INSTITUTUL DE MATEMATICA
“SIMION STOILOW”
AL ACADEMIEI ROMANE

PREPRINT SERIES OF THE INSTITUTE OF MATHEMATICS
OF THE ROMANIAN ACADEMY

ISSN 0250 3638

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- a term equivalent definition of the group coming from algebras of logic -
Part I

by

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Preprint nr.11/2011

BUCURESTI

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Abstract

In Part I, we introduce the implicative-group and the partially-ordered (lattice-ordered) implicative-group as a term equivalent definition of the group and the partially-ordered (lattice-ordered) group, respectively; two intermediary term equivalent notions are also introduced. The lattice-ordered implicative-group is the great piece which missed from the puzzle showing the connections between lattice-ordered groups and some algebras of logic. We establish "horizontal" connections at group level and "vertical" connections between the group level and the algebras of logic level. We discuss about the filters/ideals and the deductive systems of the involved notions.

In Part II, we study the normal filters/ideals and the compatible deductive systems, the representability of some of the involved algebras and we establish other "vertical" connections between the group level and the algebras of logic level. Finally, we introduce and study the implicative-states and the Bosbach-states on l-groups with strong unit.

Keywords group, implicative-group, partially-ordered group, partially-ordered implicative-group, lattice-ordered group, lattice-ordered implicative-group, pseudo-t-norm, pseudo-t-conorm, porim, pseudo-BCK(pP) lattice, pseudo-MV algebra, pseudo-Wajsberg algebra, pseudo-BL algebra, pseudo-Hájek(pP) algebra, left-algebra, right-algebra, residuated lattice

AMS classification (2010): 06F15, 06F35, 06D35

1 Introduction

Pseudo-MV algebras, the non-commutative generalizations of Chang's MV algebras, were introduced in 1999 [13] and developed in [15]. Pseudo-MV algebras are intervals [8] in l-groups and pseudo-Wajsberg algebras are termwise equivalent [3], [4] to pseudo-MV algebras. Hence, pseudo-Wajsberg algebras must be connected to a notion that is termwise equivalent to the l-group. That notion is the great piece which missed from the puzzle showing the connections between algebras of logic and l-groups and we introduce it in this paper: the l-implicative-group.

$$\begin{array}{ccc} ? & \Longleftrightarrow & \mathbf{l-groups} \\ \Updownarrow & & \Updownarrow \\ \mathbf{pseudo-Wajsberg\ algebras} & \Longleftrightarrow & \mathbf{pseudo-MV\ algebras} \end{array}$$

The paper is organized in five sections, as follows.

In Section 2, we recall some notions concerning algebras of logic and some "horizontal" connections at algebras of logic level, needed in the paper. We introduce the notion of *residoid* (Definition 2.4) as the analalogous of *monoid* and we prove that some filters coincide with some deductive systems (Theorem 2.17).

In Section 3, we recall some basic things concerning groups and po-groups (l-groups). We introduce and study two implications, $\rightarrow$ and $\rightsquigarrow$, at groups level. We introduce the intermediary notions of "X-groups" and "X-po-groups" ("X-l-groups") and prove they are termwise equivalent to groups and po-groups (l-groups) respectively.

In Section 4, we introduce and study the notions of implicative-group and po-implicative-group (l-implicative-group). We introduce and study two operations, $-$ and $+$, at implicative-groups level. We introduce the intermediary notions of "X-implicative-groups" and "X-po-implicative-groups" ("X-l-implicative-groups") and prove they are termwise equivalent to implicative-groups and po-implicative-groups (l-implicative-groups) respectively. Mainly, we prove then that groups and po-groups (l-groups) are termwise equivalent to implicative-groups and po-implicative-groups (l-implicative-groups) respectively (Theorems 4.13, 4.23, Corollary 4.30 respectively). Finally, we introduce the notion of *deductive system* of a po-implicative-group and prove that the convex subgroups of a po-group and the deductive systems of the termwise equivalent po-implicative-group coincide (Theorem 4.27).

In Section 5, we present some "vertical" connections between the po-group-level and the algebras of logic level (the negative-cone-level and the positive-cone-level). More precisely:
- in Subsection 5.1, we prove that the po-group operations $+, 0$ restricted to the negative (positive) cone determine a structure of left- (right-) partially-ordered integral monoid (*poim*) (Theorem 5.1), that the convex subgroups of the po-group restricted to the negative (positive) cone are filters/ideals of the left- (right-) poim (Theorem 5.2);
- in Subsection 5.2, we prove that the l-implicative-group operations $\vee, \wedge, \rightarrow, \rightsquigarrow, 0$ and $+$, restricted to the negative (positive) cone, determine a structure of reversed left- (right-) pseudo-BCK lattice with pseudo-product (pseudo-sum, respectively) verifying some properties (Theorem 5.3 - the main result); we prove, consequently, that by bounding in two different ways the above mentioned left- (right-) pseudo-BCK lattice with pseudo-product (pseudo-sum, respectively) verifying some properties, we obtain a left- (right-) pseudo-Wajsberg algebra or a left-pseudo-Hájek(pP) algebra with properties (pP1), (pP2) (right-pseudo-Hájek(pS) algebra with the dual properties (pP1^d), (pP2^d)) (Corollary 5.8); it follows, equivalently, that the l-group operations $\vee, \wedge, +, 0$ and $\rightarrow, \rightsquigarrow$, restricted to the negative (positive) cone, determine a structure of left- (right-) non-commutative residuated lattice verifying some properties (Theorem 5.10) and that by bounding in two different ways the above mentioned left- (right-) non-commutative residuated lattice verifying some properties, we obtain a left- (right-) pseudo-MV algebra or a left- (right-) pseudo-product algebra, as it is well known (Corollary 5.12); the deductive systems of the l-implicative-group restricted to the negative (positive) cone are $(\rightarrow^L, \rightsquigarrow^L)$-deductive systems ($(\rightarrow^R, \rightsquigarrow^R)$-deductive systems) of the reversed left- (right-) pseudo-BCK lattice with pseudo-product (pseudo-sum, respectively) (Theorem 5.9). The analysis of two important examples ends the paper.

The list of the basic properties used in the paper is the following: for all x, y, z,
(T1) $(x \odot y) \odot z = x \odot (y \odot z)$,
(T2) $x \odot 1 = x = 1 \odot x$,
(T3) $x \leq y$ implies $x \odot z \leq y \odot z$ and $z \odot x \leq z \odot y$,
(R1) $y \rightarrow^L z \leq (z \rightarrow^L x) \rightsquigarrow^L (y \rightarrow^L x)$, $y \rightsquigarrow^L z \leq (z \rightsquigarrow^L x) \rightarrow^L (y \rightsquigarrow^L x)$ (L *comes from* "*Left*"),
(R2) $1 \rightarrow^L x = x = 1 \rightsquigarrow^L x$,
(R3) $x \leq y \Longleftrightarrow x \rightarrow^L y = 1 \Longleftrightarrow x \rightsquigarrow^L y = 1$,
(R4) $x \leq y \Longrightarrow z \rightarrow^L x \leq z \rightarrow^L y$, $z \rightsquigarrow^L x \leq z \rightsquigarrow^L y$,
(R1') $y \rightarrow^R z \geq (z \rightarrow^R x) \rightsquigarrow^R (y \rightarrow^R x)$, $y \rightsquigarrow^R z \geq (z \rightsquigarrow^R x) \rightarrow^R (y \rightsquigarrow^R x)$ (R *comes from* "*Right*"),
(R2') $0 \rightarrow^R x = x = 0 \rightsquigarrow^R x$,
(R3') $x \geq y \Longleftrightarrow x \rightarrow^R y = 0 \Longleftrightarrow x \rightsquigarrow^R y = 0$,
(R4') $x \leq y \Longrightarrow z \rightarrow^R x \leq z \rightarrow^R y$, $z \rightsquigarrow^R x \leq z \rightsquigarrow^R y$,
(R1^{re}) $y \rightarrow z = (z \rightarrow x) \rightsquigarrow (y \rightarrow x)$, $\quad y \rightsquigarrow z = (z \rightsquigarrow x) \rightarrow (y \rightsquigarrow x)$,
(R2^{re}) $0 \rightarrow x = x = 0 \rightsquigarrow x$,
(R3^{re}) $x = y \Longleftrightarrow x \rightarrow y = 0 \Longleftrightarrow x \rightsquigarrow y = 0$,
(pP) (pseudo-product) $\exists\, x \odot y \overset{notation}{=} \min\{z \mid x \leq y \rightarrow^L z\} = \min\{z \mid y \leq x \rightsquigarrow^L z\}$,

(pS) (pseudo-sum) $\exists\, x \oplus y \overset{notation}{=} \max\{z \mid x \geq y \to^R z\} = \max\{z \mid y \geq x \rightsquigarrow^R z\}$,

(pR) (pseudo-residuum) $\exists\, y \to^L z \overset{notation}{=} \max\{x \mid x \odot y \leq z\}$, $\exists\, x \rightsquigarrow^L z \overset{notation}{=} \max\{y \mid x \odot y \leq z\}$,

(pcoR) (pseudo-coresiduum) $\exists\, y \to^R z \overset{notation}{=} \min\{x \mid x \oplus y \geq z\}$, $\exists\, x \rightsquigarrow^R z \overset{notation}{=} \min\{y \mid x \oplus y \geq z\}$,

(pPR)=(pRP) $x \odot y \leq z \Longleftrightarrow x \leq y \to^L z \Longleftrightarrow y \leq x \rightsquigarrow^L z$,

(pScoR)=(pcorRS) $x \oplus y \geq z \Longleftrightarrow x \geq y \to^R z \Longleftrightarrow y \geq x \rightsquigarrow^R z$,

(pDN) (pseudo-Double Negation) $(x^{\sim})^{-} = x = (x^{-})^{\sim}$,

(pC) $x \vee y = (x \rightsquigarrow^L y) \to^L y = (x \to^L y) \rightsquigarrow^L y$,

(pC^d) $x \wedge y = (x \to^R y) \rightsquigarrow^R y = (x \rightsquigarrow^R y) \to^R y$,

(*) $(x \odot z) \to^L (y \odot z) = x \to^L y$, $(z \odot x) \rightsquigarrow^L (z \odot y) = x \rightsquigarrow^L y$,

(*^{d}) $(x \oplus z) \to^R (y \oplus z) = x \to^R y$, $(z \oplus x) \rightsquigarrow^R (z \oplus y) = x \rightsquigarrow^R y$,

(pprel) (pseudo-prelinearity) $(x \to^L y) \vee (y \to^L x) = 1 = (x \rightsquigarrow^L y) \vee (y \rightsquigarrow^L x)$,

(pdiv) (pseudo-divisibility) $x \wedge y = (x \to^L y) \odot x = x \odot (x \rightsquigarrow^L y)$,

(pprel^d) $(x \to^R y) \wedge (y \to^R x) = 0 = (x \rightsquigarrow^R y) \wedge (y \rightsquigarrow^R x)$,

(pdiv^d) $x \vee y = (x \to^R y) \oplus x = x \oplus (x \rightsquigarrow^R y)$,

(pP1) $x \wedge x^{-^L} = 0 = x \wedge x^{\sim^L}$,

(pP2) $(z^{-^L})^{-^L} \odot [(x \odot z) \to^L (y \odot z)] \leq x \to^L y$, $(z^{\sim^L})^{\sim^L} \odot [(z \odot x) \rightsquigarrow^L (z \odot y)] \leq x \rightsquigarrow^L y$,

(pP1^d) $x \vee x^{-^R} = 1 = x \vee x^{\sim^R}$,

(pP2^d) $(z^{-^R})^{-^R} \oplus [(x \oplus z) \to^R (y \oplus z)] \geq x \to^R y$, $(z^{\sim^R})^{\sim^R} \oplus [(z \oplus x) \rightsquigarrow^R (z \oplus y)] \geq x \rightsquigarrow^R y$.

(G1) (see (T1) $x + (y + z) = (x + y) + z$,

(G2) (see (T2)) $x + 0 = x = 0 + x$,

(G3) $x + (-x) = 0 = (-x) + x$,

(G4) (see (T3)) $x \leq y$ implies $a + x \leq a + y$ and $x + a \leq y + a$,

(G4') $x \leq y \Longleftrightarrow a + x \leq a + y \Longleftrightarrow x + a \leq y + a$,

(G5) $-(-x) = x$,

(G6) $-0 = 0$,

(G7) $x \leq y \Longleftrightarrow -y \leq -x$,

(G8) $a + (x \vee y) + b = (a + x + b) \vee (a + y + b)$ and dually

(G9) $a + (x \wedge y) + b = (a + x + b) \wedge (a + y + b)$,

(G10) $-(x \vee y) = (-x) \wedge (-y)$ and dually

(G11) $-(x \wedge y) = (-x) \vee (-y)$,

(G12) $x \vee y = x - (x \wedge y) + y = [(x \wedge y) \to x] + y = x + [(x \wedge y) \rightsquigarrow y]$ and dually

(G13) $x \wedge y = x - (x \vee y) + y = [(x \vee y) \to x] + y = x + [(x \vee y) \rightsquigarrow y]$,

(G14) The lattice $(G, \vee, \wedge)$ is distributive.

(#) $x + y = z \Longleftrightarrow x = y \to z \Longleftrightarrow y = x \rightsquigarrow z$,

(α) $(x \to y) + x = y = x + (x \rightsquigarrow y)$,

(β) $x \to (y + x) = y = x \rightsquigarrow (x + y)$,

($\#^{\leq}$) $x + y \leq z \Longleftrightarrow x \leq y \to z \Longleftrightarrow y \leq x \rightsquigarrow z$,

($\#^{\geq}$) $x + y \geq z \Longleftrightarrow x \geq y \to z \Longleftrightarrow y \geq x \rightsquigarrow z$,

(I1) (see (R1^{re})) $y \to z = (z \to x) \rightsquigarrow (y \to x)$, $y \rightsquigarrow z = (z \rightsquigarrow x) \to (y \rightsquigarrow x)$,

(I2) $y = (y \to x) \rightsquigarrow x$, $y = (y \rightsquigarrow x) \to x$,

(I3) (see (R3^{re})) $x = y \Longleftrightarrow x \to y = 0 \Longleftrightarrow x \rightsquigarrow y = 0$,

(I4) $x \to 0 = x \rightsquigarrow 0$,

(I5) $x \leq y$ implies $z \to x \leq z \to y$, $z \rightsquigarrow x \leq z \rightsquigarrow y$,

(I5') $x \leq y \Longleftrightarrow z \to x \leq z \to y \Longleftrightarrow z \rightsquigarrow x \leq z \rightsquigarrow y$,

(I6) $x \leq y$ implies $y \to z \leq x \to z$, $y \rightsquigarrow z \leq x \rightsquigarrow z$,

(I7) (see (R2^{re})) $0 \to x = x = 0 \rightsquigarrow x$,

(I8) $z \rightsquigarrow (y \to x) = y \to (z \rightsquigarrow x)$,

(I9) $x \to x = 0 = x \rightsquigarrow x$,

(I10) $-(-x) = x$,

(I11) $-0 = 0$,

(I12) $x \rightsquigarrow (-y) = y \to (-x)$,

(I13) $(-x) \rightsquigarrow y = (-y) \to x$.
(L) $(x \to y) \to (x \to z) = (y \to x) \to (y \to z)$.

We believe that the new notions and the new connections presented in this paper (Part I) open new perspectives in the study of ordered groups, on the one hand, and of algebras of logic, on the other hand.

2 Preliminaries. "Horizontal" connections at algebra of logic level

In this Section, we recall some of the notions and the results needed in the sequel concerning algebras of logic. We introduce the notion of *residoid* as the analogous of *monoid*. A new result is Theorem 2.17.

We have classified in [23] the existing dual algebras of logic in *left-algebras* (those having a pseudo-t-norm $\odot$ and/or a (reversed) pseudo-residuum ($\to^L$, $\rightsquigarrow^L$) and the greatest element 1 as principal primitive operations) and *right-algebras* (those having a pseudo-t-conorm $\oplus$ and/or a (reversed) pseudo-coresiduum ($\to^R$, $\rightsquigarrow^R$) and the smallest element 0 among others in the signature).

Definition 2.1 [9] (see [16], [29] for t-norms on $[0, 1]$)
(1) A *pseudo-t-norm* on the poset $(A^L, \leq, 1)$ with greatest element 1 is a binary operation $\odot$ verifying (T1), (T2), (T3).
(1') A *pseudo-t-conorm* on the poset $(A^R, \leq, 0)$ with smallest element 0 is a binary operation $\oplus$ verifying (T1), (T2), (T3).

A pseudo-t-norm (pseudo-t-conorm) $\circ$ is *commutative* if it verifies: $x \circ y = y \circ x$, for all x, y. A commutative pseudo-t-norm (pseudo-t-conorm) is called a *t-norm (t-conorm* respectively).

Remark 2.2 In what follows, the sets A^L and A^R may coincide.

Definition 2.3 [20], [21] (Note that in [20], [21] we have considered the axioms (R1) - (R4), but (R4) can be obtained from (R1) - (R3).)
(1) A *pseudo-residuum* on the poset $(A^L, \leq, 1)$ with greatest element 1 is a pair of binary operations $(\to^L, \rightsquigarrow^L)$ verifying (R1), (R2), (R3).
(1')A *pseudo-coresiduum* on the poset $(A^R, \leq, 0)$ with smallest element 0 is a pair of binary operations $(\to^R, \rightsquigarrow^R)$ verifying (R1'), (R2'), (R3').

A reversed pseudo-residuum (pseudo-coresiduum) is obtained by "reversing" the implications. A pseudo-residuum (pseudo-coresiduum) $(\to, \rightsquigarrow)$ is *commutative* if it verifies $\to = \rightsquigarrow$. A commutative pseudo-residuum (pseudo-coresiduum) is called a *residuum (coresiduum* respectively) and it is denoted by $\to$.

A *monoid* is an algebra of type $(2, 0)$, denoted additively by $(A, +, 0)$ or multiplicatively by $(A, \cdot, 1)$, verifying (T1) and (T2) from Definition 2.1. The monoid is *commutative* if its binary operation is commutative.

A *partially-ordered, residuated, integral left-monoid* or a *left-porim* for short is a structure $(A^L, \leq, \odot, 1)$ such that $(A^L, \leq, 1)$ is a poset with greatest element, $\odot$ verifies (T1), (T2), (T3) from Definition 2.1, and property (pR) holds; *integral* means that the greatest element of the poset coincides with the neutral element of the monoid.
Dually, a *partially-ordered, residuated, integral right-monoid* or a *right-porim* for short is a structure $(A^R, \leq, \oplus, 0)$ such that $(A^R, \leq, 0)$ is a poset with smallest element, $\oplus$ verifies (T1), (T2), (T3), and property (pcoR) holds.

We introduce now the new notion of "residoid" as the analogous of "monoid".

Definition 2.4 A *residoid* is an algebra $(A, \to, \rightsquigarrow, 0)$ of type $(2, 2, 0)$ verifying: for all $x, y, z \in A$,
(R1^{re}) $y \to z = (z \to x) \rightsquigarrow (y \to x), \quad y \rightsquigarrow z = (z \rightsquigarrow x) \to (y \rightsquigarrow x)$,
(R2^{re}) $0 \to x = x = 0 \rightsquigarrow x$,
(R3^{re}) $x = y \Longleftrightarrow x \to y = 0 \Longleftrightarrow x \rightsquigarrow y = 0$.

The residoid is *commutative* if $\rightarrow = \rightsquigarrow$.

BCK algebras were introduced in 1966 by K. Iséki [19] (see [28]), as right-algebras. Pseudo-BCK algebras were introduced in 2001 [14], as a non-commutative generalization of BCK algebras, as right-algebras too. Their operations were $\star$ and $\#$, i.e. the "reversed" of what we have usually in non-commutative algebras of logic: $x \star y = y \rightarrow^R x$ and $x \# y = y \rightsquigarrow^R x$. Therefore, we have introduced the reversed right- and left- pseudo-BCK algebras in [22], [21].

Here we recall only "reversed left-pseudo-BCK algebras" and some of their Properties, the dual definitions and Properties (concerning "reversed right-pseudo-BCK algebras") being obtained by replacing A^L by A^R, $\leq$ by $\geq$, 1 by 0, $\rightarrow^L$ by $\rightarrow^R$, $\rightsquigarrow^L$ by $\rightsquigarrow^R$ respectively.

