THE OSOFSKY-SMITH THEOREM FOR MODULAR LATTICES, AND APPLICATIONS (I)

TOMA ALBU*
"Simion Stoilow" Institute of Mathematics of the Romanian Academy
P.O. Box 1 - 764
RO - 010145 Bucharest 1, ROMANIA
e-mail: Toma.Albu@imar.ro

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Abstract

The aim of this paper is to present a latticial version of the renown module theoretical *Osofsky-Smith Theorem*. Applications to Grothendieck categories and module categories equipped with a torsion theory will be given in a subsequent paper.

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Introduction

The aim of this paper is to state and prove a latticial version of the renown *Osofsky-Smith Theorem* [12, Theorem 1] saying that a cyclic right R-module having all of its subfactors extending (i.e., CS) is a finite direct sum of uniform submodules. Though the Osofsky-Smith Theorem is a module-theoretical result, that can be also formulated and proved in a categorical setting as it is suggested in [12] (see [6] for such a setting), our contention is that it is a result of a strong latticial nature.

In Section 0 we present the terminology, notation, and basic results on lattices which will be needed in the sequel.

Section 1 is devoted to CC lattices (acronym for *C*losed elements are *C*omplements). These are the latticial counterparts of CS modules (acronym for *C*omplements submodules are direct *S*ummands).

Section 2 is devoted to prove a key technical result concerning lattices having ACC on complements, needed in the proof of the latticial Osofsky-Smith Theorem.

In Section 3 we state and prove a version of the module-theoretical *Osofsky-Smith Theorem* for compact, compactly generated, modular lattices. Applications to Grothendieck categories and module categories equipped with a torsion theory will be given in a subsequent paper.

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0 Preliminaries

The aim of this section is to present the terminology, notation, and basic results on lattices which will be needed in the sequel.

We use $\mathbb{N}$ to denote the set $\{1, 2, \ldots\}$ of all positive integers. All lattices considered in this paper are assumed to have a least element denoted by 0 and a last element denoted by 1. We denote by $\mathcal{L}$ (resp. $\mathcal{M}, \mathcal{U}$) the class of all lattices (resp. modular, upper continuous lattices). Throughout the paper a lattice will always mean a member of $\mathcal{L}$, and $(L\,, \leqslant\,, \wedge\,, \vee\,, 0\,, 1)$, or more simply, just L, will always denote such a lattice. If the lattices L and L' are isomorphic, we denote this by $L \simeq L'$. The (uniform) Goldie dimension of L will be denoted by $u(L)$.

For a lattice L and elements $a \leqslant b$ in L we write

$$b/a := [a, b] = \{\ x \in L \mid a \leqslant x \leqslant b\ \},$$

$$[a, b[:= \{\ x \in P \mid a \leqslant x < b\ \}.$$

A *subfactor* of L is any interval b/a of L with $a < b$. We say that the interval b/a is *simple* if $a \neq b$ and $b/a = \{a, b\}$. If $a < b$ are elements of L and there is no $c \in L$ such that $a < c < b$, then we say that a is *covered* by b, and we write $a \prec b$. Thus, the interval b/a is *simple* if and only if $a \prec b$.

For a lattice L and $a, b, c \in L$, the notation $a = b \dot{\vee} c$ will mean that $a = b \vee c$ and $b \wedge c = 0$, and we say that a is a *direct join* of b and c. Also, for a nonempty subset B of L, we use the *direct join* notation $a = \dot{\bigvee}_{b \in B} b$ if B is an independent subset of L and $a = \bigvee_{b \in B} b$. Recall that a subset A of a complete lattice L is said to be *join independent*, or just *independent*, if $0 \notin A$ and $a \wedge \bigvee_{x \in A \setminus \{a\}} x = 0$ for all $a \in A$.

An element $c \in L$ is a *complement in* L if there exists an element $a \in L$ such that $a \wedge c = 0$ and $a \vee c = 1$; we say in this case that c is a *complement* of a *in* L. One denotes by $D(L)$ the set of all complements of L. The lattice L is called *indecomposable* if $L \neq \{0\}$ and $D(L) = \{0, 1\}$, and $a \in L$ is an *indecomposable* element if $a/0$ is an indecomposable lattice. The lattice L is said to be *complemented* if every element of L has a complement in L.

An element $a \in L$ is said to be an *atom* of L if the interval $a/0$ is simple, or equivalently, if $0 \prec a$. We denote by $\mathcal{A}(L)$ the set, possibly empty, of all atoms of L. As in [11], the lattice L is called *semi-atomic* if 1 is a join of atoms of L. The *socle* $\mathrm{Soc}(L)$ of L is the join of all atoms of L. If L is a semi-atomic, upper continuous, modular lattice, then L is complemented, and for every $a \leqslant b$ in L, the interval b/a of L is also a semi-atomic lattice by [11, Theorem 1.8.2 and Corollary 1.8.4].

An element $b \in L$ is a *pseudo-complement in* L if there exists an element $a \in L$ such that $a \wedge b = 0$ and b is maximal with this property; we say in this case that b is a *pseudo-complement* of a. One denotes by $P(L)$ the set of all pseudo-complement elements of L.

As in [13], L is called *pseudo-complemented* if every element of L has a pseudo-complement. Note that in [14], L is called *pseudo-complemented* if for every $a \leqslant b$ in L and for every $x \in b/a$, there exists a pseudo-complement of x in b/a. Every upper continuous modular lattice is pseudo-complemented.

An element $e \in L$ is *essential in* L if $e \wedge x \neq 0$ for every $x \neq 0$ in L. One denotes by $E(L)$ the set of all essential elements of L. The lattice L is called *uniform* if $L \neq \{0\}$ and $x \wedge y \neq 0$ for every nonzero elements $x, y \in L$. An element u of L is called *uniform* if the interval $u/0$ of L is a uniform lattice. As in [13], L is called *E-complemented* ("E" for essential) if for each $a \in L$ there exists $b \in L$ such that $a \wedge b = 0$ and $a \vee b \in E(L)$.

An element $c \in L$ is said to be *closed* if $c \notin E(a/0)$ for all $a \in L$ with $c < a$. One denotes by $C(L)$ the set of all closed elements of L. As in [13], the lattice L is called *essentially closed* if for all $a \in L$, the set $S_a = \{ e \in L \mid a \in E(e/0) \}$ has a maximal element, or equivalently, for any $a \in L$ there exists $c \in C(L)$ with $a \in E(c/0)$. Note that any modular upper continuous lattice is essentially closed.

An element $c \in L$ is *compact in* L if whenever $c \leqslant \bigvee_{x \in A} x$ for a subset A of L, there is a finite subset F of A such that $c \leqslant \bigvee_{x \in F} x$. One denotes by $K(L)$ the set of all compact elements of L. The lattice L is said to be *compact* if 1 is a compact element in L, and *compactly generated* if it is complete and every element of L is a join of compact elements. Note that an element c of an upper continuous lattice is compact if and only if the lattice $c/0$ is compact, and any compactly generated lattice is upper continuous.

For all undefined notation and terminology on lattices, the reader is referred to [4], [5], [9], and/or [14].

We list below two results that will be needed in the sequel.

