# Convexity properties of coverings of 1-convex surfaces *

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*Dedicated to Professor Cabiria Andreian Cazacu on her* $85^{th}$ *birthday*

**Abstract**

We prove that there exists a 1-convex surface whose universal covering does not satisfy the discrete disk property.

## 1 Introduction

The well-known Shafarevich Conjecture asserts that the universal covering space of a projective algebraic manifold is holomorphically convex. Although there are partial results, a complete answer to this problem is not known even for surfaces. (We remark that if instead of the universal covering one considers an arbitrary non-compact one, there are counterexamples, see [9]).

In this paper we are interested in studying convexity properties of the universal covering of 1-convex surfaces. We recall that projective algebraic manifolds are a particular case of Moishezon manifolds, that the exceptional set of a 1-convex manifold is a Moishezon space and that every Moishezon space is the exceptional set of a 1-convex space.

Suppose that $X$ is a 1-convex surface and $p : \tilde{X} \to X$ is a covering map. It is known (see [1]) that in general $\tilde{X}$ is not holomorphically convex. In fact $\tilde{X}$ might not be even weakly 1-complete (that is, $\tilde{X}$ might not carry a continuous plurisubharmonic exhaustion function). However $\tilde{X}$ can be exhausted by a sequence of strongly pseudoconvex domains and therefore

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$\tilde{X}$ satisfies the continuous disk property (see the next section for a precise definition). We investigate the *discrete* disk property for $\tilde{X}$ which definitely is a stronger property.

Our main goal is to give an example of a 1-convex surface whose universal covering does not satisfy the discrete disk property. In particular it will not be $p_5$-convex in the sense of [4]. This means that we will prove the following theorem:

**Theorem.** *There exists a 1-convex surface whose universal covering does not satisfy the discrete disk property.*

We remark that we proved in [2] that if $\tilde{X}$ does not contain an infinite Nori string of rational curves then actually $\tilde{X}$ does satisfy the discrete disk property. Therefore our example must contain such a Nori string.

We note that important convexity properties of coverings of 1-convex manifolds have been established in [9].

In the study of coverings of compact complex surfaces an important phenomenon is the appearance of rational Nori strings, see [11], section 6. For different configurations of Nori strings that can appear in the universal covering surfaces of Kodaira's class $VII_0$ see [3], Theorem 3.27 and [8].

The main point of our paper is that we construct a neighborhood of a Nori string (that appears in the covering of a 1-convex surface) that does not satisfy the discrete disk property.

# 2 Preliminaries

We denote by $\Delta$ the unit disk in $\mathbb{C}$, $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ and for $c > 0$ by $\Delta_{1+c}$ the disk $\Delta_{1+c} := \{z \in \mathbb{C} : |z| < 1 + c\}$.
For $\epsilon > 0$ we define $H_\epsilon \subset \mathbb{C} \times \mathbb{R}$ as

$$H_\epsilon = \Delta_{1+\epsilon} \times [0, 1) \bigcup \{z \in \mathbb{C} : 1 - \epsilon < |z| < 1 + \epsilon\} \times \{1\}.$$

The following is just an intrinsic version of the classical Continuity Principle (see, for example, [7] page 47).

**Definition 1.** *A complex space $X$ is said to satisfy the continuous disk property if whenever $\epsilon$ is a positive number and $F : H_\epsilon \to X$ is a continuous function such that, for every $t \in [0, 1)$, $F_t : \Delta_{1+\epsilon} \to X$, $F_t(z) = F(z, t)$, is holomorphic we have that $F(H_{\epsilon_1})$ is relatively compact in $X$ for any $0 < \epsilon_1 < \epsilon$.*

Motivated by the above definition we introduced in [2]:

**Definition 2.** *Suppose that $X$ is a complex space. We say that $X$ satisfies the discrete disk property if whenever $g_n : U \to X$ is a sequence of holomorphic functions defined on an open neighborhood $U$ of $\overline{\Delta}$ for which there exists an $\epsilon > 0$ and a continuous function $\gamma : S^1 = \{z \in \mathbb{C} : |z| = 1\} \to X$ such that $\Delta_{1+\epsilon} \subset U$, $\bigcup_{n\geq 1} g_n(\Delta_{1+\epsilon} \setminus \Delta)$ is relatively compact in $X$ and $g_{n|S^1}$ converges uniformly to $\gamma$ we have that $\bigcup_{n\geq 1} g_n(\overline{\Delta})$ is relatively compact in $X$.*

Note that if a complex space is $p_5$-convex in the sense of Docquier and Grauert [4] then it satisfies the discrete disk property. Therefore our example will not be $p_5$-convex either. $X$ is called $p_5$-convex if whenever $\{\Delta_\nu\}_{\nu\geq 0}$ is a sequence of holomorphic disks such that $\bigcup_{\nu\geq 0} \partial\Delta_\nu \Subset X$ we have that $\bigcup_{\nu\geq 0} \overline{\Delta}_\nu \Subset X$ as well.

In [5] it is constructed a complex manifold which is an increasing union of Stein open subsets, and therefore it satisfies the continuous disk property, but it does not satisfy the discrete disk property. In particular this shows that the discrete disk property is stronger that the continuous one.

We recall that a compact complex curve is called rational if its normalization is $\mathbb{P}^1$.

A complex manifold is called 1-convex if it is the modification of a Stein space at a finite set of points.

**Definition 3.** *Let $L$ be a connected 1-dimensional complex space and $\cup L_i$ be its decomposition into irreducible components. $L$ is called an infinite Nori string if all $L_i$ are compact and $L$ is not compact*

The following theorem was proved in [2].

**Theorem 1.** *Let $X$ be a 1-convex surface and $p : \tilde{X} \to X$ be a covering map. If $\tilde{X}$ does not contain an infinite Nori string of rational curves then $\tilde{X}$ satisfies the discrete disk property.*

# 3 The Results

As we mentioned in the introduction, our goal is to prove the following theorem.