We shall use the following definition of a reversed left-pseudo-BCK algebra:

Definition 2.5 [21] A *reversed left-pseudo-BCK algebra* is a structure $\mathcal{A}^L = (A^L, \leq, \rightarrow^L, \rightsquigarrow^L, 1)$ such that $(A^L, \leq, 1)$ is a poset with greatest element and (R1) - (R3) hold.

A reversed left-pseudo-BCK algebra is *commutative* if $\rightarrow^L = \rightsquigarrow^L$. A commutative reversed left-pseudo-BCK algebra is a reversed left-BCK algebra.

Definition 2.6 A *reversed left-pseudo-BCK algebra with property (pP) (i.e. with pseudo-product)* or a *reversed left-pseudo-BCK(pP) algebra* for short is a reversed left-pseudo-BCK algebra such that property (pP) holds.

We denote by **r-pBCK(pP)** the class of reversed left-pseudo-BCK(pP) algebras and the corresponding category.

Let $\mathcal{A}^L = (A^L, \leq, \rightarrow^L, \rightsquigarrow^L, 1)$ be a reversed left-pseudo-BCK algebra. If the order relation $\leq$ is a lattice order relation ($x \leq y \Leftrightarrow x \wedge y = x \Leftrightarrow x \vee y = y$, for all $x, y \in A^L$), then we say that $\mathcal{A}^L$ is a *reversed left-pseudo-BCK lattice* and it is denoted by $\mathcal{A}^L = (A^L, \wedge, \vee, \rightarrow^L, \rightsquigarrow^L, 1)$. A *reversed left-pseudo-BCK(pP) lattice* is an algebra that is simultaneously a pseudo-BCK(pP) algebra and a pseudo-BCK lattice.

If there is an element 0 of a reversed left-pseudo-BCK algebra $\mathcal{A}^L = (A^L, \leq, \rightarrow^L, \rightsquigarrow^L, 1)$, satisfying $0 \leq x$, for all $x \in A^L$, then 0 is called the *zero* of $\mathcal{A}^L$. A reversed left-pseudo-BCK algebra with zero is called to be *bounded* and it is denoted by: $(A^L, \leq, \rightarrow^L, \rightsquigarrow^L, 0, 1)$.

Let $\mathcal{A}^L = (A^L, \leq, \rightarrow^L, \rightsquigarrow^L, 0, 1)$ be a bounded reversed left-pseudo-BCK algebra. Define, for all $x \in A$, two negations, ${}^- = {}^{-L}$ and ${}^\sim = {}^{\sim L}$, by [22]: for all $x \in A$, $x^- \stackrel{def.}{=} x \rightarrow^L 0$, $\quad x^\sim \stackrel{def.}{=} x \rightsquigarrow^L 0$.

A bounded reversed left-pseudo-BCK algebra is said to be *with property (pDN)* or *involutive*, if it verifies (pDN).

Theorem 2.7 *Let $\mathcal{A}^L = (A^L, \leq, \rightarrow^L, \rightsquigarrow^L, 0, 1)$ be a bounded reversed left-pseudo-BCK algebra with property (pDN) (involutive). Then $\mathcal{A}$ is with property (pP) and we have*

$$x \odot y \stackrel{notation}{=} \min\{z \mid x \leq y \rightarrow^L z\} = \min\{z \mid y \leq x \rightsquigarrow^L z\} = (x \rightarrow^L y^-)^\sim = (y \rightsquigarrow^L x^\sim)^-, \tag{1}$$

$$x \rightarrow^L y = (x \odot y^\sim)^- = y \oplus^L x^-, \quad x \rightsquigarrow^L y = (y^- \odot x)^\sim = x^\sim \oplus^L y. \tag{2}$$

Let $\mathcal{A}^L = (A^L, \leq, \rightarrow^L, \rightsquigarrow^L, 0, 1)$ be a bounded reversed left-pseudo-BCK algebra with property (pDN) (involutive). Then we have also:

$$y^- \rightsquigarrow^L x = x^\sim \rightarrow^L y. \tag{3}$$

Remarks 2.8 [25]

(1) In a bounded reversed left-pseudo-BCK(pP) algebra (with the pseudo-product $\odot$) with property (pDN) (involutive) $\mathcal{A}^L = (A^L, \leq, \rightarrow^L, \rightsquigarrow^L, 0, 1)$, we can define the following additional "right" operations:

$$x \oplus^L y \stackrel{def.}{=} (y^- \odot x^-)^\sim = (y^\sim \odot x^\sim)^-, \tag{4}$$

$$x \Rightarrow^L y \stackrel{def.}{=} (x^- \rightsquigarrow^L y^\sim)^-, \quad x \approx>^L y = (x^\sim \rightarrow^L y^-)^\sim. \tag{5}$$

Then,
- $x^- = x \to^L 0 = x \Rightarrow^L 1$ and $x^\sim = x \rightsquigarrow^L 0 = x \approx>^L 1$;
- the connections between the "right" operations $\oplus^L$, $\Rightarrow^L$, $\approx>^L$ are:

$$x \oplus^L y = (x \Rightarrow^L y^-)^\sim = (y \approx>^L x^\sim)^-, \tag{6}$$

$$x \Rightarrow^L y = (x \oplus^L y^\sim)^- = y \odot x^-, \quad x \approx>^L y = (y^- \oplus^L x)^\sim = x^\sim \odot y; \tag{7}$$

- the "left" operations expressed in terms of "right" operations are:

$$x \odot y = (y^- \oplus^L x^-)^\sim = (y^\sim \oplus^L x^\sim)^-, \tag{8}$$

$$x \to^L y = (x^- \approx>^L y^\sim)^-, \quad x \rightsquigarrow^L y = (x^\sim \Rightarrow^L y^-)^\sim. \tag{9}$$

Consequently, the algebra $\mathcal{A}^{LR} = (A^L, \leq, \Rightarrow^L, \approx>^L, 0, 1)$ is a bounded reversed right-pseudo-BCK(pS) algebra (with the pseudo-sum $\oplus^L$) with property (pDN) (involutive), that is termwise equivalent with $\mathcal{A}^L$. We say that $\mathcal{A}^L$ is *selfdual*.

(1') Dually, in a bounded reversed right-pseudo-BCK(pS) algebra (with the pseudo-sum $\oplus$) with property (pDN) (involutive) (where now $^- = ^{-R}$ and $^\sim = ^{\sim R}$) $\mathcal{A}^R = (A^R, \leq, \to^R, \rightsquigarrow^R, 0, 1)$, we can define the following additional "left" operations:

$$x \odot^R y \stackrel{def.}{=} (y^- \oplus x^-)^\sim = (y^\sim \oplus x^\sim)^-, \tag{10}$$

$$x \Rightarrow^R y \stackrel{def.}{=} (x^- \rightsquigarrow^R y^\sim)^-, \quad x \approx>^R y = (x^\sim \to^R y^-)^\sim. \tag{11}$$

Then,
- $x^- = x \to^R 1 = x \Rightarrow^R 0$ and $x^\sim = x \rightsquigarrow^R 1 = x \approx>^R 0$
and so on.
Consequently, the algebra $\mathcal{A}^{RL} = (A^R, \leq, \Rightarrow^R, \approx>^R, 0, 1)$ is a bounded reversed left-pseudo-BCK(pP) algebra (with the pseudo-product $\odot^R$) with property (pDN) (involutive), that is termwise equivalent with $\mathcal{A}^R$. We say that $\mathcal{A}^R$ is *selfdual*.

Remarks 2.9

(1) The following four left-structures are categorically equivalent [21] ([20] for the commutative case): the reversed left-pseudo-BCK(pP) algebras (the left-potirs), the reversed left-pseudo-BCK(pRP) algebras, the left-X-pseudo-BCK(pPR) algebras and the left-porims. The corresponding definitions and these equivalences are illustrated in Figure 1 (where $A = A^L$, $\to = \to^L$, $\rightsquigarrow = \rightsquigarrow^L$):

r-pBCK(pP)	$\Longleftrightarrow$	**r-pBCK(pPR)**	$\Longleftrightarrow$	**X-pBCK(pPR)**	$\Longleftrightarrow$	**left-porim**
$(A, \leq, \to, \rightsquigarrow, 1)$		$(A, \leq, \to, \rightsquigarrow, \odot, 1)$		$(A, \leq, \odot, \to, \rightsquigarrow, 1)$		$(A, \leq, \odot, 1)$
poset with 1		poset with 1		poset with 1		poset with 1
(R1), (R2), (R3)		(R1), (R2), (R3)		(T1), (T2)		(T1), (T2), (T3)
(pP)		(pPR)		(pPR)		(pR)
(1.1)		(1.2)		(2.2)		(2.1)

Figure 1:

(2) Consequently, in lattice-order case, the following four algebras are categorically equivalent: the reversed left-pseudo-BCK(pP) lattices (the left-l-tirs), the reversed left-pseudo-BCK(pRP) lattices, the left-X-pseudo-BCK(pPR) lattices (= left-non-commutative residuated lattices) and the left-l-rims.

r-pBCK(pP)$_{inv}$	$\Longleftrightarrow$	**r-pBCK(pPR)**$_{inv}$	$\Longleftrightarrow$	**X-pBCK(pPR)**$_{inv}$	$\Longleftrightarrow$	**left-porim**$_{inv}$
$(A, \leq, \rightarrow, \rightsquigarrow, 0, 1)$		$(A, \leq, \rightarrow, \rightsquigarrow, \odot, 0, 1)$		$(A, \leq, \odot, \rightarrow, \rightsquigarrow, 0, 1)$		$(A, \leq, \odot, 0, 1)$
poset with $0, 1$		poset with $0, 1$		poset with $0, 1$		poset with $0, 1$
(R1), (R2), (R3)		(R1), (R2), (R3)		(T1), (T2)		(T1),(T2),(T3)
$x^- = x \rightarrow 0$,		$x^- = x \rightarrow 0$,		$x^- = x \rightarrow 0$,		$x^- = \max\{y \mid y \odot x = 0\}$,
$x^\sim = x \rightsquigarrow 0$,		$x^\sim = x \rightsquigarrow 0$,		$x^\sim = x \rightsquigarrow 0$,		$x^\sim = \max\{y \mid x \odot y = 0\}$
(pDN)		(pDN)		(pDN)		(pDN)
(pP) $\exists\, x \odot y = (x \rightarrow y^-)^\sim = (y \rightsquigarrow x^\sim)^-$		(pPR)		(pPR)		(pR) $\exists\, x \rightarrow y = (x \odot y^\sim)^-$ $\exists\, x \rightsquigarrow y = (y^- \odot x)^\sim$
(1.1)		(1.2)		(2.2)		(2.1)

Figure 2:

(3) By adding property (pDN) to algebras from (1), we obtain the definitions and the termwise-equivalences from Figure 2 (where $A = A^L$, $\rightarrow = \rightarrow^L$, $\rightsquigarrow = \rightsquigarrow^L$):

Hence, for examples, the left-pseudo-Wajsberg algebras $(A^L, \rightarrow^L, \rightsquigarrow^L, {}^-, {}^\sim, 1)$ (they are in the column (1.1)) are termwise equivalent to the left-pseudo-MV algebras $(A^L, \odot, \oplus^L, {}^{-^L}, {}^{\sim^L}, 0, 1)$ (they are in the column (2.1)), which by Remarks 2.8 (1) are termwise equivalent to the right-pseudo-MV algebras $(A^L, \oplus^L, \odot, {}^{-^L}, {}^{\sim^L}, 0, 1)$.

Remark 2.10 Dually, for examples, the following two right-structures from Figure 3 are termwise equivalent:

r-pBCK(pS)$_{inv}$	$\Longleftrightarrow$	...	$\Longleftrightarrow$	...	$\Longleftrightarrow$	**right-porim**$_{inv}$
$(A^R, \leq, \rightarrow^R, \rightsquigarrow^R, 0, 1)$						$(A^R, \leq, \oplus, 0, 1)$
poset with $0, 1$						poset with $0, 1$
(R1'), (R2'), (R3')						(T1), (T2), (T3)
$x^- = x \rightarrow^R 1$,						$x^- = \min\{y \mid y \oplus x = 0\}$,
$x^\sim = x \rightsquigarrow^R 1$,						$x^\sim = \min\{y \mid x \oplus y = 0\}$
(pDN)						(pDN)
(pS) $\exists\, x \oplus y = (x \rightarrow^R y^-)^\sim = (y \rightsquigarrow^R x^\sim)^-$						**(pcoR)** $\exists\, x \rightarrow^R y = (x \oplus y^\sim)^-$ $\exists\, x \rightsquigarrow^R y = (y^- \oplus x)^\sim$
(1.1)						(2.1)

Figure 3:

Hence, for examples, the right-pseudo-Wajsberg algebras $(A^R, \rightarrow^R, \rightsquigarrow^R, {}^{-^R}, {}^{\sim^R}, 0)$ (they are in the column (1.1)) are termwise equivalent to the right-pseudo-MV algebras $(A^R, \oplus, \odot^R, {}^{-^R}, {}^{\sim^R}, 0, 1)$ (they are in the column (2.1)), which by Remarks 2.8 (1) are termwise equivalent to the left-pseudo-MV algebras $(A^R, \odot^R, \oplus, {}^{-^R}, {}^{\sim^R}, 0, 1)$.

Definition 2.11 We say that a reversed left-pseudo-BCK lattice $\mathcal{A}^L = (A^L, \wedge, \vee, \to^L, \leadsto^L, 1)$ is *with property (pC)* if property (pC) holds.

Theorem 2.12 *([23], Theorem 4.4) Let $\mathcal{A}^L = (A^L, \wedge, \vee, \to^L, \leadsto^L, 1)$ be a reversed left-pseudo-BCK(pP) lattice. Then,*

$$(pC) \implies (pprel) + (pdiv).$$

Corollary 2.13 *Let $\mathcal{A}^L = (A^L, \wedge, \vee, \to^L, \leadsto^L, 0, 1)$ be a bounded reversed left-pseudo-BCK(pP) lattice with (pC) property. Then $\mathcal{A}^L$ is with (pDN) property.*

Recall that:
- a *(left-) pseudo-BL algebra* is a bounded left-non-commutative residuated lattice verifying (pprel) and (pdiv);
- a *left-pseudo-Hájek(pP) algebra* is a bounded reversed left-pseudo-BCK(pP) lattice verifying (pprel) and (pdiv);
- the left-pseudo-BL algebras and the left-pseudo-Hájek(pP) algebras are categorically equivalent.
- A *(left-) pseudo-product algebra* is a (left-) pseudo-BL algebra verifying (pP1) and (pP2).
- The (left-) pseudo-product algebras and the left-pseudo-Hájek(pP) algebras verifying (pP1) and (pP2) are categorically equivalent.
- *Left-pseudo-MV algebras* are exactly (left-) pseudo-BL algebras verifying property (pDN);
- *(left-) pseudo-Wajsberg algebras* are exactly left-pseudo-Hájek(pP) algebras verifying property (pDN).

Theorem 2.14 *The bounded reversed left-pseudo-BCK(pP) lattice with (pC) property is an equivalent definition of the left-pseudo-Wajsberg algebra.*

Finally, concerning filters and ideals, we recall the following dual definitions.

Definition 2.15
(1) Let $\mathcal{A}^L = (A^L, \leq, \odot, 1)$ be a partially-ordered integral left-monoid (left-poim). A *($\odot$)-filter* or simply a *filter*, when there is no danger of confusion, of $\mathcal{A}^L$ is a subset $F \subseteq A^L$ which satisfies: (f1) $1 \in F$, (pf2) $x, y \in F$ imply $x \odot y \in F$, (f3) $x \in F$ and $x \leq y$ imply $y \in F$.
(1') Let $\mathcal{A}^R = (A^R, \leq, \oplus, 0)$ be a partially-ordered integral right-monoid (right-poim). An *($\oplus$)-ideal* or simply an *ideal*, when there is no danger of confusion, of $\mathcal{A}^R$ is a subset $I \subseteq A^R$ which satisfies: (i1) $0 \in I$, (pi2) $x, y \in I$ imply $x \oplus y \in I$, (i3) $x \in I$ and $x \geq y$ imply $y \in I$.

Concerning the deductive systems, we recall the following dual definitions.

Definition 2.16
(1) Let $\mathcal{A}^L = (A^L, \leq, \to^L, \leadsto^L, 1)$ be a reversed left-pseudo-BCK algebra. A *($\to^L, \leadsto^L$)-deductive system* or simply a *deductive system*, when there is no danger of confusion, of $\mathcal{A}^L$ is a subset $S \subseteq A^L$ which satisfies: (ds1) $1 \in S$, (pds2) $x \in S$ and $x \to^L y \in S$ imply $y \in S$ (or $x \in S$ and $x \leadsto^L y \in S$ imply $y \in S$).
(1') Let $\mathcal{A}^R = (A^R, \leq, \to^R, \leadsto^R, 0)$ be a reversed right-pseudo-BCK algebra. A *($\to^R, \leadsto^R$)-deductive system* or simply a *deductive system*, when there is no danger of confusion, of $\mathcal{A}^R$ is a subset $S' \subseteq A^R$ which satisfies: (ds1') $0 \in S'$, (pds2') $x \in S'$ and $x \to^R y \in S'$ imply $y \in S'$ (or $x \in S'$ and $x \leadsto^R y \in S'$ imply $y \in S'$).

Now, following the ideas from [2], we prove the following result:

Theorem 2.17 *Let $\mathcal{A}^L_t = (A^L, \leq, \to^L, \leadsto^L, 1)$ be a reversed left-pseudo-BCK(pP) algebra, where:*

$$x \odot y \overset{notation}{=} \min\{z \mid x \leq y \to^L z\} = \min\{z \mid y \leq x \leadsto^L z\},$$

and let $\mathcal{A}^L_r = (A^L, \leq, \odot, 1)$ be the categorically equivalent left-porim, where:

$$y \to^L z \overset{notation}{=} \max\{x \mid x \odot y \leq z\}, \quad x \leadsto^L z \overset{notation}{=} \max\{y \mid x \odot y \leq z\}.$$

Then, the deductive systems (i.e. ($\to^L, \leadsto^L$)-deductive systems) of $\mathcal{A}^L_t$ coincide with the ($\odot$)-filters of $\mathcal{A}^L_r$.

Proof. If $\mathcal{A}_t^L = (A^L, \leq, \to^L, \rightsquigarrow^L, 1)$ is a reversed left-pseudo-BCK(pP) algebra, then $(A^L, \leq, \to^L, \rightsquigarrow^L, \odot, 1)$ is a reversed left-pseudo-BCK(pRP) algebra [21], i.e. property (pRP) holds and if $\mathcal{A}_r^L = (A^L, \leq, \odot, 1)$ is a left-porim, then $(A^L, \leq, \odot, \to^L, \rightsquigarrow^L, 1)$ is a left-X-pseudo-BCK(pPR) algebra [21], i.e. property (pPR) holds.

Let S be a deductive system of $\mathcal{A}_t^L$, i.e. (ds1) and (pds2) hold; we must prove that S is an $(\odot)$-filter of $\mathcal{A}_r^L$, i.e. (f1), (pf2), (f3) hold. Indeed, (f1) is (ds1). To prove (pf2), let $x, y \in S$; then $y \to^L (x \to^L (y \odot x)) = 1$; by (pds2), it follows that $x \to^L (y \odot x) \in S$, hence by (pds2) again, it follows that $y \odot x \in S$; thus, (pf2) holds. To prove (f3), let $x \in S$, $x \leq y$; then $x \to^L y = 1$, hence $x \to^L y \in S$; it folows by (pds2) that $y \in S$, i.e. (f3) holds.

Conversely, let F be an $(\odot)$-filter of $\mathcal{A}_r^L$, i.e. (f1), (pf2), (f3) hold; we must prove that F is a deductive system of $\mathcal{A}_t^L$, i.e. (ds1) and (pds2) hold. Indeed, (ds1) is (f1). To prove (pds2), let $x \in F$, $x \to^L y \in F$; then by (pf2), $(x \to^L y) \odot x \in F$; but we have $(x \to^L y) \odot x \leq y$; hence, by (f3), $y \in F$; (or let $x \in F$, $x \rightsquigarrow^L y \in F$; then by (pf2), $x \odot (x \rightsquigarrow^L y) \in F$; but, we have $x \odot (x \rightsquigarrow^L y) \leq y$; hence, by (f3), $y \in F$;) thus, (pds2) holds. □

Note that, dually, the $(\to^R, \rightsquigarrow^R)$-deductive systems of a reversed right-pseudo-BCK(pS) algebra coincide with the ideals of the categorically equivalent right-porim.

Note also that Theorem 2.17 and its dual are induced by Theorems 4.27, 5.2, 5.9.

