Real polynomial maps, $F : \mathbb{R}^{n+1} \to \mathbb{R}^n$.

Question: Can we detect the atypical values by variation in the topology of the fiber?

Theorem (Suzuki-Hà-Lê)

Let $F : \mathbb{C}^2 \to \mathbb{C}$ be a polynomial function and let $\lambda_0 \in \mathbb{C} \setminus F(\mathrm{Sing} F)$. Then $\lambda \notin \mathrm{Atyp}\, F$ if and only if the Euler characteristic of the fibres $\chi(F_\lambda)$ is constant for λ varying in some small neighbourhood of λ_0.

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Differences between the real and the complex case: Let $f : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{C}^2 \to \mathbb{C}$ be non-constant polynomial maps, $Z_f = \{(x, y) \in \mathbb{R}^2 : f(x, y) = 0\}$, $Z_g = \{(x, y) \in \mathbb{C}^2 : g(x, y) = 0\}$.

- $\dim_{\mathbb{R}}(Z_f) < \dim_{\mathbb{R}}(Z_g)$
- If Z_f is a smooth curve then each connected component is either diffeomorphic to $\mathbb{R}$ or S^1. The topology of Z_g could be more complicated.
- Z_g does not have compact connected components. Z_f or some of its components could be compact.
- Z_g does not have isolated points and it is never empty. Z_f might have isolated points or it might me empty.

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Example: (M. Tibăr, A. Zaharia, Asymptotic behavior of families of real curves. Manuscripta Math. 99 (1999), 383–393.)

$F : \mathbb{R}^2 \to \mathbb{R}$,
$F(x, y) =$
$x^2y^3(y^2 - 25)^2 + 2xy(y^2 - 25)(y + 25) - (y^4 + y^3 - 50y^2 - 51y + 575)$

- 0 is a regular value of F
- For $\lambda \in (-\varepsilon, \varepsilon)$, ε small enough, $F^{-1}(\lambda)$ has five connected components, each one of them diffeomorphic to $\mathbb{R}$.
- F is not a locally trivial fibration at 0.

- Why not?
- Two phenomena: "vanishing" and "splitting".
  - as $\lambda \to 0$ a connected component of $F^{-1}(\lambda)$ might "vanish".
  - as $\lambda \to 0$ a connected component of $F^{-1}(\lambda)$ might split in two connected components.
  - as $\lambda \to 0$ a compact connected component of $F^{-1}(\lambda)$ might become noncompact.

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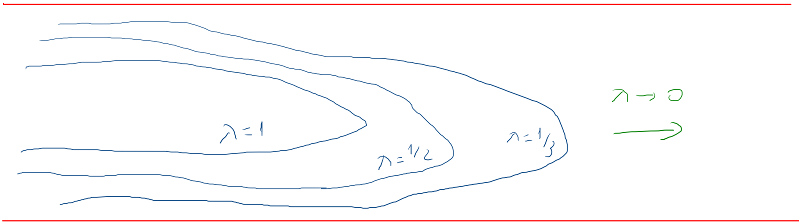
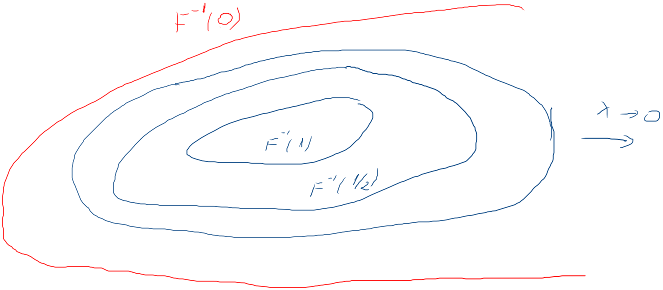
λ = 1
λ = 1/2
λ → 0


λ = 1
λ = 1/2
λ = 1/3
λ → 0



F⁻¹(0)
F⁻¹(1)
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For polynomial maps $F : \mathbb{R}^2 \to \mathbb{R}$:

• M. Tibăr, A. Zaharia, Asymptotic behavior of families of real curves. Manuscripta Math. 99 (1999), 383–393.

For polynomial maps $F : \mathbb{R}^{n+1} \to \mathbb{R}^n$, $n \geq 2$:

• C. Joița, M. Tibăr: Bifurcation values of families of real curves. Proceedings of the Royal Society of Edinburgh Section A: Mathematics 147 (2017), 1233–1242.

