INSTITUTUL DE MATEMATICA
"SIMION STOILOW"
AL ACADEMIEI ROMANE

PREPRINT SERIES OF THE INSTITUTE OF MATHEMATICS
OF THE ROMANIAN ACADEMY

ISSN 0250 3638

The implicative-group
- a term equivalent definition of the group coming from algebras of logic -
Part II

by

Afrodita Iorgulescu

Preprint nr.12/2011

BUCURESTI

The implicative-group
- a term equivalent definition of the group
caming from algebras of logic -
Part II

by

Afrodita Iorgulescu

Preprint nr.11/2011

December, 2011

Afrodita Iorgulescu
E-mail: afrodita.iorgulescu@ase.ro

The implicative-group
- a term equivalent definition of the group coming from algebras of logic -
Part II

Afrodita Iorgulescu
E-mail: afrodita.iorgulescu@ase.ro

December 9, 2011

Abstract

In Part I, we introduce the implicative-group (the partially-ordered (lattice-ordered) implicative-group) as a term equivalent definition of the group (the partially-ordered (lattice-ordered) group, respectively); two intermediary term equivalent notions are also introduced. The lattice-ordered implicative-group is the great piece which missed of the puzzle showing the connections between lattice-ordered groups and some algebras of logic. We establish "horizontal" connections at group level and "vertical" connections between the group level and the algebras of logic level. We discuss about the filters (ideals) and the deductive systems of the involved notions.

In Part II, we study the normal filters/ideals and the compatible deductive systems, the representability of some of the involved algebras and we establish other "vertical" connections between the group level and the algebras of logic level. Finally, we introduce and study the implicative-states and the Bosbach-states on l-groups with strong unit.

Keywords po-group, po-implicative-group, l-group, l-implicative-group, pseudo-BCK algebra, partially-ordered integral monoid, pseudo-Wajsberg algebra, pseudo-MV algebra, filter, ideal, deductive system, representability, state, implicative-state, Bosbach-state

AMS classification (2000): 06F15, 06F35, 06D35

1 Introduction

Pseudo-MV algebras, the non-commutative generalizations of Chang's MV algebras, were introduced in 1999 [13] and developed in [14]. Pseudo-MV algebras are intervals [8] in l-groups and pseudo-Wajsberg algebras are term equivalent [3], [4] to pseudo-MV algebras. Hence, pseudo-Wajsberg algebras had to be connected to a notion that is term equivalent to the l-group. That notion is the great piece which missed of the puzzle showing the connections between algebras of logic and l-groups and was introduced in Part I [19]: is the l-implicative-group.

$$\begin{array}{ccc} \mathbf{l-implicative-group} & \Longleftrightarrow & \mathbf{l-groups} \\ \Updownarrow & & \Updownarrow \\ \mathbf{pseudo-Wajsberg\ algebras} & \Longleftrightarrow & \mathbf{pseudo-MV\ algebras} \end{array}$$

Note that usually in the literature the case of right-pseudo-MV algebras is considered, since in po-groups and l-groups the positive cone is usually considered. Note also that if we come from logic, where the truth is represented by 1, then we arive to consider the left-pseudo-MV algebras (the left-structures in general) and the negative cone. Therefore, in Part I and in this Part II, we deal with both cases: the left- and the right- structures.

In Part I also we studied the "horizonthal" connections at group-level and also "vertical" connections between the group-level and the algebras of logic level: pseudo-BCK algebras, pseudo-product algebras and pseudo-Wajsberg (pseudo-MV) algebras.

In this paper Part II, we continue to study "horizontal" connections at group-level and "vertical" connections between the group-level and the algebras of logic level. The paper is organized as follows: in Section 2, we recall some results from Part I.

In Section 3, we study normal filters/ideals and compatible deductive systems. Thus, first we introduce the notion of *compatible* deductive system of a po-implicative-group versus the old notion of *normal* convex po-subgroup of a po-group and prove the equivalence "*compatible* if and only if *normal*" (Theorem 3.3). Then, we mainly prove that the normality (compatibility) at l-group (l-implicative-group) G level is inherited by the algebras obtained by restricting the l-group (l-implicative-group) operations to G^- and to G^+, to $[u', 0]$ and to $[0, u]$, and finally to $G^-_{-\infty} = \{-\infty\} \cup G^-$ and to $G^+_{+\infty} = G^+ \cup \{+\infty\}$ and, also, that the equivalence "*compatible* if and only if *normal*", existing at l-group (l-implicative-group) level, is preserved by the algebras obtained by restricting the l-group (l-implicative-group) operations to G^- and to G^+, to $[u', 0]$ and to $[0, u]$, and finally to $G^-_{-\infty}$ and to $G^+_{+\infty}$ (Theorems 3.11, 3.18, 3.22). Other important results, at general level, are the Theorems 3.8, 3.13, 3.20 and the Corollary 3.23.

In Section 4, we study the representability. First, we find equivalent conditions for an l-implicative-group to be representable (Theorem 4.6). Then, we prove that representability at l-implicative-group G level is inherited by the algebras obtained by restricting the operations from G to G^- and to G^+ (Theorem 4.9). Another important result at this level is Theorem 4.10. The research here must be continued: connections between the representability at l-group (l-implicative-group) G level and the representability at $[u', 0] \subset G^-$, $[0, u] \subset G^+$ level and at $G^-_{-\infty}$, $G^+_{+\infty}$ level must be found.

In Section 5, we study the states. First, we define the distance functions d^L_1, d^L_2 and d^R_1, d^R_2 on an l-group and prove some properties (Proposition 5.2), following the ideas in the pseudo-BL algebras case. Then, we introduce the notions of *additive-state*, or *state* for short, on a po-group with strong unit and *implicative-state* on a po-implicative-group with strong unit and prove they coincide (Theorem 5.7). Next, we introduce the notions of *state morphism* on an l-group with strong unit and *implicative-state morphism* on an l-implicative-group with strong unit and prove they coincide, by the same Theorem 5.7. Finally, we introduce the notion of *Bosbach-state* on an l-group with strong unit, prove some properties and prove that any state is a Bosbach-state (Theorem 5.17), following the ideas from [12]. The research from this section must continue at least with the study of the restrictions of the various kinds of states from the l-group level to the G^-, G^+ level, the $[u', 0]$, $[0, u]$ level and the $G^-_{-\infty}$, $G^+_{+\infty}$ level.

2 Preliminaries

Recall first the following notations (where ^{d} means "dual"):

(pP) (pseudo-product) $\exists\ x \odot y \overset{notation}{=} \min\{z \mid x \le y \to^L z\} = \min\{z \mid y \le x \rightsquigarrow^L z\}$,

(pS) (pseudo-sum) $\exists\ x \oplus y \overset{notation}{=} \max\{z \mid x \ge y \to^R z\} = \max\{z \mid y \ge x \rightsquigarrow^R z\}$,

(pR) (pseudo-residuum) $\exists\ y \to^L z \overset{notation}{=} \max\{x \mid x \odot y \le z\}$, $x \rightsquigarrow^L z \overset{notation}{=} \max\{y \mid x \odot y \le z\}$,

(pcoR) (pseudo-coresiduum) $\exists\ y \to^R z \overset{notation}{=} \min\{x \mid x \oplus y \ge z\}$, $x \rightsquigarrow^R z \overset{notation}{=} \min\{y \mid x \oplus y \ge z\}$,

(pPR)=(pRP) $x \odot y \le z \iff x \le y \to^L z \iff y \le x \rightsquigarrow^L z$,

(pScoR)=(pcorRS) $x \oplus y \ge z \iff x \ge y \to^R z \iff y \ge x \rightsquigarrow^R z$,

(pDN) (pseudo-Double Negation) $(x^\sim)^- = x = (x^-)^\sim$,

(pC) $x \vee y = (x \rightsquigarrow^L y) \to^L y = (x \to^L y) \rightsquigarrow^L y$,

(pC^d) $x \wedge y = (x \to^R y) \rightsquigarrow^R y = (x \rightsquigarrow^R y) \to^R y$;

(pC) implies (pprel) and (pdiv),

(pC^d) implies (pprel^d) and (pdiv^d);

(pprel) (pseudo-prelinearity) $(x \to^L y) \vee (y \to^L x) = 1 = (x \rightsquigarrow^L y) \vee (y \rightsquigarrow^L x)$,

(pdiv) (pseudo-divisibility) $x \wedge y = (x \to^L y) \odot x = x \odot (x \rightsquigarrow^L y)$,

(pprel^d) $(x \to^R y) \wedge (y \to^R x) = 0 = (x \rightsquigarrow^R y) \wedge (y \rightsquigarrow^R x)$,

(pdiv^d) $x \vee y = (x \to^R y) \oplus x = x \oplus (x \rightsquigarrow^R y)$;

(*) $(x \odot z) \to^L (y \odot z) = x \to^L y$, $\quad (z \odot x) \rightsquigarrow^L (z \odot y) = x \rightsquigarrow^L y$,

($*^d$) $(x \oplus z) \to^R (y \oplus z) = x \to^R y, \quad (z \oplus x) \rightsquigarrow^R (z \oplus y) = x \rightsquigarrow^R y$;
(pP1) $x \wedge x^{-^L} = 0 = x \wedge x^{\sim^L}$,
(pP2) $(z^{-^L})^{-^L} \odot [(x \odot z) \to^L (y \odot z)] \leq x \to^L y$, $(z^{\sim^L})^{\sim^L} \odot [(z \odot x) \rightsquigarrow^L (z \odot y)] \leq x \rightsquigarrow^L y$,
(pP1^d) $x \vee x^{-^R} = 1 = x \vee x^{\sim^R}$,
(pP2^d) $(z^{-^R})^{-^R} \oplus [(x \oplus z) \to^R (y \oplus z)] \geq x \to^R y$, $(z^{\sim^R})^{\sim^R} \oplus [(z \oplus x) \rightsquigarrow^R (z \oplus y)] \geq x \rightsquigarrow^R y$.

Recall [16] that the left-pseudo-BCK(pP) algebras (denote one by $\mathcal{A}^L = (A^L, \leq, \to^L, \rightsquigarrow^L, 1)$, where the pseudo-product is $\odot$) are categorically equivalent to the left-porims (= partially-ordered residuated integral left-monoids) (denote the corresponding one by $\mathcal{A}^L_m = (A^L, \leq, \odot, 1)$, where the pseudo-residuum is $(\to^L, \rightsquigarrow^L)$), and, dually, the right-pseudo-BCK(pS) algebras ($\mathcal{A}^R = (A^R, \leq, \to^R, \rightsquigarrow^R, 0)$, where the pseudo-sum is $\oplus$) are categorically equivalent to the right-porims (= partially-ordered residuated integral right-monoids) ($\mathcal{A}^R_m = (A^R, \leq, \oplus, 0)$, where the pseudo-coresiduum is $(\to^R, \rightsquigarrow^R)$). We write for short:

(a) left-pseudo-BCK(pP) algebras $\Longleftrightarrow$ left-porims and dually
(a') right-pseudo-BCK(pS) algebras $\Longleftrightarrow$ right-porims.

Recall also that the $(\to^L, \rightsquigarrow^L)$-deductive systems of $\mathcal{A}^L$ (the $(\to^R, \rightsquigarrow^R)$-deductive systems of $\mathcal{A}^R$) coincide with the filters of the equivalent $\mathcal{A}^L_m$ (the ideals of the equivalent $\mathcal{A}^R_m$, respectively) [19], Theorem 2.16.

Remark 2.1 Thus, we can roughly speak about the $(\to^L, \rightsquigarrow^L)$-deductive systems and the filters of $\mathcal{A}^L$ (or of $\mathcal{A}^L_m$) and say that they coincide; dually, we can roughly speak about the $(\to^R, \rightsquigarrow^R)$-deductive systems and the ideals of $\mathcal{A}^R$ (or of $\mathcal{A}^R_m$) and say that they coincide.

Moreover, we have:
(b) left-pseudo-BCK(pP) lattices $\Longleftrightarrow$ left-l-rims (= lattice-ordered residuated integral left-monoids) and
(b') right-pseudo-BCK(pS) lattices $\Longleftrightarrow$ right-l-rims.
It follows, on one hand, that
(c) left-pseudo-BCK(pP) lattices verifying condition (pC) $\Longleftrightarrow$ left-l-rims verifying condition (pC) and
(c') right-pseudo-BCK(pS) lattices verifying condition (pC^d) $\Longleftrightarrow$ right-l-rims verifying condition (pC^d), hence
(d) bounded left-pseudo-BCK(pP) lattices verifying (pC) $\Longleftrightarrow$ bounded left-l-rims verifying (pC) and
(d') bounded right-pseudo-BCK(pS) lattices verifying (pC^d) $\Longleftrightarrow$ bounded right-l-rims verifying (pC^d).

But, recall also ([18], Theorem 10.2.16) that:
the bounded left-pseudo-BCK(pP) lattice verifying (pC) is an equivalent definition of the left-pseudo-Wajsberg algebra, while the bounded left-l-rim verifying (pC) is an equivalent definition of the left-pseudo-MV algebra; and left-pseudo-Wajsberg algebras are term equivalent to left-pseudo-MV algebras. Dually,
the bounded right-pseudo-BCK(pS) lattice verifying (pC^d) is an equivalent definition of the right-pseudo-Wajsberg algebra, while the bounded right-l-rim verifying (pC^d) is an equivalent definition of the right-pseudo-MV algebra; and right-pseudo-Wajsberg algebras are term equivalent to right-pseudo-MV algebras.

On the other hand, it follows that:
(e) bounded left-pseudo-BCK(pP) lattices verifying (prel),(pdiv) (= left-pseudo-Hajék(pP) algebras) $\Longleftrightarrow$ bounded left-l-rims verifying (prel), (pdiv) ($\equiv$ left-pseudo-BL algebras) and
(e') bounded right-pseudo-BCK(pS) lattices verifying (prel^d), (pdiv^d) (= right-pseudo-Hajék(pS) algebras) $\Longleftrightarrow$ bounded right-l-rims verifying (prel^d), (pdiv^d) ($\equiv$ right-pseudo-BL algebras), hence
(f) left-pseudo-Hájek(pP) algebras verifying (pP1), (pP2) $\Longleftrightarrow$ bounded left-l-rims verifying (pprel), (pdiv) and (pP1), (pP2) ($\equiv$ left-pseudo-BL algebras verifying (pP1), (pP2) = left-pseudo-product algebras) and
(f') right-pseudo-Hájek(pS) algebras verifying (pP1^d), (pP2^d) $\Longleftrightarrow$ bounded right-l-rims verifying (pprel^d), (pdiv^d) and (pP1^d), (pP2^d) ($\equiv$ right-pseudo-BL algebras verifying (pP1^d), (pP2^d) = right-pseudo-product algebras).