3 Groups, po-groups, l-groups. Some "horizontal" connections at group level

The section has three subsections.

3.1 Groups

We consider the group as an algebra $\mathcal{G} = (G, +, -, 0)$ of type $(2, 1, 0)$ verifying (G1), (G2), (G3).

The group is said to be *commutative or abelian* if $x + y = y + x$, for all $x, y \in G$.

Proposition 3.1 *Let $\mathcal{G}$ be a group. Then properties (G5), (G6) hold.*

If $(G, +, -, 0)$ is a group, then $(G, +, 0)$ is a monoid and for all $x, y \in G$, $x + y = 0 \Longleftrightarrow x = -y \Longleftrightarrow y = -x$ and $x + y = -[-y + (-x)]$.

3.1.1 New operations in groups: $\to$ and $\rightsquigarrow$. Their Properties

Since the group is, by (G5), an involutive structure and since in the involutive algebras of logic we have (2), we introduce the new operations $\to$ and $\rightsquigarrow$ on G, called "implications", defined by: for all $x, y \in G$,

$$x \to y \stackrel{def.}{=} -[x + (-y)] = y + (-x), \quad x \rightsquigarrow y \stackrel{def.}{=} -[(-y) + x] = (-x) + y, \tag{12}$$

since $x \to y = -(x - y) = y - x$ and $x \rightsquigarrow y = -(-y + x) = -x + y$.

Note that:

(i) in multiplicative notation, the group is $(G, \cdot, ^{-1}, 1)$ and hence (12) becomes:
$x \to y \stackrel{def.}{=} (x \cdot y^{-1})^{-1}$, $\quad x \rightsquigarrow y \stackrel{def.}{=} (y^{-1} \cdot x)^{-1}$;

(ii) in ([12], pag. 160), the implication $\rightsquigarrow$ is denoted by $\backslash$ ($x \backslash y = x \rightsquigarrow y$) and the implication $\to$ is replaced by its inverse, denoted by / (i.e. $x/y = y \to x$);

(iii) if the group is commutative, then the two implications coincide: $\to = \rightsquigarrow$.

Remark 3.2 (See Remarks 2.8)

Note that, for all $x, y \in G$, we have:

$$x \cdot y \stackrel{def.}{=} -((-y) + (-x)) = x + y,$$

$$x \to^R y \overset{def.}{=} -((-x) \rightsquigarrow (-y)) = -(x + (-y)) = y - x = x \to y,$$

$$x \rightsquigarrow^R y \overset{def.}{=} -((-x) \to (-y)) = -((-y) - (-x)) = -(-y + x) = -x + y = x \rightsquigarrow y,$$

i.e. the addition $+$ is selfdual, the dual of $\rightsquigarrow$ is $\to$ and the dual of $\to$ is $\rightsquigarrow$ (the implication $\to$ can be expressed in terms of $\rightsquigarrow$ and viceversa). Consequently, one can better understand the results from papers [11] and [18] concerning algebras $(G, \circ)$ of type (2), with two (one respectively) equations, that are termwise equivalent to groups.

Remark 3.3 Note that in an involutive reversed left-BCK(P) algebra (hence commutative), (12) does not hold, i.e. $(x \odot y^-)^- \neq y \odot x^-$. Indeed, take as example the following particular case (see [26], pag. 163): the Boolean algebra $L_{2\times 2} = \{0, a, b, 1\}$, with $0 < a, b < 1$, organized as a reversed left-BCK(P) algebra with the operation $\to$ and $\odot$ as in the following tables ($x^- = x \to 0$):

$\mathcal{L}_{2\times 2}$

$\to$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	a	b	1

$\odot$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

Then, for $x = a$ and $y = 1$, we obtain $1 = 0^- = (a \odot 0)^- = (a \odot 1^-)^- \neq 1 \odot a^- = 1 \odot b = b$.

Let $(G, +, -, 0)$ be a group. Then, we have the special property: for all $x, y, z \in G$,

$$x + y = -(x \to (-y)) = (-y) \to x, \quad x + y = -(y \rightsquigarrow (-x)) = (-x) \rightsquigarrow y. \tag{13}$$

Indeed, $-(x \to (-y)) = -(-y - x) = x + y$ and $(-y) \to x = x - (-y) = x + y$; $-(y \rightsquigarrow (-x)) = -(-y - x) = x + y$ and $(-x) \rightsquigarrow y = -(-x) + y = x + y$.

Remark 3.4 (See Remark 3.3) Recall that in an involutive reversed left-pseudo-BCK algebra(pP), the properties (1) and (3) hold, but the analogous of (13) does not hold. Indeed, take the commutative case, i.e. an involutive reversed left-BCK(P) algebra, and here take the Boolean algebra $\mathcal{L}_{2\times 2}$ from the previous Remark 3.3. We prove that $(x \to (y^-))^- \neq y^- \to x$. Indeed, for $x = a$ and $y = b$ we obtain $0 = 1^- = (a \to a)^- = (a \to (b^-))^- \neq b^- \to a = a \to a = 1$.

Proposition 3.5 *Let $(G, +, -, 0)$ be a group. Then, for all $x, y, z \in G$, we have:*

$$y \to z = (z \to x) \rightsquigarrow (y \to x), \quad y \rightsquigarrow z = (z \rightsquigarrow x) \to (y \rightsquigarrow x), \tag{14}$$

$$(y \to x) \rightsquigarrow x = y = (y \rightsquigarrow x) \to x, \tag{15}$$

$$-x = x \to 0 = x \rightsquigarrow 0, \tag{16}$$

$$x = y \Longleftrightarrow x \to y = 0 \Longleftrightarrow x \rightsquigarrow y = 0. \tag{17}$$

Proof.

(14): $(z \to x) \rightsquigarrow (y \to x) = (x - z) \rightsquigarrow (x - y) =$
$-(x - z) + (x - y) = (z - x) + (x - y) \overset{(G1),(G3)}{=} z + 0 - y \overset{(G1),(G2)}{=} z - y = y \to z$ and
$(z \rightsquigarrow x) \to (y \rightsquigarrow x) = (-z + x) \to (-y + x) =$
$(-y + x) - (-z + x) = (-y + x) + (-x + z) \overset{(G1),(G3)}{=} -y + 0 + z \overset{(G1),(G2)}{=} -y + z = y \rightsquigarrow z$.

(15): $(y \to x) \rightsquigarrow x = (x - y) \rightsquigarrow x = -(x - y) + x = (y - x) + x = y$ and $(y \rightsquigarrow x) \to x = (-y + x) \to x = x - (-y + x) = x + (-x + y) = y$.

(16): $-x = 0 - x = -x + 0$, by (G2).

(17): obviously. □

Proposition 3.6 *In a group* $(G, +, -, 0)$*, the following Properties hold: for all* $x, y, z \in G$*,*

$$x + y = z \Longleftrightarrow x = y \rightarrow z \Longleftrightarrow y = x \rightsquigarrow z \quad (\textit{see } [12], \textit{ page } 160), \tag{18}$$

$$x = y \Longleftrightarrow -y = -x, \tag{19}$$

$$0 \rightarrow x = x = 0 \rightsquigarrow x, \tag{20}$$

$$z \rightsquigarrow (y \rightarrow x) = y \rightarrow (z \rightsquigarrow x), \tag{21}$$

$$z \rightarrow x = (y \rightarrow z) \rightarrow (y \rightarrow x), \quad z \rightsquigarrow x = (y \rightsquigarrow z) \rightsquigarrow (y \rightsquigarrow x), \tag{22}$$

$$x \rightarrow x = 0 = x \rightsquigarrow x, \tag{23}$$

$$x \rightsquigarrow (-y) = y \rightarrow (-x), \tag{24}$$

$$-(x \rightarrow 0) = x = -(x \rightsquigarrow 0), \tag{25}$$

$$[(y \rightarrow x) \rightsquigarrow x] \rightarrow x = y \rightarrow x, \quad [(y \rightsquigarrow x) \rightarrow x] \rightsquigarrow x = y \rightsquigarrow x, \tag{26}$$

$$x \rightarrow (y \rightarrow z) = (x + y) \rightarrow z, \quad x \rightsquigarrow (y \rightsquigarrow z) = (y + x) \rightsquigarrow z, \tag{27}$$

$$x \rightsquigarrow y = (-y) \rightarrow (-x), \quad x \rightarrow y = (-y) \rightsquigarrow (-x), \tag{28}$$

$$(-x) \rightsquigarrow y = (-y) \rightarrow x, \tag{29}$$

$$(x \rightarrow y) + x = y = x + (x \rightsquigarrow y), \tag{30}$$

$$x \rightarrow (y + x) = y = x \rightsquigarrow (x + y), \tag{31}$$

$$x \rightarrow y = (x + z) \rightarrow (y + z), \quad x \rightsquigarrow y = (z + x) \rightsquigarrow (z + y), \tag{32}$$

$$(y + x) \rightarrow x = -y = (x + y) \rightsquigarrow x, \tag{33}$$

$$y \rightarrow (x \rightarrow (y + x)) = 0 = y \rightsquigarrow (x \rightsquigarrow (x + y). \tag{34}$$

Proof.

(18): $x + y = z$ implies $x = z - y = y \rightarrow z$ and $y = -x + z = x \rightsquigarrow z$, by (G3); conversely, $x = y \rightarrow z$, i.e. $x = z - y$, implies $x + y = z$ and, similarly, $y = x \rightsquigarrow z$ implies $x + y = z$ too, by (G3).

(19): (i) If $x = y$, then, obviously, $-y = -x$; (ii) if $-y = -x$, then, by (i), $-(-x) = -(-y)$, i.e. $x = y$, by (G5).

(20): $0 \rightarrow x = x - 0 \overset{(G6)}{=} x + 0 \overset{(G2)}{=} x$ and $0 \rightsquigarrow x = -0 + x \overset{(G6)}{=} 0 + x \overset{(G2)}{=} x$.

(21): $z \rightsquigarrow (y \rightarrow x) = z \rightsquigarrow (x - y) = -z + (x - y)$; $\quad y \rightarrow (z \rightsquigarrow x) = y \rightarrow (-z + x) = (-z + x) - y \overset{(G1)}{=} -z + (x - y)$; thus (21) holds.

(22): $(y \rightarrow z) \rightarrow (y \rightarrow x) = (x - y) - (z - y) = (x - y) + (y - z) = x - z = z \rightarrow x$ and $(y \rightsquigarrow z) \rightsquigarrow (y \rightsquigarrow x) = -(-y + z) + (-y + x) = (-z + y) + (-y + x) = -z + x = z \rightsquigarrow x$, by (G1).

(23): $x \rightarrow x = x - x \overset{(G3)}{=} 0$ and $x \rightsquigarrow x = -x + x \overset{(G3)}{=} 0$.

(24): $x \rightsquigarrow (-y) = (-x) + (-y) = -x - y = y \rightarrow (-x)$.

(25): $-(x \rightarrow 0) = -(0 - x) = x - 0 \overset{(G6)}{=} x + 0 \overset{(G2)}{=} x$ and $-(x \rightsquigarrow 0) = -(-x + 0) \overset{(G2)}{=} -(-x) \overset{(G5)}{=} x$.

(26): $[(y \rightarrow x) \rightsquigarrow x] \rightarrow x = [(x - y) \rightsquigarrow x] \rightarrow x = [-(x - y) + x] \rightarrow x = [(y - x) + x] \rightarrow x = [y + (-x + x)] \rightarrow x = [y + 0] \rightarrow x = y \rightarrow x$ and
$[(y \rightsquigarrow x) \rightarrow x] \rightsquigarrow x = [(-y + x) \rightarrow x] \rightsquigarrow x = [x - (-y + x)] \rightsquigarrow x = [x + (-x + y)] \rightsquigarrow x = [(x + (-x)) + y] \rightsquigarrow x = [0 + y] \rightsquigarrow x = y \rightsquigarrow x$.

(27): $x \rightarrow (y \rightarrow z) = x \rightarrow (z - y) = (z - y) - x = z + ((-y) + (-x)) = z - (x + y) = (x + y) \rightarrow z$ and $x \rightsquigarrow (y \rightsquigarrow z) = x \rightsquigarrow (-y + z) = -x + (-y + z) = [(-x) + (-y)] + z = -(y + x) + z = (y + x) \rightsquigarrow z$.

(28): $(-y) \rightarrow (-x) = (-x) - (-y) = -x + y = x \rightsquigarrow y$ and
$(-y) \rightsquigarrow (-x) = -(-y) + (-x) = y + (-x) = x \rightarrow y$.

(29): $(-x) \rightsquigarrow y = -(-x) + y = x + y = x - (-y) = (-y) \rightarrow x$; or by (24) and (G5).

(30): $(x \rightarrow y) + x = (y - x) + x = y$; $x + (x \rightsquigarrow y) = x + (-x + y) = y$.

(31): $x \rightarrow (y + x) = (y + x) - x = y$ and $x \rightsquigarrow (x + y) = -x + (x + y) = y$.

(32): $(x + z) \rightarrow (y + z) = (y + z) - (x + z) = (y + z) + (-z - x) = y - x = x \rightarrow y$, $(z + x) \rightsquigarrow (z + y) = -(z + x) + (z + y) = (-x - z) + (z + y) = -x + y = x \rightsquigarrow y$.

(33): $(y + x) \rightarrow x = x - (y + x) = x + (-x - y) = -y$ and $(x + y) \rightsquigarrow x = -(x + y) + x = (-y - x) + x = -y$.

(34): $y \rightarrow (x \rightarrow (y + x)) = y \rightarrow y = 0$ and $y \rightsquigarrow (x \rightsquigarrow (x + y)) = y \rightsquigarrow y = 0$, by (31) and (23). □

Proposition 3.7 *In a group $(G,+,-,0)$, the following Properties hold: for all $x, y, z, x_1, x_2, \ldots, x_n \in G$,*
(a) $(x \rightsquigarrow y) + (y \rightsquigarrow z) = x \rightsquigarrow z$,
(a') $(y \to z) + (x \to y) = x \to z$,
(b) $(x_1 \rightsquigarrow x_2) + (x_2 \rightsquigarrow x_3) + \ldots + (x_{n-1} \rightsquigarrow x_n) = x_1 \rightsquigarrow x_n$,
(b') $(x_{n-1} \to x_n) + \ldots + (x_2 \to x_3) + (x_1 \to x_2) = x_1 \to x_n$.

Proof.
(a) $(x \rightsquigarrow y) + (y \rightsquigarrow z) = (-x+y) + (-y+z) = -x + (y-y) + z = -x + 0 + z = -x + z = x \rightsquigarrow z$.
(a'), (b), (b') have similar proofs. □

3.1.2 X-groups: an intermediary notion

Definition 3.8 We shall name *X-group* an algebra $(G,+,\to,\rightsquigarrow,0)$ of type $(2,2,2,0)$ such that $(G,+,0)$ is a monoid (i.e. (G1) and (G2) hold) and the following property holds: for all $x, y, z \in G$,
(#) $x + y = z \Longleftrightarrow x = y \to z \Longleftrightarrow y = x \rightsquigarrow z$, (it is (18)).

First, we present the following general lemma:

Lemma 3.9 *Let $\mathcal{A} = (A,+,\to,\rightsquigarrow)$ (or $\mathcal{A} = (A,\to,\rightsquigarrow,+)$).*
Then

$$(\#) \iff (\alpha) + (\beta),$$

where: for all $x, y \in A$,

$$(\alpha) \qquad (x \to y) + x = y = x + (x \rightsquigarrow y) \quad (\textit{it is } (30)),$$

$$(\beta) \qquad x \to (y+x) = y = x \rightsquigarrow (x+y) \quad (\textit{it is } (31)).$$

Proof. $(\#) \Longrightarrow (\alpha) + (\beta)$:
(α): $(x \to y) + x = y \overset{(\#)}{\Longleftrightarrow} x \to y = x \to y$ and $x + (x \rightsquigarrow y) = y \overset{(\#)}{\Longleftrightarrow} x \rightsquigarrow y = x \rightsquigarrow y$, which are true.
(β): $y = x \to (y+x) \overset{(\#)}{\Longleftrightarrow} y + x = y + x$ and $y = x \rightsquigarrow (x+y) \overset{(\#)}{\Longleftrightarrow} x + y = x + y$, which are true.
$(\alpha) + (\beta) \Longrightarrow (\#)$:
Suppose $x + y = z$; then $y \to z = y \to (x+y) \overset{(\beta)}{=} x$ and $x \rightsquigarrow z = x \rightsquigarrow (x+y) \overset{(\beta)}{=} y$.
Conversely, suppose $x = y \to z$; then $x + y = (y \to z) + y \overset{(\alpha)}{=} z$; suppose $y = x \rightsquigarrow z$; then $x + y = x + (x \rightsquigarrow z) \overset{(\alpha)}{=} z$. □

Proposition 3.10 *Let $(G,+,\to,\rightsquigarrow,0)$ be an X-group. Then, for all $x, y \in G$,*

$$x \to 0 = x \rightsquigarrow 0. \tag{35}$$

Proof. $x \to 0 \overset{(G2)}{=} (x \to 0) + 0 \overset{(\alpha)}{=} (x \to 0) + [x + (x \rightsquigarrow 0)] \overset{(G1)}{=} [(x \to 0) + x] + (x \rightsquigarrow 0) \overset{(\alpha)}{=} 0 + (x \rightsquigarrow 0) \overset{(G2)}{=} x \rightsquigarrow 0$. □

Now we prove that the groups are termwise equivalent to the X-groups (result which can be found in ([12], page 160), in a different form; we prefer the following form):

Theorem 3.11
(1) Let $\mathcal{G} = (G,+,-,0)$ be a group. Define $\rho(\mathcal{G}) \overset{def.}{=} (G,+,\to,\rightsquigarrow,0)$, where $x \to y = y - x$ and $x \rightsquigarrow y = -x + y$. Then, $\rho(\mathcal{G})$ is an X-group.
(1') Conversely, let $\mathcal{G} = (G,+,\to,\rightsquigarrow,0)$ be an X-group. Define $\rho^*(\mathcal{G}) \overset{def.}{=} (G,+,-,0)$, where $-x = x \to 0 \overset{(35)}{=} x \rightsquigarrow 0$. Then, $\rho^*(\mathcal{G})$ is a group.
(2) The above defined mappings ρ and ρ^* are mutually inverse.

Proof.

(1): follows by Proposition 3.6, property (18).

(1'): Since $(G,+,0)$ is monoid, it follows that (G1), (G2) hold. It remains to prove that (G3) holds too. Indeed, $(-x)+x=(x\to 0)+x\overset{(\alpha)}{=}0$ and $x+(-x)=x+(x\rightsquigarrow 0)\overset{(\alpha)}{=}0$; thus, (G3) holds.

(2): If $(G,+,-,0)\xrightarrow{\rho}(G,+,\to,\rightsquigarrow,0)\xrightarrow{\rho^*}(G,+,\sim,0)$, then $\sim x=x\to 0=0-x=-x$.
Let now $(G,+,\to,\rightsquigarrow,0)\xrightarrow{\rho^*}(G,+,-,0)\xrightarrow{\rho}(G,+,\Rightarrow,\approx>,0)$. First we prove that $x\Rightarrow y=x\to y$. Indeed, $x\Rightarrow y=y-x=y+(-x)=y+(x\to 0)\overset{(35)}{=}y+(x\rightsquigarrow 0)\overset{(\alpha)}{=}[(x\to y)+x]+(x\rightsquigarrow 0)\overset{(G1)}{=}(x\to y)+[x+(x\rightsquigarrow 0)]\overset{(\alpha)}{=}(x\to y)+0\overset{(G2)}{=}x\to y$.
Now we shall prove that $x\approx> y=x\rightsquigarrow y$. Indeed, $x\approx> y=-x+y=(x\to 0)+y\overset{(\alpha)}{=}(x\to 0)+[x+(x\rightsquigarrow y)]\overset{(G1)}{=}[(x\to 0)+x]+(x\rightsquigarrow y)\overset{(\alpha)}{=}0+(x\rightsquigarrow y)\overset{(G2)}{=}x\rightsquigarrow y$. □

3.2 Po-groups

Recall that a *partially-ordered group* or a *po-group* for short is a structure $\mathcal{G}=(G,\leq,+,-,0)$ where $(G,+,-,0)$ is a group and $\leq$ is a partial order on G compatible with $+$, i.e. we have (G4).

Corollary 3.12 *Let $\mathcal{G}$ be a po-group. If $x\leq y$ and $a\leq b$, then $x+a\leq y+b$ and $a+x\leq b+y$.*

If $\mathcal{G}$ is a po-group, then $G^-=\{x\in G\mid x\leq 0\}$ will be called the *negative cone* and $G^+=\{x\in G\mid x\geq 0\}$ will be called the *positive cone* of $\mathcal{G}$.

