Multigrid methods for variational inequalities

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Abstract

In this paper we introduce four multigrid algorithms for the constrained minimization of non-quadratic functionals. The algorithm introduced in [L. Badea, *Convergence rate of a Schwarz multilevel method for the constrained minimization of non-quadratic functionals*, SIAM J. Numer. Anal., **44**, 2, 2006, p. 449-477], has a sub-optimal computing complexity because the convex set, which is defined on the finest mesh, is used in the smoothing steps on the coarse levels. The first algorithm we introduce in this paper is a standard V-cycle multigrid iteration which improves the algorithm in the above cited paper, having an optimal computing complexity. This algorithm can be also viewed as performing a multiplicative iteration on each level and a multiplicative one over the levels, too. The three other proposed algorithms are combinations of additive or multiplicative iterations on levels with additive or multiplicative ones over the levels. These algorithms are given for the constrained minimization of non-quadratic functionals where the convex set is of two-obstacle type and have an optimal computing complexity. We give estimations of the global convergence rate in function of the number of levels, and compare our results with the estimations of the asymptotic convergence rate existing in the literature for complementary problems.

Keywords: domain decomposition methods, variational inequalities, non-quadratic minimization, multigrid and multilevel methods, nonlinear obstacle problems
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1 Introduction

The multigrid or multilevel methods for the constrained minimization of functionals have been studied almost exclusively for one-obstacle problems. Such a method has been proposed by Mandel in [19], [20] and [8]. Related methods

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have been introduced by Brandt and Cryer in [5] and Hackbush and Mittelmann in [11]. The method has been studied later by Kornhuber in [13] and extended to variational inequalities of the second kind in [14] and [15]. A variant of this method using truncated nodal basis functions has been introduced by Hoppe and Kornhuber in [12] and analyzed by Kornhuber and Yserentant in [17]. Also, versions of this method have been applied to Signorini's problem in elasticity by Kornhuber and Krause in [16] and Wohlmuth and Krause in [24]. Evidently, the above list of citations is not exhaustive and, for further information, we recommend the review article [10] written by Gräser and Kornhuber.

Regarding the convergence study of the method, an asymptotic convergence rate of $1 - 1/(1 + CJ^3)$, J being the number of levels, has been proved by Kornhuber in [13] for the complementary problem in the bidimensional space. For the two-level method, global convergence rates have been established by Badea Tai and Wang in [4], and for its additive variant by Badea in [3]. Also, a global convergence rate has been also estimated by Tai in [21] for a multilevel subset decomposition method.

In [2], we have introduced a projected multilevel method for constrained minimization problems where the convex set can be more general than of one- or two-obstacle type, for instance. The main drawback of this method, is its sub-optimal computing complexity because the convex set, which is defined on the finest mesh, is used in the smoothing steps on the coarse levels. Also, the global convergence rate we found there in $\mathbf{R}^2$ for quadratic functionals was of $1 - 1/(1 + CJ^5)$ which is weaker than the asymptotic one given in the above cited papers. We introduce in the present paper four multilevel algorithms which have an optimal computing complexity. The first algorithm is a standard V-cycle multigrid iteration which improves the algorithm in the above cited paper. This algorithm can be also viewed as performing a multiplicative iteration on each level and a multiplicative one over the levels, too. The three other proposed algorithms are combinations of additive or multiplicative iterations on levels with additive or multiplicative ones over the levels. These algorithms are given for the constrained minimization of non-quadratic functionals where the convex set is of two-obstacle type and have an optimal computing complexity. We also give estimations of the global convergence rate in function of the number of levels. We found, for instance, that, in $\mathbf{R}^2$, for the minimization of quadratic functionals, the first algorithm has a global convergence rate of $1-1/(1+CJ^3)$, like the asymptotic convergence rate existing in the literature for complementary problems. The methods are described as multigrid V-cycles, but, evidently, the results hold for W-cycle iterations, for instance.

The paper is organized as follows. In Section 2, we state four algorithms in a general framework of reflexive Banach spaces, and prove their convergence under some assumptions. In Section 3 we show that these algorithms can be viewed as multilevel methods for the constraint minimization of non quadratic functionals if we associate finite element spaces to several level meshes and consider decompositions of the domain at each level. We prove that the assumptions made in the previous section hold for convex sets of two-obstacle type. If the decompositions of the domain are made using the supports of the nodal basis functions we get, in Section 4, the multigrid methods. This particular choice of the domain decomposition allow us to obtain better bounds for the convergence rate of the methods. Finally, in Appendix, Section 5, we show that the multilevel method in [2] can be viewed as a particular case of the first algorithm introduced in this paper. We also correct an error in Proposition 5.1 in that paper.

2 Abstract convergence results

We consider a reflexive Banach space V and let $K \subset V$ be a nonempty closed convex set. Let $F : V \rightarrow \mathbf{R}$ be a Gâteaux differentiable functional, which is assumed to be coercive on K, in the sense that $\frac{F(v)}{||v||} \rightarrow \infty$, as $||v|| \rightarrow \infty$, $v \in K$, if K is not bounded. Also, we assume that there exist two real numbers $p,\ q > 1$ such that for any real number $M > 0$ there exist $\alpha_M,\ \beta_M > 0$ for which

$$\begin{array}{l} \alpha_M||v-u||^p \leq < F'(v) - F'(u), v-u > \text{ and} \\ ||F'(v) - F'(u)||_{V'} \leq \beta_M ||v-u||^{q-1} \end{array} \tag{2.1}$$

for any $u, v \in V$ with $||u||, ||v|| \leq M$. Above, we have denoted by F' the Gâteaux derivative of F, and we have marked that the constants α_M and β_M may depend on M. It is evident that if (2.1) holds, then for any $u, v \in V$, $||u||, ||v|| \leq M$, we have

$$\alpha_M||v-u||^p \leq < F'(v) - F'(u), v-u > \leq \beta_M||v-u||^q.$$

Following the way in [9], we can prove that for any $u, v \in V$, $||u||, ||v|| \leq M$, we have

$$\begin{array}{l} < F'(u), v-u > + \frac{\alpha_M}{p}||v-u||^p \leq F(v) - F(u) \leq \\ < F'(u), v-u > + \frac{\beta_M}{q}||v-u||^q. \end{array} \tag{2.2}$$

Also, using the same techniques, we can prove that if F satisfies (2.1), then

$$1 < q \leq 2 \leq p$$

We point out that since F is Gâteaux differentiable and satisfies (2.1), then F is a convex functional (see Proposition 5.5 in [7], pag. 25).

Now, let we assume that we have J closed subspaces of V, $V_1, \ldots, V_J$, and let V_{ji}, $i = 1, \ldots I_j$ be some closed subspaces of V_j, $j = J, \ldots, 1$. The subspaces V_j, $j = J, \ldots, 1$, will be associated with the grid levels, and, for each level $j = J, \ldots, 1$, V_{ji}, $i = 1, \ldots I_j$, are associated with a domain decomposition. Let us write

$$I = \max_{j=J,\ldots,1} I_j$$

In certain cases, the second equation in (2.1) can be refined, and we assume that there exist some constants $0 < \beta_{jk} \leq 1$, $\beta_{jk} = \beta_{kj}$, $j,\ k = J, \ldots, 1$, such that

$$\langle F'(v + v_{ji}) - F'(v), v_{kl} \rangle \leq \beta_M \beta_{jl} ||v_{ji}||^{q-1} ||v_{kl}|| \tag{2.3}$$

for any $v \in V$, $v_{ji} \in V_{ji}$, $v_{kl} \in V_{kl}$ with $||v||$, $||v + v_{ji}||$, $||v_{kl}|| \leq M$, $i = 1, \ldots, I_j$ and $l = 1, \ldots, I_l$. Evidently, in view of (2.1), the above inequality holds for

$$\beta_{jk} = 1,\ j,\ k = J, \ldots, 1 \tag{2.4}$$

We consider the variational inequality

$$u \in K:\ < F'(u), v - u > \geq 0, \text{ for any } v \in K, \tag{2.5}$$

and since the functional F is convex and differentiable, it is equivalent with the minimization problem

$$u \in K : F(u) \leq F(v), \text{ for any } v \in K, \tag{2.6}$$

We can use, for instance, Theorem 8.5 in [18], pag. 251, to prove that problem (2.6) has a unique solution if F has the above properties. In view of (2.2), for a given $M > 0$ such that the solution $u \in K$ of (2.6) satisfies $||u|| \leq M$, we have

$$\frac{\alpha_M}{p} ||v - u||^p \leq F(v) - F(u) \text{ for any } v \in K,\ ||v|| \leq M. \tag{2.7}$$

The algorithms we introduce will be a combination of additive or multiplicative algorithms over the levels with additive or multiplicative algorithms on each level. First, to introduce the algorithms, we make an assumption on choice of the convex sets where we look for the level corrections. The chosen level convex sets depend on the current approximation in the algorithms.

Assumption 2.1. *For a given $w \in K$, we recursively introduce the convex sets K_j, $j = J,\ J-1,\ \ldots, 1$, as*

- at level J: we assume that $0 \in K_J$, $K_J \subset \{v_J \in V_J : w + v_J \in K\}$ and consider a $w_J \in K_J$

- at a level $J - 1 \geq j \geq 1$: we assume that $0 \in K_j$, $K_j \subset \{v_j \in V_j : w + w_J + \ldots + w_{j+1} + v_j \in K\}$ and consider a $w_j \in K_j$

We can easily check that if we take, for $j = J - 1, \ldots, 1$,

$$K_j \subset \{v_j \in V_j : w_{j+1} + v_j \in K_{j+1}\}. \tag{2.8}$$

then $K_j \subset \{v_j \in V_j : w + w_J + \ldots + w_{j+1} + v_j \in K\}$. Evidently, the optimal convergence of the algorithms depends on the effective choice of these level convex sets K_j.

We first introduce the algorithm which is of the multiplicative type over the levels as well as on each level.

Algorithm 2.1. *We start the algorithm with an arbitrary $u^0 \in K$. Assuming that at iteration $n \geq 0$ we have $u^n \in K$, we successively perform the following steps:*

- at the level J, as in Assumption 2.1, with $w = u^n$, we construct the convex set K_J. Then, we first write $w_J^n = 0$, and, for $i = 1, \ldots, I_J$, we successively calculate $w_{Ji}^{n+1} \in V_{Ji}$, $w_J^{n+\frac{i-1}{I_J}} + w_{Ji}^{n+1} \in K_J$, the solution of the inequalities

$$\langle F'(u^n + w_J^{n+\frac{i-1}{I_J}} + w_{Ji}^{n+1}), v_{Ji} - w_{Ji}^{n+1} \rangle \geq 0 \tag{2.9}$$

for any $v_{Ji} \in V_{Ji}$, $w_J^{n+\frac{i-1}{I_J}} + v_{Ji} \in K_J$, and write $w_J^{n+\frac{i}{I_J}} = w_J^{n+\frac{i-1}{I_J}} + w_{Ji}^{n+1}$.

- at a level $J-1 \geq j \geq 1$, as in Assumption 2.1, we construct the convex set K_j with $w = u^n$ and $w_J = w_J^{n+1}, \ldots, w_{j+1} = w_{j+1}^{n+1}$. Then, we write $w_j^n = 0$, and for $i = 1, \ldots, I_j$, we successively calculate $w_{ji}^{n+1} \in V_{ji}$, $w_j^{n+\frac{i-1}{I_j}} + w_{ji}^{n+1} \in K_j$, the solution of the inequalities

$$\langle F'(u^n + \sum_{k=j+1}^{J} w_k^{n+1} + w_j^{n+\frac{i-1}{I_j}} + w_{ji}^{n+1}), v_{ji} - w_{ji}^{n+1} \rangle \geq 0 \tag{2.10}$$

for any $v_{ji} \in V_{ji}$, $w_j^{n+\frac{i-1}{I_j}} + v_{ji} \in K_j$, and write $w_j^{n+\frac{i}{I_j}} = w_J^{n+\frac{i-1}{I_j}} + w_{ji}^{n+1}$.

- we write $u^{n+1} = u^n + \sum_{j=1}^{J} w_j^{n+1}$.

The algorithm which is of the multiplicative type over the levels and of the additive type on each level is written as

Algorithm 2.2. *We start the algorithm with an arbitrary $u^0 \in K$. Assuming that at iteration $n \geq 0$ we have $u^n \in K$, we successively perform the following steps:*

- at the level J, as in Assumption 2.1, we construct the convex set K_J with $w = u^n$. Then, we simultaneously calculate $w_{Ji}^{n+1} \in V_{Ji} \cap K_J$, $i = 1, \ldots, I_J$, the solutions of the inequalities

$$\langle F'(u^n + w_{Ji}^{n+1}), v_{Ji} - w_{Ji}^{n+1} \rangle \geq 0 \tag{2.11}$$

for any $v_{Ji} \in V_{Ji} \cap K_J$, and write $w_J^{n+1} = \frac{r}{I} \sum_{i=1}^{I_J} w_{Ji}^{n+1}$.

- at a level $J-1 \geq j \geq 1$, as in Assumption 2.1, we construct the convex set K_j with $w = u^n$ and $w_J = w_J^{n+1}, \ldots, w_{j+1} = w_{j+1}^{n+1}$. Then, we simultaneously calculate $w_{ji}^{n+1} \in V_{ji} \cap K_j$, $i = 1, \ldots, I_j$, the solutions of the inequalities

$$\langle F'(u^n + \sum_{k=j+1}^{J} w_k^{n+1} + w_{ji}^{n+1}), v_{ji} - w_{ji}^{n+1} \rangle \geq 0 \tag{2.12}$$

for any $v_{ji} \in V_{ji} \cap K_j$, *and write* $w_j^{n+1} = \frac{r}{I} \sum_{i=1}^{I_j} w_{ji}^{n+1}$.

- *we write* $u^{n+1} = u^n + \sum_{j=1}^{J} w_j^{n+1}$.

Above, r *is a constant in the interval* $(0, 1]$.

The algorithm which is of the additive type over the levels and of the multiplicative type on each level is written as,

Algorithm 2.3. *We start the algorithm with an* $u^0 \in K$. *Assuming that at iteration* $n \geq 0$ *we have* $u^n \in K$, *for* $j = 1, \ldots, J$, *we simultaneously perform the following steps*

- *we construct the convex set* K_j *as in Assumption 2.1 with* $w = u^n$ *and* $w_J = \ldots = w_1 = 0$,

- *we write* $w_j^n = 0$, *and for* $i = 1, \ldots, I_j$, *we successively calculate* $w_{ji}^{n+1} \in V_{ji}$, $w_j^{n+\frac{i-1}{I_j}} + w_{ji}^{n+1} \in K_j$, *the solution of the inequalities*

$$\langle F'(u^n + w_j^{n+\frac{i-1}{I_j}} + w_{ji}^{n+1}), v_{ji} - w_{ji}^{n+1} \rangle \geq 0 \tag{2.13}$$

for any $v_{ji} \in V_{ji}$, $w_j^{n+\frac{i-1}{I_j}} + v_{ji} \in K_j$, *and write* $w_j^{n+\frac{i}{I_j}} = w_j^{n+\frac{i-1}{I_j}} + w_{ji}^{n+1}$,

Then, we write $u^{n+1} = u^n + \frac{s}{J} \sum_{j=1}^{J} w_j^{n+1}$, *with a fixed* $0 < s \leq 1$.

