambda_K)_1)S(d_1)xd_2(\Lambda_K)_2) = \\
&= \frac{\epsilon(b_i)}{\epsilon(d)}\alpha(Sd_1xd_2) = \\
&= \frac{\epsilon(b_i)}{\epsilon(d)}\alpha_d(x)
\end{aligned}$$

for all $d \in \mathcal{B}_i$.

This implies that if $d \sim d'$ then $\frac{\alpha_d}{\epsilon(d)} = \frac{\alpha_{d'}}{\epsilon(d')}$. □

Let N be a H-module and W an H^*-module. Then $W \otimes N$ becomes an H-module such that

$$h(w \otimes m) = w_0 \otimes (S(w_1)hw_2)m \tag{3.5.3}$$

It can be checked that $W \otimes N \cong N^{|W|}$ as H-modules. Indeed the map $\phi : W \otimes N \to {}_{\epsilon}W \otimes N$ $w \otimes n \mapsto w_0 \otimes w_1 n$ is an isomorphism of H-modules where ${}_{\epsilon}W$ is considered left

H-module with the trivial action. Its inverse is given by $w \otimes m \mapsto w0 \otimes S(w_1)m$. To check that ϕ is an H-module map one has that

$$\begin{aligned} \phi(h.(w \otimes n)) &= \phi(w_0 \otimes (S(w_1)hw_2)n) \\ &= w_0 \otimes w_1(S(w_2)hw_3)n \\ &= w_0 \otimes hw_1 n \\ &= h.(w_0 \otimes w_1 n) \\ &= h\phi(w \otimes n) \end{aligned}$$

for all $w \in W$, $m \in M$ and $h \in H$.

Proposition 3.5.3. *Let K be a normal Hopf subalgebra of H and M be an irreducible K-module with character $\alpha \in C(K)$. If $d \in \mathrm{Irr}(H^*)$ then*

$$\frac{1}{\epsilon(d)}\alpha_d \uparrow_K^H = \alpha \uparrow_K^H .$$

Proof. With the notations from subsection 3.4.1 let $\mathcal{A}_i$ be the subset of $\mathrm{Irr}(K)$ which contains α. It is enough to show that the constituents of α_d are contained in this set and then the induction formula (3.4.1) from the same subsection can be applied. For this, suppose N is an irreducible H-module and

$$N \downarrow_K = \oplus_{i=1}^s N_i \tag{3.5.4}$$

where N_i are irreducible K-modules. The above result implies that $W \otimes N \cong N^{|W|}$ as H-modules. Therefore $(W \otimes N) \downarrow_K = (N \downarrow_K)^{|W|}$ as K-modules. But $(W \otimes N) \downarrow_K = \oplus_{i=1}^s (W \otimes N_i)$ where each $W \otimes N_i$ is a K-module by Equation (3.5.3). Thus

$$\oplus_{i=1}^s N_i^{|W|} = \oplus_{i=1}^s (W \otimes N_i) \tag{3.5.5}$$

This shows that if N_i is a constituent of $N \downarrow_K$ then $W \otimes N_i$ has all the irreducible K- constituents among those of $N \downarrow_K$. The formula (3.4.1) applied for each irreducible constituent of α_d gives that

$$\frac{1}{\epsilon(d)}\alpha_d \uparrow_K^H = \alpha \uparrow_K^H \tag{3.5.6}$$

for all $\alpha \in \mathrm{Irr}(K)$ and $d \in \mathrm{Irr}(H^*)$. □

Proposition 3.5.4. *Let K be a normal Hopf subalgebra of H and M be an irreducible K-module. Then $M \uparrow_K^H \downarrow_K^H$ and $\oplus_{d \in \mathrm{Irr}(H^*)} {}^d M$ have the same irreducible constituents.*

Proof. Consider the equivalence relation $r^H_{\Bbbk,\, K}$ from Section 2 and let $\mathcal{B}_1, \cdots, \mathcal{B}_s$ be its equivalence classes. Pick an irreducible character $d_i \in \mathcal{B}_i$ in each equivalence class of $r^H_{\Bbbk,\, K}$ and let C_i be its associated simple coalgebra. Then Corollary 2.5 implies that $H = \oplus_{i=1}^s C_i K$. It follows that the induced module $M \uparrow_K^H$ is given by

$$M \uparrow_K^H = H \otimes_K M = \oplus_{i=1}^s C_i K \otimes_K M$$

Each $C_iK \otimes_K M$ is a K-module by left multiplication with elements of K since

$$k.(ck' \otimes_K m) = c_1(Sc_2kc_3)k' \otimes_K m = c_1 \otimes_K (Sc_2kc_3)k'm$$

for all $k,\ k' \in K$, $c \in C_i$ and $m \in M$. Thus $M \uparrow_K^H$ restricted to K is the sum of the K-modules $C_iK \otimes_K M$. On the other hand the composition of the canonical maps $C_i \otimes K \hookrightarrow C_iK \otimes M \to C_iK \otimes_K M$ is a surjective morphism of K-modules which implies that $C_iK \otimes_K M$ is a homeomorphic image of $\epsilon(d_i)$ copies of ${}^{d_i}M$. Therefore the irreducible constituents of $M \uparrow_K^H \downarrow_K^H$ are among those of $\oplus_{d\in \mathrm{Irr}(H^*)}\ {}^dM$. In the proof of the previous Proposition we showed the other inclusion. Thus $M \uparrow_K^H \downarrow_K^H$ and $\oplus_{d\in \mathrm{Irr}(H^*)}\ {}^dM$ have the same irreducible constituents $\square$

Chapter 4

Clifford theory for cocentral extensions

The classical Clifford correspondence for normal subgroups is considered in the more general setting of semisimple Hopf algebras. We prove that this correspondence still holds if the extension determined by the normal Hopf subalgebra is cocentral.

The starting point for Clifford theory is Clifford's paper [28] on representations of normal groups. Since then a lot of literature was written on the subject. Parallel theories for graded rings and Lie algebras were developed in [33] and [7] respectively, as well as in other papers. A unifying setting for these theories was developed by Schneider [106] for Hopf Galois extensions. The main problem with this more general theory is that usually the stabilizer is not a Hopf subalgebra and is not an extension of the based ring.

A more general approach was considered by Witherspoon in [120, 121] for any normal extension of semisimple algebras. With a certain definition of the stabilizer it was proven in [120] that the Clifford correspondence holds.

In this chapter we address an analogue of initial's Clifford approach for groups. We consider an extension of Hopf algebras A/B where B is a normal Hopf subalgebra of A and let M be an irreducible B-module. The conjugate B-modules of M are defined as in [14] and the stabilizer Z of M is a Hopf subalgebra of A containing B. We say that the Clifford correspondence holds for M if induction from Z to A provides a bijection between the sets of Z (respectively A)-modules that contain M as a B-submodule.

Since B is normal in A also in the sense of [120] the results from this paper can also be applied. It is shown that the Clifford correspondence holds for M if and only if Z is a stabilizer in the sense proposed in [120]. A necessary and sufficient condition for this to happen is given in Proposition 4.2. Our approach uses the character theory for Hopf algebras and normal Hopf subalgebras. If the extension

$$\Bbbk \longrightarrow B \xrightarrow{i} A \xrightarrow{\pi} H \longrightarrow \Bbbk$$

is cocentral then we prove that this condition is satisfied (see Corollary 6.5.1). Recall that a such extension is cocentral if $H^* \subset \mathcal{Z}(A^*)$ via π^*.

The chapter is organized as follows. First section recalls the character theory results for Hopf subalgebras that are further needed. The next section defines the conjugate module and introduces the stabilizer as a Hopf subalgebra. The necessary and sufficient condition for the Clifford correspondence to hold is proven in this section. Third section considers the case when the quotient Hopf algebra is a finite group algebra. A different approach gives in these settings another criterion for the Clifford correspondence to hold (see Theorem 4.3.1). As a corollary of this it is proven that the Clifford correspondence holds for cocentral extensions. In the last section of this chapter a counterexample of a non cocentral extension where the Clifford correspondence does not hold anymore is given.

4.1 Normal Hopf subalgebras

Throughout of this chapter A will be a finite dimensional semisimple Hopf algebra over an algebraically closed field $\Bbbk$ of characteristic zero. Then A is also cosemisimple and $S^2 = \mathrm{Id}$ [73].

Let B be a Hopf subalgebra A. By [14, Corollary 2.5] there is a coset decomposition for A

$$A = \oplus_{C/\sim} BC.$$

where $\sim$ is an equivalence relation on the set of simple subcoalgebras of A given by $C \sim C'$ if and only if $BC = BC'$. In [14] this equivalence relation is denoted by $r^{A}_{B,\,\Bbbk}$.

Since A is also cosemisimple [73] the set of simple subcoalgebras of A is in bijection with the set of irreducible characters of A^* (see [72] for this correspondence).

Suppose now that B is a normal Hopf subalgebra of A. Recall that this means $a_1BS(a_2) \subset B$ for all $a \in A$. If χ and μ are two irreducible characters of A it can be proven that their restriction to B either have the same irreducible constituents or they don't have common constituents at all. Define $\chi \sim \mu$ if and only $m_{B}(\chi \downarrow^{A}_{B},\ \mu \downarrow^{A}_{B}) > 0$. With the above notations this is the equivalence relation $r^{A^*}_{H^*,\,\Bbbk}$ for the inclusion $H^* \subset A^*$ where $H = A//B$ is the quotient Hopf algebra.

Let $\mathcal{A}_1, \cdots, \mathcal{A}_l$ the equivalence classes of the above relation and

$$a_i = \sum_{\chi \in \mathcal{A}_i} \chi(1)\chi$$

for $1 \leq i \leq l$.

This equivalence relation determines an equivalence relation on the set of irreducible characters of B. Two irreducible B-characters α and β are equivalent if and only if they are constituents of $\chi \downarrow^{A}_{B}$ for some irreducible character χ of A.

Let $\mathcal{B}_1, \cdots, \mathcal{B}_l$ be the equivalence classes of this new equivalence relation and let

$$b_i = \sum_{\alpha \in \mathcal{B}_i} \alpha(1)\alpha.$$

The induction-restriction formulae from [14] can be written as

$$\frac{\chi\downarrow_B^A}{\chi(1)} = \frac{b_i}{b_i(1)} \tag{4.1.1}$$

and

$$\frac{\alpha\uparrow_B^A}{\alpha(1)} = \frac{|A|}{|B|}\frac{a_i}{a_i(1)} \tag{4.1.2}$$

if $\chi \in \mathcal{A}_i$ and $\alpha \in \mathcal{B}_i$.

Since the regular character of A restricts to $\frac{|A|}{|B|}$ copies of the regular character of B it follows that

$$a_i\downarrow_B^A = \frac{|A|}{|B|}b_i. \tag{4.1.3}$$

In particular $a_i(1) = \frac{|A|}{|B|}b_i(1)$ for all $1 \le i \le l$(see also [14, Subsection 4.1]).

4.2 Conjugate modules and stabilizers

Let M be an irreducible B-module with character $\alpha \in C(B)$. We recall the following notion of conjugate module introduced in [14]. It was also previously considered in [106] in the cocommutative case.

If W is an A^*-module then $W \otimes M$ becomes a B-module with

$$b(w \otimes m) = w_0 \otimes (S(w_1)bw_2)m \tag{4.2.1}$$

Here we used that any left A^*-module W is a right A-comodule via $\rho(w) = w_0 \otimes w_1$. It can be checked that if $W \cong W'$ as A^*-modules then $W \otimes M \cong W' \otimes M$. Thus for any irreducible character $d \in \mathrm{Irr}(A^*)$ associated to a simple A-comodule W one can define the B-module ${}^dM \cong W \otimes M$. If α is the character of M then the character ${}^d\alpha$ of ${}^dM$ is given by

$${}^d\alpha(x) = \alpha(Sd_1xd_2) \tag{4.2.2}$$

for all $x \in B$ (see [14, Proposition 5.3]).

Remark 4.2.1. *From [14, Proposition 5.12] it follows that the equivalence class of a character $\alpha \in \mathrm{Irr}(B)$ is given by all the irreducible constituents of ${}^d\alpha$ as d runs through all irreducible characters of H^*.*

Fix $\alpha \in \mathrm{Irr}(B)$ and suppose that $\alpha \in \mathcal{B}_i$ for some index i.

Proposition 4.1. *The set $\{d \in \mathrm{Irr}(A^*) \mid {}^d\alpha = \epsilon(d)\alpha\}$ is closed under multiplication and " $*$ ". Thus it generates a Hopf subalgebra Z of A that contains B.*

Proof. Since ${}^{d}({}^{d'}\alpha) = {}^{dd'}\alpha$ it follows that the above set is closed under multiplication. Since d^* is a constituent of some power of d it also follows that the set is closed under " $*$ " too. Thus it generates a Hopf subalgebra Z of A (see [92]) with $Z = \oplus_C C$ where the sum is over all simple subcoalgebras of H whose irreducible characters d satisfy ${}^{d}\alpha = \epsilon(d)\alpha$. If $d \in B$ then ${}^{d}\alpha(x) = \alpha(Sd_1xd_2) = \alpha(xd_2S(d_1)) = \epsilon(d)\alpha(x)$ for all $x \in B$. Therefore $B \subset Z$. □

Z will be called *the stabilizer* of α in A.

Remark 4.2.2. *If C is any subcoalgebra of H then $C \otimes M$ has a structure of B-module as above using the fact that C is a right A-comodule via Δ. Then $C \otimes M \cong M^{|C|}$ as B-modules if and only if $C \subset Z$.*

Remark 4.2.3. *If $A = \Bbbk G$ and $B = \Bbbk N$ for a normal subgroup N then Z coincides with the stabilizer of α introduced in [28].*

4.2.1 On the stabilizer

Since B is normal in Z one can define as above two equivalences relations, on $\mathrm{Irr}(Z)$ respectively $\mathrm{Irr}(B)$. Let $\mathcal{Z}_1, \cdots, \mathcal{Z}_r$ be the equivalence classes in $\mathrm{Irr}(Z)$ and $\mathcal{B'}_1, \cdots, \mathcal{B'}_r$ be the corresponding equivalence classes in $\mathrm{Irr}(B)$.

Remark 4.2.1 implies that α by itself form an equivalence class of $\mathrm{Irr}(B)$, say $\mathcal{B'}_1$. Then clearly the corresponding equivalence class $\mathcal{Z}_1$ is given by

$$\mathcal{Z}_1 = \{\psi \in \mathrm{Irr}(Z) |\ \psi \downarrow_B \text{ contains } \alpha\}.$$

Formula (4.1.1) becomes in this situation $\psi \downarrow_B^Z = \frac{\psi(1)}{\alpha(1)}\alpha$ for all $\psi \in \mathcal{Z}_1$. Let $\psi_\alpha = \sum_{\psi \in \mathcal{Z}_1} \psi(1)\psi$. Then $\psi_\alpha \downarrow_B^Z = \frac{|Z|}{|B|}\alpha(1)\alpha$ by 4.1.3 and $\psi_\alpha(1) = \frac{|Z|}{|B|}\alpha(1)^2$.

Lemma 4.2.1. *With the above notations*

$$\psi_\alpha \uparrow_Z^A = \frac{\alpha(1)^2}{b_i(1)} a_i.$$

Proof. One has $\alpha \uparrow_B^Z = \frac{|Z|}{|B|}\frac{\alpha(1)}{\psi_\alpha(1)}\psi_\alpha$ by 4.1.2. But $\psi_\alpha(1) = \frac{|Z|}{|B|}\alpha(1)^2$ and the last formula becomes

$$\alpha \uparrow_B^Z = \frac{\psi_\alpha}{\alpha(1)}$$

Thus $\alpha \uparrow_B^A = (\alpha \uparrow_B^Z) \uparrow_Z^A = \frac{\psi_\alpha \uparrow_Z^A}{\alpha(1)}$. On the other hand $\alpha \uparrow_B^A = \frac{|A|\alpha(1)}{|B|a_i(1)} a_i$ and one gets that:

$$\psi_\alpha \uparrow_Z^A = \frac{|A|\alpha(1)^2}{|B|a_i(1)} a_i = \frac{\alpha(1)^2}{b_i(1)} a_i.$$

□

4.2.2 Definition of the Clifford correspondence

The above Lemma implies that for any $\psi \in \mathcal{Z}_1$ all the irreducible constituents of $\psi \uparrow_Z^A$ are in $\mathcal{A}_i$. We say that the Clifford correspondence holds for the irreducible character $\alpha \in \mathcal{B}_i$ if $\psi \uparrow_Z^A$ is irreducible for any irreducible character $\psi \in \mathcal{Z}_1$ and the induction function

$$\mathrm{ind} : \mathcal{Z}_1 \to \mathcal{A}_i$$

given by $\mathrm{ind}(\psi) = \psi \uparrow_Z^A$ is a bijection.

4.2.3 Clifford theory for normal subrings

Let $B \subset A$ an extension of $\Bbbk$-algebras. An ideal J of B is called A-invariant if $AJ = JA$. Following [120] the extension A/B is called normal if every two sided ideal of B is A-invariant. Witherspoon gave a general Clifford correspondence for normal extensions. Let M be a B-module. Then M is called A stable if the module $M \uparrow_B^A \downarrow_B^A$ is isomorphic to direct sum of copies of M. A stabilizer S of M is a semisimple algebra S such that $B \subset S \subset A$, B is a normal subring of S, M is S-stable, and $M - \mathrm{soc}(M \uparrow_B^A \downarrow_B^A) = M - \mathrm{soc}(M \uparrow_B^S \downarrow_B^S)$. Here the M-socle of a B -module is the sum of all its submodules isomorphic to M.

Next we investigate a relationship between the stabilizer Z previously defined and the notion of stabilizer defined as above for normal extensions. It is easy to see that if B is a normal Hopf subalgebra of A then the extension A/B is normal in the above sense (see also [120, Proposition 5.3]). By the same argument B is normal in Z and from Remark 4.2.1 it follows that M is Z-stable. Thus Z is a stabilizer in the above sense if and only if the socle condition is satisfied. In terms of characters this can be written as $m_{_B}(\alpha \uparrow_B^Z \downarrow_B^Z,\ \alpha) = m_{_B}(\alpha \uparrow_B^A \downarrow_B^A,\ \alpha)$ where α is the character of M.

Proposition 4.2. *With the above notations:*

1. $|Z| \leq \frac{|A|\alpha(1)^2}{b_i(1)}$.

2. *Equality holds if and only if Z is a stabilizer in the sense of [120].*

Proof. Clearly $m_{_B}(\alpha \uparrow_B^Z \downarrow_B^Z,\ \alpha) \leq m_{_B}(\alpha \uparrow_B^A \downarrow_B^A,\ \alpha)$ and equality holds if and only if Z is a stabilizer in the sense of [120].

Let as before $s = \frac{|A|}{|B|}$ be the index of B in A and $s' = \frac{|Z|}{|B|}$ be the index of B in Z.

Using formulae (4.1.2) and (4.1.3) it can be seen that :

$$\begin{aligned} m_{_B}(\alpha \uparrow_B^A \downarrow_B^A,\ \alpha) &= \frac{s\alpha(1)}{a_i(1)} m_{_B}(a_i \downarrow_B^A,\ \alpha) = \frac{s^2\alpha(1)}{a_i(1)} m_{_B}(b_i,\ \alpha) \\ &= \frac{\alpha(1)^2 s^2}{a_i(1)} \\ &= \frac{\alpha(1)^2}{b_i(1)} s \end{aligned}$$

A similar argument applied to the extension $B \subset Z$ gives

$$m_{_B}(\alpha \uparrow_B^Z \downarrow_B^Z,\ \alpha) = \frac{\alpha(1)^2 s'}{b'_1(1)} = s'$$

since in this situation $b'_1 = \alpha(1)\alpha$. Thus $s' = \frac{|Z|}{|B|} \leq s\frac{\alpha(1)^2}{b_i(1)} = \frac{|A|}{|B|}\frac{\alpha(1)^2}{b_i(1)}$ which gives the required inequality. □

Remark 4.2.1. If $A = \Bbbk G$ and $B = \Bbbk N$ for a normal subgroup N then the above inequality is equality. It states that the number of conjugate modules of α is the index of the stabilizer of α in G.

4.2.4 Clifford correspondence

Theorem 4.2.2. *The Clifford correspondence holds for α if and only if Z is a stabilizer in the sense given in [120].*

Proof. If Z is a stabilizer in the sense given in [120] then the Clifford correspondence holds by Theorem 4.6 of the same paper.

Conversely, suppose that the map

$$\mathrm{ind} : \mathcal{Z}_1 \to \mathcal{A}_i$$

given by $\mathrm{ind}(\psi) = \psi \uparrow_Z^A$ is a bijection. Thus for any $\psi \in \mathcal{Z}_1$ there is a $\chi \in \mathcal{A}_i$ such that $\psi \uparrow_Z^A = \chi$. Note that this implies $\psi(1) = \frac{|Z|}{|A|}\chi(1)$.

Since ind is a bijection one can write

$$\psi_\alpha \uparrow_Z^A = \sum_{\psi \in \mathcal{Z}_1} \psi(1)\psi \uparrow_Z^A = \sum_{\chi \in \mathcal{A}_i} \frac{|Z|}{|A|}\chi(1)\chi = \frac{|Z|}{|A|}a_i$$

which implies that $\psi_\alpha(1) = (\frac{|Z|}{|A|})^2 a_i(1)$. Lemma 4.2.1 implies $\psi_\alpha(1) = \frac{|Z|}{|A|}\frac{\alpha(1)^2}{b_i(1)}a_i(1)$ and therefore one gets $|Z| = \frac{|A|\alpha(1)^2}{b_i(1)}$. Proposition 4.2 implies that Z is a stabilizer in the sense given in [120]. □

4.3 Extensions of Hopf algebras

Let B be a normal Hopf subalgebra of A and $H = A//B$. Then we have the extension

$$\Bbbk \longrightarrow B \xrightarrow{\ i\ } A \xrightarrow{\ \pi\ } H \longrightarrow \Bbbk \tag{4.3.1}$$

and A/B is an H-Galois extension with the comodule structure $\rho : A \to A \otimes H$ given by $\rho = (\mathrm{id} \otimes \pi)\Delta$.

Remark 4.3.1. *The restriction functor form A-modules to B-modules induces a map* $\mathrm{res} : C(A) \to C(B)$. *It is easy to see that* $\mathrm{res} = i^*|_{C(A)}$, *the restriction of $i^* : A^* \to B^*$ to the subalgebra of characters $C(A)$. By duality, $\pi|_{C(A^*)}$ is the restriction map of A^*-characters to H^* (here $H^* \subset A^*$ via π^*).*

4.3.1 Results on Hopf Galois extensions

In this subsection we recall few facts about Clifford theory for Hopf Galois extensions over finite group algebras $H = \Bbbk F$ from [106]. (see also [33]). Let A/B be a Hopf Galois extension over $H = \Bbbk F$ via the comodule map $\rho : A \to A \otimes \Bbbk F$. For any $f \in F$ let $A_f = \rho^{-1}(A \otimes \Bbbk f)$. Since A/B is a Hopf Galois extension one has that $A = \oplus_{f\in F} A_f$ is a strongly graded algebra by F with $A_1 = B$. The functor $A_f \otimes_B - :_B \mathcal{M} \to_B \mathcal{M}$ is an equivalence of categories since A_f is an invertible B-bimodule, $A_f \otimes_B A_{f^{-1}} = B$. In particular for any simple B-module M then $A_f \otimes_B M$ is also a simple B-module. From this it follows that the group F acts on the irreducible representations of B by $f.M := A_f \otimes_B M$

The stabilizer H of M is defined as the set of all $f \in F$ such that $A_f \otimes_B M \cong M$ as B-modules. It is a subgroup of F. Let $S := A(H) = \rho^{-1}(A \otimes \Bbbk H) = \oplus_{h\in H} A_h$. Then the induction map ind

$$\{V \in S - mod : V \downarrow^S_B \textit{ contains } M\} \to \{P \in A - mod : P \downarrow^A_B \textit{ contains } M\}$$

given by $\mathrm{ind}(M) = S \otimes_B M$ is a bijection.

4.3.2 Extensions by $\Bbbk F$

For the rest of this section we suppose that $H = \Bbbk F$ for some finite group F. Then $H^* = \Bbbk^F$ is a normal Hopf subalgebra of A^* and one can define the same equivalence relations form the beginning of this Chapter for this new extension. Since $\mathrm{Irr}(\Bbbk^F) = F$ this gives a partition of the group $F = \bigsqcup_{j=1}^m \mathcal{F}_j$. Then by Remark 4.3.1 formula (4.1.3) applied to this situation implies that for any $d \in \mathrm{Irr}(A^*)$ there is an unique index j such that

$$\pi(d) = \frac{\epsilon(d)}{|\mathcal{F}_j|} \sum_{f\in\mathcal{F}_j} f. \tag{4.3.1}$$

4.3.3 Dimension of the orbit

Let M be an irreducible representation of B with character α and let $H \le F$ be the stabilizer of M. Since $|f.M| = |M|$ it follows that all the irreducible representations in the equivalence class of M have the same dimension. If s is their number then clearly $s = \frac{|F|}{|H|}$. Suppose now that $\mathcal{B}_i$ is the equivalence class of α. The above results implies that $\mathcal{B}_i$ coincide with the set of characters of the irreducible modules $f.M$. Thus

$$b_i(1) = s\alpha(1)^2 = \frac{|F|}{|H|}\alpha(1)^2. \tag{4.3.1}$$

4.3.4 Coset decomposition

Recall the coset decomposition for A

$$A = \oplus_{C/\sim} BC. \tag{4.3.1}$$

where $\sim$ is an equivalence relation on the set of simple subcoalgebras of A given by $C \sim C'$ if and only if $BC = BC'$. Note that $BC = CB$ for any simple subcoalgebra C of A since B is a normal Hopf subalgebra (see also [14]).