**Theorem 2.** *There exists a 1-convex surface whose universal covering does not satisfy the discrete disk property.*

We will describe first the basic idea of the proof of the theorem. We start with a basic example of a 2-dimensional complex manifold $X$ that does not satisfy the discrete disk property and contains an infinite Nori string of rational curves. We consider the complex manifold which is obtained from $\mathbb{C}^2$ after an infinite sequence of blow-ups as follows: we blow-up first $\Omega_0 := \mathbb{C}^2$ at the origin $(0,0) = a_0 \in \mathbb{C}^2$ and we denote this blow-up by $\Omega_1$. Let $l_1$ be the proper transform of $z_1 = 0$ and let $a_1$ be the intersection between $l_1$ and the exceptional divisor of $\Omega_1$. We blow-up $\Omega_1$ at $a_1$ and we obtain $\Omega_2$. We let $l_2$ to be the proper transform of $l_1$ and $a_2$ the intersection between $l_2$ and the exceptional divisor of $\Omega_2$ and we blow-up again. Inductively we obtain a sequence $\{\Omega_k\}_{k\geq 0}$ of complex manifolds and $\Omega_k \setminus \{a_k\} \subset \Omega_{k+1} \setminus \{a_{k+1}\}$. Let $X_0$ be the union (i.e. the inductive limit) of $\Omega_k \setminus \{a_k\}$. Notice now that the standard biholomorphism $(z_1, z_2) \to (\frac{\xi_1}{\xi_2}, z_2)$ between $\mathbb{C}^2$ and $\{(z_1, z_2, [\xi_1 : \xi_2]) \in \mathbb{C}^2 \times \mathbb{P}^1 : z_1\xi_2 = z_2\xi_1 \text{ and } \xi_2 \neq 0\}$ induces a biholomorphism $\iota$ between $X_0$ and an open subset of $X_0$. We let $X_k$, $k \in Z$, $k < 0$, be copies of $X_0$ and $X_k \hookrightarrow X_{k-1}$ be the inclusion given by $\iota$. For details see Step 1. (it is a going back process which is possible since we consider the blow-up at a point of $\mathbb{C}^2$ not of $\mathbb{P}^2$). We define $X$ as the union $\bigcup_{k\leq 0} X_k$. It is not difficult to see that $X$ does not satisfy the discrete disk property: we let $f_n : \mathbb{C} \to \mathbb{C}^2$, $n \geq 1$, $f_n(\lambda) = ((\frac{\lambda}{2})^n, \lambda)$ and $g_n : \mathbb{C} \to X_0$ the proper transform of $f_n$. Then $\bigcup_{n\geq 1} g_n(\Delta_2 \setminus \Delta)$ is relatively compact in $X$ and $\{g_n(0)\}_{n\geq 1}$ is discrete.

Notice that $X$ contains a Nori string $\{L_k\}_{k\in\mathbb{Z}}$ of curves isomorphic to $\mathbb{P}^1$. Then $\bigcup_{k\in\mathbb{Z}} L_k$ will cover $F_0 \cup F_1$ where $F_0$ and $F_1$ are isomorphic to $\mathbb{P}^1$ and $F_0 \cap F_1$ has exactly two points. An appropriately chosen neighborhood $U$ of $\bigcup_{k\in\mathbb{Z}} L_k$ in $X$ will cover a manifold $V$ which is a neighborhood of $F_0 \cup F_1$. It is again not very hard to prove that $U$ does not satisfy the discrete disk property. However $F_0 \cup F_1$ is not exceptional because the intersection matrix is

$$\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}$$

and then we have to blow-up again at two points, one on $F_0$ and one on $F_1$ in order to make the intersection matrix negatively defined. Then a small enough neighborhood of the proper transform of $F_0 \bigcup F_1$ is a 1-convex surface. We blow-up $U$ at the preimages of these points and then an open neighborhood, $\tilde{W}$, of the proper transform of $\bigcup_{k\in\mathbb{Z}} L_k$ is a covering of a 1-convex surface.

The core of our paper is to show that $\tilde{W}$ does not satisfy the discrete disk property.

A sequence of holomorphic disks defined in the simple-minded way as the one above will not work because their image will not stay in a small neighborhood of the proper transform of $\bigcup_{k\in\mathbb{Z}} L_k$. In fact these disks have to stay in a union of conic open subsets of $X_0$. To be able to define the sequence of holomorphic disks needed we will work in local coordinates.

We move now to the proof of Theorem 2.

**Step 1.** We construct a 1-convex manifold $W$ and a covering $\tilde{p} : \tilde{W} \to W$. In the second step we will show that $\tilde{W}$ does not have the discrete disk property.

As we said, we let $\Omega_0 = \mathbb{C}^2$, $(z_1^{(0)}, z_2^{(0)})$ the coordinate functions and $a_0 = (0,0)$. Let $\Omega_1$ be the blow-up of $\Omega_0$ in $a_0$, that is $\Omega_1 = \{(z_1^{(0)}, z_2^{(0)}, [\xi_1^{(0)} : \xi_2^{(0)}]) \in \Omega_0 \times \mathbb{P}^1 : z_1^{(0)}\xi_2^{(0)} = z_2^{(0)}\xi_1^{(0)}\}$ and $a_1 = (0,0,[0:1]) \in \Omega_1$. Let $\Omega_2$ be the blow up of $\Omega_1$ in $a_1$ and let $L_0$ be the proper transform of the exceptional set of $\Omega_1$. The open subset of $\Omega_1$ given by $\xi_2^{(0)} \neq 0$ is biholomorphic to $\mathbb{C}^2$ with the coordinate functions $z_1^{(1)} := \frac{\xi_1^{(0)}}{\xi_2^{(0)}}$ and $z_2^{(1)} := z_2^{(0)}$. In these coordinates $a_1$ is given by $z_1^{(1)} = 0$, $z_2^{(1)} = 0$. We continue this procedure $k$ times and we obtain $\Omega_k$. In doing so we obtain also $L_0, \dots L_{k-1}$, which are complex curves each one of them isomorphic to $\mathbb{P}^1$, and $a_0, a_1, \dots, a_k$ the points where we are blowing up. Note that $\Omega_j \setminus \{a_j\}$ is an open subset of $\Omega_{j+1} \setminus \{a_{j+1}\}$. We set

$$X_0 := \cup_{j=0}^{\infty} \Omega_j \setminus \{a_j\}.$$

We call $X_0$ *the infinite blow-up* of $\mathbb{C}^2$ at the origin. Notice that we have also a canonical map $\pi : X_0 \to \mathbb{C}^2$ such that $\pi^{-1}(0) = \bigcup_{k\geq 0} L_k$ and $\pi : X_0 \setminus \bigcup_{k\geq 0} L_k \to \mathbb{C}^2 \setminus \{0\}$ is a biholomorphism.

As this is clearly a local construction it can carried out around any point of a smooth complex surface once that we have chosen a system of coordinates around this point.

We let $M$ be the blow-up of $\mathbb{C}^2$ at the origin, written in coordinates as follows: $M = \{(z_1^{(-1)}, z_2^{(-1)}, [\xi_1^{(-1)} : \xi_2^{(-1)}]) \in \mathbb{C}^2 \times \mathbb{P}^1 : z_1^{(-1)}\xi_2^{(-1)} = z_2^{(-1)}\xi_1^{(-1)}\}$. Then $\{(z_1^{(-1)}, z_2^{(-1)}, [\xi_1^{(-1)} : \xi_2^{(-1)}]) \in M : \xi_2^{(-1)} \neq 0\}$ is an open set of $M$, biholomorphic to $\mathbb{C}^2$ with coordinate functions $z_1^{(0)} := \frac{\xi_1^{(-1)}}{\xi_2^{(-1)}}$ and $z_2^{(0)} := z_2^{(-1)}$.

For this open subset of $M$ and this system of coordinates we let $X_{-1}$ be the infinite blow-up of $M$ at the point $(0,0,[0:1])$. We let $L_{-1}$ to be the (proper transform of) the exceptional set of $M$. Notice then that $X_0$ is an open subset of $X_{-1}$ and that $X_{-1}$ is biholomorphic to $X_0$. Similarly we construct $X_k$ and $L_k$ for $k \leq -2$. We have that $X_k$ is an open subset of $X_{k-1}$ (in fact $X_k$ is the complement of a line in $X_{k-1}$). We put $X = \bigcup_{k=0}^{-\infty} X_k$ and $L = \bigcup_{k=-\infty}^{\infty} L_k$. Notice that if $|j-k| \geq 2$ then $L_j \cap L_k = \emptyset$.

