COHERENT STATE MAP QUANTIZATION IN A HERMITIAN-LIKE SETTING

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Abstract. For vector bundles having an involution on the base space, Hermitian-like structures are defined in terms of such an involution. We prove a universality theorem for suitable self-involutive reproducing kernels on Hermitian-like vector bundles. This result relies on pullback operations involving the tautological bundle on the Grassmann manifold of a Hilbert space and exhibits the aforementioned reproducing kernels as pullbacks of universal reproducing kernels that live on the Hermitian-like tautological bundle. To this end we use a certain type of classifying morphisms, which are geometric versions of the coherent state maps from quantum theory. As a consequence of that theorem, we obtain some differential geometric properties of these reproducing kernels in this setting.

1. Introduction

This paper belongs to a line of research ([BG08], [BG09], [BG11], [BG14], [BG15]) on differential geometric aspects of reproducing kernels and their related structures. As explained below, the present development was suggested by some problems that claim their origin in the broad interaction between the reproducing kernels and the quantum theory, more precisely in the applications of the coherent state method such as it is proposed and carried out in [Od88], [Od92], [Od07], [HO13], among other papers.

Coherent state maps. By coherent state map we mean a smooth (symplectic) map $\zeta\colon Z \to \mathbb{CP}(\mathcal{H})$ of a (finite-dimensional Kähler) manifold Z into a complex projective Hilbert space $\mathbb{CP}(\mathcal{H})$, for some complex Hilbert space $\mathcal{H}$. (Recall that $\mathbb{CP}(\mathcal{H})$ may well be viewed as the Grassmannian manifold on $\mathcal{H}$ formed by the one-dimensional subspaces of $\mathcal{H}$.) In usual physical interpretations, Z is to be regarded as the classical phase space of a mechanical system and $\mathbb{CP}(\mathcal{H})$ as the space of pure quantum states. The transition probability amplitudes in Z can be expressed in terms of a reproducing kernel K defined on the line bundle $\mathbb{L} \to Z$ obtained as the pullback of the tautological bundle $\mathbb{E} \to \mathbb{CP}(\mathcal{H})$, through the map ζ. Indeed, the existence of the kernel K is equivalent to the existence of the map ζ, and this fact can be formulated in terms of appropriate equivalent categories. In this

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way, the (Fubini-Study) Hermitian and complex structures on $\mathbb{E} \to \mathbb{CP}(\mathcal{H})$ induce corresponding Hermitian and complex structures, and associated Chern covariant derivative, on the bundle $\mathbb{L} \to Z$; see [Od88], [Od92]. The area of application of the coherent state method has been widened to abstract settings of quantization involving either polarized C^*-algebras ([Od07]) or positive kernels on bundles acted on by compact groups ([HO13]). In the latter case the space $\mathbb{CP}(\mathcal{H})$ is replaced by the n-Grassmannian on $\mathcal{H}$ formed by all the n-dimensional subspaces of $\mathcal{H}$, with fixed $n \in \mathbb{N}$. For links between coherent state maps and algebraic geometry, see [Be97], [BeSc00], and also [MP97]. Other applications of reproducing kernels in physics can be seen in [ABG13], [Wi89], for example.

Primary motivation of the present research. Our aim has been the study of quantization problems related to the above circle of ideas, in an infinite-dimensional setting, from the perspective of operator theory or operator algebras and their physical implications. The mathematical framework of this approach is provided by differential geometry of infinite-dimensional manifolds and Banach-Lie groups acting on these manifolds. In this connection, some references related in spirit to the present investigation are [Tu07], [DGP13], [PS11], [AL09], [ALRS97], and also the measure-free approach of [HzSz12], since the absence of suitable measures for defining L^2-spaces is one of the difficulties encountered in infinite dimensions, which can be addressed by using reproducing kernel Hilbert spaces instead. It is worth mentioning that techniques from this area of infinite-dimensional geometry also turned out to be relevant for applications in quantum chemistry ([ChMe12]), the study of Berry's geometrical phase factor ([CoMa01]), quantum optics ([HOT03]), etc.

One motivation for the proposed study relies on the general observation that the notions involved in the coherent state method have significant physical meaning in infinite dimensions. See for example [FKN92], [MR88], [Wi88] and references therein, for infinite-dimensional Grassmannians. More specific and direct motivation for the present paper comes from the geometric models of Borel-Weil type for representations of unitary groups of C^*-algebras, which were obtained in [BR07] by using reproducing kernels on suitable Hermitian homogeneous vector bundles. Such kernels yield the representation Hilbert spaces in the geometric models. In order to include holomorphy and full groups of invertible elements of C^*-algebras in that theory, the notion of like-Hermitian vector bundle was introduced in [BG08] in close relationship with the existence of involutions on the base spaces of those bundles and with the corresponding complexifications. An approach to these topics in the framework of category theory was developed in [BG11], where several universality results were established which allow us to recover the reproducing kernels from the tautological ones associated with universal Grassmann vector bundles. In this respect, the above-mentioned projective manifold $\mathbb{CP}(\mathcal{H})$ is substitued by the full Grassmannian manifold $\mathrm{Gr}(\mathcal{H})$ on $\mathcal{H}$, the line bundle $\mathbb{E} \to \mathbb{CP}(\mathcal{H})$ by the tautological bundle $\mathcal{T}(\mathcal{H}) \to \mathrm{Gr}(\mathcal{H})$ and the coherent state map $\zeta\colon Z \to \mathbb{CP}(\mathcal{H})$ by a classifying morphism $\zeta\colon Z \to \mathrm{Gr}(\mathcal{H})$; (it is to be noticed that infinite-dimensional Grassmannian manifolds are usually considered in physics as classical phase spaces, whereas they play here the role of a quantization tool) see [BG11, Sect. 5].

Technical aspects: universality and geometry of reproducing kernels. On the other hand, reproducing kernels and connections or covariant derivatives and associated geometric objects often occur simultaneously on vector bundles, as it happens in the setting worked on in [Od88] and [Od92], for instance. Thus it sounds sensible to find out possible explanations for such an occurrence in general. In the Hermitian case, we have recently approached that question by relying on the basic idea of transferring the geometry of the tautological bundle $\mathcal{T}(\mathcal{H}) \to \mathrm{Gr}(\mathcal{H})$ to the given bundle, say $D \to Z$, on the basis of the universality theorems and corresponding classifying morphisms. See [BG14] for the construction of the natural connection induced by a given reproducing kernel K, and the calculation of the corresponding covariant derivative ∇_K. It is shown in [BG15] that ∇_K is essentially the Chern covariant derivative for $D \to Z$; that is, the connection which is compatible both with the Hermitian and the complex structure (if there is one) in $D \to Z$. Some possible links with algebraic geometry are also implicit in [BG15].

As is well known, complex structures are very important both in quantization of mechanical systems and in Borel-Weil realizations. Some examples of the homogeneous spaces and vector bundles considered in [BR07], in relation with unitary representations, are indeed holomorphic, but this is not true in general. In order to include holomorphy and *full* groups of invertible elements of C^*-algebras in the theory of [BR07], the notion of like-Hermitian vector bundle was introduced and studied in [BG08] in close relationship with the existence of involutions on the base spaces of those bundles and with the corresponding complexifications. Moreover, the like-Hermitian bundles have been discussed from a categorial viewpoint in [BG11]. However, the universality theorem for reproducing kernels on like-Hermitian bundles, given in [BG11, Th. 5.1], is not entirely satisfactory in the sense that its proof required a somehow unpleasant *additional assumption* in the statement of the theorem —namely, that the classifying morphism should commute *a priori* with the involutions on the base spaces of the bundles under consideration. Since dealing with involutions in the base spaces of vector bundles is essential to introduce holomorphy in the theory via the complexifications (so is essential to the level of generality, examples and applications given in [BG08] in particular) one would like to find an improvement of [BG11, Th. 5.1] which in turn allows us to transfer geometry from the tautological bundles to bundles with involution.

The present paper is not intended as a merely formal generalization of results of [BG11] or [BG14]. For the sake of precision, one must insist that the purpose of this paper is two-fold. Firstly, we give an improvement of [BG11, Th. 5.1] which seems to be the right theorem of universality for bundles endowed with an involutive structure (see Theorem 4.1 below). Then a new structure emerges in a natural manner for the bundles involved -so for the tautological ones in particular-, which is slightly more general than the like-Hermitian one, in the sense that bilinear forms must be admitted besides the usual sesquilinear forms of the (like-) Hermitian counterpart. We will refer to these vector bundles in the sequel as having a *Hermitian-like* structure. Secondly, we define the proper reproducing kernels associated with such bundles, and then we construct their basic differential geometry by extending results of [BG14]. More advanced aspects of that geometry, such as compatibility of covariant derivatives with Hermitian-like structures and complex structures, or positivity of curvatures, in analogy to [BG15] for the Hermitian case, are left for a forthcoming paper.

Organization of the present paper. In Section 2 we briefly provide a few ideas and results on reproducing kernels on Hermitian vector bundles and their geometry. These facts are mentioned here for the sake of reader's motivation, since their extensions to a Hermitian-like setting are the main results that will be established in the next sections. We also make some remarks in this respect. This section is thought of a continuation of the introduction, to explain the basic symbols and fix notation. Section 3 contains basic definitions, properties and examples of vector bundles, morphisms in-between and reproducing kernels in the Hermitian-like setting. In Section 4 we prove the universality theorem for *self-involutive* kernels on Hermitian-like bundles (Theorem 4.1) and characterize the key notion of *admissible* self-involutive kernel (see Definition 4.6, based on Theorem 4.4 and Corollary 4.5). Section 5 is devoted to point out the intrinsic differential geometry of admissible self-involutive kernels. To make the paper as self-contained as possible, for convenience of the reader, we include definitions and results (without proofs) of [BG14] about linear connections, their pull-backs and corresponding covariant derivatives. The new result in this section is Theorem 5.8, which extends [BG14, Theorem 4.2]. Finally, in Section 6, we apply our general results to the study of some specific examples, in particular to *two-sided* Stinespring dilations of tracial completely positive maps. This example extends [BG11, Proposition 6.1].

2. By way of motivation: the Hermitian setting

Some references for this section are [BG11] and [BG14], but many of these ideas in the case of line bundles actually go back to [Od88] and [Od92].

If Z is a Banach manifold, then a *Hermitian structure* on a smooth Banach vector bundle $\Pi\colon D \to Z$ is a family $\{(\cdot \mid \cdot)_z\}_{z\in Z}$ such that for every $z \in Z$, $(\cdot \mid \cdot)_z\colon D_z \times D_z \to \mathbb{C}$ is a scalar product ($\mathbb{C}$-linear in the first variable and conjugate $\mathbb{C}$-linear in the second variable) that turns the fiber D_z into a complex Hilbert space, and this family of scalar products is assumed to depend smoothly on $z \in Z$, in the sense that the following condition is satisfied: If V is any open subset of Z, and $\Psi_V\colon V \times \mathcal{E} \to \Pi^{-1}(V)$ is a trivializations (whose typical fiber is the complex Hilbert space $\mathcal{E}$) of the vector bundle Π over V, then the function $(z, x, y) \mapsto (\Psi_V(z, x) \mid \Psi_V(z, y))_z$, $V \times \mathcal{E} \times \mathcal{E} \to \mathbb{C}$ is smooth. A *Hermitian bundle* is simply any bundle endowed with a Hermitian structure as above.

Remark 2.1. In this paper we deal with vector bundles endowed with an involution $z \in Z \mapsto z^{-*} \in Z$ in the base space Z, and introduce an extension of the Hermitian notions which depends explicitly on such an involution. Thus the above structure $\{(\cdot \mid \cdot)_z\}_{z\in Z}$ is replaced by $\{(\cdot \mid \cdot)_{z,z^{-*}}\}_{z\in Z}$ and it is allowed to be bilinear as well, to include natural examples (see Definition 3.1).

In the above setting, if we also denote by $p_1, p_2\colon Z \times Z \to Z$ the Cartesian projections, then a *reproducing kernel* on Π is any positive definite section K of the bundle $\mathrm{Hom}(p_2^*\Pi, p_1^*\Pi) \to Z \times Z$. That is, K is a family of bounded linear operators $K(s, t)\colon D_t \to D_s$ depending on $s, t \in Z$ and satisfying the following positivity condition: For every $n \geq 1$ and $t_j \in Z$, $\eta_j \in D_{t_j}$ $(j = 1, \dots, n)$,

$$\sum_{j,l=1}^{n} \big(K(t_l, t_j)\eta_j \mid \eta_l\big)_{t_l} \geq 0.$$

For every $\xi \in D$ we set $K_\xi := K(\cdot, \Pi(\xi))\xi \colon Z \to D$, which is a section of the bundle Π. For $\xi, \eta \in D$, the prescriptions

$$(K_\xi \mid K_\eta)_{\mathcal{H}^K} := (K(\Pi(\eta), \Pi(\xi))\xi \mid \eta)_{\Pi(\eta)},$$

define an inner product $(\cdot \mid \cdot)_{\mathcal{H}^K}$ on $\mathrm{span}\{K_\xi : \xi \in D\}$ whose completion gives rise to a Hilbert space denoted by $\mathcal{H}^K$, which consists of sections of the bundle Π. The reproducing kernel K naturally gives rise to the mappings

$$\widehat{K} \colon D \to \mathcal{H}^K, \quad \widehat{K}(\xi) = K_\xi, \tag{2.1}$$

$$\zeta_K \colon Z \to \mathrm{Gr}(\mathcal{H}^K), \quad \zeta_K(s) = \overline{\widehat{K}(D_s)}, \tag{2.2}$$

where the bar over $\widehat{K}(D_s)$ indicates the topological closure.

Remark 2.2. The above definition of reproducing kernels must be adapted to the Hermitian-like framework accordingly, relying on the involution $z \in Z \mapsto z^{-*} \in Z$. This is done in Definition 3.5. Central to our approach here is the notion of self-involutive reproducing kernel, given in terms of involutive quasimorphisms, whose definition is given prior to Example 3.7. For involutive quasimorphisms, see Definition 3.4.

