Complex polynomial maps, $F : \mathbb{C}^{n+1} \to \mathbb{C}^n$, $n \geq 2$.

- Question: Can we detect the atypical values of F using the variation in the topology of the fiber?

- You have seen that for $n = 1$, i.e. for polynomial functions $F : \mathbb{C}^2 \to \mathbb{C}$ the atypical values can be detected using the variation of the Euler characteristic of the fibers (Suzuki-Hà-Le Theorem). We assume that $n \geq 2$.

- Question: If $\chi(F^{-1}(\lambda))$ is constant in a neighborhood of λ_0, does it follow that $\lambda_0 \notin \operatorname{Atyp} F$?

- Note that if λ is a regular value then $F^{-1}(\lambda)$ is an open Riemann surface, i.e. $\dim_{\mathbb{C}} F^{-1}(\lambda) = 1$, $\dim_{\mathbb{R}} F^{-1}(\lambda) = 2$. Hence we are dealing with only two Betti numbers $b_0(F^{-1}(\lambda)$ and $b_1(F^{-1}(\lambda)$.

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Example: Hà H.V., Nguyen T.T., On the topology of polynomial mappings from $\mathbb{C}^n$ to $\mathbb{C}^{n-1}$. Internat. J. Math. 22 (2011), 435–448.

$F : \mathbb{C}^3 \to \mathbb{C}^2$, $F(x, y, z) = (xy + 1, (xyz + 1)(xyz + z - 1))$.
Then:
- $(0, 0) \in \mathbb{C}^2$ is a regular value;
- $F^{-1}(0, 0) = \mathbb{C}^*$,
- for a generic λ, $F^{-1}(\lambda) = \mathbb{C}^* \sqcup \mathbb{C}^*$

Exercise: prove it!

$\chi(\mathbb{C}^*) = \chi(\mathbb{C}^* \sqcup \mathbb{C}^*) = 0$

- This shows that the Euler characteristic is a too "coarse" invariant. What about the Betti numbers?

- Example: C. Joița, M. Tibăr: Bifurcation set of multi-parameter families of complex curves. Journal of Topology 11 (2018), 739–751.
$F : \mathbb{C}^3 \to \mathbb{C}^2$,
$F(x, y, z) = \big(x, [(x-1)(xz + y^2) + 1][x(xz + y^2) - 1]\big)$.

  Then $(0, 0) \in \mathbb{C}^2$ is a regular value and all fibers are isomorphic

$$F^{-1}(\lambda) \simeq \mathbb{C} \sqcup \mathbb{C}$$

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- $(0,0)$ is not a typical value for F!

- Why not?
- Similar phenomenon as in the real case:
- As $\lambda \to (0,0)$, one connected component of $F^{-1}(\lambda)$ (i.e. one copy of $\mathbb{C}$) "vanishes".
- The other connected component of $F^{-1}(\lambda)$ "splits" in the two connected components of $F^{-1}(0,0)$.

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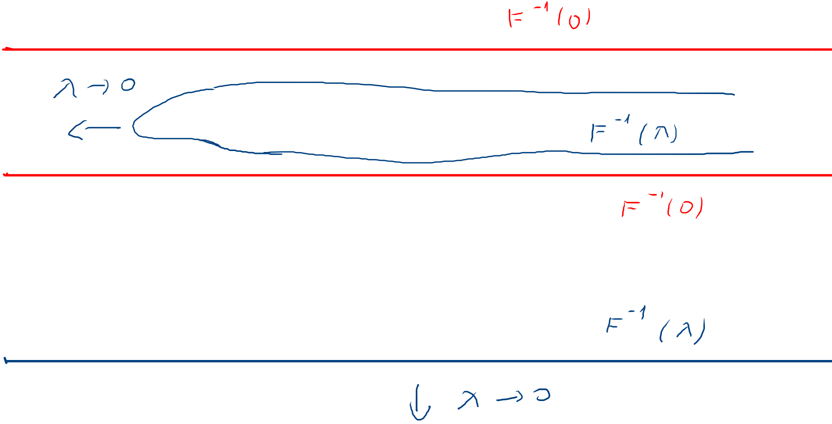

F^{-1}(0)
λ → 0
F^{-1}(λ)
F^{-1}(0)
F^{-1}(λ)
λ → 0

Question: What if we do not have vanishing?