Definition (Vanishing)

We say that there are vanishing components at infinity when λ tends to λ_0 if there is a sequence of points $\lambda_k \in \mathbb{R}^n$, $\lambda_k \to \lambda_0$, such that for some choice of a connected component C_k of $F^{-1}(\lambda_k)$ the sequence of sets $\{C_k\}_{k\in\mathbb{N}}$ is locally finite.

(Locally finite: for any compact $K \subset M$, there is an integer $p_K \in \mathbb{N}$ such that $\forall q \geq p_K,\ C_q \cap K = \emptyset$).

If there are no vanishing components at infinity when λ tends to λ_0 we say that "there is no vanishing at λ_0" and we abbreviate by $NV(\lambda_0)$.

Exercise: If we have $NV(a)$ for some $a \in \mathbb{R}^n$ there exists a neighborhood U of a such that we have $NV(b)$ for every $b \in U$.

Definition

Let $\{M_k\}_k$ be a sequence of subsets of $\mathbb{R}^m$. A point $x \in \mathbb{R}^m$ is called a limit point of $\{M_k\}_k$ if there exist
- a sequence of integers $\{k_i\}_i \subset \mathbb{N}$ with $\lim_{i\to\infty} k_i = \infty$,
- a sequence of points $\{x_i\}_{i\in\mathbb{N}}$ with $x_i \in M_{k_i}$ such that $\lim_{i\to\infty} x_i = x$.

The set of all limit points of $\{M_k\}_k$ is denoted by $\lim M_k$.

Proposition

Suppose that $a \in \operatorname{Im} F \setminus \overline{F(\operatorname{Sing} F)}$ and $\{b_k\}_{k\in\mathbb{N}}$ is a sequence of points in $\mathbb{R}^n$ such that $b_k \to a$. For each k, let $C^j_{b_k}$ be a connected component of $F^{-1}(b_k)$. Then $\lim C^j_{b_k}$ is either empty or a union of connected components of $F^{-1}(a)$.

Definition

a) We say that there is no splitting at infinity at $a \in \mathbb{R}^n$, and we abbreviate this by $NS(a)$, if:

- for every sequence $\{b_k\}_{k\in\mathbb{N}}$ in $\mathbb{R}^n$ such that $b_k \to a$,
- for every sequence $\{p_k\}_{k\in\mathbb{N}}$ in $\mathbb{R}^{n+1}$ such that $F(p_k) = b_k$,

if $C^j_{b_k}$ denotes the connected component of $F^{-1}(b_k)$ which contains p_k, then the limit set $\lim C^j_{b_k}$ is connected.

b) We say that there is strong non-splitting at infinity at $a \in \mathbb{R}^n$, and we abbreviate this by $SNS(a)$, if in addition to the definition of $NS(a)$ we ask the following: if all the components $C^j_{b_k}$ are compact then the limit $\lim C^j_{b_k}$ is compact as well.

Theorem

Let $F : \mathbb{R}^{n+1} \to \mathbb{R}^n$ *be a polynomial map. Let* a *be an interior point of the set* $\mathrm{Im}\, F \setminus \overline{F(\mathrm{Sing} F)} \subset \mathbb{R}^n$ *. Then* $a \notin \mathrm{Atyp}\, F$ *if and only if the following two conditions are satisfied:*

- *the Euler characteristic* $\chi(F^{-1}(\lambda))$ *is constant when* λ *varies within some neighbourhood of* a*,*
- *there is no component of* $F^{-1}(\lambda)$ *which vanishes at infinity as* λ *tends to* a*.*

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Let $F : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be a polynomial map. Let a be an interior point of the set $\operatorname{Im} F \setminus \overline{F(\operatorname{Sing} F)} \subset \mathbb{R}^n$. Then $a \notin \operatorname{Atyp} F$ if and only if the following two conditions are satisfied:

- *the Betti numbers of $F^{-1}(\lambda)$ are constant for λ in some neighbourhood of a,*
- *there is no splitting at infinity at a.*

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- *there is no component of* $F^{-1}(\lambda)$ *which vanishes at infinity as* λ *tends to* a*.*
- *there is strong no splitting at infinity at* a*.*

To summarise:

- χ constant + $NV \implies$ locally trivial
- b_0 and b_1 constant + $NS \implies$ locally trivial
- $NV + SNS \implies$ locally trivial

A few words about the proofs.

