Natural local times

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Abstract

In the abstract frame given by a filtered complete probability space we consider a random set M and we consider one of the local times of M which enjoyes a natural measurability property, starting from a local time of 0 on the space $W = \{w : [0,\infty) \to [0,\infty]; w\ is\ r.c.l.l.\}$ endowed with canonical structures.

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Introduction

The local time in 0 for the brownian motion, discovered by P.Levy, was the object of various generalizations in unexpected directions in Probability. Our approach, although in a general frame, deals however with the "markovian" local time of a point which is regular for "itself" and so our main result theorem 2.1 should be considered together with remark 2.3. The idea of considering the visits in 0 of the coordinate process on the canonical space $W = \{w : [0,\infty) \to [0,\infty]; w\ is\ r.c.l.l.\}$ was suggested by the papers of Krylov - Juškevič [9], Hoffmann - Jorgensen [8] and especially of Maisonneuve [11], by considering the *rest of life* process (R_t) defined by $R_t = D_t - t$, although this process is generally not adapted.

1. Preliminaries

The frame of our paper is intended to be as general as possible for our purpose. The main application of our considerations, presented in remark 2.3. is in the frame given by a sufficiently general Markov process, and we assume that the reader is familiar with the considered notions. If not, see [6] or [7] for definitions and basic results in connection to our paper.

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a filtered complete probability space such that $(\mathcal{F}_t)$ satisfies the usual conditions, that is $(\mathcal{F}_t)$ is right continuous and $\mathcal{F}_0$ contains the P - null sets, called negligible in the sequel.

We consider a progressive set $M \subset R_+ \times \Omega$. For any $\omega \in \Omega$, we denote $M(\omega) = \{t \geq 0; (t, \omega) \in M\}$, and we consider the following sets: $\overline{M}(\omega)$ the closure of $M(\omega)$ in R_+, $\overleftarrow{M}(\omega)$ the set of enter points in $M(\omega)$, and $\overrightarrow{M}(\omega)$ the set of exit points from $M(\omega)$. More precisely, if $\bigcup(a_n, b_n)$ is the representation of the complement in R_+ of $\overline{M}(\omega)$ as a countable or finite union of disjoint open intervals, then $\overrightarrow{M}(\omega) = \bigcup a_n$, and $\overleftarrow{M}(\omega) = \bigcup b_n$. If *one* of these intervals is right closed, for $a_n = 0$, then $a_n \notin \overrightarrow{M}(\omega)$ but $b_n \in \overleftarrow{M}(\omega)$, and if *one* of the b_n is ∞, then still $a_n \in \overrightarrow{M}(\omega)$.

For any $t \geq 0$ (resp. $t > 0$), one defines the random variables $D_t(\omega) = \inf\{s > t; s \in M(\omega)\}$ (resp. $g_t(\omega) = \sup\{s < t; s \in M(\omega)\}$). Then one can see that the process (D_t) is r.c.l.l. but generally *not* $(\mathcal{F}_t)$ adapted, and the process (g_t) is left continuous and $(\mathcal{F}_t)$ adapted.

The *natural* filtration of M is defined as follows: for $t \geq 0$ one considers $\mathcal{G}_t^0 = \sigma(g_s; s \leq t)$ and one takes $\mathcal{G}_t = \mathcal{G}_{t+}^0 \bigvee \mathcal{N}$, where $\mathcal{N}$ denotes the family of negligible sets from $\mathcal{F}$. (of course $\mathcal{G}_{t+}^0 = \bigcap_{s>t} \mathcal{G}_s^0$).