Next we want to define a fundamental system of open neighborhoods of $L_k$ for each $k \in \mathbb{Z}$. To do that we notice that, by construction, $L_k$ is obtained as follows: we have $\mathbb{C}^2$ with coordinate functions $(z_1^{(k)}, z_2^{(k)})$ we blow it up at the origin and then we blow it up again at the point $(0, 0, [0:1])$. The manifold thus obtained is denoted by $\widehat{\mathbb{C}}^2$. Then $L_k$ is the proper transform of the exceptional set of the first blow-up. That is we have that $\widehat{\mathbb{C}}^2$ is given in $\mathbb{C}^2 \times \mathbb{P}^1 \times \mathbb{P}^1$ with coordinates $(z_1^{(k)}, z_2^{(k)}, [\xi_1^{(k)} : \xi_2^{(k)}], [\xi_1^{(k+1)} : \xi_2^{(k+1)}])$ by

$$z_1^{(k)} \xi_2^{(k)} = z_2^{(k)} \xi_1^{(k)}, \quad \xi_1^{(k)} \xi_2^{(k+1)} = \xi_1^{(k+1)} \xi_2^{(k)} z_2^{(k)}$$

In $\widehat{\mathbb{C}}^2$, $L_k$ is given by the equations $z_1^{(k)} = 0$, $\xi_2^{(k+1)} = 0$.

For $r \in (0, 1]$ we define $U_r^{(k)} := \{|\xi_2^{(k+1)}| < r|\xi_1^{(k+1)}|,\ |z_1^{(k)}| < r\}$ and we notice that $\{U_r^{(k)}\}_{r>0}$ is indeed a fundamental system of open neighborhoods of $L_k$. Obviously $U_r^{(j)}$ and $U_r^{(k)}$ are biholomorphic for every $j$ and $k$.

We want to show that if $|j-k| \geq 2$ then $U_r^{(j)} \cap U_r^{(k)} = \emptyset$. It is clear from our construction that without loss of generality we can assume that $j = 0$ and $k \geq 2$. As $U_r^{(j)} \cap U_r^{(k)}$ is an open set, it suffices to show that $(U_r^{(0)} \setminus L) \cap (U_r^{(k)} \setminus L) = \emptyset$. We recall that we have defined $z_1^{(k+1)} = \frac{\xi_1^{(k)}}{\xi_2^{(k)}}$ and $z_2^{(k+1)} = z_2^{(k)}$. Hence, outside $L$ and for $k \geq 0$, we have that $[z_1^{(k+1)} : z_2^{(k+1)}] = [\xi_1^{(k)} : \xi_2^{(k)} z_2^{(k)}] = [z_1^{(k)} : z_2^{(k)} z_2^{(0)}]$. Inductively we get $[z_1^{(k+1)} : z_2^{(k+1)}] = [z_1^{(0)} : (z_2^{(0)})^{k+2}]$. The inequality $|z_1^{(k)}| < r$ is equivalent to $|\xi_1^{(k-1)}| < r|\xi_2^{(k-1)}|$. As $[\xi_1^{(j)} : \xi_2^{(j)}] = [z_1^{(j)} : z_2^{(j)}]$ for every $j \in \mathbb{Z}$ and every point in $X \setminus L$ it follows that

$$U_r^{(k)} \setminus L = \{(z_1^{(0)}, z_2^{(0)}) \in \mathbb{C}^2 :\ |z_2^{(0)}|^{k+2} < r|z_1^{(0)}|, |z_1^{(0)}| < r|z_2^{(0)}|^k\} \qquad (1)$$

We have that $U_r^{(0)} \setminus L = \{(z_1^{(0)}, z_2^{(0)}) \in \mathbb{C}^2 :\ |z_2^{(0)}|^2 < r|z_1^{(0)}|, |z_1^{(0)}| < r\}$. In particular every point of $U_r^{(0)} \setminus L$ satisfies $|z_2^{(0)}|^2 < r|z_1^{(0)}| < r^2$, hence $|z_2^{(0)}| < r$. Then a point in the intersection $(U_r^{(0)} \setminus L) \cap (U_r^{(k)} \setminus L)$ would satisfy $|z_2^{(0)}|^2 < r|z_1^{(0)}| < r^2 |z_2^{(0)}|^k$. As $k \geq 2$ we get $1 < r^2 |z_2^{(0)}|^{k-2} < r^k$ and this contradicts our choice of $r \leq 1$.

It is clear that the mapping $(z_1^{(k)}, z_2^{(k)}, [\xi_1^{(k)} : \xi_2^{(k)}], [\xi_1^{(k+1)} : \xi_2^{(k+1)}]) \to (z_1^{(j)}, z_2^{(j)}, [\xi_1^{(j)} : \xi_2^{(j)}], [\xi_1^{(j+1)} : \xi_2^{(j+1)}])$ induces a biholomorphism of $q_{k,j} : U_r^{(k)} \to U_r^{(j)}$. Moreover $q_{k,k+2|U_r^{(k)} \cap U_r^{(k+1)}} = q_{k+1,k+3|U_r^{(k)} \cap U_r^{(k+1)}}$.

Let $\mathcal{U} = \bigcup_{k\in\mathbb{Z}} U_1^{(k)}$. We have then a biholomorphism $q : \mathcal{U} \to \mathcal{U}$ defined by $q_{|U_1^{(k)}} = q_{k,k+2}$ which induces an action of $\mathbb{Z}$ on $\mathcal{U}$. If we set $Y := \mathcal{U}/\mathbb{Z}$ and we let $p : \mathcal{U} \to Y$ be the canonical projection then $p$ is a covering map. Namely if we set $\mathcal{U}^{(0)} = p(U_1^{(0)}) = p(U_1^{(2k)})$ for every $k \in \mathbb{Z}$ then $p^{-1}\mathcal{U}^{(0)} = \bigcup_{k\in\mathbb{Z}} U_1^{(2k)}$, and $U_1^{(2k)}$, $k \in \mathbb{Z}$, are pairwise disjoint and biholomorphic via $p$ to $\mathcal{U}^{(0)}$. The same thing for $\mathcal{U}^{(1)} = p(U_1^{(1)}) = p(U_1^{(2k+1)})$.