Example 2.3 (universal reproducing kernel)**.** For any complex Hilbert space $\mathcal{H}$, its tautological bundle $\Pi_{\mathcal{H}} \colon \mathcal{T}(\mathcal{H}) \to \mathrm{Gr}(\mathcal{H})$ has a natural Hermitian structure given by $((\mathcal{S}, x) \mid (\mathcal{S}, y))_{\mathcal{S}} := (x \mid y)_{\mathcal{H}}$ for all $\mathcal{S} \in \mathrm{Gr}(\mathcal{H})$ and $x, y \in \mathcal{S}$, where $\mathcal{S} \in \mathrm{Gr}(\mathcal{H})$. Recall that by definition $\mathcal{T}(\mathcal{H}) = \{(\mathcal{S}, x) \in \mathrm{Gr}(\mathcal{H}) \times \mathcal{H} : x \in \mathcal{S}\}$ and $\mathrm{Gr}(\mathcal{H})$ is the Grassmann manifold of all closed linear subspaces of $\mathcal{H}$. This Hermitian bundle carries a natural reproducing kernel $Q_{\mathcal{H}}$, called *universal reproducing kernel*,

$$Q_{\mathcal{H}}(\mathcal{S}_1, \mathcal{S}_2) := p_{\mathcal{S}_1}|_{\mathcal{S}_2} \colon \mathcal{S}_2 \to \mathcal{S}_1 \quad \text{for } \mathcal{S}_1, \mathcal{S}_2 \in \mathrm{Gr}(\mathcal{H}).$$

The orthogonal projection $p_{\mathcal{S}}$ on any $\mathcal{S} \in \mathrm{Gr}(\mathcal{H})$ can be regarded as an operator between different Hilbert spaces $p_{\mathcal{S}} \colon \mathcal{H} \to \mathcal{S}$, whose adjoint operator is the inclusion operator $\iota_{\mathcal{S}} \colon \mathcal{S} \hookrightarrow \mathcal{H}$ that is, $(p_{\mathcal{S}})^* = \iota_{\mathcal{S}} \colon \mathcal{S} \to \mathcal{H}$. This shows that

$$Q_{\mathcal{H}}(\mathcal{S}_1, \mathcal{S}_2) = p_{\mathcal{S}_1}(p_{\mathcal{S}_2})^* \quad \text{for } \mathcal{S}_1, \mathcal{S}_2 \in \mathrm{Gr}(\mathcal{H})$$

which resembles the expressions of other reproducing kernels, for instance the reproducing kernel $(1 - z_1\overline{z_2})^{-1}$ of the Hardy space on the unit disk.

In this exemple, the reproducing kernel Hilbert space associated with $Q_{\mathcal{H}}$ can be identified with $\mathcal{H}$, and the maps (2.1) and (2.2) are

$$\widehat{K} \colon D = \mathcal{T}(\mathcal{H}) \to \mathcal{H}, \quad (\mathcal{S}, x) \mapsto x,$$
$$\zeta_K \colon Z = \mathrm{Gr}(\mathcal{H}) \to \mathrm{Gr}(\mathcal{H}), \quad \zeta_K(\mathcal{S}) = \mathcal{S}.$$

The above terminology is motivated by the universality properties of these kernels, established in [BG11]. More precisely, we defined an operation of pull-back of reproducing kernels via maps between vector bundles, and using this operation, one has the following theorem:

Theorem 2.4 (universality)**.** *Let $\Pi \colon D \to Z$ be a Hermitian vector bundle endowed with a reproducing kernel K. If we define*

$$\delta_K := (\zeta_K \circ \Pi, \widehat{K}) \colon D \to \mathcal{T}(\mathcal{H}^K), \quad \xi \mapsto (\zeta_K(\Pi(\xi)), \widehat{K}(\xi))$$

then the pair of maps $\Delta_K := (\delta_K, \zeta_K)$ defines a map of vector bundles from Π to the tautological vector bundle of $\mathcal{H}^K$ which recovers K as the pull-back of the universal reproducing kernel $Q_{\mathcal{H}^K}$, which we denote by $K = (\Delta_K)^* Q_{\mathcal{H}^K}$.

Remark 2.5. In order to extend Theorem 2.4 to a universal theorem for Hermitian-like vector bundles, so that it entails the natural correspondence between involutions in particular, one needs to appeal to self-involutive kernels and also to consider Hermitian-like structures in the tautological bundle defined by involutive, isometric, linear or conjugate-linear operators $C\colon \mathcal{H} \to \mathcal{H}$ on the Hilbert space $\mathcal{H}$. Then the definition of the universal kernel $Q_{\mathcal{H}}$ in Example 2.3 is widened to kernels $Q_{\mathcal{H},C}$ given by $Q_{\mathcal{H},C}(S_1, S_2) := p_{S_1} \circ C \mid_{C(S_2)}\colon C(S_2) \to S_1$ for $S_1, S_2 \in \mathrm{Gr}(\mathcal{H})$; see Example 3.7. Conjugate-linear isometries C arise in natural examples like [BG11, Proposition 6.1], for instance. As said before, the wished-for extension of Theorem 2.4 is Theorem 4.1 below.

Since there the tautological vector bundles carry canonical linear connections, it is natural to consider their pull-backs through the maps of type Δ_K from Theorem 2.4. In order to realize that idea, we must ensure that the components of Δ_K are smooth maps. We proved in [BG14] that this is the case if the reproducing kernel K is *admissible*, in the sense that if it has the following properties:

(a) The kernel K is smooth as a section of the bundle $\mathrm{Hom}(\mathrm{p}_2^*\Pi, \mathrm{p}_1^*\Pi)$.
(b) For every $s \in Z$ the operator $K(s, s) \in \mathcal{B}(D_s)$ is invertible.
(c) The mapping $\zeta_K\colon Z \to \mathrm{Gr}(\mathcal{H}^K)$ is smooth.

As proved in [BG14, Ex. 3.7], if the fibers of the Hermitian vector bundle $\Pi\colon D \to Z$ are finite dimensional (for instance, if Π is a line bundle), then for every reproducing kernel K on Π which satisfies the above conditions (a)–(b), the condition (c) is automatically satisfied hence it is an admissible reproducing kernel.

The conclusion is that on any Hermitian vector bundle Π there exists a canonical correspondence from admissible reproducing kernels to linear connections, namely to admissible K there corresponds the pull-back, say Φ_K, via Δ_K of the universal connection on the tautological bundle of $\mathcal{H}^K$. Every linear connection Φ_K has indeed associated a covariant derivative ∇_K (Definition 5.1 below). This derivative was computed in [BG14, Theorem 4.2]. We recall that the aforementioned tautological bundle is a holomorphic Hermitian vector bundle and its universal connection is precisely the Chern connection, that is, the unique linear connection that is compatible both with the Hermitian and with the holomorphic structure.

Remark 2.6. Admissible kernels can be also introduced in the Hermitian-like context almost verbatim; see Definition 4.6. This definition relies here on Corollary 4.5. Then to any admissible self-involutive reproducing kernel on a Hermitian-like vector bundle one can associate a canonical linear connection Φ_K with corresponding covariant derivative ∇_K. The computation of ∇_K is made in Theorem 5.8.

3. Hermitian-like vector bundles and reproducing kernels

3.1. **Hermitian-like structures.** Here we propose a generalization of the notion of Hermitian structure in vector bundles endowed with an involution in the base space. This structure depends substantially of the given involution.

Definition 3.1. Let Z be a real or complex Banach manifold with an involutive diffeomorphism $z \mapsto z^{-*}$, $Z \to Z$, that is, $(z^{-*})^{-*} = z$ for all $z \in Z$. A *Hermitian-like structure* on a smooth vector bundle $\Pi\colon D \to Z$ is a family $\{(\cdot \mid \cdot)_{z,z^{-*}}\}_{z \in Z}$ with the following properties:

(a) For every $z \in Z$, $(\cdot \mid \cdot)_{z,z^{-*}}\colon D_z \times D_{z^{-*}} \to \mathbb{C}$ is a sesquilinear or bilinear strong duality pairing.

(b) For all $z \in Z$, $\xi \in D_z$, and $\eta \in D_{z^{-*}}$ we have

$$\overline{(\xi \mid \eta)_{z,z^{-*}}} = (\eta \mid \xi)_{z^{-*},z}, \text{ if the pairing is sesquilinear,}$$

or

$$(\xi \mid \eta)_{z,z^{-*}} = (\eta \mid \xi)_{z^{-*},z}, \text{ if the pairing is bilinear.}$$

(c) If V is any open subset of Z, and

$$\Psi_V : V \times \mathcal{E} \to \Pi^{-1}(V) \text{ and } \Psi_{V^{-*}} : V^{-*} \times \mathcal{E} \to \Pi^{-1}(V^{-*})$$

are trivializations (whose typical fiber is a complex Banach space $\mathcal{E}$) of the vector bundle Π over V and V^{-*} $(:= \{z^{-*} \mid z \in V\})$, respectively, then the function $(z, x, y) \mapsto (\Psi_V(z, x) \mid \Psi_{V^{-*}}(z^{-*}, y))_{z,z^{-*}}$, $V \times \mathcal{E} \times \mathcal{E} \to \mathbb{C}$ is smooth.

We call *Hermitian-like vector bundle* any bundle endowed with a Hermitian-like structure as before.

Condition (a) in Definition 3.1 means that the functional $(\cdot \mid \cdot)_{z,z^{-*}} : D_z \times D_{z^{-*}} \to \mathbb{C}$ is continuous, is linear in the first variable and conjugate-linear in the second variable when the functional is sesquilinear, and both the mappings

$$\xi \mapsto (\xi \mid \cdot)_{z,z^{-*}}, \quad D_z \to (\overline{D}_{z^{-*}})^*, \quad \text{and} \quad \eta \mapsto (\cdot \mid \eta)_{z,z^{-*}}, \quad \overline{D}_{z^{-*}} \to D_z^*,$$

are (not necessarily isometric) isomorphisms of complex Banach spaces in this case. (Here we denote, for any complex Banach space $\mathcal{Z}$, by $\mathcal{Z}^*$ its dual (complex) Banach space and by $\overline{\mathcal{Z}}$ the complex-conjugate Banach space.) If $(\cdot \mid \cdot)_{z,z^{-*}} : D_z \times D_{z^{-*}} \to \mathbb{C}$ is bilinear then the properties are similar with the only exception that in the duality one must remove complex conjugation.

Remark 3.2. The difference between Definition 3.1 and the definition in [BG08] and [BG11] of the so-called there like-Hermitian vector bundles is that we now allow the pairing $(\cdot \mid \cdot)_{z,z^{-*}}$ to be bilinear. The usual Hermitian vector bundles are particular cases of the above definition for $-* = \mathrm{id}_Z$.

Next, we define a suitable class of morphisms between Hermitian-like bundles. Let $\Pi : D \to Z$ and $\widetilde{\Pi} : \widetilde{D} \to \widetilde{Z}$ be Hermitian-like vector bundles (which we assume not necessarily of the same sesquilinear or bilinear type), and assume that each of the manifolds Z and $\widetilde{Z}$ is endowed with an involutive diffeomorphism denoted by $z \mapsto z^{-*}$ for both manifolds.

Definition 3.3. A *quasimorphism* of Π into $\widetilde{\Pi}$ is a pair $\Theta = (\delta, \zeta)$ such that $\delta : D \to \widetilde{D}$ and $\zeta : Z \to \widetilde{Z}$ are mappings such that:

(i) $\Pi \circ \zeta = \delta \circ \widetilde{\Pi}$;

(ii) the mapping $\delta_z := \delta|_{D_z} : D_z \to \widetilde{D}_{\zeta(z)}$ is a bounded operator which is allowed to be either linear for all $z \in Z$ or conjugate-linear for all $z \in Z$;

(iii) for all $z \in Z$ we have $\zeta(z^{-*}) = \zeta(z)^{-*}$.

By Definition 3.1, for every $z \in Z$ there exists a unique bounded operator $(\delta_z)^{-*} : \widetilde{D}_{\zeta(z)^{-*}} \to D_{z^{-*}}$ defined by

$$(\forall \xi \in D_z, \eta \in \widetilde{D}_{\zeta(z)^{-*}}) \quad ((\delta_z)^{-*}\eta \mid \xi)_{z^{-*},z} = (\eta \mid \delta_z \xi)_{\zeta(z)^{-*},\zeta(z)} \tag{3.1}$$

in the cases: 1) Π and $\widetilde{\Pi}$ have sesquilinear structures and δ is linear on fibers (then δ^{-*} is also linear on fibers), 2) Π and $\widetilde{\Pi}$ have bilinear structures and δ is linear on fibers (then δ^{-*} is also linear on fibers), 3) Π sesquilinear, $\widetilde{\Pi}$ bilinear and δ

conjugate-linear (then δ^{-*} linear), 4) Π bilinear, $\widetilde{\Pi}$ sesquilinear and δ conjugate-linear (then δ^{-*} linear); or by

$$(\forall \xi \in D_z, \eta \in \widetilde{D}_{\zeta(z)^{-*}}) \quad ((\delta_z)^{-*}\eta \mid \xi)_{z^{-*},z} = \overline{(\eta \mid \delta_z \xi)_{\zeta(z)^{-*},\zeta(z)}} \tag{3.2}$$

in the cases: 1) Π and $\widetilde{\Pi}$ sesquilinear, δ conjugate-linear (then δ^{-*} conjugate-linear), 2) Π and $\widetilde{\Pi}$ bilinear, δ conjugate-linear (then δ^{-*} conjugate-linear), 3) Π sesquilinear, $\widetilde{\Pi}$ bilinear and δ linear (then δ^{-*} conjugate-linear), 4) Π bilinear, $\widetilde{\Pi}$ sesquilinear and δ linear (then δ^{-*} conjugate-linear), depending on the (not necessarily simultaneous) sesquilinear or bilinear character of the Hermitian-like structures in Π and $\widetilde{\Pi}$. Notice that the linearity or conjugate-linearity of the mapping $(\delta_z)^{-*}$ also depends on these structures and the given linearity or conjugate-linearity of δ_z.

Definition 3.4. An *involutive quasimorphism* on a Hermitian-like bundle $\Pi\colon D \to Z$ is any quasimorphism of the form $\Theta = (\tau, -*)$, where $-*$ is the involutive diffeomorphism in Z and $\tau\colon D \to D$ is a smooth map such that $\tau^2 = \mathrm{id}_D$, with $\tau(D_s) \subseteq D_{s^{-*}}$ $(s \in Z)$, and for all $s \in Z$, $\xi \in D_s$, $\eta \in D_{s^{-*}}$ we have

(i) $(\tau(\xi) \mid \tau(\eta))_{s^{-*},s} = (\eta \mid \xi)_{s^{-*},s}$ if τ is fiberwise linear and Π is bilinear, or τ is fiberwise conjugate-linear and Π is sesquilinear; and

(ii) $(\tau(\xi) \mid \tau(\eta))_{s^{-*},s} = \overline{(\eta \mid \xi)_{s^{-*},s}}$, if τ is fiberwise linear and and Π is sesquilinear, or τ is fiberwise conjugate-linear and Π is bilinear.

