Real polynomial maps
$F : \mathbb{R}^2 \to \mathbb{R}$

Algorithmic Aspects

L. R. G. Dias, C. Joița, M. Tibăr: *Atypical points at infinity and algorithmic detection of the bifurcation locus of real polynomials*
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“Localisation at infinity”

We regard $\mathbb{R}^2$ as an open subset of the projective space $\mathbb{P}^2$:
- (x, y) are the coordinates in $\mathbb{R}^2$

- A point in $\mathbb{P}^2$ is an equivalence class $[x : y : z]$

- $\mathbb{R}^2$ is identified with $\{[x : y : z] \in \mathbb{P}^2 \mid z \neq 0\}$ through the map $(x, y) \to [x : y : 1]$

- $L^\infty := \{[x : y : z] \in \mathbb{P}^2 \mid z = 0\} \simeq \mathbb{P}^1$ denotes the line at infinity.

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A fiber of $F : \mathbb{R}^2 \to \mathbb{R}$, $\{(x, y) \in \mathbb{R}^2 : F(x, y) = t\}$, is compactified and we obtain $\{[x : y : z] \in \mathbb{P}^2 : \tilde{F}(x, y, z) = tz^d\}$.
Here $\tilde{F}$ denotes the homogenization of F in the variables z.

Note that all $\{[x : y : z] \in \mathbb{P}^2 : \tilde{F}(x, y, z) = tz^d\}$ are compact.

Even if $\{(x, y) \in \mathbb{R}^2 : F(x, y) = t\}$ was smooth, the compactification might introduce singularities at infinity.

We will use the theorem that says $(NV)+(SNS)$ is equivalent to local triviality.

Instead of talking about

vanishing and splitting "at infinity"

we would like to talk about

vanishing and splitting "at a given point infinity".

Remarks:

- In $\mathbb{R}^2$, compact connected components do not vanish.
- The two types of splitting: the splitting of a line component $F^{-1}(t)$ as $t \to a$ or the transformation of a compact component of $F^{-1}(t)$ into a line component are similar phenomena when seen from "infinity".

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We fix a point $(p, \lambda) \in L^\infty \times \mathbb{R}$. Without loss of generality we may assume that $p = [0 : 1 : 0]$.
We consider a local chart $U \simeq \mathbb{R}^2 \subset \mathbb{P}^2$ with origin at p.
Let $g_t : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$, $g_t(x, z) := \tilde{F}(x, 1, z) - tz^d$ and $C_t := \{g_t = 0\} = \{[x : y : z] \in \mathbb{P}^2 : \tilde{F}(x, y, z) = tz^d\} \cap U$.

Definition

a) We say that F has a splitting at (p, λ), shortly $(\mathrm{S}_{(p,\lambda)})$, if there is a small disk D_ε at $p \in U \simeq \mathbb{R}^2$ such that, for all $t \neq \lambda$ close enough to λ, $C_t \cap \overline{D}_\varepsilon$ has a connected component C_t^i such that $p \notin C_t^i$, $C_t^i \cap \partial D_\varepsilon \neq \emptyset$ and the Euclidean distance $\mathrm{dist}(C_t^i, p) \to 0$ as $t \to \lambda$.

b) We say that F has a *vanishing loop at* (p, λ), shortly $(\mathrm{V}_{(p,\lambda)})$, if there is a small disk D_ε at $p \in \mathbb{R}^2$ such that, for all $t \neq \lambda$ close enough to λ, $C_t \cap D_\varepsilon \setminus \{p\}$ has a non-empty connected component $C_t^i \setminus \{p\}$ with $C_t^i \cap \partial D_\varepsilon = \emptyset$ and $\lim_{t \to \lambda} C_t^i \cap D_\varepsilon = \{p\}$.

Theorem

A regular value λ is an atypical value of F if and only if there exists $p \in L^\infty$ such that we have either $(S_{(p,\lambda)})$ or $(V_{(p,\lambda)})$.