Consequently, in all these algebras also, the $(\to^L, \rightsquigarrow^L)$-deductive systems (the $(\to^R, \rightsquigarrow^R)$-deductive systems) coincide with the filters (ideals, respectively).

We shall not recall definitions and other properties of algebras of logic, because of lack of space. In this paper, note that "reversed" pseudo-BCK algebras will be simply called "pseudo-BCK algebras", left-ones or right-ones.

We now recall from Part I some of the necessary results needed in the sequel concerning the (implicative-) groups.

2.1 Groups, po-groups, l-groups

• Let $\mathcal{G} = (G, +, -, 0)$ be a group, in additive notation in this paper. We introduced the new operations $\rightarrow$ and $\rightsquigarrow$ on G, called "implications", defined by: for all $x, y \in G$,

$$x \rightarrow y \stackrel{def.}{=} -[x + (-y)] = y + (-x), \quad x \rightsquigarrow y \stackrel{def.}{=} -[(-y) + x] = (-x) + y. \tag{1}$$

The two implications satisfy the following properties: for all $x, y, z \in G$,

$$x + y = -(x \rightarrow (-y)) = (-y) \rightarrow x, \quad x + y = -(y \rightsquigarrow (-x)) = (-x) \rightsquigarrow y, \tag{2}$$

$$y \rightarrow z = (z \rightarrow x) \rightsquigarrow (y \rightarrow x), \quad y \rightsquigarrow z = (z \rightsquigarrow x) \rightarrow (y \rightsquigarrow x), \tag{3}$$

$$(y \rightarrow x) \rightsquigarrow x = y = (y \rightsquigarrow x) \rightarrow x, \tag{4}$$

$$-x = x \rightarrow 0 = x \rightsquigarrow 0, \tag{5}$$

$$x = y \Longleftrightarrow x \rightarrow y = 0 \Longleftrightarrow x \rightsquigarrow y = 0, \tag{6}$$

$$x + y = z \Longleftrightarrow x = y \rightarrow z \Longleftrightarrow y = x \rightsquigarrow z \quad (see\ [11],\ page\ 160), \tag{7}$$

$$x = y \Longleftrightarrow -y = -x, \tag{8}$$

$$0 \rightarrow x = x = 0 \rightsquigarrow x, \tag{9}$$

$$z \rightsquigarrow (y \rightarrow x) = y \rightarrow (z \rightsquigarrow x), \tag{10}$$

$$z \rightarrow x = (y \rightarrow z) \rightarrow (y \rightarrow x), \quad z \rightsquigarrow x = (y \rightsquigarrow z) \rightsquigarrow (y \rightsquigarrow x), \tag{11}$$

$$x \rightarrow x = 0 = x \rightsquigarrow x, \tag{12}$$

$$x \rightsquigarrow (-y) = y \rightarrow (-x), \tag{13}$$

$$-(x \rightarrow 0) = x = -(x \rightsquigarrow 0), \tag{14}$$

$$[(y \rightarrow x) \rightsquigarrow x] \rightarrow x = y \rightarrow x, \quad [(y \rightsquigarrow x) \rightarrow x] \rightsquigarrow x = y \rightsquigarrow x, \tag{15}$$

$$x \rightarrow (y \rightarrow z) = (x + y) \rightarrow z, \quad x \rightsquigarrow (y \rightsquigarrow z) = (y + x) \rightsquigarrow z, \tag{16}$$

$$x \rightsquigarrow y = (-y) \rightarrow (-x), \quad x \rightarrow y = (-y) \rightsquigarrow (-x), \tag{17}$$

$$(-x) \rightsquigarrow y = (-y) \rightarrow x, \tag{18}$$

$$(x \rightarrow y) + x = y = x + (x \rightsquigarrow y), \tag{19}$$

$$x \rightarrow (y + x) = y = x \rightsquigarrow (x + y), \tag{20}$$

$$x \rightarrow y = (x + z) \rightarrow (y + z), \quad x \rightsquigarrow y = (z + x) \rightsquigarrow (z + y), \tag{21}$$

$$(y + x) \rightarrow x = -y = (x + y) \rightsquigarrow x, \tag{22}$$

$$y \rightarrow (x \rightarrow (y + x)) = 0 = y \rightsquigarrow (x \rightsquigarrow (x + y). \tag{23}$$

Proposition 2.2 *In a group $\mathcal{G}$, the following properties hold: for all $x, y, z, x_1, x_2, \ldots, x_n \in G$ ($n \geq 2$),*
(a) $(x \rightsquigarrow y) + (y \rightsquigarrow z) = x \rightsquigarrow z$,
(a') $(y \rightarrow z) + (x \rightarrow y) = x \rightarrow z$,
(b) $(x_1 \rightsquigarrow x_2) + (x_2 \rightsquigarrow x_3) + \ldots + (x_{n-1} \rightsquigarrow x_n) = x_1 \rightsquigarrow x_n$,
(b') $(x_{n-1} \rightarrow x_n) + \ldots + (x_2 \rightarrow x_3) + (x_1 \rightarrow x_2) = x_1 \rightarrow x_n$.

• Let now $\mathcal{G} = (G, \leq, +, -, 0)$ be a partially-ordered group or a po-group for short. Then the following properties hold: for all $x, y, z \in G$,

$$(i) \quad x + y \leq z \Leftrightarrow x \leq y \to z \Leftrightarrow y \leq x \rightsquigarrow z, \qquad \textit{and dually} \tag{24}$$

$$(ii) \quad x + y \geq z \Leftrightarrow x \geq y \to z \Leftrightarrow y \geq x \rightsquigarrow z,$$

$$x \leq y \implies z \to x \leq z \to y \text{ and } z \rightsquigarrow x \leq z \rightsquigarrow y, \tag{25}$$

$$x \leq y \implies y \to z \leq x \to z \text{ and } y \rightsquigarrow z \leq x \rightsquigarrow z. \tag{26}$$

Corollary 2.3 *Let $\mathcal{G}$ be a po-group. For all $x, y \in G$, if $x \leq y$ then:*
$x \to y \geq 0$, $x \rightsquigarrow y \geq 0$ *and* $y \to x \leq 0$, $y \rightsquigarrow x \leq 0$.

• Let finally $\mathcal{G} = (G, \vee, \wedge, +, -, 0)$ be a lattice-ordered group or an l-group for short. Then we have, for all $x, y, a, b \in G$:
(G8) $a + (x \vee y) + b = (a + x + b) \vee (a + y + b)$ and dually
(G9) $a + (x \wedge y) + b = (a + x + b) \wedge (a + y + b)$;
(G10) $-(x \vee y) = (-x) \wedge (-y)$ and dually
(G11) $-(x \wedge y) = (-x) \vee (-y)$;
(G12) $x \vee y = x - (x \wedge y) + y, \quad x \wedge y = x - (x \vee y) + y$,
(G13) The lattice $(G, \vee, \wedge)$ is distributive;

$$(x \vee z) \to y = (x \to y) \wedge (z \to y), \quad (x \vee z) \rightsquigarrow y = (x \rightsquigarrow y) \wedge (z \rightsquigarrow y) \quad \textit{and dually} \tag{27}$$

$$(x \wedge z) \to y = (x \to y) \vee (z \to y), \quad (x \wedge z) \rightsquigarrow y = (x \rightsquigarrow y) \vee (z \rightsquigarrow y); \tag{28}$$

$$y \to (x \vee z) = (y \to x) \vee (y \to z), \quad y \rightsquigarrow (x \vee z) = (y \rightsquigarrow x) \vee (y \rightsquigarrow z) \quad \textit{and dually} \tag{29}$$

$$y \to (x \wedge z) = (y \to x) \wedge (y \to z), \quad y \rightsquigarrow (x \wedge z) = (y \rightsquigarrow x) \wedge (y \rightsquigarrow z); \tag{30}$$

$$[(x \wedge 0) \rightsquigarrow 0] \wedge 0 = 0, \quad [(x \wedge 0) \to 0] \wedge 0 = 0 \quad \textit{and dually} \tag{31}$$

$$[(x \vee 0) \rightsquigarrow 0] \vee 0 = 0, \quad [(x \vee 0) \to 0] \vee 0 = 0; \tag{32}$$

$$(x \vee y) \to (x \wedge y) = (x \to y) \wedge (y \to x) \wedge 0 \leq 0, \quad (x \vee y) \rightsquigarrow (x \wedge y) = (x \rightsquigarrow y) \wedge (y \rightsquigarrow x) \wedge 0 \leq 0 \tag{33}$$

and dually

$$(x \wedge y) \to (x \vee y) = (x \to y) \vee (y \to x) \vee 0 \geq 0, \quad (x \wedge y) \rightsquigarrow (x \vee y) = (x \rightsquigarrow y) \vee (y \rightsquigarrow x) \vee 0 \geq 0; \tag{34}$$

$$x \to (x \wedge y) = 0 \wedge (x \to y), \quad x \rightsquigarrow (x \wedge y) = 0 \wedge (x \rightsquigarrow y), \tag{35}$$

$$(x \wedge y) \to x = 0 \vee (y \to x), \quad (x \wedge y) \rightsquigarrow x = 0 \vee (y \rightsquigarrow x). \tag{36}$$

2.2 Implicative-groups, po-implicative-groups, l-implicative-groups

• An *implicative-group* ([19], Definition 4.1) is an algebra $\mathcal{G} = (G, \to, \rightsquigarrow, 0)$ of type $(2, 2, 0)$ such that the following axioms hold: for all $x, y, z \in G$,
(I1) $y \to z = (z \to x) \rightsquigarrow (y \to x), \quad y \rightsquigarrow z = (z \rightsquigarrow x) \to (y \rightsquigarrow x)$,
(I2) $y = (y \to x) \rightsquigarrow x, \quad y = (y \rightsquigarrow x) \to x$,
(I3) $x = y \iff x \to y = 0 \iff x \rightsquigarrow y = 0$,
(I4) $x \to 0 = x \rightsquigarrow 0$.

The implicative-group is said to be *commutative or abelian* if $x \to y = x \rightsquigarrow y$, for all $x, y \in G$.

Let $\mathcal{G}$ be an implicative-group. Then, we have, for all $x, y, z \in G$:
(I7) $0 \to x = x = 0 \rightsquigarrow x$,
(I8) $z \rightsquigarrow (y \to x) = y \to (z \rightsquigarrow x)$,
(I9) $x \to x = 0 = x \rightsquigarrow x$,

$$z \to x = (y \to z) \to (y \to x), \quad z \rightsquigarrow x = (y \rightsquigarrow z) \rightsquigarrow (y \rightsquigarrow x). \tag{37}$$

An **equivalent definition of the implicative-group** is the following: an *implicative-group* is an algebra $\mathcal{G} = (G, \to, \rightsquigarrow, 0)$ of type $(2, 2, 0)$ such that (I1), (I7), (I3), (I4) hold.

The groups and the implicative-groups are termwise equivalent:

Theorem 2.4 *([19], Theorem 4.13)*

(1) Let $\mathcal{G} = (G, +, -, 0)$ be a group. Define $\Phi(\mathcal{G}) = (G, \to, \rightsquigarrow, 0)$ by: for all $x, y \in G$,

$$x \to y \stackrel{def.}{=} -(x + (-y)) = -(x - y) = y - x,$$

$$x \rightsquigarrow y \stackrel{def.}{=} -((-y) + x) = -(-y + x) = -x + y.$$

Then $\Phi(\mathcal{G})$ is an implicative-group.

(1') Conversely, let $\mathcal{G} = (G, \to, \rightsquigarrow, 0)$ be an implicative-group. Define $\Psi(\mathcal{G}) = (G, +, -, 0)$ by: for all $x, y \in G$,

$$-x \stackrel{def.}{=} x \to 0 \stackrel{(I4)}{=} x \rightsquigarrow 0,$$

$$x + y \stackrel{def.}{=} -(x \to (-y)) \stackrel{(2)}{=} -(y \rightsquigarrow (-x)).$$

Then $\Psi(\mathcal{G})$ is a group.

(2) The maps Φ and Ψ are mutually inverse.

The implicative-group is commutative if and only if the term equivalent group is commutative, i.e. $x \to y = x \rightsquigarrow y$ for all x, y if and only if $x + y = y + x$ for all x, y.

• A *partially-ordered implicative-group* or a *po-implicative-group* for short ([19], Definition 4.17) is a structure $\mathcal{G} = (G, \leq, \to, \rightsquigarrow, 0)$, where $(G, \to, \rightsquigarrow, 0)$ is an implicative-group and $\leq$ is a partial order on G compatible with $\to$, $\rightsquigarrow$, i.e. we have: for all $x, y, z \in G$,
(I5) $x \leq y$ implies $z \to x \leq z \to y$ and $z \rightsquigarrow x \leq z \rightsquigarrow y$.

The po-groups and the po-implicative-groups are termwise equivalent ([19], Theorem 4.23).

Let $\mathcal{G}$ be a po-implicative-group. A *deductive system* of $\mathcal{G}$ ([19], Definition 4.27) is a subset $S \subseteq G$ which satisfies: for all $x, y, a, b \in G$,
(DS1) $0 \in S$,
(pDS2)(a) $x \in S$, $x \to y \in S$ imply $y \in S$ (or $x \in S$, $x \rightsquigarrow y \in S$ imply $y \in S$); (b) $x \in S$ implies $x \to 0 = x \rightsquigarrow 0 \in S$,
(pDS3) $a, b \in S$ and $a \leq x \leq b$ imply $x \in S$.

Theorem 2.5 *([19], Theorem 4.28)*

Let $\mathcal{G}_g = (G, \leq, +, -, 0)$ be a po-group and let $\mathcal{G}_{ig} = (G, \leq, \to, \rightsquigarrow, 0)$ be the term equivalent po-implicative-group. Then, the convex po-subgroups of $\mathcal{G}_g$ coincide with the deductive systems of $\mathcal{G}_{ig}$.

Remark 2.6 Thus, we can roughly speak about the convex po-subgroups and the deductive systems of $\mathcal{G}_g$ (or of $\mathcal{G}_{ig}$) and say that they coincide.

• If the partial order relation $\leq$ is a lattice order relation, with the lattice operations $\wedge$ and $\vee$ defined by: $x \leq y \Leftrightarrow x \wedge y = x \Leftrightarrow x \vee y = y$, then $\mathcal{G}$ is a *lattice-ordered implicative-group* or an *l-implicative-group* for short, denoted $\mathcal{G} = (G, \vee, \wedge, \to, \rightsquigarrow, 0)$.

The l-groups and the l-implicative-groups are termwise equivalent ([19], Corollary 4.31).