Hermitian-like vector bundles admit a suitable notion of reproducing kernels on them.

Definition 3.5. Let $\Pi\colon D \to Z$ be a Hermitian-like bundle. A *reproducing* $(-*)$-*kernel* on Π is a smooth section of the bundle $\mathrm{Hom}(\mathrm{p}_2^*\Pi, \mathrm{p}_1^*\Pi) \to \mathrm{Z} \times \mathrm{Z}$ such that the mappings

$$K(s,t)\colon D_{t^{-*}} \to D_s \quad (s,t \in Z)$$

are linear if the Hermitian-like structure is sesquilinear (respectively, they are conjugate-linear if that structure is bilinear), and such that it is $(-*)$-positive definite in the following sense: For every $n \geq 1$ and $t_j \in Z$, $\eta_j \in D_{t_j^{-*}}$ $(j = 1, \dots, n)$,

$$\sum_{j,l=1}^{n} \bigl(K(t_l, t_j)\eta_j \mid \eta_l\bigr)_{t_l, t_l^{-*}} \geq 0.$$

Here $p_1, p_2\colon Z \times Z \to Z$ are the natural projection mappings.

Here the main difference we have in relation with the definition of reproducing $(-*)$-kernel given in [BG08] and [BG11] is that the domain of the bounded linear mapping $K(s,t)$ is supposed to be now $D_{t^{-*}}$, whereas it was D_t in the quoted references. We prefer to take domains $D_{t^{-*}}$ in the present work to better reflect formally formulae and properties concerning classical reproducing kernels. The theory developed for $(-*)$-kernels in [BG08] and [BG11] is readily adapted to domains $D_{t^{-*}}$. Throughout the paper we will use properties of the kernels in accordance with the new notation, without any further explanation, unless it seems to be necessary for a better understanding.

Reproducing kernels are important because they yield Hilbert spaces. In our case, this is as follows. For every $\xi \in D$ we set $K_\xi := K(\cdot, \Pi(\xi)^{-*})\xi\colon Z \to D$. The functions K_ξ (for $\xi \in D$) are smooth sections of the bundle Π. For $\xi \in D_s, \eta \in D_t$ with $s, t \in Z$, the prescriptions

$$(K_\xi \mid K_\eta)_{\mathcal{H}^K} := (K(\Pi(\eta)^{-*}, \Pi(\xi)^{-*})\xi \mid \eta)_{t^{-*},t},$$

when Π is sesquilinear Hermitian-like, and

$$(K_\xi \mid K_\eta)_{\mathcal{H}^K} := \overline{(K(\Pi(\eta)^{-*}, \Pi(\xi)^{-*})\xi \mid \eta)}_{t^{-*},t},$$

when Π is bilinear Hermitian-like, define an inner product $(\cdot \mid \cdot)_{\mathcal{H}^K}$ on $\mathrm{span}\{K_\xi : \xi \in D\}$ whose completion gives rise to a Hilbert space denoted by $\mathcal{H}^K$.

In analogy with the notion of pull-back of kernels introduced in [BG11], we now make the following definition.

Definition 3.6. Let $\Pi\colon D \to Z$ and $\widetilde{\Pi}\colon \widetilde{D} \to \widetilde{Z}$ be two smooth Hermitian-like vector bundles such that there is a quasimorphism $\Theta = (\delta, \zeta)$ from Π to $\widetilde{\Pi}$. Assume that $\widetilde{K}$ is a reproducing $(-*)$-kernel on $\widetilde{\Pi}$. The *pull-back of the reproducing* $(-*)$-*kernel* $\widetilde{K}$ through Θ is the reproducing $(-*)$-kernel $\Theta^*\widetilde{K}$ on Π defined by

$$(\forall s, t \in Z) \quad \Theta^*\widetilde{K}(s,t) = (\delta_{s^{-*}})^{-*} \circ \widetilde{K}(\zeta(s), \zeta(t)) \circ \delta_{t^{-*}}; \tag{3.3}$$

that is, the diagram

$$\begin{array}{ccc} \widetilde{D}_{\zeta(t^{-*})} & \xrightarrow{\widetilde{K}(\zeta(s),\zeta(t))} & \widetilde{D}_{\zeta(s)} \\ {\scriptstyle\delta_{t^{-*}}}\uparrow & & \downarrow{\scriptstyle(\delta_{s^{-*}})^{-*}} \\ D_{t^{-*}} & \xrightarrow{\Theta^*\widetilde{K}(s,t)} & D_s \end{array}$$

is commutative for all $s, t \in Z$.

Given an involutive quasimorphism $\Theta_\tau = (\tau, -*)$ of the Hermitian-like bundle Π, we say that a reproducing $(-*)$-kernel K on Π is *self-involutive* if $K = \Theta_\tau^* K$, that is, if

$$(\forall s, t \in Z) \quad K(s,t) = \tau^{-1} \circ K(s^{-*}, t^{-*}) \circ \tau|_{D_{t^{-*}}}, \tag{3.4}$$

One basic example of Hermitian-like vector bundle is the universal bundle of a Hilbert space $\mathcal{H}$ provided with an involutive isometry on $\mathcal{H}$:

Example 3.7. *Universal bundles.* Put $\mathcal{G} = \mathrm{GL}(\mathcal{H})$, the full group of invertible bounded operators on $\mathcal{H}$, and $\mathcal{U} = \mathrm{U}(\mathcal{H})$, the subgroup of unitary bounded operators. Let $\Pi_{\mathcal{H}}\colon \mathcal{T}(\mathcal{H}) \to \mathrm{Gr}(\mathcal{H})$ be the bundle of Example 2.3. This is a holomorphic vector bundle on which $\mathcal{G}$ and $\mathcal{U}$ act holomorphically through the inclusion $\mathcal{G} \hookrightarrow \mathcal{B}(\mathcal{H})$; see [Up85, Ex. 3.11 and 6.20]. We call $\Pi_{\mathcal{H}}$ the *universal (tautological) vector bundle* associated with the Hilbert space $\mathcal{H}$.

For every $\mathcal{S} \in \mathrm{Gr}(\mathcal{H})$ we denote by $p_{\mathcal{S}}\colon \mathcal{H} \to \mathcal{S}$ the corresponding orthogonal projection. Take $\mathcal{S}_0 \in \mathrm{Gr}(\mathcal{H})$ and put $p := p_{\mathcal{S}_0}$. Let $\mathrm{Gr}_{\mathcal{S}_0}(\mathcal{H})$ denote the orbit of $\mathcal{S}_0$ in $\mathrm{Gr}(\mathcal{H})$ under the usual operator action by $\mathcal{G}$. The orbit $\mathrm{Gr}_{\mathcal{S}_0}(\mathcal{H})$ coincides with the unitary orbit of $\mathcal{S}_0$ and with the connected component of $\mathcal{S}_0$ in $\mathrm{Gr}(\mathcal{H})$, so it is given by

$$\begin{aligned} \mathrm{Gr}_{\mathcal{S}_0}(\mathcal{H}) &= \{g\mathcal{S}_0 \mid g \in \mathcal{G}\} = \{u\mathcal{S}_0 \mid u \in \mathcal{U}\} \\ &= \{\mathcal{S} \in \mathrm{Gr}(\mathcal{H}) \mid \dim \mathcal{S} = \dim \mathcal{S}_0 \text{ and } \dim \mathcal{S}^\perp = \dim \mathcal{S}_0^\perp\} \\ &\simeq \mathcal{U}/(\mathrm{U}(\mathcal{S}_0) \times \mathrm{U}(\mathcal{S}_0^\perp)) = \mathcal{U}/\mathcal{U}(p) \simeq \mathcal{G}/\mathcal{G}([p]), \end{aligned}$$

where $\mathcal{U}(p) := \{u \in \mathcal{U} \mid u\mathcal{S}_0 = \mathcal{S}_0\}$ and $\mathcal{G}([p]) := \{g \in \mathcal{G} \mid g\mathcal{S}_0 = \mathcal{S}_0\}$. (See for instance [Up85, Prop. 23.1] or [BG09, Lemma 4.3].)

Set $\mathcal{T}_{\mathcal{S}_0}(\mathcal{H}) := \{(\mathcal{S}, x) \in \mathcal{T}(\mathcal{H}) \mid \mathcal{S} \in \mathrm{Gr}_{\mathcal{S}_0}(\mathcal{H})\}$. The *universal vector bundle at* $\mathcal{S}_0$ is the holomorphic vector bundle $\Pi_{\mathcal{S}_0} \colon \mathcal{T}_{\mathcal{S}_0}(\mathcal{H}) \to \mathrm{Gr}_{\mathcal{S}_0}(\mathcal{H})$ obtained by restriction of $\Pi_{\mathcal{H}}$ to $\mathcal{T}_{\mathcal{S}_0}(\mathcal{H})$. The maps $\mathcal{G} \times_{\mathcal{G}([p])} \mathcal{S}_0 \ni [(g, x)] \mapsto (g\mathcal{S}_0, gx) \in \mathcal{T}_{\mathcal{S}_0}(\mathcal{H})$ and $\mathcal{U} \times_{\mathcal{U}(p)} \mathcal{S}_0 \ni [(u, x)] \mapsto (u\mathcal{S}_0, ux) \in \mathcal{T}_{\mathcal{S}_0}(\mathcal{H})$ induce biholomorphic diffeomorphisms between $\Pi_{\mathcal{H},\mathcal{S}_0} \colon \mathcal{T}_{\mathcal{S}_0}(\mathcal{H}) \to \mathrm{Gr}_{\mathcal{S}_0}(\mathcal{H})$ and the vector bundles $\mathcal{G} \times_{\mathcal{G}([p])} \mathcal{S}_0 \to \mathcal{G}/\mathcal{G}([p])$ and $\mathcal{U} \times_{\mathcal{U}(p)} \mathcal{S}_0 \to \mathcal{U}/\mathcal{U}(p)$, respectively. See [BG09, Prop. 4.5].

Let $C \colon \mathcal{H} \to \mathcal{H}$ be an involutive isometric, linear or conjugate-linear, operator on $\mathcal{H}$. Then the mapping

$$\mathcal{S} \mapsto \mathcal{S}^{-*} := C(\mathcal{S}), \quad \mathrm{Gr}(\mathcal{H}) \to \mathrm{Gr}(\mathcal{H})$$

is a smooth involution in the Grassmann manifold $\mathrm{Gr}(\mathcal{H})$. The universal bundle $\Pi_{\mathcal{H}}$ is Hermitian-like if we endow it with the Hermitian-like pairings

$$(x \mid y)_{S,S^{-*}} := (x \mid Cy)_{\mathcal{H}} \qquad (S \in \mathrm{Gr}(\mathcal{H}), x \in S, y \in C(S)).$$

(Notice that these pairings are bilinear if C is conjugate-linear). Further, this structure admits a reproducing kernel $Q_{\mathcal{H},C}$ defined by

$$Q_{\mathcal{H},C}(S_1, S_2) := p_{S_1} \circ C \mid_{C(\mathcal{S}_2)} \colon S_2^{-*} \to S_1 \quad \text{for } S_1, S_2 \in \mathrm{Gr}(\mathcal{H}).$$

We will use the notation $\Pi_{\mathcal{H},C}$ to refer to this Hermitian-like universal bundle in the sequel.

Fix an element $\mathcal{S}_0$ in $\mathrm{Gr}(\mathcal{H})$ such that $C(\mathcal{S}_0) = \mathcal{S}_0$. Then the orbit $\mathrm{Gr}_{\mathcal{S}_0}(\mathcal{H})$ is invariant under the involution $\mathcal{S} \mapsto C(\mathcal{S})$ of $\mathrm{Gr}(\mathcal{H})$, and therefore by restriction we get the Hermitian-like vector $\Pi_{\mathcal{S}_0,C} \colon \mathcal{T}_{\mathcal{S}_0}(\mathcal{H}) \to \mathrm{Gr}_{\mathcal{S}_0}(\mathcal{H})$ as a subbundle of $\Pi_{\mathcal{H},C}$. Denote by $Q_{\mathcal{S}_0,C}$ the restriction of the kernel $Q_{\mathcal{H},C}$ to the bundle $\Pi_{\mathcal{S}_0,C}$. For every $\mathcal{S} \in \mathrm{Gr}_{\mathcal{S}_0}(\mathcal{H})$ there exists $u \in \mathcal{U}$ such that $u\mathcal{S}_0 = \mathcal{S}$ and $u\mathcal{S}_0^{\perp} = \mathcal{S}^{\perp}$. Then $up_{\mathcal{S}_0} = p_{\mathcal{S}}u$, that is, $p_{\mathcal{S}} = up_{\mathcal{S}_0}u^{-1}$. Thus for all $u_1, u_2 \in \mathcal{U}$ and $x_1, x_2 \in \mathcal{S}_0$ we have

$$Q_{\mathcal{S}_0,C}(u_1\mathcal{S}_0, u_2\mathcal{S}_0)(Cu_2x_2) = p_{u_1\mathcal{S}_0}(u_2x_2) = u_1p_{\mathcal{S}_0}(u_1^{-1}u_2x_2).$$

For the above assertions, see [BG11, Def. 4.2 and Rem. 4.3].

Any of the bundles $\Pi_{\mathcal{H},C}$ or $\Pi_{\mathcal{S}_0,C}$ will be called here *universal vector bundle.* The Hermitian ones correspond to the choice $C = \mathrm{id}_{\mathcal{H}}$.

Just defining $\tau^C \colon (\mathcal{S}, x) \mapsto (C(\mathcal{S}), C(x))$, $\mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{H})$ one obtains an involutive quasimorphism on the universal vector bundles. Note that the kernels $Q_{\mathcal{H},C}$ and $Q_{\mathcal{S}_0,C}$ are self-involutive with respect to C and τ^C.

4. Universality theorem and admissible kernels

Here we prove the universality theorem for self-involutive kernels. By following [BG11, Th. 5.1], let us define $\widehat{K}(\xi) := K_\xi = K(\cdot, \Pi(\xi)^{-*})\xi$ for $\xi \in D$, and

$$\zeta_K \colon s \mapsto \overline{\widehat{K}(D_s)},\ Z \to \mathrm{Gr}(\mathcal{H}^K),$$

where the bar over $\widehat{K}(D_s)$ means topological closure. Put $\delta_K := (\zeta_K \circ \Pi, \widehat{K})$.