Lemma 4.3.1. *Suppose that $\pi : A \to \Bbbk F$ is a surjective map of Hopf algebras where F is a finite group. Let C be a simple subcoalgebra of A with irreducible character d and suppose $\pi(d) = \sum_{g\in\mathcal{A}} a_g g$ where $\mathcal{A} \subset F$ and a_g are positive integers for all $g \in \mathcal{A}$. Then $\pi(C) = \oplus_{g\in\mathcal{A}} \Bbbk g$.*

Proof. $\pi(C)$ is a subcoalgebra of $\Bbbk F$ and therefore $\pi(C) = \oplus_{g\in\mathcal{B}} \Bbbk g$. It is enough to show $\mathcal{A} = \mathcal{B}$. Clearly $\mathcal{A} \subset \mathcal{B}$. For any $g \in \mathcal{B}$ let $\Bbbk_g$ be a copy of the field $\Bbbk$. By duality π^* induces an embedding of the semisimple algebra $R = \prod_{g\in\mathcal{B}} \Bbbk_g$ in the matrix algebra $C^* = M_{\epsilon(d)}(\Bbbk)$. Writing the primitive idempotents of R in terms of the primitive idempotents of C^* it follows that $\mathcal{B} \subset \mathcal{A}$. □

Lemma 4.3.2. *Assume that $H = \Bbbk F$ for some finite group F. Let $d \in \mathrm{Irr}(A^*)$ associated to the simple subcoalgebra C. If*

$$\pi(d) = \frac{\epsilon(d)}{|\mathcal{F}_j|} \sum_{f\in\mathcal{F}_j} f$$

then the coset $BC = \oplus_{f\in\mathcal{F}_j} A(f)$.

Proof. Let $A_s = \oplus_{f\in\mathcal{F}_s} A(f)$ for all $1 \le s \le m$. Then $A = \oplus_{s=1}^m A_s$. The above lemma implies that $\pi(C) = \oplus_{f\in\mathcal{F}_j} \Bbbk f$. Since $\pi(BC) = \pi(C)$ this shows $BC \subset A_j$. The coset decomposition formula (4.3.1) forces $BC = A_j$. □

Theorem 4.3.1. *Suppose that $H = \Bbbk F$ for some finite group F. Let M be an irreducible representation of B with character α and let $H \le F$ be the stabilizer of M. Then $Z \subset S := A(H)$ and the Clifford correspondence holds for α if and only if $Z = S$.*

Proof. Since A/B is an Hopf Galois extension over $H = \Bbbk F$ it follows as above that A is strongly F-graded with $A = \oplus_{f\in F} A_f$. First we will show that $Z \subset S = A(H)$. Recall the definition of Z as the sum of all simple subcoalgebras C whose irreducible characters d verify the property ${}^d\alpha = \epsilon(d)\alpha$. Let C be such an algebra with character d. As above there is a j such that $\pi(d) = \frac{\epsilon(d)}{|\mathcal{F}_j|} \sum_{f\in\mathcal{F}_j} f$. It is easy to see that the canonical map $CB \otimes M \to CB \otimes_B M$ is a morphism of B-modules. Since $BC \subset Z$ is a subcoalgebra Remark 4.2.2 implies $CB \otimes M \cong M^{|CB|}$ as B-modules. Thus $CB \otimes_B M$ is a sum of copies of M. By Lemma 4.3.2 $BC \otimes_B M = \oplus_{f\in\mathcal{F}_j} A(f) \otimes_B M$ which shows that $\mathcal{F}_j \subset H$ and therefore $C \subset A(H)$ by Lemma 4.3.1. Thus $Z = \sum_{C\subset Z} C \subset A(H)$.

Since S/B is a $\Bbbk H$-Hopf Galois extension it follows that $|S| = |B||H|$. Using formula (4.3.1) it follows that $|S| = \frac{|A|\alpha(1)^2}{b_i(1)}$ if $\alpha \in \mathcal{B}_i$. Then Theorem 4.2.2 shows that the Clifford correspondence holds if and only if $|Z| = |S|$. □

It is easy to see that $\Delta_A(S) \subset A \otimes S$.

Corollary 4.3.2. *Suppose that $H = \Bbbk F$ for some finite group F. Let M be an irreducible representation of B with character α and let $H \leq F$ be the stabilizer of M. Then the Clifford correspondence holds for α if and only S is a Hopf subalgebra of A.*

Proof. Any S-module which restricted to B contains M is a direct sum of copies of M as a B-module by [106, Corollary 2.2]. If S is a Hopf algebra then Remark 4.2.1 applied to the extension S/B implies that $S \subset Z$. Thus $S = Z$. □

Corollary 4.3.3. *Suppose that the extension* (4.3.1) *is cocentral. Then the Clifford correspondence holds for any irreducible B-module M.*

Proof. Since H^* is commutative there is a finite group F such that $H = \Bbbk F$. It is easy to see that $H^* \subset \mathcal{Z}(A^*)$ via π^* if and only if $\pi(a_1) \otimes a_2 = \pi(a_2) \otimes a_1$ for all $a \in A$. This last relation implies that S is a Hopf subalgebra of A and the previous corollary finishes the proof. □

4.4 A Counterexample

Let $\Sigma = FG$ be an an exact factorization of finite groups. This gives a right action $\triangleleft : G \times F \to G$ of F on the set G, and a left action $\triangleright : G \times F \to F$ of G on the set F subject to the following two conditions:

$$s \triangleright xy = (s \triangleright x)((s \triangleleft x) \triangleright y) \qquad st \triangleleft x = (s \triangleleft (t \triangleright x))(t \triangleleft x)$$

The actions $\triangleright$ and $\triangleleft$ are determined by the relations $gx = (g \triangleright x)(g \triangleleft x)$ for all $x \in F$, $g \in G$. Note that $1 \triangleright x = x$ and $s \triangleleft 1 = s$.

Consider the Hopf algebra $A = \Bbbk^G \# \Bbbk F$ [81] which is a smashed product and coproduct using the above two action. The structure of A is given by:

$$(\delta_g x)(\delta_h y) = \delta_{g \triangleleft x, h} \delta_g xy$$
$$\Delta(\delta_g x) = \sum_{st=g} \delta_s (t \triangleright x) \otimes \delta_t x$$

Then A fits into the abelian extension

$$\Bbbk \longrightarrow \Bbbk^G \xrightarrow{i} A \xrightarrow{\pi} \Bbbk F \longrightarrow \Bbbk \tag{4.4.1}$$

As above F acts on $\mathrm{Irr}(\Bbbk^G) = G$. It is easy to see that this action is exactly $\triangleleft$. Let $g \in G$ and H be the stabilizer of g under $\triangleleft$. Using the above notations it follows that $S = A(H) = \Bbbk^G \# \Bbbk H$. We will construct an example where S is not a Hopf algebra and therefore the Clifford correspondence does not hold $g \in G$. Remark that the above comultiplication formula implies S is a Hopf subalgebra if and only if $G \triangleright H \subset H$.

$\mathbb{C}_4 \triangleleft \mathbb{S}_3$	g	g^2	g^3
t	g	g^3	g^2
s	g^2	g^3	g
s^2	g^3	g	g^2
st	g^3	g^2	g
ts	g^2	g	g^3

Table 4.1: The right action of $\mathbb{S}_3$ on $\mathbb{C}_4$

$\mathbb{C}_4 \triangleright \mathbb{S}_3$	t	s	s^2	st	ts
g	ts	t	s	st	s^2
g^2	s^2	ts	t	st	s
g^3	s	s^2	ts	st	t

Table 4.2: The left action of $\mathbb{C}_4$ on $\mathbb{S}_3$

Consider the exact fact factorization $\mathbb{S}_4 = \mathbb{C}_4\mathbb{S}_3$ where $\mathbb{C}_4$ is generated by the four cycle $g = (1234)$ and $\mathbb{S}_3$ is given by the permutations that leave 4 fixed. If $t = (12)$ and $s = (123)$ then the actions $\triangleleft$ and $\triangleright$ are given in Tables 1 and 2.

The stabilizer of the element g is the subgroup $\{1, t\}$ which is not invariant by the action of $\mathbb{C}_4$. Thus the Clifford correspondence does not hold for g.

Part III

Fusion categories: Group actions on fusion categories

Chapter 5

Fusion rules of equivariantization

5.1 Introduction

Throughout this chapter we shall work over an algebraically closed field $\Bbbk$ of characteristic zero.

Let $\mathcal{C}$ be a fusion category over $\Bbbk$, that is, $\mathcal{C}$ is a semisimple rigid monoidal category over $\Bbbk$ with finitely many isomorphism classes of simple objects, finite-dimensional Hom spaces, and such that the unit object $\mathbf{1}$ is simple (see [27, 43, 44]).

Consider an action $\rho : \underline{G} \to \underline{\mathrm{Aut}}_{\otimes}\mathcal{C}$ of a finite group G by tensor autoequivalences of $\mathcal{C}$ and let $\mathcal{C}^G$ be the equivariantization of $\mathcal{C}$ with respect to this action. Equivariantization under a finite group action, as well as its applications, generalizations and related constructions, have been intensively studied in the last years by several authors. See for instance [3, 13, 12, 36, 37, 45, 50, 57, 98, 111].

In the sense of the notions introduced in [13, 12], equivariantization gives rise in a canonical way to a *central exact sequence of tensor categories*

$$\mathrm{Rep}\ G \to \mathcal{C}^G \to \mathcal{C},$$

where Rep G is the category of finite-dimensional representations of G. On the other hand, combined with the dual notion of (graded) group extension of a fusion category, equivariantization underlies the notion of *solvability* of a fusion category developed in [45].

An important invariant of a fusion category $\mathcal{C}$ is its Grothendieck ring, $\mathrm{gr}(\mathcal{C})$. For instance, the knowledge of the structure of the Grothendieck ring allows to determine all fusion subcategories of $\mathcal{C}$, which correspond to the so-called *based subrings*.

Let $\mathrm{Irr}(\mathcal{C}) = \{\mathbf{1} = S_0, \dots, S_n\}$ denote the set of isomorphism classes of simple objects of $\mathcal{C}$. Then $\mathrm{Irr}(\mathcal{C})$ is a basis of $\mathrm{gr}(\mathcal{C})$ and, for all $0 \le i, j \le n$, we have a relation

$$S_i S_j = \sum_{l=0}^{n} N_{i,j}^{l}\, S_l, \tag{5.1.1}$$

where $N_{i,j}^l$ are non-negative integers given by $N_{i,j}^l = \dim \mathrm{Hom}_{\mathcal{C}}(S_l, S_i \otimes S_j)$, $0 \leq l \leq n$. The relations (5.1.1) are known as the *fusion rules* of $\mathcal{C}$. They are determined by the set $\mathrm{Irr}(\mathcal{C})$ and the multiplicities $N_{i,j}^l \in \mathbb{Z}_{\geq 0}$.

The main result of this chapter is the determination of the fusion rules of $\mathcal{C}^G$ in terms of the fusion rules of $\mathcal{C}$ and certain canonical group-theoretical data associated to the group action. This is contained in Theorem 6.1.3. As it turns out, the structure of the Grothendieck ring of $\mathcal{C}^G$ resembles that of the rings introduced by Witherspoon in [122]. See Remark 5.3.4.

As an example, consider a semisimple cocentral Hopf algebra extension H of a Hopf algebra A by a finite group G, that is, H fits into a cocentral exact sequence

$$\Bbbk \to A \to H \to \Bbbk G \to \Bbbk.$$

As shown in [94] the category Rep H of finite-dimensional representations of H is an equivariantization $(\mathrm{Rep}\ A)^G$ with respect to an appropriate action of G on Rep A. Thus Theorem 6.1.3 implies that the fusion rules of the category Rep H can be described in terms of the fusion rules of Rep A and the action of G. In particular, Theorem 6.1.3 generalizes the results obtained for cocentral abelian extensions of Hopf algebras in [61] and [122].

We discuss in detail the case where $\mathcal{C}$ is a *pointed* fusion category, that is, when all simple objects of $\mathcal{C}$ are invertible. In this case the fusion rules of $\mathcal{C}^G$ are described completely in terms of group-theoretical data. See Theorem 5.4.1.

It is known that every braided group-theoretical fusion category is an equivariantization of a pointed fusion category [87, 86]. Therefore, our results entail the determination of the fusion rules in any braided group-theoretical fusion category.

In order to establish Theorem 6.1.3, we give an explicit description of the simple objects of the equivariantization $\mathcal{C}^G$. This is done, more generally, for any action $\rho : \underline{G} \to \underline{\mathrm{Aut}}\mathcal{C}$ of the group G by autoequivalences of a $\Bbbk$-linear finite semisimple category $\mathcal{C}$. Such an action induces naturally an action of G on the set $\mathrm{Irr}(\mathcal{C})$ of isomorphism classes of simple objects of $\mathcal{C}$. Let $Y \in \mathrm{Irr}(\mathcal{C})$ and let $G_Y \subseteq G$ denote the inertia subgroup of Y, that is,

$$G_Y = \{g \in G \mid \rho^g(Y) \simeq Y\}. \tag{5.1.2}$$

We show that isomorphism classes of simple objects of $\mathcal{C}^G$ are parameterized by pairs (Y, π), where Y runs over the orbits of the action of G on $\mathrm{Irr}(\mathcal{C})$, and π is the equivalence class of an irreducible projective representation of the inertia subgroup G_Y with a certain factor set $\widetilde{\alpha}_Y \in Z^2(G_Y, \Bbbk^*)$. This result is analogous to the parameterization of irreducible representations of a finite group in terms of those of a normal subgroup given by Clifford Theorem. It extends the description obtained in [83] for the case where $\mathcal{C}^G$ is the category of representations of an (algebra) group crossed product (see [94, Subsection 3.1]).

In the case where $\mathcal{C}$ is a fusion category, the duality in $\mathcal{C}$ gives rise to a ring involution $^* : \mathrm{gr}(\mathcal{C}) \to \mathrm{gr}(\mathcal{C})$. We describe this ring involution for the Grothendieck ring of $\mathcal{C}^G$ in

Subsection 5.3.4, more precisely, we use Theorem 6.1.3 in order to determine the dual of the simple object in $\mathcal{C}^G$ corresponding to a pair (Y, π) as above.

We may regard Rep G as a fusion subcategory of $\mathcal{C}^G$ under a canonical embedding. In this way $\mathcal{C}^G$ becomes a Rep G-bimodule category under the action given by tensor product. As another consequence of Theorem 6.1.3, we give a decomposition of $\mathcal{C}^G$ into indecomposable Rep G-module categories. See Theorem 5.3.8.

This chapter is organized as follows. In Section 5.2 we recall the definition of the equivariantization of a semisimple abelian category over $\Bbbk$ under a finite group action and give a parameterization of its simple objects. With respect to this parameterization, we determine in Section 5.3 the fusion rules in an equivariantization of a fusion category. Appart from the main result, Theorem 6.1.3, we also present in this section the above mentioned applications to the determination of the dual of a simple object and the decomposition of $\mathcal{C}^G$ as a Rep G-bimodule category. In Section 5.4 we specialize our main result to the case of an equivariantization of a pointed fusion category and in particular, to braided group-theoretical fusion categories. We include an Appendix at the end of the chapter, where we give an account of the relevant facts about projective group representations needed throughout.

5.2 Simple objects of an equivariantization

The goal of this section is to describe a (mostly well-known) Clifford correspondence entailing a classification of isomorphism classes of simple objects of $\mathcal{C}^G$ in terms of the action of G on the set $\mathrm{Irr}(\mathcal{C})$ of isomorphism classes of simple objects of $\mathcal{C}$. Some instances of this correspondence appear for instance in [45, Proof of Proposition 6.2] and [86, Proposition 5.5].

By abuse of notation, we often indicate an object of a category $\mathcal{C}$ and its isomorphism class by the same letter.

5.2.1 Equivariantization under a finite group action

Let $\mathcal{C}$ be a finite semisimple $\Bbbk$-linear category and let G be a finite group. Let also $\rho : \underline{G} \to \underline{\mathrm{Aut}}\,\mathcal{C}$ be an action of G on $\mathcal{C}$ by $\Bbbk$-linear autoequivalences. Thus, for every $g \in G$, we have a $\Bbbk$-linear functor $\rho^g : \mathcal{C} \to \mathcal{C}$ and natural isomorphisms

$$\rho_2^{g,h} : \rho^g \rho^h \to \rho^{gh}, \quad g, h \in G,$$

and $\rho_0 : \mathrm{id}_{\mathcal{C}} \to \rho^e$, subject to the following conditions:

$$(\rho_2^{ab,c})_X \, (\rho_2^{a,b})_{\rho^c(X)} = (\rho_2^{a,bc})_X \, \rho^a((\rho_2^{b,c})_X), \tag{5.2.1}$$

$$(\rho_2^{a,e})_X \rho^a_{\rho_0(X)} = (\rho_2^{e,a})_X (\rho_0)_{\rho^a(X)}, \tag{5.2.2}$$

for all objects $X \in \mathcal{C}$, and for all $a, b, c \in G$. By the naturality of $\rho_2^{g,h}$, $g, h \in G$, we have the following relation:

$$\rho^{gh}(f)\,(\rho_2^{g,h})_Y = (\rho_2^{g,h})_X\,\rho^g\rho^h(f), \tag{5.2.3}$$

for every morphism $f : Y \to X$ in $\mathcal{C}$. For simplicity, we shall assume in what follows that $\rho^e = \mathrm{id}_{\mathcal{C}}$ and ρ_0, $\rho_2^{g,e}$, $\rho_2^{e,g}$ are identities.

Let $\mathcal{C}^G$ denote the corresponding *equivariantization*. Recall that $\mathcal{C}^G$ is a finite semisimple $\Bbbk$-linear category whose objects are G-equivariant objects of $\mathcal{C}$, that is, pairs (X, μ), where X is an object of $\mathcal{C}$ and $\mu = (\mu^g)_{g\in G}$, such that $\mu^g : \rho^g X \to X$ is an isomorphism, for all $g \in G$, satisfying

$$\mu^g\rho^g(\mu^h) = \mu^{gh}(\rho_2^{g,h})_X, \quad \forall g, h \in G, \qquad \mu_e\rho_{0X} = \mathrm{id}_X\,. \tag{5.2.4}$$

A morphism $f : (X, \mu) \to (X', \mu')$ in $\mathcal{C}^G$ is a morphism $f : X \to X'$ in $\mathcal{C}$ such that $f\mu^g = \mu'^g\rho^g(f)$, for all $g \in G$.

We shall also say that an object X of $\mathcal{C}$ is G-*equivariant* if there exists such a collection $\mu = (\mu^g)_{g\in G}$ so that $(X, \mu) \in \mathcal{C}^G$. Note that μ is not necessarily unique.

The forgetful functor $F : \mathcal{C}^G \to \mathcal{C}$, $F(X, \mu) = X$, is a dominant functor. The functor F has a left adjoint $L : \mathcal{C} \to \mathcal{C}^G$, defined by $L(X) = (\oplus_{t\in G}\rho^t X, \mu_X)$, where $(\mu_X)_g : \oplus_{t\in G}\rho^g\rho^t X \to \oplus_{t\in G}\rho^t X$ is given componentwise by the isomorphisms $(\rho_2^{g,t})_X$. The composition $T_\rho = FL : \mathcal{C} \to \mathcal{C}$ is a faithful $\Bbbk$-linear monad on $\mathcal{C}$ such that $\mathcal{C}^G$ is equivalent to the category $\mathcal{C}^{T_\rho}$ of T_ρ-modules in $\mathcal{C}$. See [13, Subsection 5.3].

5.2.2 Frobenius-Perron dimensions of simple objects of $\mathcal{C}^G$

Let $X, Y \in \mathcal{C}$. Then $\mathrm{Hom}_{\mathcal{C}}(\rho^g X, \rho^g Y) \simeq \mathrm{Hom}_{\mathcal{C}}(X, Y)$, for all $g \in G$. Therefore, for all $g \in G$, and for all objects M of $\mathcal{C}^G$, we have

$$\mathrm{Hom}_{\mathcal{C}}(F(M), \rho^g Y) \simeq \mathrm{Hom}_{\mathcal{C}}(F(M), Y). \tag{5.2.1}$$

The action of the group G on $\mathcal{C}$ permutes isomorphism classes of simple objects of $\mathcal{C}$. Let $Y \in \mathrm{Irr}(\mathcal{C})$. We shall denote $G_Y := \mathrm{St}_G(Y) \subseteq G$ the *inertia* subgroup of Y, that is,

$$G_Y = \{g \in G|\ \rho^g(Y) \simeq Y\}.$$

Then Y has exactly $n = [G : G_Y]$ mutually nonisomorphic G-conjugates $Y = Y_1, \ldots, Y_n$. For every $1 \leq j \leq n$, we have $Y_j \simeq \rho^{g_j} Y$, where $g_1 = e, \ldots, g_n$ is a complete set of representatives of the left cosets of G_Y in G.

Proposition 5.2.1. *Let $M = (X, \mu)$ be a simple object of $\mathcal{C}^G$ and let Y be a simple constituent of X in $\mathcal{C}$. Let $Y = Y_1, \ldots, Y_n$, $n = [G : G_Y]$, be the mutually nonisomorphic G-conjugates of Y. Then $X \simeq m \oplus_{i=1}^n Y_i$, where $m = \dim \mathrm{Hom}_{\mathcal{C}}(X, Y)$.*

Proof. Consider the object $T(Y) = FL(Y) = \oplus_{g\in G}\rho^g Y$. Let Z be a simple object of $\mathcal{C}$ such that $Z \not\simeq Y_j$, $j = 1, \dots, n$. Then $\operatorname{Hom}_{\mathcal{C}}(Z, T(Y)) = 0$. By adjointness, we have $\operatorname{Hom}_{\mathcal{C}^G}(L(Y), M) \simeq \operatorname{Hom}_{\mathcal{C}}(Y, X) \neq 0$. Then M is a simple direct summand of $L(Y)$ in $\mathcal{C}^G$. This implies that $X = F(M)$ is a direct sum of simple subobjects of $FL(Y) = T(Y)$. Therefore $\operatorname{Hom}_{\mathcal{C}}(Z, X) = 0$.

Hence $X \simeq \oplus_{i=1}^n m_i Y_i$, where $m_i = \dim \operatorname{Hom}_{\mathcal{C}}(Y_i, X)$, for all i. By (5.2.1), we have $m_i = m_1 = m$, for all $i = 1, \dots, n$. This proves the proposition. □

Corollary 5.2.2. *Let $M = (X, \mu)$ be a simple object of $\mathcal{C}^G$ and let Y be a simple constituent of X in $\mathcal{C}$. Then* $\operatorname{FPdim} M = m[G : G_Y] \operatorname{FPdim} Y$, *where $m = \dim \operatorname{Hom}_{\mathcal{C}}(Y, X)$.* □

5.2.3 Equivariantization and projective group representations

Let $Y \in \mathcal{C}$ be a fixed simple object. The action ρ of G on $\mathcal{C}$ induces by restriction an action of G_Y on $\mathcal{C}$ by autoequivalences. We may thus consider the equivariantization $\mathcal{C}^{G_Y}$.

By definition of G_Y, there exist isomorphisms $c^g : \rho^g(Y) \to Y$, for all $g \in G_Y$. For all $g, h \in G_Y$, the composition $c^g \rho^g(c^h)(\rho^{g,h}_{2_Y})^{-1}(c^{gh})^{-1}$ defines an isomorphism $Y \to Y$. Since Y is a simple object, there exist nonzero $\widetilde{\alpha}_Y(g, h) \in \Bbbk$ such that

$$\widetilde{\alpha}_Y(g, h)^{-1} \operatorname{id}_Y = c^g \rho^g(c^h)(\rho^{g,h}_{2_Y})^{-1}(c^{gh})^{-1} : Y \to Y. \tag{5.2.1}$$

This defines a map $\widetilde{\alpha}_Y : G_Y \times G_Y \to \Bbbk^*$ which is a 2-cocycle on G_Y.

Remark 5.2.1. *The cocycle $\widetilde{\alpha}_Y$ measures the possible obstruction for (Y, c) to be a G_Y-equivariant object, where $c = (c^g)_{g\in G_Y}$.*

Consider another choice of isomorphisms $v^g : \rho^g(Y) \to Y$, $g \in G_Y$. Since Y is a simple object, the composition $c^g(v^g)^{-1} : Y \to Y$ is given by scalar multiplication by some $f(g) \in \Bbbk^*$, for all $g \in G_Y$. Denoting by $\widetilde{\beta}_Y$ the 2-cocycle related to $(v^g)_g$, it easy to see that $\widetilde{\alpha}_Y$ and $\widetilde{\beta}_Y$ differ by the coboundary of the cochain $f : G_Y \to \Bbbk^*$. This implies that the cohomology class $\alpha_Y \in H^2(G_Y, \Bbbk^*)$ of $\widetilde{\alpha}_Y$ depends only on the isomorphism class of the simple object Y.

Lemma 5.2.1. *Let $(X, \mu) \in \mathcal{C}^G$ and let $Y \in \operatorname{Irr}(\mathcal{C})$. Consider, for every $g \in G_Y$, isomorphisms $c^g : \rho^g(Y) \to Y$ and let $\widetilde{\alpha}_Y$ be the associated 2-cocycle on G_Y. Then the space* $\operatorname{Hom}_{\mathcal{C}}(Y, X)$ *carries a projective representation of G_Y with factor set $\widetilde{\alpha}_Y$, defined in the form*

$$\pi(g)(f) = \mu^g \rho^g(f)(c^g)^{-1} : Y \to X, \tag{5.2.2}$$

for all $f \in \operatorname{Hom}_{\mathcal{C}}(Y, X)$.

Proof. The relation $\pi(g)\pi(h) = \widetilde{\alpha}_Y(g, h)\pi(gh)$, $g, h \in G$, follows by straightforward computation, using the compatibility conditions for ρ. □

Remark 5.2.2. *Suppose that* $\phi : (X, \mu) \to (X', \mu')$ *is an isomorphism in* $\mathcal{C}^G$*. Then the induced isomorphism* $\operatorname{Hom}_{\mathcal{C}}(Y, X) \to \operatorname{Hom}_{\mathcal{C}}(Y, X')$, $f \mapsto \phi f$, *is an isomorphism of projective representations. Similarly, if* $Y' \simeq Y$ *is another representative of the isomorphism class of* Y *and* $c'^g : \rho^g(Y') \to Y'$, $g \in G_Y$, *is a collection of isomorphisms, then the projective representations* $\operatorname{Hom}_{\mathcal{C}}(Y, X)$ *and* $\operatorname{Hom}_{\mathcal{C}}(Y', X)$ *are projectively equivalent.*

Proposition 5.2.3. *Let* $Y \in \operatorname{Irr}(\mathcal{C})$*. There is a bijective correspondence between isomorphism classes of simple objects* $L = (N, \nu)$ *of* $\mathcal{C}^{G_Y}$ *such that* $N \simeq \operatorname{Hom}_{\mathcal{C}}(Y, N) \otimes Y$ *and equivalence classes of irreducible* α_Y*-projective representations of the group* G_Y*. If the simple object* $L = (N, \nu)$ *corresponds to the projective representation* π*, then* $\pi \simeq \operatorname{Hom}_{\mathcal{C}}(Y, N)$ *and* $\operatorname{FPdim} L = \dim \pi \ \operatorname{FPdim} Y$.