Let $F_0 := p(L_0)$ and $F_1 := p(L_1)$. Then $F_0$ and $F_1$ are both biholomorphic to $\mathbb{P}^1$ and, moreover, we have $F_0 \cdot F_0 = -2$, $F_1 \cdot F_1 = -2$, $F_0 \cdot F_1 = 2$. Let $\alpha_k \in L_k$ be the point given by $(z_1^{(k)}, z_2^{(k)}, [\xi_1^{(k)} : \xi_2^{(k)}], [\xi_1^{(k+1)} : \xi_2^{(k+1)}]) = (0, 0, [1 : 1], [1 : 0])$ and $\beta_0, \beta_1 \in Y$ the points $\beta_0 = p(\alpha_{2k})$, $\beta_1 = p(\alpha_{2k+1})$. We let $\pi : \tilde{Y} \to Y$ to be the blow up of $Y$ at $\beta_0$ and $\beta_1$ and we denote by $\tilde{F}_0$ and $\tilde{F}_1$ respectively the proper transforms of $F_0$ and $F_1$. Note that $\tilde{F}_0 \cdot \tilde{F}_0 = -3$, $\tilde{F}_1 \cdot \tilde{F}_1 = -3$, $\tilde{F}_0 \cdot \tilde{F}_1 = 2$. As the intersection matrix

$$\begin{bmatrix} -3 & 2 \\ 2 & -3 \end{bmatrix}$$

is negative definite, it follows, see [6], that $\tilde{F} := \tilde{F}_0 \cup \tilde{F}_1$ is exceptional. We consider the following diagram:

$$\begin{array}{ccc} \tilde{\mathcal{U}} & \xrightarrow{\tilde{\pi}} & \mathcal{U} \\ {\scriptstyle \tilde{p}}\downarrow & & \downarrow{\scriptstyle p} \\ \tilde{Y} & \xrightarrow{\pi} & Y \end{array}$$

We let $\tilde{p} : \tilde{\mathcal{U}} \to \tilde{Y}$ be the pull-back of $p$. Clearly $\tilde{p}$ is a covering map and $\tilde{\pi} : \tilde{\mathcal{U}} \to \mathcal{U}$ is obtained by blowing-up $\mathcal{U}$ at every $\alpha_k$, $k \in \mathbb{Z}$. We choose now $W$ a 1-convex neighborhood of $\tilde{F}$ and we put $\tilde{W} := \tilde{p}^{-1}(W)$, $\tilde{L} := \tilde{p}^{-1}(\tilde{F})$. If $\tilde{L}_k$ is the proper transform of $L_k$ in $\tilde{\mathcal{U}}$ then $\tilde{L} = \bigcup \tilde{L}_k$. We will show that $\tilde{W}$ does not have the discrete disk property. In our construction of the sequence of holomorphic discs we want to make sure that their image stays in $\tilde{W}$. To do that we need a "concrete" open neighborhood of $\tilde{L}$ in $\tilde{W}$. To obtain it we consider $\{\tilde{W}_{r,\rho}^{(k)}\}$ a fundamental system of neighborhoods for $\tilde{L}_k$, each one of them being actually the preimage via $\tilde{\pi}$ of a cone centered at $\alpha_k$. Moreover $q_{k,j}$ induces a biholomorphism $\tilde{W}_{r,\rho}^{(k)} \to \tilde{W}_{r,\rho}^{(j)}$. The construction is as follows.

We have the following description of the blow-up of $U_1^{(k)}$ in $\alpha_k$: it is the set $\tilde{U}_1^{(k)}$ of all

$$(z_1^{(k)}, z_2^{(k)}, [\xi_1^{(k)} : \xi_2^{(k)}], [\xi_1^{(k+1)} : \xi_2^{(k+1)}], [w_1 : w_2]) \in \mathbb{C}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$$

such that

$$z_1^{(k)}\xi_2^{(k)} = z_2^{(k)}\xi_1^{(k)},\ \xi_1^{(k)}\xi_2^{(k+1)} = \xi_1^{(k+1)}\xi_2^{(k)}z_2^{(k)},\ w_2 z_1^{(k)}\xi_1^{(k)} = w_1(\xi_1^{(k)} - \xi_2^{(k)})$$

and

$$|z_1^{(k)}| < 1,\ |\xi_2^{(k+1)}| < |\xi_1^{(k+1)}|$$

The proper transform of $L_k$ is given by $z_1^{(k)} = 0$, $\xi_2^{(k+1)} = 0$, $w_1 = 0$. A fundamental system of neighborhoods for $\tilde{L}_k$ is given by

$$\tilde{W}_{r,\rho}^{(k)} = \{|z_1^{(k)}| < r,\ |\xi_2^{(k+1)}| < r|\xi_1^{(k+1)}|,\ |w_1| < \rho|w_2|\} \subset \tilde{U}_1^{(k)}.$$

There exist then $\rho > 0$ and $r > 0$ such that $\tilde{W}_r^\rho = \bigcup_{k\in\mathbb{Z}} \tilde{W}_{r,\rho}^{(k)} \subset \tilde{W}$.
If we denote by $W_r^\rho \subset \mathcal{U}$ the set

$$\bigcup_{k\in\mathbb{Z}} \{(z_1^{(k)}, z_2^{(k)}, [\xi_1^{(k)} : \xi_2^{(k)}], [\xi_1^{(k+1)} : \xi_2^{(k+1)}]) \in U_r^{(k)} : |z_1^{(k)}\xi_1^{(k)}| < \rho|\xi_2^{(k)} - \xi_1^{(k)}|\}$$

we have that $\tilde{W} \setminus \tilde{L} \supset \tilde{W}_r^\rho \setminus \tilde{L} \supset W_r^\rho \setminus L$. We notice at the same time that keeping $\rho \in (0,1)$ fixed and choosing a small enough $r > 0$ we have that $\tilde{W}_{r,\rho}^{(k)} \cap \tilde{W}_{r,\rho}^{(k+1)} = U_r^{(k)} \cap U_r^{(k+1)}$ for every $k \in \mathbb{Z}$. We fix such an $r \in (0,1)$ that satisfies also $r \le \frac{\rho}{2}(1-r)$.

**Step 2.** We construct a sequence of holomorphic discs that proves that $\tilde{W}$ does not have the discrete disk property.

**We fix** $n \in \mathbb{N}$. To define our $n^{th}$ holomorphic disk, $g_n$, we will start with two polynomial functions $f_1 = f_1^{(n)}$ and $f_2 = f_2^{(n)}$ and $g_n$ will be the proper transform of $(f_1, f_2) : \mathbb{C} \to \Omega_0$ restricted to a neighborhood of $\overline{\Delta}_2$ (we recall that $\Omega_0$ was defined as $\mathbb{C}^2$ with coordinate functions $(z_1^{(0)}, z_2^{(0)})$). This proper transform is considered after all the blow-ups we made, i.e. first at the points $\{a_j\}_{j\in\mathbb{Z}}$ and then $\{\alpha_j\}_{j\in\mathbb{Z}}$.

**Lemma 1.** *Let* $\mathcal{Z}^{(n)} = \mathcal{Z} := \{\lambda \in \mathbb{C} : f_1(\lambda) = f_2(\lambda) = 0\}$. *Suppose that* $f_1$ *and* $f_2$ *satisfy the following properties:*

*a)* $(f_1, f_2)(\overline{\Delta}_2 \setminus \mathcal{Z}) \subset \bigcup_{k\geq 0}^{n-1} U_r^{(k)} \setminus L$,
*b) for every* $k \in \{0, 1, \ldots, n-1\}$ *if* $\lambda \in \overline{\Delta}_2 \setminus \mathcal{Z}$ *satisfies*
$|f_1(\lambda)| < r|f_2(\lambda)|^k$ *and*
$|f_2(\lambda)|^{k+2} < r|f_1(\lambda)|$
*then it satisfies* $|f_1(\lambda)|^2 < \rho|f_2(\lambda)^{k+1} - f_1(\lambda)| \cdot |f_2(\lambda)|^k$.

*Then* $g_n(\overline{\Delta}_2) \subset \tilde{W}$.

