Scientific report

for the project

PN-II-ID-PCE-2011-3-0533

for the period

December 2013 – December 2014

In this project we investigate the space of all Alexandrov surfaces. The importance of Alexandrov spaces stems mainly from the generality of the concept, which allows both differentiable and non-differentiable manifolds to be included in the investigation. An important class of Alexandrov surfaces is that of convex surfaces. Part of our research is devoted to them. In many applications, precisely the non-differentiable case, particularly the theory of polyhedra, including the graphs which are their 1-skeleta, is the prevailing one. Thus, the study of graphs essentially appearing in computer networks is also included in our project. Generalized Halin graphs and Toeplitz graphs are among those studied. Moreover, generalized convexity is also investigated. Concerning the space of Alexandrov surfaces, we tackle approximation issues and undertake a generic study (via Baire categories).

In the period December 2013 – December 2014:
- three scientific articles have been published, accepted before 2014, in ISI journals, summing up to an Impact Factor of $\text{FI} = 1.476$ and a Relative Influence Score of $\text{SRI} = 2.704$;
- eight scientific articles have been accepted in ISI journals, summing up to an Impact Factor of $\text{FI} = 5.389$ and a Relative Influence Score of $\text{SRI} = 11.089$;
- an article has been accepted in a BDI journal, and other two in volumes to appear in Springer series Proc. in Math. & Stat.;
- other four papers have been submitted.

All articles are mentioning the grant.

The members of the research team had 3 talks at 2 international conferences, in the mentioned period. A part of the obtained results have been discussed at the geometry seminar of the host institute (IMAR), as well as in a talk at Université de Haute Alsace (Mulhouse).

Other details concerning the scientific activity of the research team, not included in this report, can be found on the grant’s web page, „http://imar.ro/~cvilcu/Web_0533.html”. For example, one can find there links for articles (to the journals which published them or to other places for downloading) and for talks (to the corresponding conferences).

Articles published in 2014, accepted before 2014

1. T. Zamfirescu: Right convexity,
J. Convex Analysis 21 (2014), 253-260 ; ISI, FI= 0,592, SRI=0,850.
A convex set is F-convex if every pair of points in the set lie in a right triangle included in the set. We characterize F-convex sets, find some classes of F-convex sets, investigate F-convexity for cones and cylinders, and find out that most convex bodies are F-convex. On our way, we also describe the curvature at the endpoints of diameters of most convex bodies.


We establish that certain classes of simple, closed, polygonal curves on the surface of a convex polyhedron develop in the plane without overlap. Our primary proof technique shows that such curves “live on a cone,” and then develops the curves by cutting the cone along a “generator” and flattening the cone in the plane. The conical existence results support a type of source unfolding of the surface of a polyhedron, described elsewhere.


In this note we describe some geometrical properties that simplicially convex bodies typically enjoy. It is shown, for example, that they are nowhere dense and of measure zero. Moreover, they look at least half-dense from any of their points.

**Articles accepted in 2014**


No hypohamiltonian graphs are embeddable in the planar square lattice. This lattice contains, however, graphs in which every vertex is missed by some longest cycle. In this paper we present graphs with this property, embeddable in various lattices, and of remarkably small order.


We demonstrate that, in most Alexandrov surfaces of curvature bounded below, most points are not interior to any geodesic. Thus, these surfaces are not Riemannian, in contrast to the “almost Riemannian” structure found by Otsu–Shioya in any Alexandrov space.


We prove that the complement of any ane 2-arrangement in Rd is minimal, that is, it is homotopy equivalent to a cell complex with as many i-cells as its i-th rational Betti number. For the proof, we provide a Lefschetz-type hyperplane theorem for complements of 2-arrangements, and introduce Alexander duality for combinatorial Morse functions. Our results greatly generalize previous work by Falk, Dimca--Papadima, Hattori, Randell, and Salvetti--Settepanella and others, and they demonstrate that in contrast to previous investigations, a purely combinatorial approach suces to show minimality and the Lefschetz Hyperplane Theorem for complements of complex hyperplane arrangements.
We prove an Alexander-type duality for valuations for certain subcomplexes in the boundary of polyhedra. These strengthen and simplify results of Stanley (1974) and Miller-Reiner (2005). We give a generalization of Brion’s theorem for this relative situation and we discuss the topology of the possible subcomplexes for which the duality relation holds.