Proof. Let $c^g : \rho^g(Y) \to Y$, $g \in G_Y$, be any fixed choice of isomorphisms, and let $\widetilde{\alpha}_Y$ be the associated 2-cocycle. Let also π be a projective representation of G_Y on the vector space V with factor set $\widetilde{\alpha}_Y$. Then the pair $(V \otimes Y, \nu)$ is a G_Y-equivariant object, where

$$\nu^g = \pi(g) \otimes c^g : \rho^g(V \otimes Y) = V \otimes \rho^g(Y) \to V \otimes Y. \tag{5.2.3}$$

Conversely, if $L = (N, \nu)$ is an object of $\mathcal{C}^{G_Y}$ with $N \simeq \operatorname{Hom}_{\mathcal{C}}(Y, N) \otimes Y$, then $V = \operatorname{Hom}_{\mathcal{C}}(Y, N)$ carries a projective representation π of G_Y with factor set $\widetilde{\alpha}_Y$ defined by (5.2.2).

These assignments are functorial and mutually inverse up to isomorphisms. Then $L = (N, \nu)$ is a simple object of $\mathcal{C}^{G_Y}$ if and only if $V = \operatorname{Hom}_{\mathcal{C}}(Y, N)$ is an irreducible projective representation. This implies the proposition. □

5.2.4 The relative adjoint

Consider the forgetful functor $F_Y : \mathcal{C}^G \to \mathcal{C}^{G_Y}$. We discuss in this subsection the left adjoint $L_Y : \mathcal{C}^{G_Y} \to \mathcal{C}^G$ of the functor F_Y.

Let $\mathcal{R}$ be a set of representatives of the left cosets of G_Y in G. So that G is a disjoint union $G = \cup_{t \in \mathcal{R}} t G_Y$.

Set, for all $(N, \nu) \in \mathcal{C}^{G_Y}$, $L_Y(N, \nu) = L_Y^{\mathcal{R}}(N, \nu) = (\oplus_{t \in \mathcal{R}} \rho^t(N), \mu)$, where, for all $g \in G$, $\mu^g : \oplus_{t \in \mathcal{R}} \rho^g \rho^t(N) \to \oplus_{t \in \mathcal{R}} \rho^t(N)$ is defined componentwise by the formula

$$\mu^{g,t} = \rho^s(\nu^h)(\rho_2^{s,h})^{-1} \rho_2^{g,t} : \rho^g \rho^t(N) \to \rho^s(N), \tag{5.2.1}$$

where the elements $h \in G_Y$, $s \in \mathcal{R}$ are uniquely determined by the relation

$$gt = sh. \tag{5.2.2}$$

Remark 5.2.3. *We shall show in Proposition 5.2.4 below that the functor* $L_Y^{\mathcal{R}}$ *thus defined is left adjoint of the functor* F_Y*. By uniqueness of the adjoint, it will follow that, up to isomorphism,* $L_Y^{\mathcal{R}}$ *does not depend on the particular choice of the set of representatives* $\mathcal{R}$*.*

Lemma 5.2.2. *Let* $(N,\nu)\in\mathcal{C}^{G_Y}$. *Then* $L_Y(N,\nu)\in\mathcal{C}^G$.

Proof. For every $g\in G$, $t\in\mathcal{R}$, let $s(g,t)\in\mathcal{R}$, $h(g,t)\in G_Y$ be the elements uniquely determined by the relation $gt=s(g,t)h(g,t)$. Note that, for all $a,b\in G$, $t\in\mathcal{R}$, the following relations hold:

$$s(ab,t)=s(a,s(b,t)), \tag{5.2.3}$$

$$h(ab,t)=h(a,s(b,t))h(b,t), \tag{5.2.4}$$

$$s(ab,t)h(a,s(b,t))=as(b,t). \tag{5.2.5}$$

In order to prove the lemma we shall show that, for all objects $(N,\nu)\in\mathcal{C}^{G_Y}$, and for all $a,b\in G$, $t\in\mathcal{R}$, the following diagram is commutative:

$$\begin{array}{ccc} \rho^a\rho^b\rho^t(N) & \xrightarrow{(\rho_2^{a,b})_{\rho^t(N)}} & \rho^{ab}\rho^t(N) \\ \downarrow{\scriptstyle \rho^a(\mu^{b,t})} & & \downarrow{\scriptstyle \mu^{ab,t}} \\ \rho^a\rho^{s(b,t)}(N) & \xrightarrow[\mu^{a,s(b,t)}]{} & \rho^{s(ab,t)}(N). \end{array} \tag{5.2.6}$$

This is done as follows. By (5.2.1), the relevant maps in diagram (5.2.6) are given by

$$\mu^{ab,t}=\rho^{s(ab,t)}(\nu^{h(ab,t)})\,(\rho_2^{s(ab,t),h(ab,t)})^{-1}\,\rho_2^{ab,t}, \tag{5.2.7}$$

$$\mu^{a,s(b,t)}=\rho^{s(a,s(b,t))}(\nu^{h(a,s(b,t))})\,(\rho_2^{s(a,s(b,t)),h(a,s(b,t))})^{-1}\rho_2^{a,s(b,t)}, \tag{5.2.8}$$

$$\rho^a(\mu^{b,t})=\rho^a\rho^{s(b,t)}(\nu^{h(b,t)})\,\rho^a(\rho_2^{s(b,t)h(b,t)})^{-1}\,\rho^a(\rho_2^{b,t}). \tag{5.2.9}$$

Using (5.2.3) and the fact that (N,ν) is G_Y-equivariant, we compute

$$\begin{aligned}\rho^{s(ab,t)}(\nu^{h(ab,t)}) &= \rho^{s(a,s(b,t))}(\nu^{h(a,s(b,t))h(b,t)})\\ &= \rho^{s(a,s(b,t))}\left(\nu^{h(a,s(b,t))}\,\rho^{h(a,s(b,t))}(\nu^{h(b,t)})\,(\rho_2^{h(a,s(b,t)),h(b,t)})^{-1}\right)\\ &= \rho^{s(a,s(b,t))}\left(\nu^{h(a,s(b,t))}\right)\rho^{s(a,s(b,t))}\rho^{h(a,s(b,t))}\left(\nu^{h(b,t)}\right)\\ &\qquad\times\rho^{s(a,s(b,t))}\left(\rho_2^{h(a,s(b,t)),h(b,t)}\right)^{-1}\end{aligned}$$

Similarly, relations (5.2.4) and (5.2.5), together with the defining condition (5.2.1) on the isomorphisms ρ_2, give

$$\begin{aligned}\left(\rho_2^{s(ab,t),h(ab,t)}\right)^{-1} &= \rho^{s(ab,t)}\left(\rho_2^{h(a,s(b,t)),h(b,t)}\right)\left(\rho_2^{s(ab,t),h(a,s(b,t))}\right)^{-1}_{\rho^{h(b,t)}(N)}\\ &\qquad\times\left(\rho_2^{as(b,t),h(b,t)}\right)^{-1}.\end{aligned}$$

The naturality condition (5.2.3) on $\rho_2^{g,h}$ and relation (5.2.5) imply that

$$\begin{aligned}\left(\rho_2^{s(ab,t),h(a,s(b,t))}\right)^{-1}_{\rho^{h(b,t)}(N)} &= \left(\rho^{s(ab,t)}\rho^{h(a,s(b,t))}(\nu^{h(b,t)})\right)^{-1}\left(\rho_2^{s(ab,t),h(a,s(b,t))}\right)^{-1}\\ &\qquad\times\rho^{s(ab,t),h(a,s(b,t))}(\nu^{h(b,t)}).\end{aligned}$$

Hence we get

$$\rho^{s(ab,t)}(\nu^{h(ab,t)})\left(\rho_2^{s(ab,t),h(ab,t)}\right)^{-1} = \rho^{s(a,s(b,t))}\left(\nu^{h(a,s(b,t))}\right)\left(\rho_2^{s(ab,t),h(a,s(b,t))}\right)^{-1}$$
$$\times\, \rho^{as(b,t)}(\nu^{h(b,t)})\left(\rho_2^{as(b,t)),h(b,t)}\right)^{-1}.$$

Composing this resulting morphism with the inverse of

$$\rho^{s(a,s(b,t))}(\nu^{h(a,s(b,t))})\,(\rho_2^{s(a,s(b,t)),h(a,s(b,t))})^{-1},$$

and using (5.2.5), we obtain the expression

$$\rho^{as(b,t)}(\nu^{h(b,t)})\left(\rho_2^{as(b,t),h(b,t)}\right)^{-1}. \tag{5.2.10}$$

Using relation (5.2.1), we see that commutativity of the diagram (5.2.6) is equivalent to

$$\rho^{as(b,t)}(\nu^{h(b,t)})\left(\rho_2^{as(b,t),h(b,t)}\right)^{-1}\rho_2^{a,bt}\,\rho^a(\rho_2^{b,t}) = \mu^{a,s(b,t)}\,\rho^a(\mu^{b,t}). \tag{5.2.11}$$

Finally, we compute

$$\begin{aligned}\rho_2^{a,bt} &= \rho_2^{a,s(b,t)h(b,t)}\\ &= \rho_2^{as(b,t),h(b,t)}(\rho_2^{a,s(b,t)})_{\rho^{h(b,t)}(N)}\,\rho^a(\rho_2^{s(b,t),h(b,t)})^{-1}\\ &= \rho_2^{as(b,t),h(b,t)}\rho^{as(b,t)}(\nu^{h(b,t)})^{-1}\rho_2^{a,s(b,t)}\,\rho^a\rho^{s(b,t)}(\nu^{h(b,t)})\,\rho^a(\rho_2^{s(b,t),h(b,t)})^{-1}.\end{aligned}$$

Combining this with (5.2.9) we get relation (5.2.11). This shows that the diagram (5.2.6) is commutative, as claimed, and finishes the proof of the lemma. □

In view of Lemma 5.2.2 there is a well defined functor $L_Y = L_Y^{\mathcal{R}} : \mathcal{C}^{G_Y} \to \mathcal{C}^G$.

Proposition 5.2.4. *The functor $L_Y^{\mathcal{R}}$ is left adjoint of the functor F_Y.*

Proof. We define natural transformations $\eta\colon \mathrm{id}_{\mathcal{C}^{G_Y}} \to F_Y L_Y^{\mathcal{R}}$ and $\epsilon\colon L_Y^{\mathcal{R}} F_Y \to \mathrm{id}_{\mathcal{C}^G}$, in the form

$$\begin{aligned}\eta_{(N,\nu)} &= i_e : N = \rho^e(N) \to F_Y L_Y^{\mathcal{R}}(N,\nu) = F_Y(\oplus_{t\in\mathcal{R}}\rho^t(N),\mu),\\ \epsilon_{(M,u)} &= \oplus_{t\in\mathcal{R}} u^t : L_Y^{\mathcal{R}} F_Y(M,u) = (\oplus_{t\in\mathcal{R}}\rho^t(M),\mu) \to (M,u),\end{aligned}$$

for every $(N,\nu) \in \mathcal{C}^{G_Y}$, $(M,u) \in \mathcal{C}^G$. It is straightforward to verify that η and ϵ are well-defined and that the compositions

$$F_Y \overset{\eta F_Y}{\to} F_Y L_Y^{\mathcal{R}} F_Y \overset{F_Y\epsilon}{\to} F_Y, \quad L_Y^{\mathcal{R}} \overset{L_Y^{\mathcal{R}}\eta}{\to} L_Y^{\mathcal{R}} F_Y L_Y^{\mathcal{R}} \overset{\epsilon L_Y^{\mathcal{R}}}{\to} L_Y^{\mathcal{R}}$$

are identities. This implies the proposition. □

Remark 5.2.4. *Note that the restriction of L_Y to the fusion subcategory* Rep G_Y *of $\mathcal{C}^{G_Y}$ is isomorphic to the induction functor* Rep $G_Y \to$ Rep $G \subseteq \mathcal{C}^G$.

As pointed out in Remark 5.2.3, we have the following:

Corollary 5.2.5. *The functor $L_Y^{\mathcal{R}}$ is, up to isomorphism, uniquely determined by the subgroup G_Y.* □

5.2.5 Parameterization of simple objects

The following theorem is the main result of this section.

Theorem 5.2.6. *Let $Y \in \mathcal{C}$ be a simple object. Then the functor $L_Y : \mathcal{C}^{G_Y} \to \mathcal{C}^G$ induces a bijective correspondence between isomorphism classes of:*

(a) Simple objects $(N, \nu) \in \mathcal{C}^{G_Y}$ such that $\operatorname{Hom}_{\mathcal{C}}(Y, N) \neq 0$, and

(b) Simple objects $(X, \mu) \in \mathcal{C}^G$ such that $\operatorname{Hom}_{\mathcal{C}}(Y, X) \neq 0$.

If (X, μ) in $\mathcal{C}^G$ as in (b) corresponds to (N, ν) in $\mathcal{C}^{G_Y}$ as in (a), then we have $\operatorname{Hom}_{\mathcal{C}}(Y, N) \simeq \operatorname{Hom}_{\mathcal{C}}(Y, X)$ as projective representations of G_Y. Moreover, $N \simeq \operatorname{Hom}_{\mathcal{C}}(Y, N) \otimes Y$.

Proof. Let $(N, \nu) \in \mathcal{C}^{G_Y}$ be a simple object as in (a). By Proposition 5.2.1 applied to G_Y, $N \simeq mY$, where $m = \dim \operatorname{Hom}_{\mathcal{C}}(Y, N)$. Thus $N \simeq \operatorname{Hom}_{\mathcal{C}}(Y, N) \otimes Y$.

Let $(X, \mu) \in \mathcal{C}^G$ be a simple object such that (N, ν) is a simple direct summand of $F_Y(X, \mu)$ in $\mathcal{C}^{G_Y}$. By adjunction, (X, μ) is a simple direct summand of $L_Y(N, \nu)$ in $\mathcal{C}^G$. Then X is a direct summand of $\oplus_{t \in G/G_Y} \rho^t(N)$ in $\mathcal{C}$. Since X is G-equivariant, then $\operatorname{Hom}_{\mathcal{C}}(X, N) \neq 0$.
Therefore $\operatorname{Hom}_{\mathcal{C}}(Y, X) \neq 0$ and (X, μ) satisfies the condition in (b).

Again by Proposition 5.2.1, we get that $X \simeq e \oplus_{i=1}^n Y_i$, where $Y = Y_1, \ldots, Y_n$, $n = [G : G_Y]$, are the mutually nonisomorphic G-conjugates of Y. Note that $e = \dim \operatorname{Hom}_{\mathcal{C}}(Y, X) \leq \dim \operatorname{Hom}_{\mathcal{C}}(Y, N) = m$, because the multiplicity of Y in $\rho^t(N)$ is 0, for all $t \notin G_Y$.

Since (N, ν) is a direct summand of $F_Y(X, \mu)$ in $\mathcal{C}^{G_Y}$, comparing Frobenius-Perron dimensions, we obtain

$$[G : G_Y] m \operatorname{FPdim} Y = \operatorname{FPdim} L_Y(N, \nu) \leq \operatorname{FPdim}(X, \mu) = e[G : G_Y] \operatorname{FPdim} Y.$$

Therefore $e = m$, and necessarily $(X, \mu) = L_Y(N, \nu)$. This implies surjectivity of the map induced by L_Y from (a) to (b).

Suppose $(N', \nu') \not\simeq (N, \nu)$ is a simple summand of $F_Y(X, \mu)$, with (N', ν') as in (b). Applying the forgetful functor $\mathcal{C}^{G_Y} \to \mathcal{C}$ and comparing the multiplicity of Y we get

$$e = \dim \operatorname{Hom}_{\mathcal{C}}(Y, X) \geq \dim \operatorname{Hom}_{\mathcal{C}}(Y, N \oplus N') = m + \dim \operatorname{Hom}_{\mathcal{C}}(Y, N') > m.$$

This is a contradiction since $e = m$. Hence (N, ν) is the unique, up to isomorphisms, simple object as in (b) such that

$$\operatorname{Hom}_{\mathcal{C}^G}(L_Y(N, \nu), (X, \mu)) \simeq \operatorname{Hom}_{\mathcal{C}^{G_Y}}((N, \nu), F_Y(X, \mu)) \neq 0.$$

This proves injectivity of the map induced by L_Y. Thus this map is bijective, as claimed.

Now suppose that the class of the simple object (X, μ) of $\mathcal{C}^G$ as in (b) corresponds to the class of the simple object (N, ν) of $\mathcal{C}^{G_Y}$ as in (a). The proof above shows that $N \simeq \operatorname{Hom}_{\mathcal{C}}(Y, N) \otimes Y$.

As we have shown, $\dim \operatorname{Hom}_{\mathcal{C}}(Y, X) = \dim \operatorname{Hom}_{\mathcal{C}}(Y, N)$. Since N is a direct summand of X in $\mathcal{C}$, then $\operatorname{Hom}_{\mathcal{C}}(Y, N) \subseteq \operatorname{Hom}_{\mathcal{C}}(Y, X)$, thus these spaces are equal. Furthermore, the corresponding projective representations given by Formula (5.2.2) clearly coincide on both spaces. This finishes the proof of the theorem. □

Combining Theorem 5.2.6 with Proposition 5.2.3 we obtain the following:

Corollary 5.2.7. *There is a bijective correspondence between the set of isomorphism classes of simple objects (X, μ) of $\mathcal{C}^G$ and the set of pairs (Y, π), where Y runs over the orbits of the action of G on $\operatorname{Irr}(\mathcal{C})$ and π runs over the equivalence classes of irreducible α_Y-projective representations of the inertia subgroup $G_Y \subseteq G$, where $\alpha_Y \in H^2(G_Y, \Bbbk^*)$ is the cohomology class of the cocycle $\widetilde{\alpha}_Y$ determined by (5.2.1).*

Let (X, μ) be the simple object corresponding to the pair (Y, π). Then we have $X \simeq \oplus_{t \in G/G_Y} \rho^t(V_\pi \otimes Y)$. In particular, $\operatorname{FPdim}(X, \mu) = \dim \pi [G : G_Y] \operatorname{FPdim} Y$. □

Let $Y \in \operatorname{Irr}(\mathcal{C})/G$ and let π be an irreducible α_Y-projective representation of the group G_Y. We shall use the notation $S_{Y,\pi}$ to indicate the isomorphism class of the simple object of $\mathcal{C}^G$ corresponding to the pair (Y, π). We shall also say that such simple object $S_{Y,\pi}$ *lies over* Y.

Remark 5.2.5. *For every set $\mathcal{R}$ of left coset representatives of G_Y in G and for every collection of isomorphisms $\{c^g : \rho^g(Y) \to Y\}_{g \in G_Y}$, the class $S_{Y,\pi}$ is represented by the simple object $S^{\mathcal{R},c}_{Y,\pi} := L^{\mathcal{R}}_Y(\pi \otimes Y)$, with the G_Y-equivariant structure on $\pi \otimes Y$ given by* (5.2.3).

Let us describe more explicitly the dependence of the simple object $S^{\mathcal{R},c}_{Y,\pi}$ on the choice of the isomorphisms $c^h : \rho^h(Y) \to Y$. Suppose we are given another collection of isomorphisms $c' = \{c'^g\}$. Then, Y being simple, for any $g \in G_Y$ we can write $c'^g = d_{c,c'}(g) c^g$, for some scalar $d_{c,c'}(g) \in \Bbbk^*$. It follows from (5.2.3) *that $\pi \otimes Y = d^{-1}_{c,c'} \pi \otimes Y$ as objects of $\mathcal{C}^{G_Y}$. Hence*

$$S^{\mathcal{R},c}_{Y,\pi} = L^{\mathcal{R}}_Y(\pi \otimes Y) = L^{\mathcal{R}}_Y(d^{-1}_{c,c'} \pi \otimes Y) = S^{\mathcal{R},c'}_{Y, d^{-1}_{c,c'} \pi}. \tag{5.2.1}$$

Theorem 5.2.6 implies the following:

Lemma 5.2.3. *Let $Y \in \operatorname{Irr}(\mathcal{C})$ and let π be an α_Y-projective representation of G_Y. Then*

$$\pi \simeq \operatorname{Hom}_{\mathcal{C}}(Y,\ S_{Y,\pi}) \tag{5.2.2}$$

as G_Y-projective representations.

Proof. Let V_π denote the vector space of the representation π. We have $S_{Y,\pi} \simeq L_Y(V_\pi \otimes Y, (\pi(g) \otimes c^g)_{g \in G})$, where $c^g : \rho^g(Y) \to Y$ is a collection of isomorphisms. It follows from Proposition 5.2.3 and Theorem 5.2.6 that $\pi \simeq \operatorname{Hom}_{\mathcal{C}}(Y, V_\pi \otimes Y) \simeq \operatorname{Hom}_{\mathcal{C}}(Y, S_{Y,\pi})$. □

As a consequence we now get:

Proposition 5.2.8. *Let $Y \in \operatorname{Irr}(\mathcal{C})$ and let π be an irreducible α_Y-projective representation of G_Y. Then, for all $(X, \mu) \in \mathcal{C}^G$, we have*

$$\dim \operatorname{Hom}_{\mathcal{C}^G}(S_{Y,\pi},\ (X, \mu)) = m_{G_Y}(\pi,\ \operatorname{Hom}_{\mathcal{C}}(Y, X)).$$

In particular, the simple object $S_{Y,\pi}$ is a constituent of (X, μ) if and only if π is a constituent of $\operatorname{Hom}_{\mathcal{C}}(Y, X)$.

Here, $m_{G_Y}(\pi,\ \operatorname{Hom}_{\mathcal{C}}(Y, X))$ denotes the multiplicity of π in $\operatorname{Hom}_{\mathcal{C}}(Y, X)$. See Section 5.5.

Proof. We have a decomposition $(X, \mu) \simeq \oplus_{(Z,\gamma)} \operatorname{Hom}_{\mathcal{C}^G}(S_{Z,\gamma}, (X, \mu)) \otimes S_{Z,\gamma}$, where Z runs over a set of representatives of the orbits of the action of G on $\operatorname{Irr}(\mathcal{C})$ and γ is an irreducible α_Z-projective representation of G_Z. Since $\operatorname{Hom}_{\mathcal{C}}(Y, S_{Z,\gamma}) = 0$, for all $Z \neq Y$, then, as projective G_Y-representations,

$$\begin{aligned} \operatorname{Hom}_{\mathcal{C}}(Y, X) &\simeq \oplus_{(Z,\gamma)} \operatorname{Hom}_{\mathcal{C}^G}(S_{Z,\gamma}, (X, \mu)) \otimes \operatorname{Hom}_{\mathcal{C}}(Y, S_{Z,\gamma}) \\ &\simeq \oplus_{\gamma} \operatorname{Hom}_{\mathcal{C}^G}(S_{Y,\gamma}, (X, \mu)) \otimes \gamma, \end{aligned}$$

the last isomorphism by Lemma 5.2.3. This implies the proposition. □

5.2.6 On the choice of isomorphisms in a fixed orbit

Let $Y \in \operatorname{Irr}(\mathcal{C})$ and let $t \in G$. Since $\rho^t(Y)$ is a constituent of $F(S_{Y,\pi})$, it follows from Theorem 5.2.6 that $S_{Y,\pi} \simeq S_{\rho^t(Y),\delta}$, for some irreducible projective representation δ of $G_{\rho^t(Y)}$. In this subsection we discuss the dependence of δ upon π and the choice of the sets of isomorphisms c_Y, $c_{\rho^t(Y)}$.

Let π be a projective representation of G_Y with factor set $\widetilde{\alpha}_Y$ and let ${}^t\pi$ be the conjugate projective representation of $G_{\rho^t(Y)} = tG_Yt^{-1} =: {}^tG_Y$. That is, $V_{{}^t\pi} = V_\pi$ and the action is defined as ${}^t\pi(h) = \pi(t^{-1}ht)$, for all $h \in G_Y$. Denote by ${}^t\widetilde{\alpha}_Y$ the 2-cocycle of $G_{\rho^t(Y)}$ given by

$${}^t\widetilde{\alpha}_Y(tht^{-1}, th't^{-1}) = \widetilde{\alpha}_Y(h, h'), \quad h, h' \in G_Y. \tag{5.2.1}$$

Then ${}^t\pi$ is a projective representation of $G_{\rho^t(Y)}$ with factor set ${}^t\widetilde{\alpha}_Y$.

Note that a given collection of isomorphisms $c^g : \rho^g(Y) \to Y$, $g \in G_Y$, determines canonically a collection of isomorphisms $({}^tc)^g : \rho^g(\rho^t(Y)) \to \rho^t(Y)$, $g \in {}^tG_Y$, in the form

$$({}^tc)^g := \rho^t(c_Y^{t^{-1}gt})\,(\rho_2^{t,t^{-1}gt})_Y^{-1}\,(\rho_2^{g,t})_Y. \tag{5.2.2}$$

Indeed, $t^{-1}gt \in G_Y$ since $G_{\rho^t(Y)} = tG_Yt^{-1}$.