We say that a right continuous, increasing, null in 0, and $(\mathcal{F}_t)$ adapted process (A_t) is a *local time* of M if for P - almost any $\omega \in \Omega$ it is true that

$A.(\omega)$ is continuous on R_+ and supp $dA.(\omega) = \overline{M}(\omega)$. Our definition relaxes a little the usual definition of the local time which requires that $A.(\omega)$ be continuous for *any* $\omega \in \Omega$. We recall that there exists at least a local time of M if $M(\omega)$ has no isolated point for P - almost any $\omega \in \Omega$ (but the local time of M is not unique), and this hypothesis is assumed for the sequel. We also assume that M satisfies the following (well known) hypothesis:
(P) For any *predictable* stopping time S, the set $\{\omega \in \Omega; S(\omega) \in \vec{M}(\omega)\}$ is negligible.

Suggested by the proof of [6, XV, 86], we consider the space $W = \{w : [0, \infty) \to [0, \infty]; w \ is \ r.c.l.l.\}$, and the mapping $\rho : \Omega \to W$ defined by $\rho(\omega) = R.(\omega)$, where $R_t = D_t - t$ for any $t \geq 0$. We denote $W' = \rho(\Omega)$, and we note that for the sequel the set W' is of interest (and not W), endowed with the traces of some "canonical" structures on W. If (Y_t) denotes the canonical coordinate process on W', we denote $\mathcal{T}_t^0 = \sigma(Y_s; s \leq t)$ for any $t \geq 0$, $\mathcal{T}_\infty^0 = \sigma(Y_s; s \geq 0)$, and $Q = \rho \circ P$, the image of P throughout ρ, considered as a probability of reference on $(W', \mathcal{T}_\infty^0)$. If we complete $\mathcal{T}_\infty^0$ with respect to Q and we denote by $\mathcal{N}'$ the correponding negligible sets, we finally take $\mathcal{T}_t = \mathcal{T}_{t+}^0 \bigvee \mathcal{N}'$, and clearly the filtration $(\mathcal{T}_t)$ satisfies the usual conditions with respect to Q. Returning to Ω, we denote $\mathcal{H}_t^0 = \sigma(R_s; s \leq t)$ for any $t \geq 0$, and we take $\mathcal{H}_t = \mathcal{H}_{t+}^0 \bigvee \mathcal{N}$. We note that $\mathcal{H}_t \supset \mathcal{G}_t$, whereas $\mathcal{H}_t \not\subseteq \mathcal{F}_t$ in general.

PROPOSITION 1.1. *Let $h : W' \to R_+$ be $\mathcal{T}_\infty$ measurable. Then, for any $t \geq 0$ the following relation holds:*

$$E_P[h \circ \rho \mid \mathcal{H}_t] = E_Q[h \mid \mathcal{T}_t] \circ \rho. \tag{1.1}$$

Proof. First, we note that the conditional expectation from the left makes sense. This follows from the obvious remarks that $\mathcal{H}_t^0 = \rho^{-1}(\mathcal{T}_t^0)$ for any $t \geq 0$, by taking $t = \infty$, and the fact that $\rho^{-1}(\mathcal{N}') \subset \mathcal{N}$. If we fix now $t < \infty$, to prove (1.1) we may suppose that h is bounded, and it clearly suffices to check that for any $\mathcal{A} \in \mathcal{H}_{t+}^0$ the following relation holds:

$$\int_{\mathcal{A}} h \circ \rho \, dP = \int_{\mathcal{A}} E_Q[h \mid \mathcal{T}_t] \circ \rho \, dP. \tag{1.2}$$

If we fix now $n \in \mathbb{N}$, then $\mathcal{A} \in \mathcal{H}_{t+\frac{1}{n}}^0$, and clearly we have that

$$\int_{\mathcal{A}} h \circ \rho \, dP = \int_{\mathcal{A}} E_Q[h \mid \mathcal{T}_{t+\frac{1}{n}}] \circ \rho \, dP. \tag{1.3}$$

Using now the well known fact that $E_Q[h \mid \mathcal{H}_{t+\frac{1}{n}}]$ converges to $E_Q[h \mid \mathcal{T}_t]$ in $L^1(Q)$ and Q - a.s., a passing to the limit in (1.3) for $n \to \infty$ gives (1.2), and the proof is finished. □

2. The main result

Considering the space W' defined above, we denote $M' = \{(t, w) \in R_+ \times W'; w(t) = 0\}$, and the remark we need is that for any $\omega \in \Omega$ we have $M'(\rho(\omega)) = \overline{M}(\omega) \setminus \vec{M}(\omega)$, and if in addition $M(\omega)$ has no isolated point this is the smallest right closed subset of R_+ whose closure is $\overline{M}(\omega)$.