Theorem (C. Joița, M. Tibăr)

Let λ_0 be an interior point of the set $\operatorname{Im} F \setminus \overline{F(\operatorname{Sing} F)}$. Then $\lambda_0 \notin \operatorname{Atyp} F$ if and only if the Euler characteristic of the fibres is constant for λ varying in some neighborhood of λ_0 and no connected component of $F^{-1}(\lambda)$ is vanishing at infinity when $\lambda \to \lambda_0$.

- The proof is more difficult and requires a more sophisticated machinery.

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One important step in the proof:

Theorem

Let $p : M \to B$ be a holomorphic map between connected complex manifolds, $\dim_{\mathbb{C}} M = \dim_{\mathbb{C}} B + 1$, and λ_0 be an interior point of the set $\mathrm{Im}\, p \setminus \overline{p(\mathrm{Sing}\, p)}$. We asume that:

- *the fibers are connected,*
- *the Betti numbers $b_0(\lambda)$ and $b_1(\lambda)$ of every fiber $p^{-1}(\lambda)$ are finite,*
- *M is Stein.*

Then $\lambda_0 \notin \mathrm{Atyp}\, p$ if and only if the Euler characteristic of the fibres is constant for λ varying in some neighborhood of λ_0.

Main ingredients of the proof

C.F.B. Palmeira, Open manifolds foliated by planes, Ann. of Math. 107 (1978), 109–131.
G. Meigniez: Submersions, fibrations and bundles. Trans. Amer. Math. Soc. 354 (2002), 3771–3787.

Theorem

A surjective smooth submersion with all fibres diffeomorphic to $\mathbb{R}^p$ is a locally trivial fibration.

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Theorem

A surjective smooth submersion with all fibres diffeomorphic to $\mathbb{R}^p$ *is a locally trivial fibration.*

- In our case the fibers of $p : M \to B$ are smooth and have complex dimension 1. Hence they are Riemann surfaces.
- In general they are not diffeomorphic to $\mathbb{R}^2$ because they are not simply connected.
- We can look at their universal cover.
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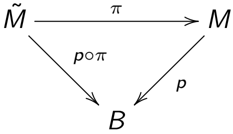
Y. S. Ilyashenko, Covering manifolds for analytic families of leaves of foliations by analytic curves. Topol. Methods Nonlinear Anal. 11 (1998), 361–373.
Y. S. Ilyashenko, Foliations by analytic curves. Mat. Sb. (N.S.) 88 (130) (1972), 558–577.

Theorem

Let $p : M \to B$ be a holomorphic surjective submersion between connected complex manifolds such that $\dim_{\mathbb{C}} M = \dim_{\mathbb{C}} B + 1$ and M is Stein.
Then there exist a complex manifold $\widetilde{M}$ and a locally biholomorphic map $\pi : \widetilde{M} \to M$ such that the restriction $\pi_{|} : \pi^{-1}(p^{-1}(x)) \to p^{-1}(x)$ is the universal cover of $p^{-1}(x)$, for every $x \in B$.

In particular the fibers of $p \circ \pi : \widetilde{M} \to B$ are diffeomorphic to $\mathbb{R}^2$.

Therefore we have the following diagram



We know that $p \circ \pi$ is a smooth submersion with fibers diffeomorphic to $\mathbb{R}^2$.
By Palmeira-Meigniez theorem, $p \circ \pi$ is a locally trivial fibration.

We want to prove that p is a locally trivial fibration.

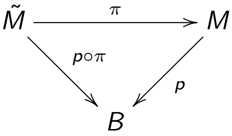
Definition

A map $\pi : E \to B$ between two manifolds is called a Serre fibration if for any simplex X, any homotopy $f : X \times [0,1] \to B$ and $\tilde{f}_0$ a lifting of $f_{|X\times\{0\}}$, there exists a lifting $\tilde{f} : X \times [0,1] \to E$ of f such that $\tilde{f}_{|X\times\{0\}} = \tilde{f}_0$.