Finally, the algorithm which is of the additive type over the levels as well as on each level, is written as,

Algorithm 2.4. *We start the algorithm with an* $u^0 \in K$. *Assuming that at iteration* $n \geq 0$ *we have* $u^n \in K$, *we simultaneously perform, for* $j = 1, \ldots, J$, *the following steps*

- *we construct the convex sets* K_j *as in Assumption 2.1 with* $w = u^n$ *and* $w_J = \ldots = w_1 = 0$,

- *we simultaneously calculate, for* $i = 1 \ldots, I_j$, $w_{ji}^{n+1} \in V_{ji} \cap K_j$, *the solutions of the inequalities*

$$\langle F'(u^n + w_{ji}^{n+1}), v_{ji} - w_{ji}^{n+1} \rangle \geq 0 \tag{2.14}$$

for any $v_{ji} \in V_{ji} \cap K_j$, *and write* $w_j^{n+1} = \frac{r}{I} \sum_{i=1}^{I_j} w_{ji}^{n+1}$, *with a fixed* $0 < r \leq 1$.

Then, we write $u^{n+1} = u^n + \frac{s}{J} \sum_{j=1}^{J} w_j^{n+1}$, *with a fixed* $0 < s \leq 1$.

Evidently, inequalities (2.10), (2.12), (2.13) and (2.14) are equivalent, respectively, with the following minimization problems

– find $w_{ji}^{n+1} \in V_{ji}$, $w_j^{n+\frac{i-1}{I_j}} + w_{ji}^{n+1} \in K_j$,

$$F(u^n + \sum_{k=j+1}^{J} w_k^{n+1} + w_j^{n+\frac{i-1}{I_j}} + w_{ji}^{n+1}) \leq F(u^n + \sum_{k=j+1}^{J} w_k^{n+1} + w_j^{n+\frac{i-1}{I_j}} + v_{ji}) \tag{2.15}$$

for any $v_{ji} \in V_{ji}$, $w_j^{n+\frac{i-1}{I_j}} + v_{ji} \in K_j$,

– find $w_{ji}^{n+1} \in V_{ji} \cap K_j$,

$$F(u^n + \sum_{k=j+1}^{J} w_k^{n+1} + w_{ji}^{n+1}) \leq F(u^n + \sum_{k=j+1}^{J} w_k^{n+1} + v_{ji}) \tag{2.16}$$

for any $v_{ji} \in V_{ji} \cap K_j$,

– find $w_{ji}^{n+1} \in V_{ji}$, $w_j^{n+\frac{i-1}{I_j}} + w_{ji}^{n+1} \in K_j$,

$$F(u^n + w_j^{n+\frac{i-1}{I_j}} + w_{ji}^{n+1}) \leq F(u^n + w_j^{n+\frac{i-1}{I_j}} + v_{ji}) \tag{2.17}$$

for any $v_{ji} \in V_{ji}$, $w_j^{n+\frac{i-1}{I_j}} + v_{ji} \in K_j$,

– find $w_{ji}^{n+1} \in V_{ji} \cap K_j$,

$$F(u^n + w_{ji}^{n+1}) \leq F(u^n + v_{ji}) \tag{2.18}$$

for any $v_{ji} \in V_{ji} \cap K_j$.

In order to prove the convergence of the above algorithms, we shall make new assumptions. First we fix a constant σ satisfying

$$\frac{p}{p-q+1} \leq \sigma \leq p$$

and assume that there exists a constant C_1 such that

$$||\sum_{j=1}^{J} \sum_{i=1}^{I_j} w_{ji}|| \leq C_1 (\sum_{j=1}^{J} \sum_{i=1}^{I_j} ||w_{ji}||^{\sigma})^{\frac{1}{\sigma}} \tag{2.19}$$

for any $w_{ji} \in V_{ji}$, $j = J, \ldots, 1$, $i = 1, \ldots, I_j$. Evidently, we can take, for instance,

$$C_1 = (IJ)^{\frac{\sigma-1}{\sigma}} \tag{2.20}$$

but sharper estimations can be available in certain cases. For Algorithms 2.1 and 2.3, we assume

Assumption 2.2. *There exists two constants $C_2,\ C_3>0$ such that for any $w\in K$, $w_{ji}\in V_{ji}$, $w_{j1}+\ldots+w_{ji}\in K_j$, $j=J,\ldots,1$, $i=1,\ldots,I_j$, and $u\in K$, there exist $u_{ji}\in V_{ji}$, $j=J,\ldots,1$, $i=1,\ldots,I_j$, which satisfy*

$$\begin{aligned}
&u_{j1}\in K_j \text{ and } w_{j1}+\ldots+w_{ji-1}+u_{ji}\in K_j,\ i=2,\ldots,I_j,\ j=J,\ldots,1\\
&u-w=\sum_{j=1}^{J}\sum_{i=1}^{I_j}u_{ji}\\
&\sum_{j=1}^{J}\sum_{i=1}^{I_j}||u_{ji}||^\sigma\le C_2^\sigma||u-w||^\sigma+C_3^\sigma\sum_{j=1}^{J}\sum_{i=1}^{I_j}||w_{ji}||^\sigma
\end{aligned}$$

The convex sets K_j, $j=J,\ldots,1$, are constructed as in Assumption 2.1 with the above w and $w_j=\sum_{i=1}^{I_j}w_{ji}$, $j=J,\ldots,1$, for Algorithm 2.1, and with w and $w_J=\ldots=w_1=0$, for Algorithm 2.3.

For Algorithms 2.2 and 2.4, we assume

Assumption 2.3. *There exists two constants $C_2,\ C_3>0$ such that for any $w\in K$, $w_{ji}\in V_{ji}\cap K_j$, $j=J,\ldots,1$, $i=1,\ldots,I_j$, and $u\in K$, there exist $u_{ji}\in V_{ji}\cap K_j$, $j=J,\ldots,1$, $i=1,\ldots,I_j$, which satisfy*

$$\begin{aligned}
&u-w=\sum_{j=1}^{J}\sum_{i=1}^{I_j}u_{ji}\\
&\sum_{j=1}^{J}\sum_{i=1}^{I_j}||u_{ji}||^\sigma\le C_2^\sigma||u-w||^\sigma+C_3^\sigma\sum_{j=1}^{J}\sum_{i=1}^{I_j}||w_{ji}||^\sigma
\end{aligned}$$

The convex sets K_j, $j=J,\ldots,1$, are constructed as in Assumption 2.1 with the above w and $w_j=\frac{r}{I}\sum_{i=1}^{I_j}w_{ji}$, $j=J,\ldots,1$, for Algorithm 2.2, and with $w\in K$ and $w_J=\ldots=w_1=0$, for Algorithm 2.4.

The convergence result is given by

Theorem 2.1. *We consider that V is a reflexive Banach space, V_j, $j=1,\ldots,J$, are closed subspaces of V, and V_{ji}, $i=1,\ldots,I_j$, are closed subspaces of V_j. Also, let K be a non empty closed convex subset of V, and K_j, $j=1,\ldots,J$, be non empty closed subsets of V_j given by Assumption 2.1. We consider a Gâteaux differentiable functional F on V which is supposed to be coercive if K is not bounded, and which satisfies (2.1). Also, we assume that Assumption 2.2 or 2.3 holds if we refer to Algorithms 2.1 and 2.3, or to Algorithms 2.2 and 2.4, respectively. On these conditions, if u is the solution of problem (2.5) and u^n, $n\ge0$, are its approximations obtained from one of*

Algorithms 2.1–2.4, then there exists $M > 0$ such that $||u||$, $||u^n|| \leq M$, $n \geq 0$, and the following error estimations hold:

(i) if $p = q = 2$ we have

$$F(u^n) - F(u) \leq (\frac{\tilde{C}_1}{\tilde{C}_1 + 1})^n [F(u^0) - F(u)], \tag{2.21}$$

$$||u^n - u||^2 \leq \frac{2}{\alpha_M} (\frac{\tilde{C}_1}{\tilde{C}_1 + 1})^n [F(u^0) - F(u)], \tag{2.22}$$

where $\tilde{C}_1$ is given in (2.45), and

(ii) if $p > q$ we have

$$F(u^n) - F(u) \leq \frac{F(u^0) - F(u)}{[1 + n\tilde{C}_2(F(u^0) - F(u))^{\frac{p-q}{q-1}}]^{\frac{q-1}{p-q}}}, \tag{2.23}$$

$$||u - u^n||^p \leq \frac{p}{\alpha_M} \frac{F(u^0) - F(u)}{[1 + n\tilde{C}_2(F(u^0) - F(u))^{\frac{p-q}{q-1}}]^{\frac{q-1}{p-q}}}, \tag{2.24}$$

where $\tilde{C}_2$ is given in (2.49).

Proof. Step 1. We first prove the boundedness of the approximations u^n of u as well as of the corrections w_{ji}^{n+1} obtained from the above algorithms. For Algorithm 2.1, from (2.15), we get

$$F(u^n + \sum_{k=j}^{J} w_k^{n+1}) \leq F(u^n + \sum_{k=j+1}^{J} w_k^{n+1} + w_j^{n+\frac{i}{I_j}}) \leq$$
$$F(u^n + \sum_{k=j+1}^{J} w_k^{n+1} + w_j^{n+\frac{i-1}{I_j}}) \leq F(u^n + \sum_{k=j+1}^{J} w_k^{n+1})$$

for any $j = J-1, \ldots, 1$ and $i = 1, \ldots, I_j$. Also, from (2.9), we have

$$F(u^n + w_J^{n+\frac{i}{I_J}}) \leq F(u^n + w_J^{n+\frac{i-1}{I_J}}) \leq F(u^n)$$

for any $i = 1, \ldots, I_J$. Since $u^{n+1} = u^n + \sum_{j=1}^{J} w_j^{n+1}$, from these inequalities, we get

$$\begin{aligned} F(u^{n+1}) \leq F(u^n + \sum_{k=j}^{J} w_k^{n+1}) \leq F(u^n + \sum_{k=j+1}^{J} w_k^{n+1} + w_j^{n+\frac{i}{I_j}}) \leq \\ F(u^n + \sum_{k=j+1}^{J} w_k^{n+1}) \leq F(u^n) \leq F(u^0) \end{aligned} \tag{2.25}$$

for any $n \geq 0$, $j = J, \ldots, 1$ and $i = 1, \ldots, I_j$. For the writing simplicity, above, as well as in the following, we make the convention that the sums $\sum_{J+1}^{J}$ are zero.

For Algorithm 2.2, form the convexity of F and (2.16), we get

$$F(u^n + \sum_{k=j}^{J} w_k^{n+1}) = F(u^n + \sum_{k=j+1}^{J} w_k^{n+1} + \frac{r}{I}\sum_{i=1}^{I_j} w_{ji}^{n+1}) \leq$$
$$(1 - \frac{rI_j}{I})F(u^n + \sum_{k=j+1}^{J} w_k^{n+1}) + \frac{r}{I}\sum_{i=1}^{I_j} F(u^n + \sum_{k=j+1}^{J} w_k^{n+1} + w_{ji}^{n+1}) \leq$$
$$F(u^n + \sum_{k=j+1}^{J} w_k^{n+1})$$

Consequently, we have

$$\begin{aligned} F(u^{n+1}) &\leq F(u^n + \sum_{k=j}^{J} w_k^{n+1}) \leq F(u^n + \sum_{k=j+1}^{J} w_k^{n+1} + w_{ji}^{n+1}) \leq \\ F(u^n &+ \sum_{k=j+1}^{J} w_k^{n+1}) \leq F(u^n) \leq F(u^0) \end{aligned} \tag{2.26}$$

for any $n \geq 0$, $j = J, \ldots, 1$ and $i = 1, \ldots, I_j$.

For Algorithm 2.3, form the convexity of F and (2.17), we get

$$F(u^{n+1}) = F(u^n + \frac{s}{J}\sum_{j=1}^{J} w_j^{n+1}) \leq$$
$$(1 - s)F(u^n) + \frac{s}{J}\sum_{j=1}^{J} F(u^n + w_j^{n+1}) \leq F(u^n)$$

Consequently, we have

$$F(u^{n+1}) \leq F(u^n) \leq F(u^0) \text{ and } F(u^n + w_j^{n+\frac{i}{I_j}}) \leq F(u^n) \tag{2.27}$$

for any $n \geq 0$, $j = J, \ldots, 1$ and $i = 1, \ldots, I_j$.

For Algorithm 2.4, form the convexity of F and (2.18), we get

$$F(u^{n+1}) = F(u^n + \frac{s}{J}\sum_{j=1}^{J} w_j^{n+1}) \leq (1 - s)F(u^n) +$$
$$\frac{s}{J}\sum_{j=1}^{J} F(u^n + w_j^{n+1}) = (1 - s)F(u^n) + \frac{s}{J}\sum_{j=1}^{J} F(u^n + \frac{r}{I}\sum_{i=1}^{I_j} w_{ji}^{n+1}) \leq$$
$$(1 - s)F(u^n) + s(1 - \frac{rI_j}{I})F(u^n) + \frac{rs}{IJ}\sum_{j=1}^{J}\sum_{i=1}^{I_j} F(u^n + w_{ji}^{n+1}) \leq F(u^n)$$

Consequently, we have

$$F(u^{n+1}) \le F(u^n) \le F(u^0) \text{ and } F(u^n + w_{ji}^{n+1}) \le F(u^n) \tag{2.28}$$

for any $n \ge 0$, $j = J, \dots, 1$ and $i = 1, \dots, I_j$.

If K is not bounded, from (2.25)–(2.28) and the coerciveness of F, it follows that there exists a $M > 0$, such that $||u||$, $||u^n||$, $||w_{ji}^{n+1}|| \le M$, $n \ge 0$, $j = J, \dots, 1$, $i = 1, \dots, I_j$, for the four Algorithms 2.1–2.4.