Remark 5.2.6. *Assume that Y is a simple object representing a class in a fixed orbit of the action of G. For the objects $\rho^t(Y)$, let the isomorphisms ${}^tc$ be given as in* (5.2.2)*. Then formula* (5.2.1) *gives the 2-cocycle $\widetilde{\alpha}_{\rho^t(Y)}(tht^{-1}, th't^{-1}) = \widetilde{\alpha}_Y(h, h')$ on the inertia subgroup $G_{\rho^t(Y)}$.*

Lemma 5.2.1. *Let H be a subgroup of G and let $(M,\nu) \in \mathcal{C}^H$. Then, for all $x \in G$, $(\rho^x(M), {}^x\nu) \in \mathcal{C}^{{}^xH}$ with equivariant structure $({}^x\nu)^{xhx^{-1}} : \rho^{xhx^{-1}}\rho^x(M) \to \rho^x(M)$ defined, for every $h \in H$, as the composition*

$$\rho^{xhx^{-1}}(\rho^x(M)) \xrightarrow{\rho_2^{xhx^{-1},x}} \rho^{xh}(M) \xrightarrow{(\rho_2^{x,h})^{-1}} \rho^x(\rho^h(M)) \xrightarrow{\rho^x(\nu^h)} \rho^x(M). \tag{5.2.3}$$

Proof. Consider the 2-cocycle $\widetilde{\alpha}_{\rho^x(M)} \in Z^2(G_{\rho^x(M)}, \Bbbk^*)$ associated to the collection of isomorphisms ${}^x\nu$. Since ν is an equivariant structure on M, it has a trivial associated 2-cocycle. It follows from Remark 5.2.6 that $\widetilde{\alpha}_{\rho^x(M)} = 1$ and therefore ${}^x\nu$ does define an equivariant structure on $\rho^x(M)$. □

Corollary 5.2.9. *Let ${}^tc$ be the collection of isomorphisms given by equation* (5.2.2) *and let $V_{{}^t\pi} \otimes \rho^t(Y)$ be the associated object of $\mathcal{C}^{G_{\rho^t(Y)}}$. Then $\rho^t(V_\pi \otimes Y) = V_{{}^t\pi} \otimes \rho^t(Y)$ in $\mathcal{C}^{G_{\rho^t(Y)}}$.*

In particular we have an isomorphism of ${}^tG_Y$-equivariant objects $\rho^t(V_\pi \otimes Y) \simeq V_{{}^t\pi} \otimes \rho^t(Y)$, where the ${}^tG_Y$-equivariant structure on $V_{{}^t\pi} \otimes \rho^t(Y)$ is induced by any choice of isomorphisms $c_{\rho^t(Y)}$ for $\rho^t(Y)$.

Proof. By Lemma 5.2.1, a $G_{\rho^t(Y)}$-equivariant structure on $\rho^t(V_\pi \otimes Y)$ is given by

$$\rho^{tht^{-1}}\rho^t(V_\pi \otimes Y) \xrightarrow{\rho_2^{tht^{-1},t}} \rho^{th}(V_\pi \otimes Y) \xrightarrow{(\rho_2^{t,h})^{-1}} \rho^t\rho^h(V_\pi \otimes Y) \xrightarrow{\pi(h)\otimes\rho^t(c_Y^h)} V_\pi \otimes \rho^t(Y),$$

for every $h \in H$. Since ${}^t\pi(tht^{-1}) = \pi(h)$, for all $h \in H$, this coincides with the equivariant structure of $V_{{}^t\pi} \otimes \rho^t(Y)$ induced by ${}^tc$. □

Suppose now that for every $h \in G_Y$ we have arbitrary isomorphisms

$$c_{\rho^t(Y)}^{tht^{-1}} : \rho^{tht^{-1}}(\rho^t(Y)) \to \rho^t(Y)$$

These give rise to isomorphisms

$$\rho^t(Y) \xrightarrow{(c_{\rho^t(Y)}^{tht^{-1}})^{-1}} \rho^{tht^{-1}}(\rho^t(Y)) \xrightarrow{\rho_2^{tht^{-1},t}} \rho^{th}(Y) \xrightarrow{(\rho_2^{t,h})^{-1}} \rho^t(\rho^h(Y)) \xrightarrow{\rho^t(c_Y^h)} \rho^t(Y).$$

Since $\rho^t(Y)$ is a simple object, there exist scalars $d_Y(t,h) \in \Bbbk^*$ such that

$$\rho^t(c_Y^h)(\rho_2^{t,h})^{-1}\rho_2^{tht^{-1},t} = d_Y(t,h)c_{\rho^t(Y)}^{tht^{-1}}, \tag{5.2.4}$$

for all $h \in H$.

5.3 Fusion rules for $\mathcal{C}^G$

In this section we shall assume that $\mathcal{C}$ is a fusion category over $\Bbbk$ and $\rho : \underline{G} \to \underline{\mathrm{Aut}_{\otimes}}\mathcal{C}$ is an action of G on $\mathcal{C}$ by *tensor autoequivalences*, that is, $\rho_2^{g,h} : \rho^g \rho^h \to \rho^{gh}$ are natural isomorphisms of tensor functors, for all $g, h \in G$. Thus, for all $g \in G$, ρ^g is endowed with a monoidal structure $(\rho_2^g)_{X,Y} : \rho^g(X \otimes Y) \to \rho^g(X) \otimes \rho^g(Y)$, $X, Y \in \mathcal{C}$, and the following relation holds:

$$\rho_2^{gh}{}_{X,Y} \rho_2^{g,h}{}_{X \otimes Y} = (\rho_2^{g,h}{}_X \otimes \rho_2^{g,h}{}_Y)\, \rho^g_{2\,\rho^h X, \rho^h Y}\, \rho^g(\rho^h_{2\,X,Y}), \tag{5.3.1}$$

for all $g, h \in G$, $X, Y \in \mathcal{C}$.

Then $\mathcal{C}^G$ is also a fusion category with tensor product $(X, \mu_X) \otimes (Y, \mu_Y) = (X \otimes Y, (\mu_X \otimes \mu_Y)\rho_{2\,X,Y})$, where for all $g \in G$, $\rho_2{}^g_{X,Y} : \rho^g(X \otimes Y) \to \rho^g(X) \otimes \rho^g(Y)$ is the monoidal structure on ρ^g.

Let $\pi : G \to \mathrm{GL}(V)$ be a finite dimensional representation of G on the vector space V. Then the (trivial) object $V \otimes \mathbf{1} \in \mathcal{C}$ has a G-equivariant structure defined by $\pi(g) \otimes \mathrm{id}_{\mathbf{1}} : \rho^g(V \otimes \mathbf{1}) \to V \otimes \mathbf{1}$. This induces an embedding of fusion categories $\mathrm{Rep}\, G \to \mathcal{C}^G$ that gives rise to an exact sequence of fusion categories

$$\mathrm{Rep}\, G \to \mathcal{C}^G \to \mathcal{C}. \tag{5.3.2}$$

See [13, Subsection 5.4].

Remark 5.3.1. *Let* $\mathrm{G}(\mathcal{C})$ *be the set of isomorphism classes of invertible objects of* $\mathcal{C}$*. The exact sequence* (5.3.2) *induces an exact sequence of groups*

$$1 \to \widehat{G} \to \mathrm{G}(\mathcal{C}^G) \to \mathrm{G}_0(\mathcal{C}) \to 1,$$

where $\widehat{G} \simeq G/[G, G]$ *denotes the group of invertible characters of* G *and* $\mathrm{G}_0(\mathcal{C})$ *is the subgroup of* $\mathrm{G}(\mathcal{C})$ *consisting of isomorphism classes of invertible objects which are* G*-equivariant. Indeed,* F *preserves Frobenius-Perron dimensions, and thus it induces a group homomorphism* $F : \mathrm{G}(\mathcal{C}^G) \to \mathrm{G}_0(\mathcal{C})$*, which is clearly surjective. The kernel of* F *coincides with the invertible objects of* $\mathfrak{Ker}_F = \mathrm{Rep}\, G$*.*

Remark 5.3.2. *Note that if* π *is an irreducible representation of* $G = G_{\mathbf{1}}$ *on* V*, then the simple object* $(V \otimes \mathbf{1}, (\pi(g) \otimes \mathrm{id}_{\mathbf{1}})_g)$ *of* $\mathcal{C}^G$ *is isomorphic to the simple object* $S_{\mathbf{1},\pi}$ *corresponding to the pair* $(\mathbf{1}, \pi)$ *as in Corollary 5.2.7.*

5.3.1 Orbit formula for the tensor product of two simple objects

Let $Y, Z, U \in \mathrm{Irr}(\mathcal{C})$ and let π, γ, δ, be projective representations of the corresponding inertia subgroups with factor sets determined by (5.2.1). Let also $S_{Y,\pi}$, $S_{Z,\gamma}$ and $S_{U,\delta}$ be the associated simple objects of $\mathcal{C}^G$.

The multiplicity of $S_{U,\delta}$ in the tensor product $S_{Y,\pi} \otimes S_{Z,\gamma}$ is given by the dimension of the vector space $\mathrm{Hom}_{\mathcal{C}^G}(S_{U,\delta},\ S_{Y,\pi} \otimes S_{Z,\gamma})$. In view of Proposition 5.2.8, this multiplicity is the same as the multiplicity of δ in the space

$$\mathrm{Hom}_{\mathcal{C}}(U,\ S_{Y,\pi} \otimes S_{Z,\gamma}), \tag{5.3.1}$$

regarded as an α_U-projective representation of G_U.

Consider the diagonal action of G on $G/G_Y \times G/G_Z$ coming from the natural actions by left multiplication of G on G/G_Y and G/G_Z. The stabilizer of a pair (t, s) is the subgroup ${}^tG_Y \cap {}^sG_Z \subseteq G$.

As objects of $\mathcal{C}$, we have that

$$\begin{aligned} S_{Y,\pi} \otimes S_{Z,\gamma} &\simeq \Big(\bigoplus_{u\in G/G_Y} \rho^u(V_\pi \otimes Y)\Big) \otimes \Big(\bigoplus_{v\in G/G_Z} \rho^v(V_\gamma \otimes Z)\Big) \\ &\simeq \bigoplus_{(u,v)\in G/G_Y\times G/G_Z} \rho^u(V_\pi \otimes Y) \otimes \rho^v(V_\gamma \otimes Z) \\ &= \bigoplus_{\mathcal{O}} S_{\mathcal{O}}, \end{aligned}$$

where the last summation is over the distinct G-orbits $\mathcal{O}$ in $G/G_Y \times G/G_Z$, and for every G-orbit $\mathcal{O}$,

$$S_{\mathcal{O}} := \oplus_{(u,v)\in\mathcal{O}}\, \rho^u(V_\pi \otimes Y) \otimes \rho^v(V_\gamma \otimes Z). \tag{5.3.2}$$

The subgroup G_U acts on $G/G_Y \times G/G_Z$ by restriction and every G-orbit in $G/G_Y \times G/G_Z$ is a disjoint union of G_U-orbits. Note that the stabilizer of $(t, s) \in G/G_Y \times G/G_Z$ under the action of G_U is the subgroup $T = G_U \cap tG_Yt^{-1} \cap sG_Zs^{-1}$.

Lemma 5.3.1. *For every G-orbit (respectively, G_U-orbit) $\mathcal{O} \subseteq G/G_Y \times G/G_Z$, $S_{\mathcal{O}}$ is an equivariant subobject (respectively, a G_U-equivariant subobject) of the tensor product $S_{Y,\pi} \otimes S_{Z,\gamma}$.*

Proof. We shall prove the statement for G-orbits, the proof for G_U-orbits being analogous. Let $g \in G$. The equivariant structure $\mu^g := \mu^g_{S_{Y,\pi}\otimes S_{Z,\gamma}}$ of $S_{Y,\pi} \otimes S_{Z,\gamma}$ is given componentwise by

$$(\mu^g)^{u,v} = (\mu^{g,u}_{S_{Y,\pi}} \otimes \mu^{g,v}_{S_{Z,\gamma}})(V_\pi \otimes V_\gamma \otimes (\rho^g_2)_{\rho^u Y, \rho^v Z}),$$

where, for every $(u, v) \in G/G_Y \times G/G_Z$, $\mu^{g,u}_{S_{Y,\pi}}$ and $\mu^{g,v}_{S_{Z,\gamma}}$ are given by formula (5.2.1). It follows that

$$(\mu^g)^{u,v}(\rho^g(V_\pi \otimes V_\delta \otimes \rho^u(Y) \otimes \rho^v(Z)) \subseteq V_\pi \otimes V_\delta \otimes \rho^{u'}(Y) \otimes \rho^{v'}(Z),$$

where $(u', v') \in G/G_Y \times G/G_Z$ are uniquely determined by the relations $gu = u'h_Y$ and $gv = v'h_Z$, with $h_Y \in G_Y$ and $h_Z \in G_Z$. Therefore, $\mu^g(S_{\mathcal{O}}) \subseteq S_{\mathcal{O}}$, for all $g \in G$. This implies the lemma. □

The map $G \times G \to G$, $(a, b) \mapsto a^{-1}b$, induces a surjective map $p : G/G_Y \times G/G_Z \to G_Y\backslash G/G_Z$, such that $p(tG_Y, sG_Z) = G_Yt^{-1}sG_Z$.

Let $\mathcal{O}_G(t,s)$ denote the G-orbit of an element $(t,s) \in G/G_Y \times G/G_Z$. Observe that for all $g \in G$, we have $p^{-1}(G_Y g G_Z) = \mathcal{O}_G(e,g)$. Therefore, p induces an identification between the orbit space of $G/G_Y \times G/G_Z$ under the action of G and the space of double cosets $G_Y \backslash G/G_Z$. Combining this with (5.3.2), we obtain:

Corollary 5.3.1. *We have a decomposition* $S_{Y,\pi} \otimes S_{Z,\gamma} \simeq \bigoplus_{D \in G_Y \backslash G/G_Z} S_D$, *where* $S_D := \bigoplus_{t^{-1}s \in D} \rho^t(V_\pi \otimes Y) \otimes \rho^s(V_\gamma \otimes Z)$. *Moreover, for all* $D \in G_Y \backslash G/G_Z$, S_D *is a* G*-equivariant subobject of* $S_{Y,\pi} \otimes S_{Z,\gamma}$. □

5.3.2 Projective representation on multiplicity spaces.

It follows from Lemma 5.3.1 that for every G-orbit $\mathcal{O} \subseteq G/G_Y \times G/G_Z$, the space $\mathcal{H}_{\mathcal{O}} := \mathrm{Hom}_{\mathcal{C}}(U, S_{\mathcal{O}})$ is an α_U-projective representation of G_U.

Let $\mathcal{O} = \mathcal{O}_1 \cup \cdots \cup \mathcal{O}_{n_{\mathcal{O}}}$ be the decomposition of $\mathcal{O}$ into disjoint G_U-orbits $\mathcal{O}_1, \ldots, \mathcal{O}_{n_{\mathcal{O}}}$. Then, for all $1 \leq i \leq n_{\mathcal{O}}$, the space $\mathcal{H}(i) = \mathrm{Hom}_{\mathcal{C}}(U, S_{\mathcal{O}_i})$ is also an α_U-projective representation of G_U, where $S_{\mathcal{O}_i} := \oplus_{(u,v) \in \mathcal{O}_i} \rho^u(V_\pi \otimes Y) \otimes \rho^v(V_\gamma \otimes Z)$.

Furthermore, as G_U-projective representations,

$$\mathrm{Hom}_{\mathcal{C}}(U,\ S_{Y,\pi} \otimes S_{Z,\gamma}) \simeq \oplus_{\mathcal{O}} \mathcal{H}_{\mathcal{O}} \simeq \oplus_{\mathcal{O}} \oplus_{i=1}^{n_{\mathcal{O}}} \mathcal{H}(i),$$

where summation is understood to run over all orbits $\mathcal{O} = \mathcal{O}_G(t,s)$ such that $\mathrm{Hom}_{\mathcal{C}}(U, \rho^t(Y) \otimes \rho^s(Z)) \neq 0$.

For every $(t,s) \in G/G_Y \times G/G_Z$, let

$$\mathcal{H}_{t,s} := \mathrm{Hom}_{\mathcal{C}}(U,\ V_{{}^t\pi} \otimes V_{{}^s\gamma} \otimes \rho^t(Y) \otimes \rho^s(Z)).$$

Lemma 5.3.2. *Let* $t,s \in G$. *Then* $\mathcal{H}_{t,s}$ *is an* $\alpha_U|_T$*-projective representation of* $T = G_U \cap {}^tG_Y \cap {}^sG_Z$. *Moreover, for all* $1 \leq i \leq n$, *we have*

$$\mathcal{H}(i) = \mathrm{Hom}_{\mathcal{C}}(U, S_{\mathcal{O}_i}) = \bigoplus_{(t,s) \in \mathcal{O}_i} \mathcal{H}_{t,s}, \tag{5.3.1}$$

as projective representations of T.

Proof. Since $V_{{}^t\pi} \otimes \rho^t(Y)$ and $V_{{}^s\gamma} \otimes \rho^s(Z)$ are ${}^tG_Y \cap {}^sG_Z$-equivariant objects, so is their tensor product. Lemma 5.2.1 implies that $\mathcal{H}_{t,s}$ is an $\alpha_U|_T$-projective representation of T. The decomposition (5.3.1) follows from the definition of $S_{\mathcal{O}_i}$. □

Proposition 5.3.2. *Let* $U, Y, Z \in \mathrm{Irr}(\mathcal{C})$ *and let* $t, s \in G$. *Then the vector space* $\tau_U^{t,s}(Y,Z) := \mathrm{Hom}_{\mathcal{C}}(U, \rho^t(Y) \otimes \rho^s(Z))$ *carries an* α*-projective representation of the subgroup* $T := G_U \cap G_{\rho^t(Y)} \cap G_{\rho^s(Z)}$, *where* $\alpha := \alpha_U|_T \alpha^{-1}_{\rho^t(Y)}|_T \alpha^{-1}_{\rho^s(Z)}|_T$. *The action of* $g \in T$ *is given by*

$$g.f = (c^g_{\rho^t(Y)} \otimes c^g_{\rho^s(Z)})\,(\rho^g_2)_{\rho^t(Y),\,\rho^s(Z)}\,\rho^g(f)\,(c^g_U)^{-1}, \tag{5.3.2}$$

for all $f \in \mathrm{Hom}_{\mathcal{C}}(U, \rho^t(Y) \otimes \rho^s(Z))$. *Furthermore,*

$$\mathcal{H}_{t,s} \simeq {}^t\pi|_T \otimes {}^s\gamma|_T \otimes \tau_U^{t,s}(Y,Z),$$

as projective representations of T.

Remark 5.3.3. *Observe that the equivalence class of the projective representation $\tau_U^{t,s}(Y,Z)$ is independent on the choice of the isomorphism classes of U, Y, Z as well as on the choice of isomorphisms $c_{\rho^t(Y)}$, $c_{\rho^s(Z)}$ and c_U.*

Proof. Given $X \in \operatorname{Irr}(\mathcal{C})$, we shall consider in what follows a fixed (but arbitrary) collection of isomorphisms $c_X = \{c_X^g : \rho^g(X) \to X\}_{g\in G_X}$. Let also $\widetilde{\alpha}_X \in Z^2(G_X, \Bbbk^*)$ be the associated 2-cocycles.

We first show that formula (5.3.2) does define a projective representation of T with factor set $\widetilde{\alpha}_U|_T \widetilde{\alpha}^{-1}_{\rho^t(Y)}|_T \widetilde{\alpha}^{-1}_{\rho^s(Z)}|_T$. Let $g, h \in T$, $f \in \operatorname{Hom}_{\mathcal{C}}(U, \rho^t(Y) \otimes \rho^s(Z))$. Using the definition of the cocycles $\widetilde{\alpha}$ given by (5.2.1) and relation (5.3.1), we compute:

$$
\begin{aligned}
g.(h.f) &= (c^g_{\rho^tY} \otimes c^g_{\rho^sZ})\,(\rho_2^g)_{\rho^tY,\,\rho^sZ}\rho^g\left((c^h_{\rho^tY} \otimes c^h_{\rho^sZ})\,(\rho_2^h)_{\rho^tY,\,\rho^sZ}\,\rho^h(f)\,c_U^{h\,-1}\right) c_U^{g\,-1}\\
&= (c^g_{\rho^tY} \otimes c^g_{\rho^sZ})\,(\rho_2^g)_{\rho^tY,\,\rho^sZ}\,\rho^g(c^h_{\rho^tY} \otimes c^h_{\rho^sZ})\,\rho^g((\rho_2^h)_{\rho^tY,\,\rho^sZ})\\
&\qquad \rho^g\rho^h(f)\,\rho^g(c_U^h)^{-1}\,c_U^{g\,-1}\\
&= (c^g_{\rho^tY} \otimes c^g_{\rho^sZ})\left(\rho^g(c^h_{\rho^tY}) \otimes \rho^g(c^h_{\rho^sZ})\right)(\rho_2^g)_{\rho^tY,\,\rho^sZ}\rho^g((\rho_2^h)_{\rho^tY,\,\rho^sZ})\,\rho^g\rho^h(f)\\
&\qquad \rho^g(c_U^h)^{-1}c_U^{g\,-1}\\
&= (c^g_{\rho^tY} \otimes c^g_{\rho^sZ})\left(\rho^g(c^h_{\rho^tY}) \otimes \rho^g(c^h_{\rho^sZ})\right)\left((\rho_2^{g,h})^{-1}_{\rho^tY} \otimes (\rho_2^{g,h})^{-1}_{\rho^sZ}\right)(\rho_2^{gh})_{\rho^tY,\,\rho^sZ}\\
&\qquad (\rho_2^{g,h})_{\rho^tY\otimes\rho^sZ}\,\rho^g\rho^h(f)\,\rho^g(c_U^h)^{-1}\,c_U^{g\,-1}\\
&= \widetilde{\alpha}^{-1}_{\rho^tY}(g,h)\widetilde{\alpha}^{-1}_{\rho^sZ}(g,h)(c^{gh}_{\rho^tY} \otimes c^{gh}_{\rho^sZ})\,(\rho_2^{gh})_{\rho^tY,\,\rho^sZ}\rho^{gh}(f)(\rho_2^{g,h})_U\,\rho^g(c_U^h)^{-1}\,c_U^{g\,-1}\\
&= \widetilde{\alpha}^{-1}_{\rho^tY}(g,h)\widetilde{\alpha}^{-1}_{\rho^sZ}(g,h)\widetilde{\alpha}_U(g,h)\,(c^{gh}_{\rho^tY} \otimes c^{gh}_{\rho^sZ})\,(\rho_2^{gh})_{\rho^tY,\,\rho^sZ}\rho^{gh}(f)\,c_U^{gh\,-1}\\
&= \widetilde{\alpha}^{-1}_{\rho^tY}(g,h)\widetilde{\alpha}^{-1}_{\rho^sZ}(g,h)\widetilde{\alpha}_U(g,h)\,(gh.f).
\end{aligned}
$$

On the other hand, with respect to the given choice of isomorphisms $\{c_X^g\}_{g\in G_X}$, $X \in \operatorname{Irr}(\mathcal{C})$, the ${}^tG_Y \cap {}^sG_Z$-equivariant structures on $V_{{}^t\pi} \otimes \rho^t(Y)$ and $V_{{}^s\gamma} \otimes \rho^s(Z)$ are given, respectively, by ${}^t\pi(g) \otimes c^g_{\rho^t(Y)} : \rho^g(V_{{}^t\pi} \otimes \rho^t(Y)) \to V_{{}^t\pi} \otimes \rho^t(Y)$, and ${}^s\gamma(g) \otimes c^g_{\rho^s(Z)} : \rho^g(V_{{}^s\gamma} \otimes \rho^s(Z)) \to V_{{}^s\gamma} \otimes \rho^s(Z)$, for all $g \in T$.

Thus, the action of $g \in T$ on $f \in \mathcal{H}_{t,s} = \operatorname{Hom}_{\mathcal{C}}(U, V_{{}^t\pi} \otimes V_{{}^s\gamma} \otimes \rho^t(Y) \otimes \rho^s(Z))$ is determined by

$$
\begin{aligned}
g.f &= ({}^t\pi(g) \otimes {}^s\gamma(g) \otimes c^g_{\rho^t(Y)} \otimes c^g_{\rho^s(Z)})\,(V_{{}^t\pi} \otimes V_{{}^s\gamma} \otimes (\rho_2^g)_{\rho^t(Y),\rho^s(Z)})\,\rho^g(f)\,(c_U^g)^{-1}\\
&= {}^t\pi(g) \otimes {}^s\gamma(g) \otimes \left((c^g_{\rho^t(Y)} \otimes c^g_{\rho^s(Z)})\,(\rho_2^g)_{\rho^t(Y),\rho^s(Z)}\right)\rho^g(f)\,(c_U^g)^{-1}.
\end{aligned}
$$

In view of the $\Bbbk$-linearity of the functors ρ^g, $g \in G$, this implies that the canonical isomorphism

$$V_{{}^t\pi} \otimes V_{{}^s\gamma} \otimes \operatorname{Hom}_{\mathcal{C}}(U, \rho^t(Y) \otimes \rho^s(Z)) \simeq \operatorname{Hom}_{\mathcal{C}}(U, V_{{}^t\pi} \otimes V_{{}^s\gamma} \otimes \rho^t(Y) \otimes \rho^s(Z))$$

is indeed an isomorphism of projective representations of T. This finishes the proof of the proposition. $\square$

Proposition 5.3.3. *Let $(t_i, s_i) \in \mathcal{O}_i$, $1 \le i \le n$, and let $T_i = G_U \cap {}^{t_i}G_Y \cap {}^{s_i}G_Z$ be its stabilizer in G_U. Then $\mathcal{H}(i) \simeq \operatorname{Ind}_{T_i}^{G_U} \mathcal{H}_{t_i,s_i}$, as projective G_U-representations.*

Proof. The proposition follows from Lemma 5.3.2, in view of Lemma 5.5.1. Note that the group G_U permutes the set $\mathcal{O}_i$ transitively. □

5.3.3 Fusion rules

The following theorem gives the fusion rules for the category $\mathcal{C}^G$.

Theorem 5.3.4. *Let $U, Y, Z \in \operatorname{Irr}(\mathcal{C})$ and let δ, π, γ be irreducible projective representations of the inertia subgroups G_U, G_Y, G_Z with factor sets determined by (5.2.1). Then the multiplicity of $S_{U,\delta}$ in the tensor product $S_{Y,\pi} \otimes S_{Z,\gamma}$ is given by the formula*

$$\sum_{D \in G_Y \backslash G / G_Z} \sum_{\substack{1 \le i \le n \\ t_i^{-1} s_i \in D \\ \operatorname{Hom}_{\mathcal{C}}(U, \rho^{t_i} Y \otimes \rho^{s_i} Z) \neq 0}} m_{T_i}(\delta|_{T_i},\ {}^{t_i}\pi|_{T_i} \otimes {}^{s_i}\gamma|_{T_i} \otimes \tau_U^{s_i,t_i}(Y, Z)), \tag{5.3.1}$$

where $(t_1, s_1), \ldots, (t_n, s_n)$ are representatives of the distinct G_U-orbits $\mathcal{O}_1, \ldots, \mathcal{O}_n$ in $G/G_Y \times G/G_Z$ and, for all $1 \le i \le n$, $T_i = G_U \cap {}^{t_i}G_Y \cap {}^{s_i}G_Z$, and m_{T_i} denotes the multiplicity form of projective T_i-representations.