Using an intuition from metric geometry, we prove that any ag normal simplicial complex satisfies the non-revisiting path conjecture. As a consequence, the diameter of its facet-ridge graph is smaller than the number of vertices minus the dimension, as in the Hirsch conjecture. This proves the Hirsch conjecture for all ag polytopes, and more generally, for all (connected) flag homology manifolds.

We prove that every polytope described by algebraic coordinates is the face of a projectively unique polytope. This provides a universality property for projectively unique polytopes. Using a closely related result of Below, we construct a combinatorial type of 5-dimensional polytope that is not realizable as a subpolytope of any stacked polytope. This disproves a classical conjecture in polytope theory, first formulated by Shephard in the seventies.

7. L. Yuan, T. Zamfirescu: Right triple convex completion, J. Convex Analysis, to appear; ISI, FI= 0,592, SRI= 0,850.
A set M in a Hilbert space is rt-convex if every pair of its points is included in a 3-point subset {x; y; z} of M making a right angle. We find here for various families of sets the minimal number of points necessary to add to the sets in order to render them rt-convex. For example, for convex bodies this number is at most 2.

We show that, in the sense of Baire categories, a typical Alexandrov surface with curvature bounded below by κ has no conical points. We use this result to prove that, on such a surface (unless it is flat), at a typical point, the lower and the upper Gaussian curvatures are equal to κ and ∞ respectively.

We show that the Theorem of the Three Perpendiculars holds in any finite-dimensional space form.

Denote by \( A(\kappa) \) the set of all compact Alexandrov surfaces with curvature bounded below by \( \kappa \) without boundary, endowed with the topology induced by the Gromov–Hausdorff metric. We determine the connected components of \( A(\kappa) \) and of its closure.


A Toeplitz graph is a graph with a Toeplitz adjacency matrix. In this paper we investigate the property of hamiltonian connectedness for some undirected Toeplitz graphs.

**Articles submitted for publication in 2014**


   In this paper we settle long-standing questions regarding the combinatorial complexity of Minkowski sums of polytopes: We give a tight upper bound for the number of faces of a Minkowski sum, including a characterization of the case of equality. We similarly give a (tight) upper bound theorem for mixed faces of Minkowski sums. This has a wide range of applications and generalizes the classical the Upper Bound Theorems of McMullen and Stanley. Our main tool is relative Stanley–Reisner theory, a powerful generalization of the algebraic theory of simplicial complexes inaugurated by Hochster, Reisner, and Stanley. A key feature of our theory is the ability to accomodate topological as well as combinatorial restrictions. We illustrate this by providing several simplicial isoperimetric and reverse isoperimetric inequalities.


   The purpose of this paper is to establish analogues of the classical Lefschetz Section Theorem for smooth tropical varieties. More precisely, we prove tropical analogues of the section theorems of Lefschetz, Andreotti–Frankel, Bott–Milnor–Thom, Hamm–Lê and Kodaira–Spencer, and the vanishing theorems of Andreotti–Frankel and Akizuki–Kodaira–Nakano. We start the paper by resolving a conjecture of Mikhalkin and Ziegler (2008) concerning the homotopy Cohen–Macaulayness of certain filtrations of geometric lattices, generalizing earlier work on full geometric lattices by Folkman and others. This translates to a crucial index estimate for the stratified Morse data at critical points of the tropical variety, and it can also by itself be interpreted as a Lefschetz-type theorem for matroids.


   We study the existence of simple closed geodesics on most (in the sense of Baire category) Alexandrov surfaces with curvature bounded below, compact and without boundary. We show that it depends on both the curvature bound and the topology of the surfaces.
4. J. Rouyer, C. Vîlcu: *Farthest points on most Alexandrov surfaces*,
We study global maxima of distance functions on most Alexandrov surfaces with curvature bounded below, where most is used in the sense of Baire categories.

**Talks at international conferences and workshops**

1. J. Rouyer, *Geometry of most Alexandrov surfaces*,
   4th Workshop for Young Researchers in Mathematics, Constanta, Romania, May 2014.

2. C. Vîlcu, *Farthest points on most Alexandrov surfaces*,
   4th Workshop for Young Researchers in Mathematics, Constanta, Romania, May 2014.

3. J. Rouyer, *Geometry of most Alexandrov surfaces*,
   Geometry Conference, Mulhouse, France, September 2014.

Project Director,
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