2.3 "Vertical" connections (between group level and algebras of logic level)

• At partial order level, we have:

Theorem 2.7 *([19], Theorem 5.1) Let $\mathcal{G} = (G, \leq, +, -, 0)$ be a po-group. Then*

(1) $\mathcal{G}^- = (G^-, \leq, \odot = +, \mathbf{1} = 0)$ is a left-poim (= partially-ordered, integral left-monoid);

(1') $\mathcal{G}^+ = (G^+, \leq, \oplus = +, \mathbf{0} = 0)$ is a right-poim (= partially-ordered, integral right-monoid).

Theorem 2.8 *([19], Theorem 5.2) Let $\mathcal{G} = (G, \leq, +, -, 0)$ be a po-group and S be a convex po-subgroup of $\mathcal{G}$. Then,*

(1) $S^L = S \cap G^-$ is a filter of the left-poim $\mathcal{G}^- = (G^-, \leq, \odot = +, \mathbf{1} = 0)$ from Theorem 2.7 (1).

(1') $S^R = S \cap G^+$ is an ideal of the right-poim $\mathcal{G}^+ = (G^+, \leq, \oplus = +, \mathbf{0} = 0)$ from Theorem 2.7 (1').

Recall from part I that, by above theorem, **a better name for a "convex po-subgroup" should be that of "filter-ideal"**.

- At lattice order level, we have:

Theorem 2.9 *([19], Theorem 5.3) Let $\mathcal{G} = (G, \vee, \wedge, \rightarrow, \rightsquigarrow, 0)$ be an l-implicative-group.*
(1) Define, for all $x, y \in G^-$:

$$x \rightarrow^L y \stackrel{def.}{=} (x \rightarrow y) \wedge 0, \quad x \rightsquigarrow^L y \stackrel{def.}{=} (x \rightsquigarrow y) \wedge 0. \tag{38}$$

Then, $\mathcal{G}^L = (G^-, \wedge, \vee, \rightarrow^L, \rightsquigarrow^L, \mathbf{1} = 0)$ is a distributive left-pseudo-BCK(pP) lattice (with the pseudo-product $\odot = +$) verifying conditions (pC) and (*) (see condition (pP2) from the definition of a (left-) pseudo-product algebra and (21)).
(1') Define, for all $x, y \in G^+$:

$$x \rightarrow^R y \stackrel{def.}{=} (x \rightarrow y) \vee 0, \quad x \rightsquigarrow^R y \stackrel{def.}{=} (x \rightsquigarrow y) \vee 0. \tag{39}$$

Then, $\mathcal{G}^R = (G^+, \vee, \wedge, \rightarrow^R, \rightsquigarrow^R, \mathbf{0} = 0)$ is a distributive right-pseudo-BCK(pS) lattice (with the pseudo-sum $\oplus = +$) verifying the dual conditions (pC^d) and ($*^d$).

Theorem 2.10 *([19], Theorem 5.9) Let $\mathcal{G} = (G, \vee, \wedge, \rightarrow, \rightsquigarrow, 0)$ be an l-implicative-group and S be a deductive system of $\mathcal{G}$. Then,*
(1) $S^L = S \cap G^-$ is a $(\rightarrow^L, \rightsquigarrow^L)$-deductive system of $\mathcal{G}^L = (G^-, \wedge, \vee, \rightarrow^L, \rightsquigarrow^L, \mathbf{1} = 0)$ from Theorem 2.9 (1).
(1') $S^R = S \cap G^+$ is a $(\rightarrow^R, \rightsquigarrow^R)$-deductive system of $\mathcal{G}^R = (G^+, \vee, \wedge, \rightarrow^R, \rightsquigarrow^R, \mathbf{0} = 0)$ from Theorem 2.9 (1').

In Part I, we "bounded" the algebras $\mathcal{G}^L$ and $\mathcal{G}^R$ from Theorem 2.9 in two different ways: first, with an "internal" element, second, with an "external" element; we obtained the equivalent of known results, as follows:

Corollary 2.11 *([19], Corollary 5.8)*
(i) Let $\mathcal{G}^L = (G^-, \wedge, \vee, \rightarrow^L, \rightsquigarrow^L, \mathbf{1} = 0)$ from Theorem 2.9 (1). Let us "bound" this algebra in two different ways:
1) Let us take $u' < 0$ from G^- and consider the interval $[u', 0]$. Then the algebra

$$\mathcal{G}_1^L = ([u', 0], \wedge, \vee, \rightarrow^L, \rightsquigarrow^L, \mathbf{0} = u', \mathbf{1} = 0)$$

is a bounded left-pseudo-BCK(pP) lattice (with the pseudo-product $x \odot^L y \stackrel{def.}{=} (x \odot y) \vee u' = (x + y) \vee u'$) with condition (pC), i.e. is an equivalent definition of the **left-pseudo-Wajsberg algebra** *(see [17] and [21], [10] for the commutative case)*

$$\mathcal{G}_{1'}^L = ([u', 0], \rightarrow^L, \rightsquigarrow^L, {}^{-^L}, {}^{\sim^L}, \mathbf{0} = u', \mathbf{1} = 0).$$

2) Let us consider a symbol $-\infty$ distinct from the elements of G. Define $G^-_{-\infty} \stackrel{def.}{=} \{-\infty\} \cup G^-$ and extend the operations $\rightarrow^L, \rightsquigarrow^L, \odot$ from G^- to $G^-_{-\infty}$ as follows:

$$x \rightarrow_2^L y \stackrel{def.}{=} \begin{cases} x \rightarrow^L y, & if \quad x, y \in G^- \\ -\infty, & if \quad x \in G^-,\ y = -\infty \\ 0, & if \quad x = -\infty, \end{cases} \qquad x \rightsquigarrow_2^L y \stackrel{def.}{=} \begin{cases} x \rightsquigarrow^L y, & if \quad x, y \in G^- \\ -\infty, & if \quad x \in G^-,\ y = -\infty \\ 0, & if \quad x = -\infty, \end{cases}$$

$$x \odot_2 y \stackrel{def.}{=} \begin{cases} x \odot y = x + y, & if \quad x, y \in G^- \\ -\infty, & if \quad otherwise. \end{cases}$$

We extend the lattice order relation $\leq$ as follows: we put $-\infty \leq x$, for any $x \in G^-_{-\infty}$. Then, the algebra

$$\mathcal{G}_2^L = (G^-_{-\infty}, \wedge, \vee, \rightarrow_2^L, \rightsquigarrow_2^L, \mathbf{0} = -\infty, \mathbf{1} = 0)$$

is a **left-pseudo-Hájek(pP) algebra (with the pseudo-product $\odot_2$) verifying conditions (pP1) and (pP2)** *(see [17]).*

(i') Let $\mathcal{G}^R = (G^+, \vee, \wedge, \to^R, \rightsquigarrow^R, \mathbf{0} = 0)$ from Theorem 2.9 (1'). Let us "bound" this algebra in two different ways:

1') Let us take $u > 0$ from G^+ and consider the interval $[0, u] = \{x \in G^+ \mid 0 \leq x \leq u\}$. Then the algebra

$$\mathcal{G}_1^R = ([0, u], \vee, \wedge, \to^R, \rightsquigarrow^R, \mathbf{0} = 0, \mathbf{1} = u)$$

is a bounded right-pseudo-BCK(pS) lattice (with the pseudo-sum $x \oplus^R y \stackrel{def.}{=} (x \oplus y) \wedge u = (x + y) \wedge u$) with condition ($pC^d$), i.e. is an equivalent definition of the **right-pseudo-Wajsberg algebra**

$$\mathcal{G}_{1'}^R = ([0, u], \to^R, \rightsquigarrow^R, {}^{-^R}, {}^{\sim^R}, \mathbf{0} = 0, \mathbf{1} = u).$$

2') Let us consider a symbol $+\infty$ distinct from the elements of G. Define $G^+_{+\infty} \stackrel{def.}{=} G^+ \cup \{+\infty\}$ and extend the operations $\to^R, \rightsquigarrow^R$, $\oplus$ from G^+ to $G^+_{+\infty}$ as follows:

$$x \to_2^R y \stackrel{def.}{=} \begin{cases} x \to^R y, & if \quad x, y \in G^+ \\ +\infty, & if \quad x \in G^+,\ y = +\infty \\ 0, & if \quad x = +\infty, \end{cases} \qquad x \rightsquigarrow_2^R y \stackrel{def.}{=} \begin{cases} x \rightsquigarrow^R, & if \quad x, y \in G^+ \\ +\infty, & if \quad x \in G^+,\ y = +\infty \\ 0, & if \quad x = +\infty, \end{cases}$$

$$x \oplus_2 y \stackrel{def.}{=} \begin{cases} x \oplus y = x + y, & if \quad x, y \in G^+ \\ +\infty, & if \quad otherwise. \end{cases}$$

We extend the lattice order relation $\geq$ as follows: we put $+\infty \geq x$, for any $x \in G^+_{+\infty}$. Then, the algebra

$$\mathcal{G}_2^R = (G^+_{+\infty}, \vee, \wedge, \to_2^R, \rightsquigarrow_2^R, \mathbf{0} = 0, \mathbf{1} = +\infty)$$

is a **right-pseudo-Hájek(pS) algebra (with the pseudo-sum $\oplus_2$) verifying the dual conditions (pP1^d) and (pP2^d)**.

Note that in Part I, Theorem 2.9 was formulated equivalently ([19], Theorem 5.10) for l-groups and (1) left-l-rims $\mathcal{G}_m^L$ (equivalent to noncommutative left-residuated lattices) and (1') right-l-rims $\mathcal{G}_m^R$. Also in Part I, the above corollary was formulated equivalently ([19], Corollary 5.12) for: (i) left-l-rims, left-pseudo-MV algebras $\mathcal{G}_{m1'}^L = ([u', 0], \odot^L, \oplus^L, {}^{-^L}, {}^{\sim^L}, \mathbf{0} = u', \mathbf{1} = 0)$, where, for every $x, y \in [u', 0]$:

$$x \odot^L y \stackrel{def.}{=} (x \odot y) \vee u' = (x + y) \vee u', \quad x \oplus^L y \stackrel{def.}{=} (x - u' + y) \wedge 0, \quad x^{-^L} \stackrel{def.}{=} u' - x, \quad x^{\sim^L} \stackrel{def.}{=} -x + u',$$

and left-pseudo-product algebras and, dually, for: (i') right-l-rims, right-pseudo-MV algebras $\mathcal{G}_{m1'}^R = ([0, u], \oplus^R, \odot^R, {}^{-^R}, {}^{\sim^R}, \mathbf{0} = 0, \mathbf{1} = u)$, where, for every $x, y \in [0, u]$:

$$x \oplus^R y \stackrel{def.}{=} (x \oplus y) \wedge u = (x + y) \wedge u, \quad x \odot^R y \stackrel{def.}{=} (x - u + y) \vee 0, \quad x^{-^R} \stackrel{def.}{=} u - x, \quad x^{\sim^R} \stackrel{def.}{=} -x + u$$

and right-pseudo-product algebras.

Recall also from Part I that the left-pseudo-MV algebra $\mathcal{G}_{m1'}^L$ is term equivalent to the right-pseudo-MV algebra $\mathcal{G}_{m1'}^{LR} = ([u', 0], \oplus^L, \odot^L, {}^{-^L}, {}^{\sim^L}, \mathbf{0} = u', \mathbf{1} = 0)$ and dually, the right-pseudo-MV algebra $\mathcal{G}_{m1'}^R$ is term equivalent to the left-pseudo-MV algebra $\mathcal{G}_{m1'}^{RL} = ([0, u], \odot^R, \oplus^R, {}^{-^R}, {}^{\sim^R}, \mathbf{0} = 0, \mathbf{1} = u)$.

3 Normal filters/ideals and compatible deductive systems

3.1 po-groups (po-implicative groups) and associated algebras on G^-, G^+

Recall the following definition:

Definition 3.1 Let $\mathcal{G}_g = (G, \leq, +, -, 0)$ be a po-group. A convex po-subgroup S of $\mathcal{G}_g$ is *normal* if the following condition (N_g) holds:

$$(N_g) \qquad for\ any\ g \in G, \;\; S + g = g + S.$$

Recall that $S + g = g + S$ means:
(i) for each $h \in S$, there exists $h' \in S$ such that $h + g = g + h'$;
(ii) for each $h' \in S$, there exists $h \in S$ such that $g + h' = h + g$.

We introduce now the following definition:

Definition 3.2 Let $\mathcal{G}_{ig} = (G, \leq, \rightarrow, \rightsquigarrow, 0)$ be a po-implicative-group. A deductive system S of $\mathcal{G}_{ig}$ is *compatible* if the following condition (C_{ig}) holds:

$$(C_{ig}) \qquad for\ any\ x, y \in G, \;\; x \rightarrow y \in S \Longleftrightarrow x \rightsquigarrow y \in S.$$

Using Remark 2.6, we formulate the following complex result (which brings together more results):

Theorem 3.3
*Let $\mathcal{G}_{ig} = (G, \leq, \rightarrow, \rightsquigarrow, 0)$ be a po-implicative-group (or let $\mathcal{G}_g = (G, \leq, +, -, 0)$ be a po-group). Let S be a deductive system of $\mathcal{G}_{ig}$ (or, equivalently, a convex po-subgroup of $\mathcal{G}_g$).
Then, S is compatible if and only if S is normal, i.e. $(C_{ig}) \Longleftrightarrow (N_g)$.*

Proof. (C_{ig}) $\Longrightarrow$ (N_g): Suppose that (C_{ig}) holds and let $g \in G$.
(i) Let $h \in S$; denote $y \overset{notation}{=} h + g$ and remark that by (19) we have

$$(g \rightarrow y) + g = y = g + (g \rightsquigarrow y).$$

Hence, $y = h + g = (g \rightarrow y) + g$, which implies that $h = g \rightarrow y$. Hence, $g \rightarrow y \in S$. By (C_{ig}), it follows that $g \rightsquigarrow y \in S$ also. Hence, there exists $h' = g \rightsquigarrow y \in S$ such that
$h + g = (g \rightarrow y) + g = y = g + (g \rightarrow y) = g + h'$.
(ii) Similarly, let $h' \in S$; denote $x \overset{notation}{=} g + h'$ and remark that by (19) we have

$$(g \rightarrow x) + g = x = g + (g \rightsquigarrow x).$$

Hence, $x = g + h' = g + (g \rightsquigarrow x)$, which implies $h' = g \rightsquigarrow x$. Hence, $g \rightsquigarrow x \in S$. By ($\mathrm{C}_{ig}$), it follows that $g \rightsquigarrow x \in S$ too. Hence, there exists $h = g \rightarrow x \in S$ such that
$g + h' = g + (g \rightsquigarrow x) = x = (g \rightarrow x) + g = h + g$.
By (i) and (ii), we obtain that (N_g) holds.
(N_g) $\Longrightarrow$ (C_{ig}): Suppose that (N_g) holds and that $x \rightarrow y \in S$. By (19), we have

$$(x \rightarrow y) + x = y = x + (x \rightsquigarrow y).$$

Put then $h = x \rightarrow y \in S$; hence, $h + x = y = x + (x \rightsquigarrow y)$. By ($\mathrm{N}_g$), there exists $h' \in S$ such that $h + x = y = x + h'$. Hence, $y = x + (x \rightsquigarrow y) = x + h'$. It follows that $x \rightsquigarrow y = h'$. Thus, $x \rightsquigarrow y \in S$.
Similarly, $x \rightsquigarrow y \in S$ implies that $x \rightarrow y \in S$.
Thus, (C_{ig}) holds. □

We introduce now the following definition:

Definition 3.4
(1) Let $\mathcal{M}^L = (M^L, \leq, \odot, 1)$ be a left-poim. A filter S^L of $\mathcal{M}^L$ is *normal* if the following condition (N^L) holds:

$$(N^L) \qquad for\ any\ x \in M^L, \;\; S^L \odot x = x \odot S^L.$$

(1') Let $\mathcal{M}^R = (M^R, \leq, \oplus, 0)$ be a right-poim. An ideal S^R of $\mathcal{M}^R$ is *normal* if the following condition (N^R) holds:

$$(N^R) \qquad for\ any\ x \in M^R, \;\; S^R \oplus x = x \oplus S^R.$$

Then we have the following result:

Proposition 3.5 *Let $\mathcal{G} = (G, \leq, +, -, 0)$ be a po-group and S be a normal convex po-subgroup of $\mathcal{G}$. Then,*
(1) $S^L = S \cap G^-$ is a normal filter of the left-poim $\mathcal{G}^- = (G^-, \leq, \odot = +, \mathbf{1} = 0)$ from the Theorem 2.7 (1).
(1') $S^R = S \cap G^+$ is a normal ideal of the right-poim $\mathcal{G}^+ = (G^+, \leq, \oplus = +, \mathbf{0} = 0)$ from the Theorem 2.7 (1').