Lemma 0.1. ([1, Lemma 2.1].) *The following statements hold for a lattice L.*

(1) *If $c \in K(L)$ and $a \in L$, then $c \vee a \in K(1/a)$.*

(2) *If L is compactly generated and $a < b$ are elements of L, then b/a is compactly generated, and for every $k \in K(b/a)$, there exists $c \in K(L)$ such that $k = c \vee a$.* □

We do not know whether the last part of Lemma 0.1, i.e.,

$$\forall\, a < b \text{ in } L \text{ and } \forall\, k \in K(b/a) \Longrightarrow \exists\, c \in K(L) \text{ such that } k = c \vee a,$$

does hold without the assumption that L is compactly generated.

Lemma 0.2. (The Latticial Krull's Lemma [1, Corollary 2.3]). *If L is a complete compact lattice and $x < 1$ in L, then there exists $m \in L$ such that $x \leqslant m$ and m is maximal in $[0, 1[$.* □

1 CC lattices, compact lattices, and CEK lattices

The purpose of this section is to present the concept of a CC lattice introduced in [2], as well as its basic properties. We also discuss some properties of compact lattices and introduce the concept of a CEK (acronym for *C*losed are *E*ssentially *C*ompact) lattice and establish its main properties.

Recall that for a lattice L we use throughout this paper the following notation.

$P(L) =$ the set of all *pseudo-complement* elements of L (P for "*Pseudo*"),
$E(L) =$ the set of all *essential* elements of L (E for "*Essential*"),
$C(L) =$ the set of all *closed* elements of L (C for "*Closed*"),
$D(L) =$ the set of all *complement* elements of L (D for "*Direct summand*"),
$K(L) =$ the set of all *compact* elements of L (K for "*Kompakt*"),

Definitions 1.1. ([2].) *A lattice L is called* CC (*or* extending) *if for every $x \in L$ there exists $d \in D(L)$ such that $x \in E(d/0)$, and L is called* completely CC (*or* completely extending) *if the interval $1/a$ is a* CC (*or* extending) *lattice for every $a \in L$.* □

Lemma 1.2. ([2, Lemma 1.3].) *The following assertions hold for a lattice $L \in \mathcal{M}$.*

(1) $D(L) \subseteq C(L) \cap P(L)$.

(2) $D(L) \cap (d/0) = D(d/0)$ *for every* $d \in D(L)$.

(3) $d \in D(L)$ & $d \leqslant a \in L \Longrightarrow d \in D(a/0)$. □

The next result explain the term of CC lattice, acronym for Closed elements are Complements.

Proposition 1.3. ([2, Proposition 1.5].) *The following statements hold for a lattice $L \in \mathcal{M}$.*

(1) *L is uniform $\Longrightarrow$ L is* CC, *and, if additionally L is indecomposable, then inverse implication "$\Longleftarrow$" also holds.*

(2) *If additionally L is essentially closed* (*in particular, if L is upper continuous*) *then*

$$L \text{ is } CC \Longleftrightarrow C(L) \subseteq D(L) \Longleftrightarrow C(L) = D(L).$$

(3) *If additionally L is essentially closed and E-complemented* (*in particular, if L is upper continuous*) *then*

$$L \text{ is } CC \Longleftrightarrow C(L) \subseteq D(L) \Longleftrightarrow C(L) = D(L) \Longleftrightarrow P(L) \subseteq D(L) \Longleftrightarrow P(L) = D(L).$$ □

Proposition 1.4. ([2, Proposition 1.9].) *Let $L \in \mathcal{M}$ be an essentially closed lattice* (*in particular an upper continuous lattice*). *If L is a* CC *lattice then so is also $d/0$ for any $d \in D(L)$, in other words, the* CC *condition is inherited by "complement intervals".* □

Corollary 1.5. *Let $L \in \mathcal{M}$ be an essentially closed* CC *lattice. Then L has finite Goldie dimension if and only if 1 is a finite direct join of uniform elements of L.*

Proof. One implication is clear. For the other one, assume that L has finite Goldie dimension. Then L contains a uniform element v. Let $c \in C(L)$ such that $v \in E(c/0)$. Then $c \in D(L)$ because L is CC, so $1 = c \dot{\vee} c'$, for some $c' \in L$. It follows that $c'/0$ is also CC by Proposition 1.4. Now observe that $u(c'/0) < u(L)$, so the proof proceeds by induction on $u(L)$. □

Definitions 1.6. *Let L be a lattice.*

(1) *An element $a \in L$ is called* essentially compact *if there exists $e \leqslant a$ such that $e \in E(a/0) \cap K(L)$. We denote by $E_k(L)$ the set of all essentially compact elements of L.*

(2) *L is called* CEK *(for* Closed *are* Essentially *Compact) if every closed element of L is essentially compact, i.e., $C(L) \subseteq E_k(L)$.* □

The next result provides large classes of CEK lattices.

Proposition 1.7. *Let $L \in \mathcal{M}$ be a nonzero complete lattice having the following property:*

$$(\dagger) \qquad \text{for every } 0 \neq x \in L \text{ there exists } 0 \neq k \in K(L) \text{ with } k \leqslant x,$$

in particular, L can be any compactly generated lattice.

Then L has finite Goldie dimension if and only if each element of L is essentially compact, i.e., $L = E_k(L)$. In particular, any modular lattice with finite Goldie dimension satisfying $(\dagger)$ is CEK.

Proof. Assume that L has finite Goldie dimension, and let $a \in L$. Then the interval $a/0$ has also finite Goldie dimension, so there exists an independent family $(u_i)_{1 \leqslant i \leqslant n}$ of uniform elements of $a/0$ such that $\bigvee_{1 \leqslant i \leqslant n} u_i \in E(a/0)$. By hypothesis, for every i, $1 \leqslant i \leqslant n$, there exist $0 \neq k_i \in K(L)$ such that $k_i \leqslant u_i$. Then, if we set $k := \bigvee_{1 \leqslant i \leqslant n} k_i$ and $u := \bigvee_{1 \leqslant i \leqslant n} u_i$, we have $k \in E(u/0) \cap K(L)$, so $k \in E(a/0) \cap K(L)$, as desired.

Conversely, assume that L has infinite Goldie dimension. Then $L \setminus \{0\}$ contains an infinite independent set $\{x_1, x_2, \dots\}$. Since L is a complete lattice, we may consider the element $x := \bigvee_{i \in \mathbb{N}} x_i$. Then $x \notin E_k(L)$, for otherwise, it would exist $c \leqslant x$ such that $c \in E(x/0) \cap K(L)$. There exists $m \in \mathbb{N}$ with $c \leqslant \bigvee_{1 \leqslant i \leqslant m} x_i$, and then

$$c \wedge x_{m+1} \leqslant \big(\bigvee_{1 \leqslant i \leqslant m} x_i \big) \wedge x_{m+1} = 0,$$

so $c \notin E(x/0)$. This means that $x \notin E_k(L)$, and we are done. □

We do not know whether the condition $(\dagger)$ implies that L is compactly generated, but we guess no.