Remark: One can prove actually a more general statement in which one drops the regularity of the fibre $F^{-1}(\lambda)$ and allow $\mathrm{Sing} F^{-1}(\lambda)$ to be a compact set. In exchange we replace "atypical value" with the notion of "atypical values at infinity".

Searching for atypical points at infinity.

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Another useful way to look at the picture:

$\mathbb{X} := \{([x:y:z],t) \in \mathbb{P}^2 \times \mathbb{R} \mid \tilde{F}(x,y,z) - tz^d = 0\}$

$\mathbb{X}^\infty := \mathbb{X} \cap (L^\infty \times \mathbb{R})$

$\sigma : \mathbb{X} \to \mathbb{P}^2$ and $\tau : \mathbb{X} \to \mathbb{R}$ are the canonical projections

Note that $\sigma(\mathbb{X}^\infty)$ is a finite set.

Also note that the fiber of τ above t, $\tau^{-1}(t)$ is precisely $\{[x:y:z] \in \mathbb{P}^2 : \tilde{F}(x,y,z) = tz^d\}$, i.e. the compactification of $F^{-1}(t)$.

We identify $\mathbb{R}^2$ with an open subset of $\mathbb{X}$ via

$$(x,y) \mapsto (x,y,f(x,y)) \mapsto ([x:y:1], f(x,y)) \in \mathbb{X}$$

Definition

Let $\rho : \mathbb{R}^2 \to \mathbb{R}$, $\rho(x,y) = x^2 + y^2$, be the square of the Euclidean distance from the origin. The Milnor set of F, denoted by $M(F)$, is the critical locus of the mapping $(F,\rho)\colon \mathbb{R}^2 \to \mathbb{R}^2$.

Let $\overline{M(F)}^{\mathbb{X}} \subset \mathbb{X}$ be the closure in $\mathbb{X}$ of the Milnor set. It is an analytic set.

Theorem

The set of atypical points at infinity (i.e. those points (p,λ) where we have $(S_{(p,\lambda)})$ or $(V_{(p,\lambda)})$) is contained in the set $\mathcal{M} := \overline{M(F)}^{\mathbb{X}} \cap L^\infty \times \mathbb{R}$, in particular it is finite.

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Suppose that we have a point $(p, \lambda) \in \mathcal{M} \subset L^\infty \times \mathbb{R}$ and we want to decide if it is atypical or not.

We can choose a small disk D_ε at p such that:
- $(C_\lambda \cap D_\varepsilon) \setminus \{p\}$ is smooth
- all connected components of $C_\lambda \cap D_\varepsilon$ pass through p
- no connected components of $C_\lambda \cap D_\varepsilon$ is a loop.

Then, notice that:
- If there exists a sequence of points $t_k \to \lambda$ and for each k a connected component $C^i_{t_k}$ with $C^i_{t_k} \cap \partial D_\varepsilon \neq \emptyset$, $p \notin C^i_{t_k}$, then we must have $\mathrm{dist}(C^i_{t_k}, p) \to 0$. Hence splitting. If we do not have such a sequence, obviously we do not have splitting.
- If there exists a sequence of points $t_k \to \lambda$ such that each $C_{t_k} \cap D_\varepsilon \setminus \{p\}$ contains a loop that passes through p, then this loop has to "shrink" to p as $t_k \to \lambda$. Hence vanishing. If we do not have such a sequence, obviously we do not have vanishing.

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- If there exists a sequence of points $t_k \to \lambda$ such that each $C_{t_k} \cap D_\varepsilon \setminus \{p\}$ contains a loop that passes through p, then this loop has to "shrink" to p as $t_k \to \lambda$. Hence vanishing. If we do not have such a sequence, obviously we do not have vanishing.