Corollary 3.13 *G^+ and G^- are closed under $+$.*

Proof. Let $x,y\in G^+$, i.e. $x\geq 0$, $y\geq 0$. Then, by (G4), (G2), $x+y\geq 0+y=y$; since $y\geq 0$, it follows, by transitivity of $\leq$, that $x+y\geq 0$, i.e. $x+y\in G^+$. Similarly, G^- is closed under $+$. □

Proposition 3.14 *Let $\mathcal{G}$ be a po-group. Then, for all $x,y,z\in G$, we have:*
(G7) $x\leq y\Longleftrightarrow -y\leq -x$,
(G4') $x\leq y\Longleftrightarrow a+x\leq a+y\Longleftrightarrow x+a\leq y+a$.

$$(i)\quad x+y\leq z\Leftrightarrow x\leq y\to z\Leftrightarrow y\leq x\rightsquigarrow z\quad \textit{and dually} \tag{36}$$

$$(ii)\quad x+y\geq z\Leftrightarrow x\geq y\to z\Leftrightarrow y\geq x\rightsquigarrow z.$$

Proof.

(36) (i): $x\leq y\to z\Leftrightarrow x\leq z-y\Leftrightarrow x+y\leq z$, by (G4), and $y\leq x\rightsquigarrow z\Leftrightarrow y\leq -x+z\Leftrightarrow x+y\leq z$, by (G4); (ii) has a similar proof. □

Corollary 3.15 *Let $\mathcal{G}$ be a po-group. Then, for all $x,y\in G$, we have:*
(i) $y\leq 0\Longleftrightarrow x\leq y\to x\Longleftrightarrow x\leq y\rightsquigarrow x$ and dually
(ii) $y\geq 0\Longleftrightarrow x\geq y\to x\Longleftrightarrow x\geq y\rightsquigarrow x$.

Proof. Take $z=x$ in (36) and then take z=y in (36). □

Note that the important Properties (G5) and (G7) make the operation $-$ be an *involution.*

Proposition 3.16 *Let $\mathcal{G}$ be a po-group. Then, for all $x,y,z\in G$,*

$$x\leq y\implies z\to x\leq z\to y \textit{ and } z\rightsquigarrow x\leq z\rightsquigarrow y, \tag{37}$$

$$x\leq y\implies y\to z\leq x\to z \textit{ and } y\rightsquigarrow z\leq x\rightsquigarrow z. \tag{38}$$

Proof.

(37): Let $x \le y$; then $z \to x = x - z \overset{(G4)}{\le} y - z = z \to y$ and $z \rightsquigarrow x = -z + x \overset{(G4)}{\le} -z + y = z \rightsquigarrow y$.

(38): Let $x \le y$; then by (G7), $-y \le -x$; hence, $y \to z = z - y \overset{(G4)}{\le} z - x = x \to z$ and $y \rightsquigarrow z = -y + z \overset{(G4)}{\le} -x + z = x \rightsquigarrow z$. □

Corollary 3.17 *Let $\mathcal{G}$ be a po-group. For all $x, y \in G$:*
if $x \le y$ then $x \to y \ge 0$, $x \rightsquigarrow y \ge 0$ and $y \to x \le 0$, $y \rightsquigarrow x \le 0$.

Proof. Let $x \le y$; then,
- by (37), $x \to x \le x \to y$, i.e., by (23), $0 \le x \to y$; similarly, $0 \le x \rightsquigarrow y$;
- by (38), $y \to x \le x \to x = 0$, i.e. $y \to x \le 0$; similarly, $y \rightsquigarrow x \le 0$. □

3.2.1 X-po-groups: the intermediary notion

Definition 3.18 We shall name *X-po-group* a structure $(G, \le, +, \to, \rightsquigarrow, 0)$ such that $(G, +, 0)$ is a monoid (i.e. (G1), (G2) hold), $\le$ is an order relation on G and the following two properties holds: for all $x, z \in G$,
$(\#^{\le})$ $x + y \le z \Longleftrightarrow x \le y \to z \Longleftrightarrow y \le x \rightsquigarrow z$ (it is (36) (i)),
$(\#^{\ge})$ $x + y \ge z \Longleftrightarrow x \ge y \to z \Longleftrightarrow y \ge x \rightsquigarrow z$ (it is (36) (ii)).

First, we give some general results as a general lemma:

Lemma 3.19 *(see Lemma 3.9) Let $\mathcal{A} = (\le, +, \to, \rightsquigarrow)$ (or $\mathcal{A} = (A, \le, \to, \rightsquigarrow, +)$) such that $\le$ is a partial order on A. Then*

(i) properties $(\#^{\le})$ and $(\#^{\ge})$ imply the following properties: for all $x, y, z \in A$,
(#) $x + y = z \Longleftrightarrow x = y \to z \Longleftrightarrow y = x \rightsquigarrow z$,
(G4') $x \le y \Longleftrightarrow z + x \le z + y \Longleftrightarrow x + z \le y + z$,
(I5) $x \le y$ implies $z \to x \le z \to y$, $z \rightsquigarrow x \le z \rightsquigarrow y$,
(I6) $x \le y$ implies $y \to z \le x \to z$, $y \rightsquigarrow z \le x \rightsquigarrow z$.

(ii) the following equivalence holds

$$(\#^{\le}) \; + \; (\#^{\ge}) \iff (G4) \; + \; (I5) \; + \; (\#).$$

Proof. (i):
(#): properties $(\#^{\le})$ and $(\#^{\ge})$ imply property (#) since ($p \leftrightarrow q$ and $r \leftrightarrow s$) imply (p and r)$\leftrightarrow$(q and s) and since the order relation $\le$ is antisymmetric. Recall that (#) $\iff$ (α) + (β); we shall use this equivalence in the rest of the proof.

(G4'): Suppose $x \le y$; by (β), $y = z \rightsquigarrow (z+y)$, hence $x \le z \rightsquigarrow (z+y)$ and $x \le z \rightsquigarrow (z+y) \overset{(\#^{\le})}{\Longleftrightarrow} z+x \le z+y$, i.e. first part of (G4) holds. Conversely, $z + x \le z + y \overset{(\#^{\le})}{\Longleftrightarrow} x \le z \rightsquigarrow (z + y) \overset{(\beta)}{=} y$.

Suppose again $x \le y$; by (β), $y = z \to (y+z)$, hence $x \le z \to (y+z)$ and $x \le z \to (y+z) \overset{(\#^{\le})}{\Longleftrightarrow} x+z \le y+z$; thus (G4) holds. Conversely, $x + z \le y + z \overset{(\#^{\le})}{\Longleftrightarrow} x \le z \to (y + z) \overset{(\beta)}{=} y$.

(I5): Suppose $x \le y$; by (α), $(z \to x) + z = x$, hence $(z \to x) + z \le y$ and $(z \to x) + z \le y \overset{(\#^{\le})}{\Longleftrightarrow} z \to x \le z \to y$; by (α) also, $z + (z \rightsquigarrow x) = x$, hence $z + (z \rightsquigarrow x) \le y$ and $z + (z \rightsquigarrow x) \le y \overset{(\#^{\le})}{\Longleftrightarrow} z \rightsquigarrow x \le z \rightsquigarrow y$.

(I6): Suppose that $x \le y$; by (G4'), $(y \to z) + x \le (y \to z) + y \overset{(\alpha)}{=} z$ and $(y \to z) + x \le z \overset{(\#^{\le})}{\Longleftrightarrow} y \to z \le x \to z$; by (G4') also, $x + (y \rightsquigarrow z) \le y + (y \rightsquigarrow z) \overset{(\alpha)}{=} z$ and $x + (y \rightsquigarrow z) \le z \overset{(\#^{\le})}{\Longleftrightarrow} y \rightsquigarrow z \le x \rightsquigarrow z$.

(ii): By (i), it remains to prove that (G4) + (I5) + (#) imply $(\#^{\le})$ + $(\#^{\ge})$. Indeed, (G4) + (I5) + (#) imply $(\#^{\le})$:
- Suppose $x + y \le z$; then by (I5), $y \to (x + y) \le y \to z$ and $x \rightsquigarrow (x + y) \le x \rightsquigarrow z$; hence, by (β), we obtain that $x \le y \to z$ and $y \le x \rightsquigarrow z$.
- Conversely, suppose $x \le y \to z$; then by (G4), $x + y \le (y \to z) + y \overset{(\alpha)}{=} z$; suppose $y \le x \rightsquigarrow z$; then by (G4), $x + y \le x + (x \rightsquigarrow z) \overset{(\alpha)}{=} z$.

Similarly, (G4) + (I5) + (#) imply ($\#^{\geq}$). $\Box$

Note that if $(G, \leq, +, \to, \rightsquigarrow, 0)$ is an X-po-group, then $(G, +, \to, \rightsquigarrow, 0)$ is an X-group, by above Lemma. Note also that (36) is ($\#^{\leq}$) + ($\#^{\geq}$), (37) is (I5) and (38) is (I6).

Now we prove that the po-groups are termwise equivalent to X-po-groups.

Theorem 3.20 *(See Theorem 3.11)*

(1) Let $\mathcal{G} = (G, \leq, +, -, 0)$ be a po-group. Define $\rho'(\mathcal{G}) \stackrel{def.}{=} (G, \leq, +, \to, \rightsquigarrow, 0)$, with $(G, +, \to, \rightsquigarrow, 0) = \rho(G, +, -, 0)$ from Theorem 3.11 (1). Then, $\rho'(\mathcal{G})$ is an X-po-group.

(1') Conversely, let $\mathcal{G} = (G, \leq, +, \to, \rightsquigarrow, 0)$ be an X-po-group. Define ${\rho^*}'(\mathcal{G}) \stackrel{def.}{=} (G, \leq, +, -, 0)$, with $(G, +, -, 0) = \rho^*(G, +, \to, \rightsquigarrow, 0)$ from Theorem 3.11 (1'). Then, ${\rho^*}'(\mathcal{G})$ is a po-group.

(2) The above defined mappings ρ' and ${\rho^*}'$ are mutually inverse.

Proof.

(1): follows by Theorem 3.11 (1) and by properties (36).

(1'): by Theorem 3.11 (1'), $(G, +, -, 0)$ is a group; (G4') holds by above Lemma.

(2): follows by Theorem 3.11 (2). $\Box$

If the partial order $\leq$ is linear (total), then $\mathcal{G}$ is a *linearly-ordered group* or a *totally-ordered group*. Note that in a linearly-ordered group $\mathcal{G}$ we have either $x \leq 0$ or $x \geq 0$, for any $x \in G$, hence $G = G^- \cup G^+$.

Note that the presence of the order relation implies the presence of the *Duality Principle*. It follows that there are two dual po-groups. If their support sets differ ($G_1 \neq G_2$), then their unit elements differ and suppose that $0_1 \leq 0_2$ in the union set $G_1 \cup G_2$; we then shall call $\mathcal{G}_1$ as *left-po-group* and $\mathcal{G}_2$ as *right-po-group*. If their support sets coincide ($G = G_1 = G_2$), we shall say that G is *self-dual*, i.e. $(G, \leq, +, -, 0)$ is in the same time left-po-group and right-po-group.

3.3 l-groups

If the partial order $\leq$ is a lattice order, then the po-group $\mathcal{G}$ is called *lattice-ordered group* or *l-group* for short, denoted additively by $\mathcal{G} = (G, \vee, \wedge, +, -, 0)$. An introduction in l-groups is [1], see also [6].

Note that an l-group may be linearly-ordered or not, while a linearly-ordered group is an l-group.

Proposition 3.21 *Let $\mathcal{G}$ be an l-group. Then the properties (G8), (G9), (G10), (G11), (G12), (G13), (G14) hold.*

Corollary 3.22 *Let $\mathcal{G}$ be an l-group. Then we have, for all $x, y \in G$:*
(i) $x, y \in G^-$ imply $x + y \leq x \wedge y$,
(i') $x, y \in G^+$ imply $x + y \geq x \vee y$.

Proof. By (G12), $x \wedge y = [(x \vee y) \to x] + y$ and $x \vee y = [(x \wedge y) \to x] + y$.
(i): if $x, y \leq 0$, then $x \vee y \leq 0$, hence by Corollary 3.15, $x \leq (x \vee y) \to x$; then, by (G4), we obtain that $x + y \leq [(x \vee y) \to x] + y = x \wedge y$.
(i'): if $x, y \geq 0$, then $x \wedge y \geq 0$, hence by Corollary 3.15, $x \geq (x \wedge y) \to x$; then, by (G4), we obtain that $x + y \geq [(x \wedge y) \to x] + y = x \vee y$. $\Box$

Proposition 3.23 *In an l-group $(G, \vee, \wedge, +, -, 0)$, the following properties hold, for all $x, y, z \in G$:*

$$(x \vee z) \to y = (x \to y) \wedge (z \to y), \quad (x \vee z) \rightsquigarrow y = (x \rightsquigarrow y) \wedge (z \rightsquigarrow y) \text{ and dually} \tag{39}$$

$$(x \wedge z) \to y = (x \to y) \vee (z \to y), \quad (x \wedge z) \rightsquigarrow y = (x \rightsquigarrow y) \vee (z \rightsquigarrow y); \tag{40}$$

$$y \to (x \vee z) = (y \to x) \vee (y \to z), \quad y \rightsquigarrow (x \vee z) = (y \rightsquigarrow x) \vee (y \rightsquigarrow z) \text{ and dually} \tag{41}$$

$$y \to (x \wedge z) = (y \to x) \wedge (y \to z), \quad y \rightsquigarrow (x \wedge z) = (y \rightsquigarrow x) \wedge (y \rightsquigarrow z); \tag{42}$$

$$[(x \wedge 0) \rightsquigarrow 0] \wedge 0 = 0, \quad [(x \wedge 0) \to 0] \wedge 0 = 0 \textit{ and dually} \tag{43}$$

$$[(x \vee 0) \rightsquigarrow 0] \vee 0 = 0, \quad [(x \vee 0) \to 0] \vee 0 = 0. \tag{44}$$

Proof.

(39): $(x \vee z) \to y = y - (x \vee z) = y + [(-x) \wedge (-z)] = (y - x) \wedge (y - z) = (x \to y) \wedge (z \to y)$ and $(x \vee z) \rightsquigarrow y = -(x \vee z) + y = [(-x) \wedge (-z)] + y = (-x + y) \wedge (-z + y) = (x \rightsquigarrow y) \wedge (z \rightsquigarrow y)$.

(40): $(x \wedge z) \to y = y - (x \wedge z) = y + [(-x) \vee (-z)] = (y - x) \vee (y - z) = (x \to y) \vee (z \to y)$ and $(x \wedge z) \rightsquigarrow y = -(x \wedge z) + y = [(-x) \vee (-z)] + y = (-x + y) \vee (-z + y) = (x \rightsquigarrow y) \vee (z \rightsquigarrow y)$.

(41): $y \to (x \vee z) = (x \vee z) - y = (x - y) \vee (z - y) = (y \to x) \vee (y \to z)$ and $y \rightsquigarrow (x \vee z) = -y + (x \vee z) = (-y + x) \vee (-y + z) = (y \rightsquigarrow x) \vee (y \rightsquigarrow z)$.

(42): $y \to (x \wedge z) = (x \wedge z) - y = (x - y) \wedge (z - y) = (y \to x) \wedge (y \to z)$ and $y \rightsquigarrow (x \wedge z) = -y + (x \wedge z) = (-y + x) \wedge (-y + z) = (y \rightsquigarrow x) \wedge (y \rightsquigarrow z)$.

(43):

$[(x \wedge 0) \rightsquigarrow 0] \wedge 0 = [-(x \wedge 0) + 0] \wedge 0 = [(-x \vee -0) + 0] \wedge 0 = [-x \vee 0] \wedge 0 \overset{absorbtion}{=} 0$,

$[(x \wedge 0) \to 0] \wedge 0 = [0 - (x \wedge 0)] \wedge 0 = [0 + (-x \vee -0)] \wedge 0 = [-x \vee 0] \wedge 0 \overset{absorbtion}{=} 0$,

(44):

$[(x \vee 0) \rightsquigarrow 0] \vee 0 = [-(x \vee 0) + 0] \vee 0 = [(-x \wedge -0) + 0] \vee 0 = [-x \wedge 0] \vee 0 \overset{absorbtion}{=} 0$,

$[(x \vee 0) \to 0] \vee 0 = [0 - (x \vee 0)] \vee 0 = [0 + (-x \wedge -0)] \vee 0 = [-x \wedge 0] \vee 0 \overset{absorbtion}{=} 0$.

□

3.3.1 X-l-groups: the intermediary notion

If the partial order $\leq$ is a lattice order, then the X-po-group $\mathcal{G}$ is called *X-l-group* and is denoted by $\mathcal{G} = (G, \vee, \wedge, +, \to, \rightsquigarrow, 0)$.

We then obviously obtain that

Corollary 3.24 *The l-groups are termwise equivalent to X-l-groups.*

3.3.2 Examples

(0) $\mathcal{R} = (\mathbf{R}, \leq, +, -, 0)$ is a self-dual abelian linearly-ordered l-group.

(1) $\mathcal{D} = (D = (0, +\infty) = \{x \in \mathbf{R} \mid x > 0\}, \leq, \cdot, ^{-1}, 1)$ is an abelian linearly-ordered l-group.

(2) $\mathcal{G}_R = (G_R = (0, \infty) \times \mathbf{R}, \leq, +, -, 0_{G_R})$ [9] with:
$(a, b) + (c, d) \overset{def.}{=} (ac, bc + ad)$, $0_{G_R} = (1, 0)$, $-(a, b) = (\frac{1}{a}, -\frac{b}{a^2})$ and with the lexicographic order $\leq$ is a linearly ordered commutative l-group, where: $(a, b) \to (c, d) = (\frac{c}{a}, \frac{ad-bc}{a^2})$.

(2') Its dual is [27] $\mathcal{G}_L = (G_L = (-\infty, 0) \times \mathbf{R}, \leq, +, -, 0_{G_L})$ with:
$(-a, b) + (-c, d) \overset{def.}{=} (-ac, bc + ad)$, $0_{G_R} = (-1, 0)$, $-(-a, b) = (-\frac{1}{a}, -\frac{b}{a^2})$, $(-a, b) \to (-c, d) = (-\frac{c}{a}, \frac{ad-bc}{a^2})$ and with the lexicographic order $\leq$.

$\mathcal{G}_R$ is the linearly ordered abelian right-l-group and $\mathcal{G}_L$ is the linearly ordered abelian left-l-group, since $(-1, 0) < (1, 0)$ in the set $G_L \cup G_R$.

(3) Let $G = G_R = (0, \infty) \times \Re$ and define a binary operation "+" on G by:

$$(a, b) + (c, d) \overset{def.}{=} (ac, bc + d).$$

The operation "+" is associative, non-commutative and

$$0_G = (1, 0), \qquad -(a, b) \overset{def.}{=} (\frac{1}{a}, -\frac{b}{a}).$$

The order relation is the lexicographic order: $(a, b) < (c, d)$ iff $a < c$ or ($a = c$ and $b < d$). It makes G a lattice and the structure $\mathcal{G}_{\mathcal{R}} = (G, \vee, \wedge, +, -, 0_G)$ a linearly-ordered, non-abelian, l-group [9], where:

$$(a, b) \to (c, d) = (\frac{c}{a}, \frac{d-b}{a}), \quad (a, b) \rightsquigarrow (c, d) = (\frac{c}{a}, \frac{ad-bc}{a}).$$

(3') Dually, let $G = G_L = (-\infty, 0) \times \Re$ and define a binary operation "+" on G by:

$$(-a, b) + (-c, d) \stackrel{def.}{=} (-ac, bc + d).$$

The operation "+" is associative, non-commutative and

$$0_G = (-1, 0), \qquad -(-a, b) \stackrel{def.}{=} (-\frac{1}{a}, -\frac{b}{a}),$$

since $(-a, b) + (-\frac{1}{a}, -\frac{b}{a}) = (-1, \frac{b}{a} - \frac{b}{a}) = (-1, 0)$.
The order relation is the lexicographic order; it makes G a lattice and the structure $\mathcal{G}_L = (G, \wedge, \vee, +, -, 0_G)$ a linearly-ordered, non-abelian l-group [27], where:

$$(-a, b) \to (-c, d) = (-\frac{c}{a}, \frac{d-b}{a}), \quad (-a, b) \rightsquigarrow (-c, d) = (-\frac{c}{a}, \frac{ad-bc}{a}).$$

$\mathcal{G}_R$ is the right-l-group and $\mathcal{G}_L$ is the left-l-group, since $(-1, 0) < (1, 0)$ in the set $G_L \cup G_R$.