Lemma 1.8. *The following statements hold for a lattice L.*

(1) *If $L \in \mathcal{M}$ is a compact lattice, then so is $d/0$ for any $d \in D(L)$. If additionally L is upper continuous, then $D(L) \subseteq K(L)$.*

(2) *If L is a compact lattice, so is also any of its quotient intervals $1/a$.*

(3) *Assume that $L \in \mathcal{U}$, and let $a \leqslant b \leqslant c$ in L be such that the intervals b/a and c/b are both compact lattices. Then c/a is also a compact lattice.*

(4) *If L is a complete compact lattice, then any lattice isomorphic to L is also compact.*

Proof. (1) Let $d \in D(L)$. Then, there exists $d' \in L$ such that $d \vee d' = 1$ and $d \wedge d' = 0$. Let $A \subseteq L$ be such that $d = \bigvee_{x \in A} x$. Because 1 is a compact element of L and $1 = d \vee d' = \bigvee_{x \in A}(x \vee d')$, there exists a finite subset F of A such that

$$1 = \bigvee_{x \in F} (x \vee d') = \big(\bigvee_{x \in F} x\big) \vee d' = y \vee d',$$

where $y := \bigvee_{x \in F} x \leqslant d$. By modularity, we have

$$d = (y \vee d') \wedge d = y \vee (d \wedge d') = y \vee 0 = y = \bigvee_{x \in F} x,$$

which shows that d is a compact element of $d/0$, as desired.

Assume now that additionally $L \in \mathcal{U}$ and let $d \in D(L)$. In order to show that d is a compact element of L, it is sufficient to show that if $A \subseteq L$ is a directed subset such that $d \leqslant \bigvee_{a \in A} a$, then $d \leqslant a_0$ for some $a_0 \in A$. By upper continuity, we have $d = \bigvee_{a \in A}(a \wedge d)$. Continue as above, with $a \wedge d$ instead of x in $d = \bigvee_{x \in A} x$, to get a finite subset G of A such that $d = \bigvee_{a \in G}(a \wedge d)$. But A has been assumed to be a directed set, so $d \leqslant a_0$ for some $a_0 \in A$, and we are done.

(2) This is obvious.

(3) Without loss of generality we may assume that $a = 0$ and $c = 1$. So, we have to prove that if b is compact in $b/0$ and 1 is compact in $1/b$, then 1 is compact in $1/0 = L$. Let D be a directed subset of L such that $1 = \bigvee_{x \in D} x$. By upper continuity, we have

$$b = \bigvee_{x \in D} (x \wedge b),$$

so $b = y \wedge b$, i.e., $b \leqslant y$ for some $y \in D$ because b is compact in $b/0$.

On the other hand, we have $1 = \bigvee_{x \in D}(x \vee b)$, so $1 = z \vee b$ for some $z \in D$ because 1 is compact in $1/b$. There exist now $t \in D$ with $y \leqslant t$ and $z \leqslant t$ since D is a directed subset. It follows that $b \leqslant y \leqslant t$ and $1 = t \vee b \leqslant t$, so $1 = t \in D$, which shows that 1 is compact in L.

(4) Observe first that if $f : L \longrightarrow L'$ is a lattice isomorphism of complete lattices, then clearly

$$f\big(\bigvee_{i \in I} x_i\big) = \bigvee_{i \in I} f(x_i)$$

for any family $(x_i)_{i \in I}$ of elements of L. This easily implies that $f(c) \in K(L')$ for any $c \in K(L)$; in particular $1' = f(1) \in K(L')$ if 1 is a compact element of L, that is, L' is compact if so is L. □

Lemma 1.9. *Let $\{0\} \neq L \in \mathcal{M} \cap \mathcal{U}$, and let $(x_i)_{i \in I}$ be a nonempty independent family of elements of L. Assume that for every $i \in I$ there exists $m_i \in L$ such that $m_i \prec x_i$. Denote $m'_i := \bigvee_{j \in I \setminus \{i\}} m_j$ and $a_i := x_i \vee m'_i$ for every $i \in I$.*

Then $(a_i)_{i \in I}$ is an independent family of atoms of the interval $\big(\bigvee_{i \in I} x_i\big)/\big(\bigvee_{i \in I} m_i\big)$ of L whose join is $\bigvee_{i \in I} x_i$, and so, the interval $\big(\bigvee_{i \in I} x_i\big)/\big(\bigvee_{i \in I} m_i\big)$ is a semi-atomic lattice.

Proof. For simplicity, set $m := \bigvee_{i\in I} m_i$ and $x := \bigvee_{i\in I} x_i$. Then, for every $i \in I$, we have by modularity

$$a_i/m = (x_i \vee m_i')/m = (m \vee x_i)/m \simeq x_i/(m \wedge x_i) = x_i/m_i$$

because $m \wedge x_i = (m_i \vee m_i') \wedge x_i = m_i \vee (m_i' \wedge x_i)$ and $m_i' \wedge x_i \leqslant \big(\bigvee_{j\in I\setminus\{i\}} x_j\big) \wedge x_i = 0$. Thus $a_i/m \simeq x_i/m_i$ is a simple interval because $m_i \prec x_i$, so a_i is an atom of x/m. Clearly, $x = \bigvee_{i\in I} a_i$, so x/m is a semi-atomic lattice.

We are now going to show that $(a_i)_{i\in I}$ is an independent family of atoms of the interval x/m, i.e.,

$$a_i \wedge \Big(\bigvee_{j\in I\setminus\{i\}} a_j\Big) = m$$

for all $i \in I$.

For a fixed $i \in I$ we have

$$a_i \wedge \Big(\bigvee_{j\in I\setminus\{i\}} a_j\Big) = (x_i \vee m_i') \wedge \Big(\bigvee_{j\in I\setminus\{i\}} a_j\Big) = (x_i \vee m_i') \wedge (m_i \vee x_i') = (x_i \vee m_i') \wedge y_i,$$

where

$$x_i' := \bigvee_{j\in I\setminus\{i\}} x_j \quad \text{and} \quad y_i := m_i \vee x_i'.$$

Observe that $m_i' \leqslant y_i$, so, using repeatedly the modularity, we obtain

$$(x_i \vee m_i') \wedge y_i = (x_i \wedge y_i) \vee m_i' = (x_i \wedge (m_i \vee x_i')) \vee m_i' = (m_i \vee (x_i \wedge x_i')) \vee m_i' = m,$$

as desired. □

Lemma 1.10. *Let $\{0\} \neq L \in \mathcal{M} \cap \mathcal{U}$ be a semi-atomic lattice, and let $(a_i)_{i\in I}$ be an independent family of atoms of L such that $1 = \dot{\bigvee}_{i\in I} a_i$. Then, the following statements hold.*

(1) *L is compact if and only if I is finite.*

(2) *If I is infinite, then there exists an independent family $(c_i)_{i\in I}$ of non compact elements of L such that $1 = \dot{\bigvee}_{i\in I} c_i$ and each c_i is a direct join of a countable subfamily of the given family $(a_j)_{j\in I}$ of atoms of L.*

Proof. (1) If I is finite, then L is compact because 1 is a finite join of atoms, which are all compact elements.

Conversely, if I is infinite then L has infinite Goldie dimension, so $1 = \dot{\bigvee}_{i\in I} a_i$ is not a compact element, i.e., L is not compact.