Theorem 4.1. *Let $\Pi \colon D \to Z$ be a Hermitian-like vector bundle having a self-involutive reproducing kernel K with respect to an involutive smooth quasimorphism $(\tau, -*)$. Then there exist an involutive, isometric operator $C := C(\tau) \colon \mathcal{H}^K \to \mathcal{H}^K$, and a vector bundle quasimorphism $\Delta_K := (\delta_K, \zeta_K)$ from Π into the universal bundle $\Pi_{\mathcal{H}^K,C}$ such that*

$$K = (\Delta_K)^* Q_{\mathcal{H}^K,C};$$

that is, for all $s,t\in Z$*,*

$$K(s,t)=((\delta_K)_{s^{-*}})^{-*}\circ Q_{\mathcal{H}^K,C}(\zeta_K(s),\zeta_K(t))\circ(\delta_K)_{t^{-*}}\,.$$

Proof. We prove the assertion when Π has a sesquilinear structure and the quasimorphism $(\tau,-*)$ is conjugate-linear on the fibers. The other cases are similar.

Let $\mathcal{H}^K$ be the Hilbert space generated by the kernel K, which is to say, by the basic sections K_ξ, $\xi\in D$. Since K is self-involutive we can do as in [BG11, Cor. 3.7, Prop. 3.6] to get an involutive conjugate-linear isometry $\overline{\tau}\colon\mathcal{H}^K\to\mathcal{H}^K$ such that $\overline{\tau}(K_\xi)=K_{\tau(\xi)}$, $(\xi\in D)$, and

$$(\overline{\tau}F)(t)=\tau(F(t^{-*}))\text{ for all }t\in Z\text{ and }F\in\mathcal{H}^K.$$

Then take $C=\overline{\tau}$. Since $\zeta_K(s)$ is a closed subspace of $\mathcal{H}^K$, it is readily seen that $C(\zeta_K(s))=\zeta_K(s^{-*})$ for each $s\in Z$. Finally, let us check that K can be recovered from $Q_{\mathcal{H}^K,C}$ by the pull-back operation with Δ_K.

Put $\delta_z:=(\delta_K)_z$, for $z\in Z$, for short. Take $s,t\in Z$. For all $\eta\in D_{t^{-*}}$ and $\xi\in D_{s^{-*}}$ we have

$$\begin{aligned}
&((\Delta_K)^*Q_{\mathcal{H}^K,C}(\zeta_K(s),\zeta_K(t))(\eta)\mid\xi)_{s,s^{-*}}\\
&\qquad=\big((\delta_{s^{-*}})^{-*}\circ Q_{\mathcal{H}^K,C}(\zeta_K(s),\zeta_K(t))(\delta_{t^{-*}})(\eta)\mid\xi\big)_{s,s^{-*}}\\
&\qquad=\overline{\big(Q_{\mathcal{H}^K,C}(\zeta_K(s),\zeta_K(t))(K_\eta)\mid K_\xi\big)}_{\zeta_K^C(s),\zeta_K^C(s^{-*})}\\
&\qquad=\overline{\big((p_{\zeta_K(s)}\circ C)(K_\eta)\mid C(K_\xi)\big)}_{\mathcal{H}^K}\\
&\qquad=\overline{(C(K_\eta)\mid C(K_\xi))}_{\mathcal{H}^K}\\
&\qquad=(K_\eta\mid K_\xi)_{\mathcal{H}^K}=(K(s,t)\eta\mid\xi)_{s^{-*}}\,,
\end{aligned}$$

as we wanted to show. □

Remark 4.2. (i) We will call the quasimorphism Δ_K constructed in Theorem 4.1 *the classifying quasimorphism* associated with the kernel K.

(ii) Note that the *Hermitian* case, that is, the case involving Hermitian vector bundles, in Theorem 4.1 corresponds to the choice of τ and C as the respective identity maps. Thus Theorem 4.1 extends the Hermitian result contained in [BG11, Th. 5.1 and Th. 6.2].

(iii) It is also possible to construct classifying morphisms in certain cases even though the reproducing kernel is not assumed to be self-involutive. For instance, this happens when the vector bundle $\Pi\colon D\to Z$ is acted on by a Banach-Lie group and the associated kernel K is compatible with the corresponding action. The resulting morphism then takes values in a complexification of the tautological bundle, see [BG11, Th. 5.14].

With geometric applications of the above theorem in mind, we study in the next subsection when the first component of a classifying quasimorphism is a fiberwise isomorphism.

4.1. **Quantization maps and kernels.** Motivated by the significant physical interpretation given in [Od88] and [Od92] (see also [MP97] and [BG11]) to maps from manifolds into the projective space of a complex Hilbert space, we gave in [BG14] the following notion.

Definition 4.3. Let Z and $\mathcal{H}$ be a Banach manifold and a complex Hilbert space, respectively. Any smooth mapping $\zeta\colon Z \to \mathrm{Gr}(\mathcal{H})$ will be termed a *quantization map* from Z to $\mathcal{H}$.

Set $\mathcal{D}_\zeta := \{(s,x) \in Z \times \mathcal{H} : s \in Z, x \in \zeta(s)\}$. Then $\mathcal{D}_\zeta$ is a Banach manifold of the same class as Z, and the projection $\Pi^\zeta\colon (s,x) \in \mathcal{D}_\zeta \mapsto s \in Z$ defines a vector bundle, with local trivializations

$$(\Pi^\zeta)^{-1}(\Omega_s) \simeq \Omega_s \times \zeta(s) \ \text{ for } \Omega_s := \zeta^{-1}(\mathrm{Gr}_{\zeta(s)}(\mathcal{H})), \quad s \in Z,$$

where the fiber at $s \in Z$ is identified to $\zeta(s)$. Put now

$$\psi_\zeta(s,x) := (\zeta(s), x), \quad s \in Z, x \in \zeta(s) \subseteq \mathcal{H},$$

so that (ψ_ζ, ζ) is a vector bundle morphism from $\Pi_\zeta\colon \mathcal{D}_\zeta \to Z$ to the universal bundle $\mathcal{T}(\mathcal{H}) \to \mathrm{Gr}(\mathcal{H})$. In fact, by identifying $\mathcal{D}_\zeta$ with $\{(s,(\zeta(s),x)) \mid s \in Z, x \in \zeta(s)\}$ one has that $\Pi_\zeta\colon \mathcal{D}_\zeta \to Z$ is diffeomorphic to the pull-back bundle of $\mathcal{T}(\mathcal{H}) \to \mathrm{Gr}(\mathcal{H})$ defined by the mapping ζ.

Assume in addition that there exist an involutive diffeomorphism $s \mapsto s^{-*}$, $Z \to Z$ and a linear or conjugate-linear isometry $C\colon \mathcal{H} \to \mathcal{H}$ such that $\zeta(s^{-*}) = C(\zeta(s))$ for all $s \in Z$. Then we provide the bundle $\mathcal{D}_\zeta \to Z$ with the Hermitian-like structure given by $(\,\cdot \mid \cdot\,)_{s,s^{-*}} := (\,\cdot \mid C(\cdot)\,)_{\mathcal{H}}$. Such a bundle admits the reproducing kernel given by

$$K_{\zeta,C}(s,t) := p_{\zeta(s)} \circ C|_{\zeta(t^{-*})}\colon \zeta(t^{-*}) \to \zeta(s) \quad (s,t \in Z).$$

Clearly, $C \circ K_{\zeta,C}(s^{-*}, s^{-*}) = \mathrm{id}_{\zeta(s)}$ for every $s \in Z$. (The case $-* = \mathrm{id}_Z$ and $C = \mathrm{id}_{\mathcal{H}}$ was considered in [BG14].)

Thus to every quantization map ζ there corresponds a self-involutive Hermitian-like reproducing kernel $K_{\zeta,C}$ such that $K_{\zeta,C}(s,s)$ is invertible, from the fiber on s^{-*} onto the fiber on s, for all $s \in Z$, and it is natural to investigate the correspondence in the opposite direction.

Let $\Pi\colon D \to Z$ be a Hermitian-like vector bundle with an involutive (smooth) quasimorphism and a self-involutive $(-*)$-reproducing kernel K. Let $\widehat{K}(\xi) := K_\xi$, for $\xi \in D$, and ζ_K be as prior to Theorem 4.1. Note that $\widehat{K}$ is continuous on D_s for all $s \in D$, since for every $\xi \in D_s$ we have $\|\widehat{K}(\xi)\|^2 = (K_\xi \mid K_\xi)_{\mathcal{H}^K} = (K(s,s^{-*})\xi \mid \xi)_{s,s^{-*}}$. Let $C\colon \mathcal{H}^K \to \mathcal{H}^K$ be the involutive isometry whose existence is ensured by Theorem 4.1. In the following result the bundle $\Pi_{\zeta_K}\colon \mathcal{D}_{\zeta_K} \to Z$ is endowed with the C-Hermitian-like structure and kernel $K_{\zeta_K,C}$ introduced above.

Theorem 4.4. *In the above setting, suppose that for every $s \in Z$ the mapping*

$$\widehat{K}|_{D_s}\colon \xi \mapsto \widehat{K}(\xi),\ D_s \to \mathcal{H}^K$$

is injective and has closed range. Then:

(i) *The mapping $\check{K} := (\Pi, \widehat{K})$ is a bijection from D onto $\mathcal{D}_{\zeta_K}$ and therefore $\Delta_{\zeta_K} := (\check{K}, \mathrm{id}_Z)$ is an algebraic isomorphism and a fiberwise topological isomorphism*

$$\begin{array}{ccc} D & \xrightarrow{\check{K}} & \mathcal{D}_{\zeta_K} \\ {\scriptstyle\Pi}\big\downarrow & & \big\downarrow{\scriptstyle\Pi^{\zeta_K}} \\ Z & \xrightarrow{\mathrm{id}_Z} & Z \end{array}$$

of vector bundles. In addition, if $\widehat{K}$ is smooth then the isomorphism Δ_{ζ_K} is smooth.

(ii) *The quasimorphism* $\Delta_K = (\delta_K, \zeta_K)$ *of Theorem 4.1 factorizes according to the commutative diagram*

$$\begin{array}{ccccc} \delta_K\colon D & \xrightarrow{\check{K}} & \mathcal{D}_{\zeta_K} & \xrightarrow{\psi_K} & \mathcal{T}(\mathcal{H}^K) \\ \downarrow{\scriptstyle \Pi} & & \downarrow{\scriptstyle \Pi_{\zeta_K}} & & \downarrow{\scriptstyle \Pi_{\mathcal{H}^K}} \\ \zeta_K\colon Z & \xrightarrow{\mathrm{id}_Z} & Z & \xrightarrow{\zeta_K} & \mathrm{Gr}(\mathcal{H}^K), \end{array}$$

where $\psi_K := \psi_{\zeta_K}$ *is as after Definition 4.3. In particular, for* ζ_K *smooth, the vector bundle* $\Pi\colon D \to Z$ *is diffeomorphic to the pull-back vector bundle induced by* $\zeta_K\colon Z \to \mathrm{Gr}(\mathcal{H}^K)$.

(iii) *Set* $\Theta_{\zeta_K} := (\check{K}, \mathrm{id}_Z)$. *Then the pull-back relation* $K = \Delta^*_{\zeta_K} Q_{\mathcal{H},C}$ *factorizes as*

$$K = \Theta^*_K K_{\zeta_K,C} = \Theta^*_{\zeta_K} (\psi_K, \zeta_K)^* Q_{\mathcal{H},C} = \Delta^*_{\zeta_K} Q_{\mathcal{H},C}.$$

Proof. Parts (i) and (ii) can be proved by mimicking the arguments of [BG14, Theorem 3.11].

(iii) Assume that Π has a sesquilinear structure and that C is conjugate-linear. For every $s, t \in Z$, $\xi \in D_{s^{-*}}$ and $\eta \in D_{t^{-*}}$ we have

$$\begin{aligned}(\Theta^*_K K_{\zeta_K,C}(\eta) \mid \xi)_{s,s^{-*}} &= \overline{(p_{\zeta_K(s)}(C(K_\eta)) \mid C(K_\xi))_{\mathcal{H}_K}} \\ &= (K_\eta \mid K_\xi)_{\mathcal{H}^K} = (K(s,t)\eta \mid \xi)_{s,s^{-*}},\end{aligned}$$

and then it follows that $K = \Theta^*_{\zeta_K} K_{\zeta_K,C}$ as claimed. For the remaining part of the statement, just note that

$$\begin{aligned}(\psi_K, \zeta_K) \circ \Theta_{\zeta_K} &= (\psi_K, \zeta_K) \circ ((\Pi, \widehat{K}), \mathrm{id}_Z), \\ (\psi_K \circ (\Pi, \widehat{K}), \zeta_K) &= (\delta_K, \zeta_K) = \Delta_K\end{aligned}$$

and we are done. The other cases are similar. □

Corollary 4.5. *Let* $\Pi\colon D \to Z$ *be a Hermitian-like vector bundle with a self-involutive reproducing* $(-*)$*-kernel* K. *Then the following assertions are equivalent:*

(i) $\widehat{K}|_{D_s}$ *is injective and has closed range for all* $s \in Z$.

(ii) $K(s,s)$ *is invertible from* $D_{s^{-*}}$ *onto* D_s, *for all* $s \in Z$.

(iii) *For every* $s \in Z$ *there exists* $z = z(s) \in Z$ *such that the operator* $K(s,z)$ *is invertible from* $D_{z^{-*}}$ *onto* D_s.

Proof. (i) ⇒ (ii). As seen before $\check{K}$ is a fiberwise topological isomorphism on D. Moreover, for $\eta \in D_{s^{-*}}$, $\eta \neq 0$, and $\xi \in D_s$,

$$\begin{aligned}((\check{K})^{-*}((s^{-*}, K_\eta)) \mid \xi)_{s^{-*},s} &= ((s^{-*}, K_\eta) \mid (s, K_\xi))_{s^{-*},s} \\ &= (C(K_\eta) \mid K_\xi)_{\mathcal{H}^K} \left[\text{ or } = \overline{(C(K_\eta) \mid K_\xi)_{\mathcal{H}^K}} \right].\end{aligned}$$

Hence,

$$\begin{aligned}\|(\check{K})^{-*}((s^{-*}, K_\eta))\|_{D_{s^{-*}}} &= \sup_{\|\xi\|_{D_s} \le 1} |(C(K_\eta) \mid K_\xi)_{\mathcal{H}_K}| \\ &\ge \|K_\eta\|^2_{\mathcal{H}_K} / \|\eta\|_{s^{-*}} \ge c\|K_\eta\|_{\mathcal{H}_K} \equiv c\|(s^{-*}, K_\eta)\|_{(\mathcal{D}_{\zeta_K})_{s^{-*}}},\end{aligned}$$

where we have taken $\xi = \tau(\eta)/\|\tau(\eta)\|_{D_s}$ for the first inequality, and have used that $\widehat{K}$ is a fiberwise open mapping for the second inequality. It follows that $(\check{K})^{-*}$ is injective and has closed range on fibers.