4 Implicative-groups, po-implicative-groups, l-implicative-groups. Other "horizontal" connections at group level

In this section, we introduce and study these new notions which are coming from algebras of logic. Just as the po-groups are the analogous of involutive porims, we introduce the po-implicative-groups as the analogous of the involutive reversed pseudo-BCK algebras. The section has three subsections, the analogous of those from Section 3.

4.1 Implicative-groups

Definition 4.1 An *implicative-group* is an algebra $\mathcal{G} = (G, \to, \rightsquigarrow, 0)$ of type $(2, 2, 0)$, such that the following axioms hold: for all $x, y, z \in G$,
(I1) $y \to z = (z \to x) \rightsquigarrow (y \to x), \quad y \rightsquigarrow z = (z \rightsquigarrow x) \to (y \rightsquigarrow x)$,
(I2) $y = (y \to x) \rightsquigarrow x, \quad y = (y \rightsquigarrow x) \to x$,
(I3) $x = y \iff x \to y = 0 \iff x \rightsquigarrow y = 0$,
(I4) $x \to 0 = x \rightsquigarrow 0$.

The implicative-group is said to be *commutative or abelian* if $x \to y = x \rightsquigarrow y$, for all $x, y \in G$.

Proposition 4.2 *Let $(G, \to, \rightsquigarrow, 0)$ be an implicative-group. Then, we have, for all $x, y, z \in G$:*
(I7) $0 \to x = x = 0 \rightsquigarrow x$,
(I8) $z \rightsquigarrow (y \to x) = y \to (z \rightsquigarrow x)$,
(I9) $x \to x = 0 = x \rightsquigarrow x$,

$$z \to x = (y \to z) \to (y \to x), \quad z \rightsquigarrow x = (y \rightsquigarrow z) \rightsquigarrow (y \rightsquigarrow x). \tag{45}$$

Proof.
(I7): $(0 \to x) \rightsquigarrow x \stackrel{(I2)}{=} 0$, hence by (I3) we obtain $0 \to x = x$; $(0 \rightsquigarrow x) \to x \stackrel{(I2)}{=} 0$, hence $0 \rightsquigarrow x = x$.
(I8): $y \to (z \rightsquigarrow x) \stackrel{(I1)}{=} ((z \rightsquigarrow x) \to x) \rightsquigarrow (y \to x) \stackrel{(I2)}{=} z \rightsquigarrow (y \to x)$.
(I9): $x \to x \stackrel{(I3)}{=} 0$ and $x \rightsquigarrow x \stackrel{(I3)}{=} 0$.
(45): $(z \to x) \rightsquigarrow [(y \to z) \to (y \to x)] \stackrel{(I8)}{=} (y \to z) \to [(z \to x) \rightsquigarrow (y \to x)] \stackrel{(I1)}{=} (y \to z) \to (y \to z) \stackrel{(I9)}{=} 0$ and $(z \rightsquigarrow x) \to [(y \rightsquigarrow z) \rightsquigarrow (y \rightsquigarrow x)] \stackrel{(I8)}{=} (y \rightsquigarrow z) \rightsquigarrow [(z \rightsquigarrow x) \to (y \rightsquigarrow x)] \stackrel{(I1)}{=} (y \rightsquigarrow z) \rightsquigarrow (y \rightsquigarrow z) \stackrel{(I9)}{=} 0$.
□

Remark 4.3 An equivalent definition of the implicative-group is the following: an *implicative-group* is an algebra $\mathcal{G} = (G, \to, \rightsquigarrow, 0)$ of type $(2, 2, 0)$, such that (I1), (I7), (I3), (I4) hold (see Definition 2.5 of the reversed left-pseudo-BCK algebra). Indeed, (I2), (I3) imply (I7), while (I7), (I1) imply (I2).

The following obvious result is the analogous of that saying that any group is a monoid.

Proposition 4.4 *Any implicative-group is a residoid (i.e. (I1), (I7), (I3) hold).*

4.1.1 New operations in implicative-groups: $-$ and $+$. Their properties

Let $\mathcal{G} = (G, \to, \rightsquigarrow, 0)$ be an implicative-group. Define an unary operation $-$ by: for all $x \in G$,

$$-x \overset{def.}{=} x \to 0 \overset{(I4)}{=} x \rightsquigarrow 0. \tag{46}$$

Proposition 4.5 *Let $\mathcal{G}$ be an implicative-group. Then, for all $x, y \in G$:*
(I10) $-(-x) = x$,
(I11) $-0 = 0$,
(I12) $x \rightsquigarrow (-y) = y \to (-x)$,
(I13) $(-x) \rightsquigarrow y = (-y) \to x$.

Proof.
(I10): $-(-x) \overset{def.}{=} -(x \to 0) \overset{def.}{=} (x \to 0) \rightsquigarrow 0 \overset{(I2)}{=} x$.
(I11): $-0 \overset{def.}{=} 0 \to 0 \overset{(I3)}{=} 0$.
(I12): $x \rightsquigarrow (-y) \overset{def.}{=} x \rightsquigarrow (y \to 0) \overset{(I8)}{=} y \to (x \rightsquigarrow 0) = y \to (-x)$.
(I13): $(-x) \rightsquigarrow y \overset{(I10)}{=} (-x) \rightsquigarrow (-(-y)) \overset{(I12)}{=} (-y) \to (-(-x)) \overset{(I10)}{=} (-y) \to x$. □

Note that by (I12), we have, for all x, y:

$$-[x \to (-y)] = -[y \rightsquigarrow (-x)]. \tag{47}$$

Since the implicative-group is a special involutive structure, by (I10) (there is only one negation), and since in the involutive algebras of logic we have (1), we introduce the new operation $+$ on G by: for all $x, y \in G$,

$$x + y \overset{def.}{=} -[x \to (-y)] \overset{(47)}{=} -[y \rightsquigarrow (-x)]. \tag{48}$$

Remark 4.6 (See Remark 3.4). By property (13) of a group, we could define equivalently $x + y$ by:

$$x + y \overset{def.}{=} (-y) \to x \overset{(I13)}{=} (-x) \rightsquigarrow y.$$

Proposition 4.7 *Let $(G, \to, \rightsquigarrow, 0)$ be an implicative-group. Then, (G1), (G2), (G3), (α) and (β) hold.*

Proof.
(G1): $x + (y + z) = x + [-(y \to (-z)] = -[x \to (-(-(y \to (-z))))] \overset{(I10)}{=} -[x \to (y \to (-z))]$;
$(x + y) + z = [-(x \to (-y))] + z = -[z \rightsquigarrow (-(-(x \to (-y))))] \overset{(I10)}{=} -[z \rightsquigarrow (x \to (-y))] \overset{(I8)}{=}$
$-[x \to (z \rightsquigarrow (-y))] \overset{(I12)}{=} -(x \to (y \to (-z)))$; thus (G1) holds.
(G2): $x + 0 = -(x \to (-0)) \overset{(I11)}{=} -(x \to 0) = -(-x) \overset{(I10)}{=} x$ and
$0 + x = -(x \rightsquigarrow (-0)) \overset{(I11)}{=} -(x \rightsquigarrow 0) = -(-x) \overset{(I10)}{=} x$.
(G3): $x + (-x) = -(x \to (-(-x))) \overset{(I10)}{=} -(x \to x) \overset{(I9)}{=} -0 \overset{(I11)}{=} 0$ and
$(-x) + x = -(x \rightsquigarrow (-(-x))) \overset{(I10)}{=} -(x \rightsquigarrow x) \overset{(I9)}{=} -0 \overset{(I11)}{=} 0$.
(α): $(x \to y) + x = -[(x \to y) \to (-x)] = [(x \to y) \to (x \to 0)] \to 0 \overset{(45)}{=} (y \to 0) \to 0 = -(-y) \overset{(I10)}{=} y$
and $x + (x \rightsquigarrow y) = -[(x \rightsquigarrow y) \rightsquigarrow (-x)] = [(x \rightsquigarrow y) \rightsquigarrow (x \rightsquigarrow 0)] \rightsquigarrow 0 \overset{(45)}{=} (y \rightsquigarrow 0) \rightsquigarrow 0 = -(-y) \overset{(I10)}{=} y$.

(β): $x \to (y+x) \overset{(48)}{=} x \to (-[y \to (-x)]) \overset{(46)}{=} x \to ([y \to (-x)] \rightsquigarrow 0) \overset{(I8)}{=} [y \to (-x)] \rightsquigarrow (x \to 0) = [y \to (-x)] \rightsquigarrow (-x) \overset{(I2)}{=} y$ and $x \rightsquigarrow (x+y) \overset{(48)}{=} x \rightsquigarrow (-[y \rightsquigarrow (-x)]) \overset{(46)}{=} x \rightsquigarrow ([y \rightsquigarrow (-x)] \to 0) \overset{(I8)}{=} [y \rightsquigarrow (-x)] \to (x \rightsquigarrow 0) = [y \rightsquigarrow (-x)] \to (-x) \overset{(I2)}{=} y$. □

Corollary 4.8 *Let $(G, \to, \rightsquigarrow, 0)$ be an implicative-group. Then,*
(i) $(G, +, -, 0)$ is a group;
(ii) property (#) holds.

Proof. (i) follows by Proposition 4.7 and (ii) follows by (i) or by Lemma 3.9. □

4.1.2 X-implicative-groups: an intermediary notion

Definition 4.9 (see Definition 3.8) We shall name *X-implicative-group* an algebra $(A, \to, \rightsquigarrow, +, 0)$ of type $(2, 2, 2, 0)$ such that $(A, \to, \rightsquigarrow, 0)$ is a residoid (i.e. (I1), (I7), (I3) hold) and the property (#) holds (i.e. properties (α) and (β) hold, by Lemma 3.9).

First, we need some results.

Proposition 4.10 *Let $(G, \to, \rightsquigarrow, +, 0)$ be an X-implicative-group. Then, for all $x, y, z \in G$,*

$$y = (y \to x) \rightsquigarrow x, \quad y = (y \rightsquigarrow x) \to x \qquad (\textit{is (I2)}), \tag{49}$$

$$x + 0 = x = 0 + x \qquad (\textit{is (G2)}), \tag{50}$$

$$z \to x = (y \to z) \to (y \to x), \quad z \rightsquigarrow x = (y \rightsquigarrow z) \rightsquigarrow (y \rightsquigarrow x), \tag{51}$$

$$x \to (y \rightsquigarrow z) = y \rightsquigarrow (x \to z) \qquad (\textit{is (I8)}), \tag{52}$$

$$x + (y+z) = (x+y) + z \quad (\textit{associativity}) \qquad (\textit{is (G1)}), \tag{53}$$

$$x \to 0 = x \rightsquigarrow 0 \qquad (\textit{is (I4)}). \tag{54}$$

Proof.
(49): $y = (y \to x) \rightsquigarrow x \overset{(\#)}{\Longleftrightarrow} y \to x = y \to x$, which is true and $y = (y \rightsquigarrow x) \to x \overset{(\#)}{\Longleftrightarrow} y \rightsquigarrow x = y \rightsquigarrow x$, which is also true.
(50): $x + 0 = x \overset{(\#)}{\Longleftrightarrow} x = 0 \to x$ and $0 + x = x \overset{(\#)}{\Longleftrightarrow} x = 0 \rightsquigarrow x$, which are true by (I7).
(51): $z \to x = (y \to z) \to (y \to x) \overset{(\#)}{\Longleftrightarrow} y \to z = (z \to x) \rightsquigarrow (y \to x)$ and
$z \rightsquigarrow x = (y \rightsquigarrow z) \rightsquigarrow (y \rightsquigarrow x) \overset{(\#)}{\Longleftrightarrow} y \rightsquigarrow z = (z \rightsquigarrow x) \to (y \rightsquigarrow x)$, which are true by (I1).
(52): $x \to (y \rightsquigarrow z) = y \rightsquigarrow (x \to z) \overset{(\#)}{\Longleftrightarrow} y = [x \to (y \rightsquigarrow z)] \to (x \to z)$, which is true; indeed, $[x \to (y \rightsquigarrow z)] \to (x \to z) = (y \rightsquigarrow z) \to z = y$, by (51) and (49).
(53): $x + (y+z) = a \overset{(\#)}{\Longleftrightarrow} y + z = x \rightsquigarrow a \overset{(\#)}{\Longleftrightarrow} y = z \to (x \rightsquigarrow a) \overset{(52)}{=} x \rightsquigarrow (z \to a) \overset{(\#)}{\Longleftrightarrow} x + y = z \to a \overset{(\#)}{\Longleftrightarrow} (x+y) + z = a$.
(54): $x \to 0 \overset{(50)}{=} (x \to 0) + 0 \overset{(\alpha)}{=} (x \to 0) + [x + (x \rightsquigarrow 0)] \overset{(53)}{=} [(x \to 0) + x] + (x \rightsquigarrow 0) \overset{(\alpha)}{=} 0 + (x \rightsquigarrow 0) \overset{(50)}{=} x \rightsquigarrow 0$. □

Remarks 4.11
(i) properties (#) and (50) imply property (I3); indeed, $y \overset{(50)}{=} 0 + y = z \overset{(\#)}{\Longleftrightarrow} 0 = y \to z$ and $x \overset{(50)}{=} x + 0 = z \overset{(\#)}{\Longleftrightarrow} 0 = x \rightsquigarrow z$. Hence, an equivalent definition of the X-implicative-group is the following: an *X-implicative-group* is an algebra $(A, \to, \rightsquigarrow, +, 0)$ such that (I1), (I7) and (#) hold.
(ii) If $(G, \to, \rightsquigarrow, +, 0)$ is an X-implicative-group, then $(G, +, \to, \rightsquigarrow, 0)$ is an X-group, by (53), (50).

Now we prove that the implicative-groups are termwise equivalent to the X-implicative-groups:

Theorem 4.12

(1) Let $\mathcal{G} = (G, \rightarrow, \rightsquigarrow, 0)$ *be an implicative-group. Define* $\pi(\mathcal{G}) \stackrel{def.}{=} (G, \rightarrow, \rightsquigarrow, +, 0)$ *with*

$$x + y = -(x \rightarrow (-y)) \stackrel{(47)}{=} -(y \rightsquigarrow (-x)), \quad \textit{where} \quad -x \stackrel{notation}{=} x \rightarrow 0 \stackrel{(I4)}{=} x \rightsquigarrow 0.$$

Then, $\pi(\mathcal{G})$ *is an X-implicative-group.*

(1') Conversely, let $\mathcal{G} = (G, \rightarrow, \rightsquigarrow, +, 0)$ *be an X-implicative-group. Define* $\pi^*(\mathcal{G}) \stackrel{def.}{=} (G, \rightarrow, \rightsquigarrow, 0)$*. Then,* $\pi^*(\mathcal{G})$ *is an implicative-group.*

(2) The above defined mappings are mutually inverse.

Proof.

(1): follows by Proposition 4.4 and by Corollary 4.8.

(1'): By Remark 4.3, we must prove that (I1), (I7), (I3), (I4) hold. Since $(G, \rightarrow, \rightsquigarrow)$ is a residoid, it follows that (I1), (I7), (I3) hold; (I4) is (54).

(2) If $(G, \rightarrow, \rightsquigarrow, 0) \stackrel{\pi}{\longrightarrow} (G, \rightarrow, \rightsquigarrow, +, 0) \stackrel{\pi^*}{\longrightarrow} (G, \rightarrow, \rightsquigarrow, 0)$, then there is nothing to prove.
Now, let $(G, \rightarrow, \rightsquigarrow, +, 0) \stackrel{\pi^*}{\longrightarrow} (G, \rightarrow, \rightsquigarrow, 0) \stackrel{\pi}{\longrightarrow} (G, \rightarrow, \rightsquigarrow, \oplus, 0)$: we have to prove that $x \oplus y = x + y$, for all $x, y \in G$. Indeed, $x \oplus y = -(x \rightarrow (-y)) = [x \rightarrow (y \rightarrow 0)] \rightsquigarrow 0 = x + y \stackrel{(\#)}{\Longleftrightarrow} x = y \rightarrow ([x \rightarrow (y \rightarrow 0)] \rightsquigarrow 0)$, which is true, since $y \rightarrow ([x \rightarrow (y \rightarrow 0)] \rightsquigarrow 0) = [x \rightarrow (y \rightarrow 0)] \rightsquigarrow (y \rightarrow 0) = x$, by (52) and (49). □

4.1.3 The groups and the implicative-groups are termwise equivalent

The following theorem establishes the announced result:

Theorem 4.13

(1) Let $\mathcal{G} = (G, +, -, 0)$ *be a group. Define* $\Phi(\mathcal{G}) \stackrel{def.}{=} (G, \rightarrow, \rightsquigarrow, 0)$ *by: for all* $x, y \in G$,

$$x \rightarrow y \stackrel{def.}{=} -(x + (-y)) = -(x - y) = y - x,$$

$$x \rightsquigarrow y \stackrel{def.}{=} -((-y) + x) = -(-y + x) = -x + y.$$

Then $\Phi(\mathcal{G})$ *is an implicative-group.*

(1') Conversely, let $\mathcal{G} = (G, \rightarrow, \rightsquigarrow, 0)$ *be an implicative-group. Define* $\Psi(\mathcal{G}) \stackrel{def.}{=} (G, +, -, 0)$ *by: for all* $x, y \in G$,

$$-x \stackrel{def.}{=} x \rightarrow 0 \stackrel{(I4)}{=} x \rightsquigarrow 0,$$

$$x + y \stackrel{def.}{=} -(x \rightarrow (-y)) \stackrel{(47)}{=} -(y \rightsquigarrow (-x)).$$

Then $\Psi(\mathcal{G})$ *is a group.*

(2) The maps Φ *and* Ψ *are mutually inverse.*

Proof.

(1) follows by Proposition 3.5.

(1') follows by Corollary 4.8.

(2): Let $(G, +, -, 0) \stackrel{\Phi}{\longrightarrow} (G, \rightarrow, \rightsquigarrow, 0) \stackrel{\Psi}{\longrightarrow} (G, \oplus, ^*, 0)$. Then, for all $x, y \in G$,
$x^* = x \rightarrow 0 = -(x + (-0)) = -(x + 0) = -x$ and
$x \oplus y = (x \rightarrow y^*)^* = -(x \rightarrow (-y)) = -(-(x = (-(-y)))) = x + y$, by (G5).
Let now $(G, \rightarrow, \rightsquigarrow, 0) \stackrel{\Psi}{\longrightarrow} (G, +, -, 0) \stackrel{\Phi}{\longrightarrow} (G, \Rightarrow, \approx>, 0)$. Then, for all $x, y \in G$,
$x \Rightarrow y = -(x + (-y)) = -(-(x \rightarrow (-(-y)))) = x \rightarrow y$ and
$x \approx> y = -((-y) + x) = -(-(x \rightsquigarrow (-(-y)))) = x \rightsquigarrow y$. □

By Theorems 3.11, 4.12 and 4.13 we obtain:

Corollary 4.14 *The groups, the X-groups, the implicative-groups and the X-implicative-groups are all termwise equivalent, as Figure 4 illustrates.*

implic.-groups	$\Longleftrightarrow$	X-implic.-groups	$\Longleftrightarrow$	X-groups	$\Longleftrightarrow$	groups
$(G,\to,\rightsquigarrow,0)$ (I1),(I7),(I3),(I4)		$(G,\to,\rightsquigarrow,+,0)$ (I1),(I7),(I3), (#)		$(G,+,\to,\rightsquigarrow,0)$ (G1),(G2), (#)		$(G,+,-,0)$ (G1),(G2),(G3)
$-x = x \to 0,$ $= x \rightsquigarrow 0$ $x+y$ $= -(x \to (-y))$ $= -(y \rightsquigarrow (-x))$		$-x = x \to 0$ $= x \rightsquigarrow 0,$		$-x = x \to 0$ $= x \rightsquigarrow 0$		$x \to y = -(x-y)$ $= y - x,$ $x \rightsquigarrow y = -(-y+x)$ $= -x + y$
(1.1)		(1.2)		(2.2)		(2.1)

Figure 4:

Remark 4.15 (See Remarks 3.2 and 2.10) For all $x \in G$, we have:
$0 = x \to x = x \rightsquigarrow x$, $-x = x \to 0 = x \rightsquigarrow 0$ and $-(-x) = x$ (there is only one **involutive** negation).