(2) If I is infinite, then $|I| = |I| \cdot \aleph_0$, where $|I|$ means the cardinal number of the set I. So, one can partition I as a union $I = \bigcup_{\lambda\in\Lambda} I_\lambda$ of a family $(I_\lambda)_{\lambda\in\Lambda}$ of mutually disjoint sets indexed by an index set Λ with $|\Lambda| = |I|$ and such that $|I_\lambda| = \aleph_0$ for all $\lambda \in \Lambda$. Set $c_\lambda := \dot{\bigvee}_{i\in I_\lambda} a_i$ for every $\lambda \in \Lambda$. Then, by (1), all c_λ, $\lambda \in \Lambda$, are not compact elements of L, and $1 = \dot{\bigvee}_{\lambda\in\Lambda} c_\lambda$. To finish the proof, note that one can replace Λ with I because $|\Lambda| = |I|$. □

2 Lattices with ACC on complements

In this section we establish a very technical result that is the key point in proving the *Latticial Osofsky-Smith Theorem.*

Lemma 2.1. *Let L be a compact, compactly generated, modular lattice. Assume that all compact subfactors b/a of L are* CEK, *i.e., every $c \in C(b/a)$ is an essentially compact element of b/a. Then $D(L)$ is Noetherian poset.*

Proof. First, observe that the given lattice L being compactly generated is also upper continuous. We will adapt the module theoretical proofs of [7, Theorem 7.12] and [10, Theorem 6.44] to our latticial setting. Note that the adaptation is not always so straightforward because we have to avoid some module theoretical tools and results used in the module case that do not work in a latticial frame.

Step 1: Assume that $D(L)$ is not a Noetherian poset. Then, there exists an infinite ascending chain

$$a_1 < a_2 < a_3 < \dots$$

of elements of $D(L)$.

We have $1 = a_1 \dot\vee b_1$, so $a_2 = a_2 \wedge (a_1 \vee b_1) = a_1 \vee (a_2 \wedge b_1) = a_1 \dot\vee (a_2 \wedge b_1)$ by modularity. It follows that $a_2 \wedge b_1 \in D(L)$, and hence $a_2 \wedge b_1 \in D(b_1/0)$ by Lemma 1.2 (2). Let $b_2 \in b_1/0$ be such that $b_1 = b_2 \dot\vee (a_2 \wedge b_1)$. Then $b_2 \in D(L)$, so $1 = a_2 \dot\vee b_2$. Repeating this procedure we obtain an infinite descending chain

$$b_1 > b_2 > b_3 > \dots$$

of elements of $D(L)$ such that $1 = a_j \dot\vee b_j$ for all $j \geqslant 1$. By Lemma 1.2 (2), for each $j \geqslant 1$ there exists $c_j \in D(L)$ with $b_j = b_{j+1} \dot\vee c_{j+1}$ for some $0 \neq c_{j+1} \in D(L)$. If we set $c_1 = a_1$, then, by recurrence, we have

$$1 = c_1 \dot\vee c_2 \dot\vee \dots \dot\vee c_n \dot\vee b_n \quad \text{and} \quad \bigvee_{j \geqslant n+1} c_j \leqslant b_n$$

for all $n \geqslant 1$. Note that $(c_n)_{n \geqslant 1}$ is an independent family of elements of L because so is any of its finite subfamily.

Since L is compact and $c_j \in D(L)$, we deduce by Lemma 1.8 (1) that the interval $c_j/0$ is also compact for every $j \geqslant 1$. Now, apply the Latticial Krull Lemma (Corollary 0.2) to deduce that there exists $m_j \in L$ with $m_j \prec c_j$ for all $j \geqslant 1$. For simplicity, set

$$m := \bigvee_{j \in \mathbb{N}} m_j, \quad c := \bigvee_{j \in \mathbb{N}} c_j, \quad M := 1/m, \quad C := c/m.$$

By Lemma 1.9, C is a semi-atomic sublattice of the compact lattice M.

Step 2: We claim that

$$\Big(\dot{\bigvee_{1 \leqslant i \leqslant n}} c_i \Big) \vee m \in D(M)$$

for all $n \in \mathbb{N}$. More precisely, we are going to prove that

$$((\dot{\bigvee_{1 \leqslant i \leqslant n}} c_i) \vee m) \dot{\vee} (b_n \vee m) = 1.$$

To do that, for a fixed $n \in \mathbb{N}$, we set for simplicity

$$x := \dot{\bigvee_{1 \leqslant i \leqslant n}} c_i, \ \ z := \bigvee_{1 \leqslant i \leqslant n} m_i, \quad \text{and} \quad y := b_n \vee m.$$

Since

$$\bigvee_{i \geqslant n+1} m_i \leqslant \bigvee_{i \geqslant n+1} c_i \leqslant b_n, \ \ z \leqslant x, \quad \text{and} \quad x \wedge b_n = 0,$$

we have by modularity

$$x \wedge y = x \wedge (b_n \vee m) = x \wedge \big(b_n \vee z \vee (\bigvee_{i \geqslant n+1} m_i)\big) = x \wedge (b_n \vee z) = (x \wedge b_n) \vee z = 0 \vee z = z.$$

Thus, again by modularity, we deduce that

$$(x \vee m) \wedge y = (x \wedge y) \vee m = z \vee m = m,$$

which, together with

$$(x \vee m) \vee y = (x \vee b_n) \vee m = 1 \vee m = 1,$$

proves our claim.

Step 3: We are now going to prove that $K(C) \subseteq D(M)$. Let $k \in K(C)$. Then, by compactness, there exists $n \in \mathbb{N}$ such that

$$k \leqslant (\dot{\bigvee_{1 \leqslant i \leqslant n}} c_i) \vee m \in D(M) \cap (c/m) = D(c/m) = D(C)$$

by Step 2 and Lemma 1.2. But C is a semi-atomic lattice, and so is also any of its subfactors. Therefore $k \in D\big(((\dot{\bigvee}_{1 \leqslant i \leqslant n} c_i) \vee m)/m\big) \subseteq D(M)$ because any semi-atomic lattice is complemented by [11, Theorem 1.8.2]. This shows that $K(C) \subseteq D(M)$.

Step 4: Since the lattice $M = 1/m$ is upper continuous, it is essentially closed, so, there exists $e \in M$ such that $c \in E(e/m)$ is maximal, and then, $e \in C(M)$. By hypothesis, e is essentially compact in M. This means that there exists $f \in E(e/m) \cap K(M)$. We claim that $c \leqslant f$. Indeed, let $a \in \mathcal{A}(C)$, i.e., a is an atom of the semi-atomic lattice $C = c/m$. Since $f \in E(e/m)$ and $m < a \leqslant c \leqslant e$, we have $f \wedge a \neq m$. But $m \leqslant f \wedge a \leqslant a$, so $f \wedge a = a$ because a is an atom of c/m, and then $a \leqslant f$. Consequently,

$$c := \mathrm{Soc}(c/m) = \bigvee_{a \in \mathcal{A}(c/m)} a \leqslant f,$$

which proves our claim. Now, observe that necessarily $c < f$, for otherwise, if $c = f$ then the semi-atomic lattice $C = c/m = f/m$ would be compact, which is a contradiction.