Remarks:

- The connected components of $F^{-1}(\lambda)$ are lines or circles and hence the Euler characteristic counts the number of line components.
- As $\lambda \in \mathbb{R}^n$ approaches a, the compact connected components of $F^{-1}(\lambda)$ might survive or not but no new compact component is created because we are dealing with regular values.
- If we have χ constant + NV then we have also SNS: any splitting would create new line components. As no line component of $F^{-1}(\lambda)$ vanishes, the number of line components of $F^{-1}(a)$ would go up. This implies also that b_0 and b_1 are constant.
- If we have b_0 and b_1 constant + NS then we have also NV: since no new compact component is created, and b_1 is constant, no compact component vanishes. Also no compact component "mutate" into a noncompact one. This means that no new line component is created. As b_0 is constant, no line component vanishes either.

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To show local triviality around $a \in \mathbb{R}^n$, for each connected component C of $F^{-1}(a)$ we have to find an open neighborhood $U \supset C$ that does not intersect any other connected component and $F_| : U \to F(U)$ is a trivial fibration.

It easy to deal with compact components.

Proposition

Suppose that X and Y are manifolds and $f : X \to Y$ is a continuous function. If $b \in Y$ is a point such that $f^{-1}(b)$ is compact then there exists an open neighbprhood U of $f^{-1}(b)$ and an open neighborhood V of b such that $f(U) \subset V$ and the map $f_| : U \to V$ is proper.

Hence if C is compact connected component of $F^{-1}(a)$ we choose a neighborhood U of C such that $F_| : U \to F(U)$ is both a submersion and a proper map and we can apply Ehresmann fibration theorem.

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What about line components?

Proposition

Let M be a smooth manifold of dimension $m+1$ and $g : M \to \mathbb{R}^m$ be a smooth submersion without singularities and such that all its fibres $g^{-1}(t)$ are and diffeomorphic to $\mathbb{R}$. Then g is a C^∞ trivial fibration.

Let $C_a^1, \ldots, C_a^l$ be the components of $F^{-1}(a)$.
• For each $j = 1, \ldots, l$, we choose a point $z_j \in C_a^j$ and, we fix a small enough ball $B_j \ni z_j$ such that $B_j \cap F^{-1}(a)$ is connected and that the restriction of F to B_j is a trivial fibration.
• We may assume that the small balls $B_1, \ldots, B_l$ are pairwise disjoint. and hence for each $b \in \cap_j F(B_j)$, B_j intersects exactly one connected component of $F^{-1}(b)$.

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• We may assume that the small balls $B_1, \ldots, B_l$ are pairwise disjoint. and hence for each $b \in \cap_j F(B_j)$, B_j intersects exactly one connected component of $F^{-1}(b)$.

Therefore we can define a function Φ_b on the set $\{1, \ldots, l\}$ with values in the set of connected components of $F^{-1}(b)$ by setting $\Phi_b(j)$ to be the unique component of $F^{-1}(b)$ which intersects B_j.

The function Φ_b might be or might not be injective and it might be or might not be surjective.

Roughly speaking:
Failure of Φ_b to be surjective corresponds to vanishing.
Failure of Φ_b to be injective corresponds to splitting.

If b_0 is constant Φ is surjective iff it is injective iff it is bijective.

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Suspose that one of three sets of conditions holds:
- χ constant + NV,
- b_0 and b_1 constant + NS,
- NV + SNS.

Then:

- There exists an open neighborhood of a, $D \subset \mathbb{R}^n$ such that Φ_b is bijective for $b \in D$.
- We consider a line component C_a^i of $F^{-1}(a)$. Let $\mathcal{L}_i$ denote the union over all $b \in D$ of the connected components of $F^{-1}(b)$ which intersect B_i.
- Each such connected component of $F^{-1}(b)$ is a line component.

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- If C_a^i and C_a^j are two different components of $F^{-1}(a)$ then $\mathcal{L}_i \cap \mathcal{L}_j = \emptyset$.

- Each $\mathcal{L}_i$ is open.

- $F_| : \mathcal{L}_i \to F(\mathcal{L}_i)$ is a submersion that has all fibers diffeomorphic to $\mathbb{R}$.

- We deduce that $F_| : \mathcal{L}_i \to F(\mathcal{L}_i)$ is a trivial fibration.

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- If C_a^i and C_a^j are two different components of $F^{-1}(a)$ then $\mathcal{L}_i \cap \mathcal{L}_j = \emptyset$.

- Each $\mathcal{L}_i$ is open.

- $F_| : \mathcal{L}_i \to F(\mathcal{L}_i)$ is a submersion that has all fibers diffeomorphic to $\mathbb{R}$.

- We deduce that $F_| : \mathcal{L}_i \to F(\mathcal{L}_i)$ is a trivial fibration.