Proposition 4.16 *The implicative-group is commutative iff the termwise equivalent group is commutative, i.e.* $x \to y = x \rightsquigarrow y$ *for all* x, y *if and only if* $x + y = y + x$ *for all* x, y.

Proof. $x \to y = x \rightsquigarrow y$ for all x, y implies $-x \to y = -x \rightsquigarrow y \Leftrightarrow y - (-x) = -(-x) + y$, i.e. $y + x = x + y$, by (G5). Conversely, $x + y = y + x$ for all x, y implies $-x + y = y - x$, i.e. $x \rightsquigarrow y = x \to y$. $\Box$

4.2 Po-implicative-groups

Definition 4.17 A *partially-ordered implicative-group* or a *po-implicative-group* for short is a structure $\mathcal{G} = (G, \leq, \to, \rightsquigarrow, 0)$, where $(G, \to, \rightsquigarrow, 0)$ is an implicative-group and $\leq$ is a partial order on G compatible with $\to$, $\rightsquigarrow$, i.e. we have: for all $x, y, z \in G$,
(I5) $x \leq y$ implies $z \to x \leq z \to y$ and $z \rightsquigarrow x \leq z \rightsquigarrow y$.

Proposition 4.18 *Let* $(G, \leq, \to, \rightsquigarrow, 0)$ *be a po-implicative-group. Then, the following properties hold: for all* $x, y, a \in G$,
(I14) $x \leq y$ *implies* $-y \leq -x$,
(G4) $x \leq y$ *implies* $a + x \leq a + y$ *and* $x + a \leq y + a$,
(I5') $x \leq y \Longleftrightarrow z \to x \leq z \to y \Longleftrightarrow z \rightsquigarrow x \leq z \rightsquigarrow y$,
(#) $x + y = z \Longleftrightarrow x = y \to z \Longleftrightarrow y = x \rightsquigarrow z$.

Proof.
(I14): By (I6), for $z = 0$, we obtain: $x \leq y$ implies $-y = y \to 0 \leq x \to 0 = -x$.
(G4): Let $x \leq y$; then $-y \leq -x$, by (I14); by (I5), $a \to (-y) \leq a \to (-x)$, hence by (I14) again, we obtain $a + x = -(a \to (-x)) \leq -(a \to (-y)) = a + y$ and by (I5), $a \rightsquigarrow (-y) \leq a \rightsquigarrow (-x)$, hence by (I14) again, we obtain $x + a = -(a \rightsquigarrow (-x)) \leq -(a \rightsquigarrow (-y)) = y + a$.
(I5'): By (I5), it is sufficient to prove that $z \to x \leq z \to y$ implies $x \leq y$ and that $z \rightsquigarrow x \leq z \rightsquigarrow y$ implies $x \leq y$. Indeed, $z \to x \leq z \to y$ implies, by above (G4), that $x \overset{(\alpha)}{=} (z \to x) + z \leq (z \to y) + z \overset{(\alpha)}{=} y$ and $z \rightsquigarrow x \leq z \rightsquigarrow y$ implies, by above (G4), that $x \overset{(\alpha)}{=} z + (z \rightsquigarrow x) \leq z + (z \rightsquigarrow y) \overset{(\alpha)}{=} y$.
(#): follows by Corollary 4.8. $\Box$

Corollary 4.19 *Let* $(G, \leq, \to, \rightsquigarrow, 0)$ *be a po-implicative-group. Then,*
(i) $(G, \leq, +, -, 0)$ *is a po-group.*
(ii) properties $(\#^{\leq})$ *and* $(\#^{\geq})$ *hold.*

Proof. (i): By Corollary 4.8 and Proposition 4.18.
(ii): By Lemma 3.19 (ii). □

4.2.1 X-po-implicative-groups: the intermediary notion

Definition 4.20 We shall name *X-po-implicative-group* a structure $(G, \leq, \to, \rightsquigarrow, +, 0)$ such that $(G, \to, \rightsquigarrow, 0)$ is a residoid (i.e. (I1), (I7), (I3) hold), $\leq$ is an order relation on G and the properties $(\#^{\leq})$ and $(\#^{\geq})$ hold (i.e. (I5), (G4) and (#) hold, equivalently).

Corollary 4.21 *Let $(G, \leq, \to, \rightsquigarrow, +, 0)$ be an X-po-implicative-group. Then $(G, \to, \rightsquigarrow, +, 0)$ is an X-implicative-group.*

Proof. Obvious, by Lemma 3.19. □

Now we prove that the po-implicative-groups are termwise equivalent to the X-po-implicative-groups.

Theorem 4.22 *(See Theorem 4.12)*

(1) Let $\mathcal{G} = (G, \leq, \to, \rightsquigarrow, 0)$ be a po-implicative-group. Define $\pi'(\mathcal{G}) \stackrel{def.}{=} (G, \leq, \to, \rightsquigarrow, +, 0)$ where $(G, \to, \rightsquigarrow, +, 0) = \pi(G, \to, \rightsquigarrow, 0)$ from Theorem 4.12 (1). Then, $\pi'(\mathcal{G})$ is an X-po-implicative-group.

(1') Conversely, let $\mathcal{G} = (G, \leq, \to, \rightsquigarrow, +, 0)$ be an X-po-implicative-group. Define $\pi^{*'}(\mathcal{G}) \stackrel{def.}{=} (G, \leq, \to, \rightsquigarrow, 0)$. Then, $\pi^{*'}(\mathcal{G})$ is a po-implicative-group.

(2) The above defined mappings π' and $\pi^{*'}$ are mutually inverse.

Proof.
(1): follows by Theorem 4.12 (1) and Corollary 4.19.
(1'): By Corollary 4.21, $(A, \to, \rightsquigarrow, +, 0)$ is an X-implicative-group and then, by Theorem 4.12 (1'), $(G, \to, \rightsquigarrow, 0)$ is an implicative-group. It remains to prove that (I5) holds, which follows by Lemma 3.19.
(2) follows by Theorem 4.12 (2). □

4.2.2 The po-groups and the po-implicative-groups are termwise equivalent

The announced result follows by the next theorem.

Theorem 4.23 *(See Theorem 4.13)*

(1) Let $\mathcal{G} = (G, \leq, +, -, 0)$ be a po-group. Define $\Phi'(\mathcal{G}) \stackrel{def.}{=} (G, \leq, \to, \rightsquigarrow, 0)$, where $(G, \to, \rightsquigarrow, 0) = \Phi(G, +, -, 0)$, with Φ from Theorem 4.13(1). Then $\Phi'(\mathcal{G})$ is a po-implicative-group.

(1') Conversely, let $\mathcal{G} = (G, \leq, \to, \rightsquigarrow, 0)$ be a po-implicative-group. Define $\Psi'(\mathcal{G}) \stackrel{def.}{=} (G, \leq, +, -, 0)$, where $(G, +, -, 0) = \Psi(G, \to, \rightsquigarrow, 0)$, with Ψ from Theorem 4.13(1'). Then $\Psi'(\mathcal{G})$ is a po-group.

(2) The maps Φ' and Ψ' are mutually inverse.

Proof.
To prove (1), by Theorem 4.13 (1), it remains to prove (I5), which holds by Proposition 3.16.
(1') follows by Corollary 4.19.
(2) follows by Theorem 4.13 (2). □

By Theorems 3.20, 4.22 and 4.23, we obtain:

Corollary 4.24 *The po-groups, the X-po-groups, the po-implicative-groups and the X-po-implicative-groups are all termwise equivalent, as Figure 5 illustrates (see Figure 4 and the dual Figures 2,3).*

If the partial order relation $\leq$ is linear (total), then $\mathcal{G}$ is a linearly-ordered implicative-group.

The presence of the order relation implies the presence of the *Duality Principle*. It follows that there are two dual po-implicative-groups. If their support sets coincide ($G = G_1 = G_2$), we say that they are *self-dual*, i.e. $(G, \leq, \to, \rightsquigarrow, 0)$ is in the same time left-po-implicative-group and right-po-implicative-group; if their support sets differ ($G_1 \neq G_2$), then their unit elements differ and say that $0_1 \leq 0_2$ in the union set $G_1 \cup G_2$; we then call $\mathcal{G}_1$ as *left-po-implicative-group* and $\mathcal{G}_2$ as *right-po-implicative-group*.

po-imp.-groups	$\Longleftrightarrow$	**X-po-imp.-groups**	$\Longleftrightarrow$	**X-po-groups**	$\Longleftrightarrow$	**po-groups**
$(G, \leq, \rightarrow, \rightsquigarrow, 0)$ $\leq$ partial order (I1),(I7),(I3),(I4) **(I5)**		$(G, \leq, \rightarrow, \rightsquigarrow, +, 0)$ $\leq$ partial order (I1),(I7),(I3), $(\#^{\leq})$, $(\#^{\geq})$		$(G, \leq, +, \rightarrow, \rightsquigarrow, 0)$ $\leq$ partial order (G1),(G2), $(\#^{\leq})$, $(\#^{\geq})$		$(G, \leq, +, -, 0)$ $\leq$ partial order (G1),(G2),(G3) **(G4)**
$-x = x \rightarrow 0$ $= x \rightsquigarrow 0,$ $x + y$ $= -(x \rightarrow (-y))$ $= -(y \rightsquigarrow (-x))$		$-x = x \rightarrow 0$ $= x \rightsquigarrow 0$		$-x = x \rightarrow 0$ $= x \rightsquigarrow 0$		$x \rightarrow y$ $= -(x - y),$ $x \rightsquigarrow y$ $= -(-y + x)$
(1.1)		(1.2)		(2.2)		(2.1)

Figure 5:

Remarks 4.25

(1) There is a strong analogy between the po-groups and the involutive left- and right- porims and between the po-implicative-groups and the involutive reversed left- and right- pseudo-BCK algebras (see Figure 5 and the dual Figures 2,3).

(2) Since pseudo-Wajsberg algebras are termwise equivalent to pseudo-MV algebras [3], [4] and pseudo-MV algebras are intervals in l-groups with strong unit [8] and l-groups are termwise equivalent to l-implicative-groups, it follows that pseudo-Wajsberg algebras are intervals in l-implicative-groups with strong unit; find a direct proof of the last statement is an open problem (see [31]).

4.2.3 Deductive systems

Recall that a *convex po-subgroup* of a po-group $\mathcal{G} = (G, \leq, +, -, 0)$ is a subset $S \subseteq G$ which satisfies: for all $x, y, a, b \in G$,
(CS1) $0 \in S$,
(pCS2) (a) $x, y \in S$ imply $x + y \in S$ and (b) $x \in S$ implies $-x \in S$,
(pCS3) $a, b \in S$ and $a \leq x \leq b$ imply $x \in S$.

Note that a convex po-subgroup is also called a *po-ideal* of the po-group, but we shall see that a better name would be that of *filter-ideal*, because it determines both a filter and an ideal of certain structures built on G^- and G^+ respectively (see Theorem 5.2).

We introduce now the notion of "deductive system":

Definition 4.26 Let $\mathcal{G} = (G, \leq, \rightarrow, \rightsquigarrow, 0)$ be a po-implicative-group.

A *deductive system* of $\mathcal{G}$ is a subset $S \subseteq G$ which satisfies: for all $x, y, a, b \in G$,
(DS1)=(CS1) $0 \in S$,
(pDS2)(a) $x \in S$, $x \rightarrow y \in S$ imply $y \in S$ (or $x \in S$, $x \rightsquigarrow y \in S$ imply $y \in S$); (b) $x \in S$ implies $x \rightarrow 0 = x \rightsquigarrow 0 \in S$,
(pDS3)=(pCS3) $a, b \in S$ and $a \leq x \leq b$ imply $x \in S$.

Theorem 4.27 *Let $\mathcal{G}_g = (G, \leq, +, -, 0)$ be a po-group and let $\mathcal{G}_{ig} = (G, \leq, \rightarrow, \rightsquigarrow, 0)$ be the termwise equivalent po-implicative-group. Then, the convex po-subgroups of $\mathcal{G}_g$ coincide with the deductive systems of $\mathcal{G}_{ig}$.*

Proof. By Corollary 4.24, we can use Lemma 3.9.

Let S be a convex po-subgroup of $\mathcal{G}_g$, i.e. (CS1), (pCS2), (pCS3) hold. We must prove that S is a deductive system of $\mathcal{G}_{ig}$, i.e. that (DS1), (pDS2), (pDS3) hold. Indeed, (DS1) and (pDS3) hold.

(pDS2)(a): Let $x \in S$ and $x \to y \in S$; then, by (pCS2)(a), $(x \to y) + x \in S$; but, by (30) or by Lemma 3.9, $(x \to y) + x = y$, hence we obtain that $y \in S$; (similarly, let $x \in S$ and $x \rightsquigarrow y \in S$; then, by (pCS2)(a), $x + (x \rightsquigarrow y) \in S$; by (30) or by Lemma 3.9, $x + (x \rightsquigarrow y) = y$, hence, we obtain that $y \in S$;) thus, (pDS2)(a) holds. (pDS2)(b) holds by (pCS2)(b), since $-x = x \to 0 = x \rightsquigarrow 0$.

Conversely, let S be a deductive system of $\mathcal{G}_{ig}$, i.e. (DS1), (pDS2), (pDS3) hold. We must prove that S is a convex po-subgroup of $\mathcal{G}_g$, i.e. that (CS1), (pCS2), (pCS3) hold. Indeed, (CS1) and (pCS3) hold by (DS1) and (pDS3). (pCS2)(a): Let $x, y \in S$; by Lemma 3.9, $y \to (x + y) = x$; then, by (pDS2)(a), we obtain that $x + y \in S$. (pCS2)(b): holds by (pDS2)(b), since $-x = x \to 0 = x \rightsquigarrow 0$. □

Note that this theorem induces Theorem 2.17 and its dual.

4.3 l-implicative-groups

If the partial order relation $\leq$ is a lattice order relation, with the lattice operations $\wedge$ and $\vee$ defined by: $x \leq y \Leftrightarrow x \wedge y = x \Leftrightarrow x \vee y = y$, then $\mathcal{G}$ is a *lattice-ordered implicative-group* or a *l-implicative-group* for short, denoted $\mathcal{G} = (G, \vee, \wedge, \to, \rightsquigarrow, 0)$. Note that an l-implicative-group may be linearly-ordered or not, while a linearly-ordered implicative-group is an l-implicative-group.

Corollary 4.28 *Let $(G, \vee, \wedge)$ be the reduct of an l-group or of an l-implicative-group. Then, G^+ and G^- are closed under $\vee$ and $\wedge$.*

Proof.
Let $x, y \in G^+$, i.e. $x \geq 0$ and $y \geq 0$. Then, 0 is a lower bound of $\{x, y\}$; hence, $0 \leq x \wedge y$, i.e. $x \wedge y \in G^+$; since $0 \leq x \wedge y \leq x \vee y$, it follows that $0 \leq x \vee y$, i.e. $x \vee y \in G^+$ too.
Let now $x, y \in G^-$, i.e. $x \leq 0$ and $y \leq 0$. Then, 0 is an upper bound of $\{x, y\}$; hence, $0 \geq x \vee y$, i.e. $x \vee y \in G^-$; since $0 \geq x \vee y \geq x \wedge y$, it follows that $0 \geq x \wedge y$, i.e. $x \wedge y \in G^-$ too. □

4.3.1 X-l-implicative-groups: the intermediary notion

If the partial order $\leq$ is a lattice order, then:
- the po-implicative-group $\mathcal{G}$ is called an *l-implicative-group* and is denoted $\mathcal{G} = (G, \vee, \wedge, \to, \rightsquigarrow, 0)$,
- the X-po-implicative-group $\mathcal{G}$ is called an *X-l-implicative-group* and is denoted $\mathcal{G} = (G, \vee, \wedge, \to, \rightsquigarrow, +, 0)$.

We obviously obtain:

Corollary 4.29 *The l-implicative-groups are termwise equivalent to the X-l-implicative-groups.*

4.3.2 The l-groups and the l-implicative-groups are termwise equivalent

The announced result follows immediately (by Theorem 4.23):

Corollary 4.30 *l-groups are termwise equivalent to l-implicative-groups.*

By the analagous in lattice-ordered case of Theorems 3.20, 4.22 and 4.23, we obtain:

Corollary 4.31 *The l-groups, the X-l-groups, the l-implicative-groups and the X-l-implicative-groups are all termwise equivalent, as Figure 6 illustrates (see Figures 4,5).*

5 "Vertical" connections (between group level and algebras of logic level)

This section has two subsections.

l-imp.-groups	$\Longleftrightarrow$	**X-l-imp.-groups**	$\Longleftrightarrow$	**X-l-groups**	$\Longleftrightarrow$	**l-groups**
$(G, \vee, \wedge, \to, \rightsquigarrow, 0)$		$(G, \vee, \wedge, \to, \rightsquigarrow, +, 0)$		$(G, \vee, \wedge, +, \to, \rightsquigarrow, 0)$		$(G, \vee, \wedge, +, -, 0)$
(I1),(I7),(I3),(I4)		(I1),(I7),(I3),		(G1),(G2),		(G1),(G2),(G3)
(I5)		$(\#^{\leq})$, $(\#^{\geq})$		$(\#^{\leq})$, $(\#^{\geq})$		**(G4)**
$-x = x \to 0$		$-x = x \to 0$		$-x = x \to 0$		$x \to y$
$= x \rightsquigarrow 0,$		$= x \rightsquigarrow 0$		$= x \rightsquigarrow 0$		$= -(x - y),$
$x + y$						$x \rightsquigarrow y$
$= -(x \to (-y))$						$= -(-y + x)$
$= -(y \rightsquigarrow (-x))$						
(1.1)		(1.2)		(2.2)		(2.1)

Figure 6:

5.1 "Vertical" connections at partial-order level

G^- and G^+ are not closed under $\to$ and $\rightsquigarrow$, therefore we cannot connect po-implicative-groups with partially-ordered integral left- and right- residoids, in general. But G^- and G^+ are closed under $+$, by Corollary 3.13, therefore we obtain the following result.

Theorem 5.1 *Let $\mathcal{G} = (G, \leq, +, -, 0)$ be a po-group. Then*
(1) $\mathcal{G}^- = (G^-, \leq, \odot = +, \mathbf{1} = 0)$ is a left-poim (= partially-ordered, integral left-monoid);
(1') $\mathcal{G}^+ = (G^+, \leq, \oplus = +, \mathbf{0} = 0)$ is a right-poim (= partially-ordered, integral right-monoid).

Proof. It is obvious that $(G^-, \leq, \mathbf{1} = 0)$ is a poset with greatest element and that $(G^+, \leq, \mathbf{0} = 0)$ is a poset with smallest element. Then (T1), (T2), (T3) hold by (G1), (G2), (G4) respectively. □

Now, we shall analyse the connections between the convex po-subgroups (= filters-ideals) of $\mathcal{G}$, on the one hand, and the filters of $\mathcal{G}^L$ and the ideals of $\mathcal{G}^R$, on the other hand:

Theorem 5.2 *Let $\mathcal{G} = (G, \leq, +, -, 0)$ be a po-group and S be a convex po-subgroup (= filter-ideal) of $\mathcal{G}$. Then,*
(1) $S^L = S \cap G^-$ is a filter of the left-poim $\mathcal{G}^- = (G^-, \leq, \odot = +, \mathbf{1} = 0)$ from Theorem 5.1 (1).
(1') $S^R = S \cap G^+$ is an ideal of the right-poim $\mathcal{G}^+ = (G^+, \leq, \oplus = +, \mathbf{0} = 0)$ from Theorem 5.1 (1').

Proof. We prove (1): (f1): holds by (CS1). (pf2): Let $x, y \in S^L$; then, on the one hand, $x, y \in S$ and hence, by (pCS2)(a), $x + y \in S$; on the other hand, $x, y \in G^-$, hence $x + y \in G^-$. Consequently, $x \odot y = x + y \in S^L$ and thus, (pf2) holds. (f3): Let $x \in S^L$ and $x \leq y$ ($y \in G^-$); then, $x \in S$, $0 \in S$ and $x \leq y \leq 0$, hence $y \in S$, by (pCS3); consequently, $y \in S^L$ and thus (f3) holds. (1') has a similar proof. □

5.2 "Vertical" connections at lattice-order level

5.2.1 Connections between l-implicative-groups and pseudo-BCK lattices

The following theorem is the main result of this section.