Moreover, if $\xi \in D_s$ satisfies $\big((\check{K})^{-*}((s^{-*}, K_\eta)) \mid \xi\big)_{s^{-*},s} = 0$ for every $\eta \in D_{s^{-*}}$ then in particular

$$0 = \big((s^{-*}, K_\eta)) \mid (s, K_\xi)\big)_{s^{-*},s} = (C(K_\xi) \mid K_\xi)_{\mathcal{H}^K} = \|K_\xi\|^2_{\mathcal{H}^K}$$

and therefore $K_\xi = 0$, so $\xi = 0$.

Thus $(\check{K})^{-*}$ is a topological isomorphism from $(\mathcal{D}_{\zeta_K})_{s^{-*}}$ onto $D_{s^{-*}}$ for all $s \in Z$. Then the invertibility of $K(s,s)$ for all $s \in Z$ is a consequence of the commutativity of the diagram

$$\begin{array}{ccc}
(\mathcal{D}_{\zeta_K})_{s^{-*}} \equiv \zeta_K(s^{-*}) & \xrightarrow{K_{\zeta_K}(s,s)} & \zeta_K(s) \\
{\scriptstyle \check{K}\equiv\widehat{K}}\Big\uparrow & & \Big\downarrow{\scriptstyle (\check{K})^{-*}} \\
D_{s^{-*}} & \xrightarrow{K(s,s)} & D_s
\end{array}$$

(ii) $\Rightarrow$ (iii). This is obvious.

(iii) $\Rightarrow$ (i). First, observe that the order of s and z as components of $K(s,z)$ is irrelevant, see [BG11, p. 137]. Then, by hypothesis, for every $s \in Z$, $\xi \in D_{s^{-*}}$ and $z = z(s)$, we have

$$\|\xi\|_{D_s} \le \|K(z,s)^{-1}\| \|K(z,s)\xi\|$$

and moreover $K(z,s)\xi = \theta_z(K_\xi)$ for the bounded linear operator $\theta_z \colon \mathcal{H}^K \to D_z$ defined by

$$(\theta_z(h) \mid \eta)_{z,z^{-*}} := (h \mid K_\eta)_{\mathcal{H}^K}, \quad h \in \mathcal{H}^K, \eta \in D_{z^{-*}},$$

see [BG08, p. 2896]. Hence, for some constant $C(s)$, $\|\xi\|_{D_s} \le C(s)\|K_\xi\|_{\mathcal{H}^K}$, whence statement (i) follows. □

Definition 4.6. A self-involutive $(-*)$-reproducing kernel K on a vector bundle endowed with an involutive quasimorphism is called *admissible* if:
(i) The kernel K is smooth as a section of the bundle $\mathrm{Hom}(\mathrm{pr}_2^*\Pi, \mathrm{pr}_1^*\Pi)$
(ii) For every $s \in Z$ the operator $K(s,s) \colon D_{s^{-*}} \to D_s$ is invertible .
(iii) The mapping $\zeta_K \colon Z \to \mathrm{Gr}(\mathcal{H}^K)$ is smooth.

Let us remark that condition (i) of the above definition implies that the mapping $\widehat{K} \colon D \to \mathcal{H}^K$ is smooth (see [BG14, Lemma 3.3]).

5. Geometry of self-involutive kernels

It was shown in [BG14] and [BG15] that reproducing kernels on Hermitian vector bundles entail a number of differential geometric features. Namely, a manner to construct linear connections from reproducing kernels on such bundles was given in [BG14], along with the calculation of the corresponding covariant derivatives. We then proved in [BG15] that the existence of a reproducing kernel K on a Hermitian holomorphic vector bundle Π implies the Griffith positivity of the curvature associated with the (unique) Chern covariant derivative on Π.

It is our aim in this section to extend the results of [BG14] to the setting of Hermitian-like vector bundles.

5.1. **Connections on vector bundles.** Recall that a *connection* on a vector (fiber) bundle $\Pi \colon D \to Z$ is a smooth map $\Phi \colon TD \to TD$ with the following properties:

(i) $\Phi \circ \Phi = \Phi$;
(ii) the pair (Φ, id_D) is an endomorphism of the tangent bundle $\tau_D \colon TD \to D$;

(iii) the pair (Φ, id_{TZ}) is a linear endomorphism of the vector bundle $T\Pi\colon TD \to TZ$ (i.e., if Φ is linear on the fibers of the bundle $T\Pi$); see [KM97, subsect. 37.27].

(iv) for every $\xi \in D$, if we denote $\Phi_\xi := \Phi|_{T_\xi D}\colon T_\xi D \to T_\xi D$, then we have $\mathrm{Ran}\,(\Phi_\xi) = \mathrm{Ker}\,(T_\xi \Pi)$, so that we get an exact sequence

$$0 \to H_\xi D \hookrightarrow T_\xi D \xrightarrow{\Phi_\xi} T_\xi D \xrightarrow{T_\xi \Pi} T_{\Pi(\xi)} Z \to 0.$$

Here $T\Pi\colon TD \to TZ$ is the tangent map of Π and $H_\xi D := \mathrm{Ker}\,(\Phi_\xi)$ is a closed linear subspace of $T_\xi D$ called the space of *horizontal vectors* at $\xi \in D$. Similarly, the space of *vertical vectors* at $\xi \in D$ is $\mathcal{V}_\xi D := \mathrm{Ker}\,(T_\xi \Pi)$. Then we have the direct sum decomposition $T_\xi D = H_\xi D \oplus \mathcal{V}_\xi D$, for every $\xi \in D$. Let Let $\mathcal{V}D = \mathrm{Ker}\,(T\Pi)$ denote the vertical subbundle of the tangent bundle $\tau_D\colon TD \to D$ whose fibers are the spaces $\mathcal{V}_\xi D$, $\xi \in D$ (cf. [KM97, subsect. 37.2]).

Next let $\Omega^1(Z, D)$ the space of locally defined smooth differential 1-forms on Z with values in the bundle $\Pi\colon D \to Z$, hence the set of smooth mappings $\eta\colon \tau_Z^{-1}(Z_\eta) \to D$, where $\tau_Z\colon TZ \to Z$ is the tangent bundle and Z_η is a suitable open subset of Z, such that for every $z \in Z_\eta$ we have a bounded linear operator $\eta_z := \eta|_{T_z Z}\colon T_z Z \to D_z = \Pi^{-1}(z)$. (So the pair (η, id_Z) is a homomorphism of vector bundles from the tangent bundle $\tau_D|_{Z_\eta}$ to the bundle Π.) For the sake of simplicity we actually omit the subscript η in Z_η, as if the forms were always defined throughout Z; in fact, the algebraic operations are performed on the intersections of the domains, and so on. Similarly, we let $\Omega^0(Z, D)$ be the space of locally defined smooth sections of the vector bundle Π.

Definition 5.1. The *covariant derivative* associated with the linear connection Φ is the linear mapping $\nabla\colon \Omega^0(Z, D) \to \Omega^1(Z, D)$, defined for every $\sigma \in \Omega^0(Z, D)$ by the composition

$$\nabla\sigma\colon TZ \xrightarrow{T\sigma} TD \xrightarrow{\Phi} \mathcal{V}D \xrightarrow{r} D$$

that is, $\nabla\sigma = (r \circ \Phi) \circ T\sigma$. (The composition $r \circ \Phi$ is the so-called *connection map.*)

One has the following technical, fundamental, result (see [BG14, Proposition A.4]

Proposition 5.2. *Let $\Pi\colon D \to Z$ and $\widetilde{\Pi}\colon \widetilde{D} \to \widetilde{Z}$ be vector bundles endowed with the linear connections Φ and $\widetilde{\Phi}$, with the corresponding covariant derivatives ∇ and $\widetilde{\nabla}$, respectively. Assume that $\Theta = (\delta, \zeta)$ is a homomorphism of vector bundles from Π into $\widetilde{\Pi}$ (that is, the diagram*

$$\begin{array}{ccc} D & \xrightarrow{\delta} & \widetilde{D} \\ {\scriptstyle\Pi}\downarrow & & \downarrow{\scriptstyle\widetilde{\Pi}} \\ Z & \xrightarrow{\zeta} & \widetilde{Z} \end{array}$$

is commutative and both δ and ζ are smooth) such that $T\delta \circ \Phi = \widetilde{\Phi} \circ T\delta$. If $\sigma \in \Omega^0(Z, D)$ and $\widetilde{\sigma} \in \Omega^0(\widetilde{Z}, \widetilde{D})$ such that $\delta \circ \sigma = \widetilde{\sigma} \circ \zeta$, then $\delta \circ \nabla\sigma = \widetilde{\nabla}\widetilde{\sigma} \circ T\zeta$.

5.2. **Pull-backs of connections.** A definition of pull-back of connections on infinite-dimensional vector bundles has been given in [BG14] which, unlike the various notions of pull-back that one can find in the literature on finite-dimensional bundles, requires neither connection maps, nor connection forms, nor covariant derivatives, but rather the connection itself. The definition (which is also valid in the finite-dimensional case) relies on the following result.

Proposition 5.3. *Let* $\Pi\colon D\to Z$ *and* $\widetilde{\Pi}\colon \widetilde{D}\to\widetilde{Z}$ *be vector bundles, and let* $\Theta=(\delta,\zeta)$ *be a smooth vector bundle homomorphism from* Π *into* $\widetilde{\Pi}$. *In addition, assume that for every* $s\in Z$ *the mapping* δ *induces a (linear or conjugate-linear) bounded isomorphism of the fiber* $D_s:=\Pi^{-1}(\{s\})$ *onto the fiber* $\widetilde{D}_{\zeta(s)}:=\widetilde{\Pi}^{-1}(\zeta(s))$.

Then for every (linear or conjugate-linear) connection $\widetilde{\Phi}$ *on the vector bundle* $\widetilde{\Pi}\colon\widetilde{D}\to\widetilde{Z}$ *there exists a unique (linear or conjugate-linear) connection* Φ *on the bundle* $\Pi\colon D\to Z$ *such that the diagram*

$$\begin{array}{ccc} TD & \xrightarrow{T\delta} & T\widetilde{D} \\ {\scriptstyle\Phi}\downarrow & & \downarrow{\scriptstyle\widetilde{\Phi}} \\ TD & \xrightarrow{T\delta} & T\widetilde{D} \end{array}$$

is commutative.

Proof. See [BG14, Proposition A.6 and Definition A.7]. □

Definition 5.4. In the setting of Proposition 5.3 we say that the connection Φ is the *pull-back of the connection* $\widetilde{\Phi}$ and we denote $\Phi=\Theta^*(\widetilde{\Phi})$.

The intertwining property of the covariant derivatives follows at once.

Corollary 5.5. *Let* Φ *and* $\widetilde{\Phi}$ *be two connections such that* $\Phi=\Theta^*(\widetilde{\Phi})$ *as above. Let* ∇ *and* $\widetilde{\nabla}$ *be the corresponding covariant derivatives, respectively. If we have* $\sigma\in\Omega^0(Z,D)$ *and* $\widetilde{\sigma}\in\Omega^0(\widetilde{Z},\widetilde{D})$ *such that* $\delta\circ\sigma=\widetilde{\sigma}\circ\zeta$, *then* $\delta\circ\nabla\sigma=\widetilde{\nabla}\widetilde{\sigma}\circ T\zeta$.

5.3. **Linear connections induced by reproducing kernels.** Let $\Pi\colon D\to Z$ be a Hermitian-like vector bundle endowed with an *admissible* self-involutive reproducing $(-*)$-kernel K such that $K(z,s)$ is invertible from $D_{s^{-*}}$ onto D_s for all $s\in Z$ and corresponding $z=z(s)\in Z$ (see Corollary 4.5).

Let $\Delta_K=(\delta_K,\zeta_K)$ and C be the classifying quasimorphism for K and isometry of $\mathcal{H}^K$, respectively, constructed in Theorem 4.1. Assume that $\mathcal{S}_0$ in $\mathrm{Gr}(\mathcal{H}^K)$ is such that $C(\mathcal{S}_0)=\mathcal{S}_0$ and $\zeta_K(Z)\subseteq\mathrm{Gr}_{\mathcal{S}_0}(\mathcal{H}^K)$, and therefore $\delta_K(D)\subseteq\mathcal{T}_{\mathcal{S}_0}(\mathcal{H}^K)$, so we have that Δ_K is a quasimorphism from Π to the universal bundle $\Pi_{\mathcal{S}_0,C}$ at $\mathcal{S}_0\subseteq\mathcal{H}^K$:

$$\begin{array}{ccc} D & \xrightarrow{\delta_K} & \mathcal{T}_{\mathcal{S}_0}(\mathcal{H}^K) \\ {\scriptstyle\Pi}\downarrow & & \downarrow{\scriptstyle\Pi_{\mathcal{S}_0,C}} \\ Z & \xrightarrow{\zeta_K} & \mathrm{Gr}_{\mathcal{S}_0}(\mathcal{H}^K). \end{array}$$

Let E_p be the conditional expectation, associated to the orthogonal projection $p:=p_{\mathcal{S}_0}\colon\mathcal{H}^K\to\mathcal{S}_0$, given by

$$E_p(T):=pTp+(\mathbf{1}-p)T(\mathbf{1}-p),\quad T\in\mathcal{B}(\mathcal{H}^K)$$

Then the mapping $\Phi_{\mathcal{S}_0}\colon T(\mathcal{T}_{\mathcal{S}_0}(\mathcal{H}))\to T(\mathcal{T}_{\mathcal{S}_0}(\mathcal{H}))$ defined for $u\in GL(\mathcal{H}^K)$, $T\in\mathcal{B}(\mathcal{H})$, and $x,y\in\mathcal{S}_0$ by

$$[((u,T),(x,y))]\mapsto[((u,E_p(T)),(x,y))]=[((u,0),(x,E_p(T)x+y))],$$

is a connection on the tautological bundle $\Pi_{\mathcal{H},\mathcal{S}_0}\colon\mathcal{T}_{\mathcal{S}_0}(\mathcal{H})\to\mathrm{Gr}_{\mathcal{S}_0}(\mathcal{H})$. We call $\Phi_{\mathcal{S}_0}$ the *universal connection* on $\Pi_{\mathcal{H},\mathcal{S}_0}$. See [BG14, Definition 2.11].