Proposition (G. Meigniez)

Suppose that M and B are smooth manifolds such that $\dim_{\mathbb{R}} M = \dim_{\mathbb{R}} B + 2$ and $p : M \to B$ is a surjective smooth map. If p is both a submersion and a Serre fibration then $p : M \to B$ is a locally trivial fibre bundle.

- Exercise: if in the diagram



both π and $p \circ \pi$ are Serre fibrations then p is a Serre fibration.

- We need to show that π is a Serre fibration. Suffices show that π is a covering map.
- We need some sort of "control" of the fundamental group of $p^{-1}(x)$ when x moves in B.

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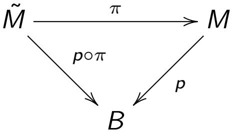

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$$
\begin{array}{ccc}
\tilde{M} & \xrightarrow{\pi} & M \\
& {\scriptstyle p\circ\pi} \searrow \quad \swarrow {\scriptstyle p} & \\
& B &
\end{array}
$$

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- Every smooth real manifold can embedded as a closed submanifold in some affine space $\mathbb{R}^N$.
- Not true for complex manifolds. E.g. compact complex manifolds cannot be embedded in $\mathbb{C}^N$.

- A complex manifold is called Stein if it can embedded as a closed complex submanifold in some affine space $\mathbb{C}^N$.

- Examples:
  - A compact manifold of positive dimension is not Stein.
  - $\mathbb{C}^n \setminus \{0\}$ is not Stein.
  - Any convex open subset of $\mathbb{C}^n$ is Stein.
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- Suppose that M is Stein complex manifold and $\Omega \subset M$ is a Stein open subset. Ω is called Runge in M if every holomorphic function $f : \Omega \to \mathbb{C}$ can be approximated uniformly on compacts with holomorphic functions defined on M.
- Suppose that M is Stein complex manifold and $\Omega \subset M$ is a Stein open subset which is Runge in M. If $N \subset M$ is a closed submanifold then $N \cap \Omega$ is Runge in N.
- If S is an open Riemann surface and D is an open subset, then D is Runge in S if and only if the morphism $H_1(D, G) \to H_1(S, G)$ induced by the inclusion $D \hookrightarrow S$ is injective for any abelian group G.

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Corollary

We consider a polynomial map $F : \mathbb{C}^{n+1} \to \mathbb{C}^n$*.*
We denote by $\overline{F}_\lambda$ *the closure of the fibre* $F^{-1}(\lambda)$ *in* $\mathbb{P}^{n+1}$ *in some fixed system of coordinates.*
Let λ_0 *be an interior point of the set* $\mathrm{Im}\, F \setminus \overline{F(\mathrm{Sing} F)} \subset \mathbb{C}^n$*.*
If the degree $\deg \overline{F}_\lambda$ *and the Euler characteristic* $\chi(F^{-1}(\lambda))$ *are constant for* λ *varying in some neighborhood of* λ_0*, then* $\lambda_0 \notin \mathrm{Atyp}\, F$*.*

Note: If $n = 1$ then F is just a polynomial function and $\deg \overline{F}_\lambda$ is equal to the degree of the polynomial F for every λ.

Corollary

Let λ_0 *be a point in the interior of the set* $\mathrm{Im}\, F \setminus \overline{F(\mathrm{Sing} F)} \subset \mathbb{C}^n$*.*

- *Assume that for* λ *in some neighborhood of* λ_0 *the fibers* $F^{-1}(\lambda)$ *have constant Euler characteristic and the Betti number* b_1 *of each of their components is at least 2. Then* $\lambda_0 \notin \mathrm{Atyp}\, F$*.*
- *Assume that for* λ *in some neighborhood of* λ_0 *the fibers* $F^{-1}(\lambda)$ *have constant Betti numbers* $b_0(t)$ *and* $b_1(t)$ *and the Betti number* b_1 *of each of their components is at least 1. Then* $\lambda_0 \notin \mathrm{Atyp}\, F$*.*