Step 5: By Lemma 1.10, we can express the top element c of the non compact semi-atomic lattice $C = c/m$ as

$$c = \dot{\bigvee_{j\in\mathbb{N}}} s_j,$$

where each s_j is a non compact element of C. As above, for every $j \in \mathbb{N}$ there exists $d_j \in f/m$ such that $s_j \in E(d_j/m)$ and $d_j \in C(f/m)$, so, by hypothesis, $d_j \in E_k(f/m)$. This implies that there exists $f_j \in E(d_j/m) \cap K(f/m)$ for every $j \in \mathbb{N}$. Now observe that $s_j < d_j$, for otherwise, if $s_j = d_j$, then $f_j \in E(s_j/m) \cap K(f/m)$, and s_j/m being a semi-atomic lattice it would follow that $s_j = f_j \in K(f/m)$, which is a contradiction. Therefore, $c < d_j \vee c$ for all $j \in \mathbb{N}$, for otherwise $d_j \leqslant c$, in which case $s_j \in E(d_j/m)$ and $d_j \in c/m$ with c/m a semi-atomic, hence complemented lattice, would imply that $s_j = d_j$, which we just showed that is not possible.

As $(s_j)_{j\in\mathbb{N}}$ is an independent family of elements of f/m, it follows that the family $(d_j)_{j\in\mathbb{N}}$ of elements of f/m is also independent because $s_j \in E(d_j/m)$.

Step 6: Now, we claim that $(d_j \vee c)_{j\in\mathbb{N}}$ is an independent family of elements of f/c. To do that, it is sufficient to show that

$$\big((d_1 \vee c) \vee \ldots \vee (d_n \vee c)\big) \wedge (d_{n+1} \vee c) \leqslant c$$

for every $n \in \mathbb{N}$. Since $s_i \leqslant d_i$ for all $i \in \mathbb{N}$ and $c = \bigvee_{i\in\mathbb{N}} s_i$, we have by modularity

$$\begin{aligned}\big((d_1 \vee c) \vee \ldots \vee (d_n \vee c)\big) \wedge (d_{n+1} \vee c) &= \big((d_1 \vee \ldots \vee d_n) \vee (s_{n+1} \vee s_{n+2} \vee \ldots)\big) \wedge \\ &\quad \wedge \big((s_1 \vee \ldots \vee s_n) \vee (d_{n+1} \vee s_{n+2} \vee s_{n+3} \vee \ldots)\big) \\ &= (d \vee a) \wedge (s \vee b),\end{aligned}$$

where, for simplicity, we have denoted

$$\begin{aligned} d &:= d_1 \vee \ldots \vee d_n \\ s &:= s_1 \vee \ldots \vee s_n \\ a &:= s_{n+1} \vee s_{n+2} \vee \ldots \\ b &:= d_{n+1} \vee s_{n+2} \vee s_{n+3} \vee \ldots \end{aligned}$$

Since $a \leqslant s \vee b$ and $s \leqslant d$ we have by modularity

$$(d\vee a)\wedge(s\vee b) = (d\wedge(s\vee b))\vee a = (s\vee(d\wedge b))\vee a = (s\vee m)\vee a = s_1\vee\ldots\vee s_n\vee s_{n+1}\vee s_{n+2}\vee\ldots = c.$$

Indeed, because $(d_i)_{i\in\mathbb{N}}$ is an independent family of f/m, we have

$$d\wedge b = (d_1\vee\ldots\vee d_n)\wedge(d_{n+1}\vee s_{n+2}\vee s_{n+3}\vee\ldots) \leqslant (d_1\vee\ldots\vee d_n)\wedge(d_{n+1}\vee d_{n+2}\vee d_{n+3}\vee\ldots) = m$$

by [8, Lemma 1.4]. This proves that $(d_j \vee c)_{j\in\mathbb{N}}$ is an independent family of elements of f/c.

Step 7: Set $N := f/c$. Since $f \in K(M)$, it follows that N is a compact lattice. By hypothesis, N is CEK. Let $n \in N$ be such that $\bigvee_{i\in\mathbb{N}}(d_i \vee c) \in E(n/c)$ and n is maximal in the set $\{\, x \in N \mid \bigvee_{i\in\mathbb{N}}(d_i \vee c) \in E(x/c)\}$. Then $n \in C(N)$, hence there exists $g \in E(n/c) \cap K(N)$.

Now, observe that $g_i' := g \wedge (d_i \vee c) > c$ for all $i \in \mathbb{N}$ because $g \in E(n/c)$, and so, we also have $g > c$. If we set $g_i := g_i' \wedge d_i = g \wedge d_i$, then clearly we have $s_i \leqslant g_i \leqslant d_i$ for all $i \in \mathbb{N}$. We claim that $g_i \nleqslant c$. Indeed, if we assume that $g_i \leqslant c$, then we have by modularity

$$g_i' = g \wedge (d_i \vee c) = (g \wedge d_i) \vee c = g_i \vee c \leqslant c,$$

which contradicts $g_i' > c$.

Since $g \in K(N)$, by Lemma 0.1 (2) we can write $g = h \vee c$ for some $h \in K(M)$, $h > m$. Now, fix $i \in \mathbb{N}$. Then $g_i \leqslant g = h \vee c = h \dot{\vee} k$ for some $0 < k \leqslant c$. Indeed, since $C = c/m$ is a semi-atomic lattice, we have $c = (c \wedge h) \dot{\vee} k$ in C, for some $m < k \leqslant c$. So, $(c \wedge h) \wedge k = h \wedge k = m$. It follows that

$$h \vee c = h \vee ((c \wedge h) \vee k) = h \vee k,$$

and $h \wedge k = m$, that is

$$h \vee c = h \dot{\vee} k$$

in $M = 1/m$.

Assume that $g_i \wedge h = m$. Then, by modularity, we have

$$g_i/m = g_i/(g_i \wedge h) \simeq (g_i \vee h)/h \subseteq (h \vee c)/h = (h \dot{\vee} k)/h \simeq k/(h \wedge k) = k/m \subseteq c/m.$$

Thus, g_i/m is a semi-atomic lattice, because $C = c/m$ is so. On the other hand, we know that $s_i \in E(d_i/m)$, and $s_i \leqslant g_i \leqslant d_i$, so $s_i \in E(g_i/m)$. We deduce that $g_i = s_i \leqslant c$, which is a contradiction. Consequently, we must have $g_i \wedge h > m$.

Step 8: As we have seen in Step 7, $s_i \in E(g_i/m)$, and then $m \neq s_i \wedge (g_i \wedge h) = s_i \wedge h$. Since $m < s_i \wedge h \leqslant s_i$ and s_i/m is a semi-atomic lattice, we deduce that, for each $i \in \mathbb{N}$ there exists an atom $t_i \in (s_i \wedge h)/m$.

If we set $t := \bigvee_{i \in \mathbb{N}} t_i = \dot{\bigvee}_{i \in \mathbb{N}} t_i$, then t/m is a semi-atomic lattice. Since h/m is essentially closed, there exists $l \in C(h/m)$ with $t \in E(l/m)$. But h/m is a compact subfactor of L, so, by hypothesis, it is CEK. It follows that there exists $p \in E(l/m) \cap K(h/m)$.

We claim that $t < p$. Indeed, for each $j \in \mathbb{N}$, $p \wedge t_j \leqslant t_j$, so, because $m \prec t_j$ we have either $p \wedge t_j = t_j$, i.e., $t_j \leqslant p$, or $p \wedge t_j = m$, which is not possible because $p \in E(l/m)$. Thus $t_j \leqslant p$, $\forall\, j \in \mathbb{N}$, and then $t \leqslant p$. Now, $t \neq p$ since $p \in K(h/m)$ by its choice and $t \notin K(h/m)$ by Lemma 1.10 (1). This proves that $t < p$. It follows that $c < p \vee c$, for otherwise, we would have $t < p \leqslant c$. Then p/m would be a semi-atomic lattice of finite Goldie dimension, and so would be also t/m, which is a contradiction.