Since K is assumed to be admissible $\check{K}$ and ζ_K in Theorem 4.4 are smooth, hence so is δ_K and Δ_K. Moreover, that K is admissible in particular implies that the map δ_K is a fiberwise linear or conjugate-linear isomorphism from D_s onto $\widehat{K}(D_s)$. Thus one can apply Definition 5.4.

Definition 5.6. Under the above conditions, we call *connection* on Π *induced by* K the pull-back connection Φ_K given by

$$\Phi_K := (\Delta_K)^*(\Phi_{\mathcal{S}_0}).$$

We compute now the covariant derivative for the connection Φ_K. First of all, one needs to find the covariant derivative corresponding to the universal connection $\Phi_{\mathcal{S}_0}$. This has been settled in [BG14], by giving a formula which emphasizes the role of the orthogonal projections on closed subspaces. The following is Proposition 2.13 of [BG14].

Proposition 5.7. *Let $\mathcal{S}_0 \in \mathrm{Gr}(\mathcal{H})$. If $\sigma \in \Omega^0(\mathrm{Gr}_{\mathcal{S}_0}(\mathcal{H}), \mathcal{T}_{\mathcal{S}_0}(\mathcal{H}))$ is a smooth section, then there exists a unique smooth function $F_\sigma \in \mathcal{C}^\infty(\mathrm{Gr}_{\mathcal{S}_0}(\mathcal{H}), \mathcal{H})$ such that $\sigma(\cdot) = (\,\cdot\,, F_\sigma(\cdot))$ and we have*

$$\nabla\sigma(X) = (\mathcal{S}, p_{\mathcal{S}}(\mathrm{d}F_\sigma(X))), \quad \mathcal{S} \in \mathrm{Gr}_{\mathcal{S}_0}(\mathcal{H}), X \in T_{\mathcal{S}}(\mathrm{Gr}_{\mathcal{S}_0}(\mathcal{H})),$$

where $p_{\mathcal{S}}$ is the orthogonal projection from $\mathcal{H}$ onto $\mathcal{S}$.

We now can establish the general result on covariant derivatives associated to kernels. The proof goes along the same argument as in [BG14, Theorem 4.2] up to the necessary adaptation to the Hermitian-like case.

Theorem 5.8. *In the setting of* Definition 5.6*, let $\nabla_K \colon \Omega^0(Z, D) \to \Omega^1(Z, D)$ be the covariant derivative for the connection induced by K.*

If $\sigma \in \Omega^0(Z, D)$ has the property that there exists $\widetilde{\sigma} \in \Omega^0(\mathrm{Gr}_{\mathcal{S}_0}(\mathcal{H}^K), \mathcal{T}_{\mathcal{S}_0}(\mathcal{H}^K))$ such that $\delta_K \circ \sigma = \widetilde{\sigma} \circ \zeta_K$, then for $s, z = z(s) \in Z$ and $X \in T_s Z$ we have

$$(\nabla_K\sigma)(X) = K(z(s), s^{-*})^{-1}\Big(\underbrace{\big((p_{\zeta_K(s)}(\mathrm{d}(\widehat{K} \circ \sigma)(X)))\big)}_{\in \mathcal{H}^K \subseteq \Omega^0(Z,D)}(z(s))\Big).$$

Proof. Recall that for every $\xi \in D$ we have $\widehat{K}(\xi) = K_\xi = K(\cdot, \Pi(\xi)^{-*})\xi$ and $\delta_K(\xi) = (\zeta_K(\Pi(\xi)), \widehat{K}(\xi))$. Let $s \in Z$ and $X \in T_s Z$ arbitrary, with $z \in Z$ such that $K(z, s^{-*})$ is invertible. Since $\delta_K \circ \sigma = \widetilde{\sigma} \circ \zeta_K$, we obtain $\delta_K \circ \nabla_K\sigma = \widetilde{\nabla_K}\widetilde{\sigma} \circ T(\zeta_K)$ by Proposition 5.2, where $\widetilde{\nabla_K}$ denotes the covariant derivative for the universal connection on the tautological vector bundle $\Pi_{\mathcal{H},\mathcal{S}_0} \colon \mathcal{T}_{\mathcal{S}_0}(\mathcal{H}^K) \to \mathrm{Gr}_{\mathcal{S}_0}(\mathcal{H}^K)$. In particular we get

$$(\zeta_K(s), \widehat{K}((\nabla\sigma)(X))) = \delta_K((\nabla_K\sigma)(X)) = \widetilde{\nabla_K}\widetilde{\sigma}(T(\zeta_K)X). \tag{5.1}$$

On the other hand, since $\widetilde{\sigma} \in \Omega^0(\mathrm{Gr}_{\mathcal{S}_0}(\mathcal{H}^K), \mathcal{T}_{\mathcal{S}_0}(\mathcal{H}^K))$, there exists a uniquely determined function $F_{\widetilde{\sigma}} \in \mathcal{C}^\infty(\mathrm{Gr}_{\mathcal{S}_0}(\mathcal{H}^K), \mathcal{H}^K)$ with $\widetilde{\sigma}(\cdot) = (\cdot, F_{\widetilde{\sigma}}(\cdot))$. Then by Proposition 5.7 we get

$$\widetilde{\nabla_K}\widetilde{\sigma}(T(\zeta_K)X) = (\zeta_K(s), p_{\zeta_K(s)}(\mathrm{d}F_{\widetilde{\sigma}}(T(\zeta_K)X))). \tag{5.2}$$

By using $\delta_K \circ \sigma = \widetilde{\sigma} \circ \zeta_K$ again, we obtain $F_{\widetilde{\sigma}} \circ \zeta_K = \widehat{K} \circ \sigma = \widehat{K} \circ \sigma \colon Z \to \mathcal{H}^K$, hence by differentiation,

$$\mathrm{d}F_{\widetilde{\sigma}} \circ T(\zeta_K) = \mathrm{d}(\widehat{K} \circ \sigma). \tag{5.3}$$

It now follows by (5.1)–(5.3) that

$$\widehat{K}((\nabla_K\sigma)(X)) = p_{\zeta_K(s)}(\mathrm{d}(\widehat{K}\circ\sigma)(X)) \in \mathcal{H}^K.$$

Both sides of the above equality are sections in the bundle $\Pi\colon D\to Z$, and moreover $(\nabla_K\sigma)X\in D_s$. By evaluating the left-hand side at the point $z\in Z$ we obtain the value $K(z,s^{-*})\big((\nabla_K\sigma)X\big)\in D_z$. Hence by evaluating both sides of the above equality at z and then applying the operator $K(z,s^{-*})^{-1}$ to both sides of the equality obtained after the evaluation we obtain

$$(\nabla_K\sigma)(X) = K(z,s^{-*})^{-1}\big((p_{\zeta_K(s)}(\mathrm{d}(\widehat{K}\circ\sigma)(X)))(z)\big),$$

as we wanted to show. □

Remark 5.9. We have seen that there is a correspondence from admissible (self-involutive) reproducing kernels to linear connections (and covariant derivatives). In the Hermitian case (for the involution $-*=\mathrm{id}$) such a correspondence takes the form of a (unique) functor between suitable categories (see [BG14, Section 4.2]). Similar results can be obtained in the more general framework of Hermitian-like vector bundles, once one has introduced the appropriate, correspondingly, categories.

6. Examples

6.1. **Kernels on trivial bundles.** Examples of linear connections and covariant derivatives associated with reproducing kernels on trivial bundles have been given in [BG14] in the hermitian case. Here we show more general Hermitian-like structures on trivial bundles.

So, let $\mathcal{X}$ be a Banach manifold and let $\mathcal{H}$ be a Hilbert space. Assume that $\kappa\colon\mathcal{X}\times\mathcal{X}\to\mathcal{B}(\mathcal{H})$ is a smooth map that, also, is the reproducing kernel of a Hilbert space denoted by $\mathcal{H}^\kappa$. This means in particular that, for every $x_i\in\mathcal{X}$, $v_i\in\mathcal{H}$, $i=1,\dots,n$,

$$\sum_{i,j=1}^{n}(\kappa(x_i,x_j)v_j\mid v_i)_{\mathcal{H}}\geq 0,$$

and that $\mathcal{H}^\kappa$ is the Hilbert space of $\mathcal{H}$-valued functions on $\mathcal{X}$ generated by the space $\mathrm{span}\{\kappa_x\otimes v : x\in\mathcal{X}, v\in\mathcal{H}\}$, where

$$\kappa_x\otimes v := \kappa(\cdot,x)v\colon\mathcal{X}\to\mathcal{H},$$

with respect to the inner product given by

$$(\kappa_x\otimes v\mid\kappa_y\otimes w)_{\mathcal{H}^\kappa} := (\kappa(y,x)v\mid w)_{\mathcal{H}},$$

see [Ne00, Theorem I.1.4, (2) and (a)].

Assume in addition that $\kappa(x,x)$ is invertible in $\mathcal{B}(\mathcal{H})$ for all x, and that there exist a diffeomorphic involution $x\in\mathcal{X}\mapsto x^{-*}\in\mathcal{X}$ and a (conjugate-)linear isometry $C\colon\mathcal{H}\to\mathcal{H}$ related through the equation

$$C\kappa(x^{-*},y^{-*})C = \kappa(x,y)\qquad\forall x,y\in\mathcal{X}.$$

We define a Hermitian-like structure on the trivial vector bundle $\mathcal{X}\times\mathcal{H}\to\mathcal{X}$ by

$$\big((x,u)\mid(x^{-*},v)\big)_{x,x^{-*}} := (u\mid v)_{\mathcal{H}}\,,\quad (x\in\mathcal{X}; u,v\in\mathcal{H}),$$

and a kernel, accordingly, given by the family of operators $K(x,y)$ defined by

$$\{y^{-*}\}\times\mathcal{H}\ni(y^{-*},v)\mapsto(x,\kappa(x,y)v)\in\{x\}\times\mathcal{H}.$$

whenever $x, y \in \mathcal{X}$. It is readily seen that K is a reproducing kernel on the aforementioned trivial bundle. Moreover, for $x, y \in \mathcal{X}$ and $v \in \mathcal{H}$, we have

$$K_{(x,v)}(y) = K(y, x^{-*})(x, v) = (y, \kappa_{x^{-*}}(y)v),$$

and therefore there is the correspondence $K_{(x,v)} \longleftrightarrow \kappa_{x^{-*}} \otimes v$. Also, it is readily seen that if $(x, v), (y, w) \in \mathcal{X} \times \mathcal{H}$ then

$$\big(K_{(x,v)} \mid K_{(y,w)}\big)_{\mathcal{H}^K} = (\kappa_{x^{-*}} \otimes v \mid \kappa_{y^{-*}} \otimes w)_{\mathcal{H}^\kappa}.$$

Thus we get that $\mathcal{H}^K = \mathcal{H}^\kappa$.

Define the involution

$$\tau\colon \mathcal{X} \times \mathcal{H} \ni (x, v) \mapsto (x^{-*}, Cv) \in \mathcal{X} \times \mathcal{H}.$$

For every $x, y \in \mathcal{X}$ and $v \in \mathcal{H}$,

$$\begin{aligned}(\tau \circ K(x^{-*}, y^{-*}) \circ \tau)(y^{-*}, v)) &= (\tau \circ K(x^{-*}, y^{-*}))(y, Cv) \\ &= \tau\left((x^{-*}, \kappa(x^{-*}, y^{-*})Cv)\right) \\ &= (x, C\kappa(x^{-*}, y^{-*})Cv) \\ &= (x, \kappa(x, y)v) = K(x, y)(y^{-*}, v),\end{aligned}$$

so that K is self-involutive with respect to τ. Further, transferring $C_\tau := C(\tau)$ from $\mathcal{H}^K$ to $\mathcal{H}^\kappa$ gives us

$$C_\tau(\kappa_{x^{-*}} \otimes v) = \kappa_x \otimes Cv, \quad (x \in \mathcal{X}, v \in \mathcal{H}).$$

The classifying quasimorphism for K is

$$\begin{array}{ccc} (x, v) \in \mathcal{X} \times \mathcal{H} & \xrightarrow{\delta_K} & (\kappa_{x*-*} \otimes \mathcal{H}, \kappa_{x*-*} \otimes v) \in \mathcal{T}(\mathcal{H}^\kappa) \\ {\scriptstyle p_X}\big\downarrow & & \big\downarrow{\scriptstyle \Pi_{\mathcal{H}^\kappa}} \\ x \in \mathcal{X} & \xrightarrow{\zeta_K} & \kappa_{x*-*} \otimes \mathcal{H} \in \mathrm{Gr}(\mathcal{H}^\kappa)\,. \end{array}$$

In the Hermitian case, which correspond to the choice $-* = \mathrm{id}_\mathcal{X}$ and $C = \mathrm{id}_\mathcal{H}$ simultaneously, the covariant derivative formula for the connection induced by the reproducing kernel K when $\mathcal{H} = \mathbb{C}$ was given in [BG14]. Then moreover it was applied to examples of classical reproducing kernels of holomorphic Hilbert spaces. We are not going ahead here with the corresponding formulas for general involutions $-*$ and C, which so are left to prospective readers.

6.2. **Hermitian-like homogeneous fiber bundles.** Let G_A and G_B be two Banach-Lie groups, G_B Banach-Lie subgroup of G_A, with an involutive diffeomorphism "$*$" in G_A for which G_B is stable. For every $g \in G_A$, put $g^{-*} := (g^*)^{-1}$ and then define the involutive diffeomorphism $-*$ in the homogeneous space G_A/G_B by $(gG_B)^{-*} := g^{-*}G_B$ for every $g \in G_A$. Let $\rho_A\colon G_A \to \mathcal{B}(\mathcal{H}_A)$ and $\rho_B\colon G_B \to \mathcal{B}(\mathcal{H}_B)$ be uniformly continuous $*$-representations into Hilbert spaces $\mathcal{H}_B$ and $\mathcal{H}_A$ respectively, such that $\mathcal{H}_B \subseteq \mathcal{H}_A$, $\rho_B(g) = \rho_A(g)|_{\mathcal{H}_B}$ for $g \in G_B$ and $\mathcal{H}_A = \overline{\mathrm{span}}\, \rho_A(G_A)\mathcal{H}_B$.