Proof. It follows from Proposition 5.2.8 that

$$\dim \operatorname{Hom}_{\mathcal{C}^G}(S_{U,\delta},\ S_{Y,\pi} \otimes S_{Z,\gamma}) = m_{G_U}(\delta,\ \operatorname{Hom}_{\mathcal{C}}(U, S_{Y,\pi} \otimes S_{Z,\gamma})).$$

In view of Corollary 5.3.1, we have a decomposition

$$\operatorname{Hom}_{\mathcal{C}}(U, S_{Y,\pi} \otimes S_{Z,\gamma}) \simeq \bigoplus_{D \in G_Y \backslash G / G_Z} \mathcal{H}_D,$$

as projective representations of G_U, where

$$\mathcal{H}_D := \bigoplus_{\substack{t^{-1}s \in D, \\ \operatorname{Hom}_{\mathcal{C}}(U, \rho^t(Y) \otimes \rho^s(Z)) \neq 0}} \operatorname{Hom}_{\mathcal{C}}(U, \rho^t(V_\pi \otimes Y) \otimes \rho^s(V_\gamma \otimes Z)).$$

Consider a decomposition $\mathcal{O}_1 \cup \cdots \cup \mathcal{O}_n$ of $G/G_Y \times G/G_Z$ into disjoint G_U-orbits, and let $\mathcal{H}(i) \simeq \operatorname{Hom}_{\mathcal{C}}(U, S_{\mathcal{O}_i})$, $1 \le i \le n$, as in (5.3.1).

Let also $(t_i, s_i) \in \mathcal{O}_i$ be a representative of the orbit $\mathcal{O}_i$ with stabilizer $T_i = G_U \cap {}^{t_i}G_Y \cap {}^{s_i}G_Z$. By Proposition 5.3.3 we have

$$\mathcal{H}_D \simeq \bigoplus_{\substack{1 \le i \le n \\ t_i^{-1} s_i \in D}} \mathcal{H}(i) \simeq \bigoplus_{\substack{1 \le i \le n \\ t_i^{-1} s_i \in D}} \operatorname{Ind}_{T_i}^{G_U} \mathcal{H}_{t_i,s_i},$$

Therefore, using Frobenius Reciprocity and Proposition 5.3.2, we get

$$\begin{aligned}
\dim \operatorname{Hom}_{\mathcal{C}^G}(S_{U,\delta}, S_{Y,\pi}\otimes S_{Z,\gamma}) &= \sum_{D\in G_Y\backslash G/G_Z} m_{G_U}(\delta,\ \mathcal{H}_D)\\
&= \sum_{D\in G_Y\backslash G/G_Z} \sum_{\substack{1\le i\le n\\ t_i^{-1}s_i\in D\\ \operatorname{Hom}_{\mathcal{C}}(U,\rho^{t_i}Y\otimes\rho^{s_i}Z)\neq 0}} m_{G_U}(\delta,\ \operatorname{Ind}_{T_i}^{G_U}\mathcal{H}_{t_i,s_i})\\
&= \sum_{D\in G_Y\backslash G/G_Z} \sum_{\substack{1\le i\le n\\ t_i^{-1}s_i\in D\\ \operatorname{Hom}_{\mathcal{C}}(U,\rho^{t_i}Y\otimes\rho^{s_i}Z)\neq 0}} m_{T_i}(\delta|_{T_i},\ \mathcal{H}_{t_i,s_i})\\
&= \sum_{D\in G_Y\backslash G/G_Z} \sum_{\substack{1\le i\le n\\ t_i^{-1}s_i\in D\\ \operatorname{Hom}_{\mathcal{C}}(U,\rho^{t_i}Y\otimes\rho^{s_i}Z)\neq 0}} m_{T_i}(\delta|_{T_i},{}^{t_i}\pi|_{T_i}\otimes{}^{s_i}\gamma|_{T_i}\otimes\tau_U^{s_i,t_i}(Y,Z)).
\end{aligned}$$

Thus we get formula (5.3.1). This finishes the proof of the theorem. □

Remark 5.3.4. *In the paper [122] a class of rings graded by conjugacy classes of finite groups is studied. More generally, suppose G and Γ are finite groups such that G acts on Γ by group automorphisms, and denote this action by $g\mapsto {}^xg$, $g\in\Gamma$, $x\in G$. Let $A=\oplus_{g\in\Gamma}A(g)$, where $A(g)$, $g\in\Gamma$, are free modules over a commutative ring R, endowed with R-linear isomorphisms $c_{g,x}: A(g)\to A({}^xg)$ and R-bilinear maps $m_{g,h}: A(g)\times A(h)\to A(gh)$, $g,h\in\Gamma$, $x\in G$, subject to certain compatibility conditions. Then there is an associative multiplication in A defined componentwise by $m_{g,h}$, $g,h\in G$, and the submodule of G-invariants: $A^G=\{a\in A: c_x(a)=a\}$ is a subring of A; here $c_x: A\to A$ is the R-linear map which is $c_{g,x}$ on the component $A(g)$. Furthermore, under weaker assumptions on the maps $c_{g,x}$, $m_{g,h}$, A^G is an associative ring with multiplication*

$$(a.b)_u := \sum_{\substack{(y,z)\in G_u\backslash(\Gamma\times\Gamma)\\ yz=u}} m_{y,z}(a_y,b_z),$$

for all $a,b\in A^G$, $u\in\Gamma$. See [122, Theorem 2.2].

It is shown in [122, Section 4] that the Grothendieck ring of the fusion category of representations of a Hopf algebra cocentral abelian extension of $\Bbbk^\Gamma$ by $\Bbbk G$ fits into the above construction. As shown in [94, Proposition 3.5], this fusion category is an equivariantization of the category $\mathcal{C}(\Gamma)$ of finite dimensional Γ-graded vector spaces under the induced action of G.

Consider now an action of the finite group G on a fusion category $\mathcal{C}$. For every $Y\in$ $\operatorname{Irr}(\mathcal{C})$ let $A(Y):=K_0(\Bbbk_{\alpha_Y}[G_Y])$ be the Grothendieck group of the twisted group algebra $\Bbbk_{\alpha_Y}[G_Y]$ and let $A:=\oplus_{Y\in\operatorname{Irr}(\mathcal{C})}A(Y)$. For every $g\in G$, let $c_{Y,g}: A(Y)\to A(\rho^g(Y))$ and $c_g: A\to A$ be given by

$$c_{Y,g}(\pi)={}^g\pi,\quad c_g:=\oplus_{Y\in G}c_{Y,g}. \tag{5.3.2}$$

Theorem 5.2.6 implies that there is an isomorphism of abelian groups $\mathrm{gr}(\mathcal{C}^G) \simeq \{a \in A \mid c_Y(a) = a\} = A^G$ *such that the simple object* $S_{Y,\pi}$ *corresponds to the irreducible projective representation* $\pi \in A(Y)$*, for every* $Y \in \mathrm{Irr}(\mathcal{C})$*.*

It follows from Theorem 6.1.3 that, for all $Y, Z \in \mathrm{Irr}(\mathcal{C})/G$*, the product* $S_{Y,\pi}S_{Z,\delta}$ *has the expression*

$$S_{Y,\pi}S_{Z,\delta} = \sum_{\substack{D\in G_Y\backslash G/G_Z \\ U\in\mathrm{Irr}(\mathcal{C})/G}} \sum_{\substack{1\leq i\leq n \\ t_i^{-1}s_i\in D \\ \mathrm{Hom}_{\mathcal{C}}(U,\rho^{t_i}Y\otimes\rho^{s_i}Z)\neq 0}} S_{U,m_U^{t_i,s_i}(Y,Z)(\pi,\delta)} \tag{5.3.3}$$

where the map $m_U^{t,s}(Y,Z) : A(Y) \times A(Z) \to A(U)$ *is defined by*

$$(\pi,\delta) \mapsto \mathrm{Ind}_T^U(\ {}^t\pi|_T \otimes\ {}^s\delta|_T \otimes \tau_U^{t,s}(Y,Z)) \tag{5.3.4}$$

Comparing (5.3.3) *with the formula given in [122, Corollary 2.5], we observe that the structure of the Grothendieck ring of the equivariantization* $\mathcal{C}^G$ *is similar to that of the rings* A^G *in [122], where the set* $\mathrm{Irr}(\mathcal{C})$ *plays now the rôle of the group* Γ*.*

Corollary 5.3.5. *A simple object* $S_{U,\delta}$ *is a constituent of a tensor product of simple objects* $S_{Y,\pi} \otimes S_{Z,\gamma}$ *if and only if there exist* $t \in G/G_Y$ *and* $s \in G/G_Z$ *such that*

(a) $\mathrm{Hom}_{\mathcal{C}}(U, \rho^t(Y) \otimes \rho^s(Z)) \neq 0$ *and*

(b) $m_T(\delta|_T,\ {}^t\pi|_T \otimes^s \gamma|_T \otimes \tau_U^{t,s}(Y,Z)) \neq 0$*, where* $T = G_U \cap tG_Yt^{-1} \cap sG_Ys^{-1}$*.*

□

5.3.4 The dual of a simple object

Let $Y \in \mathrm{Irr}(\mathcal{C})$. Then the multiplicity of the unit object of $\mathcal{C}$ in the tensor product $Y \otimes Y^*$ is one. Hence $\tau_Y := \tau_{\mathbf{1}}^{e,e}(Y,Y^*) = \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, Y \otimes Y^*)$ is a one dimensional (linear) representation of $G = G_{\mathbf{1}}$. In particular, it follows from Proposition 5.3.2 that the cohomology class of the product $\alpha_Y\alpha_{Y^*}$ is trivial on $G_Y = G_{Y^*}$.

Recall that the dual π^* of the G_Y-projective representation π is defined as V_π^* with $\pi^*(h)(f) = f \circ \pi(h)^{-1}$. This is an α_Y^{-1}-projective representation of G_Y.

Proposition 5.3.6. *The dual object of* $S_{Y,\pi} \in \mathcal{C}^G$ *is determined by*

$$S_{Y,\pi}^* \simeq S_{Y^*,\,\pi^*}.$$

Proof. Observe that $S_{Y,\pi}^* \simeq S_{Z,\gamma}$, for some $Z \in \mathrm{Irr}(\mathcal{C})/G$ and some α_Z-projective representation of G_Z. On the other hand, $S_{Y,\pi}^* \simeq S_{Z,\gamma}$ if and only if the unit object is a constituent of $S_{Y,\pi} \otimes S_{Z,\gamma}$. Since the unit object of $\mathcal{C}^G$ is isomorphic to $S_{\mathbf{1},\epsilon}$, where ϵ denotes the trivial representation of $G_{\mathbf{1}} = G$, it follows from Corollary 5.3.5 that $S_{Y,\pi}^* \simeq S_{Y^*,\,\pi^*\otimes\tau_Y^{-1}}$, where $\tau_Y = \mathrm{Hom}_{\mathcal{C}}(\mathbf{1},\ Y \otimes Y^*)$. Since τ_Y is a linear character of G_Y, then $\pi^* \otimes \tau_Y^{-1} \simeq \pi^*$ as projective G_Y-representations (see Section 5.5). Then $S_{Y^*,\,\pi^*\otimes\tau_Y^{-1}} \simeq S_{Y^*,\,\pi^*}$ and the proposition follows. □

Combining Proposition 5.3.6 with Frobenius Reciprocity we obtain:

Corollary 5.3.7. *Let $Y, Z \in \mathrm{Irr}(\mathcal{C})/G$ and let π, γ be projective representations of G_Y and G_Z with factor sets determined by* (5.2.1)*. Let also δ be an irreducible representation of G. Then $S_{\mathbf{1},\delta}$ is a constituent of $S_{Y,\pi} \otimes S^*_{Z,\gamma}$ if and only if $Z = Y^*$ and δ is a constituent of $(\pi \otimes \gamma^*) \uparrow_{G_Y}^{G}$.* □

5.3.5 $\mathcal{C}^G$ as a Rep G-bimodule category

Let us regard the category Rep G as a fusion subcategory of $\mathcal{C}^G$ via the natural embedding $\pi \mapsto (\pi \otimes \mathbf{1}, \pi(g) \otimes \mathrm{id}_{\mathbf{1}})$. So that the tensor product of $\mathcal{C}^G$ makes $\mathcal{C}^G$ into a Rep G-bimodule category.

The results in Section 5.2 imply that there is an equivalence of $\Bbbk$-linear categories $\mathcal{C}^G \simeq \oplus_{Y \in \mathrm{Irr}(\mathcal{C})/G} \mathrm{Rep}_{\alpha_Y} G_Y$, where $\mathrm{Rep}_{\alpha_Y} G_Y$ is the category of finite dimensional α_Y-projective representations of G_Y. Under this equivalence, a simple object π of $\mathrm{Rep}_{\alpha_Y} G_Y$, that is, an irreducible α_Y-projective representation of G_Y, corresponds to the simple object $S_{Y,\pi}$ of $\mathcal{C}^G$. In other words, $\mathrm{Rep}_{\alpha_Y} G_Y$ is identified with the full subcategory of $\mathcal{C}$ whose simple objects are lying over Y. An explicit equivalence is determined, for every $Y \in \mathrm{Irr}(\mathcal{C})/G$, by the functors $L_Y : \mathcal{C}^{G_Y} \to \mathcal{C}^G$ and $F_Y : \mathcal{C}^G \to \mathcal{C}^{G_Y}$.

For each $Y \in \mathrm{Irr}(\mathcal{C})$, the category $\mathrm{Rep}_{\alpha_Y} G_Y$ is in a canonical way an indecomposable Rep G-bimodule category via tensor product of projective representations; see [102, Theorem 3.2]. As a consequence of Theorem 6.1.3 we obtain:

Theorem 5.3.8. *There is an equivalence of* Rep *G-bimodule categories*

$$\mathcal{C}^G \simeq \bigoplus_{Y \in \mathrm{Irr}(\mathcal{C})/G} \mathrm{Rep}_{\alpha_Y} G_Y. \tag{5.3.1}$$

Moreover, each $\mathrm{Rep}_{\alpha_Y} G_Y$ *is an indecomposable* Rep *G-bimodule category.*

Proof. Let π be an irreducible representation of G, so that π corresponds to the simple object $S_{\mathbf{1},\pi} \in \mathrm{Rep}\ G$, and let $S_{Z,\gamma} \in \mathrm{Rep}_{\alpha_Z} G_Z$ be another simple object, where $Z \in \mathrm{Irr}(\mathcal{C})/G$. It follows from Corollary 5.3.5 that if the simple object $S_{U,\delta}$, $U \in \mathrm{Irr}(\mathcal{C})/G$, is a constituent of $S_{\mathbf{1},\pi} \otimes S_{Z,\gamma}$, then $U \simeq \rho^s(Z)$, for some $s \in G/G_Z$. Hence $U = Z$ and thus the group $T = G_Z \cap G_{\mathbf{1}} \cap G_Z$ coincides with G_Z, $\tau_U(\mathbf{1}, Z) \simeq \mathrm{Hom}_{\mathcal{C}}(Z, Z)$ is a one dimensional (linear) representation of G_Z. Therefore $\pi|_{G_Z} \otimes \gamma \otimes \tau_U(\mathbf{1}, Z) \simeq \pi|_{G_Z} \otimes \gamma$ as projective representations of G_Z.

By Theorem 6.1.3, the multiplicity of $S_{U,\delta}$ in the tensor product $S_{\mathbf{1},\pi} \otimes S_{Z,\gamma}$ equals $m_{G_Z}(\delta, \pi \otimes \gamma)$. Therefore we obtain

$$S_{\mathbf{1},\pi} \otimes S_{Z,\gamma} \simeq \bigoplus_{\delta} m_{G_Z}(\delta, \pi|_{G_Z} \otimes \gamma)\ S_{Z,\delta},$$

where δ runs over the equivalence classes of α_Z-projective representations of G_Z. Clearly this object corresponds to $\pi|_{G_Z} \otimes \gamma \in \mathrm{Rep}_{\alpha_Z} G_Z$.

Similar arguments apply for the tensor product $S_{Z,\gamma} \otimes S_{\mathbf{1},\pi}$. This completes the proof of the theorem. □

For any $U \in \operatorname{Irr}(\mathcal{C})$, let us extend the notation $S_{U,\delta} = L_U(\delta \otimes U)$ to indicate the object of $\mathcal{C}^G$ corresponding to an arbitrary α_U-projective representation δ of G_U.

Remark 5.3.5. *Let $Y, Z \in \operatorname{Irr}(\mathcal{C})$ and let $S_{Y,\pi}$, $S_{Z,\gamma}$ be simple objects of $\mathcal{C}^G$ lying over Y and Z, respectively. So that π is an irreducible α_Y-projective representation of G_Y and γ is an irreducible α_Z-projective representation of G_Z.*

According to Theorem 5.3.8, the tensor product $S_{Y,\pi} \otimes S_{Z,\gamma}$ has a decomposition

$$S_{Y,\pi} \otimes S_{Z,\gamma} \cong \bigoplus_{U \in \operatorname{Irr}(\mathcal{C})/G} S_{U,\delta}, \tag{5.3.2}$$

where, for all $U \in \operatorname{Irr}(\mathcal{C})/G$, $S_{U,\delta} \in \mathcal{C}^G$ is the sum of simple constituents of $S_{Y,\pi} \otimes S_{Z,\gamma}$ lying over U. It follows from Proposition 5.2.8 that $\delta \cong \operatorname{Hom}_{\mathcal{C}}(U, S_{Y,\pi} \otimes S_{Z,\gamma})$.

Remark 5.3.6. *The action of G on $\mathcal{C}$ induces an action of G on $\operatorname{gr}(\mathcal{C})$ by algebra automorphisms. Let $\operatorname{gr}(\mathcal{C})^G \subseteq \operatorname{gr}(\mathcal{C})$ be the subring of G-invariants in $\operatorname{gr}(\mathcal{C})$. For every $Y \in \operatorname{Irr}(\mathcal{C})$, let us consider the element*

$$\mathcal{S}(Y) := \sum_{t \in G/G_Y} \rho^t(Y) \in \operatorname{gr}(\mathcal{C}). \tag{5.3.3}$$

Clearly, we have $\mathcal{S}(Y) \in \operatorname{gr}(\mathcal{C})^G$ and $\mathcal{S}(Y) = \mathcal{S}(\rho^g(Y))$, for all $Y \in \operatorname{Irr}(\mathcal{C})$. Observe that $F^!(S_{Y,\pi}) = (\dim \pi)\mathcal{S}(Y)$, where $F^! : \operatorname{gr}(\mathcal{C}^G) \to \operatorname{gr}(\mathcal{C})$ is the ring map induced by the forgetful functor $F : \mathcal{C}^G \to \mathcal{C}$. Moreover, the set $\{\mathcal{S}(Y) : Y \in \operatorname{Irr}(\mathcal{C})/G\}$ is a basis for $\operatorname{gr}(\mathcal{C})^G$ and, for all $Y, Z \in \operatorname{Irr}(\mathcal{C})/G$, we have

$$\mathcal{S}(Y)\mathcal{S}(Z) = \sum_{U \in \operatorname{Irr}(\mathcal{C})/G} m_{Y,Z}^U \, \mathcal{S}(U), \tag{5.3.4}$$

for some nonnegative integers $m_{Y,Z}^U$.

Let $Y, Z, U \in \operatorname{Irr}(\mathcal{C})/G$. Consider any fixed simple objects $S_{Y,\pi}$ and $S_{Z,\gamma}$ of $\mathcal{C}^G$ lying over Y and Z, respectively. Applying the map $F^!$ in formula (5.3.2), we obtain that $m_{Y,Z}^U = \dim \delta/(\dim \pi)(\dim \gamma)$, where $\delta = \operatorname{Hom}_{\mathcal{C}}(U, S_{Y,\pi} \otimes S_{Z,\gamma}) \simeq V_\pi \otimes V_\gamma \otimes (\oplus_{(t,s) \in G/G_Y \times G/G_Z} \operatorname{Hom}_{\mathcal{C}}(U, \rho^t(Y) \otimes \rho^s(Z)))$. Therefore, for all $Y, Z, U \in \operatorname{Irr}(\mathcal{C})/G$, the integers $m_{Y,Z}^U$ are given by the formula

$$m_{Y,Z}^U = \sum_{(t,s) \in G/G_Y \times G/G_Z} \dim \operatorname{Hom}_{\mathcal{C}}(U, \rho^t(Y) \otimes \rho^s(Z)).$$

5.4 Application to equivariantizations of pointed fusion categories

We shall consider in this section a *pointed* fusion category $\mathcal{C}$, that is, all simple objects of $\mathcal{C}$ are invertible. Then there is an equivalence of fusion categories $\mathcal{C} \simeq \mathcal{C}(\Gamma, \omega)$, where $\Gamma = \mathrm{G}(\mathcal{C})$ is the group of isomorphism classes of invertible objects in $\mathcal{C}$, $\omega : \Gamma \times \Gamma \times \Gamma \to \Bbbk^*$ is an invertible normalized 3-cocycle and $\mathcal{C}(\Gamma, \omega) = \mathrm{Vec}_\omega^\Gamma$ is the category of finite dimensional Γ-graded vector spaces with associativity constraint induced by ω.

5.4.1 Group actions on $\mathcal{C}(\Gamma, \omega)$ and equivariantizations

Let $\mathcal{C} = \mathcal{C}(\Gamma, \omega)$ and let G be a finite group. An action $\rho : \underline{G} \to \underline{\mathrm{Aut}_\otimes}\mathcal{C}$ of G on $\mathcal{C}$ is determined by an action by group automorphisms of G on Γ, that we shall indicate by $x \mapsto {}^g x$, $x \in \Gamma$, $g \in G$, and two maps $\tau : G \times \Gamma \times \Gamma \to \Bbbk^*$, and $\sigma : G \times G \times \Gamma \to \Bbbk^*$, satisfying

$$\frac{\omega(x, y, z)}{\omega({}^g x, {}^g y, {}^g z)} = \frac{\tau(g; xy, z)\, \tau(g; x, y)}{\tau(g; y, z)\, \tau(g; x, yz)}$$

$$1 = \frac{\sigma(h, l; x)\, \sigma(g, hl; x)}{\sigma(gh, l; x)\, \sigma(g, h; {}^l x)}$$

$$\frac{\tau(gh; x, y)}{\tau(g; {}^h x, {}^h y)\, \tau(h; x, y)} = \frac{\sigma(g, h; x)\, \sigma(g, h; y)}{\sigma(g, h; xy)},$$

for all $x, y, z \in \Gamma$, $g, h, l \in G$.

We shall also assume that τ and σ satisfy the additional normalization conditions $\tau(g; x, y) = \sigma(g, h; x) = 1$, whenever some of the arguments g, h, x or y is an identity.

The action $\rho : \underline{G} \to \underline{\mathrm{Aut}_\otimes}\mathcal{C}$ determined by this data is defined by letting $\rho^g(x) = {}^g x$, for all $g \in G$, $x \in \Gamma$, and $\rho^g = \mathrm{id}$ on arrows, together with the following constraints:

$$(\rho_2^{g,h})_x = \sigma(g, h; x)^{-1}\, \mathrm{id}_{{}^{gh}x}, \quad (\rho_2^g)_{x,y} = \tau(g; x, y)^{-1}\, \mathrm{id}_{xy}, \quad \rho_0^g = \mathrm{id}_e, \tag{5.4.1}$$

for all $g, h \in G$, $x, y \in \Gamma$. See [112, Section 7].

5.4.2 Fusion rules for $\mathcal{C}(\Gamma, \omega)^G$

Let us denote $\sigma_x(g, h) := \sigma(g, h; x)$ and $\tau_{x,y}(g) := \tau(g; x, y)$, $x, y \in \Gamma$, $g, h \in G$.

For all $x \in \Gamma$ and $g \in G_x$ we let the isomorphism $c_x : {}^g x = x \to x$ to be the identity of x. Therefore, the cocycle $\widetilde{\alpha}_x : G_x \times G_x \to \Bbbk^*$ defined by (5.2.1) is given by

$$\widetilde{\alpha}_x(g, h) = \sigma_x(g, h)^{-1}, \tag{5.4.1}$$

for all $g, h \in G_x$.

It follows from Corollary 5.2.7 that the set of isomorphism classes of simple objects of $\mathcal{C}^G$ is parameterized by isomorphism classes of pairs (y, π), where y runs over the orbits of the action of G on Γ and π is an irreducible projective representation of the inertia subgroup $G_y \subseteq G$ with factor set σ_y.

Let $\mathcal{O}$ be a G-orbit in $G/G_y \times G/G_z$ corresponding to a double coset $D \in G_y\backslash G/G_z$. Then $\mathcal{O} = \mathcal{O}_G(e, g)$, for any $g \in D$, and $\mathcal{O}$ contains at most one G_U-orbit, $\mathcal{O}_{G_U}(t, s)$, $t^{-1}s \in D$, such that $\mathrm{Hom}_{\mathcal{C}}(x, {}^ty \otimes {}^sz) \neq 0$. Indeed, the condition $\mathrm{Hom}_{\mathcal{C}}(x, {}^ty \otimes {}^sz) \neq 0$ amounts in this case to $x = {}^ty{}^sz$. Thus, for all $e \neq g \in G/G_U$, $x \neq {}^gx = {}^{gt}y{}^{gs}z$.

In addition, if $x = {}^ty{}^sz$, then ${}^tG_y \cap {}^sG_z \subseteq G_x$. Therefore, $G_x \cap {}^tG_y \cap {}^sG_z = {}^tG_y \cap {}^sG_z$.

In the projective representation of ${}^tG_y \cap {}^sG_z$ in $\mathrm{Hom}_{\mathcal{C}}(x, {}^ty{}^sz) \simeq \Bbbk$, defined in Lemma 5.3.2, the action of an element $g \in {}^ty{}^sz$ is nothing but scalar multiplication by $\tau_{{}^ty, {}^sz}(g)^{-1}$.

As a consequence of Theorem 6.1.3, the following theorem gives the fusion rules for the category $\mathcal{C}(\Gamma, \omega)^G$.