Step 9: We claim that

$$p \wedge (\dot{\bigvee_{i \in \mathbb{N}}} d_i) \leqslant c.$$

To show this observe that, for every $n \geqslant 1$, we have

$$\Big(p \wedge (\dot{\bigvee_{1 \leqslant i \leqslant n}} d_i)\Big) \wedge c = p \wedge \Big((\dot{\bigvee_{1 \leqslant i \leqslant n}} d_i) \wedge c\Big) = p \wedge (\dot{\bigvee_{1 \leqslant i \leqslant n}} s_i)$$

$$= t \wedge \big(\dot{\bigvee_{1\leqslant i\leqslant n}} s_i\big) = \big(\dot{\bigvee_{i\in\mathbb{N}}} t_i\big) \wedge \big(\dot{\bigvee_{1\leqslant i\leqslant n}} s_i\big) = \dot{\bigvee_{1\leqslant i\leqslant n}} t_i.$$

Indeed, let a be an atom of the semi-atomic lattice $\big(p \wedge \big(\dot{\bigvee}_{1\leqslant i\leqslant n} s_i\big)\big)/m$. Then $m \neq a \wedge t$ because $t \in E(p/m)$ by Step 8, and so $a \leqslant t$. It follows that

$$p \wedge \big(\dot{\bigvee_{1\leqslant i\leqslant n}} s_i\big) \leqslant t.$$

Thus

$$p \wedge \big(\dot{\bigvee_{1\leqslant i\leqslant n}} s_i\big) = t \wedge \big(\dot{\bigvee_{1\leqslant i\leqslant n}} s_i\big),$$

which proves that

$$\Big(p \wedge \big(\dot{\bigvee_{1\leqslant i\leqslant n}} d_i\big)\Big) \wedge c = \dot{\bigvee_{1\leqslant i\leqslant n}} t_i.$$

But $\dot{\bigvee}_{1\leqslant i\leqslant n} t_i \in K(C) \subseteq D(M)$ by Step 3, and so, $\dot{\bigvee}_{1\leqslant i\leqslant n} t_i \in D(f/m)$ by Lemma 1.2 (3). Since $c \in E(f/m)$ and $p \wedge \big(\dot{\bigvee}_{1\leqslant i\leqslant n} d_i\big) \in E\big(\big(p \wedge \big(\dot{\bigvee}_{1\leqslant i\leqslant n} d_i\big)\big)/m\big)$, we deduce that

$$\Big(p \wedge \big(\dot{\bigvee_{1\leqslant i\leqslant n}} d_i\big)\Big) \wedge c \in E\big(\big(p \wedge \big(\dot{\bigvee_{1\leqslant i\leqslant n}} d_i\big)\big)/m\big) \cap D\big(\big(p \wedge \big(\dot{\bigvee_{1\leqslant i\leqslant n}} d_i\big)\big)/m\big).$$

This shows that

$$\Big(p \wedge \big(\dot{\bigvee_{1\leqslant i\leqslant n}} d_i\big)\Big) \wedge c = p \wedge \big(\dot{\bigvee_{1\leqslant i\leqslant n}} d_i\big),$$

i.e.,

$$p \wedge \big(\dot{\bigvee_{1\leqslant i\leqslant n}} d_i\big) \leqslant c.$$

Consequently, using the upper continuity of L we deduce that

$$p \wedge \big(\dot{\bigvee_{i\in\mathbb{N}}} d_i\big) = p \wedge \big(\bigvee_{n\in\mathbb{N}} \big(\dot{\bigvee_{1\leqslant i\leqslant n}} d_i\big)\big) = \bigvee_{n\in\mathbb{N}} \Big(p \wedge \big(\dot{\bigvee_{1\leqslant i\leqslant n}} d_i\big)\Big) \leqslant c.$$

Step 10: Since $s_i \leqslant d_i$ for all $i \in \mathbb{N}$, we have $c = \dot{\bigvee}_{i\in\mathbb{N}} s_i \leqslant \dot{\bigvee}_{i\in\mathbb{N}} d_i$, and so, by modularity we obtain

$$(p \vee c) \wedge \big(\dot{\bigvee_{i\in\mathbb{N}}} (d_i \vee c)\big) = (p \vee c) \wedge \big(\dot{\bigvee_{i\in\mathbb{N}}} d_i\big) = \big(p \wedge \big(\dot{\bigvee_{i\in\mathbb{N}}} d_i\big)\big) \vee c = c,$$

so

$$(p \vee c) \wedge \big(\dot{\bigvee_{i\in\mathbb{N}}} (d_i \vee c)\big) = c.$$

But $\dot{\bigvee}_{i\in\mathbb{N}} (d_i \vee c) \in E(n/c)$ and $p \vee c > c$, so $(p \vee c) \wedge \big(\dot{\bigvee}_{i\in\mathbb{N}} (d_i \vee c)\big) > c$, which is a contradiction. The proof of the lemma is now complete. □

Remarks 2.2. (1) The condition that the lattice L is compact is necessary in Lemma 2.1. Indeed, let M be an infinite dimensional vector space over the field F, and let L denote the lattice $\mathcal{L}_F(M)$ of all submodules of ${}_FM$. Then all compact subfactors b/a of L, i.e., all the lattices of all F-submodules of all finitely dimensional factor modules V/W, with $W \leqslant V \leqslant {}_FM$ are CEK by Proposition 1.7, but $D(L)$ is not Noetherian.

(2) We do not know whether Lemma 2.1 holds when replacing the condition that L is compactly generated with the weaker one that L is upper continuous. Note that the compact generation is used only when invoking once Lemma 0.1 (2) in the proof of Lemma 2.1 (see Step 7). Actually one uses the following property of any compactly generated lattice L:

$$(*) \qquad \forall\, a < b \ \ in \ \ L, \forall\, k \in K(b/a), \exists\, c \in K(L) \ \ with \ \ k = c \vee a,$$

and nothing else, so it seems that $(*)$ is actually the only fact from the assumption that L is compactly generated needed in the proof of Lemma 2.1. Note that the property $(*)$ is exactly the condition (P_3) in Section 3, right after Corollary 3.5, for the class of all compact, compactly generated, modular lattices. □

3 The Latticial Osofsky-Smith Theorem

In this section, based on Lemma 2.1, we prove the latticial version of the module-theoretical Osofsky-Smith Theorem. Our contention is that the natural setting for this theorem is *Lattice Theory*, being concerned as it is, with latticial concepts like essentiality, uniformity, complementarity, compactness, direct join in certain lattices.

The technical Lemma 2.1 shows that for compact, compactly generated lattices, modular lattices L having all subfactors CEK, the poset $D(L)$ is Noetherian i.e., the ACC holds for complement elements of L. The next result, asserting that for lattices L possessing ACC or DCC on complement elements, the last element of L is a direct join of finitely many indecomposable elements of L, is a latticial version of a well known result about modules (see, e.g., [3, Proposition 10.14]).

Lemma 3.1. *Let* $\{0\} \neq L \in \mathcal{M}$, *and assume that the subset* $D(L)$ *of* L *is either Noetherian or Artinian. Then* 1 *is a direct join of a finitely many indecomposable elements of* L.