Theorem 5.3 *Let $\mathcal{G} = (G, \vee, \wedge, \to, \rightsquigarrow, 0)$ be an l-implicative-group.*
(1) Define, for all $x, y \in G^-$:

$$x \to^L y \overset{def.}{=} (x \to y) \wedge 0, \tag{55}$$

$$x \rightsquigarrow^L y \overset{def.}{=} (x \rightsquigarrow y) \wedge 0. \tag{56}$$

Then, $\mathcal{G}^L = (G^-, \wedge, \vee, \rightarrow^L, \rightsquigarrow^L, \mathbf{1} = 0)$ is a distributive reversed left-pseudo-BCK(pP) lattice (with the pseudo-product $\odot = +$) verifying properties (pC) and (*) (see property (pP2) from the definition of a left-pseudo-product algebra and (32)).

(1') Define, for all $x, y \in G^+$:

$$x \rightarrow^R y \stackrel{def.}{=} (x \rightarrow y) \vee 0, \tag{57}$$

$$x \rightsquigarrow^R y \stackrel{def.}{=} (x \rightsquigarrow y) \vee 0. \tag{58}$$

Then, $\mathcal{G}^R = (G^+, \vee, \wedge, \rightarrow^R, \rightsquigarrow^R, \mathbf{0} = 0)$ is a distributive reversed right-pseudo-BCK(pS) lattice (with the pseudo-sum $\oplus = +$) verifying the dual properties (pC^d) and ($*^d$).

Proof.

- G^- is closed under the lattice operations $\wedge$ and $\vee$ of $\mathcal{G}$, by Corollary 4.28, and $(G^-, \wedge, \vee)$ is a distributive lattice, since $(G, \vee, \wedge)$ is a distributive lattice. Then G^- is closed under $\rightarrow^L$ and $\rightsquigarrow^L$.

- We prove now that $\mathcal{G}^L$ is a reversed left-pseudo-BCK algebra (see Definition 2.5), i.e. $(G^-, \leq, \mathbf{1} = 0)$ is a poset with greatest element and that properties (R1) - (R3) hold.

• Obviously, $(G^-, \leq, \mathbf{1} = 0)$ is a poset with greatest element, where $x \leq y$ iff $x \wedge y = x$ iff $x \vee y = y$.

• (R1): We must prove that for all $x, y, z \in G^-$,

$$(z \rightarrow^L x) \rightsquigarrow^L (y \rightarrow^L x) \geq y \rightarrow^L z, \quad (z \rightsquigarrow^L x) \rightarrow^L (y \rightsquigarrow^L x) \geq y \rightsquigarrow^L z. \tag{59}$$

First, we shall prove that

$$(z \rightarrow^L x) \rightsquigarrow^L (y \rightarrow^L x) = (y \rightarrow^L z) \vee (y \rightarrow^L x), \tag{60}$$

$$(z \rightsquigarrow^L x) \rightarrow^L (y \rightsquigarrow^L x) = (y \rightsquigarrow^L z) \vee (y \rightsquigarrow^L x).$$

Indeed, denote $A = (z \rightarrow^L x) \rightsquigarrow^L (y \rightarrow^L x)$. Then,
$A = ([(z \rightarrow x) \wedge 0] \rightsquigarrow [(y \rightarrow x) \wedge 0]) \wedge 0 =$
$([(z \rightarrow x) \wedge 0] \rightsquigarrow (y \rightarrow x)) \wedge ([(z \rightarrow x) \wedge 0] \rightsquigarrow 0) \wedge 0 \stackrel{(43)}{=}$
$([(z \rightarrow x) \wedge 0] \rightsquigarrow (y \rightarrow x)) \wedge 0 =$
$([(z \rightarrow x) \rightsquigarrow (y \rightarrow x)] \vee [0 \rightsquigarrow (y \rightarrow x)]) \wedge 0 =$
$((y \rightarrow z) \vee (y \rightarrow x)) \wedge 0 \stackrel{distributivity}{=}$
$[(y \rightarrow z) \wedge 0] \vee [(y \rightarrow x) \wedge 0] = (y \rightarrow^L z) \vee (y \rightarrow^L x)$.

Similarly, denote $B = (z \rightsquigarrow^L x) \rightarrow^L (y \rightsquigarrow^L x)$. Then
$B = ([(z \rightsquigarrow x) \wedge 0] \rightarrow [(y \rightsquigarrow x) \wedge 0]) \wedge 0 =$
$([(z \rightsquigarrow x) \wedge 0] \rightarrow (y \rightsquigarrow x)) \wedge ([(z \rightsquigarrow x) \wedge 0] \rightarrow 0) \wedge 0 \stackrel{(43)}{=}$
$([(z \rightsquigarrow x) \wedge 0] \rightarrow (y \rightsquigarrow x)) \wedge 0 =$
$([(z \rightsquigarrow x) \rightarrow (y \rightsquigarrow x)] \vee [0 \rightarrow (y \rightsquigarrow x)]) \wedge 0 =$
$((y \rightsquigarrow z) \vee (y \rightsquigarrow x)) \wedge 0 \stackrel{distributivity}{=}$
$[(y \rightsquigarrow z) \wedge 0] \vee [(y \rightsquigarrow x) \wedge 0] = (y \rightsquigarrow^L z) \vee (y \rightsquigarrow^L x)$.
Thus, (60) holds and therefore (59) holds.

• (R2): We must prove that for all $x \in G^-$ we have

$$\mathbf{1} \rightarrow^L x = x = \mathbf{1} \rightsquigarrow^L x. \tag{61}$$

Indeed, $\mathbf{1} \rightarrow^L x = (0 \rightarrow x) \wedge 0 \stackrel{(I7)}{=} x \wedge 0 = x$ and $\mathbf{1} \rightsquigarrow^L x = (0 \rightsquigarrow x) \wedge 0 \stackrel{(I7)}{=} x \wedge 0 = x$. Thus, (61) holds.

• (R3): We must prove that for all $x, y \in G^-$ we have

$$x \leq y \Longleftrightarrow x \rightarrow^L y = \mathbf{1} \Longleftrightarrow x \rightsquigarrow^L y = \mathbf{1}, \tag{62}$$

which follow immediately by the definitions of $\rightarrow^L$ and $\rightsquigarrow^L$.
Hence, $(G^-, \wedge, \vee, \rightarrow^L, \rightsquigarrow^L, \mathbf{1} = 0)$ is a reversed left-pseudo-BCK lattice.

- To prove that $\mathcal{G}^L$ is with the pseudo-product $\odot = +$, it is equivalent to prove that for every $x, y, z \in G^-$, properties (pRP) hold, i.e.

$$x \odot y \leq z \iff x \leq y \to^L z = (y \to z) \wedge 0, \tag{63}$$

$$x \odot y \leq z \iff y \leq x \leadsto^L z = (x \leadsto z) \wedge 0. \tag{64}$$

(63): By (36), we have $x \odot y = x + y \leq z \iff x \leq y \to z$; if $x \leq y \to z$, then, since $x \leq 0$, we obtain $x \leq (y \to z) \wedge 0 = y \to^L z$, i.e. $x \leq y \to^L z$; conversely, if $x \leq y \to^L z$, then $x \leq y \to z$, since $y \to^L z = (y \to z) \wedge 0 \leq y \to z$.
(64): similarly.

- To prove that the reversed left-pseudo-BCK(pP) lattice $\mathcal{G}^L$ satisfies property (pC) means to prove that for all $x, y \in G^-$ we have

$$(y \to^L x) \leadsto^L x = y \vee x = (y \leadsto^L x) \to^L x. \tag{65}$$

Indeed, $(y \to^L x) \leadsto^L x = ([(y \to x) \wedge 0] \leadsto x) \wedge 0 \overset{(40)}{=} ([(y \to x) \leadsto x] \vee [0 \leadsto x]) \wedge 0 = (y \vee x) \wedge 0 = y \vee x$.
And $(y \leadsto^L x) \to^L x = ([(y \leadsto x) \wedge 0] \to x) \wedge 0 \overset{(40)}{=} ([(y \leadsto x) \to x] \vee [0 \to x]) \wedge 0 = (y \vee x) \wedge 0 = y \vee x$; thus, (65) holds.

- The reversed left-pseudo-BCK(pP) lattice $\mathcal{G}^L$ satisfies property (*) by (32). Thus, (1) holds.

(1') has a similar proof, where the analogous of (63), (64) are respectively:

$$x \oplus y \geq z \iff x \geq y \to^R z = (y \to z) \vee 0, \tag{66}$$

$$x \oplus y \geq z \iff y \geq x \leadsto^R z = (x \leadsto z) \vee 0. \tag{67}$$

□

Remark 5.4
(i) By Theorem 2.12, properties (pprel) and (pdiv) are verified by $\mathcal{G}^L$.
(i') By the dual of Theorem 2.12, the dual properties (pprel^d) and (pdiv^d) are verified by $\mathcal{G}^R$.

Remarks 5.5
(0) An *L-algebra* [31] is an algebra $(X, \to, 1)$ verifying: for all $x, y, z \in X$, $x \to x = x \to 1 = 1$, $1 \to x = x$, $x \to y = y \to x = 1$ imply $x = y$ and the following property (L):
(L) $(x \to y) \to (x \to z) = (y \to x) \to (y \to z)$.
Note that if $(G, \to, \leadsto, 0)$ is an implicative-group, then $(G, \to, 0)$ and $(G, \leadsto, 0)$ are not L algebras, since (L) does not hold neither for $\to$ ni for $\leadsto$, by (22).
(i) But $(G^-, \to^L, \mathbf{1} = 0)$ and $(G^-, \leadsto^L, \mathbf{1} = 0)$ are L algebras, because (L) holds. Indeed,
$(x \to^L y) \to^L (x \to^L z) = ([(x \to y) \wedge 0] \to [(x \to z) \wedge 0]) \wedge 0 \overset{(42)}{=}$
$([(x \to y) \wedge 0] \to (x \to z)) \wedge ([(x \to y) \wedge 0] \to 0) \wedge 0 \overset{(43)}{=}$
$([(x \to y) \wedge 0] \to (x \to z)) \wedge 0 = ([(x \to y) \to (x \to z)] \vee [0 \to (x \to z)]) \wedge 0 \overset{(22)}{=}$
$((y \to z) \vee (x \to z)) \wedge 0 \overset{distributivity}{=} [(y \to z) \wedge 0] \vee [(x \to z) \wedge 0] = (y \to^L z) \vee (x \to^L z)$;
hence, $(y \to^L x) \to^L (y \to^L z) = (x \to^L z) \vee (y \to^L z)$, i.e. (L) holds for $\to^L$.
Similarly, $(x \leadsto^L y) \leadsto^L (x \leadsto^L z) = ([(x \leadsto y) \wedge 0] \leadsto [(x \leadsto z) \wedge 0]) \wedge 0 \overset{(42)}{=}$
$([(x \leadsto y) \wedge 0] \leadsto (x \leadsto z)) \wedge ([(x \leadsto y) \wedge 0] \leadsto 0) \wedge 0 \overset{(43)}{=}$
$([(x \leadsto y) \wedge 0] \leadsto (x \leadsto z)) \wedge 0 = ([(x \leadsto y) \leadsto (x \leadsto z)] \vee [0 \leadsto (x \leadsto z)]) \wedge 0 \overset{(22)}{=}$
$((y \leadsto z) \vee (x \leadsto z)) \wedge 0 \overset{distributivity}{=} [(y \leadsto z) \wedge 0] \vee [(x \leadsto z) \wedge 0] = (y \leadsto^L z) \vee (x \leadsto^L z)$;
hence, $(y \leadsto^L x) \leadsto^L (y \leadsto^L z) = (x \leadsto^L z) \vee (y \leadsto^L z)$, i.e. (L) holds for $\leadsto^L$ too.
(i') Dually, $(G^+, \to^R, \mathbf{0} = 0)$ and $(G^+, \leadsto^R, \mathbf{0} = 0)$ are L algebras.

We obtain obviously the following corollary:

Corollary 5.6 *If $\mathcal{G}$ is a linearly-ordered l-implicative-group, then:*

(1) the two implications from Theorem 5.3 (1) become, for all $x, y \in G^-$:

$$x \to^L y = \begin{cases} 0, & \text{if } x \leq y \\ x \to y, & \text{if } y < x, \end{cases} \qquad x \rightsquigarrow^L y = \begin{cases} 0, & \text{if } x \leq y \\ x \rightsquigarrow y, & \text{if } y < x. \end{cases}$$

(1') the two implications from Theorem 5.3 (1') become, for all $x, y \in G^+$:

$$x \to^R y = \begin{cases} 0, & \text{if } x \geq y \\ x \to y, & \text{if } y > x, \end{cases} \qquad x \rightsquigarrow^R y = \begin{cases} 0, & \text{if } x \geq y \\ x \rightsquigarrow y, & \text{if } y > x. \end{cases}$$

Lemma 5.7 *Let $(G, \vee, \wedge, \to, \rightsquigarrow, 0)$ be an l-implicative-group. Let $u' < 0$ and $u > 0$ from G. Then,*
(1) the interval $[u', 0] = \{x \in G^- \mid u' \leq x \leq 0\} \subset G^-$ is closed under $\to^L$ and $\rightsquigarrow^L$ in $\mathcal{G}^L$ from Theorem 5.3 (1),
(1') the interval $[0, u] \subset G^+$ is closed under $\to^R$ and $\rightsquigarrow^R$ in $\mathcal{G}^R$ from Theorem 5.3 (1').

Proof.

(1): Let $u' \in G^-$ and let $u' \leq x, y \leq 0$. By the properties of the left-pseudo-BCK algebra $\mathcal{G}^L$, we have $u' \leq y \leq x \to^L y = (x \to y) \wedge 0 \leq 0$ and $u' \leq y \leq x \rightsquigarrow^L y = (x \rightsquigarrow y) \wedge 0 \leq 0$. Hence, $u' \leq x \to^L y \leq 0$ and $u' \leq x \rightsquigarrow^L y \leq 0$.

(1'): Let $u \in G^+$ and let $0 \leq x, y \leq u$. By the properties of the right-pseudo-BCK algebra $\mathcal{G}^R$, we have $u \geq y \geq x \to^R y = (x \to y) \vee 0 \geq 0$ and $u \geq y \geq x \rightsquigarrow^R y = (x \rightsquigarrow y) \vee 0 \geq 0$. Hence, $u \geq x \to^R y \geq 0$ and $u \geq x \rightsquigarrow^R y \geq 0$. $\Box$

Let us "bound" the algebras $\mathcal{G}^L$ and $\mathcal{G}^R$ from Theorem 5.3 in two different ways: first, with an "internal" element, then, with an "external" element. We obtain the equivalent of known results, by Theorem 5.3, Remark 5.4 and Lemma 5.7:

Corollary 5.8
(i) Let $\mathcal{G}^L = (G^-, \wedge, \vee, \to^L, \rightsquigarrow^L, \mathbf{1} = 0)$ from Theorem 5.3 (1). Let us "bound" this algebra in two different ways:

1) Let us take $u' < 0$ from G^- and consider the interval $[u', 0]$. Then the algebra

$$\mathcal{G}^L_1 = ([u', 0], \wedge, \vee, \to^L, \rightsquigarrow^L, \mathbf{0} = u', \mathbf{1} = 0)$$

is a bounded reversed left-pseudo-BCK(pP) lattice (with the pseudo-product $x \odot^L y \stackrel{def.}{=} (x \odot y) \vee u' = (x + y) \vee u'$) with property (pC), i.e. is an equivalent definition of the **left-pseudo-Wajsberg algebra** *(see [23] and [30], [10] for the commutative case)*

$$\mathcal{G}^L_{1'} = ([u', 0], \to^L, \rightsquigarrow^L, ^{-^L}, ^{\sim^L}, \mathbf{0} = u', \mathbf{1} = 0).$$

2) Let us consider a symbol $-\infty$ distinct from the elements of G. Define $G^-_{-\infty} \stackrel{def.}{=} \{-\infty\} \cup G^-$ and extend the operations $\to^L, \rightsquigarrow^L, \odot$ from G^- to $G^-_{-\infty}$ as follows:

$$x \to^L_2 y \stackrel{def.}{=} \begin{cases} x \to^L y, & \text{if } x, y \in G^- \\ -\infty, & \text{if } x \in G^-,\ y = -\infty \\ 0, & \text{if } x = -\infty, \end{cases} \qquad x \rightsquigarrow^L_2 y \stackrel{def.}{=} \begin{cases} x \rightsquigarrow^L y, & \text{if } x, y \in G^- \\ -\infty, & \text{if } x \in G^-,\ y = -\infty \\ 0, & \text{if } x = -\infty, \end{cases}$$

$$x \odot_2 y \stackrel{def.}{=} \begin{cases} x \odot y = x + y, & \text{if } x, y \in G^- \\ -\infty, & \text{if otherwise.} \end{cases}$$

We extend the lattice order relation $\leq$ as follows: we put $-\infty \leq x$, for any $x \in G^-_{-\infty}$. Then, the algebra

$$\mathcal{G}^L_2 = (G^-_{-\infty}, \wedge, \vee, \to^L_2, \rightsquigarrow^L_2, \mathbf{0} = -\infty, \mathbf{1} = 0)$$

is a **left-pseudo-Hájek(pP) algebra (with the pseudo-product $\odot_2$) verifying properties (pP1) and (pP2)** *(see [23]).*

(i') Let $\mathcal{G}^R = (G^+, \vee, \wedge, \to^R, \rightsquigarrow^R, \mathbf{0} = 0)$ from Theorem 5.3 (1'). Let us "bound" this algebra in two different ways:

1') Let us take $u > 0$ from G^+ and consider the interval $[0, u] = \{x \in G^+ \mid 0 \leq x \leq u\}$. Then the algebra

$$\mathcal{G}^R_1 = ([0, u], \vee, \wedge, \to^R, \rightsquigarrow^R, \mathbf{0} = 0, \mathbf{1} = u)$$

is a bounded reversed right-pseudo-BCK(pS) lattice (with the pseudo-sum $x \oplus^R y \overset{def.}{=} (x \oplus y) \wedge u = (x + y) \wedge u$) with property (pC^d), i.e. is an equivalent definition of the **right-pseudo-Wajsberg algebra**

$$\mathcal{G}^R_{1'} = ([0, u], \to^R, \rightsquigarrow^R, {}^{-^R}, {}^{\sim^R}, \mathbf{0} = 0, \mathbf{1} = u).$$

2') Let us consider a symbol $+\infty$ distinct from the elements of G. Define $G^+_{+\infty} \overset{def.}{=} G^+ \cup \{+\infty\}$ and extend the operations $\to^R, \rightsquigarrow^R$, $\oplus$ from G^+ to $G^+_{+\infty}$ as follows:

$$x \to^R_2 y \overset{def.}{=} \begin{cases} x \to^R y, & if \quad x, y \in G^+ \\ +\infty, & if \quad x \in G^+,\ y = +\infty \\ 0, & if \quad x = +\infty, \end{cases} \qquad x \rightsquigarrow^R_2 y \overset{def.}{=} \begin{cases} x \rightsquigarrow^R, & if \quad x, y \in G^+ \\ +\infty, & if \quad x \in G^+,\ y = +\infty \\ 0, & if \quad x = +\infty, \end{cases}$$

$$x \oplus_2 y \overset{def.}{=} \begin{cases} x \oplus y = x + y, & if \quad x, y \in G^+ \\ +\infty, & if \quad otherwise. \end{cases}$$

We extend the lattice order relation $\geq$ as follows: we put $+\infty \geq x$, for any $x \in G^+_{+\infty}$. Then, the algebra

$$\mathcal{G}^R_2 = (G^+_{+\infty}, \vee, \wedge, \to^R_2, \rightsquigarrow^R_2, \mathbf{0} = 0, \mathbf{1} = +\infty)$$

is a **right-pseudo-Hájek(pS) algebra (with the pseudo-sum $\oplus_2$) verifying the dual properties (pP1^d) and (pP2^d)**.

Now we shall analyse the connections between the deductive systems of $\mathcal{G}$, on the one hand, and the deductive systems of $\mathcal{G}^L$ and $\mathcal{G}^R$, on the other hand (see Theorem 2.17 and Theorems 4.27 and 5.2).

Theorem 5.9 *Let $\mathcal{G} = (G, \vee, \wedge, \to, \rightsquigarrow, 0)$ be an l-implicative-group and S be a deductive system of $\mathcal{G}$. Then,*

(1) $S^L = S \cap G^-$ is a $(\to^L, \rightsquigarrow^L)$-deductive system of $\mathcal{G}^L = (G^-, \wedge, \vee, \to^L, \rightsquigarrow^L, \mathbf{1} = 0)$ from Theorem 5.3 (1).