*Proof.* Obviously $g_n(\mathcal{Z}) \subset \tilde{L}$. It suffices then to show that $g_n(\overline{\Delta}_2 \setminus \mathcal{Z}) \subset \tilde{W}_r^\rho \setminus \tilde{L}$. We have seen that $\tilde{W} \setminus \tilde{L} \supset W_r^\rho \setminus L$. Hence it is enough to prove that $g_n(\overline{\Delta}_2 \setminus \mathcal{Z}) \subset W_r^\rho \setminus L$.

By hypothesis we have that $g_n(\overline{\Delta}_2 \setminus \mathcal{Z}) = (f_1, f_2)(\overline{\Delta}_2 \setminus \mathcal{Z}) \subset \bigcup_{k\geq 0}^{n-1} U_r^{(k)} \setminus L$. Hence it suffices to show, for $k \in \{0, 1, \ldots, n-1\}$ and $\lambda \in \overline{\Delta}_2 \setminus \mathcal{Z}$ that if

$g_n(\lambda) = (z_1^{(k)}, z_2^{(k)}, [\xi_1^{(k)} : \xi_2^{(k)}], [\xi_1^{(k+1)} : \xi_2^{(k+1)}]) \in U_r^{(k)}$ then

$$|z_1^{(k)}\xi_1^{(k)}| < \rho|\xi_2^{(k)} - \xi_1^{(k)}| \tag{2}$$

Because $[z_1^{(k)} : z_2^{(k)}] = [\xi_1^{(k)} : \xi_2^{(k)}]$, outside $L$ this inequality is equivalent to $|z_1^{(k)}|^2 < \rho|z_2^{(k)} - z_1^{(k)}|$. At the same time $z_2^{(k)} = z_2^{(0)}$ and we have seen that $[z_1^{(k)} : z_2^{(k)}] = [z_1^{(0)} : (z_2^{(0)})^{k+1}]$. We deduce that (2) is equivalent to

$$|z_1^{(0)}|^2 < \rho|(z_2^{(0)})^{k+1} - z_1^{(0)}| \cdot |z_2^{(0)}|^k.$$

Using the description (1) of $U_r^{(k)}$ we have then to show that if $\lambda \in \overline{\Delta}_2 \setminus \mathcal{Z}$ satisfies $|f_1(\lambda)| < r|f_2(\lambda)|^k$ and $|f_2(\lambda)|^{k+2} < r|f_1(\lambda)|$ then it satisfies $|f_1(\lambda)|^2 < \rho|f_2(\lambda)^{k+1} - f_1(\lambda)| \cdot |f_2(\lambda)|^k$.

But this is exactly condition b) in our hypothesis. □

**Remark:** Let us say a few words about the the construction of $f_1$ and $f_2$. Notice that if by $h_n$ we denote the proper transform of $(f_1, f_2)$ after the blow-ups at $\{a_j\}_{j\in\mathbb{Z}}$ then in order to keep the image of $g_n$ inside $\tilde{W}$ the image of $h_n$ must contain $\alpha_j$, $0 \leq j \leq n$ and hence to intersect each $L_j$ for $0 \leq j \leq n$. This suggests the form of $f_1$ and $f_2$ bellow. At the same time we will be using Lemma 1. To prove the inequality $|f_1(\lambda)|^2 < \rho|f_2(\lambda)^{k+1} - f_1(\lambda)| \cdot |f_2(\lambda)|^k$ we have to make sure that the function on right does not have more zeros than the one on the left, counting multiplicities. These leads us to a problem of divisibility (see Lemma 3).

**The construction of $f_1$ and $f_2$:** Let $c_1, \ldots, c_{n-1}$ be integers defined recursively by $c_1 = 1$ and, for $k \geq 2$, $c_k = 2k-1+(k-1)c_1+(k-2)c_2+\cdots c_{k-1}$. We also consider $d_1, \ldots, d_{n-1}$ positive integers defined by $d_{n-1} = 1$ and, for $k \leq n-2$, $d_k = d_{k+1} + 2d_{k+2} + \cdots (n-k-1)d_{n-1} + n - k$. Let $N = 2n(d_1 + d_2 + \cdots + d_{n-1} + 1)$.

We define $f_1$ and $f_2$ as

$$f_1(\lambda) = \varepsilon P_1(\lambda) P_2^2(\lambda) \cdots P_{n-1}^{n-1}(\lambda) \cdot \lambda^n,$$

$$f_2(\lambda) = \varepsilon^2 P_1(\lambda) P_2(\lambda) \cdots P_{n-1}(\lambda) \cdot \lambda$$

where:
• $\varepsilon$ is a positive real number that satisfies $\varepsilon < (\frac{1}{6})^N \frac{1}{n+2} r$,
• $P_1, \ldots, P_{n-1}$ are polynomials defined recursively by
$P_{n-1}(\lambda) = \varepsilon^{c_{n-1}} - \lambda$ and,
$P_k(\lambda) = \varepsilon^{c_k} - P_{k+1}(\lambda) \cdot P_{k+2}^2(\lambda) \cdots P_{n-1}^{n-k-1}(\lambda) \cdot \lambda^{n-k}$, for $k \leq n-2$.

**Remarks:** 1) $P_k(0) \neq 0$ and $P_j$ and $P_k$ have no common zero for $j \neq k$. Therefore $\mathcal{Z} = \{\lambda \in \mathbb{C} : f_1(\lambda) = 0\} = \{\lambda \in \mathbb{C} : f_2(\lambda) = 0\} = \{0\} \cup \{\lambda \in \mathbb{C} : \exists k \text{ such that } P_k(\lambda) = 0\}$.
2) Each $P_k$ is a monic polynomial of degree $d_k$.

There are four conditions that we want the sequence $\{g_n\}$ to satisfy:
I) $g_n(\overline{\Delta}_2) \subset \tilde{W}$. We will prove in fact that $g_n(\overline{\Delta}_2) \subset \tilde{W}_r^\rho$.
II) $\bigcup_{n\geq 1} g_n(\Delta_2 \setminus \Delta)$ is relatively compact in $\tilde{W}$
III) $g_{n|S^1}$ is uniformly convergent
IV) $\bigcup_{n\geq 1} g_n(\overline{\Delta})$ is not relatively compact in $\tilde{W}$.

• Because $P_k(0) \neq 0$, the definition of $f_1$ and $f_2$ implies that the origin $0 \in \mathbb{C}$ is a zero of order 1 for $f_2$ and a zero of order $n$ for $f_1$. This implies that $g_n(0) \in L_{n-1}$ and this shows that $\{g_n(0)\}_{n\geq 1}$ is not relatively compact in $X$. Hence $\{g_n\}$ satisfies property IV).

• We will prove next that $\{g_n\}$ satisfies properties II) and III).
Let $K_n := \{(z_1, z_2, [\xi_1 : \xi_2] \in \Omega_1 : |z_1| \leq \frac{1}{n},\ |z_2| \leq \frac{1}{n},\ |\xi_2| \leq \frac{1}{n}|\xi_1|\}$. Note that $K_n$ is a compact subset of $X$, $K_n \supset K_{n+1}$, and $\cap_{n\geq 1} K_n = \{(0, 0, [1 : 0])\}$. Hence for $n$ large enough $K_n \subset \tilde{W}$. Therefore if we show that $g_n(\{\lambda \in \mathbb{C} : 1 \leq |\lambda| \leq 2\}) \subset K_n$ then we will prove both I) and II).