Let us consider the homogeneous vector bundle $\Pi\colon G_A \times_{G_B} \mathcal{H}_B \to G_A/G_B$, induced by the representation ρ_B, with the involutive diffeomorphism $-*$ in the base G_A/G_B. We endow Π_ρ with the Hermitian-like structure given by

$$\big([(u, f)], [(u^{-*}, h)]\big)_{s,s^{-*}} := (f \mid h)_\mathcal{H}, \quad u \in G_A,\ f, h \in \mathcal{H}.$$

Let $P\colon \mathcal{H}_A \to \mathcal{H}_B$ be the orthogonal projection. We define the reproducing $(-*)$-kernel K_ρ on the vector bundle $\Pi\colon D = G_A \times_{G_B} \mathcal{H}_B \to G_A/G_B$ by

$$K_\rho(uG_B, vG_B)[(v^{-*}, f)] = [(u, P(\rho_A(u^{-1})\rho_A(v^{-*})f))], \tag{6.1}$$

for $uG_B, vG_B \in D$ and $f \in \mathcal{H}_B$ (see [BG08]).

Proposition 6.1. *In the above setting,*

(a) *the mapping $\widehat{K_\rho}$ is smooth;*
(b) *for every $s \in G_A/G_B$ the bounded linear operator $K_\rho(s, s^{-*})$ is invertible on D_s, so $\widehat{K_\rho}|_{D_s}\colon D_s \to \mathcal{H}^{K_\rho}$ is injective and its range is closed;*
(c) *the mapping*

$$\zeta_{K_\rho}\colon G_A/G_B \to \mathrm{Gr}(\mathcal{H}^{K_\rho}), \quad \zeta_{K_\rho}(s) := \widehat{K_\rho}(D_s),$$

is smooth.

Proof. Clearly, $K_\rho(s, s^{-*}) = \mathrm{id}_{D_s}$ for all $s \in G_A/G_B$. Thus property (b) holds by Corollary 4.5. Now, by similar arguments to those of [BG08, Sect. 4] one obtains that there exists a unitary operator $W\colon \mathcal{H}^K \to \mathcal{H}_A$ such that $W(K_\eta) = \pi_A(v)g$ whenever $\eta = [(v, g)] \in D$. Then the one can prove (a) and (b) in essentially the same way as in [BG14, Proposition 5.2]. □

In general, the vector bundle Π_ρ need not have an involution τ as in Definition 3.4. It is possible to provide Π_ρ with such an involution if we assume the following.

Suppose that there exists an involutive isometry $C\colon \mathcal{H}_A \to \mathcal{H}_A$ such that

$$\text{(i)}\, C(\mathcal{H}_B) = \mathcal{H}_B, \ \text{and (ii)}\, \rho_A(u^{-*}) = C\rho_A(u)C, \quad (u \in G_A). \tag{6.2}$$

(Note that in the above assumptions we can take C as the identity whenever ρ is a unitary representation.)

Then let τ_C be the involution in $D := G_A \times_{G_B} \mathcal{H}_B$ given by

$$\tau_C\colon [(u, f)] \in D \mapsto [(u^{-*}, C(f))] \in D,$$

which is well defined because of condition 6.2 (ii). As a matter of fact, the kernel K_ρ is self-involutive with respect to the involution τ. To see this, first we prove the following simple lemma.

Lemma 6.2. *Let $\mathcal{H}$ be a Hilbert space. For any closed subspace S of $\mathcal{H}$ we denote by P_S the orthogonal projection $P\colon \mathcal{H} \to S$. Then we have*

$$P_{C(S)} = CP_SC.$$

Proof. Given any $x \in \mathcal{H}$, we know that $P_S(x)$ is characterized as the only element in S such that

$$(x - P_S(x) \mid y)_{\mathcal{H}} = 0, \qquad (y \in S).$$

Hence, if C is linear, we have for every $y \in S$,

$$\begin{aligned}\big(x - (CP_{C(S)}C)(x) \mid y\big)_{\mathcal{H}} &= \big(C(Cx - P_{C(S)}(Cx)) \mid y\big)_{\mathcal{H}} \\ &= \big(Cx - P_{C(S)}(Cx) \mid Cy\big)_{\mathcal{H}} = 0\end{aligned}$$

since $Cy \in C(S)$. (If the isometry C is conjugate-linear, then the last equality should be replaced by $\big(Cy \mid Cx - P_{C(S)}(Cx)\big)_{\mathcal{H}} = 0$.) So by uniqueness we get the wished-for equation $P_{C(S)} = CP_SC$. □

Proposition 6.3. *The reproducing kernel K_ρ is self-involutive with respect to the involution τ_C.*

Proof. (a) Write $K = K_\rho$, $\tau = \tau_C$ and recall that $P = P_{\mathcal{H}_B}$. For $s = uG_B, t = vG_B \in G_A/G_B$ and $f \in \mathcal{H}_B$, one has

$$\begin{aligned}\left(\tau \circ K(s^{-*}, t^{-*}) \circ \tau\right)\left([(v^{-*}, f)]\right) &= \left(\tau \circ K(s^{-*}, t^{-*})\right)\left([(v, Cf)]\right) \\ &= \tau\left([(u^{-*}, P(\rho_A(u^* v)Cf))]\right) \\ &= [(u, CP_{\mathcal{H}_B}(\rho(u^* v)Cf))] \\ &= \left[\left(u, P_{C(\mathcal{H}_B)}(C\rho(u^* v)Cf)\right)\right] \\ &= \left[\left(u, P(u^{-1}v^{-*})f)\right)\right] \\ &= K(s,t)\left([(v^{-*}, f)]\right),\end{aligned}$$

by Lemma 6.2 for the fourth equality and (6.2) for the next-to-last equality. □

Remark 6.4. Part (b) of Proposition 6.1 for the kernel K_ρ is in accordance with Corollary 4.5 (ii), since $K_\rho(s,s)\colon D_{s^{-*}} \to D_s$ is an invertible (surjective) operator for all $s \in G_A/G_B$: If $K_\rho(uG_B, uG_B([(u^{-*}, f)]) = 0$ for some $[(u^{-*}, f)] \in D_{s^{-*}}$, with $s = uG_B$, then $(\rho_A(u^{-1})\rho_A(u^{-*})f \mid h)_{\mathcal{H}_A} = 0$ for all $h \in \mathcal{H}_B$, and so $\|\rho_A(u^{-*})f\|^2_{\mathcal{H}_A} = 0$ by taking $h = f$. Hence $f = 0$. Thus $K(s,s)$ is injective. Similarly, if $h \in \mathcal{H}_B$ and $0 = (P_{\mathcal{H}_B}(\rho_A(u^{-1})\rho_A(u^{-*})f) \mid h)_{\mathcal{H}_A}$ for all $f \in \mathcal{H}_B$ then, taking $f = h$, we obtain $h = 0$, so the linear space $P_{\mathcal{H}_B}(\rho_A(u^{-1}u^{-*})\mathcal{H}_B)$ is dense in $\mathcal{H}_B$. Moreover, for $f \in \mathcal{H}_B$, $f \neq 0$, one has

$$\begin{aligned}\|P_{\mathcal{H}_B}(\rho_A(u^{-1}u^{-*})f\|_{\mathcal{H}_B} &= \sup_{\|h\|_{\mathcal{H}_B} \le 1} |(P_{\mathcal{H}_B}(\rho_A(u^{-1}u^{-*})f \mid h)_{\mathcal{H}_B}| \\ &\ge \frac{1}{\|f\|_{\mathcal{H}_B}} |(\rho_A(u^{-*})f \mid \rho_A(u^{-*})f)_{\mathcal{H}_A}| \\ &= \frac{1}{\|f\|_{\mathcal{H}_B}} \|\rho_A(u^{-*})f\|^2_{\mathcal{H}_A} \\ &\ge \frac{1}{\|\rho_A(u^*)\|_{\mathcal{H}_A \to \mathcal{H}_A}} \|\rho_A(u^{-*})f\|_{\mathcal{H}_A} \\ &\ge \frac{\|\rho_A(u^{-1}u^{-*})f\|_{\mathcal{H}_A}}{\|\rho_A(u^{-1})\|\,\|\rho_A(u^*)\|},\end{aligned}$$

whence it follows that $P_{\mathcal{H}_B}(\rho_A(u^{-1}u^{-*})\mathcal{H}_B)$ is closed in $\mathcal{H}_B$. In summary, $K_\rho(s,s)$ is a linear bijection.

Using the unitary operator W referred to in the proof of Proposition 6.1, one gets the identification $\mathcal{H}^{K_\rho} = \mathcal{H}_A$, so that the classifying quasimorphism $(\delta_{K_\rho}, \zeta_{K_\rho})$ takes the form

$$\delta_{K_\rho}\colon [(g,f)] \mapsto (\rho_A(g)\mathcal{H}_B, \rho_A(g)f),\ G_A \times_{G_B} \mathcal{H}_B \to \mathcal{T}_{\mathcal{H}_B}(\mathcal{H}_A),$$

and

$$\zeta_{K_\rho}\colon gG_B \mapsto \rho_A(g)\mathcal{H}_B,\ G_A/G_B \to \mathrm{Gr}_{\mathcal{H}_B}(\mathcal{H}_A).$$

Then Theorem 4.1 applies, so that the kernel K_ρ can be recovered from the tautological kernel $Q_{\mathcal{H}^K, C}$ (Notice that this is also consequence of [BG11, Theorem 5.1] since the involution $\mathcal{S} \mapsto C(\mathcal{S})$ in $\mathrm{Gr}_{\mathcal{H}_B}(\mathcal{H}_A)$ satisfies $C(\zeta_{K_\rho}(uG_B)) = C(\rho(u)\mathcal{H}_B) = \rho(u^{-*})\mathcal{H}_B = \zeta_{K_\rho}(u^{-*}G_B)$ by (6.2)(ii) and the discussion in [BG11, p. 158].)

It is straighforward to check that the map $\Phi_\rho\colon T(G_A \times_{G_B} \mathcal{H}_B) \to T(G_A \times_{G_B} \mathcal{H}_B)$ given for every $g \in G_A$, $X \in \mathfrak{g}_A$ and $f, h \in \mathcal{H}_B$ by

$$\Phi_\rho\colon [((g,X),(f,h))] \mapsto [((g,0),(f, P(d\rho(X)f) + h)]$$

is a linear connection on the homogeneous bundle $\Pi\colon G_A \times_{G_B} \mathcal{H}_B \to G_A/G_B$. In fact such a connection is equal to the connection on Π_ρ obtained as the pull-back $\Phi_{K_\rho} := \Delta^*_{K_\rho}\Phi_{E_P}$, where $\Delta_{K_\rho} = (\delta_{K_\rho}, \zeta_{K_\rho})$. This fact is immediate to prove, since the definition of both connections Φ_{K_ρ} and Φ_ρ is given by the fiberwise composition $\Phi = (T\delta_{K_\rho})^{-1} \circ \Phi_{E_P} \circ T\delta_{K_\rho}$ irrespective the Hermitian-like structure that we consider in the tautological vector bundle $\mathcal{T}(\mathcal{H}_A) \to \mathrm{Gr}(\mathcal{H}_A)$.

Definition 6.5. In the above setting, we say that $\Phi_\rho \equiv \Phi_{K_\rho}$ is the *natural connection associated with* Π_ρ.

In order to compute the covariant derivative associated with the connection Φ_ρ, note that if $\mathfrak{g}_A$ and $\mathfrak{g}_B$ are the Lie algebras of G_A and G_B, respectively, then the adjoint action of G_B on $\mathfrak{g}_A$ gives rise to a linear action on the quotient $\mathfrak{g}_A/\mathfrak{g}_B$ and we can then form the homogeneous vector bundle $G_A \times_{G_B} (\mathfrak{g}_A/\mathfrak{g}_B)$, which is isomorphic to the tangent bundle $T(G_A/G_B)$. For any closed linear subspace $\mathfrak{m}$ of $\mathfrak{g}_A$ such that $\mathfrak{g}_A = \mathfrak{g}_B \dotplus \mathfrak{m}$ we have a linear topological isomorphism $\mathfrak{m} \simeq \mathfrak{g}_A/\mathfrak{g}_B$, which gives rise to a natural linear action of G_B on $\mathfrak{m}$, hence to a homogeneous vector bundle $\mathrm{G}_A \times_{G_B} \mathfrak{m}$ which can be identified with $T(G_A/G_B)$.

Proposition 6.6. *Fix* $\mathfrak{m}$ *as above so that* $T(G_A/G_B) = G_A \times_{G_B} \mathfrak{m}$. *Let* $\phi\colon G_A \to \mathcal{H}_B$ *be any smooth function such that* $\phi(uw) = \rho_A(w)^{-1}\phi(u)$, $u \in G_A$ *and* $w \in G_B$, *and let* σ *be the associated smooth section defined by* $\sigma\colon uG_B \in G_A/G_B \mapsto [(u, \phi(u))] \in G_A \times_{G_B} \mathcal{H}_B$. *Let* $\widetilde{\sigma}\colon \mathrm{Gr}_{\mathcal{H}_B}(\mathcal{H}_A) \to \mathcal{T}_{\mathcal{H}_B}(\mathcal{H}_A)$ *be a smooth section such that* $\widetilde{\sigma}(\rho_A(u)\mathcal{H}_B) := (\rho_A(u)\mathcal{H}_B, \rho_A(u)\phi(u))$, *for all* $u \in G_A$.

Then for every tangent vector $[(u, X)] \in G_A \times_{G_B} \mathfrak{m}$ *we have*

$$(\nabla\sigma)([(u, X)]) = [(u, \mathrm{d}\phi(u, X) + \rho_A(u)^{-1} p_{\rho_A(u)\mathcal{H}_B}\rho_A(u)\mathrm{d}\rho(X)(\phi(u)))].$$

In particular, if ρ_A *is in addition unitary then*

$$(\nabla\sigma)([(u, X)]) = [(u, \mathrm{d}\phi(u, X) + P(\mathrm{d}\rho_A(X)\phi(u)))].$$

Proof. Notice first that the formula in the case when ρ_A is unitary on $\mathcal{H}_A$ follows from the general formula, since then $p_{\rho_A(u)} = \rho_A(u)P\rho_A(u)^{-1}$ for all $u \in G_A$.

Next, we prove the general formula by specializing Theorem 5.8 to our case.