Proof. (1): By Theorem 2.8 (1), S^L is a filter of $\mathcal{G}^-$. It remains to prove that is a normal filter, i.e. (N^L) holds.
(i) Let $h \in S^L = S \cap G^-$, i.e. $h \in S$ and $h \leq 0$. S being normal, by (N_g) it follows that there exists $h' \in S$ such that

$$s \overset{notation}{=} h + x = x + h'.$$

We must prove that $h' \leq 0$. First, notice that $s \leq x$, since $h \leq 0$ implies that $s = h + x \leq 0 + x = x$. Then, $s \leq x$ implies that $s = x + h' \leq x$, and therefore $h' \leq 0$.
Thus, $h' \in S^L = S \cap G^L$ and $h \odot x = x \odot h'$.
(ii) Similarly, for $h' \in S^L$, there exists $h \in S^L$, such that $x \odot h' = h \odot x$. By (i) and (ii), (N^L) holds.
(1') has a similar proof. □

Recall the following definition.

Definition 3.6 (see ([20], Definition 2.2.1)
(1) Let $\mathcal{A}^L = (A^L, \leq, \rightarrow^L, \rightsquigarrow^L, 1)$ be a left-pseudo-BCK algebra. We say that a $(\rightarrow^L, \rightsquigarrow^L)$-deductive system S^L of $\mathcal{A}^L$ is *compatible* if the following condition (C^L) holds:

$$(C^L) \qquad for\ any\ x, y \in A^L,\ \ x \rightarrow^L y \in S^L \Longleftrightarrow x \rightsquigarrow^L y \in S^L.$$

(1') Let $\mathcal{A}^R = (A^R, \leq, \rightarrow^R, \rightsquigarrow^R, 0)$ be a right-pseudo-BCK algebra. We say that a $(\rightarrow^R, \rightsquigarrow^R)$-deductive system S^R of $\mathcal{A}^R$ is *compatible* if the following condition (C^R) holds:

$$(C^R) \qquad for\ any\ x, y \in A^R,\ \ x \rightarrow^R y \in S^R \Longleftrightarrow x \rightsquigarrow^R y \in S^R.$$

3.2 l-groups (l-implicative groups) and associated algebras on G^-, G^+

In lattice order case, we have the following result:

Proposition 3.7 *Let $\mathcal{G} = (G, \vee, \wedge, \rightarrow, \rightsquigarrow, 0)$ be an l-implicative-group and S be a compatible deductive system of $\mathcal{G}$. Then,*
(1) $S^L = S \cap G^-$ is a compatible $(\rightarrow^L, \rightsquigarrow^L)$-deductive system of the left-pseudo-BCK(pP) lattice $\mathcal{G}^L = (G^-, \wedge, \vee, \rightarrow^L, \rightsquigarrow^L, \mathbf{1} = 0)$ from Theorem 2.9 (1).
(1') $S^R = S \cap G^+$ is a compatible $(\rightarrow^R, \rightsquigarrow^R)$-deductive system of the right-pseudo-BCK(pS) lattice $\mathcal{G}^R = (G^+, \vee, \wedge, \rightarrow^R, \rightsquigarrow^R, \mathbf{0} = 0)$ from Theorem 2.9 (1').

Proof. (1): By Theorem 2.10 (1), S^L is a $(\rightarrow^L, \rightsquigarrow^L)$-deductive system of $\mathcal{G}^L$. It remains to prove that S^L is compatible, i.e. for all $x, y \in G^-$:

$$x \rightarrow^L y \in S^L \Longleftrightarrow x \rightsquigarrow^L y \in S^L. \tag{40}$$

Since $x \rightarrow^L y \in G^-$, $x \rightsquigarrow^L y \in G^-$, it remains to prove that

$$x \rightarrow^L y \in S \Longleftrightarrow x \rightsquigarrow^L y \in S. \tag{41}$$

But $x \rightarrow^L y = (x \rightarrow y) \wedge 0 = (x \rightarrow y) \wedge (x \rightarrow x) \overset{(30)}{=} x \rightarrow (y \wedge x)$ and similarly $x \rightsquigarrow^L y = x \rightsquigarrow (y \wedge x)$ and since S is compatible, we have

$$x \rightarrow (y \wedge x) \in S \Longleftrightarrow x \rightsquigarrow (y \wedge x) \in S. \tag{42}$$

It follows that (41) holds, hence (40) holds.

(1') has a similar proof. □

Using Remark 2.1 and (b), (b'), we formulate the following complex result:

Theorem 3.8

(1) Let $\mathcal{A}^L = (A^L, \wedge, \vee, \rightarrow^L, \rightsquigarrow^L, 1)$ be a left-pseudo-BCK(pP) lattice verifying (pdiv) (or let $\mathcal{A}^L_m = (A^L, \wedge, \vee, \odot, 1)$ be a left-l-rim verifying (pdiv)). Let S^L be a $(\rightarrow^L, \rightsquigarrow^L)$-deductive system of $\mathcal{A}^L$ (or, equivalently, a filter of $\mathcal{A}^L_m$).
Then S^L is compatible if and only if is normal, i.e. $(C^L) \Longleftrightarrow (N^L)$.

(1') Let $\mathcal{A}^R = (A^R, \vee, \wedge, \rightarrow^R, \rightsquigarrow^R, 0)$ be a right-pseudo-BCK(pS) lattice verifying (pdiv^d) (or let $\mathcal{A}^R_m = (A^R, \vee, \wedge, \oplus, 0)$ be a right-l-rim verifying (pdiv^d)). Let S^R be a $(\rightarrow^R, \rightsquigarrow^R)$-deductive system of $\mathcal{A}^R$ (or, equivalently, an ideal of $\mathcal{A}^R_m$).
Then S^R is compatible if and only if is normal, i.e. $(C^R) \Longleftrightarrow (N^R)$.

Proof. (1): Suppose that S^L is compatible, i.e. that (C^L) holds. We must prove that S^L is normal, i.e. that (N^L) holds. Indeed, let $x \in A^L$.
(i) Let $h \in S^L$. We put $s \overset{notation}{=} h \odot x$. First, notice that $s \leq x$, since $h \odot x \leq x$. Then, by (pdiv), we have
$$s = x \wedge s = (x \rightarrow^L s) \odot x = x \odot (x \rightsquigarrow^L s).$$
But $x \rightarrow^L s = x \rightarrow^L (h \odot x) \geq h$, by ([18] page 354, property (10.3)). Hence $x \rightarrow^L s \geq h$ and $h \in S^L$. Since S^L is a filter, it follows that $x \rightarrow^L s \in S^L$. Then, by (C^L), we obtain that $x \rightsquigarrow^L s \in S^L$ too; hence, there exists $h' \overset{notation}{=} x \rightsquigarrow^L s \in S^L$ such that $h \odot x = x \odot h'$.
(ii) Similarly, for $h' \in S^L$, there exists $h \in S^L$ such that $x \odot h' = h \odot x$.
By (i) and (ii), (N^L) holds.

Suppose now that S^L is normal, i.e. that (N^L) holds. We must prove that S^L is compatible, i.e. that (C^L) holds. Indeed, let $x, y \in A^L$.
Assume $x \rightarrow^L y \in S^L$. Then putting $h = x \rightarrow^L y$, by (N^L) there exists $h' \in S^L$ such that $h \odot x = x \odot h'$. But, by (pdiv),
$$x \wedge y = (x \rightarrow^L y) \odot x = x \odot (x \rightsquigarrow^L y),$$
i.e. $x \wedge y = h \odot x = x \odot (x \rightsquigarrow^L y)$. Hence, $x \wedge y = x \odot h' = x \odot (x \rightsquigarrow^L y)$.
We must prove that $x \rightsquigarrow^L y \in S^L$. Indeed, $x \rightsquigarrow^L y = x \rightsquigarrow^L (x \wedge y) = x \rightsquigarrow^L (x \odot h') \geq h'$,
by ([18] page 367, property (10.49) and page 354, property (10.3)).
Hence, $x \rightsquigarrow^L y \geq h'$ and $h' \in S^L$; since S^L is a filter, it follows that $x \rightsquigarrow^L y \in S^L$.
Similarly, assuming $x \rightsquigarrow^L y \in S^L$, we shall obtain that $x \rightarrow^L y \in S^L$.
Thus, (C^L) holds.

(1'): similarly. □

Remark 3.9 Note that condition (pdiv) ((pdiv^d) respectively) is a necessary condition in order to have that "*compatible* property is equivalent to *normal* property". In the absence of condition (pdiv) ((pdiv^d) respectively), we may have any situation.

Open problem 3.10 Find an example of left-pseudo-BCK(pP) lattice, for instance, not verifying (pdiv), which has a filter that is normal but not compatible, or is compatible but not normal.

Finally, using the Remarks 2.6 and 2.1, we formulate the following complex result:

Theorem 3.11

Let $\mathcal{G}_{ig} = (G, \vee, \wedge, \rightarrow, \rightsquigarrow, 0)$ be an l-implicative-group (or let $\mathcal{G}_g = (G, \vee, \wedge, +, -, 0)$ be an l-group). Let S be a compatible deductive system of $\mathcal{G}_{ig}$ (or, equivalently, a normal convex l-subgroup of $\mathcal{G}_g$).
Then,
(1) $S^L = S \cap G^-$ is a compatible $(\rightarrow^L, \rightsquigarrow^L)$-deductive system of the left-pseudo-BCK(pP) lattice $\mathcal{G}^L =$

$(G^-, \wedge, \vee, \to^L, \rightsquigarrow^L, \mathbf{1} = 0)$ *(or, equivalently, S^L is a normal filter of the left-l-rim* $\mathcal{G}^L_m = (G^-, \wedge, \vee, \odot = +, \mathbf{1} = 0)$*), and S^L is compatible if and only if is normal, i.e.* $(C^L) \iff (N^L)$.

(1') $S^R = S \cap G^+$ is a compatible $(\to^R, \rightsquigarrow^R)$-deductive system of the right-pseudo-BCK(pS) lattice $\mathcal{G}^R = (G^+, \vee, \wedge, \to^R, \rightsquigarrow^R, \mathbf{0} = 0)$ *(or, equivalently, S^R is a normal ideal of the right-l-rim* $\mathcal{G}^R_m = (G^+, \vee, \wedge, \oplus = +, \mathbf{0} = 0)$*), and S^R is compatible if and only if is normal, i.e.* $(C^R) \iff (N^R)$.

Proof. (1): S^L is a compatible $(\to^L, \rightsquigarrow^L)$-deductive system of $\mathcal{G}^L$ by Proposition 3.7 (1).
S^L is a normal filter of $\mathcal{G}^L_m$ by Proposition 3.5 (1).
$(C^L) \iff (N^L)$ by Theorem 3.8 (1), since $\mathcal{G}^L$ ($\mathcal{G}^L_m$) verifies condition (pC) and (pC) implies (pdiv).

(1') has a similar proof. □

In other words, the above Theorem 3.11 says that normality (compatibility) at l-group (l-implicative-group) G level is inherited by the algebras obtained by restricting the l-group (l-implicative-group) operations to the negative cone G^- and to the positive cone G^+. Also, it says that the equivalence $(C_{ig}) \iff (N_g)$ (*compatible* if and only if *normal*), existing at l-group (l-implicative-group) level (Theorem 3.3), is preserved by the algebras obtained by restricting the l-group (l-implicative-group) operations to G^- and to G^+.

3.3 l-groups (l-implicative groups) and associated algebras on $[u', 0]$, $[0, u]$

Remarks 3.12

(a) It is proved in [13], ([14] Lemma 3.2) that in right-pseudo-MV algebras, an ideal is normal if and only if is compatible. Hence, dually, in left-pseudo-MV algebras, a filter is normal if and only if is compatible.

(b) Note that these dual results follow by Theorem 3.8 (1') and (1), since in a left-pseudo-MV algebra (right-pseudo-MV algebra) condition (pdiv) (condition (pdiv^d) respectively) is verified.

We shall clarify in this subsection how the results from the above Remark are connected with those from l-groups (l-implicative-groups) level.

First note that we have the following general theorem that generalizes the Corollary 2.11, (1),(1').