Proof. Deny. Then L is not indecomposable, so we can write

$$1 = x_1 \dot{\vee} y_1$$

with $x_1, y_1 \in D(L) \setminus \{0, 1\}$ such that y_1 cannot be written as a direct join of finitely many indecomposable elements of L. Then, we can write

$$y_1 = x_2 \dot{\vee} y_2$$

with $x_2, y_2 \in D(L) \setminus \{0, 1\}$ such that y_2 cannot be written as a direct join of finitely many indecomposable elements of L, and so on.

Thus, we obtain the following infinite chains of elements of $D(L)$:

$$x_1 < x_1 \dot{\vee} x_2 < \dots$$

and

$$1 > y_1 > y_2 < \dots$$

which is a contradiction. □

Lemma 3.2. *Any modular, upper continuous, compact,* CC *lattice is* CEK.

Proof. We have to show that $C(L) \subseteq E_k(L)$. By Proposition 1.3 (2), this means that $D(L) \subseteq E_k(L)$. So, let $d \in D(L)$. Then $d \in E(d/0) \cap K(L)$ by Lemma 1.8 (1), so L is CEK. □

Remark 3.3. More generally, any modular, upper continuous, CC lattice L having an element $k \in K(L) \cap E(L)$ is CEK.

Indeed, let $c \in C(L)$. Then, as in the proof of Lemma 3.2, $c \in D(L)$, so there exists $c' \in L$ such that $1 = c \dot{\vee} c'$. Then $k \wedge c \in E(c/0)$ because $k \in E(L)$ by assumption. Now, $k \in K(L)$, so $k/(k \wedge c')$ is a compact lattice. By modularity, we have

$$k/(k \wedge c') \simeq (k \vee c')/c' \simeq ((k \vee c') \wedge c)/(c' \wedge c) \simeq ((k \vee c') \wedge c)/0,$$

so $((k \vee c') \wedge c)/0$ is a compact lattice, and then $m := (k \vee c') \wedge c \in K(L)$ because $L \in \mathcal{U}$. On the other hand, $m \in E(c/0)$ because $k \wedge c \in E(c/0)$ and $k \wedge c \leqslant m$. Thus $m \in E(c/0) \cap K(L)$, i.e., $m \in E_k(L)$. Hence $C(L) \subseteq E_k(L)$, which shows that the lattice L is CEK, as claimed. □

Theorem 3.4. (The Latticial Osofsky-Smith Theorem). *Let L be a compact, compactly generated, modular lattice. Assume that all compact subfactors of L are* CC. *Then* 1 *is a finite direct join of uniform elements of L.*

Proof. By assumption, every compact subfactor of L is CC, so CEK by Lemma 3.2. Using now Lemma 2.1, we deduce that $D(L)$ is a Noetherian poset, so, by Lemma 3.1, $1 = \dot{\bigvee}_{1 \leqslant i \leqslant n} d_i$ is a finite direct join of indecomposable elements of d_i of L. Since L is CC, so is also any $d_i/0$ by Proposition 1.4. Finally, every d_i is uniform by Proposition 1.3 (1), and we are done. □

Corollary 3.5. *Let L be a compact, compactly generated, modular lattice such that $c/0$ is a completely* CC *lattice for every $c \in K(L)$. Then* 1 *is a finite direct join of uniform elements of L.*

Proof. This is a reformulation of Theorem 3.4. □

As in [10, Theorem 6.44], we are now going to extend the Latticial Osofsky-Smith Theorem 3.4, valid for any compact, compactly generated, modular lattice having all compact subfactors CC, to more general lattices.

Denote by $\mathcal{K}$ the class of all compact lattices, and let $\mathcal{P}$ be a nonempty subclass of $\mathcal{K} \cap \mathcal{M} \cap \mathcal{U}$ satisfying the following three conditions:

(P_1) If $L \in \mathcal{P}$, $L' \in \mathcal{L}$, and $L \simeq L'$ then $L' \in \mathcal{P}$.

(P_2) If $L \in \mathcal{P}$ then $1/a \in \mathcal{P}, \forall a \in L$.

(P_3) If $L \in \mathcal{P}$ and $b/a \in \mathcal{P}$ is a subfactor of L, then $\exists c \in L$ such that $c/0 \in \mathcal{P}$ and $b = a \vee c$.

We call $\mathcal{P}$-*compact* the members of $\mathcal{P}$. Examples of classes $\mathcal{P}$ satisfying the conditions $(P_1) - (P_3)$ above are:

- any $\varnothing \subseteq \mathcal{P} \subseteq \mathcal{K} \cap \mathcal{M} \cap \mathcal{U}$ such that $L \in \mathcal{P} \Longrightarrow 1/a \in \mathcal{P}$ & $a/0 \in \mathcal{P}, \forall a \in L$;
- the class of all compact, compactly generated, modular lattices (by Lemma 0.1);
- the class of all compact, semi-atomic, upper continuous, modular lattices;
- the class of lattices isomorphic to lattices of all submodules of all cyclic right R-modules.

For any lattice L we denote $\mathcal{P}(L) := \{ c \in L \mid c/0 \in \mathcal{P} \}$, and call $\mathcal{P}$-*compact* the elements of $\mathcal{P}(L)$. Recall that for any lattice L we have denoted by $K(L)$ the set of all compact elements of L. Note that for $c \in L$, we have $c \in K(L) \Longrightarrow c/0 \in \mathcal{K}$, and $c/0 \in \mathcal{K} \Longrightarrow c \in K(L)$ if $L \in \mathcal{U}$. It follows that $\varnothing \neq \mathcal{P}(L) \subseteq K(L)$ for any $L \in \mathcal{U}$.

Alternatively, instead of starting with a subclass $\varnothing \neq \mathcal{P} \subseteq \mathcal{K} \cap \mathcal{M} \cap \mathcal{U}$ and then defining $\mathcal{P}(L)$ for any lattice L, we may start with subsets $\varnothing \neq \mathcal{P}(L) \subseteq K(L)$ for any lattice L, satisfying easily describable versions of conditions $(P_1) - (P_3)$ above.

Let L be a lattice, and let $\varnothing \neq \mathcal{P} \subseteq \mathcal{K} \cap \mathcal{M} \cap \mathcal{U}$ satisfying the conditions $(P_1) - (P_3)$ above. An element $a \in L$ is called *essentially* $\mathcal{P}$ if there exists $e \leqslant a$ such that $e \in E(a/0) \cap \mathcal{P}(L)$. We denote by $E_{\mathcal{P}}(L)$ the set of all essentially $\mathcal{P}$ elements of L. The lattice L is said to be *CE*$\mathcal{P}$ (acronym for *C*losed are *E*ssentially $\mathcal{P}$) if any closed element of L is essentially $\mathcal{P}$, i.e., $C(L) \subseteq E_{\mathcal{P}}(L)$.