It is natural to define D'_t on W' by $D'_t(w) = \inf\{s > t; s \in M'(w)\}$, and using above remark one can see that $D'.(\rho(\omega)) = D.(\omega)$ for P - almost any $\omega \in \Omega$, since we assumed that $M(\omega)$ has no isolated point for P - almost any $\omega \in \Omega$.

For the following we say that a local time (A_t) of M (in our sense) is *natural* if (A_t) is $\mathcal{G}_t$ adapted and in addition $dA.(\omega) = dA.(\omega')$ whenever $\overline{M}(\omega) = \overline{M}(\omega')$ (if (A_t) is $B(R_+) \times \mathcal{G}_\infty^0$ measurable, the last condition is automatically satisfied, however we recall that $(\mathcal{G}_t)$ is obtained by completion with respect to negligible sets from $\mathcal{F}$).

THEOREM 2.1. *For each $p > 0$, if (A'_t) is a right continuous, increasing, null in 0, and $(\mathcal{T}_t)$ predictable process on W' satisfying the relation*

$$E_Q[e^{-pD'_t} \mid \mathcal{T}_t] = E_Q[A'_\infty - A'_t \mid \mathcal{T}_t] \quad for\ any\ t \geq 0, \tag{2.1}$$

then the process $(A_t) = (A'_t \circ \rho)$ is a natural local time of M.

Proof. First, we remark that the second condition of being "natural" is obvious from definition of (A_t), for if $\overline{M}(\omega) = \overline{M}(\omega')$, then $R.(\omega) = R.(\omega')$.

We do not assert yet that $A.(\omega)$ is continuous for P - almost any $\omega \in \Omega$, but we can assert that supp $dA'.(w) \subset \overline{M'}(w)$ and moreover $dA'.(w)$ does not charge $\overleftarrow{M}'(w)$, for Q - almost any $w \in W'$.

Indeed, since $\overline{M'}(w)$ has no isolated point for any $w \in W'$, we can use a standard argument: we have $D'_{(D'_u)} = D'_u$ for any $u \geq 0$, and from an optional stopping in (2.1) at u and D'_u, it follows that $A'_u = A'_{D'_u}$ Q - a.s. and it suffices to make u running in a countable dense subset of R_+. By transport, it follows that supp $dA(\omega) \subseteq \overline{M}(\omega)$ and moreover $dA.(\omega)$ does not charge $\overleftarrow{M}(\omega)$, for P - almost any $\omega \in \Omega$.

Next, we remark that (A_t) is a predictable process with respect to filtration $\rho^{-1}(\mathcal{T}_t)(\subseteq \mathcal{H}_t)$.

This follows from the general assertion that if (Z'_t) is a predictable (real) process with respect to $(\mathcal{T}_t)$, then the process $(Z_t) = (Z'_t \circ \rho)$ is predictable with respect to $(\rho^{-1}(\mathcal{T}_t))$, which can be proved using the monotone class theorem, starting with the defining generators of the predictable field.