Theorem 3.13

(1) Let $\mathcal{A}^L = (A^L, \wedge, \vee, \to^L, \rightsquigarrow^L, 1)$ be a left-pseudo-BCK(pP) lattice (with the pseudo-product $\odot$) verifying condition (pC). Let us "bound" this algebra with an "internal" element in the following way: let us take $u' < 1$ from A^L and consider the interval $[u', 1] = \{x \in A^L \mid u' \leq x \leq 1\} \subset A^L$. Then the algebra

$$\mathcal{A}^L_1 = ([u', 1], \wedge, \vee, \to^L, \rightsquigarrow^L, \mathbf{0} = u', 1)$$

is a bounded left-pseudo-BCK(pP) lattice (with the pseudo-product $x \odot^L y \overset{def.}{=} (x \odot y) \vee u'$) with condition (pC), i.e. is an equivalent definition of the **left-pseudo-Wajsberg algebra** *(see [17])*

$$\mathcal{A}^L_{1'} = ([u', 1], \to^L, \rightsquigarrow^L, ^{-^L}, ^{\sim^L}, \mathbf{0} = u', 1).$$

(1') Let $\mathcal{A}^R = (A^R, \vee, \wedge, \to^R, \rightsquigarrow^R, 0)$ be a right-pseudo-BCK(pS) lattice (with the pseudo-sum $\oplus$) verifying condition (pc^d). Let us "bound" this algebra in the following way: let us take $u > 0$ from A^R and consider the interval $[0, u] = \{x \in A^R \mid 0 \leq x \leq u\} \subset A^R$. Then the algebra

$$\mathcal{A}^R_1 = ([0, u], \vee, \wedge, \to^R, \rightsquigarrow^R, 0, \mathbf{1} = u)$$

is a bounded right-pseudo-BCK(pS) lattice (with the pseudo-sum $x \oplus^R y \overset{def.}{=} (x \oplus y) \wedge u$) with condition (pC^d), i.e. is an equivalent definition of the **right-pseudo-Wajsberg algebra**

$$\mathcal{A}^R_{1'} = ([0, u], \to^R, \rightsquigarrow^R, ^{-^R}, ^{\sim^R}, 0, \mathbf{1} = u).$$

Proof. (1): $[u',1]$ is obviously closed under $\wedge$ and $\vee$. $[u',1]$ is closed under $\to^L$: let $x,y \in [u',1]$; $y \leq 1$ implies $x \to^L y \leq x \to^L 1 = 1$; $u' \leq y$ implies $x \to^L u' \leq x \to^L y$ and $u' \leq x \to^L u'$; hence $u' \leq x \to^L y \leq 1$. Similarly, $[u',1]$ is closed under $\rightsquigarrow^L$. It follows that $\mathcal{A}_1^L$ is a bounded left-pseudo-BCK lattice verifying (pC). Then, it verifies (pDN), where $x^- = x^{-^L} \stackrel{def.}{=} x \to^L u'$ and $x^\sim = x^{\sim^L} \stackrel{def.}{=} x \rightsquigarrow^L u'$, for any $x \in [u',1]$. Consequently, there exists

$$x \odot^L y = (x \to^L y^-)^\sim = (x \to^L (y \to^L u')) \rightsquigarrow^L u' = ((x \odot y) \to^L u') \rightsquigarrow^L u' \stackrel{(pC)}{=} (x \odot y) \vee u'.$$

Thus, $\mathcal{A}_1^L$ is a bounded left(pseudo-BCK(pP) lattice with pseudo-product $\odot^L$, verifying (pC).

(1') has a similar proof. □

Open problem 3.14 It remains an open problem if there are other examples than l-implicative-groups producing left-pseudo-BCK(pP) lattices verifying condition (pC) (right-pseudo-BCK(pS) lattices verifying condition (pC^d), respectively).

Note that we can reformulate Theorem 3.13 equivalently, by (c), (c') and (d), (d'), as follows:

Theorem 3.15

(1) Let $\mathcal{A}_m^L = (A^L, \wedge, \vee, \odot, 1)$ be a left-l-rim (with the pseudo-residuum $(\to^L, \rightsquigarrow^L)$) verifying condition (pC). Let us "bound" this algebra with an "external" element in the following way: let us take $u' < 1$ from A^L and consider the interval $[u',1] \subset A^L$. Then the algebra

$$\mathcal{A}_{m1}^L = ([u',1], \wedge, \vee, \odot^L, \mathbf{0} = u', 1)$$

is a bounded left-l-rim (with $x \odot^L y \stackrel{def.}{=} (x \odot y) \vee u'$ and the pseudo-residuum $(\to^L, \rightsquigarrow^L)$) with condition (pC), i.e. is an equivalent definition of the **left-pseudo-MV algebra** *(see [17])*

$$\mathcal{A}_{m1'}^L = ([u',1], \odot^L, \oplus^L, {}^{-^L}, {}^{\sim^L}, \mathbf{0} = u', 1).$$

(1') Let $\mathcal{A}_m^R = (A^R, \vee, \wedge, \oplus, 0)$ be a right-l-rim (with the pseudo-coresiduum $(\to^R, \rightsquigarrow^R)$) verifying condition (pc^d). Let us "bound" this algebra in the following way: let us take $u > 0$ from A^R and consider the interval $[0,u] \subset A^R$. Then the algebra

$$\mathcal{A}_{m1}^R = ([0,u], \vee, \wedge, \oplus^R, 0, \mathbf{1} = u)$$

is a bounded right-l-rim (with $x \oplus^R y \stackrel{def.}{=} (x \oplus y) \wedge u$) with condition (pC^d), i.e. is an equivalent definition of the **right-pseudo-MV algebra**

$$\mathcal{A}_{m1'}^R = ([0,u], \oplus^R, \odot^R, {}^{-^R}, {}^{\sim^R}, 0, \mathbf{1} = u).$$

A filter of the left-pseudo-MV algebra $\mathcal{A}_{m1'}^L$ is a filter of the left-poim $([u',1], \leq, \odot^L, 1)$ and an ideal of the right-pseudo-MV algebra $\mathcal{A}_{m1'}^R$ is an ideal of the right-poim $([0,u], \leq, \oplus^R, 0)$. Then we have the following result.

Proposition 3.16 *Let $\mathcal{G} = (G, \vee, \wedge, +, -, 0)$ be an l-group and S be a convex l-subgroup of $\mathcal{G}$. Then,*

(1) for any $u' < 0$ and $S^L = S \cap G^-$, the set $S_{[]}^L = S \cap [u',0] = S^L \cap [u',0]$ is a filter of the left-pseudo-MV algebra

$$\mathcal{G}_{m1'}^L = ([u',0], \odot^L, \oplus^L, {}^{-^L}, {}^{\sim^L}, \mathbf{0} = u', \mathbf{1} = 0).$$

(1') for any $u > 0$ and $S^R = S \cap G^+$, the set $S_{[]}^R = S \cap [0,u] = S^R \cap [0,u]$ is an ideal of the right-pseudo-MV algebra

$$\mathcal{G}_{m1'}^R = ([0,u], \oplus^R, \odot^R, {}^{-^R}, {}^{\sim^R}, \mathbf{0} = 0, \mathbf{1} = u).$$

Proof. (1): First, note that by Theorem 2.8(1), S^L is a filter of $\mathcal{G}^-$.
We prove that $S^L_{[]} = S^L \cap [u', 0]$ is a filter of $\mathcal{G}^L_{m1'}$:
(F1): $0 \in S^L_{[]}$ since $0 \in S^L$ and $0 \in [u', 0]$.
(F2): Let $x, y \in S^L_{[]}$, i.e. $x, y \in S^L$ and $x, y \in [u', 0]$. We must prove that $x \odot^L y \in S^L_{[]}$. Since S^L is a filter of $\mathcal{G}^-$, it follows that $x \odot y \in S^L$; but $x \odot^L y = (x \odot y) \vee u' \geq x \odot y$, hence $x \odot^L y \in S^L$. On the other hand, $u' \leq x, y \leq 0$ imply that

$$u' \leq x \odot^L y = (x \odot y) \vee u' = (x + y) \vee u' \leq 0,$$

hence $x \odot^L y \in [u', 0]$. Thus, $x \odot^L y \in S^L \cap [u', 0] = S^L_{[]}$.
(F3): Let $x \in S^L_{[]}$, $x \leq y$ ($y \in [u', 0]$). We must prove that $y \in S^L_{[]}$. But S^L is a filter of $\mathcal{G}^-$, hence $y \in S^L$. Since $y \in [u', 0]$, we obtain that $y \in S^L \cap [u', 0] = S^L_{[]}$.

(1'): has a similar proof. □

Proposition 3.17 *Let $\mathcal{G} = (G, \vee, \wedge, \rightarrow, \rightsquigarrow, 0)$ be an l-implicative-group and S be a compatible deductive system of $\mathcal{G}$. Then,*

(1) for any $u' < 0$ and $S^L = S \cap G^-$, the set $S^L_{[]} = S \cap [u', 0] = S^L \cap [u', 0]$ is a compatible $(\rightarrow^L, \rightsquigarrow^L)$-deductive system of the bounded left-pseudo-BCK(pP) lattice verifying condition (pC)

$$\mathcal{G}^L_1 = ([u', 0], \wedge, \vee, \rightarrow^L, \rightsquigarrow^L, \mathbf{0} = u', \mathbf{1} = 0)$$

from Corollary 2.11(i) (i.e. of the left-pseudo-Wajsberg algebra $\mathcal{G}^L_{1'}$).

(1') for any $u > 0$ and $S^R = S \cap G^+$, the set $S^R_{[]} = S \cap [0, u] = S^R \cap [0, u]$ is a compatible $(\rightarrow^R, \rightsquigarrow^R)$-deductive system of the bounded right-pseudo-BCK(pS) lattice verifying condition (pC^d)

$$\mathcal{G}^R_1 = ([0, u], \vee, \wedge, \rightarrow^R, \rightsquigarrow^R, \mathbf{0} = 0, \mathbf{1} = u)$$

from Corollary 2.11(i') (i.e. of the right-pseudo-Wajsberg algebra $\mathcal{G}^R_{1'}$).

Proof. (1): First, note that by Proposition 3.5 (1), S^L is a compatible $(\rightarrow^L, \rightsquigarrow^L)$-deductive system of $\mathcal{G}^L$.
Second, $S^L_{[]}$ is a $(\rightarrow^L, \rightsquigarrow^L)$-deductive system of $[u', 0]$ since:
(ds1): $\mathbf{1} = 0 \in S^L, [u', 0]$, hence $0 \in S^L_{[]}$.
(ds2): Let $x \in S^L_{[]}$ and $x \leq y$ ($y \in [u', 0]$). It follows that $y \in S^L$. Since we also have that $y \in [u', 0]$, it follows that $y \in S^L \cap [u', 0] = S^L_{[]}$.
Third, we prove that $S^L_{[]}$ is compatible. Indeed, let $x, y \in [u', 0]$.
Assume that $x \rightarrow^L y \in S^L_{[]}$, i.e. $x \rightarrow^L y \in S^L$ and $x \rightarrow^L y \in [u', 0]$. S^L being compatible, it follows that $x \rightsquigarrow^L y \in S^L$ too. Since $[u', 0]$ is closed under $\rightarrow^L$ and $\rightsquigarrow^L$ ([19], Lemma 5.7), it follows that $x \rightsquigarrow^L y \in [u', 0]$. Hence, $x \rightsquigarrow^L y \in S^L \cap [u', 0] = S^L_{[]}$.
Assume that $x \rightsquigarrow^L y \in S^L_{[]}$. We obtain similarly that $x \rightarrow^L y \in S^L_{[]}$. Thus, $S^L_{[]}$ is compatible.

(1'): has a similar proof. □

Finally, using the Remarks 2.6 and 2.1, we formulate the following complex result:

Theorem 3.18

Let $\mathcal{G}_{ig} = (G, \vee, \wedge, \rightarrow, \rightsquigarrow, 0)$ be an l-implicative-group (or let $\mathcal{G}_g = (G, \vee, \wedge, +, -, 0)$ be an l-group). Let S be a compatible deductive system of $\mathcal{G}_{ig}$ (or, equivalently, a normal convex l-subgroup of $\mathcal{G}_g$).
Then,
(1) for any $u' < 0$ and $S^L = S \cap G^-$, the set $S^L_{[]} = S \cap [u', 0] = S^L \cap [u', 0]$ is a compatible $(\rightarrow^L, \rightsquigarrow^L)$-deductive system of the bounded left-pseudo-BCK(pP) lattice verifying condition (pC)

$$\mathcal{G}^L_1 = ([u', 0], \wedge, \vee, \rightarrow^L, \rightsquigarrow^L, \mathbf{0} = u', \mathbf{1} = 0)$$

from Corollary 2.11(i), i.e. of the left-pseudo-Wajsberg algebra $\mathcal{G}^L_{1'}$ (or, equivalently, $S^L_{[]}$ is a normal filter of the bounded left-l-rim verifying condition (pC)

$$\mathcal{G}^L_{m1} = ([u', 0], \wedge, \vee, \odot^L, \mathbf{0} = u', \mathbf{1} = 0),$$

i.e. of the left-pseudo-MV algebra $\mathcal{G}^L_{m1'}$), and $S^L_{[]}$ is compatible if and only if is normal, i.e. $(C^L) \Longleftrightarrow (N^L)$.

(1') for any $u > 0$ and $S^R = S \cap G^+$, $S^R_{[]} = S \cap [0, u] = S^R \cap [0, u]$ is a compatible $(\rightarrow^R, \rightsquigarrow^R)$-deductive system of the bounded right-pseudo-BCK(pS) lattice verifying condition (pC^d)

$$\mathcal{G}^R_1 = ([0, u], \vee, \wedge, \rightarrow^R, \rightsquigarrow^R, \mathbf{0} = 0, \mathbf{1} = u)$$

from Corollary 2.11(i'), i.e. of the right-pseudo-Wajsberg algebra $\mathcal{G}^R_{1'}$ (or, equivalently, $S^R_{[]}$ is a normal ideal of the bounded right-l-rim verifying condition (pC^d)

$$\mathcal{G}^R_{m1} = ([0, u], \vee, \wedge, \oplus^R, \mathbf{0} = 0, \mathbf{1} = u),$$

i.e. of the right-pseudo-MV algebra $\mathcal{G}^R_{m1'}$), and $S^R_{[]}$ is compatible if and only if is normal, i.e. $(C^R) \Longleftrightarrow (N^R)$.

Proof. (1): $S^L_{[]}$ is a compatible $(\rightarrow^L, \rightsquigarrow^L)$-deductive system of $\mathcal{G}^L_1$ by Proposition 3.17 (1).
$S^L_{[]}$ is a filter of $\mathcal{G}^L_{m1}$ (i.e. $[u', 0]$) by Proposition 3.16(1).
$(C^L) \Longleftrightarrow (N^L)$ by Theorem 3.8 (1), since $\mathcal{G}^L_1$ verifies condition (pC) and (pC) implies (pdiv); hence $S^L_{[]}$ is a normal filter of $[u', 0]$.