Step 2. Now, we evaluate $\sum_{j=1}^{J} \sum_{i=1}^{I_j} ||w_{ji}^{n+1}||^p$ for the four algorithms. For Algorithm 2.1, in view of (2.2), (2.9) and (2.10), we have

$$\tfrac{\alpha_M}{p} ||w_{ji}^{n+1}||^p \le F(u^n + \sum_{k=j+1}^{J} w_k^{n+1} + w_j^{n+\frac{i-1}{I_j}}) - F(u^n + \sum_{k=j+1}^{J} w_k^{n+1} + w_j^{n+\frac{i}{I_j}})$$

for $i = 1, \dots, I_J$, ie.,

$$\frac{\alpha_M}{p} \sum_{i=1}^{I_J} ||w_{ji}^{n+1}||^p \le F(u^n + \sum_{k=j+1}^{J} w_k^{n+1}) - F(u^n + \sum_{k=j}^{J} w_k^{n+1})$$

for $j = J, \dots, 1$, or

$$\frac{\alpha_M}{p} \sum_{j=1}^{J} \sum_{i=1}^{I_j} ||w_{ji}^{n+1}||^p \le F(u^n) - F(u^{n+1}) \tag{2.29}$$

For Algorithm 2.2, in view of (2.2), (2.11) and (2.12), we have

$$\frac{\alpha_M}{p} ||w_{ji}^{n+1}||^p \le F(u^n + \sum_{k=j+1}^{J} w_k^{n+1}) - F(u^n + \sum_{k=j+1}^{J} w_k^{n+1} + w_{ji}^{n+1})$$

But,

$$F(u^n + \sum_{k=j+1}^{J} w_k^{n+1} + w_j^{n+1}) = F(u^n + \sum_{k=j+1}^{J} w_k^{n+1} + \frac{r}{I} \sum_{i=1}^{I_j} w_{ji}^{n+1}) \le$$
$$(1 - \frac{rI_j}{I}) F(u^n + \sum_{k=j+1}^{J} w_k^{n+1}) + \frac{r}{I} \sum_{i=1}^{I_j} F(u^n + \sum_{k=j+1}^{J} w_k^{n+1} + w_{ji}^{n+1})$$

From the above two equations, we get

$$\frac{r}{I} \frac{\alpha_M}{p} \sum_{i=1}^{I_j} ||w_{ji}^{n+1}||^p \le F(u^n + \sum_{k=j+1}^{J} w_k^{n+1}) - F(u^n + \sum_{k=j}^{J} w_k^{n+1})$$

for $j = J, \ldots, 1$, or

$$\frac{r}{I}\frac{\alpha_M}{p}\sum_{j=1}^{J}\sum_{i=1}^{I_j}||w_{ji}^{n+1}||^p \leq F(u^n) - F(u^{n+1}) \tag{2.30}$$

Similar equations are obtained for Algorithm 2.3. Using (2.2) and (2.13), we get

$$\frac{\alpha_M}{p}\sum_{i=1}^{I_J}||w_{ji}^{n+1}||^p \leq F(u^n) - F(u^n + w_j^{n+1})$$

and, in view of the definition of u^{n+1}, we have,

$$F(u^{n+1}) = F(u^n + \frac{s}{J}\sum_{j=1}^{J} w_j^{n+1}) \leq (1-s)F(u^n) + \frac{s}{J}\sum_{j=1}^{J} F(u^n + w_j^{n+1})$$

From the above two equations, we get

$$\frac{s}{J}\frac{\alpha_M}{p}\sum_{j=1}^{J}\sum_{i=1}^{I_j}||w_{ji}^{n+1}||^p \leq F(u^n) - F(u^{n+1}) \tag{2.31}$$

Finally, using (2.2) and (2.14), we get for Algorithm 2.4,

$$\frac{r}{I}\frac{\alpha_M}{p}\sum_{i=1}^{I_j}||w_{ji}^{n+1}||^p \leq F(u^n) - F(u^n + w_j^{n+1})$$

and

$$F(u^{n+1}) \leq (1-s)F(u^n) + \frac{s}{J}\sum_{j=1}^{J} F(u^n + w_j^{n+1})$$

From these two equations, we get

$$\frac{s}{J}\frac{r}{I}\frac{\alpha_M}{p}\sum_{j=1}^{J}\sum_{i=1}^{I_j}||w_{ji}^{n+1}||^p \leq F(u^n) - F(u^{n+1}) \tag{2.32}$$

Therefore, the above equations (2.29)–(2.32) can be written as

$$t\frac{\alpha_M}{p}\sum_{j=1}^{J}\sum_{i=1}^{I_j}||w_{ji}^{n+1}||^p \leq F(u^n) - F(u^{n+1}) \tag{2.33}$$

where

$$t = \begin{cases} 1 & \text{for Algorithm 2.1} \\ \frac{r}{I} & \text{for Algorithm 2.2} \\ \frac{s}{J} & \text{for Algorithm 2.3} \\ \frac{s}{J}\frac{r}{I} & \text{for Algorithm 2.4} \end{cases} \tag{2.34}$$

Step 3. We now estimate $F(u^{n+1}) - F(u)$. First, we evaluate $F(u^{n+1})$ for each algorithm. Evidently, for Algorithm 2.1, we have

$$F(u^{n+1}) = F(u^n + \sum_{j=1}^{J} \sum_{i=1}^{I_j} w_{ji}^{n+1}) \tag{2.35}$$

Using the convexity of F, for Algorithm 2.2, we get

$$\begin{aligned} F(u^{n+1}) = F(u^n + \frac{r}{I} \sum_{j=1}^{J} \sum_{i=1}^{I_j} w_{ji}^{n+1}) \leq \\ (1 - \frac{r}{I})F(u^n) + \frac{r}{I} F(u^n + \sum_{j=1}^{J} \sum_{i=1}^{I_j} w_{ji}^{n+1}) \end{aligned} \tag{2.36}$$

Also, for Algorithm 2.3, we have

$$\begin{aligned} F(u^{n+1}) = F(u^n + \frac{s}{J} \sum_{j=J}^{J} \sum_{i=1}^{I_j} w_{ji}^{n+1}) \leq \\ (1 - \frac{s}{J})F(u^n) + \frac{s}{J} F(u^n + \sum_{j=1}^{J} \sum_{I=1}^{I_j} w_{ji}^{n+1}) \end{aligned} \tag{2.37}$$

Finally, for Algorithm 2.4, we have

$$\begin{aligned} F(u^{n+1}) = F(u^n + \frac{s}{J}\frac{r}{I} \sum_{j=J}^{J} \sum_{i=1}^{I_j} w_{ji}^{n+1}) \leq \\ (1 - \frac{s}{J}\frac{r}{I})F(u^n) + \frac{s}{J}\frac{r}{I} F(u^n + \sum_{j=1}^{J} \sum_{I=1}^{I_j} w_{ji}^{n+1}) \end{aligned} \tag{2.38}$$

From the above four equations we conclude that

$$F(u^{n+1}) \leq (1-t)F(u^n) + tF(u^n + \sum_{j=1}^{J} \sum_{I=1}^{I_j} w_{ji}^{n+1}) \tag{2.39}$$

where t is given in (2.34). Therefore, we can write

$$\begin{aligned} F(u^{n+1}) - F(u) \leq (1-t)(F(u^n) - F(u)) + \\ t(F(u^n + \sum_{j=1}^{J} \sum_{I=1}^{I_j} w_{ji}^{n+1}) - F(u)) \end{aligned} \tag{2.40}$$

With u, the solution of problem (2.5), $w = u^n$ and $w_{ji} = w_{ji}^{n+1}$, $j = J, \ldots, 1$, $I = 1, \ldots, I_j$, we consider the decomposition u_{ji}, $j = J, \ldots, 1$, $I =$

$1,\dots,I_j$, of $u-u^n$ as in Assumption 2.2, if we apply Algorithm 2.1 or 2.3, or as in Assumption 2.3, if we apply Algorithm 2.2 or 2.4. In view of (2.2), we have

$$\begin{aligned}
&F(u^n+\sum_{j=1}^{J}\sum_{i=1}^{I_j}w_{ji}^{n+1})-F(u)+\frac{\alpha_M}{p}||u^n+\sum_{j=1}^{J}\sum_{i=1}^{I_j}w_{ji}^{n+1}-u||^p\le\\
&\langle F'(u^n+\sum_{j=1}^{J}\sum_{i=1}^{I_j}w_{ji}^{n+1}),u^n+\sum_{j=1}^{J}\sum_{i=1}^{I_j}w_{ji}^{n+1}-u\rangle=\\
&\sum_{j=1}^{J}\sum_{i=1}^{I_j}\langle -F'(u^n+\sum_{j=1}^{J}\sum_{i=1}^{I_j}w_{ji}^{n+1}),u_{ji}-w_{ji}^{n+1}\rangle
\end{aligned}\tag{2.41}$$

For Algorithm 2.1, in view of (2.3), (2.9) and (2.10), we get

$$\begin{aligned}
&\langle -F'(u^n+\sum_{j=1}^{J}\sum_{i=1}^{I_j}w_{ji}^{n+1}),u_{ji}-w_{ji}^{n+1}\rangle\le\langle F'(u^n+\sum_{k=j+1}^{J}\sum_{l=1}^{I_k}w_{kl}^{n+1}+\sum_{l=1}^{i}w_{jl}^{n+1})-\\
&F'(u^n+\sum_{j=1}^{J}\sum_{i=1}^{I_j}w_{ji}^{n+1}),u_{ji}-w_{ji}^{n+1}\rangle\le\beta_M\sum_{k=1}^{J}\beta_{kj}\sum_{l=1}^{I_k}||w_{kl}^{n+1}||^{q-1}||u_{ji}-w_{ji}^{n+1}||
\end{aligned}$$

Above, we have added and subtracted the missing terms between $F'(u^n+\sum_{k=j+1}^{J}\sum_{i=1}^{I_k}w_{ki}^{n+1}+\sum_{l=1}^{i}w_{jl}^{n+1})$ and $F'(u^n+\sum_{j=1}^{J}\sum_{i=1}^{I_j}w_{ji}^{n+1})$.

For Algorithm 2.2, in view of (2.3), (2.11) and (2.12), we get

$$\begin{aligned}
&\langle -F'(u^n+\sum_{j=1}^{J}\sum_{i=1}^{I_j}w_{ji}^{n+1}),u_{ji}-w_{ji}^{n+1}\rangle\le\langle F'(u^n+\frac{r}{I}\sum_{k=j+1}^{J}\sum_{l=1}^{I_k}w_{kl}^{n+1}+w_{ji}^{n+1})-\\
&F'(u^n+\sum_{j=1}^{J}\sum_{i=1}^{I_j}w_{ji}^{n+1}),u_{ji}-w_{ji}^{n+1}\rangle\le 2\beta_M\sum_{k=1}^{J}\beta_{kj}\sum_{l=1}^{I_k}||w_{kl}^{n+1}||^{q-1}||u_{ji}-w_{ji}^{n+1}||
\end{aligned}$$

Here, we have added and subtracted the missing terms between $F'(u^n)$ and $F'(u^n+\frac{r}{I}\sum_{k=j+1}^{J}\sum_{i=1}^{I_k}w_{ki}^{n+1}+w_{ji}^{n+1})$, between $F'(u^n)$ and $F'(u^n+\sum_{j=1}^{J}\sum_{i=1}^{I_j}w_{ji}^{n+1})$, and used the fact that $\frac{r}{I}\le 1$.

Similarly, for Algorithm 2.3, using (2.13), we have

$$\begin{aligned}
&\langle -F'(u^n+\sum_{j=1}^{J}\sum_{i=1}^{I_j}w_{ji}^{n+1}),u_{ji}-w_{ji}^{n+1}\rangle\le\langle F'(u^n+\sum_{l=1}^{i}w_{jl}^{n+1})-\\
&F'(u^n+\sum_{j=1}^{J}\sum_{i=1}^{I_j}w_{ji}^{n+1}),u_{ji}-w_{ji}^{n+1}\rangle\le 2\beta_M\sum_{k=1}^{J}\beta_{kj}\sum_{l=1}^{I_k}||w_{kl}^{n+1}||^{q-1}||u_{ji}-w_{ji}^{n+1}||
\end{aligned}$$

Finally, using (2.14), we get the same inequality for Algorithm 2.4,

$$\langle -F'(u^n+\sum_{j=1}^{J}\sum_{i=1}^{I_j}w_{ji}^{n+1}), u_{ji}-w_{ji}^{n+1}\rangle \leq \langle F'(u^n+w_{ji}^{n+1})-$$
$$F'(u^n+\sum_{j=1}^{J}\sum_{i=1}^{I_j}w_{ji}^{n+1}), u_{ji}-w_{ji}^{n+1}\rangle \leq 2\beta_M\sum_{k=1}^{J}\beta_{kj}\sum_{l=1}^{I_k}||w_{kl}^{n+1}||^{q-1}||u_{ji}-w_{ji}^{n+1}||$$

Consequently, in view of (2.41) and Assumptions 2.2 and 2.3, we can write for all the four algorithms,

$$F(u^n+\sum_{j=1}^{J}\sum_{i=1}^{I_j}w_{ji}^{n+1})-F(u)+\frac{\alpha_M}{p}||u^n+\sum_{j=1}^{J}\sum_{i=1}^{I_j}w_{ji}^{n+1}-u||^p \leq$$
$$2\beta_M\sum_{j=1}^{J}\sum_{k=1}^{J}\beta_{jk}\sum_{l=1}^{I_k}||w_{kl}^{n+1}||^{q-1}\sum_{i=1}^{I_j}||u_{ji}-w_{ji}^{n+1}|| \leq$$
$$2\beta_M I^{\frac{\sigma-1}{\sigma}+\frac{p-q+1}{p}}\sum_{k=1}^{J}\left(\sum_{j=1}^{J}\beta_{jk}(\sum_{i=1}^{I_j}||u_{ji}-w_{ji}^{n+1}||^{\sigma})^{\frac{1}{\sigma}}\right)\left(\sum_{l=1}^{I_k}||w_{kl}^{n+1}||^p\right)^{\frac{q-1}{p}} \leq$$
$$2\beta_M I^{\frac{\sigma-1}{\sigma}+\frac{p-q+1}{p}}\left[\sum_{k=1}^{J}\left(\sum_{j=1}^{J}\beta_{jk}(\sum_{i=1}^{I_j}||u_{ji}-w_{ji}^{n+1}||^{\sigma})^{\frac{1}{\sigma}}\right)^{\sigma}\right]^{\frac{1}{\sigma}}\cdot$$
$$\left(\sum_{k=1}^{J}(\sum_{l=1}^{I_k}||w_{kl}^{n+1}||^p)^{\frac{q-1}{p}\frac{\sigma}{\sigma-1}}\right)^{\frac{\sigma-1}{\sigma}} \leq 2\beta_M I^{\frac{\sigma-1}{\sigma}+\frac{p-q+1}{p}}J^{\frac{\sigma-1}{\sigma}-\frac{q-1}{p}}(\max_{k=1,\cdots,J}\sum_{j=1}^{J}\beta_{kj})\cdot$$
$$(\sum_{j=1}^{J}\sum_{i=1}^{I_j}||u_{ji}-w_{ji}^{n+1}||^{\sigma})^{\frac{1}{\sigma}}(\sum_{j=1}^{J}\sum_{i=1}^{I_j}||w_{ji}^{n+1}||^p)^{\frac{q-1}{p}}$$