(1') $S^R = S \cap G^+$ is a $(\to^R, \rightsquigarrow^R)$-deductive system of $\mathcal{G}^R = (G^+, \vee, \wedge, \to^R, \rightsquigarrow^R, \mathbf{0} = 0)$ from Theorem 5.3 (1').

Proof. We prove (1): (ds1) holds by (DS1). (pds2): Let $x \in S^L$ and $x \to^L y \in S^L$ ($y \in G^-$). Then, $x \in S$ and $x \to^L y = \begin{cases} 0, & \text{if} \quad x \leq y \\ (x \to y) \wedge 0, & \text{if} \quad x \not\leq y, \end{cases} \in S$:

- if $x \leq y$, then $x \leq y \leq 0$ and $x, 0 \in S$; then, by (pDS3), we obtain that $y \in S$, hence $y \in S^L$;
- if $x \not\leq y$, then $x \to^L y = (x \to y) \wedge 0 \leq x \to y$. But $y \leq 0$ implies by (I5) that $x \to y \leq x \to 0$.

Hence, we have $x \to^L y \leq x \to y \leq x \to 0$ and $x \to^L y, x \to 0 \in S$; hence, by (pDS3), $x \to y \in S$. Thus, we obtained that $x \in S$ and $x \to y \in S$, hence, by (pDS2)(a), $y \in S$, hence $y \in S^L$.

(1') has a similar proof. □

Note that Theorems 4.27, 5.2, 5.9 imply Theorem 2.17 and its dual.

5.2.2 Connections between l-groups and left- and right- l-rims

Recall that a *left-l-rim* is a *lattice-ordered, residuated, integral left-monoid*, while a *right-l-rim* is a *lattice-ordered, residuated, integral right-monoid.*

By Corollary 4.30, we obtain the following analogous of Theorem 5.3 and Corollary 5.8.

Theorem 5.10 *Let $\mathcal{G} = (G, \vee, \wedge, +, -, 0)$ be an l-group.*

(1) Define, for all $x, y \in G^-$:

$$x \odot y \stackrel{def.}{=} x + y, \quad x \to^L y \stackrel{def.}{=} (x \to y) \wedge 0, \quad x \rightsquigarrow^L y \stackrel{def.}{=} (x \rightsquigarrow y) \wedge 0.$$

Then, $\mathcal{G}^L_m = (G^-, \wedge, \vee, \odot, \mathbf{1} = 0)$ is a distributive left-l-rim (with the pseudo-residuum $(\to^L, \rightsquigarrow^L)$) verifying properties (pC) and (*) (while the equivalent algebra $\mathcal{G}^L_{m'} = (G^-, \wedge, \vee, \odot, \to^L, \rightsquigarrow^L, \mathbf{1} = 0)$ is a non-commutative distributive left-residuated lattice verifying properties (pC) and (*)).

(1') Define, for all $x, y \in G^+$:

$$x \oplus y \stackrel{def.}{=} x + y, \quad x \to^R y \stackrel{def.}{=} (x \to y) \vee 0, \quad x \rightsquigarrow^R y \stackrel{def.}{=} (x \rightsquigarrow y) \vee 0.$$

Then, $\mathcal{G}^R_m = (G^+, \vee, \wedge, \oplus, \mathbf{0} = 0)$ is a distributive right-l-rim (with the pseudo-coresiduum $(\to^R, \rightsquigarrow^R)$) verifying properties (pC^d) and (^{d}) (while the equivalent algebra $\mathcal{G}^R_{m'} = (G^+, \vee, \wedge, \oplus, \to^R, \rightsquigarrow^R, \mathbf{0} = 0)$ is a non-commutative distributive right-residuated lattice verifying properties (pC^d) and (*^{d})).*

Remark 5.11 $\mathcal{G}^L_m$ verifies properties (pprel) and (pdiv), by Theorem 2.12, and $\mathcal{G}^R_m$ verifies the dual properties (pprel^d) and (pdiv^d).

We shall "bound" the above algebras $\mathcal{G}^L_m$ and $\mathcal{G}^R_m$ in two different ways; we obtain known results:

Corollary 5.12
(i) Let $\mathcal{G}^L_m = (G^-, \wedge, \vee, \odot, \mathbf{1} = 0)$ from above Theorem 5.10 (1). Let us "bound" this algebra in two different ways:

1) Let us take $u' < 0$ from G^- and consider the interval $[u', 0] = \{x \in G^- \mid u' \leq x \leq 0\}$. Define for all $x, y \in G^-$:

$$x \odot^L y \stackrel{def.}{=} (x \odot y) \vee u' = (x + y) \vee u', \quad x \oplus^L y \stackrel{def.}{=} (x - u' + y) \wedge 0, \quad x^{-^L} \stackrel{def.}{=} u' - x, \quad x^{\sim^L} \stackrel{def.}{=} -x + u'.$$

Then the algebra

$$\mathcal{G}^L_{m_1} = ([u', 0], \wedge, \vee, \odot^L, \mathbf{0} = u', \mathbf{1} = 0)$$

is a bounded left-l-rim verifying property (pC), i.e. is an equivalent definition of the **left-pseudo-MV algebra** *([7] and [30] for the commutative case)*

$$\mathcal{G}^L_{m'_1} = ([u', 0], \odot^L, \oplus^L, {}^{-^L}, {}^{\sim^L}, \mathbf{0} = u', \mathbf{1} = 0).$$

2) Let us consider a symbol $-\infty$ distinct from the elements of G. Define $G^-_{-\infty} \stackrel{def.}{=} \{-\infty\} \cup G^-$ and extend the operations $\odot$, $\to^L, \rightsquigarrow^L$ from G^- to $G^-_{-\infty}$ as follows:

$$x \odot_2 y \stackrel{def.}{=} \begin{cases} x \odot y = x + y, & \text{if } x, y \in G^- \\ -\infty, & \text{if otherwise.} \end{cases}$$

$$x \to^L_2 y \stackrel{def.}{=} \begin{cases} x \to^L y, & \text{if } x, y \in G^- \\ -\infty, & \text{if } x \in G^-,\ y = -\infty \\ 0, & \text{if } x = -\infty, \end{cases} \qquad x \rightsquigarrow^L_2 y \stackrel{def.}{=} \begin{cases} x \rightsquigarrow^L y, & \text{if } x, y \in G^- \\ -\infty, & \text{if } x \in G^-,\ y = -\infty \\ 0, & \text{if } x = -\infty. \end{cases}$$

We extend the lattice order relation $\leq$ as follows: we put $-\infty \leq x$, for any $x \in G^-_{-\infty}$.
Then, the algebra

$$\mathcal{G}^L_{m_2} = (G^-_{-\infty}, \wedge, \vee, \odot_2, \mathbf{0} = -\infty, \mathbf{1} = 0)$$

is a bounded left-l-rim verifying properties (pprel),(pdiv) and (pP1), (pP2), while the equivalent algebra

$$\mathcal{G}^L_{m'_2} = (G^-_{-\infty}, \wedge, \vee, \odot_2, \to^L_2, \rightsquigarrow^L_2, \mathbf{0} = -\infty, \mathbf{1} = 0)$$

is a left-pseudo-BL algebra verifying (pP1), (pP2), i.e. is a **left-pseudo-product algebra** *(due to property (*)) (see [7] and [17] for the commutative case).*

(i') Let $\mathcal{G}^R_m = (G^+, \vee, \wedge, \oplus, \mathbf{0} = 0)$ *from above Theorem 5.10 (1'). Let us "bound" this algebra in two different ways:*

1') Let us take $u > 0$ *from* G^+ *and consider the interval* $[0, u] = \{x \in G^+ \mid 0 \leq x \leq u\}$*. Define for all* $x, y \in G^+$*:*

$$x \oplus^R y \stackrel{def.}{=} (x \oplus y) \wedge u = (x + y) \wedge u, \quad x \odot^R y \stackrel{def.}{=} (x - u + y) \vee 0, \quad x^{-^R} \stackrel{def.}{=} u - x, \quad x^{\sim^R} \stackrel{def.}{=} -x + u.$$

Then the algebra

$$\mathcal{G}^R_{m_1} = ([0, u], \vee, \wedge, \oplus^R, \mathbf{0} = 0, \mathbf{1} = u)$$

is a bounded right-l-rim verifying property (pC^d)*, i.e. is an equivalent definition of the* **right-pseudo-MV algebra**

$$\mathcal{G}^R_{m'_1} = ([0, u], \oplus^R, \odot^R, {}^{-^R}, {}^{\sim^R}, \mathbf{0} = 0, \mathbf{1} = u).$$

2') Let us consider a symbol $+\infty$ *distinct from the elements of* G*. Define* $G^+_{+\infty} \stackrel{def.}{=} G^+ \cup \{+\infty\}$ *and extend the operations* $\oplus$, $\to^R, \rightsquigarrow^R$ *from* G^+ *to* $G^+_{+\infty}$ *as follows:*

$$x \oplus_2 y \stackrel{def.}{=} \begin{cases} x \oplus y = x + y, & \text{if} \quad x, y \in G^+ \\ +\infty, & \text{if} \quad \text{otherwise.} \end{cases}$$

$$x \to^R_2 y \stackrel{def.}{=} \begin{cases} x \to^R y, & \text{if} \quad x, y \in G^+ \\ +\infty, & \text{if} \quad x \in G^+,\ y = +\infty \\ 0, & \text{if} \quad x = +\infty, \end{cases} \qquad x \rightsquigarrow^R_2 y \stackrel{def.}{=} \begin{cases} x \rightsquigarrow^R y, & \text{if} \quad x, y \in G^+ \\ +\infty, & \text{if} \quad x \in G^+,\ y = +\infty \\ 0, & \text{if} \quad x = +\infty. \end{cases}$$

We extend the lattice order relation $\leq$ *as follows: we put* $+\infty \geq x$*, for any* $x \in G^+_{+\infty}$*.*
Then, the algebra

$$\mathcal{G}^R_{m_2} = (G^+_{+\infty}, \vee, \wedge, \oplus_2, \mathbf{0} = 0, \mathbf{1} = +\infty)$$

is a bounded left-l-rim verifying properties $(pprel^d)$*,* $(pdiv^d)$ *and* $(pP1^d)$*,* $(pP2^d)$*, while the equivalent algebra*

$$\mathcal{G}^R_{m'_2} = (G^+_{+\infty}, \vee, \wedge, \oplus_2, \to^R_2, \rightsquigarrow^R_2, \mathbf{0} = 0, \mathbf{1} = +\infty)$$

is a right-pseudo-BL algebra verifying $(pP1^d)$*,* $(pP2^d)$*, i.e. is a* **right-pseudo-product algebra***.*

5.2.3 The analysis of two important examples

Example 1. The additive l-group $\Re_g = (\mathbf{R}, \vee, \wedge, +, -, 0)$

Note that the additive l-group $\Re_g$ is commutative and has 1 as strong unit.

By Theorem 4.13, the termwise equivalent l-implicative-group $\Re_{ig} = (\mathbf{R}, \vee, \wedge, \to, 0)$ has the implication $\to$ defined by: $x \to y \stackrel{def.}{=} y - x$, for all $x, y \in \mathbf{R}$.

• By Theorem 5.3,

(1) $\mathcal{R}^L = (R^-, \wedge, \vee, \to^L, \mathbf{1} = 0)$ is a distributive reversed left-BCK(P) lattice verifying properties (C) and (*), where:

$$x \to^L y \stackrel{def.}{=} (x \to y) \wedge 0 = \min(0, y - x);$$

(1') $\mathcal{R}^R = (R^+, \vee, \wedge, \rightarrow^R, \mathbf{0} = 0)$ is a distributive reversed right-BCK(S) lattice verifying properties (C^d) and ($*^d$), where:

$$x \rightarrow^R y \stackrel{def.}{=} (x \rightarrow y) \vee 0 = \max(0, y - x).$$

By Corollary 5.8, by taking $u' = -1$ and $u = 1$, we obtain:

(i) (1) $\mathbf{R}_1^L = ([-1, 0], \wedge, \vee, \rightarrow^L, \mathbf{0} = -1, \mathbf{1} = 0)$ is an equivalent definition of left-Wajsberg algebra,
(2) $\mathbf{R}_2^L = (\mathbf{R}_{-\infty}^-, \wedge, \vee, \rightarrow_2^L, \mathbf{0} = -\infty, \mathbf{1} = 0)$ is an equivalent definition of left-product algebra.

(i') (1') $\mathbf{R}_1^R = ([0, 1], \vee, \wedge, \rightarrow^R, \mathbf{0} = 0, \mathbf{1} = 1)$ is an equivalent definition of right-Wajsberg algebra,
(2') $\mathbf{R}_2^R = (\mathbf{R}_{+\infty}^+, \vee, \wedge, \rightarrow_2^R, \mathbf{0} = 0, \mathbf{1} = +\infty)$ is an equivalent definition of right-product algebra.

• By Theorem 5.10,

(1) $\mathcal{R}_{m'}^L = (R^-, \wedge, \vee, \odot, \rightarrow^L, \mathbf{1} = 0)$ is a distributive left-residuated lattice verifying properties (C) and (*), where:

$$x \odot y \stackrel{def.}{=} x + y, \quad x \rightarrow^L y \stackrel{def.}{=} (x \rightarrow y) \wedge 0 = \min(0, y - x);$$

(1') $\mathcal{R}_{m'}^R = (R^+, \vee, \wedge, \oplus, \rightarrow^R, \mathbf{0} = 0)$ is a distributive right-residuated lattice verifying properties (C^d) and ($*^d$), where:

$$x \oplus y \stackrel{def.}{=} x + y, \quad x \rightarrow^R y \stackrel{def.}{=} (x \rightarrow y) \vee 0 = \max(0, y - x).$$

By Corollary 5.12, for $u' = -1$ and $u = 1$:

(i) (1) $\mathbf{R}_{m_1'}^L = ([-1, 0], \odot^L, \oplus^L, ^-, \mathbf{0} = -1, \mathbf{1} = 0)$ is a left-MV algebra, where:

$$x \odot^L y \stackrel{def.}{=} (x+y) \vee (-1) = \max(-1, x+y), \quad x^- \stackrel{def.}{=} -1-x, \quad x \oplus^L y \stackrel{def.}{=} (x-(-1)+y) \wedge 0 = \min(0, x+y+1).$$

(2) $\mathbf{R}_{m_2'}^L = (\mathbf{R}_{-\infty}^-, \wedge, \vee, \odot_2, \rightarrow_2^L, \mathbf{0} = -\infty, \mathbf{1} = 0)$ is a left-product algebra.

(i') (1') $\mathbf{R}_{m_1'}^R = ([0, 1], \oplus^R, \odot^R, ^-, \mathbf{0} = 0, \mathbf{1} = 1)$ is **the standard (right-) MV algebra**, where:

$$x \oplus^R y \stackrel{def.}{=} (x + y) \wedge 1 = \min(1, x + y), \quad x^- \stackrel{def.}{=} 1 - x, \quad x \odot^R y \stackrel{def.}{=} (x - 1 + y) \vee 0 = \max(0, x - 1 + y).$$

(2') $\mathbf{R}_{m_2'}^R = (\mathbf{R}_{+\infty}^+, \vee, \wedge, \oplus_2, \rightarrow_2^R, \mathbf{0} = 0, \mathbf{1} = +\infty)$ is a right-product algebra.

Consequently, in the right-MV algebra $\mathbf{R}_{m_1'}^R = ([0, 1], \oplus^R, \odot^R, ^-, \mathbf{0} = 0, \mathbf{1} = 1)$, we have:
$x \oplus^R y = \min(1, x + y)$, that is the basical t-conorm, with the associated coresiduum: $x \rightarrow^R y = \max(0, y - x)$;
$x \odot^R y = \max(0, x - 1 + y)$, that is the derivately t-norm ($x \odot^R y = (x^- \oplus^R y^-)^-$), with the associated residuum: $x \rightsquigarrow^R y = (x^- \rightarrow^R y^-)^- = ((1 - x) \rightarrow^R (1 - y))^- = [\max(0, (1 - y) - (1 - x))]^- = 1 - \max(0, x - y) = 1 + \min(0, y - x) = \min(1, y - x + 1)$.

On, the other hand, in the left-MV algebra $\mathbf{R}_{m_1'}^L = ([-1, 0], \odot^L, \oplus^L, ^-, \mathbf{0} = -1, \mathbf{1} = 0)$ we have:
$x \odot^L y = \max(-1, x+y)$, that is the basical t-norm, with the associated residuum: $x \rightarrow^L y = \min(0, y-x)$;
$x \oplus^L y = \min(0, x + y + 1)$, that is the derivately t-conorm ($x \oplus^L y = (x^- \odot^L y^-)^-$), with the associated coresiduum: $x \rightsquigarrow^L y = (x^- \rightarrow^L y^-)^- = ((-1 - x) \rightarrow^L (-1 - y))^- = [\min(0, (-1 - y) - (-1 - x))]^- = [\min(0, -y + x)]^- = -1 - \min(0, x - y) = -1 + \max(0, y - x) = \max(-1, y - x - 1)$.

Note now that, if we make the following translation t of interval $[-1, 0]$ to interval $[0, 1]$: $t(x) = x' = x + 1$, then we obtain that:

• $z' = z + 1 = t(z) = t(x \odot^L y) = t(\max(-1, x + y)) = \max(-1, x + y) + 1 = \max(0, x + y + 1) = \max(0, (x + 1) + (y + 1) - 1) = \max(0, x' + y' - 1) = x' \odot^R y'$, i.e. $\odot^L$ from $[-1, 0]$ becomes $\odot^R$ from $[0, 1]$; **this is the Łukasiewicz t-norm**.

• $z' = z + 1 = t(z) = t(x \rightarrow^L y) = t(\min(0, y - x)) = \min(0, y - x) + 1 = \min(1, y - x + 1) = \min(1, (y + 1) - (x + 1) + 1) = \min(1, y' - x' + 1) = x' \rightsquigarrow^R y'$, i.e. $\rightarrow^L$ from $[-1, 0]$ becomes $\rightsquigarrow^R$ from $[0, 1]$; **this is the associated Łukasiewicz residuum**.

• $z' = z + 1 = t(z) = t(x \oplus^L y) = t(\min(0, x + y + 1)) = \min(0, x + y + 1) + 1 = \min(1, x + y + 2) = \min(1, (x + 1) + (y + 1)) = \min(1, x' + y') = x' \oplus^R y'$, i.e. $\oplus^L$ from $[-1, 0]$ becomes $\oplus^R$ from $[0, 1]$.
• $z' = z + 1 = t(z) = t(x \rightsquigarrow^L y) = t(\max(-1, y - x - 1) = \max(-1, y - x - 1) + 1 = \max(0, y - x) = \max(0, (y + 1) - (x + 1)) = \max(0, y' - x') = x' \to^R y'$, i.e. $\rightsquigarrow^L$ from $[-1, 0]$ becomes $\to^R$ from $[0, 1]$.

Example 2. The multiplicative l-group $\mathcal{D}_g = (D = (0, +\infty), \vee, \wedge, \cdot, ^{-1}, 1)$

By Theorem 4.13, the termwise equivalent l-implicative-group $\mathcal{D}_{ig} = (D, \vee, \wedge, \to, 1)$ has the implication $\to$ defined by: $x \to y \overset{def.}{=} y \cdot x^{-1} = \frac{y}{x}$, for all $x, y \in D$.
We can make the same analysis as for the first example. We point only that:

By Corollary 5.12,
(i) (1) $\mathcal{D}^L_{m'_1} = ([u', 1], \odot^L, \oplus^L, ^-, \mathbf{0} = u', \mathbf{1} = 1)$ is a left-MV algebra.
(2) $\mathcal{D}^L_{m'_2} = ([0, 1] = D^- \cup \{0\}, \wedge, \vee, \odot_2, \to^L_2, \mathbf{0} = 0, \mathbf{1} = 1)$ is **the standard (left-) product algebra**.
(i') (1') $\mathcal{D}^R_{m'_1} = ([1, u], \oplus^R, \odot^R, ^-, \mathbf{0} = 1, \mathbf{1} = u)$ is a right-MV algebra.
(2') $\mathcal{D}^R_{m'_2} = (D^+ \cup \{+\infty\}, \vee, \wedge, \oplus_2, \to^R_2, \mathbf{0} = 1, \mathbf{1} = +\infty)$ is a right-product algebra.

Open problem 5.13 We have thus obtained the standard MV algebra and the standard product algebra. Can we obtain in a similar way the standard Gödel algebra?

Acknowledgements

All my gratitude to Sergiu Rudeanu and Paul Flondor for their valuable and useful remarks and suggestions that helped me to improve the presentation.
I am specially grateful to George Georgescu for his encouragements in doing this research.

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