(1'): has a similar proof. □

In other words, the above Theorem 3.18 says that normality (compatibility) at l-group (l-implicative-group) G level is inherited by the algebras obtained by restricting the l-group (l-implicative-group) operations to any segment $[u', 0] \subset G^-$ and to any segment $[0, u] \subset G^+$. Also, it says that the equivalence $(C_{ig}) \Longleftrightarrow (N_g)$ (*compatible* if and only if *normal*), existing at l-group (l-implicative-group) level (Theorem 3.3), is preserved by the algebras obtained by restricting the l-group (l-implicative-group) operations to $[u', 0]$ and to $[0, u]$ (see also Remarks 3.12 (a)).

Open problem 3.19 Find a direct proof that a normal convex l-group of an l-group $\mathcal{G}$ produces a normal filter of $\mathcal{G}^L_{m1'}$ (a normal ideal of $\mathcal{G}^R_{m1'}$) (see Proposition 3.16 and Theorem 3.18).

3.4 l-groups (l-implicative groups) and associated algebras on $\{-\infty\} \cup G^-$, $G^+ \cup \{+\infty\}$

First note that we have the following general theorem that generalizes the Corollary 2.11, (2),(2'):

Theorem 3.20

(i) Let $\mathcal{A}^L = (A^L, \wedge, \vee, \rightarrow^L, \rightsquigarrow^L, 1)$ be a left-pseudo-BCK(pP) lattice (with the pseudo-product $\odot$) verifying (pC) and (*). Let us "bound" this algebra in the following way (the usual way of "bounding" the Hilbert algebras - in the commutative case): let us consider a symbol $-\infty$ distinct from the elements of A^L. Define $A^L_{-\infty} \overset{def.}{=} \{-\infty\} \cup A^L$ and extend the operations $\rightarrow^L, \rightsquigarrow^L$, $\odot$ from A^L to $A^L_{-\infty}$ as follows:

$$x \rightarrow^L_2 y \overset{def.}{=} \begin{cases} x \rightarrow^L y, & if \quad x, y \in A^L \\ -\infty, & if \quad x \in A^L,\ y = -\infty \\ 1, & if \quad x = -\infty, \end{cases} \qquad x \rightsquigarrow^L_2 y \overset{def.}{=} \begin{cases} x \rightsquigarrow^L y, & if \quad x, y \in A^L \\ -\infty, & if \quad x \in A^L,\ y = -\infty \\ 1, & if \quad x = -\infty, \end{cases}$$

$$x \odot_2 y \overset{def.}{=} \begin{cases} x \odot y, & if \quad x, y \in A^L \\ -\infty, & if \quad otherwise. \end{cases}$$

We extend the lattice order relation $\leq$ as follows: we put $-\infty \leq x$, for any $x \in A^L_{-\infty}$. Then, the algebra

$$\mathcal{A}^L_2 = (A^L_{-\infty}, \wedge, \vee, \rightarrow^L_2, \rightsquigarrow^L_2, \mathbf{0} = -\infty, 1)$$

is a **left-pseudo-Hájek(pP) algebra (with the pseudo-product $\odot_2$) verifying conditions (pP1) and (pP2)** *(see [17]).*

(i') Let $\mathcal{A}^R = (A^R, \vee, \wedge, \to^R, \rightsquigarrow^R, 0)$ be a right-pseudo-BCK(pS) lattice (with the pseudo-sum $\oplus$) verifying conditions (pC^d) and (^{d}). Let us "bound" this algebra in the following way: let us consider a symbol $+\infty$ distinct from the elements of A^R. Define $A^R_{+\infty} \stackrel{def.}{=} A^R \cup \{+\infty\}$ and extend the operations $\to^R, \rightsquigarrow^R$, $\oplus$ from A^R to $A^R_{+\infty}$ as follows:*

$$x \to_2^R y \stackrel{def.}{=} \begin{cases} x \to^R y, & if \;\; x, y \in A^R \\ +\infty, & if \;\; x \in A^R,\; y = +\infty \\ 0, & if \;\; x = +\infty, \end{cases} \qquad x \rightsquigarrow_2^R y \stackrel{def.}{=} \begin{cases} x \rightsquigarrow^R, & if \;\; x, y \in A^R \\ +\infty, & if \;\; x \in A^R,\; y = +\infty \\ 0, & if \;\; x = +\infty, \end{cases}$$

$$x \oplus_2 y \stackrel{def.}{=} \begin{cases} x \oplus y, & if \;\; x, y \in A^R \\ +\infty, & if \;\; otherwise. \end{cases}$$

We extend the lattice order relation $\geq$ as follows: we put $+\infty \geq x$, for any $x \in A^R_{+\infty}$. Then, the algebra

$$\mathcal{A}_2^R = (A^R_{+\infty}, \vee, \wedge, \to_2^R, \rightsquigarrow_2^R, 0, \mathbf{1} = +\infty)$$

is a **right-pseudo-Hájek(pS) algebra (with the pseudo-sum $\oplus_2$) verifying the dual conditions (pP1^d) and (pP2^d).**

Proof. (i): Obviously, $\mathcal{A}_2^L$ is a bounded left-pseudo-BCK(pP) lattice, with the pseudo-product $\odot_2$ (by condition (pP)):

$x \odot_2 y \stackrel{def.}{=} \min\{z \in A^L_{-\infty} \mid x \leq y \to_2^L z\} =$

$$= \min\{z \in A^L_{-\infty} \mid x \leq \begin{cases} y \to^L z, & \text{if} \;\; y, z \in A^L \\ -\infty, & \text{if} \;\; y \in A^L,\; z = -\infty\} \\ 1, & \text{if} \;\; y = -\infty \end{cases} = \begin{cases} x \odot y, & \text{if} \;\; x, y \in A^L \\ -\infty, & \text{if} \;\; otherwise. \end{cases}$$

$\mathcal{A}^L$ verifies condition (pC), hence it verifies conditions (pprel) and (pdiv). We shall prove that $\mathcal{A}_2^L$ does not verifies (pC) anymore, but it satisfies (pprel) and (pdiv). Indeed,

- $\mathcal{A}_2^L$ does not verifies (pC), since for $-\infty$ and any $x \neq 1$, we have:
$x = -\infty \vee x \neq (x \to^L -\infty) \rightsquigarrow^L -\infty = -\infty \rightsquigarrow^L -\infty = 1$.
- $\mathcal{A}_2^L$ verifies (pprel): since $\mathcal{A}^L$ verifies (pprel), it is sufficient to prove that

$$(x \to_2^L y) \vee (y \to_2^L x) = 1 = (x \rightsquigarrow_2^L y) \vee (y \rightsquigarrow_2^L x), \quad for \;\; x, y \notin A^L.$$

Indeed, for $x = -\infty$ and $y \in A^L$, we have $(-\infty \to_2^L y) \vee (y \to_2^L -\infty) = 1 \vee -\infty = 1$ and
for $x, y = -\infty$, we have $(-\infty \to_2^L -\infty) \vee (-\infty \to_2^L -\infty) = 1 \vee 1 = 1$,
and similarly for $\rightsquigarrow_2^L$.
- $\mathcal{A}_2^L$ verifies (pdiv): since $\mathcal{A}^L$ verifies (pdiv), it is sufficient to prove that

$$x \wedge y = (x \to_2^L y) \odot_2 x = x \odot_2 (x \rightsquigarrow_2^L y), \quad for \;\; x, y \notin A^L.$$

Indeed,
· for $x = -\infty$ and $y \in A^L$, we have $-\infty \wedge y = -\infty$, $(-\infty \to_2^L y) \odot_2 -\infty = 1 \odot_2 -\infty = -\infty$ and $-\infty \odot_2 (-\infty \rightsquigarrow_2^L y) = -\infty \odot_2 1 = -\infty$;
· for $x \in A^L$ and $y = -\infty$, we have $x \wedge -\infty = -\infty$, $(x \to_2^L -\infty) \odot_2 x = -\infty \odot_2 x = -\infty$ and $x \odot_2 (x \rightsquigarrow_2^L -\infty) = x \odot_2 -\infty = -\infty$;
· for $x, y = -\infty$, we have $x \wedge y = -\infty$, $(-\infty \to_2^L -\infty) \odot_2 -\infty = 1 \odot_2 -\infty = -\infty$ and $-\infty \odot_2 (-\infty \rightsquigarrow_2^L -\infty) = -\infty \odot_2 1 = -\infty$.
Thus, $\mathcal{A}_2^L$ is a left-pseudo-Hájek(pP) algebra.
-$\mathcal{A}_2^L$ verifies (pP1). Indeed,

$$x^- = x \to_2^L -\infty = \begin{cases} x \to_2^L -\infty, & \text{if} \;\; x \in A^L \\ -\infty \to_2^L -\infty, & \text{if} \;\; x = -\infty \end{cases} = \begin{cases} -\infty, & \text{if} \;\; x \in A^L \\ 1, & \text{if} \;\; x = -\infty. \end{cases}$$

Similarly, $x^\sim = x \rightsquigarrow_2^L -\infty = \begin{cases} x \rightsquigarrow_2^L -\infty, & \text{if} \;\; x \in A^L \\ -\infty \rightsquigarrow_2^L -\infty, & \text{if} \;\; x = -\infty \end{cases} = \begin{cases} -\infty, & \text{if} \;\; x \in A^L \\ 1, & \text{if} \;\; x = -\infty. \end{cases}$

Note that $x^- = x^\sim$, for all $x \in \mathcal{A}_2^L$.
Then, $x \wedge x^- = x \wedge \begin{cases} -\infty, & \text{if} \quad x \in A^L \\ 1, & \text{if} \quad x = -\infty. \end{cases} = \begin{cases} x \wedge -\infty, & \text{if} \quad x \in A^L \\ x \wedge 1, & \text{if} \quad x = -\infty. \end{cases} = -\infty$
and similarly $x \wedge x^\sim = -\infty$.
-$\mathcal{A}_2^L$ verifies (pP2). Denote first

$$F \overset{notation}{=} (x \odot_2 z) \to_2^L (y \odot_2 z) \quad and \quad E \overset{notation}{=} (z^-)^- \odot_2 F.$$

We must prove that

$$E \leq x \to_2^L y.$$

There are eight cases:
1. $x \in A^L$, $y \in A^L$, $z \in A^L$,
2. $x \in A^L$, $y \in A^L$, $z = -\infty$,
3. $x \in A^L$, $y = -\infty$, $z \in A^L$,
4. $x \in A^L$, $y = -\infty$, $z = -\infty$,
5. $x = -\infty$, $y \in A^L$, $z \in A^L$,
6. $x = -\infty$, $y \in A^L$, $z = -\infty$,
7. $x = -\infty$, $y = -\infty$, $z \in A^L$,
8. $x = -\infty$, $y = -\infty$, $z = -\infty$.

Recall that $z^- = \begin{cases} -\infty, & \text{if} \quad z \in A^L \\ 1, & \text{if} \quad z = -\infty, \end{cases}$ hence
$(z^-)^- = \begin{cases} -\infty^-, & \text{if} \quad z \in A^L \\ 1^-, & \text{if} \quad z = -\infty \end{cases} = \begin{cases} 1, & \text{if} \quad z \in A^L \\ -\infty, & \text{if} \quad z = -\infty. \end{cases}$
We then obtain:
1: $(z^-)^- = 1$, $E = 1 \odot_2 F = F \overset{(*)}{=} x \to^L y = x \to_2^L y$.
2: $(z^-)^- = -\infty$, $E = -\infty \odot_2 F = -\infty < x \to_2^L y$.
3: $(z^-)^- = 1$, $E = 1 \odot_2 F = F = (x \odot z) \to_2^L (-\infty \odot_2 z) = (x \odot z) \to_2^L -\infty = -\infty = x \to_2^L -\infty$.
4: $(z^-)^- = -\infty$, $E = -\infty \odot_2 F = -\infty = x \to_2^L -\infty$.
5: $(z^-)^- = 1$, $E = 1 \odot_2 F = F = (-\infty \odot_2 z) \to_2^L (y \odot_2 z) = -\infty \to_2^L (y \odot_2 z) = 1 = -\infty \to_2^L y$.
6: $(z^-)^- = -\infty$, $E = -\infty \odot_2 F = -\infty < -\infty \to_2^L y = 1$.
7: $(z^-)^- = 1$, $E = 1 \odot_2 F = F = (-\infty \odot_2 z) \to_2^L (-\infty \odot_2 z) = -\infty \to_2^L -\infty = 1 = x \to_2^L y$.
8: $(z^-)^- = -\infty$, $E = -\infty \odot_2 F = -\infty < x \to_2^L y = 1$.
Similarly one can prove the second inequality. (i') has a similar proof. □

Open problem 3.21 It remains an open problem if there are other examples than l-implicative-groups producing left-pseudo-BCK(pP) lattices verifying condition (pC) and (*) (right-pseudo-BCK(pS) lattices verifying condition (pC^d) and ($*^d$), respectively).

Note also that we can reformulate Theorem 3.20 equivalently, by using (c), (c') and (f), (f'); this is left to the reader.

Let now $\mathcal{G}_{ig} = (G, \vee, \wedge, \to, \rightsquigarrow, 0)$ be an l-implicative-group (or let $\mathcal{G}_g = (G, \vee, \wedge, +, -, 0)$ be an l-group). Let $S \subseteq G$ be a deductive system of $\mathcal{G}_{ig}$ (or, equivalently, a convex l-subgroup of $\mathcal{G}_g$). Note that:

(1) if $S^L = S \cap G^-$, $-\infty \notin G$ and $G^-_{-\infty} = \{-\infty\} \cup G^-$ (see Corollary 2.11(i)(2)), then
$S \cap G^-_{-\infty} = S \cap (\{-\infty\} \cup G^-) \overset{distributivity}{=} (S \cap \{-\infty\}) \cup (S \cap G^-) = \emptyset \cup S^L = S^L$.

(1') similarly, if $S^R = S \cap G^+$, $-\infty \notin G$ and $G^+_{+\infty} = G^+ \cup \{+\infty\}$ (see Corollary 2.11(i')(2')), then $S \cap G^+_{+\infty} = S^R$.
Hence, using the Remarks 2.6 and 2.1, we formulate the following complex result:

Theorem 3.22
Let $\mathcal{G}_{ig} = (G, \vee, \wedge, \to, \rightsquigarrow, 0)$ be an l-implicative-group (or let $\mathcal{G}_g = (G, \vee, \wedge, +, -, 0)$ be an l-group). Let S be a compatible deductive system of $\mathcal{G}_{ig}$ (or, equivalently, a normal convex l-subgroup of $\mathcal{G}_g$).
Then,

(1) $S \cap G^-_{-\infty} = S^L$ is a compatible $(\to^L_2, \rightsquigarrow^L_2)$-deductive system of the left-pseudo-Hájek(pP) algebra (with the pseudo-product $\odot_2$) verifying conditions (pP1), (pP2) (see Corollary 2.11(i)(2))

$$\mathcal{G}^L_2 = (G^-_{-\infty}, \wedge, \vee, \to^L_2, \rightsquigarrow^L_2, \mathbf{0} = -\infty, \mathbf{1} = 0)$$

(or, equivalently, S^L is a normal filter of the left-pseudo-product algebra (see [19], Corollary 5.12(i)(2))

$$\mathcal{G}^L_{m2'} = (G^-_{-\infty}, \wedge, \vee, \odot_2, \to^L_2, \rightsquigarrow^L_2, \mathbf{0} = -\infty, \mathbf{1} = 0)),$$

and S^L is compatible if and only if is normal, i.e. $(C^L) \iff (N^L)$.