Theorem 5.4.1. *Let $x, y, z \in \Gamma$ and let δ, π, γ be irreducible projective representations of the inertia subgroups G_x, G_y, G_z with factor sets σ_x, σ_y, σ_z, respectively. Then the multiplicity of $S_{x,\delta}$ in the tensor product $S_{y,\pi} \otimes S_{z,\gamma}$ is given by the formula*

$$\sum_{D \in G_y\backslash G/G_z} \sum_{\substack{t^{-1}s \in D \\ x = {}^ty{}^sz}} m_{{}^tG_y \cap {}^sG_z}(\delta|_{{}^tG_y \cap {}^sG_z},\ {}^t\pi|_{{}^tG_y \cap {}^sG_z} \otimes\ {}^s\gamma|_{{}^tG_y \cap {}^sG_z}\ \tau^{-1}_{{}^ty, {}^sz}).$$

Example 5.4.1. *Consider a cocentral abelian exact sequence of Hopf algebras $\Bbbk \to \Bbbk^\Gamma \to H \to \Bbbk G \to \Bbbk$, where Γ and F are finite groups. As special case of [94, Proposition 3.5], there is an action of G on the category $\mathcal{C} = \mathcal{C}(\Gamma, 1)$ of finite dimensional representations of $\Bbbk^\Gamma$ such that* Rep $H \simeq \mathcal{C}^G$ *as fusion categories (see [94, Remark 2.1] and [90]). In this situation, the formula for the fusion rules of* Rep H *given by Theorem 5.4.1 specializes to the formula obtained by C. Goff in [61, Theorem 4.5]. See Section 5.5.*

5.4.3 Braided group-theoretical fusion categories

Recall that a fusion category is called *group-theoretical* if it is Morita equivalent to a pointed fusion category [4, 43, 56, 58]. In view of [87, Theorem 7.2], a *braided* fusion category is group-theoretical if and only if it is an equivariantization of a pointed fusion category. More precisely, it was shown in [86, Theorem 5.3] that every braided group-theoretical fusion category is equivalent to an equivariantization $\mathcal{C}(\xi)^G$ of a crossed pointed fusion category $\mathcal{C}(\xi)$ associated to a quasi-abelian 3-cocycle ξ on a finite crossed module (G, X, ∂), under a canonical action of G on $\mathcal{C}(\xi)$.

Recall that a finite crossed module (G, X, ∂) consists of a finite group G acting by automorphisms on a finite group X, and a group homomorphism $\partial : X \to G$ such that

$${}^{\partial(x)}y = xyx^{-1}, \quad \partial({}^gx) = g\partial(x)g^{-1}, \qquad g \in G,\ x, y \in X,$$

where $x \mapsto^g x$, $x \in X$, $g \in G$, denotes the action of g on X.

A quasi-abelian 3-cocycle ξ on (G, X, ∂) is a quadruple $\xi = (\omega, \gamma, \mu, c)$, where $\omega : X \times X \times X \to \Bbbk^*$ is a 3-cocycle, $\gamma : G \times G \times X \to \Bbbk^*$, $\mu : G \times X \times X \to \Bbbk^*$ and $c : X \times X \to \Bbbk^*$ are maps satisfying the compatibility conditions in [86, Definition 3.4].

As a fusion category $\mathcal{C}(\xi) = \mathcal{C}(X, \omega)$, and the action of G on $\mathcal{C}(\xi)$ is determined by the action of G on X and formulas (5.4.1), with respect to $\sigma_x(g, h) := \gamma(g, h; x)$, $\tau_{x,y}(g) := \mu(g; x, y)^{-1}$, $x, y \in X$, $g, h \in G$. See [86, Subsection 4.1].

Theorem 5.4.1 gives thus the fusion rules in the category $\mathcal{C}(\xi)^G$ in terms of group-theoretical data determined by the crossed module (G, X, ∂) and the quasi-abelian 3-cocycle ξ, entailing the determination of the fusion rules in any braided group-theoretical fusion category.

Example 5.4.2. *Let $\omega : G \times G \times G \to \Bbbk^*$ be a 3-cocycle on G. Consider the crossed module (G, G, id) with respect to the adjoint action of G on itself. The quadruple $\xi = (\omega^{-1}, \gamma^{-1}, \mu^{-1}, 1)$ is a quasi-abelian 3-cocycle on (G, G, id), where γ and μ are defined in the form*

$$\gamma(g, h; x) = \frac{\omega(g, h, x)\omega(ghxh^{-1}g^{-1}, g, h)}{\omega(g, hxh^{-1}, y)},$$
$$\mu(g; x, y) = \frac{\omega(gxg^{-1}, g, y)}{\omega(gxg^{-1}, gyg^{-1}, g)\omega(g, x, y)},$$

for all $g, h, x, y \in G$.

The equivariantization $\mathcal{C}(\xi)^G$ is equivalent to the category Rep $D^\omega G$ *of finite dimensional representations of the twisted quantum double $D^\omega G$ introduced in [35]. See [86, Lemma 6.3] and [70].*

Simple objects of $\mathcal{C}(\xi)^G$ are parameterized by $S_{x,\pi}$, where x runs over a set of representatives of conjugacy classes of G and π is an irreducible projective representation of the centralizer $Z(x)$ of x in G with factor set γ_x. Theorem 5.4.1 gives the following formula for the multiplicity of $S_{x,\delta}$ in the tensor product $S_{y,\pi} \otimes S_{z,\gamma}$:

$$\sum_{D \in Z(y)\backslash G/Z(z)} \sum_{\substack{t^{-1}s \in D \\ x = tyt^{-1}szs^{-1}}} m_{tZ(y)t^{-1} \cap sZ(z)s^{-1}}(\delta, \ {}^t\pi \otimes {}^s\gamma \ \mu(-; tyt^{-1}, szs^{-1})^{-1}).$$

We point out that the fusion rules for the category Rep $D^\omega G$ *were also determined in [61, Section 5] See also [118, 119] for some preliminary results.*

5.5 Appendix of Chapter 5

In this Appendix we give a brief account of the results on projective representations used in this chapter. See for instance [67].

Let G be a finite group and let $\widetilde{\alpha} : G \times G \to \Bbbk^*$ be a (normalized) 2-cocycle on G, that is,

$$\widetilde{\alpha}(g,h)\widetilde{\alpha}(gh,t) = \widetilde{\alpha}(g,ht)\widetilde{\alpha}(h,t), \quad \widetilde{\alpha}(g,e) = 1 = \widetilde{\alpha}(e,g), \qquad \forall g,h,t \in G.$$

A *projective representation* π of G with *factor set* $\widetilde{\alpha}$ on a vector space V is a map $\pi : G \to \mathrm{GL}(V)$, such that

$$\pi(e) = \mathrm{id}_V, \quad \pi(gh) = \widetilde{\alpha}(g,\ h)\pi(g)\pi(h), \qquad \forall g,h \in G.$$

In other words, π is a representation of the twisted group algebra $\Bbbk_{\widetilde{\alpha}}G$ on the vector space V. We shall also use the notation $V_\pi = V$ to indicate such a projective representation.

Two projective representations π and π' of G are called *(projectively) equivalent* if there is a linear isomorphism $\phi : V_\pi \to V_{\pi'}$ and a map $f : G \to \Bbbk^*$ such that $\phi\pi(g) = f(g)\pi'(g)\phi$, for all $g \in G$. In this case we shall use the notation $\pi' \simeq \pi$.

If $\pi' \simeq \pi$, then the associated cocycles $\widetilde{\alpha}$ and $\widetilde{\alpha}'$ are related by

$$\widetilde{\alpha}(g,h) = \widetilde{\alpha}'(g,h)f(g)f(h)f(gh)^{-1}, \quad g,h \in G,$$

that is, $\widetilde{\alpha}$ and $\widetilde{\alpha}'$ are cohomologous cocycles, and thus they belong to the same cohomology class $\alpha \in H^2(G,\Bbbk^*)$. We shall also call π an α-projective representation. Note that the map $f : G \to \Bbbk^*$ induces an algebra isomorphism $\tilde{f} : \Bbbk_{\widetilde{\alpha}}G \to \Bbbk_{\widetilde{\alpha}'}G$ in the form $\tilde{f}(g) = f(g)g$, for all $g \in G$. Thus π and π' are equivalent projective representations if and only if $V_\pi \simeq \tilde{f}^*(V_{\pi'})$ as $\Bbbk_{\widetilde{\alpha}}G$-modules.

Let π and π' be projective representations of G with factor sets $\widetilde{\alpha}$ and $\widetilde{\alpha}'$, respectively. The *tensor product* $\pi \otimes \pi'$ is the projective $\widetilde{\alpha}\widetilde{\alpha}'$-representation on the vector space $V_\pi \otimes V_{\pi'}$ defined by $(\pi \otimes \pi')(g)(u \otimes v) = \pi(g)u \otimes \pi'(g)v$. In particular, if π is a representation of G, then $\pi \otimes \pi'$ is again a projective representation with factor set $\widetilde{\alpha}$.

If π_1 and π_1' are projective representations projectively equivalent to π and π', respectively, then the tensor products $\pi_1 \otimes \pi_1'$ and $\pi \otimes \pi'$ are projectively equivalent. Further, suppose that π' is a one-dimensional representation, that is, a linear character of G. Then π and $\pi \otimes \pi'$ are projectively equivalent via the canonical isomorphism $\phi : V_\pi \to V_\pi \otimes \Bbbk$, $v \mapsto v \otimes 1$, and the map $f : G \to \Bbbk^*$ given by $f(g) = \pi'(g)^{-1}$, for all $g \in G$.

A nonzero projective representation $\pi : G \to \mathrm{GL}(V)$ of G is called *irreducible* if 0 and V are the only subspaces of V which are invariant under $\pi(g)$, for all $g \in G$. Hence, π is irreducible if and only if it is not projectively equivalent to a projective representation ρ of the form

$$\rho(g) = \begin{pmatrix} \pi_1(g) & * \\ 0 & \pi_2(g) \end{pmatrix}, \quad g \in G,$$

where π_1 and π_2 are nonzero projective representations or, equivalently, if V is a simple $\Bbbk_{\widetilde{\alpha}}G$-module, where $\widetilde{\alpha}$ is the factor set of π [67, Theorem 3.2.5].

Let $\pi : G \to \mathrm{GL}(V)$ be a projective representation of G with factor set $\widetilde{\alpha}$. Since the group algebra $\Bbbk_{\widetilde{\alpha}}G$ is semisimple, then $V = V_\pi$ is *completely reducible*, that is, $V_\pi \simeq V_{\pi_1} \oplus \cdots \oplus V_{\pi_n}$, where V_{π_i} is a simple $\Bbbk_{\widetilde{\alpha}}G$-module, for all $i = 1, \ldots, n$. If π' is an irreducible projective representation with factor set $\widetilde{\alpha}'$, then π' is called a *constituent* of π if π' is projectively equivalent to π_i for some $1 \leq i \leq n$. In this case, the *multiplicity* (or *intertwining number*) of π' in π is defined as

$$m_G(\pi', \pi) := \dim \mathrm{Hom}_{\Bbbk_{\widetilde{\alpha}}G}(V_{\pi_i}, V_\pi).$$

Observe that if π' is a constituent of π, then the cocycles $\widetilde{\alpha}'$ and $\widetilde{\alpha}$ belong to the same class in $H^2(G, \Bbbk^*)$. Letting $\widetilde{\alpha}'df = \widetilde{\alpha}$, with $f : G \to \Bbbk^*$, we have that $m_G(\pi', \pi) := \dim \mathrm{Hom}_{\Bbbk_{\widetilde{\alpha}}G}(\tilde{f}^*(V_{\pi'}), V_\pi)$, where $\tilde{f} : \Bbbk_{\widetilde{\alpha}}G \to \Bbbk_{\widetilde{\alpha}'}G$ is the isomorphism associated to f.

The *character* of a projective representation $\pi : G \to \mathrm{GL}(V)$ is defined as the map $\chi = \chi_V : G \to \Bbbk$ given by $\chi(g) = \mathrm{Tr}(\pi(g))$, for all $g \in G$. Let $\widetilde{\alpha}$ be the factor set of π. If π' is an irreducible projective representation of G with factor set $\widetilde{\alpha}$ and character χ', then the multiplicity of π' in π can be computed by the formula

$$\begin{aligned} m_G(\pi', \pi) = \langle \chi', \chi \rangle : &= \frac{1}{|G|} \sum_{g \in G} \frac{1}{\widetilde{\alpha}(g^{-1}, g)} \chi'(g) \chi(g^{-1}) \\ &= \frac{1}{|G|} \sum_{g \in G^0} \frac{1}{\widetilde{\alpha}(g^{-1}, g)} \chi'(g) \chi(g^{-1}), \end{aligned}$$

$G^0 \subseteq G$ is the subset of $\widetilde{\alpha}$-regular elements of G. See [67, Chapter 5].

Let $\widetilde{\alpha} : G \times G \to \Bbbk^*$ be a 2-cocycle and let $H \subseteq G$ be a subgroup. Consider a projective representation W of H with factor set $\widetilde{\alpha}|_H$. The *induced projective representation* of G is defined as $\mathrm{Ind}_H^G W = \Bbbk_{\widetilde{\alpha}}G \otimes_{\Bbbk_{\widetilde{\alpha}}H} W$. This is a projective representation of G with factor set $\widetilde{\alpha}$. By Frobenius reciprocity, we have natural isomorphisms

$$\mathrm{Hom}_{\Bbbk_{\widetilde{\alpha}}G}(\mathrm{Ind}_H^G W, V) \simeq \mathrm{Hom}_{\Bbbk_{\widetilde{\alpha}}H}(W, V|_H),$$

for every projective representation V of G with factor set $\widetilde{\alpha}$, where $V|_H$ denotes the restricted projective representation of H.

The following lemma gives a characterization of those projective representations which are induced from a subgroup.

Lemma 5.5.1. *Let $\widetilde{\alpha} : G \times G \to \Bbbk^*$ be a cocycle and let V be a $\Bbbk_{\widetilde{\alpha}}G$-module. Suppose $V = \oplus_{x \in X} V_x$ is a grading of V by a set X and assume that there is transitive action of G on X, $G \times X \to X$, $(g, x) \mapsto {}^g x$, such that $g.V_x = V_{{}^g x}$, for all $g \in G$, $x \in X$.*

Let also $y \in X$ and $G_y \subseteq G$ the inertia subgroup of y. Then V_y is a $\Bbbk_{\widetilde{\alpha}}G_y$-module and $V \simeq \mathrm{Ind}_{G_y}^G V_y$ as $\Bbbk_{\widetilde{\alpha}}G$-modules.

Proof. See [67, Theorem 5.2.1]. □

Chapter 6

Green functors arising from semisimple Hopf algebras

6.1 Main results of the Chapter

Mackey's decomposition theorem of induced modules from subgroups is a very important tool in the representations theory of finite groups. This decomposition describes the process of an induction composed with a restriction in terms of the reverse processes consisting of restrictions followed by inductions. More precisely, if G is a finite group, M and N two subgroups of G and V a finite dimensional $\Bbbk$-linear representation of M then the well known Mackey's decomposition states that there is an isomorphism of $\Bbbk N$-modules:

$$V\uparrow_{\Bbbk M}^{\Bbbk G}\downarrow_{\Bbbk N}^{\Bbbk G}\xrightarrow{\delta_V}\bigoplus_{x\in M\backslash G/N}\Bbbk[N]\otimes_{\Bbbk[\,^xM\cap N]}\,^xV. \tag{6.1.1}$$

Here $^xM := xMx^{-1}$ is the conjugate subgroup and $^xV := V$ is the conjugate ^xM-representation defined by $(xmx^{-1}).v := m.v$ for all $m \in M$ and $v \in V$. The direct sum is indexed by a set of representative group elements of G for all double cosets $M\backslash G/N$ of G relative to the two subgroups M and N. Note that the inverse isomorphism of δ_V is given on each direct summand by the left multiplication operator $n\otimes_{\Bbbk N\cap\Bbbk\,^xM}v\mapsto nx\otimes_{\Bbbk M}v$, see [108, Proposition 22].

The goal of this chapter is to investigate a similar Mackey type decomposition for the induced modules from Hopf subalgebras of semisimple Hopf algebras and restricted back to other Hopf subalgebras. In order to do this, we use the corresponding notion of a double coset relative to a pair of Hopf subalgebras of a semisimple Hopf algebra that was introduced by the author in [14] and also discussed in details in Chapter 3. We also have to define a conjugate Hopf subalgebra corresponding to the notion of a conjugate subgroup. For any Hopf subalgebra $K \subseteq H$ of a semisimple Hopf algebra H and any simple subcoalgebra C of H we define the conjugate Hopf subalgebra ^CK of K in Proposition 6.3.1. This notion corresponds to the notion of conjugate subgroup from the above decomposition.

In order to deduce that ${}^CK$ is a Hopf subalgebra of H we use several crucial results from [96] concerning the product of two subcoalgebras of a semisimple Hopf algebra as well as Frobenius-Perron theory for nonnegative matrices.

Using these tools we can prove one of the following main results of this Chapter:

Theorem 6.1.1. *Let $K \subseteq H$ be a Hopf subalgebra of a semisimple Hopf algebra and M a finite dimensional K-module. Then for any subgroup $G \subseteq G(H)$ one has a canonical isomorphism of $\Bbbk G$-modules*

$$M \uparrow_K^H \downarrow_{\Bbbk G}^H \xrightarrow{\delta_M} \bigoplus_{C \in \Bbbk G \backslash H/K} (\Bbbk G \otimes_{\Bbbk G_C} {}^CM). \tag{6.1.2}$$

Here $G(H)$ is the group of grouplike elements of H and the subgroup $G_C \subseteq G$ is determined by $\Bbbk G \cap {}^CK = \Bbbk G_C$. The conjugate module ${}^CM$ is defined by ${}^CM := CK \otimes_K M$.

As in the classical group case the homomorphism δ_M is the inverse of a natural homomorphism π_M which is constructed by the left multiplication on each direct summand. It is not difficult to check (see Theorem 6.1.2 below) that in general, for any two Hopf subalgebras $K, L \subseteq H$ the left multiplication homomorphism π_M is always an epimorphism:

Theorem 6.1.2. *Let K and L be two Hopf subalgebras of a semisimple Hopf algebra H. For any finite dimensional left K-module M there is a canonical epimorphism of L-modules*

$$\oplus_{C \in L\backslash H/K} (L \otimes_{L \cap {}^CK} {}^CM) \xrightarrow{\pi_M} M \uparrow_K^H \downarrow_L^H \tag{6.1.3}$$

given on components by $l \otimes_{L \cap {}^CK} v \mapsto lv$ for any $l \in L$ and any $v \in {}^CM$. Here the conjugate module ${}^CM$ is defined as above by ${}^CK := CK \otimes_K M$.

We remark that there is a similar direction in the literature in the paper [49]. In this paper the author considers a similar decomposition but for pointed Hopf algebras instead of semisimple Hopf algebras. Also, in [74] the author proves a similar result for some special Hopf subalgebras of quantum groups at roots of 1.

Another particular situation of Mackey's decomposition can be found in [14]. In this paper it is proven that for pairs of Hopf subalgebras that generate just one double coset subcoalgebra, the above epimorphism π_M from Theorem 6.1.2 is in fact an isomorphism. In both papers, the above homomorphism π_M is given by left multiplication.

Definition 6.1.1. *We say that (L, K) is a Mackey pair of Hopf subalgebras of H if the above left mutliplication homomorphism π_M from Theorem 6.1.2 is an isomorphism for any finite dimensional left K-module M.*

Then Theorem 6.1.1 above states that $(\Bbbk G, K)$ is a Mackey pair for any Hopf subalgebra $K \subset H$ and any subgroup $G \subset G(H)$. Moreover in Theorem 6.6.1 it is shown that for any normal Hopf subalgebra K of H the pair (K, K) is a Mackey pair. This allows us to prove a new formula (see Proposition 6.6.2) for the restriction of an induced module from a normal Hopf subalgebra which substantially improves [14, Proposition 5.12]. It also gives a criterion for an induced module from a normal Hopf subalgebra to be irreducible generalizing a well known criterion for group representations, see for example [108, Corollary 7.1].

For any semisimple Hopf algebra H, using the universal grading of the fusion category $\mathrm{Rep}(H^*)$ we construct in Section 6.5 new Mackey pairs of Hopf subalgebras of H. In turn, this allows us to define a Green functor on the universal group G of the category of representations of H^*. For $H = \Bbbk G$ one obtains in this way the usual Green functor [62]. As in group theory, this new Green functor can be used to determine new properties of the Grothendieck ring of a semisimple Hopf algebra.

In the last section we prove the following tensor product formula for two induced modules from a Mackey pair of Hopf subalgebras.

Theorem 6.1.3. *Suppose that (L, K) is a Mackey pair of Hopf subalgebras of a semisimple Hopf algebra H. Then for any K-module M and any L-module N one has a canonical isomorphism:*

$$M\uparrow_K^H \otimes N\uparrow_L^H \overset{\cong}{\to} \bigoplus_{C\in L\backslash H/K} ((CK\otimes_K M)\downarrow^{{}^CK}_{L\cap {}^CK} \otimes N\downarrow^{L}_{L\cap {}^CK})\uparrow^H_{L\cap {}^CK} \tag{6.1.4}$$

This generalizes a well known formula for the tensor product of two induced group representations, see for example [6].

This chapter is structured as follows. In the first section we recall the basic results on coset decomposition for Hopf algebras. The second section contains the construction for the conjugate Hopf subalgebra generalizing the conjugate subgroup of a finite group. These results are inspired from the treatment given in [24]. A general characterization for the conjugate Hopf subalgebra is given in Theorem 6.3.5. This theorem is automatically satisfied in the group case. In the third section we prove Theorem 6.1.2. In the next section we prove Theorem 6.1.1. We also show that for any semisimple Hopf algebra there are some canonical associated Mackey pairs arising from the universal grading of the category of finite dimensional corepresentations (see Theorem 6.5.3). Necessary and sufficient conditions for a given pair to be a Mackey pair are given in terms of the dimensions of the two Hopf subalgebras of the pair and their conjugate Hopf subalgebras. In Section 6.6 we prove that for a normal Hopf subalgebra K the pair (K, K) is always a Mackey pair. In the last subsection 6.6.2 we prove the tensor product formula from Theorem 6.1.3.

In this chapter we work over an algebraically closed field $\Bbbk$ of characteristic zero.

6.2 Double coset decomposition for Hopf subalgebras of semisimple Hopf algebras

Let L and K be two Hopf subalgebras of a semisimple Hopf algebra H. Recall from Chapter 3 that one can define an equivalence relation $r^H_{L,\,K}$ on the set of simple subcoalgebras of H as following: $C \sim D$ if $C \subset$ LDK. We also have the following:

Proposition 6.1. *If C and D are two simple subcoalgebras of H then the following are equivalent:*
1) $C \sim D$
2) $LCK = LDK$
3) $\Lambda_L C \Lambda_K = \Lambda_L D \Lambda_K$

Proof. First assertion is equivalent to the second by Corollary 3.2.3 from Chapter 3. Clearly (2) $\Rightarrow$ (3) by left multiplication with Λ_K and right multiplication with Λ_L. It will be shown that (3) $\Rightarrow$ (1). One has the following decomposition:

$$H = \bigoplus_{i=1}^{l} LC_iK$$

where $C_1, \cdots, C_l$ are representative subcoalgebras for each equivalence class of $r^H_{K,\,L}$.

It follows that $\Lambda_L H \Lambda_K = \bigoplus_{i=1}^{l} \Lambda_L C_i \Lambda_K$. Thus if $C \nsim D$ then $\Lambda_L C \Lambda_K \cap \Lambda_L D \Lambda_K = 0$ which proves (1). □

Remark 6.2.1. *The above Proposition shows that for any two simple subcoalgebras C and D of H then either $LCK = LDK$ or $LCK \cap LDK = 0$. Therefore for any subcoalgebra $D \subset LCK$ one has that $LCK = LDK$. In particular, for $L = \Bbbk$, the trivial Hopf subalgebra, one has that $D \subset CK$ if and only if $DK = CK$.*

Notations

We denote by $L \backslash H/K$ the set of double cosets LCK of H with respect to L and K. Thus the elements LCK of $L \backslash H/K$ are given by a choice of a representative of simple subcoalgebras in each equivalence class of $r^H_{L,\,K}$. Similarly, we denote by H/K be the set of right cosets CK of H with respect to K. This corresponds to a choice of a representative simple subcoalgebra in each equivalence class of $r^H_{\Bbbk,\,K}$.

Remark 6.2.2. *As noticed in [14] one has that $LCK \in \mathcal{M}^H_K$ and therefore LCK is a free right K-module. Similarly $LCK \in {}^H_L\mathcal{M}$ and therefore LCK is also a free left L-module.*

By Corollary 3.2.4 it follows that two simple subcoalgebras C and D are in the same double coset of H with respect to L and K if and only if

$$\Lambda_L \frac{c}{\epsilon(c)} \Lambda_K = \Lambda_L \frac{d}{\epsilon(d)} \Lambda_K. \tag{6.2.1}$$

where c and d are the irreducible characters of H^* associated to the simple subcoalgebras C and D. In particular for $L = \Bbbk$, the trivial Hopf subalgebra, it follows that $CK = DK$ if and only if

$$c\Lambda_K = \frac{\epsilon(c)}{\epsilon(d)} d\Lambda_K. \tag{6.2.2}$$

6.2.1 Principal eigenspace for $< C >$

For a simple subcoalgebra C we denote by $< C >$ the Hopf subalgebra of H generated by C. If d is the character associated to C we also denote this Hopf subalgebra by $< d >$.

Frobenius-Perron theory for nonnegative matrices

Next we will use the Frobenius-Perron theorem for matrices with nonnegative entries (see [54]). If $A \geq 0$ is such a matrix then A has a positive eigenvalue λ which has the biggest absolute value among all the other eigenvalues of A. The eigenspace corresponding to λ has a unique vector with all entries positive. λ is called the principal eigenvalue of A and the corresponding positive vector is called the principal vector of A. Also the eigenspace of A corresponding to λ is called the principal eigenspace of the matrix A.

For an irreducible character $d \in \mathrm{Irr}(H^*)$ let L_d be the linear operator on $C(H^*)$ given by left multiplication by d. By Theorem 3.2.2 it follows that $\epsilon(d)$ is the Frobenius-Perron eigenvalue of the nonnegative matrix associated to the operator L_d with respect to the basis given by the irreducible characters of H^*. In analogy with Frobenius-Perron theory, for a subcoalgebra C with associated character d we call the space of eigenvectors of L_d corresponding to the eigenvalue $\epsilon(d)$ as the principal eigenspace for L_d.