We assert now that (A_t) is $(\mathcal{G}_t)$ adapted and moreover it is predictable with respect to this filtration. From above considerations, it suffices to check that if (Z_t) is an arbitrary (real) process predictable with respect to $(\mathcal{H}_t)$, then the process $(Z_{g_t} I_{\{g_t>0\}})$ is predictable with respect to $(\mathcal{G}_t)$. This also follows from the monotone class theorem, not using the defining generators but using the "standard" generators of the form $Z_t = U_{I\times(s,\infty)}(t)$ for $U \in \mathcal{H}_{s-}$ (where $s > 0$ is fixed) or $Z_t = I_{U\times\{0\}}(t)$ for $U \in \mathcal{H}_0$. The second case is obvious, and for the first we recall a previously used fact that for any $0 \le u \le \infty$ we have $\rho^{-1}(\mathcal{T}_u^0) = \sigma(R_v; v \le u)$, and therefore there exists a set $V \in \sigma(R_u; u \le s)$ such that the set $V \triangle U$ is P - negligible. Hence, it follows that it suffices to check that the process $(I_V I_{\{g_t>s\}})$ is adapted with respect to $(\mathcal{G}_{t+}^0)$. For this, we remark that in fact $\sigma(R_u; u \le s) = \sigma(D_u; u \le s)$ because R_u and D_u differ by a constant for any $u \ge 0$, and hence it suffices to check that the restriction of D_u on $\{g_t > s\}$ is $\mathcal{G}_t^0$ measurable, for any fixed $u \le s$. Indeed, on $\{g_t > s\}$ we have $D_u < g_t \le t$, hence it suffices to check that for any $r < u \le t$, the set $\{D_u < r\}$ belongs to $\mathcal{G}_t$, and this follows from the obvious basic relation:

$$\{D_u < r\} = pr_2((u, r) \times \Omega) \cap M = \{g_r > u\}, \tag{2.2}$$

which also implies that $\mathcal{G}_t \subset \mathcal{H}_t$ for any $t \ge 0$.

Now, we show that (A_t) coincides (up to an evanescent set) with the "canonical" prediactable (increasing, null in 0) with respect to $(\mathcal{G}_t)$ process (A_t) defined by the property (Doob - Meyer decomposition)

$$E_P[e^{-pD_t} \mid \mathcal{G}_t] = E_P[A_\infty - A_t \mid \mathcal{G}_t] \quad for\ any\ t \ge 0. \tag{2.3}$$

Thanks to prop. 1.1 and from the preceding considerations, we can write:

$$\begin{aligned} &E_P[e^{-pD_t} \mid \mathcal{G}_t] = E_P[e^{-pD'_t\circ\rho} \mid \mathcal{H}_t \mid \mathcal{G}_t] = \\ &E_P[E_Q[e^{-pD'_t} \mid \mathcal{T}_t] \circ \rho \mid \mathcal{G}_t] = E_P[E_Q[A'_\infty - A'_t \mid \mathcal{T}_t] \circ \rho \mid \mathcal{G}_t] = \\ &E_P[A_\infty - A_t \mid \mathcal{H}_t \mid \mathcal{G}_t] = E_P[A_\infty - A_t \mid \mathcal{G}_t]. \end{aligned} \tag{2.3'}$$

Finally, we can invoque [6, XV, 53] applied for $(\mathcal{G}_t)$ and $\overline{M}$, after we remark that we have $\overline{M} \cap (0, \infty) \times \Omega = \{g_{t+} = t\}$ and so $\overline{M} \cap (0, \infty) \times \Omega$ is optional with respect to $(\mathcal{G}_t)$. Therefore it follows that $A.(\omega)$ is continuous and supp $dA.(\omega) = \overline{M}(\omega)$, for P - almost any $\omega \in \Omega$, which finish the proof. $\square$

Remark 2.3. The last part of above proof shows that we already have a version of a natural local time of M, furnished by the Doob - Meyer decomposition applied to the potentials $E_P[e^{-pD_t} \mid \mathcal{G}_t]$ (a r.c.l.l. version), therefore our result applies whenever it is easier to determine (A'_t), acting however on a canonical space, although the probability Q is obtained by transporting the given probability P, and we present below an important case of this situation.