(1') $S \cap G^+_{+\infty} = S^R$ is a compatible $(\to^R_2, \rightsquigarrow^R_2)$-deductive system of the right-pseudo-Hájek(pS) algebra (with the pseudo-sum $\oplus_2$) verifying conditions (pP1^d), (pP2^d) (see Corollary 2.11(i')(2'))

$$\mathcal{G}^R_2 = (G^+_{+\infty}, \vee, \wedge, \to^R_2, \rightsquigarrow^R_2, \mathbf{0} = 0, \mathbf{1} = +\infty)$$

(or, equivalently, S^R is a normal filter of the right-pseudo-product algebra (see [19], Corollary 5.12(i')(2'))

$$\mathcal{G}^R_{m2'} = (G^+_{+\infty}, \vee, \wedge, \oplus_2, \to^R_2, \rightsquigarrow^R_2, \mathbf{0} = 0, \mathbf{1} = +\infty)),$$

and S^R is compatible if and only if is normal, i.e. $(C^R) \iff (N^R)$.

Proof. (1): $S^L \subseteq G^-$ is a compatible $(\to^L, \rightsquigarrow^L)$-deductive system of $\mathcal{G}^L$, by Proposition 3.7 (1). It follows obviously that S^L is a compatible $(\to^L_2, \rightsquigarrow^L_2)$-deductive system of $\mathcal{G}^L_2$.
S^L is a normal filter of $\mathcal{G}^L_m$, by Proposition 3.5 (1). It follows obviously that S^L is a normal filter of $\mathcal{G}^L_{m2'}$.
$(C^L) \iff (N^L)$ by Theorem 3.8 (1), since $\mathcal{G}^L_2$ $(\mathcal{G}^L_{m2'})$ verifies condition (pdiv).
(1') has a similar proof. □

In other words, the above Theorem 3.22 says that normality (compatibility) at l-group (l-implicative-group) G level is inherited by the algebras obtained by restricting the l-group (l-implicative-group) operations to $G^-_{-\infty}$ and to $G^+_{+\infty}$. Also, it says that the equivalence $(C_{ig}) \iff (N_g)$ (*compatible* if and only if *normal*), existing at l-group (l-implicative-group) level (Theorem 3.3), is preserved by the algebras obtained by restricting the l-group (l-implicative-group) operations to $G^-_{-\infty}$ and to $G^+_{+\infty}$.

Finally, we have the following general complex result:

Corollary 3.23
(1) Let $\mathcal{A}^L = (A^L, \wedge, \vee, \to^L, \rightsquigarrow^L, 0, 1)$ be a left-pseudo-Hájek(pP) algebra (with the pseudo-product $\odot$) verifying (pP1), (pP2) (or let $\mathcal{A}^L_{m'} = (A^L, \wedge, \vee, \odot, \to^L, \rightsquigarrow^L, 0, 1)$ be a left-pseudo-product algebra). Let S^L be a $(\to^L, \rightsquigarrow^L)$-deductive system of $\mathcal{A}^L$ (or, equivalently, a filter of $\mathcal{A}^L_{m'}$).
Then S^L is compatible if and only if is normal, i.e. $(C^L) \iff (N^L)$.
(1') Let $\mathcal{A}^R = (A^R, \vee, wedge, \to^R, \rightsquigarrow^R, 0, 1)$ be a right-pseudo-Hájek(pS) algebra (with the pseudo-sum $\oplus$) verifying (pP1^d), (pP2^d) (or let $\mathcal{A}^R_{m'} = (A^R, \vee, \wedge, \oplus, \to^R, \rightsquigarrow^R, 0, 1)$ be a right-pseudo-product algebra). Let S^R be a $(\to^R, \rightsquigarrow^R)$-deductive system of $\mathcal{A}^R$ (or, equivalently, an ideal of $\mathcal{A}^R_{m'}$).
Then S^R is compatible if and only if is normal, i.e. $(C^R) \iff (N^R)$.

Proof. (1): It follows by Theorem 3.8, because both $\mathcal{A}^L$ and $\mathcal{A}^L_{m'}$ satisfy condition (pdiv).
(1'): similarly. □

4 Representability

4.1 Representable l-groups, l-implicative-groups

Recall (see [1], for example) that an l-group is *representable* if it is a subdirect product of totally-ordered groups. Recall also the following theorem that gives characterizations of representable l-groups, some of them needed in the sequel.

Theorem 4.1 *(see [1], Theorem 4.1.1)*

The following are equivalent for an l-group $\mathcal{G} = (G, \vee, \wedge, +, -, 0)$:
(a) $\mathcal{G}$ is representable.
(b) For all $a, b \in G$, $2(a \wedge b) = 2a \wedge 2b$;
(b^d) For all $a, b \in G$, $2(a \vee b) = 2a \vee 2b$.
(c) For all $a, b \in G$, $a \wedge (-b - a + b) \leq 0$;
(c^d) For all $a, b \in G$, $a \vee (-b - a + b) \geq 0$.
(d) Each polar subgroup is normal.
(e) Each minimal prime subgroup is normal.
(f) For each $a \in G$, $a > 0$, $a \wedge (-b + a + b) > 0$, for all $b \in G$;
(f^d) For each $a \in G$, $a < 0$, $a \vee (-b + a + b) < 0$, for all $b \in G$.
Note that ^{d} means "dual".

Remarks 4.2 Note that in commutative l-groups we have, for all $a, b \in G$:

$$2(a \wedge b) = 2a \wedge 2b \Longleftrightarrow (b \to a) \wedge (a \to b) \leq 0.$$
$$2(a \vee b) = 2a \vee 2b \Longleftrightarrow (b \to a) \vee (a \to b) \geq 0.$$

Indeed, for example:
$2(a \vee b) = 2a \vee 2b \Longleftrightarrow$
$(a \vee b) + (a \vee b) = 2a \vee 2b \Longleftrightarrow$
$2a \vee 2b = [a + (a \vee b)] \vee [b + (a \vee b)] \Longleftrightarrow$
$2a \vee 2b = 2a \vee (a + b) \vee (b + a) \vee 2b \Longleftrightarrow$
$2a \vee 2b = 2a \vee 2b \vee (a + b) \Longleftrightarrow$
$2a \vee 2b \geq a + b \Longleftrightarrow$
$(2a \vee 2b) - b \geq a \Longleftrightarrow$
$(2a - b) \vee b \geq a \Longleftrightarrow$
$[(2a - b) \vee b] - a \geq 0 \Longleftrightarrow$
$(a - b) \vee (b - a) \geq 0 \Longleftrightarrow$
$(b \to a) \vee (a \to b) \geq 0.$

We obtain in the non-commutative case the following results.

Proposition 4.3 *Let $\mathcal{G} = (G, \vee, \wedge, +, -, 0)$ be an l-group. Then*

$$(b) \iff (b1) \iff (b2), \qquad (b^d) \iff (b1^d) \iff (b2^d),$$

where:
(b1) for all $a, b \in G$, $(b \to a) \wedge (a \rightsquigarrow b) \leq 0 \wedge [(b \rightsquigarrow a) \rightsquigarrow (b \to a)]$,
(b2) for all $a, b \in G$, $(b \rightsquigarrow a) \wedge (a \to b) \leq 0 \wedge [(b \to a) \to (b \rightsquigarrow a)]$;
($b1^d$) for all $a, b \in G$, $(b \to a) \vee (a \rightsquigarrow b) \geq 0 \vee [(b \rightsquigarrow a) \rightsquigarrow (b \to a)]$,
($b2^d$) for all $a, b \in G$, $(b \rightsquigarrow a) \vee (a \to b) \geq 0 \vee [(b \to a) \to (b \rightsquigarrow a)]$.

Proof.

(b^d) $\Longleftrightarrow$ ($b1^d$): $2(a \vee b) = 2a \vee 2b \Longleftrightarrow$
$(a \vee b) + (a \vee b) = 2a \vee 2b \Longleftrightarrow$
$[a + (a \vee b)] \vee [b + (a \vee b)] = 2a \vee 2b \Longleftrightarrow$
$2a \vee (a + b) \vee (b + a) \vee 2b = 2a \vee 2b \Longleftrightarrow$
$2a \vee 2b \vee (a + b) \vee (b + a) = 2a \vee 2b \Longleftrightarrow$
$2a \vee 2b \geq (a + b) \vee (b + a) \Longleftrightarrow$
$(2a \vee 2b) - b \geq [(a + b) \vee (b + a)] - b \Longleftrightarrow$
$(2a - b) \vee b \geq a \vee (b + a - b) \Longleftrightarrow$
$-a + [(2a - b) \vee b] \geq -a + [a \vee (b + a - b)] \Longleftrightarrow$
$(a - b) \vee (-a + b) \geq 0 \vee (-a + b + a - b) \Longleftrightarrow$
$(b \to a) \vee (a \rightsquigarrow b) \geq -a + b + [(-b + a) \vee (a - b)] = -(-b + a) + [(b \rightsquigarrow a) \vee (b \to a)] \Longleftrightarrow$

$(b \to a) \vee (a \rightsquigarrow b) \geq (b \rightsquigarrow a) \rightsquigarrow [(b \rightsquigarrow a) \vee (b \to a)] \overset{(29)}{=} 0 \vee [(b \rightsquigarrow a) \rightsquigarrow (b \to a)]$.

(b^d) $\Longleftrightarrow$ ($b2^d$): $2(a \vee b) = 2a \vee 2b \Longleftrightarrow \ldots \Longleftrightarrow$
$2a \vee 2b \geq (b + a) \vee (a + b) \Longleftrightarrow$
$[a \vee (2b - a)] + a \geq [b \vee (a + b - a)] + a \Longleftrightarrow$
$a \vee (2b - a) \geq b \vee (a + b - a) \Longleftrightarrow$
$b + [(-b + a) \vee (b - a)] \geq b + [0 \vee (-b + a + b - a)] \Longleftrightarrow$
$(-b + a) \vee (b - a) \geq 0 \vee (-b + a + b - a) \Longleftrightarrow$
$(b \rightsquigarrow a) \vee (a \to b) \geq [(a - b) \vee (-b + a)] + b - a \Longleftrightarrow$
$(b \rightsquigarrow a) \vee (a \to b) \geq [(a - b) \vee (-b + a)] - (a - b) \Longleftrightarrow$
$(b \rightsquigarrow a) \vee (a \to b) \geq (b \to a) \to [(b \to a) \vee (b \rightsquigarrow a)] = 0 \vee [(b \to a) \to (b \rightsquigarrow a)]$.

The rest of the proof is similar. □

Remarks 4.4 (see Remarks 4.2)

Note that

$$(b1) \implies (b1"), \quad (b2) \implies (b2"); \qquad (b1^d) \implies (b1^{d}"), \quad (b2^d) \implies (b2^{d}"),$$

where:
(b1") for all $a, b \in G$, $(b \to a) \wedge (a \rightsquigarrow b) \leq 0$,
(b2") for all $a, b \in G$, $(b \rightsquigarrow a) \wedge (a \to b) \leq 0$;
($b1^{d}$") for all $a, b \in G$, $(b \to a) \vee (a \rightsquigarrow b) \geq 0$,
($b2^{d}$") for all $a, b \in G$, $(b \rightsquigarrow a) \vee (a \to b) \geq 0$.

Note that the converse implications are not true.

Note also that (b1") and (b2") coincide and that ($b1^{d}$") and ($b2^{d}$") coincide.

Proposition 4.5 *Let $\mathcal{G} = (G, \vee, \wedge, +, -, 0)$ be an l-group. Then*

$$(c) \iff (c1) \iff (c2), \qquad (c^d) \iff (c1^d) \iff (c2^d),$$

where:
(c1) for all $x, y, z, w \in G$, $(x \rightsquigarrow y) \wedge (([((y \rightsquigarrow x) \rightsquigarrow z) \rightsquigarrow z] \to w) \to w) \leq 0$,
(c2) for all $x, y, z, w \in G$, $(x \to y) \wedge (([((y \to x) \to z) \to z] \rightsquigarrow w) \rightsquigarrow w) \leq 0$;
($c1^d$) for all $x, y, z, w \in G$, $(x \rightsquigarrow y) \vee (([((y \rightsquigarrow x) \rightsquigarrow z) \rightsquigarrow z] \to w) \to w) \geq 0$,
($c2^d$) for all $x, y, z, w \in G$, $(x \to y) \vee (([((y \to x) \to z) \to z] \rightsquigarrow w) \rightsquigarrow w) \geq 0$.

Proof.

(c^d) $\implies$ ($c1^d$): $(x \rightsquigarrow y) \vee (([((y \rightsquigarrow x) \rightsquigarrow z) \rightsquigarrow z] \to w) \to w) =$
$(-x + y) \vee (([-(-(-y + x) + z) + z] \to w) \to w) =$
$(-x + y) \vee (([-(-x + y + z) + z] \to w) \to w) =$
$(-x + y) \vee (([-z - y + x + z] \to w) \to w) =$
$(-x + y) \vee ((w - [-z - y + x + z]) \to w) =$
$(-x + y) \vee ((w - z - x + y + z) \to w) =$
$(-x + y) \vee (w - (w - z - x + y + z)) =$
$(-x + y) \vee (w - z - y + x + z - w) =$
$(-x + y) \vee ((w - z) - (-x + y) + (z - w)) =$
$a \vee (-b - a + b) \geq 0$, by ($c^d$).

($c1^d$) $\implies$ (c^d): Take $x = 0$, $y = a$, $z = 0$, $w = -b$ in ($c1^d$); we obtain:
$(0 \rightsquigarrow a) \vee (([((a \rightsquigarrow 0) \rightsquigarrow 0) \rightsquigarrow 0] \to -b) \to -b) \geq 0 \Longleftrightarrow$
$a \vee ((-a \to -b) \to -b) \geq 0 \Longleftrightarrow$
$a \vee ((-b - (-a)) \to -b) \geq 0 \Longleftrightarrow$
$a \vee ((-b + a) \to -b) \geq 0 \Longleftrightarrow$
$a \vee (-b - (-b + a)) \geq 0 \Longleftrightarrow$
$a \vee (-b - a + b) \geq 0$. Thus $(c^d) \iff (c1^d)$.