**Lemma 2.** *For $k \in \{1, \dots, n-1\}$, if $P_k(\lambda) = 0$ then $|\lambda| < \frac{1}{2^k}$.*

*Proof.* We will prove our assertion by backward induction on $k$. For $k = n-1$ the statement is obvious. We assume that we have proved our assertion for $j \geq k+1$ and we prove it for $k$. For $j \geq k+1$, as $P_j$ are monic polynomials and all they zeros are inside the disk $\{|\lambda \in \mathbb{C} : |\lambda| < \frac{1}{2^j}\} \subset \{\lambda \in \mathbb{C} : |\lambda| < \frac{1}{2^{k+1}}\}$, we have that, for every $\lambda \in \mathbb{C}$ with $|\lambda| = \frac{1}{2^k}$, $|P_j(\lambda)| \geq (\frac{1}{2})^{d_j(k+1)}$ (see for example the proof of the next Corollary). It follows that $|P_{k+1}(\lambda) \cdot P_{k+2}^2(\lambda) \cdots P_{n-1}^{n-k-1}(\lambda) \cdot \lambda^{n-k}| \geq \frac{1}{2^N} > \varepsilon > \varepsilon^{c_k}$ for $|\lambda| = \frac{1}{2^k}$. Rouché's theorem (see e.g. [10] page 106) implies that $P_k(\lambda)$ and $P_{k+1}(\lambda) \cdot P_{k+2}^2(\lambda) \cdots P_{n-1}^{n-k-1}(\lambda) \cdot \lambda^{n-k}$ have the same number of zeros inside the disk $\{\lambda \in \mathbb{C} : |\lambda| < \frac{1}{2^k}\}$. As the two polynomials have the same degree and all the zeros of the second one are in this disk, it follows that all the zeros of $P_k$ are in there as well. □

**Corollary 1.** *If $\lambda \in \mathbb{C}$, $|\lambda| \leq 2$ then $|P_k(\lambda)| < 3^{d_k}$.*
*If $1 \leq |\lambda| \leq 2$ then $(\frac{1}{2})^{d_k} < |P_k(\lambda)| < 3^{d_k}$.*

*Proof.* Because $P_k$ is a monic polynomial of degree $d_k$ we have that it is of the form $P_k(\lambda) = (\lambda - \lambda_1^{(k)}) \cdots (\lambda - \lambda_{d_k}^{(k)})$ where $\lambda_j^{(k)}$ are its roots (counted with multiplicity). Lemma 2 implies that $|\lambda_j^{(k)}| < \frac{1}{2^k} \leq \frac{1}{2}$ and therefore for $|\lambda| \leq 2$ we have that $|\lambda_j^{(k)} - \lambda| < 2 + \frac{1}{2} < 3$ and for $1 \leq |\lambda| \leq 2$ we have that $\frac{1}{2} < |\lambda_j^{(k)} - \lambda| < 3$.

□

Given our choice of $\varepsilon$ and Corollary 1, a simple computation shows:

**Corollary 2.** *If $\lambda \in \mathbb{C}$ satisfies $|\lambda| \leq 2$ then we have:*
*a) $|f_1(\lambda)| < \frac{1}{n} r \leq \frac{1}{n}$,*
*b) $|f_2(\lambda)| < \frac{1}{n} r^2 \leq \frac{1}{n}$,*
*and if $1 \leq |\lambda| \leq 2$, then:*
*c) $|f_2(\lambda)| < \frac{1}{n}|f_1(\lambda)|$,*
*d) $|f_1(\lambda)| > |f_2(\lambda)|^k$ for every $k \geq 1$.*

As $f_1$ and $f_2$ have no zero inside $\{\lambda \in \mathbb{C} : 1 \leq |\lambda| \leq 2\}$ this last Corollary implies that $g_n(\{\lambda \in \mathbb{C} : 1 \leq |\lambda| \leq 2\}) \subset K_n$.

• We move now to the proof of property I).

We will use Lemma 1. Therefore we have to check the two hypothesis, a) and b). We will start with a).

We we will show first that $(f_1, f_2)(\overline{\Delta}_2 \setminus \mathcal{Z}) \subset \bigcup_{k\geq 0} U_r^{(k)} \setminus L \subset \mathcal{U} \setminus L$. We prove that

$$\bigcup_{k\geq 0} U_r^{(k)} \setminus L \supset \{(z_1^{(0)}, z_2^{(0)}) \in \mathbb{C}^2 : 0 < |z_1^{(0)}| < r,\ |z_2^{(0)}| < r^2\}.$$

This inclusion together with the first two inequalities of Corollary 2 implies that indeed $(f_1, f_2)(\overline{\Delta}_2 \setminus \mathcal{Z}) \subset \mathcal{U} \setminus L$. Let $(z_1^{(0)}, z_2^{(0)}) \in \mathbb{C}^2$ be such that $0 < |z_1^{(0)}| < r$ and $|z_2^{(0)}| < r^2$. If $z_2^{(0)} = 0$ then obviously $(z_1^{(0)}, z_2^{(0)}) \in U_r^{(0)} \setminus L$. Suppose that $z_2^{(0)} \neq 0$. We have seen that

$$U_r^{(k)} \setminus L = \{(z_1^{(0)}, z_2^{(0)}) \in \mathbb{C}^2 : |z_1^{(0)}| < r|z_2^{(0)}|^k,\ |z_2^{(0)}|^{k+2} < r|z_1^{(0)}|\}.$$

Hence we have to show that there exists $k \geq 0$ such that $\frac{|z_2^{(0)}|^{k+2}}{r} < |z_1^{(0)}| < r|z_2^{(0)}|^k$ (notice that $\frac{|z_2^{(0)}|^{k+2}}{r} < r|z_2^{(0)}|^k$ because $|z_2^{(0)}| < r^2$ and we assumed that $r < 1$). We let $I_k := (\frac{|z_2^{(0)}|^{k+2}}{r}, r|z_2^{(0)}|^k) \subset \mathbb{R}$. As $\frac{|z_2^{(0)}|^{k+2}}{r} < r|z_2^{(0)}|^{k+1}$ it follows that $I_k \cap I_{k+1} \neq \emptyset$. At the same time $I_0 = (\frac{|z_2^{(0)}|^2}{r}, r)$ and $\lim_{k\to\infty} \frac{|z_2^{(0)}|^{k+2}}{r} = 0$. This implies that $\bigcup_{k\geq 0} I_k = (0, r)$ and therefore $|z_1^{(0)}| \in \bigcup_{k\geq 0} I_k$.

We prove now that $(f_1, f_2)(\overline{\Delta}_2 \setminus \mathcal{Z}) \subset \bigcup_{k=0}^{n-1} U_r^{(k)} \setminus L$. To prove this it is enough to show that for $k \geq n$ one has $|f_1(\lambda)| \geq r|f_2(\lambda)|^k$ (and therefore $(f_1, f_2)(\lambda) \notin U_r^{(k)}$ for $k \geq n$). However from Corollary 2, d) we have that $|f_1(\lambda)| > |f_2(\lambda)|^k > r|f_2(\lambda)|^k$ for $1 \leq |\lambda| \leq 2$. As $\frac{f_2(\lambda)^k}{f_1(\lambda)}$ is a holomorphic function for $k \geq n$, the maximum modulus principle implies that the inequality is valid on $\overline{\Delta}_2$.