Next Corollary can also be deduced directly from Theorem 3.2.2.

Corollary 6.2.1. *The principal eigenspace of L_{Λ_K} is $\Lambda_K C(H^*)$ and it has a $\Bbbk$-linear basis given by $\Lambda_K d$ where d are the characters of a set of representative simple coalgebras for the right cosets of K inside H.*

Using this we can prove the following:

Theorem 6.2.2. *Let C be a subcoalgebra of a semisimple Hopf algebra H with associated character $d \in C(H^*)$. Then the principal eigenspaces of L_d and $L_{\Lambda_{<d>}}$ coincide.*

Proof. Let V be the principal eigenspace of $L_{\Lambda_{<d>}}$ and W be the principal eigenspace of L_d. Then by Corollary 6.2.1 one has that $V = \Lambda_{<d>} C(H^*)$. Since $d\Lambda_{<d>} = \epsilon(d)\Lambda_{<d>}$ then clearly $V \subseteq W$. On the other hand since $\Lambda_{<d>}$ is a polynomial with rational coefficients in d (see [92, Corollary 19]) it also follows that $W \subseteq V$. $\square$

6.2.2 Rank of cosets

Let K be a Hopf subalgebra of a semisimple Hopf algebra H. Consider the equivalence relation $r^H_{\Bbbk,\,K}$ on the set $\mathrm{Irr}(H^*)$ of simple subcoalgebras of H. As above one has $C \sim D$ if and only if $CK = DK$. Therefore

$$H = \oplus_{C \in H/K} CK. \tag{6.2.1}$$

Lemma 6.2.1. *The equivalence class under $r^H_{\Bbbk,K}$ of the trivial subcoalgebra $\Bbbk$ is the set of all simple subcoalgebras of K.*

Proof. Indeed suppose that C is a simple subcoalgebra of H equivalent to the trivial subcoalgebra $\Bbbk$. Then $CK = \Bbbk K = K$ by Proposition 6.4.1. Therefore $C \subset CK = K$. Conversely, if $C \subset K$ then $CK \subset K$ and since $CK \in \mathcal{M}^H_K$ it follows that $CK = K$. Thus $C \sim \Bbbk$. □

Proposition 6.2. *If D is a simple subcoalgebra of a semisimple Hopf algebra H and $e \in K$ is an idempotent then*

$$DK \otimes_K Ke \cong DKe$$

as vector spaces.

Proof. Since H is free right K-module one has that the map

$$\phi : H \otimes_K Ke \to He, \quad h \otimes_K re \mapsto hre$$

is an isomorphism of H-modules. Using the above decomposition (6.2.1) of H and the fact that DK is a free right K-module note that ϕ sends $DK \otimes_K Ke$ to DKe. □

Corollary 6.2.3. *Let K be a Hopf subalgebra of a semisimple Hopf algebra H. For any simple subcoalgebra C of H one has that the rank of CK as right K-module is $\dim_{\Bbbk} C\Lambda_K$.*

Proof. Put $e = \Lambda_K$ the idempotent integral of K in the above Proposition. □

6.2.3 Frobenius-Perron eigenvectors for cosets

Let T be the linear operator given by right multiplication with Λ_K on the character ring $C(H^*)$.

Remark 6.2.3. *By Theorem 3.2.2 it follows that the largest (in absolute value) eigenvalue of T equals $\dim K$. Moreover a basis of eigenvectors corresponding to this eigenvalue is given by $c\Lambda_K$ where the character $c \in \mathrm{Irr}(H^*)$ runs through a set of irreducible characters representative for all the right cosets $CK \in H/K$.*

6.3 The conjugate Hopf subalgebra ${}^{C}K$

Let as above K be a Hopf subalgebra of a semisimple Hopf algebra H. For any simple subcoalgebra C of H in this section we construct the conjugate Hopf subalgebra ${}^{C}K$ appearing in Theorem 6.1.3. If $c \in \mathrm{Irr}(H^*)$ is the associated irreducible character of C then consider the following subset of $\mathrm{Irr}(H^*)$:

$$ {}^{c}K = \{ d \in \mathrm{Irr}(H^*) \mid dc\Lambda_K = \epsilon(d)c\Lambda_K \} \tag{6.3.1} $$

where as above $\Lambda_K \in K$ is the idempotent integral of K.

Proposition 6.3.1. *The set ${}^{c}K \subset \mathrm{Irr}(H^*)$ is closed under multiplication and "*" and it generates a Hopf subalgebra ${}^{C}K$ of H. Thus*

$$ {}^{C}K = \oplus_{d \in {}^{c}K} C_d \tag{6.3.2} $$

Proof. Suppose that D and D' are two simple subcoalgebras of H whose irreducible characters satisfy $d, d' \in {}^{c}K$. Then one has $dd'c\Lambda_K = \epsilon(dd')c\Lambda_K$. On the other hand suppose that

$$ dd' = \sum_{e \in \mathrm{Irr}(H^*)} m^{e}_{d,d'} e. \tag{6.3.3} $$

Then $\epsilon(dd')c\Lambda_K = dd'c\Lambda_K = \sum_{e \in \mathrm{Irr}(H^*)} m^{e}_{d,d'} ec\Lambda_K$ and Remark 6.2.3 implies that $ec\Lambda_K$ is a scalar multiple of $c\Lambda_K$ for any e with $m^{e}_{d,d'} \neq 0$. Therefore $ec\Lambda_K = \epsilon(e)c\Lambda_K$ and $e \in {}^{c}K$. This shows that ${}^{C}K$ is a subbialgebra of H and by Remark 1.1.2 a Hopf subalgebra of H. □

Sometimes the notation ${}^{C}K$ will also be used for ${}^{c}K$ where $c \in \mathrm{Irr}(H^*)$ is the irreducible character associated to the simple subcoalgebra C.

The notion of conjugate Hopf subalgebra ${}^{C}K$ is motivated by the following Proposition:

Proposition 6.3.2. *Let H be a semisimple Hopf algebra over $\Bbbk$. If the simple subcoalgebra C is of the form $C = \Bbbk g$ with $g \in G(H)$ a group-like element of H then ${}^{C}K = gKg^{-1}$.*

Proof. Indeed, suppose that $D \in {}^{C}K$. If d is the associated irreducible character of D then by definition it follows that $dg\Lambda_K = g\Lambda_K$. Thus $g^{-1}dg\Lambda_K = \Lambda_K$. Therefore the simple subcoalgebra $g^{-1}Dg$ of H is equivalent to the trivial subcoalgebra $\Bbbk$. Then using Lemma 6.2.1 one has that $g^{-1}Dg \subset K$ and therefore ${}^{C}K \subset gKg^{-1}$. The other inclusion $gKg^{-1} \subset {}^{C}K$ is obvious. □

Remark 6.3.1. *In particular for $H = \Bbbk G$ one has that ${}^{C}\Bbbk[M] = \Bbbk[{}^{x}M]$ where $x \in G$ is given by $C = \Bbbk x$.*

Remark 6.3.2. *1. Using Remark 1.1.1 it follows from the definition of conjugate Hopf subalgebra that CK is always a left ${}^{C}K$-module.*

2. *Note that if $C(H^*)$ is commutative then ${}^C K \supseteq K$. Indeed for any $d \in \mathrm{Irr}(K^*)$ one has $d\Lambda_K = \epsilon(d)\Lambda_K$ and therefore $dc\Lambda_K = cd\Lambda_K = \epsilon(d)c\Lambda_K$.*

3. *If K is a normal Hopf subalgebra of H then since Λ_K is a central element in H by same argument it also follows that ${}^C K \supseteq K$.*

6.3.1 Some properties of the conjugate Hopf subalgebra

Proposition 6.3.3. *Let H be a semisimple Hopf algebra and K be a Hopf subalgebra of H. Then for any simple subcoalgebra C of H one has that ${}^C K$ coincides to the maximal Hopf subalgebra L of H with the property $LCK = CK$.*

Proof. The equality ${}^C KCK = CK$ follows from the character equality $\Lambda_{{}^C K} c\Lambda_K = \epsilon(\Lambda_{{}^C K})c\Lambda_K$ and Reamark 1.1.1. Conversely, if $LCK = CK$ by passing to the regular H^*-characters and using Equation (6.2.1) it follows that $\Lambda_L c\Lambda_K = \epsilon(\Lambda_L)c\Lambda_K$ which shows that $L \subset {}^C K$. □

Note that Remark 6.2.1 together with the previous proposition implies that ${}^C KC \subseteq CK$.

Corollary 6.3.4. *One has that ${}^C K \subseteq CKC^*$.*

Proof. Since $S(C) = C^*$ by applying the antipode S to the above inclusion one obtains that $C^* \, {}^C K \subseteq KC^*$. Therefore $CC^* \, {}^C K \subseteq CKC^*$ and then one has ${}^C K \subseteq CC^* \, {}^C K \subseteq CKC^*$. □

Theorem 6.3.5. *One has that ${}^C K$ is the largest Hopf subalgebra L of H with the property $LC \subseteq CK$.*

Proof. We have seen above that ${}^C KC \subseteq CK$. Suppose now that $LC \subseteq CK$ for some Hopf subalgebra L of H. Then by Remark 6.2.1 it follows that $LCK = CK$. Thus by passing to regular characters one has that $\Lambda_L c\Lambda_K = \epsilon(\Lambda_L)c\Lambda_K$ which shows the inclusion $L \subseteq {}^C K$. □

Proposition 6.3.6. *Let H be a semismple Hopf algebra and K be a Hopf subalgebra of H. Then for any subcoalgebra D with $DK = CK$ one has that ${}^D K = {}^C K$.*

Proof. One has that ${}^C KCK = CK$. If $D \subset CK$ then by Remark 6.2.1 one has that ${}^C KDK = {}^C KCK = CK = DK$ which shows that ${}^C K = {}^D K$. □

6.4 Mackey type decompositions for representations of Hopf algebras

Let K be a Hopf subalgebra of a semisimple Hopf algebra H and M be a finite dimensional K-module. Note that for any simple subcoalgebra C of H one has by Proposition 6.3.3

that ${}^{C}M := CK \otimes_K M$ is a left ${}^{C}K$-module via the left multiplication with elements of ${}^{C}K$.

Remark 6.4.1. *Let $H = \Bbbk G$ be a group algebra of a finite group G and $K = \Bbbk A$ for some subgroup A of G. Then note that ${}^{C}M := CK \otimes_K M$ coincides to the usual conjugate module ${}^{g}M$ if $C = \Bbbk g$ for some $g \in G$. Recall that ${}^{g}M = M$ as vector spaces and $(gag^{-1}).m = a.m$ for all $a \in A$ and all $m \in M$.*

6.4.1 Proof of Theorem 6.1.2.

Proof. Since by definition of the double cosets one has $H = \oplus_{C \in L\backslash H/K} LCK$ and each LCK is a free K-module, the following decomposition of L-modules follows:

$$M \uparrow_K^H \downarrow_L^H = H \otimes_K M \cong \oplus_{C \in L\backslash H/K}(LCK \otimes_K M). \tag{6.4.1}$$

Consider now the $\Bbbk$-linear map $\pi_M^{(C)} : L \otimes_{L\cap {}^{C}K} (CK \otimes_K M) \to LCK \otimes_K M$ given by

$$l \otimes_{L\cap {}^{C}K} (cx \otimes_K m) \mapsto lcx \otimes_K m$$

for all $l \in L$, $x \in K$, $c \in C$ and $m \in M$. It is easy to see that $\pi_M^{(C)}$ is a well defined map and clearly a surjective morphism of L-modules. Then $\pi_M := \oplus_{C \in L\backslash H/K} \pi_M^{(C)}$ is surjective morphism of L-modules and the proof is complete. $\square$

Remark 6.4.2. *Suppose that for $M = \Bbbk$ one has that $\pi_\Bbbk$ isomorphism in Theorem 6.1.2. Then using a dimension argument it follows that the same epimorphism π_M from Theorem 6.1.2 is in fact an isomorphism for any finite dimensional left H-module M.*

6.4.2 Mackey pairs

It follows from the proof above that (L, K) is a Mackey pair if and only if $\pi_\Bbbk$ is an isomorphism, i.e. if and only if each $\pi_\Bbbk^{(C)}$ is isomorphism for any simple subcoalgebra C of H. Since $\pi_\Bbbk^{(C)}$ is surjective passing to dimensions one has that (L, K) is a Mackey pair if and only if

$$\dim LCK = \frac{(\dim L)\,(\dim CK)}{\dim L \cap {}^{C}K} \tag{6.4.1}$$

for any simple subcoalgebra C of H.

Note that for $C = \Bbbk 1$ the above condition can be written as

$$\dim\ LK = \frac{(\dim L)(\dim K)}{\dim(L \cap K)}.$$

Remark 6.4.3. *Note also that for any Mackey pair it follows that*

$$\frac{\dim\ L \cap {}^{D}K}{\dim DK} = \frac{\dim\ L \cap {}^{C}K}{\dim CK} \tag{6.4.2}$$

if $LCK = LDK$.

Example 6.4.1. *Suppose that L, K are Hopf subalgebras of H with $LK = KL$. Then (L, K) is a Mackey pair of Hopf subalgebras of LK by [14, Proposition 3.3].*

6.4.3 Proof of Theorem 6.1.1

Let L, K be two Hopf subalgebras of a semisimple Hopf algebra H and let C be a simple subcoalgebra of H. Note that equations (6.2.1) and (6.2.2) implies that LCK can be written as a direct sum of right K-cosets,

$$LCK = \bigoplus_{DK \in \mathcal{S}} DK, \tag{6.4.1}$$

for a subset $\mathcal{S} \subset H/K$ of right cosets of K inside H. Note that always one has $CK \in \mathcal{S}$.

Next we give a proof for the main result Theorem 6.1.1.

Proof. Suppose that $L = \Bbbk G$. By Equation (6.4.1) one has to verify

$$\dim(\Bbbk G)CK = \frac{|G| \, (\dim CK)}{\dim \Bbbk G \cap \ {}^{C}K} \tag{6.4.2}$$

for any subcoalgebra C of H. Since $\Bbbk G \cap \ {}^{C}K$ is a Hopf subalgebra of $\Bbbk G$ it follows that $\Bbbk G \cap \ {}^{C}K = \Bbbk G_C$ for some subgroup G_C of G. By Equation (6.3.1) it follows that $G_C = \{g \in G \mid gd\Lambda_K = d\Lambda_K\}$ where $d \in \mathrm{Irr}(H^*)$ is the character associated to C. In terms of subcoalgebras this can be written as $G_C = \{g \in G \mid gCK = CK\}$.

With the above notations Equation (6.4.2) becomes

$$\dim(\Bbbk G)CK = \frac{|G|}{|G_C|} \dim CK \tag{6.4.3}$$

Note that the group G acts transitively on the set $\mathcal{S}$ from Equation (6.4.1). The action is given by $g.DK = gDK$ for any $g \in G$ and any $DK \in \mathcal{S}$. Let St_C be the stabilizer of the right coset CK. Thus the subgroup St_C of G is defined by $St_C = \{g \in G \mid gCK = CK\}$ which shows that $St_C = G_C$. Note that $\dim DK = \dim CK$ for any $DK \in \mathcal{S}$ since $DK = gCK$ for some $g \in G$. Thus $\dim(\Bbbk G)CK = |\mathcal{S}|(\dim CK)$ and Equation (6.4.3) becomes

$$|\mathcal{S}| = \frac{|G|}{|G_C|} \tag{6.4.4}$$

which is the same as the formula for the size of the orbit $\mathcal{S}$ of CK under the action of the finite group G. □

6.5 New examples of Green functors

In this section we construct new examples of Green functors arising from gradings on the category of corepresentations of semisimple Hopf algebras.

6.5.1 Gradings of fusion categories

In this subsection we recall few basic results on gradings of fusion categories from [58] that will be further used in the chapter.

Let $\mathcal{C}$ be a fusion category and $\mathcal{O}(\mathcal{C})$ be the set of isomorphism classes of simple objects of $\mathcal{C}$. Recall that the fusion category $\mathcal{C}$ is graded by a finite group G if there is a function $\deg : \mathcal{O}(\mathcal{C}) \to G$ such that for any two simple objects $X, Y \in \mathcal{O}(\mathcal{C})$ one has that $\deg(Z) = \deg(X)\deg(Y)$ whenever $Z \in \mathcal{O}(\mathcal{C})$ is a simple object such that Z is a constituent of $X \otimes Y$. Alternatively, there is a decomposition $\mathcal{C} = \oplus_{g\in G}\mathcal{C}_g$ such that the the tensor functor of $\mathcal{C}$ sends $\mathcal{C}_g \otimes \mathcal{C}_h$ into $\mathcal{C}_{gh}$. Here $\mathcal{C}_g$ is defined as the full abelian subcategory of $\mathcal{C}$ generated by the simple objects X of $\mathcal{C}$ satisfying $\deg(X) = g$. Recall that a grading is called universal if any other grading of $\mathcal{C}$ is arising as a quotient of the universal grading. The universal grading always exists and its grading group denoted by $U_{\mathcal{C}}$ is called the universal grading group.

Remark 6.5.1. *If $\mathcal{C} = \mathrm{Rep}(H)$ for a semisimple Hopf algebra H then by [58, Theorem 3.8] it follows that the Hopf center (i.e. the largest central Hopf subalgebra) of H is $\Bbbk G^*$ where G is the universal grading group of $\mathcal{C}$. We denote this Hopf center by $\mathcal{HZ}(H)$. Therefore one has $\mathcal{HZ}(H) = \Bbbk G^*$ where $G = U_{\mathrm{Rep}(H)}$. Moreover, in this case, by the universal property any other grading on $\mathcal{C} = \mathrm{Rep}(H)$ is given by a quotient group G/N of G. The corresponding graded components of $\mathcal{C}$ are given by*

$$\mathcal{C}_{\bar{g}} = \{M \in \mathrm{Irr}(H) \mid M \downarrow^{H}_{\Bbbk^{G/N}} = (\dim M)\bar{g}\} \tag{6.5.1}$$

for all $g \in G$. Here $\Bbbk^{G/N} \subset \Bbbk^G$ is regarded as a central Hopf subalgebra of H. Also note that in this situation one has a central extension of Hopf algebras:

$$\Bbbk \to \Bbbk^{G/N} \to H \to H//\Bbbk^{G/N} \to \Bbbk. \tag{6.5.2}$$

6.5.2 Gradings on $\mathrm{Rep}(H^*)$ and cocentral extensions

Suppose that H is a semisimple Hopf algebra such that the fusion category $\mathrm{Rep}(H^*)$ is graded by a finite group G. Then the dual version of Remark 6.5.1 implies that H fits into a cocentral extension

$$\Bbbk \to B \to H \xrightarrow{\pi} \Bbbk G \to \Bbbk. \tag{6.5.1}$$

Recall that such an exact sequence of Hopf algebras is called cocentral if $\Bbbk G^* \subset \mathcal{Z}(H^*)$ via the dual map π^*. On the other hand, using the reconstruction theorem from [2] it follows that

$$H \cong B\,{}^{\tau}\#_{\sigma}\,\Bbbk F \tag{6.5.2}$$

for some cocycle $\sigma : B \otimes B \to \Bbbk F$ and some dual cocyle $\tau : \Bbbk F \to B \otimes B$.

For any such cocentral sequence it follows that G acts on $\mathrm{Rep}(B)$ and by [94, Proposition 3.5] that $\mathrm{Rep}(H) = \mathrm{Rep}(B)^G$, the equivariantized fusion category. For the main properties of group actions and equivariantized fusion categories one can consult for example [98]. Recall that the above action of G on $\mathrm{Rep}(B)$ is given by $T : G \to \mathrm{Aut}_{\otimes}(\mathrm{Rep}(B))$, $g \mapsto T^g$. For any $M \in \mathrm{Rep}(B)$ one has that $T^g(M) = M$ as vector spaces and the action of B is given by $b.^g m := (g.b).m$ for all $g \in G$ and all $b \in B$, $m \in M$. Here the weak action of G on B is the action used in the crossed product from Equation (6.5.2).

For any subgroup M of G it is easy to check that $H(M) = B\#_{\sigma}\Bbbk M$, i.e. $H(M)$ is the unique Hopf subalgebra of H fitting the exact cocentral sequence

$$\Bbbk \to B \to H(M) \to \Bbbk M \to \Bbbk. \tag{6.5.3}$$

Lemma 6.5.1. *Let H be a semisimple Hopf algebra. Then gradings on the fusion category* $\mathrm{Rep}(H^*)$ *are in one-to one correspondence with cocentral extensions*

$$\Bbbk \to B \to H \xrightarrow{\pi} \Bbbk G \to \Bbbk. \tag{6.5.4}$$

Proof. We have shown at the beginning of this subsection how to associate a cocentral extension to any G-grading on $\mathrm{Rep}(H^*)$.

Conversely, suppose one has a cocentral exact sequence as in Equation (6.5.1). Then $\mathrm{Rep}(H^*)$ is graded by G where the graded component of degree $g \in G$ is given by

$$\mathrm{Rep}(H^*)_g = \{d \in \mathrm{Irr}(H^*) \mid \pi(d) = \epsilon(d)g\}. \tag{6.5.5}$$

Indeed, since $\Bbbk^G \subset \mathcal{Z}(H^*)$ via π^* it follows that $\Bbbk^G$ acts by scalars on each irreducible representation of H^*. Therefore for any $d \in \mathrm{Irr}(H^*)$ one has $d \downarrow_{\Bbbk G}^{H^*} = \epsilon(d)g$ for some $g \in G$. It follows then by [58, Theorem 3.8]) that $\mathrm{Rep}(H^*)$ is G-graded and

$$\mathrm{Rep}(H^*)_g = \{d \in \mathrm{Irr}(H^*) \mid d \downarrow_{\Bbbk G}^{H^*} = \epsilon(d)g\} \tag{6.5.6}$$

On the other hand it is easy to check that one has $\pi(d) = d \downarrow_{\Bbbk G}^{H^*}$ for any $d \in \mathrm{Irr}(H^*)$ (see also Remark 3.2 of [16].)

Clearly the two constructions are inverse one to the other. □

6.5.3 New examples of Mackey pairs of Hopf subalgebras

Let H be a semisimple Hopf algebra and $\mathcal{C} = \mathrm{Rep}(H^*)$. Since H^* is also a semisimple Hopf algebra [73] it follows that $\mathcal{C}$ is a fusion category. For the rest of this section fix an arbitrary G-grading $\mathcal{C} = \oplus_{g\in G}\mathcal{C}_g$ on $\mathcal{C}$.

For any subset $M \subset G$ define $\mathcal{C}_M := \oplus_{m\, \in M}\mathcal{C}_m$ as a full abelian subcategory of $\mathcal{C}$. Thus $\mathcal{O}(\mathcal{C}_M) = \sqcup_{m\in M}\mathcal{O}(\mathcal{C}_m)$. Let also $H(M)$ to be the subcolagebra of H generated by all the simple subcoalgebras of H whose irreducible H^*-characters belong to $\mathcal{O}(\mathcal{C}_M)$.

For any subcoalgebra C of H denote by $\mathrm{Irr}(C^*)$ the irreducible characters of the dual algebra C^*. Therefore by its definition $H(M)$ verifies the equality $\mathrm{Irr}(H(M)^*) = \mathcal{O}(\mathcal{C}_M)$ and as a coalgebra can be written as $H(M) = \bigoplus_{\{d\in\mathcal{O}(C_m)\,\mid\, m\in M\}} C_d$. Note that if M is a subgroup of G then $H(M)$ is a Hopf subalgebra of H by Remark 1.1.1.

For any simple subcoalgebra C of H whose associated irreducible character $d \in \mathrm{Irr}(H^*)$ has degree g we will also write for shortness that $\deg(C) = g$.

Proposition 6.5.1. *Let H be semisimple Hopf algebra and G be the universal grading group of $\mathrm{Rep}(H^*)$. Then for any arbitrary two subgroups M and N of G, the set of double cosets $H(M)\backslash H/H(N)$ is canonically bijective to the set of group double cosets $M\backslash N/G$. Moreover, the bijection is given by $H(M)CH(N) \mapsto M\deg(C)N$.*

Proof. By Remark 1.1.1 one has the following equality in terms of irreducible H^*-characters:

$$\mathrm{Irr}(H(M)CH(N)^*) = \mathcal{O}(\mathcal{C}_{M\deg(C)N})$$

Thus if $H(M)CH(N) = H(M)DH(N)$ then $\deg(C) = \deg(D)$ which shows that the above map is well defined. Clearly the map $H(M)CH(N) \mapsto M\deg(C)N$ is surjective. The injectivity of this map also follows from Remark 1.1.1. □

Note that the proof of the previous Proposition implies that the coset $H_x = H(M)CH(N)$ with $\deg(C) = x$ is given by

$$H_x = \bigoplus_{\{d\in\mathcal{O}(\mathcal{C}_{mxn})\ |\ m\in M,\ n\in N\}} C_d. \tag{6.5.1}$$

Proposition 6.5.2. *Suppose that $V \in H(M) - mod$, i.e. V is a $B\#_\sigma\Bbbk M$-mdule. Then as B-modules one has that ${}^C V \cong T^{g^{-1}}(\mathrm{Res}_B^{H(M)}(V))$ where $g \in G$ is chosen such that $\deg(C) = g$. Moreover ${}^C(V \otimes W) \cong {}^C V \otimes {}^C W$ for any two left $H(M)$-modules V and W.*

Proof. Note that in this situation one has that ${}^C H(M) = H({}^g M) = B\#_\sigma\Bbbk^g M$. By definition one has ${}^C V = CH(M) \otimes_{H(M)} V = H(gM) \otimes_{H(M)} V$. Thus

$${}^C V = (B\#\Bbbk gM) \otimes_{B\#\Bbbk M} V \cong \Bbbk gM \otimes_{\Bbbk M} V \tag{6.5.2}$$

where the inverse of the last isomorphism is given by $g \otimes_{\Bbbk M} v \mapsto (1\#g) \otimes_{B\#\Bbbk M} v$. Note that B acts on $\Bbbk gM \otimes_{\Bbbk M} V$ via $b.(g \otimes_{\Bbbk M} v) = g \otimes_{\Bbbk M} (g^{-1}.b)m$ for all $b \in B$, $v \in V$. This shows that indeed ${}^C V \cong T^{g^{-1}}(\mathrm{Res}_B^{H(M)}(V))$ as B-modules. Moreover it follows that ${}^C V$ can be identified to V as vector spaces with the $B\#_\sigma\Bbbk gMg^{-1}$-module structure given by $b.v = (g^{-1}.b)v$ and $(ghg^{-1}).v = ([g^{-1}.(\sigma(ghg^{-1}, g)\sigma^{-1}(g, h))]\#_\sigma h).v.$ for all $g \in G$, $h \in M$ and $v \in V$. Then it can be checked by direct computation that the map $v \otimes w \mapsto \tau^{-1}(g)(v \otimes w)$ from [94, Proposition 3.5] is in this case a morphism of $B\#_\sigma\Bbbk gMg^{-1}$-modules. In order to do that one has to use the compatibility conditions from [2, Theorem 2.20].