We have used above the inequality (see Corollary 4.1 in [22])

$$||Ax||_{l^\sigma} \leq (\max_i\sum_j|A_{ij}|)||x||_{l^\sigma} \tag{2.42}$$

where $A = (A_{ij})_{ij}$ is a symmetric matrix. In view of (2.19), we have

$$(\sum_{j=1}^{J}\sum_{i=1}^{I_j}||u_{ji} - w_{ji}^{n+1}||^{\sigma})^{\frac{1}{\sigma}} \leq (\sum_{j=1}^{J}\sum_{i=1}^{I_j}||u_{ji}||^{\sigma})^{\frac{1}{\sigma}} + (\sum_{j=1}^{J}\sum_{i=1}^{I_j}||w_{ji}^{n+1}||^{\sigma})^{\frac{1}{\sigma}} \leq$$
$$(C_2^{\sigma}||u - u^n||^{\sigma} + C_3^{\sigma}\sum_{j=1}^{J}\sum_{i=1}^{I_j}||w_{ji}^{n+1}||^{\sigma})^{\frac{1}{\sigma}} + (\sum_{j=1}^{J}\sum_{i=1}^{I_j}||w_{ji}^{n+1}||^{\sigma})^{\frac{1}{\sigma}} \leq$$
$$C_2||u - u^n|| + (1 + C_3)(\sum_{j=1}^{J}\sum_{i=1}^{I_j}||w_{ji}^{n+1}||^{\sigma})^{\frac{1}{\sigma}} \leq$$
$$C_2||u - u^n - \sum_{j=1}^{J}\sum_{i=1}^{I_j}w_{ji}^{n+1}|| + (1 + C_1C_2 + C_3)(IJ)^{\frac{p-\sigma}{p\sigma}}(\sum_{j=1}^{J}\sum_{i=1}^{I_j}||w_{ji}^{n+1}||^{p})^{\frac{1}{p}}$$

Therefore, we get

$$F(u^n + \sum_{j=1}^{J}\sum_{i=1}^{I_j}w_{ji}^{n+1}) - F(u) + \frac{\alpha_M}{p}||u^n + \sum_{j=1}^{J}\sum_{i=1}^{I_j}w_{ji}^{n+1} - u||^p \leq$$
$$2\beta_M I^{\frac{\sigma-1}{\sigma}+\frac{p-q+1}{p}} J^{\frac{\sigma-1}{\sigma}-\frac{q-1}{p}}(\max_{k=1,\cdots,J}\sum_{j=1}^{J}\beta_{kj})\cdot$$
$$\Bigg[C_2||u - u^n - \sum_{j=1}^{J}\sum_{i=1}^{I_j}w_{ji}^{n+1}||(\sum_{j=1}^{J}\sum_{i=1}^{I_j}||w_{ji}^{n+1}||^{p})^{\frac{q-1}{p}} +$$
$$(1 + C_1C_2 + C_3)(IJ)^{\frac{p-\sigma}{p\sigma}}(\sum_{j=1}^{J}\sum_{i=1}^{I_j}||w_{ji}^{n+1}||^{p})^{\frac{q}{p}}\Bigg]$$

But, for any $\varepsilon > 0$, $p > 1$ and $x, y \geq 0$, we have $xy \leq \varepsilon x^p + \frac{1}{\varepsilon^{\frac{1}{p-1}}}y^{\frac{p}{p-1}}$. Consequently, we have

$$F(u^n + \sum_{j=1}^{J}\sum_{i=1}^{I_j}w_{ji}^{n+1}) - F(u) + \frac{\alpha_M}{p}||u^n + \sum_{j=1}^{J}\sum_{i=1}^{I_j}w_{ji}^{n+1} - u||^p \leq$$
$$2\beta_M I^{\frac{\sigma-1}{\sigma}+\frac{p-q+1}{p}} J^{\frac{\sigma-1}{\sigma}-\frac{q-1}{p}}(\max_{k=1,\cdots,J}\sum_{j=1}^{J}\beta_{kj})\cdot$$
$$\Bigg[C_2\varepsilon||u - u^n - \sum_{j=1}^{J}\sum_{i=1}^{I_j}w_{ji}^{n+1}||^p + C_2\frac{1}{\varepsilon^{\frac{1}{p-1}}}(\sum_{j=1}^{J}\sum_{i=1}^{I_j}||w_{ji}^{n+1}||^{p})^{\frac{q-1}{p-1}} +$$
$$(1 + C_1C_2 + C_3)(IJ)^{\frac{p-\sigma}{p\sigma}}(\sum_{j=1}^{J}\sum_{i=1}^{I_j}||w_{ji}^{n+1}||^{p})^{\frac{q}{p}}\Bigg]$$

for any $\varepsilon > 0$. With

$$\varepsilon = \frac{\alpha_M}{p} \frac{1}{2C_2\beta_M I^{\frac{\sigma-1}{\sigma}+\frac{p-q+1}{p}} J^{\frac{\sigma-1}{\sigma}-\frac{q-1}{p}} (\max_{k=1,\cdots,J} \sum_{j=1}^{J} \beta_{kj})} \tag{2.43}$$

the above equation becomes,

$$F(u^n + \sum_{j=1}^{J}\sum_{i=1}^{I_j} w_{ji}^{n+1}) - F(u) \le \frac{\frac{\alpha_M}{p}}{C_2\varepsilon} \cdot$$
$$\left[\frac{C_2}{\varepsilon^{\frac{1}{p-1}}} (\sum_{j=1}^{J}\sum_{i=1}^{I_j} ||w_{ji}^{n+1}||^p)^{\frac{q-1}{p-1}} + (1 + C_1C_2 + C_3)(IJ)^{\frac{p-\sigma}{p\sigma}} (\sum_{j=1}^{J}\sum_{i=1}^{I_j} ||w_{ji}^{n+1}||^p)^{\frac{q}{p}} \right]$$

From this equation and (2.33)

$$F(u^n + \sum_{j=1}^{J}\sum_{i=1}^{I_j} w_{ji}^{n+1}) - F(u) \le \frac{\frac{\alpha_M}{p}}{C_2\varepsilon} \cdot \left[\frac{C_2}{\varepsilon^{\frac{1}{p-1}} (t\frac{\alpha_M}{p})^{\frac{q-1}{p-1}}} (F(u^n) - F(u^{n+1}))^{\frac{q-1}{p-1}} + \right.$$
$$\left. \frac{(1 + C_1C_2 + C_3)(IJ)^{\frac{p-\sigma}{p\sigma}}}{(t\frac{\alpha_M}{p})^{\frac{q}{p}}} (F(u^n) - F(u^{n+1}))^{\frac{q}{p}} \right]$$

with t in (2.34) and ε in (2.43). In view of the above equation and (2.40), we have

$$\begin{aligned} &F(u^{n+1}) - F(u) \le \tfrac{1-t}{t}(F(u^n) - F(u^{n+1})) + \tfrac{\frac{\alpha_M}{p}}{C_2\varepsilon} \cdot \\ &\left[\frac{C_2}{\varepsilon^{\frac{1}{p-1}} (t\frac{\alpha_M}{p})^{\frac{q-1}{p-1}}} (F(u^n) - F(u^{n+1}))^{\frac{q-1}{p-1}} + \right. \\ &\left. \frac{(1 + C_1C_2 + C_3)(IJ)^{\frac{p-\sigma}{p\sigma}}}{(t\frac{\alpha_M}{p})^{\frac{q}{p}}} (F(u^n) - F(u^{n+1}))^{\frac{q}{p}} \right] \end{aligned} \tag{2.44}$$

Step 4. We prove error estimations (2.21)–(2.24). First, using (2.7), we see that error estimations in (2.22) and (2.24) can be obtained from (2.21) and (2.23), respectively. Now, if $p = q = 2$, then $\sigma = 2$, and from the above equation, we easily get equation (2.21), where

$$\begin{aligned} &\tilde{C}_1 = \frac{1-t}{t} + \frac{1}{C_2 t\varepsilon} \left[\frac{C_2}{\varepsilon} + 1 + C_1C_2 + C_3 \right] \text{ with} \\ &\varepsilon = \frac{\frac{\alpha_M}{2}}{2C_2\beta_M I(\max_{k=1,\cdots,J} \sum_{j=1}^{J} \beta_{kj})} \end{aligned} \tag{2.45}$$

Finally, if $p > q$, from (2.44), we have

$$F(u^{n+1}) - F(u) \le \tilde{C}_3 (F(u^n) - F(u^{n+1}))^{\frac{q-1}{p-1}} \tag{2.46}$$

where

$$\begin{aligned}\tilde{C}_3 = \frac{1-t}{t}(F(u^0) - F(u))^{\frac{p-q}{p-1}} + \frac{\frac{\alpha_M}{p}}{C_2 \varepsilon} \Bigg[\frac{C_2}{\varepsilon^{\frac{1}{p-1}} (t\frac{\alpha_M}{p})^{\frac{q-1}{p-1}}} + \\ \frac{(1 + C_1 C_2 + C_3)(IJ)^{\frac{p-\sigma}{p\sigma}}}{(t\frac{\alpha_M}{p})^{\frac{q}{p}}} (F(u^0) - F(u))^{\frac{p-q}{p(p-1)}} \Bigg]\end{aligned} \tag{2.47}$$

with ε in (2.43). From (2.46), we get

$$F(u^{n+1}) - F(u) + \frac{1}{\tilde{C}_3^{\frac{p-1}{q-1}}} (F(u^{n+1}) - F(u))^{\frac{p-1}{q-1}} \le F(u^n) - F(u),$$

and we know (see Lemma 3.2 in [22]) that for any $r > 1$ and $c > 0$, if $x \in (0, x_0]$ and $y > 0$ satisfy $y + cy^r \le x$, then $y \le (\frac{c(r-1)}{crx_0^{r-1}+1} + x^{1-r})^{\frac{1}{1-r}}$. Consequently, we have $F(u^{n+1}) - F(u) \le [\tilde{C}_2 + (F(u^n) - F(u))^{\frac{q-p}{q-1}}]^{\frac{q-1}{q-p}}$, from which,

$$F(u^{n+1}) - F(u) \le [(n+1)\tilde{C}_2 + (F(u^0) - F(u))^{\frac{q-p}{q-1}}]^{\frac{q-1}{q-p}}, \tag{2.48}$$

where

$$\tilde{C}_2 = \frac{p-q}{(p-1)(F(u^0) - F(u))^{\frac{p-q}{q-1}} + (q-1)\tilde{C}_3^{\frac{p-1}{q-1}}}. \tag{2.49}$$

Equation (2.48) is another form of equation (2.23). □

3 Multilevel Schwarz methods

We consider a family of regular meshes $\mathcal{T}_{h_j}$ of mesh sizes h_j, $j = 1, \ldots, J$ over the domain $\Omega \subset \mathbf{R}^d$. We write $\Omega_j = \cup_{\tau \in \mathcal{T}_{h_j}} \tau$ and we assume that $\mathcal{T}_{h_{j+1}}$ is a refinement of $\mathcal{T}_{h_j}$ on Ω_j, $j = 1, \ldots, J-1$, and $\Omega_1 \subset \Omega_2 \subset \ldots \subset \Omega_J = \Omega$. Also, we assume that, if a node of $\mathcal{T}_{h_j}$ lies on $\partial\Omega_j$, then it lies on $\partial\Omega_{j+1}$, too, that is, it lies on $\partial\Omega$. Besides, we suppose that $\text{dist}_{x_{j+1} \text{node of } \mathcal{T}_{h_{j+1}}}(x_{j+1}, \Omega_j) \le Ch_j$, $j = 1, \ldots, J-1$. In this section, C denotes a generic positive constant independent of the mesh sizes, the number of meshes, as well as of the overlapping parameters and the number of subdomains in the domain decompositions which will be considered later. Since the mesh $\mathcal{T}_{h_{j+1}}$ is a refinement of $\mathcal{T}_{h_j}$, we have $h_{j+1} \le h_j$, and assume that there exists a constant γ, independent of the number of meshes or their sizes, such that

$$1 < \gamma \le \frac{h_j}{h_{j+1}} \le C\gamma, \quad j = 1, \ldots, J-1. \tag{3.1}$$

Since $h_{j+1} \leq \delta_{j+1}$, we also have

$$\frac{h_j}{\delta_{j+1}} \leq C\gamma, \quad j = 1, \ldots, J-1. \tag{3.2}$$

At each level $j = 1, \ldots, J$, we consider an overlapping decomposition $\{\Omega_j^i\}_{1 \leq i \leq I_j}$ of Ω, and assume that the mesh partition $\mathcal{T}_{h_j}$ of Ω_j supplies a mesh partition for each Ω_j^i, $1 \leq i \leq I_j$. Also, we assume that the overlapping size for the domain decomposition at the level $1 \leq j \leq J$ is δ_j. In addition, we suppose that if ω_{j+1}^i is a connected component of Ω_{j+1}^i, $j = 1, \ldots, J-1$, $i = 1, \ldots, I_j$, then

$$\text{diam}(\omega_{j+1}^i) \leq C h_j \tag{3.3}$$

Finally, we assume that $I_1 = 1$.

At each level $j = 1, \ldots, J$, we introduce the linear finite element spaces,

$$V_{h_j} = \{v \in C(\bar{\Omega}_j) : v|_\tau \in P_1(\tau),\ \tau \in \mathcal{T}_{h_j},\ v = 0 \text{ on } \partial\Omega_j\}, \tag{3.4}$$

and, for $i = 1, \ldots, I_j$, we write

$$V_{h_j}^i = \{v \in V_{h_j} : v = 0 \text{ in } \Omega_j \backslash \Omega_j^i\}. \tag{3.5}$$

The functions in V_{h_j} $j = 1, \ldots, J-1$, will be extended with zero outside Ω_j and the spaces will be considered as subspaces of $W^{1,\sigma}$, $1 \leq \sigma \leq \infty$. We denote by $||\cdot||_{0,\sigma}$ the norm in L^σ, and by $||\cdot||_{1,\sigma}$ and $|\cdot|_{1,\sigma}$ the norm and seminorm in $W^{1,\sigma}$, respectively.

We consider the two sided obstacle problem

$$u \in K : \ < F'(u), v - u > \geq 0, \text{ for any } v \in K, \tag{3.6}$$

where

$$K = \{v \in V_{h_J} : \varphi \leq v \leq \psi\}, \tag{3.7}$$

with $\varphi,\ \psi \in V_{h_J}$, $\varphi \leq \psi$. We shall prove that Assumptions 2.1–2.3 hold for this type of convex set, and explicitly write the constants C_2 and C_3 in function of the mesh and overlapping parameters. We can then conclude from Theorem 2.1 that if the functional F has the asked properties, then Algorithms 2.1–2.4 are globally convergent.