We will verify that the hypothesis b) of Lemma 1 is satisfied. Let $k \in \{0, 1 \ldots, n-1\}$ and let $\lambda \in \overline{\Delta}_2 \setminus \mathcal{Z}$ such that $|f_1(\lambda)| < r|f_2(\lambda)|^k$ and $|f_2(\lambda)|^{k+2} < r|f_1(\lambda)|$. We must show that $|f_1(\lambda)|^2 < \rho|f_2(\lambda)^{k+1} - f_1(\lambda)| \cdot |f_2(\lambda)|^k$. We will distinguish two cases: $k \geq 1$ and $k = 0$.

For $k \geq 1$ we let $A_k = \{\lambda \in \Delta_2 : |f_1(\lambda)| < r|f_2(\lambda)|^k\}$ which is an open subset of $\mathbb{C}$. We will prove something stronger. Namely we will prove that $|f_1(\lambda)|^2 \leq \frac{\rho}{2}|f_2(\lambda)^{k+1} - f_1(\lambda)| \cdot |f_2(\lambda)|^k$ on $\lambda \in \overline{A}_k$.

To prove this inequality we will show that the quotient $f_1(\lambda)^2/(f_2(\lambda)^{k+1} - f_1(\lambda))f_2(\lambda)^k$ is a holomorphic function on a neighborhood of $\overline{A}_k$, we will check the inequality on $\partial A_k$ and we will apply the maximum modulus theorem.

Notice that due to Corollary 2 we have that $A_k$ is relatively compact in $\Delta_2$ and therefore on $\partial A_k$ we have that $|f_1(\lambda)| = r|f_2(\lambda)|^k$. (It is not true, however, that $\partial A_k = \{\lambda \in \mathbb{C} : |f_1(\lambda)| = r|f_2(\lambda)|^k\}$.)

If $l \leq k-1$ and $P_l(\mu) = 0$, then $\mu \notin \overline{A}_k$. Indeed, as the polynomials $P_j$ have no common zero, given the definition of $f_1$ and $f_2$, the order of vanishing of $f_2^k$ at $\mu$ is greater than the order of vanishing of $f_1$. Therefor there exists a neighborhood $U$ of $\mu$ such that on $U \setminus \{\mu\}$ we have $|f_1(\lambda)| > r|f_2(\lambda)|^k$. We deduce that $\frac{1}{P_l}$ is holomorphic on a neighborhood of $\overline{A}_k$ and hence we do not have to worry about the zeros of $P_1, P_2, \ldots, P_{k-1}$ when proving that $f_1(\lambda)^2/(f_2(\lambda)^{k+1} - f_1(\lambda))f_2(\lambda)^k$ is holomorphic. Using the definition of $f_1$ and $f_2$ we see that we have to deal with the roots of $\varepsilon^{2k+1} \cdot P_1^k(\lambda) \cdot P_2^{k-1}(\lambda) \cdots P_k(\lambda) - P_{k+2}(\lambda) \cdot P_{k+3}^2(\lambda) \cdots P_{n-1}^{n-k-2}(\lambda) \cdot \lambda^{n-k-1}$. For $\varepsilon$ small enough this polynomial has exactly $d_{k+1}$ roots (counting multiplicity) inside $\overline{\Delta}_2$. The purpose of the Lemma 3 is to show that they are precisely the roots of $P_{k+1}$. Then in Lemma 4 we will prove the inequality $|f_1(\lambda)|^2 \leq \frac{\varrho}{2}|f_2(\lambda)^{k+1} - f_1(\lambda)| \cdot |f_2(\lambda)|^k$.

**Lemma 3.** *$P_{k+1}(\lambda)$ is a divisor of $\varepsilon^{2k+1} \cdot P_1^k(\lambda) \cdot P_2^{k-1}(\lambda) \cdots P_k(\lambda) - P_{k+2}(\lambda) \cdot P_{k+3}^2(\lambda) \cdots P_{n-1}^{n-k-2}(\lambda) \cdot \lambda^{n-k-1}$.*

*Proof.* For $k = 0$ we have to show that $P_1(\lambda)$ is a divisor of $\varepsilon - P_2(\lambda) \cdots P_{n-1}^{n-2}(\lambda) \cdot \lambda^{n-1}$. However, by definition $c_1 = 1$ and hence $P_1(\lambda) = \varepsilon - P_2(\lambda) \cdots P_{n-1}^{n-2}(\lambda) \cdot \lambda^{n-1}$ and therefore there is nothing to prove. Suppose that $k \geq 1$. Notice that for $j \leq k$ we have $P_j \equiv \varepsilon^{c_j} \pmod{P_{k+1}}$. It follows that $\varepsilon^{2k+1} \cdot P_1^k \cdot P_2^{k-1} \cdots P_k - P_{k+2} \cdot P_{k+3}^2 \cdots P_{n-1}^{n-k-2} \cdot \lambda^{n-k-1} \equiv \varepsilon^{2k+1} \cdot \varepsilon^{(k+1)c_1} \cdots \varepsilon^{c_k} - P_{k+2} \cdot P_{k+3}^2 \cdots P_{n-1}^{n-k-2} \cdot \lambda^{n-k-1} \pmod{P_{k+1}}$. However $2k + 1 + kc_1 + (k-1)c_2 + \cdots c_k = c_{k+1}$ and therefore $\varepsilon^{2k+1} \cdot P_1^k \cdot P_2^{k-1} \cdots P_k - P_{k+2} \cdot P_{k+3}^2 \cdots P_{n-1}^{n-k-2} \cdot \lambda^{n-k-1} \equiv \varepsilon^{c_{k+1}} - P_{k+2} \cdot P_{k+3}^2 \cdots P_{n-1}^{n-k-2} \cdot \lambda^{n-k-1} \equiv 0 \pmod{P_{k+1}}$. □

**Lemma 4.** *$|f_1(\lambda)|^2 \leq \frac{\varrho}{2}|f_2(\lambda)^{k+1} - f_1(\lambda)| \cdot |f_2(\lambda)|^k$ for every $\lambda \in \overline{A}_k$ and every $k$ with $1 \leq k \leq n-1$.*

*Proof.* We claim that on a neighborhood of $\overline{A}_k$ the meromorphic function

$$\frac{f_1^2(\lambda)}{(f_2^{k+1}(\lambda) - f_1(\lambda)) \cdot f_2^k(\lambda)}$$

is actually holomorphic. We consider first the case $k \leq n-2$ and we notice that

$f_2^{k+1}(\lambda) - f_1(\lambda) = \varepsilon P_1(\lambda) \cdot P_2^2(\lambda) \cdots P_{k+1}^{k+1}(\lambda) \cdot P_{k+2}^{k+1}(\lambda) \cdots P_{n-1}^{k+1}(\lambda) \cdot \lambda^{k+1} \big(\varepsilon^{2k+1} \cdot P_1^k(\lambda) \cdot P_2^{k-1}(\lambda) \cdots P_k(\lambda) - P_{k+2}(\lambda) \cdot P_{k+3}^2(\lambda) \cdots P_{n-1}^{n-k-2}(\lambda) \cdot \lambda^{n-k-1}\big)$.