This shows that we do not have to worry about checking $\mathrm{dist}(C_t^i, p) \to 0$ or $\lim_{t\to\lambda} C_t^i \cap D_\varepsilon = \{p\}$.
We have only to check "the structure" of each $C_t \cap D_\varepsilon$.

Problem: we cannot write an algorithm that checks this structure for every t in some interval.

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Theorem

Let $\lambda \in \mathbb{R}$, let $(p, \lambda) \in \mathcal{M}$ and let $\mathbb{R}^2$ be an affine chart of $\mathbb{P}^2$ containing $p \in L^\infty$. There exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon \leq \varepsilon_0$ the maps:

$$\tau_| : (\mathbb{X} \cap D_\varepsilon) \times (\lambda, \lambda + \delta) \longrightarrow (\lambda, \lambda + \delta)$$

$$\tau_| : (\mathbb{X} \cap D_\varepsilon) \times (\lambda - \delta, \lambda) \longrightarrow (\lambda - \delta, \lambda)$$

are locally trivial fibrations for any small enough δ depending on ε.

The choice of ε_0 and δ:
- $\varepsilon_0 > 0$ such that the circles ∂D_ε are transversal to C_λ for every $0 < \varepsilon \leq \varepsilon_0$
- If we fix ε (for example for $\varepsilon = \varepsilon_0$), we choose $\delta = \delta(\varepsilon)$ such that ∂D_ε is transversal to C_t for every $t \in (\lambda - \delta, \lambda + \delta)$.

Corollary

Let ε_0 *and* $\delta(\varepsilon_0) > 0$ *be such that the previous theorem holds. Let* $t_0 \in (\lambda - \delta, \lambda) \cup (\lambda, \lambda + \delta)$ *Then:*

- $C_{t_0} \cap D_\varepsilon$ *has a loop at 0 if and only if* $C_t \cap D_\varepsilon$ *has a loop at 0, for all* t *in the same interval as* t_0*.*
- $C_{t_0} \cap D_\varepsilon$ *has a connected component not containing* p *if and only if* $C_t \cap D_\varepsilon$ *has a connected component not containing* p*, for all* t *in the same interval as* t_0*.*

Consequence: we have to check for loops or connected components not containing p only at one point in the interval $(\lambda - \delta, \lambda)$ and one point in the interval $(\lambda, \lambda + \delta)$

The main steps of the algorithmic procedures:

• Find a finite set of points $\mathcal{B}$ containing the set $\mathcal{M}$.

• For each such point $(p, \lambda) \in \mathcal{B}$, find an effective tube neighbourhood $D_\varepsilon \times (-\delta + \lambda, \delta + \lambda)$

• Test the existence of vanishing or of splitting at some point $(p, \lambda) \in \mathcal{B}$.

- Finding the set $\mathcal{B}$ containing $\mathcal{M}$.

- We first determine the intersection of $\overline{M(F)}^{\mathbb{P}^2} \subset \mathbb{P}^2$ with the line at infinity L^∞.
(Homogenise with the variable z the equation $y\frac{\partial F}{\partial x}(x,y) - x\frac{\partial F}{\partial y}(x,y) = 0$ and then take $z = 0$.)

- For each point $p \in \overline{M(F)}^{\mathbb{P}^2} \cap L^\infty$ we determine the finite subset $\tau(\overline{M(F)}^{\mathbb{P}^2} \cap \{p\} \times \mathbb{R}) \subset \mathbb{R}$.

This set is:
$S_p(F) = \{t \mid \exists\{(x_j, y_j)\}_{j\in\mathbb{N}} \subset M(F), [x_j : y_j : 1] \to p,\ F(x_j, y_j) \to t\}$.

Let $p = [0 : 1 : 0] \in L^\infty$ and work in the chart $\{y \neq 0\}$ with coordinates (x, z).

Let $\hat{F}(x,z) := \tilde{F}(x, 1, z)$. Then $\lim_{[x:y:1]\to p} F(x,y) = \lim_{z\to 0} \frac{\hat{F}(x,z)}{z^d}$.