We first introduce the operators $I_{h_j} : V_{h_{j+1}} \rightarrow V_{h_j}$, $j = 1, \ldots, J-1$, defined as follows (see [2]). Let us denote by x_{ji} a node of $\mathcal{T}_{h_j}$, by ϕ_{ji} the linear nodal basis function associated with x_{ji} and $\mathcal{T}_{h_j}$, and by ω_{ji} the support of ϕ_{ji}. Given a $v \in V_{h_{j+1}}$, we write $I_{ji}^- v = \min_{x \in \omega_{ji}} v(x)^-$ and $I_{ji}^+ v = \min_{x \in \omega_{ji}} v(x)^+$, where $v(x)^- = \max(0, -v(x))$ and $v(x)^+ = \max(0, v(x))$. We notice that, since v is piecewise linear, $I_{ji}^- v$ or $I_{ji}^+ v$ are attained at a node of $\mathcal{T}_{h_{j+1}}$. Next, we define $I_{h_j}^- v := \sum_{x_{ji} \text{node of } \mathcal{T}_{h_j}} (I_{ji}^- v)\phi_{ji}(x)$ and $I_{h_j}^+ v := \sum_{x_{ji} \text{ node of } \mathcal{T}_{h_j}} (I_{ji}^+ v)\phi_{ji}(x)$, and write $I_{h_j}v = I^+_{h_j}v - I^-_{h_j}v$. It is simple to check that if $v(x) = 0$ at a point $x \in \Omega$, then $I_{h_j}v$ vanishes in a neighborhood of x, composed by the elements τ of $\mathcal{T}_{h_j}$ containing that point. Also,

$$\begin{array}{l} 0 \le I_{h_j}v(x) \le v(x) \text{ if } v(x) \ge 0 \text{ and} \\ 0 \ge I_{h_j}v(x) \ge v(x) \text{ if } v(x) \le 0 \end{array} \tag{3.8}$$

at any point $x \in \Omega$. Consequently, the function

$$\theta_v(x) = \begin{cases} \frac{I_{h_j}v(x)}{v(x)} & \text{if } v(x) \neq 0 \\ 0 & \text{if } v(x) = 0 \end{cases}$$

is well defined, continuous and satisfies

$$0 \le \theta_v(x) \le 1 \text{ for any } x \in \Omega \tag{3.9}$$

Also, for any $v,\ w \in V_{h_{j+1}}$, we have

$$v \le w \text{ in } \Omega \text{ implies } I_{h_j}v \le I_{h_j}w \text{ in } \Omega \tag{3.10}$$

We shall use these properties of the operator I_{h_j} in the following. We also recall the estimations of Lemma 4.2 in [2]: for any $v \in V_{h_{j+1}}$, we have

$$||I_{h_j}v - v||_{0,\sigma} \le C h_j C_{d,\sigma}(h_j, h_{j+1})|v|_{1,\sigma} \tag{3.11}$$

and

$$||I_{h_j}v||_{0,\sigma} \le ||v||_{0,\sigma} \text{ and } |I_{h_j}v|_{1,\sigma} \le C C_{d,\sigma}(h_j, h_{j+1})|v|_{1,\sigma} \tag{3.12}$$

where

$$C_{d,\sigma}(H,h) = \begin{cases} 1 & \text{if } d = \sigma = 1 \\ & \text{or } 1 \le d < \sigma < \infty \\ (\ln \frac{H}{h} + 1)^{\frac{d-1}{d}} & \text{if } 1 < d = \sigma < \infty \\ (\frac{H}{h})^{\frac{d-\sigma}{\sigma}} & \text{if } 1 \le \sigma < d < \infty, \end{cases} \tag{3.13}$$

It is proved in Lemma 4.2 in [2] that $||I_{h_j}v||_{0,\sigma} \le C||v||_{0,\sigma}$, but in view of (3.8), we can take $C = 1$.

Now, we define the level convex sets $K_j \subset V_{h_j}$, $j = J, \ldots, 1$, satisfying Assumption 2.1. Let K be the convex set defined in (3.7), and a $w \in K$. For the level J, we define

$$\begin{array}{l} \varphi_J = \varphi - w,\ \psi_J = \psi - w, \\ K_J = [\varphi_J, \psi_J], \text{ and consider an arbitrary } w_J \in K_J \end{array} \tag{3.14}$$

At a level $j = J-1, \ldots, 1$, we define

$$\begin{array}{l} \varphi_j = I_{h_j}(\varphi_{j+1} - w_{j+1}),\ \psi_j = I_{h_j}(\psi_{j+1} - w_{j+1}), \\ K_j = [\varphi_j, \psi_j], \text{ and consider an arbitrary } w_j \in K_j \end{array} \tag{3.15}$$

We have

Proposition 3.1. *Assumption 2.1 holds for the convex sets K_j, $j = J, \ldots, 1$, defined in (3.14) and (3.15), for any $w \in K$.*

Proof. Evidently, $0 \in K_J$. Also, in view of (3.8), we recurrently get that $0 \in K_j$ for $j = J-1, \ldots, 1$. Form the definition of K_J, we have $w + v_J \in K$ for any $v_J \in K_J$. Finally, we prove (2.8) for $j = J-1, \ldots, 1$. Let $v_j \in K_j$. Using again (3.8), we get

$$\begin{aligned} &\varphi_{j+1} - w_{j+1} \leq I_{h_j}(\varphi_{j+1} - w_{j+1}) = \varphi_j \leq v_j \leq \\ &\psi_j = I_{h_j}(\psi_{j+1} - w_{j+1}) \leq \psi_{j+1} - w_{j+1} \end{aligned}$$

ie., $w_{j+1} + v_j \in K_{j+1}$ □

Now, in order to prove that Assumptions 2.2 and 2.3 hold for the convex sets defined in (3.14) and (3.15), we consider $u,\ w \in K$ and some $w_j \in K_j$, $j = J, \ldots, 1$. First, we define

(3.16) $$v_J = u - w \text{ and } v_j = I_{h_j}(v_{j+1} - w_{j+1}) \text{ for } j = J-1, \ldots, 1$$

and then,

(3.17) $$\begin{aligned} &u_j = v_j - v_{j-1} = v_j - I_{h_{j-1}}(v_j - w_j) \text{ for } j = J, \ldots, 2 \\ &u_1 = v_1 = I_{h_1}(v_2 - w_2) \end{aligned}$$

With these notations, we have

Lemma 3.1. *If K_j are defined in (3.14) and (3.15), and v_j and u_j are defined in (3.16) and (3.17), respectively, then $v_j,\ u_j \in K_j$, $j = J, \ldots, 1$, and*

(3.18) $$u - w = \sum_{j=1}^{J} u_j$$

Proof. The writing of $u - w$ as in (3.18) is evident from (3.16) and (3.17). We prove that $v_j \in K_j,\ j = J, \ldots, 1$ by induction. First,

$$\varphi_J = \varphi - w \leq u - w \leq \psi - w = \psi_J$$

and therefore, $v_J \in K_J$. For a $j = J-1, \ldots, 1$, assuming that $v_{j+1} \in K_{j+1}$, from (3.10), we have

$$\varphi_j = I_{h_j}(\varphi_{j+1} - w_{j+1}) \leq I_{h_j}(v_{j+1} - w_{j+1}) \leq I_{h_j}(\psi_{j+1} - w_{j+1}) = \psi_j$$

or $v_j \in K_j$. For $j = J, \ldots, 2$, using (3.9), we have

$$u_j = v_j - I_{h_{j-1}}(v_j - w_j) = (1 - \theta_{v_j - w_j})v_j + \theta_{v_j - w_j} w_j$$

and therefore, $u_j \in K_j$. □

Another result we use is given by

Lemma 3.2. *If u_j are defined in (3.17), then*

$$|u_j|_{1,\sigma}^\sigma \le C(J-1)^{\sigma-1} C_{d,\sigma}(h_{j-1}, h_J)^\sigma [\sum_{k=2}^{J} |w_k|_{1,\sigma}^\sigma + |u-w|_{1,\sigma}^\sigma] \tag{3.19}$$

for $j = J, \dots, 1$, where we take $h_0 = h_1$ for $j = 1$, and

$$\begin{aligned} &||u_j||_{0,\sigma}^\sigma \le ||w_j||_{0,\sigma}^\sigma + C(J-1)^{\sigma-1} h_{j-1}^\sigma C_{d,\sigma}(h_j, h_J)^\sigma \cdot \\ &[\sum_{k=2}^{J} |w_k|_{1,\sigma}^\sigma + |u-w|_{1,\sigma}^\sigma], \text{ for } j = J, \dots, 2, \text{ and} \\ &||u_1||_{0,\sigma}^\sigma \le C(J-1)^{\sigma-1}[||u-w||_{0,\sigma}^\sigma + \sum_{j=2}^{J} ||w_j||_{0,\sigma}^\sigma] \end{aligned} \tag{3.20}$$

Proof. With v_j in (3.16), we write

$$v_j - w_j = -w_j + I_{h_j}(v_{j+1} - w_{j+1}),\ j = J-1, \dots, 1$$

and, using Lemma 5.1 in [2] for $v_j - w_j$, we get

$$\begin{aligned} &|v_j|_{1,\sigma}^\sigma = |I_{h_j}(v_{j+1} - w_{j+1})|_{1,\sigma}^\sigma \le C(J-j)^{\sigma-1} \cdot \\ &[\sum_{k=j+1}^{J-1} C_{d,\sigma}(h_j, h_k)^\sigma |w_k|_{1,\sigma}^\sigma + C_{d,\sigma}(h_j, h_J)^\sigma |v_J - w_J|_{1,\sigma}^\sigma] \end{aligned}$$

Consequently, we have

$$|v_j|_{1,\sigma}^\sigma \le C(J-j)^{\sigma-1} C_{d,\sigma}(h_j, h_J)^\sigma (\sum_{k=j+1}^{J} |w_k|_{1,\sigma}^\sigma + |u-w|_{1,\sigma}^\sigma) \tag{3.21}$$

for $j = J-1, \dots, 1$. Since $u_j = v_j - v_{j-1}$, for $j = J-1, \dots, 2$, we get

$$|u_j|_{1,\sigma}^\sigma \le C(J-j+1)^{\sigma-1} C_{d,\sigma}(h_{j-1}, h_J)^\sigma (\sum_{k=j}^{J} |w_k|_{1,\sigma}^\sigma + |u-w|_{1,\sigma}^\sigma) \tag{3.22}$$

Since $u_1 = v_1$, we have

$$|u_1|_{1,\sigma}^\sigma \le C(J-1)^{\sigma-1} C_{d,\sigma}(h_1, h_J)^\sigma (\sum_{k=2}^{J} |w_k|_{1,\sigma}^\sigma + |u-w|_{1,\sigma}^\sigma) \tag{3.23}$$

Also, from (3.16), (3.17) and (3.12), we have

$$\begin{aligned} &|u_J|_{1,\sigma} = |u - w - I_{h_{J-1}}(u - w - w_J)|_{1,\sigma} \le \\ &(1 + CC_{d,\sigma}(h_{J-1}, h_J))|u-w|_{1,\sigma} + CC_{d,\sigma}(h_{J-1}, h_J)|w_J|_{1,\sigma} \end{aligned}$$

ie., we have

$$|u_J|_{1,\sigma}^\sigma \le CC_{d,\sigma}(h_{J-1},h_J)^\sigma(|w_J|_{1,\sigma}^\sigma + |u-w|_{1,\sigma}^\sigma) \tag{3.24}$$

From (3.22), (3.23) and (3.24), we get (3.19). Now, for $j = J,\ldots,2$, from (3.11) and (3.17), we get

$$\begin{aligned}&||u_j||_{0,\sigma} \le ||v_j - w_j - I_{h_{j-1}}(v_j-w_j)||_{0,\sigma} + ||w_j||_{0,\sigma} \le \\ &Ch_{j-1}C_{d,\sigma}(h_{j-1},h_j)|v_j-w_j|_{1,\sigma} + ||w_j||_{0,\sigma} \le \\ &Ch_{j-1}(|v_j|_{1,\sigma} + |w_j|_{1,\sigma}) + ||w_j||_{0,\sigma}\end{aligned}$$

where we have used (3.1) and the definition of $C_{d,\sigma}(H,h)$, (3.13). From this equation, we get the first equation in (3.20) for $j = J$. Also, using (3.21), in view of $h_j \le h_1$, $j = J,\ldots,1$, we get the first equation in (3.20) for $j = J-1,\ldots,2$. For $j = 1$, from (3.12), we have

$$\begin{aligned}&||u_1||_{0,\sigma} = ||I_{h_1}(v_2-w_2)||_{0,\sigma} \le ||v_2-w_2||_{0,\sigma} \le \\ &||I_{h_2}(v_3-w_3)||_{0,\sigma} + ||w_2||_{0,\sigma} \le \cdots \le ||v_J||_{0,\sigma} + \sum_{j=2}^{J}||w_j||_{0,\sigma}\end{aligned}$$

ie., the second equation in (3.20) holds. □

To prove that Assumption 2.2 holds, we associate to the decomposition $\{\Omega_j^i\}_{1\le i\le I_j}$ of Ω_j, some functions $\theta_j^i \in C(\bar{\Omega}_j)$, $\theta_j^i|_\tau \in P_1(\tau)$ for any $\tau \in \mathcal{T}_{h_j}$, $i = 1,\cdots,I_j$, such that

$$\begin{aligned}&0 \le \theta_j^i \le 1 \text{ on } \Omega_j,\\ &\theta_j^i = 0 \text{ on } \cup_{l=i+1}^{I_j}\Omega_j^l\backslash\Omega_j^i,\ \theta_j^i = 1 \text{ on } \Omega_j^i\backslash\cup_{l=i+1}^{I_j}\Omega_j^l\end{aligned} \tag{3.25}$$

Also, for Assumption 2.3, we consider a unity partition to each domain decomposition $\{\Omega_j^i\}_{1\le i\le I_j}$, $j = J,\ldots,1$,

$$0 \le \theta_j^i \le 1 \text{ and } \sum_{i=1}^{I_j}\theta_j^i = 1 \text{ on } \Omega_j \tag{3.26}$$

with $\theta_j^i \in C(\bar{\Omega}_j)$, $\theta_j^i|_\tau \in P_1(\tau)$ for any $\tau \in \mathcal{T}_{h_j}$, $i = 1,\cdots,I_j$. Such functions θ_j^i with the above properties exist (see [2] or [23] p. 59, for instance). Moreover, since the overlapping size of the domain decomposition on a level $j = J,\ldots,1$ is δ_j, the above functions θ_j^i can be chosen to satisfy

$$|\partial_{x_k}\theta_j^i| \le C/\delta_j, \text{ a.e. in } \Omega_j, \text{ for any } k = 1,\ldots,d \tag{3.27}$$

Finally, we recall some interpolation properties. For a $v \in V_{h_j}$ and a continuous functions θ which is of polynomial form on the elements of $\tau \in \mathcal{T}_{h_j}$, we have (see [6] and [25]),

$$||\theta v - L_{h_j}(\theta v)||_{0,\sigma} \le Ch_j|\theta v|_{1,\sigma} \text{ and } |L_{h_j}(\theta v)|_{1,\sigma} \le C|\theta v|_{1,\sigma}$$

where L_{h_j} is the P_1-Lagrangian interpolation operator which uses the function values at the nodes of the mesh $\mathcal{T}_{h_j}$. Therefore, we have

$$||L_{h_j}(\theta v)||_{1,\sigma} \leq C||\theta v||_{1,\sigma} \tag{3.28}$$

Now, we can prove

Proposition 3.2. *Assumption 2.2 holds for the convex sets K_j, $j = J, \ldots, 1$, defined in (3.14) and (3.15). The constants C_2 and C_3 are given in (3.34) for Algorithm 2.1, and in (3.35) for Algorithm 2.3.*

Proof. Let us consider $u,\ w \in K$ and $w_{ji} \in V_{h_j}^i$ such that $w_{j1}+\ldots+w_{ji} \in K_j$, $j = J, \ldots, 1$, $i = 1, \ldots, I_j$. In the construction of the convex sets K_j, we take $w_j = \sum_{i=1}^{I_j} w_{ji}$ for Algorithm 2.1, and $w_J = \ldots = w_1 = 0$ for Algorithm 2.3. Then, from Lemma 3.1, there exist $u_j \in K_j$, $j = J, \ldots, 1$, defined in (3.17), such that (3.18) holds. Now, for each u_j, $j = J, \ldots, 1$, we define

$$u_{j1} = L_{h_j}(\theta_j^1 u_j + (1-\theta_j^1)w_{j1})$$
$$u_{ji} = L_{h_j}(\theta_j^i(u_j - \sum_{l=1}^{i-1} u_{jl}) + (1-\theta_j^i)w_{ji}), \quad i = 2, \ldots, I_j$$

with θ_j^i in (3.25). Like in Proposition 3.1 in [2], where we take $v = u_j$ and $w = 0$, we can prove that

$$\begin{aligned} & u_{ji} \in V_{h_j}^i, \quad w_{j1} + \ldots + w_{ji-1} + u_{ji} \in K_j,\ i = 1, \ldots, I_j \\ & u_j = \sum_{i=1}^{I_j} u_{ji} \end{aligned} \tag{3.29}$$

for any $j = J, \ldots, 1$. We point out that here, the condition $w_{j1}+\ldots+w_{ji-1}+u_{ji} \in K_j$ can be proved by verifying that it is satisfied only at the nodes of $\mathcal{T}_{h_j}$. From (3.18) and (3.29), we get that the first two conditions of Assumption 2.2 are satisfied.