Remark 2.2. In the frame given by a Ray markovian semigroup (P_t) on a compact metric space F, and the canonical Markov process $(\Omega, \mathcal{F}, \mathcal{F}_t, \Theta_t, X_t, P^X)$ with transition semigroup (P_t), one considers a set $J \subset N$ (the set of nonbranching points) closed in F such that $J \subset J^r$ (the set of regular points for J) and one takes $M = \{(t, \omega); X_t(\omega) \in J\}$. We recall that $\mathcal{F}^\mu_\infty$ denote the completion with respect to P^μ of $\mathcal{F}^0_\infty = \sigma(X_s; s \geq 0)$ and $\mathcal{F}^\mu_t = \mathcal{F}^0_t \bigvee \mathcal{N}^\mu$, where $\mathcal{N}^\mu$ denotes the family of P^μ - negligible sets from $\mathcal{F}^\mu_\infty$.

A local time of M is called a local time of J, and one defines the *normalized* local time (L_t) of J by te relation $L_t = \int_0^t e^s dA_s$, where (A_t) denotes a common version with respect to all systems $(\Omega, \mathcal{F}^\mu_\infty, \mathcal{F}^\mu_t, P^\mu)$ of the unique predictable process given by the Doob - Meyer decomposition satisfying the relation

$$(2.4) \qquad E[e^{-D_t} \mid \mathcal{F}_t^\mu] = E[A_\infty - A_t \mid \mathcal{F}_t^\mu] = e^{-t} e_J^1(X_t),$$

where e_J^1 denotes the 1 - balance potential of J.

Then in the particular case where J consists of a single point a, it follows that above (A_t) is exactly the 1 - natural local time of a in our sense such that $(A_t) = (A'_t \circ \rho)$, were (A'_t) is a common version with respect to all considered probabilities Q of the unique predictable process given by the Doob - Meyer decomposition satisfying the relation

$$(2.5) \qquad E_Q[e^{-D'_t} \mid \mathcal{T}_t] = E_Q[A'_\infty - A'_t \mid \mathcal{T}_t], \ for \ any \ t \geq 0.$$

The proof of above assertion is essentially contained in [6, XV, 86] where in fact is considered the process $L_t = \int_o^t e^s dA_s$ as an additive functional on Ω, and it is shown that (L_t) is a natural local time of a. In order to fix some ideas, we only sketch the proof of above assertion, based on the fact that the process (R_t) is in this particular case a Markov process with state space $[0, \infty]$, whose transition semigroup is moreover fellerian, denoted by (Π_t). If one considers the canonical Markov process (Y_t) on W with transition semigroup (Π_t), let (A'_t) denote a common version wit respect to all systems $(W, \mathcal{T}_\infty^\nu, \mathcal{T}_t^\nu, P^\nu)$ (ν is a law on $[0, \infty]$ here) of the predictable process given by the Doob - Meyer decomposition satisfying the relation

$$(2.6) \qquad E[e^{-D'_t} \mid \mathcal{T}_t^\nu] = E[A'_\infty - A'_t \mid \mathcal{T}_t^\nu] = e^{-t} e_0^1(Y_t),$$

where e_0^1 denotes the 1 - balance potential of $\{0\}$.

One can see that (A'_t) satisfies (2.5), on the other hand it is known that the process (L'_t) defined by $L'_t = \int_0^t e^s dA'_s$ (the normalized local time of $\{0\}$) is a *continuous* additive functional on W, and in the same time it is a local time of 0.

Then proceeding as in the proof of theorem 2.1, it follows that the process $(L'_t \circ \rho)$ is a $(\mathcal{G}_t)$ adapted ($\mathcal{G}_t = \bigcap_\mu \mathcal{G}_t^\mu$ here) continuous additive functional on Ω, and in the same time it is a local time of a. Using the theorem of Motoo it follows that $(L'_t \circ \rho) = c(L_t)$, where c is a positive constant, which implies that $(A'_t \circ \rho) = c(A_t)$, in particular it follows that (A_t) is $(\mathcal{G}_t)$ adapted. Taking the conditional expectation in (2.4) with respect to $(\mathcal{G}_t)$ and using remark 2.2, it follows that in fact $(A'_t \circ \rho) = (A_t)$, that is $c = 1$, which finishes the proof.

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