□

6.5.4 Examples of Mackey pairs arising from group gradings on the category $\mathrm{Rep}(H^*)$.

Let as above H be a semisimple Hopf algebra with $\mathcal{C} = \oplus_{g\in G}\mathcal{C}_g$ be a group grading of $\mathcal{C} := \mathrm{Rep}(H^*)$. It follows that

$$\mathrm{FPdim}(\mathcal{C}_g) = \frac{\dim H^*}{\dim \mathcal{HZ}(H^*)} \tag{6.5.1}$$

for all $g \in G$ where $\mathrm{FPdim}(\mathcal{C}_g) := \sum_{V\in\mathcal{O}(\mathcal{C}_G)}(\dim V)^2$ is the Perron-Frobenius dimension of the full abelian subcategory $\mathcal{C}_g$ of $\mathcal{C}$.

Theorem 6.5.3. *Let H be a semisimple Hopf algebra and M, N be any two subgroups of G. With the above notations the pair $(H(M), H(N))$ is a Mackey pair of Hopf subalgebras of H.*

Proof. Put $L := H(M)$ and $K := H(N)$. Therefore $\mathrm{Irr}(L^*) = \mathcal{O}(\mathcal{C}(M))$ and $\mathrm{Irr}(K^*) = \mathcal{O}(\mathcal{C}(N))$. Then we have to verify Equation (6.4.1) for any simple subcoalgebra C. Fix a simple subcoalgebra C of H with $\deg(C) = x$. As above one has ${}^{C}H(M) = H({}^{x}M)$.

It is easy to verify that $H(M) \cap H(N) = H(M \cap N)$ for any two subgroups M and N of G. This implies that $L\cap {}^{C}K = H(N\cap {}^{x}M)$. On the other hand from Equation (6.5.1) note that $\dim LCK = |MxN|\mathrm{FPdim}(\mathcal{C}_1)$.

Then Equation (6.4.1) is equivalent to the well known formula for the size of a double coset relative to two subgroups:

$$|MxN| = \frac{|M||N|}{|M\cap {}^{x}N|}, \tag{6.5.2}$$

for any $x \in G$. □

Remark 6.5.2. *The fact that $(H(M), H(N))$ is a Mackey pair also follows in this case from a more general version of Mackey's decomposition theorem that holds for the action of any finite group on a fusion category. This results were recently sent to publication by the author, see [20].*

Remark 6.5.3. *It also should be noticed that the author is not aware of any pair of Hopf subalgebras that is not a Mackey pair. It would be interesting to construct such counterexamples if they exist.*

6.5.5 Mackey and Green functors

For a finite group G denote by $\mathcal{S}(G)$ the lattice of all subgroups of G. Following [113] a Mackey functor for G over a ring R can be regarded as a collection of vector spaces $M(H)$ for any $H \subset \mathcal{S}(G)$ together with a family of morphisms $I_K^L : M(K) \to M(L)$, $R_K^L : M(L) \to M(K)$, and $c_{K,g} : M(K) \to M({}^{g}K)$ for all subgroups K and L of G with $K \subset L$ and for all $g \in G$. This family of morphisms has to satisfy the following compatibility conditions:

1. $I_H^H,\ R_H^H, c_{H,h} : M(H) \to M(H)$ are the identity morphisms for all subgroups H of G and any $h \in H$,

2. $R_K^J R_H^K = R_K^J$, for all subgroups $J \subset K \subset H$,

3. $I_H^K I_J^H = I_J^K$, for all subgroups $J \subset K \subset H$,

4. $c_{K,g} c_{K,h} = c_{K,gh}$ for all elements $g, h \in G$.

5. For any three subgroups $J, L \subseteq K$ of G and any $a \in M(J)$ one has the following Mackey axiom:
$$R_L^J(I_J^K(a)) = \sum_{x \in J\backslash K/L} I^L_{L\cap\, {}^xJ}(R^{\,{}^xJ}_{\,{}^xJ\cap L}(c_{J,x}(a)))$$

Moreover, a Green functor is a Mackey functor M such that for any subgroup K of G one has

that $M(K)$ is an associative R-algebra with identity and the following conditions are satisfied:

6. R_K^L and $c_{K,g}$ are always unitary R-algebra homomorphisms,

7. $I_K^L(aR_K^L(b)) = I_K^L(a)b$,

8. $I_K^L(R_K^L(b)a) = bI_K^L(a)$ for all subgroups $K \subseteq L \subseteq G$ and all $a \in M(K)$ and $b \in M(L)$.

Green functors play an important role in the representation theory of finite groups (see for example [113]).

6.5.6 New examples of Green functors

Next Theorem allows us to construct new examples of Green functors from semisimple Hopf algebras.

Theorem 6.5.4. *Let H be a semisimple Hopf algebra and G be a grading group for the fusion category* $\mathrm{Rep}(H^*)$. *Then the functor $M \mapsto K_0(H(M))$ is a Green functor.*

Proof. By Proposition 6.5.1 there is a cocentral extension

$$\Bbbk \to B \to H \xrightarrow{\pi} \Bbbk G \to \Bbbk \tag{6.5.1}$$

for some Hopf subalgebra $B \subset H$. Then as above, for a simple subcoalgebra C of H with associated character $d \in H^*$ one has that if $\pi(d) = g$ for some $g \in G$ then $\pi(C) = \Bbbk g$.

The map $R_K^L : K_0(H(L)) \to K_0(H(K))$ is induced by the restriction map $\mathrm{Res}^{H(L)}_{H(K)}$: $H(L) - \mathrm{mod} \to H(K) - \mathrm{mod}$. Similarly, the map I_K^L is induced by the induction functor between the same two categories of modules. Clearly R_K^L is a unital algebra map and the compatibility conditions 7 and 8 follow from the adjunction of the two functors. Moreover conditions 2 and 3 are automatically satisfied.

Define $c_{L,g} : K_0(L) \to K_0({}^gL)$ by $[M] \mapsto [{}^CM]$ where C is any simple subcoalgebra of H chosen with the property that $\deg(C) = g$. It follows by Proposition 6.5.2 that $c_{L,g}$ is a well defined algebra map. Condition 4 is equivalent to $T^{gh}(M) \cong T^gT^h(M)$ which is automatically satisfied for a group action on a fusion category.

It is easy to see that all other axioms from the definition of a Green functor are satisfied. For example, the Mackey decomposition axiom 5 is satisfied by Theorem 6.5.3. □

6.6 On normal Hopf subalgebras of semisimple Hopf algebras

Proposition 6.6.1. *Suppose that H is a semisimple Hopf algebra. Then for any normal Hopf subalgebra K of H one has that (K, K) is a Mackey pair of Hopf algebras.*

Proof. Note $KC = CK$ for any subcoalgebra C of K since K is a normal Hopf subalgebra of H. Then for any simple subcoalgebra C of H the dimension condition from Equation (6.4.1) can be written as

$$\dim CK = \frac{(\dim K)\,(\dim CK)}{\dim K \cap {}^CK} \tag{6.6.1}$$

which is equivalent to $K \cap {}^CK = K$. This equality follows by the third item of Remark (6.3.2). □

6.6.1 Irreducibility criterion for an induced module

Remark 6.6.1. *Let G be a finite group and H be a normal subgroup of G Then [108, Corollary 7.1] implies that an induced module $M\uparrow_H^G$ is irreducible if and only if M is irreducible and M is not isomorphic to any of its conjugate module ${}^gM$.*

Previous Theorem allows us to prove the following Proposition which is an improvement of [14, Proposition 5.12]. The second item is also a generalization of [108, Corrolary 7.1].

Proposition 6.6.2. *Let K be a normal Hopf subalgebra of a semisimple Hopf algebra H and M be a finite dimensional K-module.*

1. *Then*
$$M\uparrow_K^H\downarrow_K^H \cong \bigoplus_{C \in H/K} {}^CM$$
as K-modules.

2. *$M\uparrow_K^H$ is irreducible if and only if M is an irreducible K-module which is not a direct summand of any conjugate module ${}^CM$ for any simple subcoalgebra C of H with $C \not\subset K$.*

Proof. 1. Previous Proposition implies that

$$M \uparrow_K^H \downarrow_K^H \cong \bigoplus_{C \in K\backslash H/K} K \otimes_{K\cap {}^C K} {}^C M \tag{6.6.1}$$

as K-modules. On the other hand since K is normal note that $CK = KC$ and therefore the space $K\backslash H/K$ of double cosets coincides to the space H/K of left (right) cosets (see also Subsection 6.2 for the notation). In the the proof of the same Proposition 6.6.1 it was also remarked that $K \cap {}^C K = K$.

2. One has that $M \uparrow_K^H$ is an irreducible H-module if and only if

$$\dim_{\Bbbk} \mathrm{Hom}_H(M \uparrow_K^H,\ M \uparrow_K^H) = 1.$$

Note that by the Frobenius reciprocity one has the following $\mathrm{Hom}_H(M \uparrow_K^H,\ M \uparrow_K^H) = \mathrm{Hom}_K(M,\ M \uparrow_K^H \downarrow_K^H)$. Then previous item implies that

$$\mathrm{Hom}_K(M,\ M \uparrow_K^H) \cong \bigoplus_{C \in H/K} \mathrm{Hom}_K(M,\ {}^C M) \tag{6.6.2}$$

Since for $C = \Bbbk$ one has ${}^{\Bbbk} M = M$ it follows that $\mathrm{Hom}_K(M,\ {}^C M) = 0$ for all $C \not\subset K$. □

6.6.2 A tensor product formula for induced representations

We need the following preliminary tensor product formula for induced representations which appeared in [25].

Proposition 6.6.3. *Let K be a Hopf subalgebra of a semisimple Hopf algebra H. Then for any K-module M and any H-module V one has that*

$$M \uparrow_K^H \otimes V \cong (M \otimes V \downarrow_K^H) \uparrow_K^H . \tag{6.6.1}$$

Proof of Theorem 6.1.3: Applying Proposition 6.6.3 one has that

$$M \uparrow_K^H \otimes N \uparrow_L^H \cong (M \uparrow_K^H \downarrow_L^H \otimes N) \uparrow_L^H . \tag{6.6.2}$$

On the other hand, by Theorem 6.1.2 one has

$$M \uparrow_K^H \downarrow_L^H \cong \oplus_{C \in L\backslash H/K} (L \otimes_{L\cap {}^C K} (CK \otimes_K M)). \tag{6.6.3}$$

Thus,

$$\begin{aligned} M \uparrow_K^H \otimes N \uparrow_L^H &\cong (M \uparrow_K^H \downarrow_L^H \otimes N) \uparrow_L^H \\ &\overset{\cong}{\to} \oplus_{C \in L\backslash H/K} ((L \otimes_{L\cap {}^C K} (CK \otimes_K M)) \otimes N) \uparrow_L^H . \end{aligned}$$

Applying again Proposition 6.1.3 for the second tensor product one obtains that

$$\begin{aligned} M \uparrow_K^H \otimes N \uparrow_L^H &\overset{\cong}{\to} ((CK \otimes_K M) \otimes N \downarrow_{L\cap {}^C K}^L) \uparrow_{L\cap {}^C K}^L \uparrow_L^H \\ &\overset{\cong}{\to} \oplus_{C \in L\backslash H/K} H \otimes_{L\cap {}^C K} ((CK \otimes_K M) \otimes N \downarrow_{L\cap {}^C K}^L). \end{aligned}$$

Remark 6.6.2. *Note that the above Theorem always applies for* $K = L$ *a normal Hopf subalgebra of* H.

Chapter 7

Future plans and research directions

The author of this thesis has obtained a Ph.D. from Syracuse University in 2005. In the next year he held a postdoctoral position to the same University. In 2007 the author returned to Romania as a Scientific Researcher at the Simion Stoilow Institute of Mathematics of Romanian Academy.

In the period 2010-2012 the author held a postdoctoral grant PN II-RU-PD-168/2010 of the Romanian National Authority for Scientific Research, CNCS - UEFISCDI. Since May 2012, Sebastian Burciu is the director of a grant for young researchers PN II-RU-TE-168/2012 of the Romanian National Authority for Scientific Research, CNCS - UEFISCDI. He was also member of other national research grants. For other research grants where the author was a team member one can consult author's CV. He intends in the future to apply for other national and international grants or fellowships.

Recently the author started a scientific research seminar on Hopf algebras and tensor categories in the institute. The seminar is related with the ongoing activities of the grant PN-II-RU-TE 168/2012. One of the goals of the seminar is to bring together researchers interested in applying the the new methods of fusion categories to the study of semisimple Hopf algebras.

The future research directions of the author are to obtain new results in the classification of semisimple Hopf algebras and fusion categories.

Next we will give some details on few of these future directions.

7.1 Classification of semisimple Hopf algebras

The study of semisimple Hopf algebras became recently a very dynamic field of research. The main motivation of this studies consists in the fact that the category of finite dimensional representations of a semisimple Hopf algebra is a fusion category. Fusion categories are used in various branches of mathematics such as quantum filed theory, invariants of knots and 3-manifolds and representation theory.

If H is a finite dimensional semisimple Hopf algebra over an algebraically closed field $\Bbbk$ then Kaplansky's sixth conjecture states that the dimension of an irreducible H-module divides the dimension of H. For finite group algebras this conjecture is known as a classical result of Frobenius (see [108]). The conjecture was proven so far for some important particular cases of Hopf algebras for quasitriangular Hopf algebras, see [38, 107], and for cotriangular Hopf algebras, [41] semi-solvable Hopf algebras, see [83].

Note also that there are already several results in the literature which classify Hopf algebras in certain dimensions based on the assumption thatKaplansky's sixth conjecture is true, [39, 55, 59, 60].

Recently many properties from group representations were transferred to semisimple Hopf algebras. The starting point of this process was the paper of Larson [72] which proved the orthogonality relation for characters of (co)semisimple Hopf algebras. Next step was done by Zhu [125] who proved the class equation for semisimple Hopf algebras. This allowed people to make substantial progress in the classification of semisimple Hopf algebras in various low dimensions, see for example [78, 77, 80, 79, 88, 89, 91].

Next we will describe some future projects of the author in this direction. It is worth emphasising that some of these projects could lead to reasonably difficult subjects for a Ph.D. thesis with a good impact in the field of study.

7.1.1 Hecke algebras arising from semisimple Hopf subalgebras

In [14] the author constructed double cosets associated to Hopf subalgebras K of a semisimple Hopf algebra A. It was observed by D. Nikshych that two such double cosets can be multiplied obtaining an algebra of double cosets similar to the Hecke algebra associated to a subgroup of finite group, [32, Section 11.21]. We denote this algebra by $\mathcal{H}(K, A)$. It is easy to show that a similar formula to [32, Proposition 11.30] for the central idempotents of these Hecke algebras hold. Moreover an idempotent is primitive in $\mathcal{H}(K, A)$ if and only if it is primitive in A. We intend to investigate some other properties of Hecke algebras, especially the correspondence between the irreducible representations of these Hecke algebras and the irreducible constituents of some induced representations of A.

7.1.2 Kernels of representations of semisimple Hopf algebras

As we already have noticed in Chapter 1 a new method in the study of the semisimple Hopf algebras is by investigating the Hopf kernels associated to their representations. In [21] we have shown that the kernels of all representations of Drinfeld doubles of finite groups are all normal Hopf subalgebras. At this time there are not known examples of Drinfeld doubles admitting not normal Hopf kernels of representations. It is our intention in the future to explore the Hopf kernels of representations of semisimple Drinfeld double. For this study we will use the recent classification of irreducible representations of Drinfeld doubles given by the author in [19].

In [23] I have obtained new information concerning the left kernel of any normal coideal subalgebra of a semisimple Hopf algebra, regarded as a left module over the Drinfeld double via the adjoint action, see [126, Lemma 1]. In the future we would like to explore this information with respect to the Müger centralizer of these modules.

7.1.3 Character theory for semisimple Hopf algebras

The Mackey type decomposition from Chapter 6 suggests that the character algebra of a semisimple Hopf algebra has many analogue properties to the character algebra of a finite group. In Chapter 6 we have seen that for an equivariantized fusion category one can construct new Green rings (associated to cocentral extensions of Hopf algebras).

In a recent preprint [20] the author has shown that these Green rings have a similar ring structure as the rings introduced by Bouc in [11] and Witherspoon in [122]. These rings appear in may branches of mathematics as group representations, Hochschild cohomology and K-theory of crossed products. A classical example of such rings is given by the double (crossed) Burnside ring, see [11]. It is our intention to further investigate the algebraic structure of these rings using Hopf algebraic methods.

We also mention that a new direction in the character theory is that of (weak) orders for semisimple Hopf algebras and fusion categories that was recently developed in [31, 42]. Our results from [26] shows that if $\mathcal{C} = \mathrm{Rep}(H)$ has a (weak) Hopf order then $\mathcal{C}^G$ has also a (weak) Hopf order, for H a semisimple Hopf algebra. In the future we would like to extend the known class of Hopf algebras which posses a Hopf order, enlarging in this way the class of Hopf algebras satisfying Kaplansky's sixth conjecture.

7.1.4 Nilpotent and solvable semisimple Hopf algebras

Recently the notion of a solvable and nilpotent fusion category was introduced by Etingof, Nikshych and Ostrik in [45]. It is a well known challenging problem to characterize those semisimple Hopf algebras whose category of representations is nilpotent or solvable.

Using notions of character theory for Hopf algebras we proved in [23] that for any two normal coideal subalgebras L, K of a semisimple Hopf algebra A with a commutative character ring $C(A)$ one has that

$$\dim LK = \frac{(\dim L)(\dim K)}{\dim(L \cap K)}. \tag{7.1.1}$$

Using the notion of commutator introduced by the author in [18], (see also [29]), I intend in the future to obtain new results in this direction. For example, it is an open problem to characterize nilpotency of the category of representations of a semisimple Hopf algebra in terms of central series. Note that in [30] the authors succeeded to characterize nilpotent Hopf subalgebras in terms of an upper central series introduced in loc. cit. Formula (7.1.1) concerning the dimension of a product of two coideal subalgebras, suggests that a similar lower central series can be defined for a semisimple Hopf algebra. It is expected that characterization of nilpotency in terms of this series should also hold.

In [36] it is shown that any nilpotent fusion categories is a direct product of fusion categories of Frobenius-Perron dimensions power of primes. We would like to investigate a similar result for semisimple Hopf algebras. All known nilpotent Hopf algebras at this moment are tensor product of normal coideal subalgebras of prime power dimension.

In a recent ongoing project, the author defines a new algebraic notion of solvability of semisimple Hopf algebras in terms of an abelian series introduced using left coideal subalgebras. In [83] a stronger notion is defined insisting on the terms of the abelian series to be Hopf subalgebras. There are examples of semisimple Hopf algebras that are solvable in the new algebraic sense but neither semisolvable or solvable in the categorical sense. It is not known yet any counterexample in the opposite direction for the categorical notion of solvability. It is expected that this new class of solvable Hopf algebras to enlarge the class of semisimple Hopf algebras satisfying Kaplansky's sixth conjecture.

Another open problem related to those mentioned above is related to the unicity of composition factors of a composition series for semisimple Hopf algebras and the existence of an analogue of Jordan Hölder Theorem for semisimple Hopf algebras.

7.2 Classification of fusion categories

A new method to construct fusion categories is proposed in [43]. This method consists of constructing new Morita equivalent categories.

P. Etingof, D. Nikshych and V. Ostrik defined the notion of *group-theoretical fusion categories* consisting of those fusion categories which are Morita equivalent to a pointed fusion category. Recall that a fusion category is called pointed if all its simple objects are invertible.

In [43] it was conjectured that any fusion category is group theoretical. A counterexample to this conjecture was constructed by D. Nikshych in [98]. The counterexample consists of a category of representations of a semisimple Hopf algebra which is a $\mathbb{Z}_2-$extension of a group theoretical fusion category. More recently, the same three authors conjectured in [45] that any fusion category is *weakly group theoretical.* Recall that a fusion category is called weakly group theoretical if it is Morita equivalent to a nilpotent fusion category.

It is an open problem in the field if any fusion category is weakly group theoretical. Following results from [45] an affirmative answer to this question would imply that any fusion category has the strong Frobenius property. Thus, in particular, this would imply the sixth conjecture of Kaplansky for semisimple Hopf algebras. Müger has introduced in [85] the notion of centralizer of a fusion subcategory of a braided fusion category. One of the most remarkable features of this notion is that the centralizer of a nondegenerate fusion subcategory of a modular category is a categorical complement of the modular category. This principle is the basis of many classification results of fusion categories, see for example [36, 37, 45].

Despite its importance, in general it is a difficult task to give a concrete description for the centralizer of all fusion subcategories of a given fusion category. Only few cases are know in the literature. For instance, in the same aforementioned paper, [85], Müger described the centralizer of all fusion subcategories of the category of finite dimensional representations of a Drinfeld double of a finite abelian group. More generally, for the category of representations of a (twisted) Drinfeld double of an arbitrary finite group a similar formula was then given in [87]. Note also that for the braided center of Tambara-Yamagami categories, this centralizer was described in [57] by computing completely the S-matrix of the modular category.

The author intends to study the centralizer for two classes of braided fusion categories.

7.2.1 Müger centralizer for semisimple Drinfeld doubles

It is well known that the category of representations of the Drinfeld double of a semisimple Hopf algebra is a modular fusion category. Recall that this means that the S-matrix of a such category is an invertible matrix. Müeger's inequality inequality

$$|s_{X,Y}| \leq d_X d_Y$$

for the maximum values of the entries of S-matrix suggests that these values might be related with the values obtained by evaluating characters at cocharacters, see Proposition 1.1 from Chapter 1.

In a recent preprint, [23], we investigated for a factorizable Hopf algebra such a possible relation. More precisely, we computed the centralizer of the fusion subcategory $\mathrm{HKer}_{A^*}(d)$ in terms of the central idempotents of the character algebra of A.

In the future we would like to investigate this relation of Müger's centralizer for other fusion subcategories of $\mathrm{Rep}(A)$. The description given in [19] for the irreducible representations of a semisimple Drinfeld double $D(A)$ suggests that results similar to the one from [87] are expected.

7.2.2 Müger centralizer for equivariantized fusion categories

Suppose that a finite group G acts by tensor autoequivalences on a G - crossed braided fusion category $\mathcal{C}$, [116]. It is well known that in this case $\mathcal{C}^G$ is braided.

Using the fusion rules for equivariantizations from [26] (see also Chapter 5) one can parameterize all fusion subcategories of the equivariantized fusion category $\mathcal{C}^G$. Similar to the results from [87] these fusion subcategories are parameterized by a fusion subcategory $\mathcal{D}$

of $\mathcal{C}$, a pair of elementwise commuting subgroups H, K of G, and some twisted bicharacter $\lambda : H \times K \to K_0(\mathcal{D})$ satisfying certain compatibilities. We denote by $\mathcal{S}(H, K, \lambda, \mathcal{D})$ the subcategory corresponding to this datum. In a future project, together with S. Natale the author will study Müger's centralizer for these fusion subcategories $\mathcal{C}^G$. It is expected that a symmetry similar to the one from [87] to hold. In particular it is not difficult to see that the subgroups of the centralizer of a fusion subcategory of $\mathcal{C}^G$ are interchanged, i.e $\mathcal{S}(K, H, \mathcal{D}, \lambda) = \mathcal{S}(H, K, \mathcal{D}', \lambda')$. It remains to investigate the relationship between $\mathcal{D}$ and $\mathcal{D}'$ and between λ and λ'.

7.2.3 Mackey theory for equivariantized fusion categories

As it was already mentioned in Chapter 6 one can extend the Mackey decomposition for cocentral extensions obtained by the author in [22] to a Mackey decomposition for equivariantized fusion categories.

Such a decomposition will enable us to define a categorized notion of a Green functor [20]. This study will provide new information abut the Grothendieck group of an equivariantized fusion category. Since the Drinfeld center of a fusion category can also be described as an equivariantization (of the relative center), see [57], this new Mackey decomposition provides new properties on the structure of the Grothendieck rings of Drinfeld doubles.

7.2.4 Fusion subcategories from II_1 subfactors

It was recently observed that a categorical Morita equivalence between two fusion categories is precisely a finite-depth subfactor (the two fusion categories are the $A - A$ and $B - B$ bimodules, the Morita equivalence and its inverse are the $A-B$ and $B-A$ bimodules). This remark provided a new method for studying fusion categories in the literature intimately related to the study of II_1 subfactors, see [47, 48, 8, 101, 99, 100].

Given a fusion category $\mathcal{C}$, an important question is to understand all quantum subgroups of $\mathcal{C}$ in the sense given by Ocneanu in [101]. This equivalent to understanding all indecomposable module categories over $\mathcal{C}$. When $\mathcal{C}$ is a fusion category coming from quantum su_2 at a root of unity, then the quantum subgroups are given by the ADE Dynkin diagrams. (see for example [99, 100, 8].) For the corresponding results results into the language of fusion categories and module categories see [46, 71, 102].

It is well known now that a subfactor whose principal even part is $\mathcal{C}$ is almost the same thing as a simple algebra object in $\mathcal{C}$ (see for example [75, 76, 84, 123]). Note that all simple algebra objects in $\mathcal{C}$ can be realized as the internal endomorphisms of a simple object in some module category over $\mathcal{C}$ [102].

The author intends to extend the results from [26, Theorem 3.14] in order to obtain new information about the indecomposable module categories over an equivariantization $\mathcal{C}^G$ and to further analyze the Grothendieck rings generated by them.

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