We estimate now the constants C_2 and C_3. The above u_{ji}, $j = J, \ldots, 1$, $i = 1, \ldots, I_j$, can be written as

$$u_{ji} = L_{h_j}(\theta_{ji}^0 u_j + \sum_{k=1}^{i} \theta_{ji}^k w_{jk}),\ i = 1, \ldots, I_j \tag{3.30}$$

where

$\theta_{ji}^0 = \theta_j^1$, $\theta_{ji}^1 = 1 - \theta_j^1$ and
$\theta_{ji}^0 = \theta_j^i(1-\theta_j^{i-1})\cdots(1-\theta_j^1)$, $\theta_{ji}^i = 1-\theta_j^i$, $\theta_{ji}^k = -\theta_j^i(1-\theta_j^{i-1})\cdots(1-\theta_j^k)$,
for $i = 2, \ldots, I_j$, $k = 1, \ldots, i-1$

In view of (3.25), we have

$$|\theta_{ji}^l| \leq 1 \text{ and } |\partial_{x_k}\theta_{ji}^l| \leq C(I_j-1)/\delta_j,\ i = 1, \ldots, I_j,\ l = 0, \ldots, i,\ k = 1, \ldots, d.$$

It follows that

$$||\theta_{ji}^l v||_{0,\sigma} \le ||v||_{0,\sigma}, \quad |\theta_{ji}^l v|_{1,\sigma} \le C(|v|_{1,\sigma} + \frac{I_j - 1}{\delta_j}||v||_{0,\sigma})$$

and therefore, using (3.28), we get

$$||L_{h_j}(\theta_{ji}^l v)||_{1,\sigma} \le C(|v|_{1,\sigma} + (1 + \frac{I_j - 1}{\delta_j})||v||_{0,\sigma})$$

for any $v \in V_{h_j}$, $j = J, \dots, 1$, $i = 1, \dots, I_j$ and $l = 0, 1, \dots i$. We use this inequality to estimate the norm of the terms in (3.30).

First, since w_{ji} have the support included Ω_{ji} and vanishes at least on a part of its boundary, in view of (3.2), (3.3) and the classical Friedrichs-Poincaré inequality, we get

$$||L_{h_j}(\theta_{ji}^k w_{jk})||_{1,\sigma}^\sigma \le C(1 + (I_j - 1)\frac{h_{j-1}}{\delta_j})^\sigma |w_{jk}|_{1,\sigma}^\sigma \le CI^\sigma |w_{jk}|_{1,\sigma}^\sigma$$

for any $j = J, \dots, 2$, $i = 1, \dots, I_j$ and $k = 1, \dots, i$. Now we use Lemma 3.2. For $j = J, \dots, 2$, we have

$$\begin{aligned}
&||L_{h_j}(\theta_{ji}^k u_j)||_{1,\sigma}^\sigma \le C(|u_j|_{1,\sigma}^\sigma + (1 + \tfrac{I_j-1}{\delta_j})^\sigma ||u_j||_{0,\sigma}^\sigma) \le \\
&C(J-1)^{\sigma-1}[1 + (I_j - 1)\frac{h_{j-1}}{\delta_j}]^\sigma C_{d,\sigma}(h_{j-1}, h_J)^\sigma \cdot \\
&[\sum_{k=2}^J |w_k|_{1,\sigma}^\sigma + |u - w|_{1,\sigma}^\sigma] + C(1 + \frac{I_j - 1}{\delta_j})^\sigma ||w_j||_{0,\sigma}^\sigma \le \\
&C(J-1)^{\sigma-1} I^\sigma C_{d,\sigma}(h_{j-1}, h_J)^\sigma [\sum_{k=2}^J |w_k|_{1,\sigma}^\sigma + |u - w|_{1,\sigma}^\sigma] + \\
&C(1 + \frac{I - 1}{\delta_j})^\sigma ||w_j||_{0,\sigma}^\sigma
\end{aligned}$$

Consequently, from (3.30) and the last two equations, we have

$$\begin{aligned}
&||u_{ji}||_{1,\sigma}^\sigma \le C(I+1)^{\sigma-1} \Bigg\{ I^\sigma \sum_{k=1}^{I_j} |w_{jk}|_{1,\sigma}^\sigma + \\
&I^\sigma (J-1)^{\sigma-1} C_{d,\sigma}(h_{j-1}, h_J)^\sigma [\sum_{k=2}^J |w_k|_{1,\sigma}^\sigma + |u - w|_{1,\sigma}^\sigma] + \\
&[1 + \frac{I-1}{\delta_j}]^\sigma ||w_j||_{0,\sigma}^\sigma \Bigg\}
\end{aligned} \tag{3.31}$$

for any $j = J, \dots, 2$ and $i = 1, \dots, I_j$. At the level $j = 1$, we do not have a domain decomposition, $I_1 = 1$, and we take $u_{11} = u_1$. In this way, from Lemma 3.2, we have

$$||u_{11}||_{1,\sigma}^\sigma \le C(J-1)^{\sigma-1} C_{d,\sigma}(h_1, h_J)^\sigma (\sum_{k=2}^J ||w_k||_{1,\sigma}^\sigma + ||u - w||_{1,\sigma}^\sigma) \tag{3.32}$$

From (3.31) and (3.32), we get

$$
(3.33)\quad \begin{aligned}
&\sum_{j=1}^{J}\sum_{i=1}^{I_j} ||u_{ji}||_{1,\sigma}^{\sigma} \le CI^{\sigma+1}(I+1)^{\sigma-1}\left\{\sum_{j=2}^{J}\sum_{i=1}^{I_j}|w_{ji}|_{1,\sigma}^{\sigma}+\right.\\
&\left.(J-1)^{\sigma-1}\left[\sum_{j=2}^{J} C_{d,\sigma}(h_{j-1},h_J)^{\sigma}\right]\left[\sum_{j=2}^{J}||w_j||_{1,\sigma}^{\sigma}+||u-w||_{1,\sigma}^{\sigma}\right]\right\}+\\
&CI(I+1)^{\sigma-1}\sum_{j=2}^{J}[1+\frac{I-1}{\delta_j}]^{\sigma}||w_j||_{0,\sigma}^{\sigma}
\end{aligned}
$$

In the case of Algorithm 2.1, the convex sets K_j, $j = J,\dots,1$, are constructed in Assumption 2.1 with $w_j = \sum_{i=1}^{I_j} w_{ji}$, $j = J,\dots,1$. Consequently, we get from (3.33) that the constants C_2 and C_3 can be written as

$$
(3.34)\quad \begin{aligned}
C_2 &= CI^{\frac{\sigma+1}{\sigma}}(I+1)^{\frac{\sigma-1}{\sigma}}(J-1)^{\frac{\sigma-1}{\sigma}}[\sum_{j=2}^{J} C_{d,\sigma}(h_{j-1},h_J)^{\sigma}]^{\frac{1}{\sigma}}\\
C_3 &= CI^{2}(I+1)^{\frac{\sigma-1}{\sigma}}(J-1)^{\frac{\sigma-1}{\sigma}}[\sum_{j=2}^{J} C_{d,\sigma}(h_{j-1},h_J)^{\sigma}]^{\frac{1}{\sigma}}
\end{aligned}
$$

For Algorithm 2.3, the convex sets K_j, $j = J,\dots,1$, are constructed with $w_J = \dots = w_1 = 0$. Therefore, it follows from (3.33) that the constants C_2 and C_3 can be written as

$$
(3.35)\quad \begin{aligned}
C_2 &= CI^{\frac{\sigma+1}{\sigma}}(I+1)^{\frac{\sigma-1}{\sigma}}(J-1)^{\frac{\sigma-1}{\sigma}}[\sum_{j=2}^{J} C_{d,\sigma}(h_{j-1},h_J)^{\sigma}]^{\frac{1}{\sigma}}\\
C_3 &= CI^{\frac{\sigma+1}{\sigma}}(I+1)^{\frac{\sigma-1}{\sigma}}
\end{aligned}
$$

□

Concerning Assumption 2.3 we have

Proposition 3.3. *Assumption 2.3 holds for the convex sets K_j, $j = J,\dots,1$, defined in (3.14) and (3.15). The constants C_2 and C_3 are given in (3.39) for Algorithm 2.2, and in (3.40) for Algorithm 2.4.*

Proof. Let us consider $u,\ w \in K$ and $w_{ji} \in V_{h_j}^i \cap K_j$, $j = J,\dots,1$, $i = 1,\dots,I_j$. For Algorithm 2.2 we take $w_j = \frac{r}{I_j}\sum_{i=1}^{I_j} w_{ji}$, and $w_J = \dots = w_1 = 0$ for Algorithm 2.4, in the construction of the convex sets K_j. Then, from Lemma 3.1, there exist $u_j \in K_j$, $j = J,\dots,1$, defined in (3.17), such that (3.18) holds. Now, for each u_j, $j = J,\dots,1$, we define

$$
(3.36)\quad u_{ji} = L_{h_j}(\theta_j^i u_j),\quad i = 1,\dots,I_j \text{ for } j = J,\dots,2, \text{ and } u_{11} = u_1
$$

with θ_j^i in (3.26). It is clear that

$$u_{ji} \in V_{h_j}^i \cap K_j,\ i = 1, \dots, I_j,\ \text{and}\ u_j = \sum_{i=1}^{I_j} u_{ji} \tag{3.37}$$

for any $j = J, \dots, 1$. From (3.18) and (3.37), we get that the first condition of Assumption 2.3 holds.

We estimate now the constants C_2 and C_3. From (3.27) and (3.28), we get

$$||u_{ji}||_{1,\sigma}^{\sigma} \leq C(|u_j|_{1,\sigma}^{\sigma} + (1 + \frac{1}{\delta_j})^{\sigma} ||u_j||_{0,\sigma}^{\sigma})$$

Using this equation, the proof is similar with that of the previous proposition. For $j = J, \dots, 2$, in view of (3.19) and (3.20), we have

$$\begin{aligned} &||u_{ji}||_{1,\sigma}^{\sigma} \leq C(1 + \frac{1}{\delta_j})^{\sigma} ||w_j||_{0,\sigma}^{\sigma} + \\ &C(J-1)^{\sigma-1} C_{d,\sigma}(h_{j-1}, h_J)^{\sigma} [\sum_{k=2}^{J} |w_k|_{1,\sigma}^{\sigma} + |u - w|_{1,\sigma}^{\sigma}] \end{aligned}$$

and we use (3.32) for the estimation of $||u_{11}||_{1,\sigma}$. From these equations, we get

$$\begin{aligned} &\sum_{j=1}^{J} \sum_{i=1}^{I_j} ||u_{ji}||_{1,\sigma}^{\sigma} \leq CI \sum_{j=2}^{J} (1 + \frac{1}{\delta_j})^{\sigma} ||w_j||_{0,\sigma}^{\sigma} + \\ &CI(J-1)^{\sigma-1} [\sum_{j=2}^{J} C_{d,\sigma}(h_{j-1}, h_J)^{\sigma}] [\sum_{j=2}^{J} ||w_j||_{1,\sigma}^{\sigma} + ||u - w||_{1,\sigma}^{\sigma}] \end{aligned} \tag{3.38}$$

The convex sets K_j, $j = J, \dots, 1$, are constructed with $w_j = \frac{r}{I} \sum_{i=1}^{I_j} w_{ji}$ in the case of Algorithm 2.2. Consequently, the constants C_2 and C_3, can be written as

$$\begin{aligned} C_2 &= CI^{\frac{1}{\sigma}} (J-1)^{\frac{\sigma-1}{\sigma}} [\sum_{j=2}^{J} C_{d,\sigma}(h_{j-1}, h_J)^{\sigma}]^{\frac{1}{\sigma}} \\ C_3 &= C(J-1)^{\frac{\sigma-1}{\sigma}} [\sum_{j=2}^{J} C_{d,\sigma}(h_{j-1}, h_J)^{\sigma}]^{\frac{1}{\sigma}} \end{aligned} \tag{3.39}$$

For Algorithm 2.4, the convex sets K_j, $j = J, \dots, 1$, are constructed with $w_J = \ldots = w_1 = 0$. Therefore, we can take

$$C_2 = CI^{\frac{1}{\sigma}} (J-1)^{\frac{\sigma-1}{\sigma}} [\sum_{j=2}^{J} C_{d,\sigma}(h_{j-1}, h_J)^{\sigma}]^{\frac{1}{\sigma}} \text{ and } C_3 = 0 \tag{3.40}$$

□

The constants C_1 and β_{jk}, $j,\ k = J, \dots, 1$, can be taken in (2.20) and (2.4), but better choices are available in the case of the multigrid methods in the next section. As we see form the above estimations, the convergence rates given in Theorem 2.1 depend on the functional F, the maximum number of the subdomains on each level, I, and the number of levels J. The number of subdomains on levels can be associated with the number of colors needed to mark the subdomains such that the subdomains with the same color do not intersect with each other. Since this number of colors depends in general on the dimension of the Euclidean space where the domain lies, we can conclude that our convergence rate essentially depends on the number of levels J.