Lemma 3.6. *Let $\varnothing \neq \mathcal{P} \subseteq \mathcal{K} \cap \mathcal{M} \cap \mathcal{U}$ satisfying the conditions $(P_1) - (P_3)$ above, and let $L \in \mathcal{P}$. Assume that all subfactors $b/a \in \mathcal{P}$ of L are* CE$\mathcal{P}$, *i.e., every $c \in C(b/a)$ is an essentially $\mathcal{P}$ element of b/a. Then $D(L)$ is Noetherian poset.*

Proof. If we follow thoroughly the proof of Lemma 2.1 and keep the notation there, we observe that

- $c_i \in \mathcal{P}(L)$, because $1 \in \mathcal{P}(L)$ and $c_i \in D(L)$, in *Step 1*,
- $f \in E(e/m) \cap \mathcal{P}(M)$, so $f/m \in \mathcal{P}$, in *Step 4*,
- $f_j \in E(s_j/m) \cap \mathcal{P}(f/m)$, so $f_j/m \in \mathcal{P}$, in *Step 5*,
- $N = f/c \in \mathcal{P}$ because $f/m \in \mathcal{P}$, in *Step 7*,
- $g \in E(n/c) \cap \mathcal{P}(N)$, so $g/c \in \mathcal{P}$, in *Step 7*,
- $h \in \mathcal{P}(L)$ in view of condition (P_3), in *Step 7*,
- $p \in E(l/m) \cap \mathcal{P}(h/m)$, so $p/m \in \mathcal{P}$, in *Step 8*.

Consequently, the proof of Lemma 2.1 can be adapted word by word to the more general hypotheses of Lemma 3.6. □

Theorem 3.7. (The Latticial $\mathcal{P}$-Osofsky-Smith Theorem). *Let* $\varnothing \neq \mathcal{P} \subseteq \mathcal{K} \cap \mathcal{M} \cap \mathcal{U}$ *satisfying the conditions* $(P_1) - (P_3)$ *above, and let* $L \in \mathcal{P}$. *Assume that all subfactors of* L *in* $\mathcal{P}$ *are* CC. *Then* 1 *is a finite direct join of uniform elements of* L.

Proof. The proof is similar to that of Theorem 3.4 using Lemma 3.6 instead of Lemma 2.1. □

Corollary 3.8. *Let* $\varnothing \neq \mathcal{P} \subseteq \mathcal{K} \cap \mathcal{M} \cap \mathcal{U}$ *satisfying the conditions* $(P_1) - (P_3)$ *above, and let* $L \in \mathcal{P}$. *Assume that* $c/0$ *is a completely* CC *lattice for every* $c \in \mathcal{P}(L)$. *Then* 1 *is a finite direct join of uniform elements of* L.

Proof. This is a reformulation of Theorem 3.7. □

Corollary 3.9. *Let* $\varnothing \neq \mathcal{P} \subseteq \mathcal{K} \cap \mathcal{M} \cap \mathcal{U}$ *satisfying the conditions* $(P_1) - (P_3)$ *above. Then, the following statements are equivalent for a compactly generated modular lattice* L.

(1) L *is semi-atomic.*

(2) F *is* CC *and* $K(F) \subseteq D(F)$ *for every subfactor* $F \in \mathcal{P}$ *of* L.

Proof. (1) $\Longrightarrow$ (2): As already mentioned in Section 0, any subfactor F of L is semi-atomic, so complemented. If follows that F is CC and $K(F) \subseteq F = D(F)$.

(2) $\Longrightarrow$ (1): Let $c \in K(L)$ with $C := c/0 \in \mathcal{P}$, in other words, $c \in \mathcal{P}(L)$. Then c is a finite direct join of uniform elements of L by the Latticial $\mathcal{P}$-Osofsky-Smith Theorem (Theorem 3.7) applied to C. Let $d \leqslant c$ with $d \in D(C)$ and d uniform. Then, for every $d' \leqslant d$ with $0 \neq d' \in K(L)$ one has $d' \in D(L)$ by hypothesis, so $d' \in D(d/0)$ by Lemma 1.2 (2). Since d is uniform, we deduce that $d' = d$, so $d \in K(L)$. Let $0 \neq b \leqslant d$, and let $0 \neq b' \leqslant b$ with $b' \in K(L)$. It follows that $b' \in D(d/0)$ and so, $d = b' \leqslant b \leqslant d$. Thus, for any $0 \neq b \leqslant d$, one has $b = d$. Consequently, $d \in \mathcal{A}(L)$, which implies that $C = c/0$ is a semi-atomic lattice. Because L is a compactly generated lattice, 1 is a join of compact elements of L, so 1 is a join of atoms of L, i.e., L is a semi-atomic lattice, as desired. □

Remarks 3.10. (1) It is not clear whether the condition "F is CC" can be removed in Corollary 3.9.

(2) Corollary 3.9 is a latticial version of the following module-theoretical result: "*A right* R*-module* M *is semisimple if and only if every cyclic subfactor of* M *is* M*-injective*" (see [7, Corollary 7.14]), which, in turn is a "modularization" of the well-known *Osofsky's Theorem* [**?**] saying that a ring R is semisimple if and only if every cyclic right R-module is injective. Because we do not have in hand a good latticial substitute for the notion of an injective module, the result above seems to be the best latticial version of the Osofsky's Theorem. □

Theorems 3.4 and 3.7 suggest the following

Definitions 3.11. *A lattice L is said to be an* Osofsky-Smith lattice *if it is a compact, compactly generated, modular lattice such that all compact subfactors of L are* CC. *A module M_R is said to be an* Osofsky-Smith module *if the lattice $\mathcal{L}(M_R)$ of all submodules of M is an Osofsky-Smith lattice.*

For any class $\varnothing \neq \mathcal{P} \subseteq \mathcal{K} \cap \mathcal{M} \cap \mathcal{U}$ satisfying the conditions $(P_1) - (P_3)$ above, a lattice L is said to be a $\mathcal{P}$-Osofsky-Smith lattice *if $L \in \mathcal{P}$ and all subfactors of L in $\mathcal{P}$ are* CC. □

It is clear how can be defined the concept of a *$\mathcal{D}$-Osofsky-Smith module*, where $\mathcal{D}$ is a nonempty class of modules satisfying the conditions $(D_1) - (D_3)$ similar to the conditions $(P_1) - (P_3)$ defined above. By Theorem 3.4 (resp. Theorem 3.7) any Osofsky-Smith lattice (resp. $\mathcal{P}$-Osofsky-Smith lattice) L has the property that 1 is a finite direct join of uniform elements of L, in particular has finite Goldie dimension, but not conversely: indeed, for any prime number p, the lattice L of all submodules of the $\mathbb{Z}$-module $(\mathbb{Z}/p\,\mathbb{Z}) \oplus (\mathbb{Z}/p^3\,\mathbb{Z})$ is not CC, so it is not an Osofsky-Smith lattice, but the greatest element $(\mathbb{Z}/p\,\mathbb{Z}) \oplus (\mathbb{Z}/p^3\,\mathbb{Z})$ of L is the direct join of two uniform elements of L.

Remarks 3.12. (1) Clearly, if $\mathcal{S}$ denotes the class of all compact, semi-atomic, upper continuous, modular lattices, a lattice L is an $\mathcal{S}$-Osofsky-Smith lattice if and only if 1 is a finite direct union of uniform elements of L.

(2) It would be interesting to find other classes $\varnothing \neq \mathcal{P} \subseteq \mathcal{K} \cap \mathcal{M} \cap \mathcal{U}$ of lattices for which there is an identity between $\mathcal{P}$-Osofsky-Smith lattices and lattices for which 1 a finite direct union of uniform elements of L. □

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