Let $\hat{h}(x,z) = 0$ be the equation that defines $\overline{M(F)}$ around p and let d be the degree of F.

Finding $S_p(F)$ is equivalent to the following:
find all limits $\lim \frac{\hat{F}(x,z)}{z^d}$ for $(x,z) \to 0$ along $\hat{h}(x,z) = 0$.

We pass to complex variables. At the end we will select only the real values.
We find all the Puiseux roots of $\hat{h}(x,z) = 0$ using the Newton-Puiseux algorithm.

A Puiseux parametrisation of some root of $\hat{h}(x,z) = 0$ looks like $z = T^n$, $x = \sum_{j \geq 1} \lambda_j T^j$, and the series may be infinite even if $\hat{h}(x,z)$ is a polynomial.

However

$$\lim_{T\to 0} \frac{\hat{F}(\sum_{j\geq 1} \lambda_k T^j, T^n)}{T^{dn}} = \lim_{T\to 0} \frac{\hat{F}(\sum_{1\leq j\leq dn} \lambda_j T^j, T^n)}{T^{dn}}.$$

Hence, we only need the first dn terms in the expansion of x of it to compute the limit.

We still need to find the value of n, hence to give a bound for the number of steps in the Newton-Puiseux process.

Let $\hat{h}(x, z)$ be general in x of order $k > 0$ (i.e. k is the lowest point on the x-axis in the Newton polygon of $\hat{h}$).

One can show that $n = \min\{k!, d\}$ works and hence we have to run at most $d \cdot \min\{k!, d\}$ steps in the Newton-Puiseux algorithm.

• Finding an effective tube neighbourhood $D_\varepsilon \times (-\delta + \lambda, \delta + \lambda)$ for $(p, \lambda) \in \mathcal{B}$.

Again we assume that we work in an affine chart with $p = [0 : 1 : 0]$ as the origin.

We have to find $\varepsilon_0 > 0$ and $\delta = \delta(\varepsilon_0) > 0$ such that:

(A) for every $0 < \varepsilon \leq \varepsilon_0$, the circle ∂D_ε intersects $\{g_\lambda = 0\} \subset \mathbb{R}^2$ transversally, and

(B) $\partial D_{\varepsilon_0}$ intersects $\{g_t = 0\}$ transversally, for every $t \in (\lambda - \delta, \lambda + \delta)$.

For (A) we find the distance:

$$\Delta_{(p,\lambda)} := \operatorname{dist}\left((0,0), \left\{z\frac{\partial g_\lambda}{\partial x}(x,z) - x\frac{\partial g_\lambda}{\partial z}(x,z) = 0\right\} \cap \{g_\lambda = 0\}\right),$$

and we may then take $\varepsilon_0 := \Delta_{(p,\lambda)}/2 > 0$.

After fixing ε_0 we compute minimum $s > 0$ of the function $h(x,z) := g_\lambda^2 + \left(z\frac{\partial g_\lambda}{\partial x}(x,z) - x\frac{\partial g_\lambda}{\partial z}(x,z)\right)^2$ on $\partial D_{\varepsilon_0}$.

One can show that $\delta(\varepsilon_0) := \frac{\sqrt{s}}{\varepsilon_0^d(d+1)}$ satisfies condition (B).

• Testing the existence of vanishing or of splitting at some point $(p, \lambda) \in \mathcal{B}$.

Once that we have ε_0 and δ we may now choose any value t in $(\lambda - \delta, \lambda)$ and in $(\lambda, \lambda + \delta)$. For example, the mid-points.

For these points we have to check for loops or connected components not containing p.

One can write an algorithm doing that by following closely the one given in Section 11.6 of

S. Basu, R. Pollack, M-F. Roy, Algorithms in real algebraic geometry. Second edition. Algorithms and Computation in Mathematics, 10. Springer-Verlag, Berlin, 2006.