We first estimate the constants C_1–C_3 as functions of J. To this end, in the remainder of this section, C will be a generic constant which does not depend on J. Writing $S_{d,\sigma}(J) = \left[\sum_{j=2}^{J} C_{d,\sigma}(h_{j-1}, h_J)^\sigma\right]^{\frac{1}{\sigma}}$ from (3.1) and (3.13), we get

$$S_{d,\sigma}(J) = \begin{cases} (J-1)^{\frac{1}{\sigma}} & \text{if } d = \sigma = 1 \\ & \text{or } 1 \le d < \sigma < \infty \\ CJ & \text{if } 1 < d = \sigma < \infty \\ C^J & \text{if } 1 \le \sigma < d < \infty, \end{cases} \tag{3.41}$$

In this general framework, we take C_1, and β_{jk}, $j,\ k = J, \dots, 1$, as in (2.20) and (2.4),

$$C_1 = CJ^{\frac{\sigma-1}{\sigma}} \text{ and } \max_{k=1,\cdots,J} \sum_{j=1}^{J} \beta_{kj} = J \tag{3.42}$$

Also, from (3.34), (3.35), (3.39) and (3.40), we get

$$C_2 = C(J-1)^{\frac{\sigma-1}{\sigma}} S_{d,\sigma}(J) \tag{3.43}$$

$$C_3 = \begin{cases} C(J-1)^{\frac{\sigma-1}{\sigma}} S_{d,\sigma}(J) & \text{for Algorithms 2.1 and 2.2} \\ C & \text{for Algorithm 2.3} \\ 0 & \text{for Algorithm 2.4} \end{cases} \tag{3.44}$$

Now, we shall write the convergence rate of the multilevel Algorithms 2.1–2.4 in function of the number of levels J. In order to be more conclusive, we shall limit ourselves to a typical example where

$$F(v) = \frac{1}{\sigma}||v||_{1,\sigma}^\sigma - L(v), \quad v \in W^{1,\sigma}(\Omega) \tag{3.45}$$

where L is a linear and continuous functional on $W^{1,\sigma}(\Omega)$, $\sigma > 1$. In this case (see [1], for instance),

$$p = 2,\ q = \sigma \text{ if } \sigma < 2; \quad p = 2,\ q = 2 \text{ if } \sigma = 2; \quad p = \sigma,\ q = 2 \text{ if } \sigma > 2$$

Evidently, we can use the same procedure for other problems, too.

For $\sigma = 2$, $p = q = 2$ and $d = 1, 2, 3$, in view of (2.45), (2.34) and (3.42)–(3.44), we get

$$\tilde{C}_1(J) = \begin{cases} CJ^3 S_{d,2}(J)^2 & \text{for Algorithms 2.1 and 2.2} \\ CJ^4 S_{d,2}(J)^2 & \text{for Algorithms 2.3 and 2.4} \end{cases} \tag{3.46}$$

and, from Theorem 2.1, we have

$$||u^n - u||_{1,2}^2 \le \tilde{C}_0 \left(1 - \frac{1}{1 + \tilde{C}_1(J)}\right)^n \tag{3.47}$$

where $\tilde{C}_0$ is a constant independent of J.

For $1 < q = \sigma < 2$, $p = 2$ and $d = 1$, in view of (2.47), (2.34) and (3.42)–(3.44), we get

$$\tilde{C}_3(J) = \begin{cases} CJ^{\frac{4\sigma-1}{\sigma}} & \text{for Algorithms 2.1 and 2.2} \\ CJ^{\frac{7\sigma-2-\sigma^2}{\sigma}} & \text{for Algorithms 2.3 and 2.4} \end{cases} \tag{3.48}$$

Also, for $d = 2, 3$, we can take

$$\tilde{C}_3(J) = C^J \text{ for Algorithms 2.1 − 2.4} \tag{3.49}$$

From Theorem 2.1, we get that

$$||u^n - u||_{1,\sigma}^2 \le \tilde{C}_0 \frac{1}{\left(1 + n\tilde{C}_2(J)\right)^{\frac{\sigma-1}{2-\sigma}}} \tag{3.50}$$

where, in view of (2.49), we can take

$$\tilde{C}_2(J) = \frac{1}{1 + \tilde{C}_3(J)^{\frac{1}{\sigma-1}}} \tag{3.51}$$

For $p = \sigma > 2$, $q = 2$, $d = 1, 2, 3$ and $\sigma \le 3$, we get

$$\tilde{C}_3(J) = \begin{cases} CJ^3 S_{d,\sigma}(J) & \text{for Algorithms 2.1 and 2.2} \\ CJ^{\frac{3\sigma-1}{\sigma-1}} S_{d,\sigma}(J) & \text{for Algorithms 2.3 and 2.4} \end{cases} \tag{3.52}$$

Also, for $\sigma > 3$, we have

$$\tilde{C}_3(J) = \begin{cases} CJ^{\frac{2\sigma-1}{\sigma}} & \text{for Algorithms 2.1 and 2.2} \\ CJ^{\frac{2\sigma+1}{\sigma}} & \text{for Algorithms 2.3 and 2.4} \end{cases} \tag{3.53}$$

Finally, in this case, we have

$$||u^n - u||_{1,\sigma}^{\sigma} \le \tilde{C}_0 \frac{1}{\left(1 + n\tilde{C}_2(J)\right)^{\frac{1}{\sigma-1}}} \tag{3.54}$$

where

$$\tilde{C}_2(J) = \frac{1}{1 + \tilde{C}_3(J)^{\sigma-1}} \tag{3.55}$$

Remark 3.1. 1) The results of this section have referred to problems in $W^{1,\sigma}$ with Dirichlet boundary conditions, and the functions corresponding to the coarse levels have been extended with zero outside the domains Ω_j, $j = J-1,\ldots,1$. Let us assume that the problem has mixed boundary conditions: $\partial\Omega_J = \Gamma_d \cup \Gamma_n$, with Dirichlet conditions on Γ_d and Neumann conditions on Γ_n. In this case, if a node of $\mathcal{T}_{h_j}$, $j = J-1,\ldots,1$, lies in $\mathrm{Int}(\Gamma_n)$, we have to assume that all the sides of the elements $\tau \in \mathcal{T}_{h_j}$ having that node are included in Γ_n.

2) Similar convergence results with those ones presented in this section can be obtained for problems in $(W^{1,s})^d$.

4 Multigrid methods

In the above multilevel methods a mesh is the refinement of that on the previous level, but the domain decompositions are almost independent from one level to another. We obtain similar multigrid methods by decomposing the level domains by the supports of the nodal basis functions. Consequently, the subspaces $V_{h_j}^i$, $i = 1,\ldots,I_j$, are one-dimensional spaces generated by the nodal basis functions associated with the nodes of $\mathcal{T}_{h_j}$, $j = J,\ldots,1$. We point out that Algorithm 2.1 represents a classical V-cycle multigrid iteration. Algorithms 2.2–2.4 are some variants in which the smoothing steps are performed by a combination of multiplicative methods with additive ones. Evidently, similar results can be given for the W-cycle multigrid iterations.

In this section, we derive sharper estimations than those given in (2.20) and (2.4) for the constants C_1 and β_{jk}, $j,\ k = J,\ldots,1$. Finally, we summarize the previous results by writing the convergence rates of the four algorithms as functions of the number J of the levels, for the varied values of the constants p, q, σ and d. In this section, we denote by $V_{h_j}^i$, $j = J,\ldots,1$, $i = 1,\ldots,I_j$, the above defined one-dimensional spaces generated by the nodal basis functions associated with the nodes of the meshes.

The proof of (2.3) can be found in [22] and it essentially stands on the simple inequalities

$$||v_{ji}||_{0,\sigma,supp(v_{kl})} \le C(\tfrac{h_k}{h_j})^{\frac{d}{\sigma}} ||v_{ji}||_{0,\sigma} \text{ and } |v_{ji}|_{1,\sigma,supp(v_{kl})} \le C(\tfrac{h_k}{h_j})^{\frac{d}{\sigma}} |v_{ji}|_{1,\sigma}$$

for any $d \ge 1$, $\sigma \ge 1$, $v_{ji} \in V_{h_j}^i$, $v_{kl} \in V_{h_k}^l$ with $j \le k$, $j,k = J,\ldots,1$, $i = 1,\ldots,I_j$ and $l = 1,\ldots,I_k$. Writing

$$\gamma_{kj} = \frac{1}{\gamma^{|k-j|\frac{d}{\sigma}}} \tag{4.1}$$

in view of (3.1), we get

$$||v_{ji}||_{0,\sigma,supp(v_{kl})} \leq C\gamma_{kj}||v_{ji}||_{0,\sigma},\ |v_{ji}|_{1,\sigma,supp(v_{kl})} \leq C\gamma_{kj}|v_{ji}|_{1,\sigma} \tag{4.2}$$

where C is independent of the meshes or their number. In this way, we get that (2.3) holds for

$$\beta_{kj} = \gamma_{kj}^{q-1},\ k,j = J,\ldots,1 \tag{4.3}$$

We point out that

$$\sum_{j=1}^{J}\gamma_{kj} \leq \frac{\gamma^{\frac{d}{\sigma}}}{\gamma^{\frac{d}{\sigma}}-1} \text{ and } \sum_{j=1}^{J}\beta_{kj} \leq \frac{\gamma^{\frac{1}{q-1}\frac{d}{\sigma}}}{\gamma^{\frac{1}{q-1}\frac{d}{\sigma}}-1}$$

for any $k = J,\ldots,1$.

The constant C_1 in (2.19) is given by the following

Lemma 4.1. *Let us consider $\sigma \geq 1$, $n \in \mathbf{N}$, $n-1 < \sigma \leq n$, and $w_{ji} \in V_{h_j}^i$, $j = J,\ldots,1$, $i = 1,\ldots,I_j$. Then,*

$$C_1 = (n!)^{\frac{1}{\sigma}} C^{\frac{n-1}{n}} \left(I \frac{\gamma^{\frac{d}{n}}}{\gamma^{\frac{d}{n}}-1} \right)^{\frac{n-1}{\sigma}}$$

where C is the constant in (4.2).

Proof. For the writing simplicity, we prove the lemma for the norm $||\cdot||_{0,\sigma}$. The proof for the derivatives in the seminorm $|\cdot|_{0,\sigma}$ is identical. With γ_{jk} defined in (4.1), in view of (2.42), for any $1 \leq m \leq n-2$, we have the following recurrent inequality

$$\left\{ \sum_{j_{n-m+1}=1}^{J} \left[\sum_{i_{n-m+1}=1}^{I_{j_{n-m+1}}} (\int_\Omega |w_{j_{n-m+1}i_{n-m+1}}|^\sigma)^{\frac{1}{n}} \cdot \right.\right.$$
$$\sum_{j_{n-m}=1}^{J} \gamma_{j_{n-m+1}j_{n-m}}^{\frac{\sigma}{n}} \sum_{i_{n-m}=1}^{I_{j_{n-m}}} (\int_\Omega |w_{j_{n-m}i_{n-m}}|^\sigma)^{\frac{1}{n}} \cdots$$
$$\left.\left. \cdots \sum_{j_1=1}^{J} \gamma_{j_2j_1}^{\frac{\sigma}{n}} \sum_{i_1=1}^{I_{j_1}} (\int_\Omega |w_{j_1i_1}|^\sigma)^{\frac{1}{n}} \right]^{\frac{n}{n-m+1}} \right\}^{\frac{n-m+1}{n}} \leq$$
$$\left\{ \sum_{j_{n-m+1}=1}^{J} \left[\sum_{i_{n-m+1}=1}^{I_{j_{n-m+1}}} (\int_\Omega |w_{j_{n-m+1}i_{n-m+1}}|^\sigma)^{\frac{1}{n}} \right]^n \right\}^{\frac{1}{n}} \cdot$$
$$\left\{ \sum_{j_{n-m+1}=1}^{J} \left[\sum_{j_{n-m}=1}^{J} \gamma_{j_{n-m+1}j_{n-m}}^{\frac{\sigma}{n}} \sum_{i_{n-m}=1}^{I_{j_{n-m}}} (\int_\Omega |w_{j_{n-m}i_{n-m}}|^\sigma)^{\frac{1}{n}} \cdots \right.\right.$$
$$\left.\left. \cdots \sum_{j_1=1}^{J} \gamma_{j_2j_1}^{\frac{\sigma}{n}} \sum_{i_1=1}^{I_{j_1}} (\int_\Omega |w_{j_1i_1}|^\sigma)^{\frac{1}{n}} \right]^{\frac{n}{n-m}} \right\}^{\frac{n-m}{n}} \leq$$

$$I^{\frac{n-1}{n}}\left[\sum_{j=1}^{J}\sum_{i=1}^{I_j}||w_{ji}||_{0,\sigma}^{\sigma}\right]^{\frac{1}{n}}\cdot$$

$$\left\{\sum_{j_{n-m+1}=1}^{J}\left[\sum_{j_{n-m}=1}^{J}\gamma_{j_{n-m+1}j_{n-m}}^{\frac{\sigma}{n}}\sum_{i_{n-m}=1}^{I_{j_{n-m}}}(\int_{\Omega}|w_{j_{n-m}i_{n-m}}|^{\sigma})^{\frac{1}{n}}\cdots\right.\right.$$

$$\left.\left.\cdots\sum_{j_1=1}^{J}\gamma_{j_2j_1}^{\frac{\sigma}{n}}\sum_{i_1=1}^{I_{j_1}}(\int_{\Omega}|w_{j_1i_1}|^{\sigma})^{\frac{1}{n}}\right]^{\frac{n}{n-m}}\right\}^{\frac{n-m}{n}}\leq$$

$$I^{\frac{n-1}{n}}\frac{\gamma^{\frac{d}{n}}}{\gamma^{\frac{d}{n}}-1}\left[\sum_{j=1}^{J}\sum_{i=1}^{I_j}||w_{ji}||_{0,\sigma}^{\sigma}\right]^{\frac{1}{n}}\left\{\sum_{j_{n-m}=1}^{J}\left[\sum_{i_{n-m}=1}^{I_{j_{n-m}}}(\int_{\Omega}|w_{j_{n-m}i_{n-m}}|^{\sigma})^{\frac{1}{n}}\cdot\right.\right.$$

$$\sum_{j_{n-m-1}=1}^{J}\gamma_{j_{n-m}j_{n-m-1}}^{\frac{\sigma}{n}}\sum_{i_{n-m-1}=1}^{I_{j_{n-m-1}}}(\int_{\Omega}|w_{j_{n-m-1}i_{n-m-1}}|^{\sigma})^{\frac{1}{n}}\cdots$$

$$\left.\left.\cdots\sum_{j_1=1}^{J}\gamma_{j_2j_1}^{\frac{\sigma}{n}}\sum_{i_1=1}^{I_{j_1}}(\int_{\Omega}|w_{j_1i_1}|^{\sigma})^{\frac{1}{n}}\right]^{\frac{n}{n-m}}\right\}^{\frac{n-m}{n}}$$