We have seen that all zeros of $P_{k+2} \cdot P_{k+3}^2 \cdots P_{n-1}^{n-k-2} \cdot \lambda^{n-k-1}$ are inside the disk $\{\lambda \in \mathbb{C} : |\lambda| < \frac{1}{2}\} \subset \Delta_2$. At the same time from the definition of $\varepsilon$ and Corollary 1 it follows that on $\{\lambda \in \mathbb{C} : 1 \leq |\lambda| \leq 2\}$ we have $|\varepsilon^{2k+1} \cdot P_1^k \cdot P_2^{k-1} \cdots P_k| < |P_{k+2} \cdot P_{k+3}^2 \cdots P_{n-1}^{n-k-2} \cdot \lambda^{n-k-1}|$. Rouché's theorem implies that $\varepsilon^{2k+1} \cdot P_1^k \cdot P_2^{k-1} \cdots P_k - P_{k+2} \cdot P_{k+3}^2 \cdots P_{n-1}^{n-k-2} \cdot \lambda^{n-k-1}$ has exactly $d_{k+2} + 2d_{k+3} + \cdots + (n-k-1)d_{n-1} + n - k - 1 = d_{k+1}$ zeros inside $\Delta_2$. Then Lemma 3 implies that $\varepsilon^{2k+1} \cdot P_1^k \cdot P_2^{k-1} \cdots P_k - P_{k+2} \cdot P_{k+3}^2 \cdots P_{n-1}^{n-k-2} \cdot \lambda^{n-k-1} = P_{k+1}Q$ where $Q$ is a polynomial which is nonvanishing on a neighborhood of $\overline{\Delta}_2$. We have seen that on a neighborhood of $\overline{A}_k$ we have that $P_1 \cdot P_2^2 \cdots P_{k-1}^{k-1}$ is nonvanishing. Hence we it remains to show that

$$\frac{f_1^2(\lambda)}{P_k^k \cdot P_{k+1}^{k+1} \cdot P_{k+2}^{k+1} \cdots P_{n-1}^{k+1} \cdot \lambda^{k+1} \cdot P_{k+1} \cdot P_k^k \cdot P_{k+1}^k \cdots P_{n-1}^k \cdot \lambda^k}$$

is holomorphic and this follows from the definition of $f_1$.

For $k = n-1$ Rouché's theorem implies as above that $f_2^n - f_1 = f_1 \cdot Q_1$ where $Q_1 = \varepsilon^{2n-1} P_1^{n-1} \cdot P_2^{n-2} \cdots P_{n-1} - 1$ is nonvanishing on a neighborhood of $\overline{\Delta}_2$. It remains to notice that $\frac{f_1}{f_2^{n-1}}$ is holomorphic on a neighborhood of $\overline{A}_{n-1}$ and our claim is proved.

The maximum modulus principle implies that it is enough to check our inequality on $\partial A_k$, hence we may assume that $|f_1(\lambda)| = r|f_2(\lambda)|^k$. Then it suffices to show that $r^2|f_2(\lambda)|^{2k} \leq \frac{\rho}{2}(r|f_2(\lambda)|^k - |f_2(\lambda)|^{k+1}) \cdot |f_2(\lambda)|^k$. Therefore it is enough to show that $r^2 \leq \frac{\rho}{2}(r - |f_2(\lambda)|)$. We have seen in Corollary 1 that $|f_2(\lambda)| \leq r^2$. This means that it is enough to show that $r^2 \leq \frac{\rho}{2}(r - r^2)$ and this follows from our choice of $r$. □

This Lemma takes care of the case $1 \leq k \leq n-1$. It remains to deal with $k = 0$. That means that we have to show that for every $\lambda \in \overline{\Delta}_2 \setminus \mathcal{Z}$ that satisfies $|f_1(\lambda)| < r$ and $|f_2(\lambda)|^2 < r|f_1(\lambda)|$ we have $|f_1(\lambda)|^2 < \rho|f_2(\lambda) - f_1(\lambda)|$. This follows from the next Lemma.

**Lemma 5.** *For every $\lambda \in \overline{\Delta}_2$ we have $|f_1(\lambda)|^2 \leq \frac{\rho}{2}|f_2(\lambda) - f_1(\lambda)|$*

*Proof.* Exactly as in the proof of Lemma 4 we get that $\frac{f_1^2(\lambda)}{f_2(\lambda) - f_1(\lambda)}$ is holomorphic on a neighborhood of $\overline{\Delta}_2$. Hence we have to check the inequality only on $\partial\Delta_2$. That is, it suffices to show that $|f_1|^2 + \frac{\rho}{2}|f_2| \leq \frac{\rho}{2}|f_1|$ on $\partial\Delta_2$. This follows from Corollary 1 (note that the two terms appearing on the left-hand side of the inequality contain $\varepsilon^2$ and the one on right contains $\varepsilon$). □

**Step 3.** We show that the universal covering of $\tilde{W}$ (hence of $W$) does not satisfy the discrete disk property.

We will show first that $\tilde{W}_r^\rho$ is simply connected. As each $\tilde{W}_{r,\rho}^{(k)}$ is simply connected, it suffices to show that $\tilde{W}_{r,\rho}^{(k)} \cap \tilde{W}_{r,\rho}^{(k+1)} = U_r^{(k)} \cap U_r^{(k+1)}$ is connected for every $k \in \mathbb{Z}$. Note that for points in $U_r^{(k)} \cap U_r^{(k+1)}$ we have that $\xi_2^{(k)} \neq 0$, $\xi_1^{(k+1)} \neq 0$, $\xi_1^{(k+2)} \neq 0$. Hence $U_r^{(k)} \cap U_r^{(k+1)} \subset \mathbb{C}^2$ where the coordinate functions on $\mathbb{C}^2$ are $x := \frac{\xi_1^{(k)}}{\xi_2^{(k)}}$ and $y = \frac{\xi_2^{(k+1)}}{\xi_1^{(k+1)}}$. In this coordinates we have the following: $z_2^{(k)} = z_2^{(k+1)} = xy$, $z_1^{(k)} = x^2y$, $z_1^{(k+1)} = x$, $\frac{\xi_2^{(k+2)}}{\xi_1^{(k+2)}} = xy^2$. Therefore

$$U_r^{(k)} \cap U_r^{(k+1)} = \{(x,y) \in \mathbb{C}^2 : |y| < r,\ |x^2y| < r\} \cap \{(x,y) \in \mathbb{C}^2 : |xy^2| < r,\ |x| < r\}.$$

If $|x| < r$ and $|y| < r$ then $|x^2y| < r^3 < r$ and $|xy^2| < r^3 < r$ because we have assumed that $r < 1$. it follows that

$$U_r^{(k)} \cap U_r^{(k+1)} = \{(x,y) \in \mathbb{C}^2 : |y| < r,\ |x| < r\}.$$

In particular $U_r^{(k)} \cap U_r^{(k+1)}$ is connected (even contractible).

We proved that $g_n(\overline{\Delta}_2) \subset \tilde{W}_r^\rho$. It follows that the universal cover of $\tilde{W}$ (which contains $\tilde{W}_r^\rho$) does not satisfy the discrete disk property.

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