Recall there is the identification $\mathcal{H}_A = \mathcal{H}^{K_\rho}$ given by $\iota(h) = [(\cdot, P(\rho_A(\cdot)^{-1}h))]$ for every $h \in \mathcal{H}_A$. Hence

$$(\forall u \in G_A) \quad \iota(h)(u\mathrm{G}_B) = [(u, P(\rho_A(u)^{-1}h))]. \tag{6.3}$$

Then, for all $[(v, f)] \in G_A \times_{G_B} \mathcal{H}_B$,

$$K_{[(v,f)]=}\widehat{K}([(v, f)]) = K(\cdot, v^{-*})[(v, f)] = \iota(\rho(v)f) \in \mathcal{H}_A.$$

Now pick any $[(u, X)] \in G_A \times_{G_B} \mathfrak{m}$ and define the curves $\Gamma_X(t) := \exp_{G_A}(tX)$, $\gamma_X(t) := \Gamma_X(t)G_B$, and $u\gamma_X(t) := u\Gamma_X(t)G_B$ for all $t \in \mathbb{R}$. For every $t \in \mathbb{R}$ we have $\sigma(u\gamma_X(t)) = [(u\Gamma_X(t), \phi(u\Gamma_X(t)))]$, so

$$\widehat{K}(\sigma(u\gamma_X(t))) = \iota(\rho_A(u)\rho_A(\Gamma_X(t))\phi(u\Gamma_X(t))).$$

Therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\widehat{K}(\sigma(u\gamma_X(t))) = \iota(\rho_A(u)(\mathrm{d}\rho_A(X)\phi(u) + \mathrm{d}\phi(u, X))),$$

which entails

$$p_{\rho_A(u)\mathcal{H}_B}(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\widehat{K}(\sigma(u\gamma_X(t)))) = \iota(p_{\rho_A(u)\mathcal{H}_B}(\rho_A(u)\mathrm{d}\rho_A(X)\phi(u)) + \rho_A(u)\mathrm{d}\phi(u, X))$$

since $\mathrm{d}\phi(u,X) \in \mathcal{H}_B$. Finally, since $K(s,s^{-*}) = \mathrm{id}_{D_s}$ with $D = G_A \times_{G_B} \mathcal{H}_B$, for every $s \in G_A/G_B$, an application of Theorem 5.8 along with (6.3) concludes the proof. ☐

6.3. **Involutive homogeneous bundles and CP mappings.** We discuss here an example that illustrates Proposition 6.3 by using completely positive mappings (CP mappings, for short) and conditional expectations suitably related in between. We first give a result on dilations of CP maps acting on the right.

Lemma 6.7. *Let A be any unital C^*-algebra, $\mathcal{H}_0$ be any complex Hilbert space, and $\Psi\colon A \to \mathcal{B}(\mathcal{H}_0)$ be any unital CP map. Assume that Ψ is tracial, that is,*

$$\Psi(ab) = \Psi(ba), \quad (a,b \in A).$$

Then there exist a Hilbert space $\mathcal{H}$, an isometry $V\colon \mathcal{H}_0 \to \mathcal{H}$, and a unital $*$-representation $\rho\colon \mathrm{G}_A \to \mathcal{B}(\mathcal{H})$ such that

$$\Psi(a) = V^*\rho(a^{-1})V, \quad (a \in \mathrm{G}_A).$$

Proof. First we sketch briefly how to find $\mathcal{H}$ and V, by using the method of proof of Stinespring's theorem as given for instance in [Pa02, Theorem 4.1].

Define a nonnegative sesquilinear form on $A \otimes \mathcal{H}_0$ by the formula

$$\Big(\sum_{j=1}^{n} b_j \otimes \eta_j \mid \sum_{i=1}^{n} a_i \otimes \xi_i\Big) = \sum_{i,j=1}^{n} (\Phi(a_i^* b_j)\eta_j \mid \xi_i)$$

for all $a_1,\dots,a_n,b_1,\dots,b_n \in A$, $\xi_1,\dots,\xi_n,\eta_1,\dots,\eta_n \in \mathcal{H}_0$ and $n \geq 1$. Consider the linear space $N = \{x \in A \otimes \mathcal{H}_0 \mid (x \mid x) = 0\}$ and denote by $\mathcal{K}_0$ the Hilbert space obtained as the completion of $(A\otimes\mathcal{H}_0)/N$ with respect to the scalar product defined by $(\cdot \mid \cdot)$ on this quotient space. Then denote by $V\colon \mathcal{H}_0 \to \mathcal{K}_0$ the bounded linear map obtained as the composition

$$V\colon \mathcal{H}_0 \to A\otimes\mathcal{H}_0 \to (A\otimes\mathcal{H}_0)/N \hookrightarrow \mathcal{K}_0,$$

where the first map is defined by $A \ni h \mapsto \mathbf{1}\otimes h \in A\otimes\mathcal{H}_0$ and the second map is the natural quotient map. It is readily seen that V is an isometry since Ψ is unital.

Next we are going to construct ρ by linear maps on $A\otimes\mathcal{H}_0$. So define

$$(\forall x \in A, a \in \mathrm{G}_A)(\forall \eta \in \mathcal{H}_0) \qquad \rho(a)(x\otimes\eta) = xa^{-1}\otimes\eta.$$

For $y_i \in A$, $i = 1,\dots,n$, and $b \in A$ the following inequality between positive matrices in $M_n(A)$ holds:

$$(y_i^* b^* b y_j)_{i,j} \leq \|b^*b\|\,(y_i^* y_j)_{i,j}\,; \tag{6.4}$$

see [Pa02, p. 44].

Take then $x_i \in A$, $\xi_i \in \mathcal{H}_0$, $i = 1,\dots,n$, and $a \in \mathrm{G}_A$. Since Ψ is tracial we have $\Psi(a^{-*}x_j^*x_ia^{-1}) = \Psi(x_ia^{-1}a^{-*}x_j^*)$. Hence,

$$\begin{aligned}(\sum_{i=1}^{n} x_ia^{-1}\otimes\xi_i \mid \sum_{j=1}^{n} x_ja^{-1}\otimes\xi_j) &= \sum_{i,j=1}^{n} \big(\Psi(a^{-*}x_j^*x_ia^{-1})\xi \mid \xi_j\big)_{\mathcal{H}_0} \\ &= \sum_{i,j=1}^{n} \big(\Psi(x_ia^{-1}a^{-*}x_j^*)\xi \mid \xi_j\big)_{\mathcal{H}_0} \\ &\leq \|a^{-1}a^{-*}\| \sum_{i,j=1}^{n} \big(\Psi(x_ix_j^*)\xi \mid \xi_j\big)_{\mathcal{H}_0}\end{aligned}$$

$$= \|a^{-1}\|^2 \,(\sum_{i=1}^{n} x_i a^{-1} \otimes \xi_i \mid \sum_{j=1}^{n} x_j a^{-1} \otimes \xi_j),$$

where, for the above inequality, we used (6.4) and the complete positivity of Ψ.

Thus in particular N is invariant under the action of ρ, so that the linear map $\rho(a)$ extends to a linear map (which we continue denoting in the same way) on $A \times \mathcal{H}_0$ defined by

$$\rho(a)\colon x \otimes \xi \in A \otimes \mathcal{H}_0 \mapsto x a^{-1} \otimes \xi \in A \otimes \mathcal{H}_0,$$

which extends in turn as a continuous linear mapping from $\mathcal{H}$ into itself because of the preceding inequality.

Now, it is straightforward to check that the map $\rho\colon \mathrm{G}_A \to \mathcal{B}(\mathcal{H})$ defined as above is a $*$-representation and satisfies the equation $\Psi(a) = V^* \rho(a^{-1}) V$, $(a \in \mathrm{G}_A)$. $\square$

Recall that if, instead of the inverse-right multiplication $x \otimes \xi \mapsto x a^{-1} \otimes \xi$ in $A \otimes \mathcal{H}_0$, one considers the left multiplication $x \otimes \xi \mapsto ax \otimes \xi$, $a \in \mathrm{G}_A$, then one gets precisely the Stinespring dilation λ of Ψ on $\mathcal{H}$, which satisfies $\Psi(a) = V^* \lambda(a) V$ for all $a \in A$.

We call *two-sided Stinespring representation* of a *tracial* completely positive mapping $\Psi\colon A \to \mathcal{B}(\mathcal{H})$ the group representation $\sigma\colon \mathrm{G}_A \to \mathcal{B}(\mathcal{H})$ which is the composition of the above representations λ and ρ:

$$\sigma(a) = \lambda(a) \circ \rho(a) = \rho(a) \circ \lambda(a)\,, \quad (a \in \mathrm{G}_A).$$

Proposition 6.8. *Suppose that $\mathbf{1} \in B \subseteq A$ are C^*-algebras with a conditional expectation $E\colon A \to B$ and that there exists a tracial, unital completely positive map $\Psi\colon A \to \mathcal{B}(\mathcal{H}_0)$ satisfying $\Psi \circ E = \Psi$, where $\mathcal{H}_0$ is a complex Hilbert space. Suppose also that there exists a conjugate-linear isometry $C_0\colon \mathcal{H}_0 \to \mathcal{H}_0$ such that*

$$\Psi(a^*) = C_0 \circ \Psi(a) \circ C_0, \quad (a \in A).$$

Denote by $\sigma_A\colon A \to \mathcal{B}(\mathcal{H}_A)$ and $\sigma_B\colon B \to \mathcal{B}(\mathcal{H}_B)$ the two-sided Stinespring representations associated with Φ and $\Phi|_B$, respectively. Let $C\colon \mathcal{H}_A \to \mathcal{H}_A$ be the conjugate-linear isometry from $\mathcal{H}_A$ onto itself induced by the mapping

$$a \otimes f \mapsto a^* \otimes C_0(f), A \otimes \mathcal{H}_0 \to A \otimes \mathcal{H}_0.$$

Then $\mathcal{H}_B \subseteq \mathcal{H}_A$, $C(\mathcal{H}_B) = \mathcal{H}_B$, and the kernel K_σ given, for $u, v \in \mathrm{G}_A$ and $f \in \mathcal{H}_B$, by

$$K_\sigma(u\mathrm{G}_B, v\mathrm{G}_B)\big([(v^{-*}, f)]\big) := \big[(u, P_{\mathcal{H}_B}(\sigma_A(uv^{-*})f))\big]$$

is self-involutive with respect to the involution τ_C in $D_\sigma := \mathrm{G}_A \times_{\mathrm{G}_B,\sigma} \mathcal{H}_B$ defined as

$$\tau_C\colon [(a,h)] \mapsto [(a^{-*}, Ch)], \quad (a \in \mathrm{G}_A, h \in \mathcal{H}_B).$$

Proof. In [BG08, Lemma 6.7], it has been proven that $\mathcal{H}_B \subseteq \mathcal{H}_A$ and that for every $h_0 \in \mathcal{H}_0$ and $b \in B$ we have the commutative diagrams

$$\begin{array}{ccccc}
A & \xrightarrow{\iota_{h_0}} & \mathcal{H}_A & \xrightarrow{\lambda_A(b)} & \mathcal{H}_A \\
{\scriptstyle E}\downarrow & & {\scriptstyle P_{\mathcal{H}_B}}\downarrow & & \downarrow{\scriptstyle P_{\mathcal{H}_B}} \\
B & \xrightarrow{\iota_{h_0}} & \mathcal{H}_B & \xrightarrow{\lambda_B(b)} & \mathcal{H}_B
\end{array}$$

where $\iota_{h_0}\colon A \to \mathcal{H}_A$ is the map induced by $a \mapsto a \otimes h_0$. In fact, the commutativity of the right diagram is the simple part, and by a similar argument it can be shown that $\rho_B(b) \circ P_{\mathcal{H}_B} = P_{\mathcal{H}_B} \circ \rho_B(b)$ for all $b \in B$. Having in mind that $\sigma = \rho \circ \lambda$ we get the commutativity, for every $b \in B$, of the diagram

$$\begin{array}{ccccc} A & \xrightarrow{\iota_{h_0}} & \mathcal{H}_A & \xrightarrow{\sigma_A(b)} & \mathcal{H}_A \\ {\scriptstyle E}\downarrow & & {\scriptstyle P_{\mathcal{H}_B}}\downarrow & & \downarrow{\scriptstyle P_{\mathcal{H}_B}} \\ B & \xrightarrow{\iota_{h_0}} & \mathcal{H}_B & \xrightarrow{\sigma_B(b)} & \mathcal{H}_B \end{array}$$

This enables us to conclude that the kernel K_σ is well defined.

As for the definition of the isometry C we do observe that for $a_i \in A$, $h_i \in \mathcal{H}_0$, with $i = 1, \dots, n$, we have

$$\begin{aligned} (\sum_{i=1}^n a_i^* \otimes C_0 h_i \mid \sum_{j=1}^n a_j^* \otimes C_0 h_j)_{\mathcal{H}_\Psi} &= \sum_{i,j=1}^n (\Psi(a_j a_i^*) C_0 h_i \mid C_0 h_j)_{\mathcal{H}_0} \\ &= \sum_{i,j=1}^n (\Psi(a_i^* a_j) C_0 h_i \mid C_0 h_j)_{\mathcal{H}_0} = \sum_{i,j=1}^n \big(C_0 \Psi(a_j^* a_i) h_i \mid C_0 h_j\big)_{\mathcal{H}_0} \\ &= \sum_{i,j=1}^n \big(h_j \mid \Psi(a_j^* a_i) h_i\big)_{\mathcal{H}_0} = (\sum_{j=1}^n a_j \otimes h_j \mid \sum_{i=1}^n a_j \otimes h_i)_{\mathcal{H}_\Psi}, \end{aligned}$$

whence it follows that the conjugate-linear mapping $C \colon A \otimes \mathcal{H}_0 \to A \otimes \mathcal{H}_0$ defined by the prescription $a \otimes h \mapsto a^* \otimes C_0 h$ is an isometry and leaves invariant the null subspace $N := \{\alpha \in A \otimes \mathcal{H}_0 : (\alpha \mid \alpha)_{\mathcal{H}_\Psi} = 0\}$. Thus it extends as a conjugate-linear isometry to $\mathcal{H}$.

To end the proof we are going to apply Proposition 6.3, so it only remains to show that $\sigma_A(a^{-*}) = C\sigma_A(a)C$ for each $a \in \mathrm{G}_A$. To do this, take $f := (x \otimes h) + N$ in $\mathcal{H}$ with $x \in A$ and $h \in \mathcal{H}_0$. Then,

$$\begin{aligned} C\sigma_A(a)Cf &= (C\sigma_A(a)C)\,((x \otimes h) + N) \\ &= (C\sigma_A(a))\,((x^* \otimes C_0 h) + N) \\ &= C((ax^*a^{-1} \otimes C_0 h) + N) \\ &= ((ax^*a^{-1})^* \otimes h) + N = \sigma_A(a^{-*})f. \end{aligned}$$

Now the desired equality follows by linearity and density. □

Remark 6.9. Tracial completely positive maps from C^*-algebras into the algebra of bounded operators on a separable Hilbert space are characterized in [ChTs83]; see the theorem on page 59 of the above reference.

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