Now, since $\sigma\leq n$, we have

$$||\sum_{j=1}^{J}\sum_{i=1}^{I_j}w_{ji}||_{0,\sigma}^{\sigma}\leq\int_{\Omega}(\sum_{j=1}^{J}\sum_{i=1}^{I_j}|w_{ji}|)^{\sigma}=\int_{\Omega}\left[(\sum_{j=1}^{J}\sum_{i=1}^{I_j}|w_{ji}|)^{\frac{\sigma}{n}}\right]^{n}\leq$$

$$\int_{\Omega}\left(\sum_{j=1}^{J}\sum_{i=1}^{I_j}|w_{ji}|^{\frac{\sigma}{n}}\right)^{n}=\sum_{k_n=1}^{J}\sum_{i_n=1}^{I_{k_n}}\cdots\sum_{k_1=1}^{J}\sum_{i_1=1}^{I_{k_1}}\int_{\Omega}|w_{k_ni_n}|^{\frac{\sigma}{n}}\cdots|w_{k_1i_1}|^{\frac{\sigma}{n}}$$

By a permutation, $(k_n,\ldots,k_1)$ can be transformed in $(j_n,\ldots,j_1)$ with $j_n\leq\ldots\leq j_1$. Also, by permutations, we get $n!$ terms $(k_n,\ldots,k_1)$ from each such a $(j_n,\ldots,j_1)$. Therefore, we can write

$$||\sum_{j=1}^{J}\sum_{i=1}^{I_j}w_{ji}||_{0,\sigma}^{\sigma}\leq n!\sum_{j_n=1}^{J}\sum_{i_n=1}^{I_{j_n}}\sum_{j_{n-1}=j_n}^{J}\sum_{i_{n-1}=1}^{I_{j_{n-1}}}\cdots\sum_{j_1=j_2}^{J}\sum_{i_1=1}^{I_{j_1}}\int_{\Omega}|w_{j_ni_n}|^{\frac{\sigma}{n}}\cdots|w_{j_1i_1}|^{\frac{\sigma}{n}}$$

From here, in view of (4.2) and the recursive inequality from the beginning of this proof with $m=1,\ldots,n-2$, we get

$$||\sum_{j=1}^{J}\sum_{i=1}^{I_j}w_{ji}||_{0,\sigma}^{\sigma}\leq n!C^{\sigma\frac{n-1}{n}}\sum_{j_n=1}^{J}\sum_{i_n=1}^{I_{j_n}}\sum_{j_{n-1}=j_n}^{J}\sum_{i_{n-1}=1}^{I_{j_{n-1}}}\cdots\sum_{j_1=j_2}^{J}\sum_{i_1=1}^{I_{j_1}}$$

$$\left[\gamma_{j_nj_{n-1}}^{\frac{\sigma}{n}}(\int_{\Omega}|w_{j_ni_n}|^{\sigma})^{\frac{1}{n}}\cdots\gamma_{j_2j_1}^{\frac{\sigma}{n}}(\int_{\Omega}|w_{j_2i_2}|^{\sigma})^{\frac{1}{n}}(\int_{\Omega}|w_{j_1i_1}|^{\sigma})^{\frac{1}{n}}\right]\leq$$

$$n!C^{\sigma\frac{n-1}{n}}\sum_{j_n=1}^{J}\left[\sum_{i_n=1}^{I_{j_n}}(\int_\Omega |w_{j_ni_n}|^\sigma)^{\frac{1}{n}}\cdot\right.$$
$$\left.\sum_{j_{n-1}=1}^{J}\gamma_{j_nj_{n-1}}^{\frac{\sigma}{n}}\sum_{i_{n-1}=1}^{I_{j_{n-1}}}(\int_\Omega |w_{j_{n-1}i_{n-1}}|^\sigma)^{\frac{1}{n}}\cdots\sum_{j_1=1}^{J}\gamma_{j_2j_1}^{\frac{\sigma}{n}}\sum_{i_1=1}^{I_{j_1}}(\int_\Omega |w_{j_1i_1}|^\sigma)^{\frac{1}{n}}\right]\le$$
$$n!C^{\sigma\frac{n-1}{n}}\left(I^{\frac{n-1}{n}}\frac{\gamma^{\frac{d}{n}}}{\gamma^{\frac{d}{n}}-1}\right)^{n-2}\left[\sum_{j=1}^{J}\sum_{i=1}^{I_j}||w_{ji}||_{0,\sigma}^{\sigma}\right]^{\frac{n-2}{n}}\cdot$$
$$\left\{\sum_{j_2=1}^{J}\left[\sum_{i_2=1}^{I_{j_2}}(\int_\Omega |w_{j_2i_2}|^\sigma)^{\frac{1}{n}}\sum_{j_1=1}^{J}\gamma_{j_2j_1}^{\frac{\sigma}{n}}\sum_{i_1=1}^{I_{j_1}}(\int_\Omega |w_{j_1i_1}|^\sigma)^{\frac{1}{n}}\right]^{\frac{n}{2}}\right\}^{\frac{2}{n}}\le$$
$$n!C^{\sigma\frac{n-1}{n}}I^{n-1}\left(\frac{\gamma^{\frac{d}{n}}}{\gamma^{\frac{d}{n}}-1}\right)^{n-1}\sum_{j=1}^{J}\sum_{i=1}^{I_j}||w_{ji}||_{0,\sigma}^{\sigma}$$

□

As in the previous section, we can estimate the convergence rate of the Algorithms 2.1–2.4 as functions of the number of levels J for the example in (3.45). From the above proofs, we can conclude that, in the case of the multigrid methods, we can consider C_1 and $\max_{k=J,\dots,1}\sum_{j=1}^{J}\beta_{kj}$ as some constants independent of J and mesh parameters. Also, using the estimations of C_2 and C_3 in (3.43) and (3.44), respectively, we can write the error estimations in Theorem 2.1 of the four algorithms in function of J.

For $\sigma = 2$, $p = q = 2$ and $d = 1, 2, 3$, in view of (2.45), (2.34) and (3.42)–(3.44), we get

$$\tilde{C}_1(J) = \begin{cases} CJS_{d,2}(J)^2 & \text{for Algorithms 2.1 and 2.2} \\ CJ^2S_{d,2}(J)^2 & \text{for Algorithms 2.3 and 2.4} \end{cases} \tag{4.4}$$

and the error estimation is given in (3.47).

For $1 < q = \sigma < 2$, $p = 2$ and $d = 1, 2, 3$, in view of (2.47), (2.34) and (3.42)–(3.44), we get

$$\tilde{C}_3(J) = \begin{cases} CJ^{\frac{(4-\sigma)(\sigma-1)}{\sigma}}S_{d,\sigma}(J)^2 & \text{for Algorithms 2.1 and 2.2} \\ CJ^{\frac{4(\sigma-1)}{\sigma}}S_{d,\sigma}(J)^2 & \text{for Algorithms 2.3 and 2.4} \end{cases} \tag{4.5}$$

and the error estimation is given in (3.50) with $\tilde{C}_2(J)$ in (3.51).

For $p = \sigma > 2$, $q = 2$ and $d = 1, 2, 3$, we get

$$\tilde{C}_3(J) = \begin{cases} CJ^{\frac{2\sigma-3}{\sigma-1}}S_{d,\sigma}(J)^{\frac{\sigma}{\sigma-1}} & \text{for Algorithms 2.1 and 2.2} \\ CJ^2S_{d,\sigma}(J)^{\frac{\sigma}{\sigma-1}} & \text{for Algorithms 2.3 and 2.4} \end{cases} \tag{4.6}$$

and the error estimation is given in (3.54) with $\tilde{C}_2(J)$ in (3.55).

We make now some remarks on the above error estimations of the four algorithms.

First, as we have expected, the multiplicative (over the levels) Algorithms 2.1 and 2.2 converge better, with a $1/J$ factor than their additive variants, Algorithms 2.3 and 2.4. Since the totally or partly additive Algorithms 2.2–2.4 are more parallelizable, they can be sometimes used successfully instead of Algorithm 2.1. Excepting Algorithm 2.1, which is a standard monotone multigrid method in the sense of Kornhuber (see [13] and [10]), to our knowledge, Algorithms 2.2–2.4 are new, at least in point of view of their convergence analysis. Also, we point out that the above convergence results give global rate estimations. Moreover, our analysis refer to two sided obstacle problems which arise from the minimization of non quadratic functionals. Consequently, only in the case $p = q = \sigma = d = 2$, we can compare the convergence rates we have obtained with similar ones in the literature. In this case, from (3.47) and (4.4), we get that the global convergence of Algorithm 2.1 is $1 - \frac{1}{1+CJ^3}$. For the truncated monotone multigrid method, an asymptotic convergence rate of $1 - \frac{1}{1+CJ^4}$, and under some conditions, of $1 - \frac{1}{1+CJ^3}$, is found for the complementary problem in [13] and [10]. The same estimate, of $1 - \frac{1}{1+CJ^3}$, is obtained in [13] for the asymptotic convergence rate of the standard monotone multigrid methods for the complementary problem. In [10], it is mentioned that this asymptotic rate may be of $1 - \frac{1}{1+CJ^2}$, or even of $1 - \frac{1}{1+CJ}$, under some conditions.

5 Appendix

If we construct the convex sets K_j, $j = J, \ldots, 1$, as in the following definition,

Definition 5.1. *For a given $w \in K$, we recursively introduce the convex sets K_j, $j = J,\ J-1,\ \ldots, 1$, as*

- at level J: $K_J = \{v_J \in V_J : w + v_J \in K\}$ and consider a $w_J \in K_J$

- at a level $J - 1 \geq j \geq 1$: $K_j = \{v_j \in V_j : w + w_J + \ldots + w_{j+1} + v_j \in K\}$ and consider a $w_j \in K_j$

then, Assumption 2.1 is satisfied. The resulting Algorithm 2.1 with this definition is that one presented in [2]. Since, in this case, we use the convex set K for the iterates on the coarse levels, the algorithm has a sub-optimal computing complexity. Algorithm 2.1 in the present paper, in which the convex sets K_j are defined in (3.14) and (3.15), has an optimal computing complexity and therefore, it improves the algorithm in [2].

In [2], the convex set K is assumed to satisfy a little more general Property 3.1. If K is of two-obstacle type, then it has this property. For the one- and two-level methods, Propositions 3.1 and 4.1 in [2] shows that the assumption made to prove the general convergence theorem is verified for such convex sets. However, this property does not suffice to prove that this assumption hold in the case of the method with more than two levels, and equations (5.22) and

(5.26) in the proof of Proposition 5.1 are not true. By the introduction of the level convex sets in (5.22) and (5.26) in the present paper, we have avoided this difficulty because the conditions $w_{j1} + \ldots + w_{ji-1} + u_{ji} \in K_j$ in the proof of Proposition 3.2, and $u_{ji} \in K_j$ in the proof of Proposition 3.3, can be proved by verifying that they are satisfied only at the nodes of $\mathcal{T}_{h_j}$.

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References

[1] L. Badea, Convergence rate of a multiplicative Schwarz method for strongly nonlinear variational inequalities, in *Analysis and Optimization of Differential Systems*, V.Barbu et al. (eds.), Kluwer Academic Publishers, 2003, p. 31-42 (also available at http://www.imar.ro/ lbadea/).

[2] L. Badea, Convergence rate of a Schwarz multilevel method for the constrained minimization of non-quadratic functionals, *SIAM J. Numer. Anal.*, **44**, 2, 2006, p. 449-477.

[3] L. Badea, Additive Schwarz method for the constrained minimization of functionals in reflexive Banach spaces, in U. Langer et al. (eds.), *Domain decomposition methods in science and engineering XVII*, LNSE 60, Springer, 2008, p. 427-434.

[4] L. Badea, X.-C. Tai and J. Wang, Convergence rate analysis of a multiplicative Schwarz method for variational inequalities, *SIAM J. on Num. Anal*, **41**, 3, 2003, p. 1052-1073.

[5] A. Brandt and C. Cryer, Multigrid algorithms for the solution of linear complementary problems arising from free boundary problems, *SIAM J. Sci. Stat. Comput.*, **4**, 1983, p. 655-684.

[6] M. Dryja and O. B. Widlund, Domain decomposition algorithms with small overlap, *SIAM J. Sci. Comput.*, **15**, 3, 1994, p. 604-620.

[7] I. Ekeland and R. Temam, *Analyse convexe et problèmes variationnels*, Dunod, Paris, 1974.

[8] E. Gelman and J. Mandel, On multilevel iterative method for optimization problems, *Math. Program.*, **48**, 1990, p. 1-17.

[9] R. Glowinski, J. L. Lions and R. Trémolières, *Analyse numérique des inéquations variationnelles*, Dunod, Paris, 1976.

[10] C. Gräser and R. Kornhuber, Multigrid methods for obstacle problems, *J. Comput. Math.*, **27**, 1, 2009, p. 1-44.

[11] W. Hackbusch and H. Mittelman, On multigrid methods for variational inequalities, *Numer. Math.*, **42**, 1983, p. 65-76.

[12] R. Hoppe and R. Kornhuber, Adaptive multilevel methods for obstacle problems, *SIAM J. Numer. Anal.*, **31**, 2, 1994, p. 301-323.

[13] R. Kornhuber, Monotone multigrid methods for elliptic variational inequalities I. Numer. Math. **69**, (1994) pp. 167-184.

[14] R. Kornhuber, Monotone multigrid methods for elliptic variational inequalities II. Numer. Math. **72**, (1996) pp. 481-499

[15] Kornhuber, R., *Adaptive monotone multigrid methods for nonlinear variational problems.* Teubner-Verlag, Stuttgart (1997)

[16] R. Kornhuber and R. Krause, Adaptive multigrid methods for Signorini's problem in linear elasticity, *Comp. Visual. Sci.*, **4**, 2001, p. 9-20.

[17] R. Kornhuber and H. Yserentant, Multilevel methods for elliptic problems on domains not resolved by the coarse grid, *Contemporary Mathematics*, **180**, 1994, p. 49-60.

[18] J. L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, Paris, 1969.

[19] Mandel, J., A multilevel iterative method for symmetric, positive definite linear complementary problems, *Appl. Math. Opt.*, **11**, (1984), pp. 77-95.

[20] Mandel, J., Etude algébrique d'une méthode multigrille pour quelques problèmes de frontière libre, *C. R. Acad. Sci.*, **298**, Ser. I, (1984), pp. 469-472.

[21] X. C. Tai Rate of convergence for some constraint decomposition methods for nonlinear variational inequalities, *Numer. Math.*, **93**, 4, 2003, p. 755-786.

[22] X.-C. Tai and J. Xu, Global and uniform convergence of subspace correction methods for some convex optimization problems, *Math. Comp.*, 71, 2002, p. 105–124.

[23] A. Toselli and O. Widlund, *Domains Decomposition Methods - Algorithms and Theory.* Springer-Verlag, Berlin (2005).

[24] B. Wohlmuth and R. Krause, Monotone multigrid methods on nonmatching grids for nonlinear multibody contact problems, *SIAM Sci. Comput.*, **25**, 2003, p. 324-347.

[25] J. Xu, *Theory of multilevel methods*, PhD thesis, Cornell University, 1989