(c^d) $\Longrightarrow$ (c2^d): $(x \to y) \vee (([((y \to x) \to z) \to z] \rightsquigarrow w) \rightsquigarrow w) =$
$(y - x) \vee (([z - (z - (x - y))] \rightsquigarrow w) \rightsquigarrow w) =$
$(y - x) \vee (([z - (z + y - x)] \rightsquigarrow w) \rightsquigarrow w) =$
$(y - x) \vee (([z + x - y - z] \rightsquigarrow w) \rightsquigarrow w) =$
$(y - x) \vee ((-[z + x - y - z] + w) \rightsquigarrow w) =$
$(y - x) \vee ((z + y - x - z + w) \rightsquigarrow w) =$
$(y - x) \vee (-(z + y - x - z + w) + w) =$
$(y - x) \vee (-w + z + x - y - z + w) =$
$a \vee (-b - a + b) \geq 0$, by (c^d).
(c2^d) $\Longrightarrow$ (c^d): Take $x = 0$, $y = a$, $z = 0$, $w = b$ in (c2^d); we obtain:
$(0 \to a) \vee (([((a \to 0) \to 0) \to 0] \rightsquigarrow b) \rightsquigarrow b) \geq 0 \Longleftrightarrow$
$a \vee ((-a \rightsquigarrow b) \rightsquigarrow b) \geq 0 \Longleftrightarrow$
$a \vee ((a + b) \rightsquigarrow b) \geq 0 \Longleftrightarrow$
$a \vee (-b - a + b) \geq 0$. Thus $(c^d) \iff (c2^d)$.
The rest of the proof is similar. □

We shall say that an l-implicative-group is *representable* if it is a subdirect product of totally-ordered implicative-groups. Consequently, an l-implicative-group is representable if and only if its term equivalent l-group is representable. Then we have the following result, needed in the sequel.

Theorem 4.6
The following are equivalent for an l-implicative-group $\mathcal{G} = (G, \vee, \wedge, \to, \rightsquigarrow, 0)$:
(a) $\mathcal{G}$ is representable, (b1), (b2), (b1^d), (b2^d), (c1), (c2), (c1^d), (c2^d).

Proof. By Theorem 4.1 and Propositions 4.3, 4.5. □
We can put together Theorems 4.1 and 4.6 in the following resuming statement:

Theorem 4.7 *Let $\mathcal{G} = (G, \vee, \wedge, +, -, 0)$ be an l-group or, equivalently, let $\mathcal{G} = (G, \vee, \wedge, \to, \rightsquigarrow, 0)$ be an l-implicative-group. The following are equivalent:*
(a) $\mathcal{G}$ is representable.

(b) For all $a, b \in G$, $2(a \wedge b) = 2a \wedge 2b$,
(b1) For all $a, b \in G$, $(b \to a) \wedge (a \rightsquigarrow b) \leq 0 \wedge [(b \rightsquigarrow a) \rightsquigarrow (b \to a)]$,
(b2) For all $a, b \in G$, $(b \rightsquigarrow a) \wedge (a \to b) \leq 0 \wedge [(b \to a) \to (b \rightsquigarrow a)]$.

(b^d) For all $a, b \in G$, $2(a \vee b) = 2a \vee 2b$,
(b1^d) For all $a, b \in G$, $(b \to a) \vee (a \rightsquigarrow b) \geq 0 \vee [(b \rightsquigarrow a) \rightsquigarrow (b \to a)]$,
(b2^d) For all $a, b \in G$, $(b \rightsquigarrow a) \vee (a \to b) \geq 0 \vee [(b \to a) \to (b \rightsquigarrow a)]$.

(c) For all $a, b \in G$, $a \wedge (-b - a + b) \leq 0$,
(c1) For all $x, y, z, w \in G$, $(x \rightsquigarrow y) \wedge (([((y \rightsquigarrow x) \rightsquigarrow z) \rightsquigarrow z] \to w) \to w) \leq 0$,
(c2) For all $x, y, z, w \in G$, $(x \to y) \wedge (([((y \to x) \to z) \to z] \rightsquigarrow w) \rightsquigarrow w) \leq 0$.

(c^d) For all $a, b \in G$, $a \vee (-b - a + b) \geq 0$,
(c1^d) For all $x, y, z, w \in G$, $(x \rightsquigarrow y) \vee (([((y \rightsquigarrow x) \rightsquigarrow z) \rightsquigarrow z] \to w) \to w) \geq 0$,
(c2^d) For all $x, y, z, w \in G$, $(x \to y) \vee (([((y \to x) \to z) \to z] \rightsquigarrow w) \rightsquigarrow w) \geq 0$.

(d) Each polar subgroup is normal.
(e) Each minimal prime subgroup is normal.
(f) For each $a \in G$, $a > 0$, $a \wedge (-b + a + b) > 0$, for all $b \in G$;
(f^d) For each $a \in G$, $a < 0$, $a \vee (-b + a + b) < 0$, for all $b \in G$.

4.2 Connections between the representability at l-implicative-group G level and the representability at G^-, G^+ level

• Recall that in the **commutative case**:

A left-residuated lattice $\mathcal{A}^L = (A^L, \wedge, \vee, \odot, \to^L, 1)$ or, equivalently, a left-BCK(P) lattice $\mathcal{A}^L = (A^L, \wedge, \vee, \to^L, 1)$ with product:

(P) there exist $x \odot y \overset{notation}{=} \min\{z \mid x \le y \to^L z\}$, for all $x, y \in A^L$,

is *representable* if it is a subdirect product of linearly-ordered ones. It is known that representable such algebras are characterized by the prelinearity condition:

$$(prel) \qquad (x \to^L y) \vee (y \to^L x) = 1.$$

Dually, a right-residuated lattice $\mathcal{A}^R = (A^R, \vee, \wedge, \oplus, \to^R, 0)$ or, equivalently, a right-BCK(S) lattice $\mathcal{A}^R = (A^R, \vee, \wedge, \to^R, 0)$ with sum:

(S) there exist $x \oplus y \overset{notation}{=} \max\{z \mid x \ge y \to^R z\}$, for all $x, y \in A^R$,

is *representable* if it is a subdirect product of linearly-ordered ones; representable such algebras are characterized by the dual prelinearity condition:

$$(prel^d) \qquad (x \to^R y) \wedge (y \to^R x) = 0.$$

Then we have the following result:

Theorem 4.8 *Let $\mathcal{G} = (G, \vee, \wedge, \to, 0)$ be a representable commutative l-implicative-group.*

(1) Define, for all $x, y \in G^-$:

$$x \to^L y \overset{def.}{=} (x \to y) \wedge 0. \tag{43}$$

Then, $\mathcal{G}^L = (G^-, \wedge, \vee, \to^L, \mathbf{1} = 0)$ is a representable left-BCK(P) lattice.

(1') Define, for all $x, y \in G^+$:

$$x \to^R y \overset{def.}{=} (x \to y) \vee 0. \tag{44}$$

Then, $\mathcal{G}^R = (G^+, \vee, \wedge, \to^R, \mathbf{0} = 0)$ is a representable right-BCK(S) lattice.

Proof.

(1): By Theorem 2.9, $\mathcal{G}^L$ is a left-BCK(P) lattice. To prove that it is representable, we must prove that (prel) holds. Indeed, $(x \to^L y) \vee (y \to^L x) = [(x \to y) \wedge 0] \vee [(y \to x) \wedge 0] = [(x \to y) \vee (y \to x)] \wedge 0 = 0$, by Theorem 4.1 and Remarks 4.2.

(1') By Theorem 2.9, $\mathcal{G}^R$ is a right-BCK(S) lattice. To prove that it is representable, we must prove that (prel^d) holds. Indeed, $(x \to^R y) \wedge (y \to^R x) = [(x \to y) \vee 0] \wedge [(y \to x) \vee 0] = [(x \to y) \wedge (y \to x)] \vee 0 = 0$, by Theorem 4.1 and Remarks 4.2. □

• Recall that in the **non-commutative case**, a non-commutative left-residuated lattice $\mathcal{A}^{\mathcal{L}} = (A^L, \wedge, \vee, \odot, \to^L, \rightsquigarrow^L, 1)$ or, equivalently, a left-pseudo-BCK(pP) lattice $\mathcal{A}^L = (A^L, \wedge, \vee, \to^L, \rightsquigarrow^L, 1)$ (with the pseudo-product $\odot$) is *representable* if it is a subdirect product of linearly-ordered ones. C.J. van Alten [2] proved that such non-commutative algebras are representable if and only if they satisfy the identity:

$$(x \rightsquigarrow^L y) \vee (([((y \rightsquigarrow^L x) \rightsquigarrow^L z) \rightsquigarrow^L z] \to^L w) \to^L w) = 1, \tag{45}$$

or the identity

$$(x \to^L y) \vee (([((y \to^L x) \to^L z) \to^L z] \rightsquigarrow^L w) \rightsquigarrow^L w) = 1. \tag{46}$$

Dually, a non-commutative right-residuated lattice $\mathcal{A}^R = (A^R, \vee, \wedge, \oplus, \to^R, \rightsquigarrow^R, 0)$ or, equivalently, a right-pseudo-BCK(pS) lattice $\mathcal{A}^R = (A^R, \vee, \wedge, \to^R, \rightsquigarrow^R, 0)$ (with the pseudo-sum $\oplus$) is *representable* if it is a subdirect product of linearly-ordered ones; representable such algebras are characterized then by the dual condition:

$$(x \rightsquigarrow^R y) \wedge (([((y \rightsquigarrow^R x) \rightsquigarrow^R z) \rightsquigarrow^R z] \to^R w) \to^R w) = 0, \tag{47}$$

or

$$(x \to^R y) \wedge (([((y \to^R x) \to^R z) \to^R z] \rightsquigarrow^R w) \rightsquigarrow^R w) = 0. \tag{48}$$

We shall prove the following result:

Theorem 4.9 *(see Theorem 2.9)*
Let $\mathcal{G} = (G, \vee, \wedge, \to, \rightsquigarrow, 0)$ be a representable l-implicative-group. Then,
(1) $\mathcal{G}^L = (G^-, \wedge, \vee, \to^L, \rightsquigarrow^L, \mathbf{1} = 0)$ is a representable left-pseudo-BCK(pP) lattice (with the pseudo-product $\odot = +$).
(1') $\mathcal{G}^R = (G^+, \vee, \wedge, \to^R, \rightsquigarrow^R, \mathbf{0} = 0)$ is a representable right-pseudo-BCK(pS) lattice (with the pseudo-sum $\oplus = +$).

Proof.
(1): By Theorem 2.9, $\mathcal{G}^L$ is a left-pseudo-BCK(pP) lattice. To prove that $\mathcal{G}^L$ is representable, we must prove that condition (45), for example, holds. First denote:

$$A \overset{notation}{=} ((y \rightsquigarrow^L x) \rightsquigarrow^L z) \rightsquigarrow^L z,$$

$$B \overset{notation}{=} (A \to^L w) \to^L w,$$

$$C \overset{notation}{=} (x \rightsquigarrow^L y) \vee B.$$

We must prove, by (45), that $C = \mathbf{1}$. Indeed,

• **First proof**:
$A = ((y \rightsquigarrow^L x) \rightsquigarrow^L z) \rightsquigarrow^L z = ([(-y + x) \wedge 0] \rightsquigarrow^L z) \rightsquigarrow^L z =$
$[(-[(-y + x) \wedge 0] + z) \wedge 0] \rightsquigarrow^L z =$
$[([(-x + y) \vee 0] + z) \wedge 0] \rightsquigarrow^L z =$
$[[(-x + y + z) \vee z] \wedge 0] \rightsquigarrow^L z =$
$(-[[(-x + y + z) \vee z] \wedge 0] + z) \wedge 0 =$
$([-((-x + y + z) \vee z) \vee 0] + z) \wedge 0 =$
$([[(-z - y + x) \wedge (-z)] \vee 0] + z) \wedge 0 =$
$(([(-z - y + x) \wedge (-z)] + z) \vee z) \wedge 0 =$
$(((-z - y + x + z) \wedge 0) \vee z) \wedge 0 =$
$[(-z - y + x + z) \wedge 0] \vee z =$
$[(-z - y + x + z) \vee z] \wedge 0.$
$B = (A \to^L w) \to^L w =$
$[(w - A) \wedge 0] \to^L w =$
$(w - [(w - A) \wedge 0]) \wedge 0 =$
$(w + [(A - w) \vee 0]) \wedge 0 =$
$((w + A - w) \vee w) \wedge 0 =$
$[(w + ([(-z - y + x + z) \vee z] \wedge 0) - w) \vee w] \wedge 0 =$
$[([(w + [(-z - y + x + z) \vee z]) \wedge w] - w) \vee w] \wedge 0 =$
$[([[(w - z - y + x + z) \vee (w + z)] \wedge w] - w) \vee w] \wedge 0 =$
$[([(w - z - y + x + z - w) \vee (w + z - w)] \wedge 0) \vee w] \wedge 0 =$
$[[(w - z - y + x + z - w) \wedge 0] \vee [(w + z - w) \wedge 0] \vee w] \wedge 0 =$
$[(w - z - y + x + z - w) \wedge 0] \vee [(w + z - w) \wedge 0] \vee w \geq$
$(w - z - y + x + z - w) \wedge 0.$
Hence,
$C = (x \rightsquigarrow^L y) \vee B \geq$
$[(-x + y) \wedge 0] \vee [(w - z - y + x + z - w) \wedge 0] =$
$[(-x + y) \vee (w - z - y + x + z - w)] \wedge 0 =$
$[a \vee (-b - a + b)] \wedge 0$, with $a = -x + y$, $b = z - w$.
But $\mathcal{G}$ is representable, hence by Theorem 4.1 (c^d), for all $a, b \in G$, $a \vee (-b - a + b) \geq 0$. Hence $C \geq 0$ and thus $C = 0$, i.e. $C = \mathbf{1}$.

• **Second proof**: Denote

$$D \overset{notation}{=} ((y \rightsquigarrow x) \rightsquigarrow z) \rightsquigarrow z,$$

$$E \overset{notation}{=} (D \to w) \to w.$$

By Theorem 4.6 ($c1^d$), we have

$$(x \rightsquigarrow y) \vee E \geq 0. \tag{49}$$

Then,

$A = ((y \rightsquigarrow^L x) \rightsquigarrow^L z) \rightsquigarrow^L z = [([(y \rightsquigarrow x) \wedge 0] \rightsquigarrow z) \wedge 0] \rightsquigarrow^L z \overset{(28)}{=}$
$[(((y \rightsquigarrow x) \rightsquigarrow z) \vee (0 \rightsquigarrow z)) \wedge 0] \rightsquigarrow^L z =$
$[(((y \rightsquigarrow x) \rightsquigarrow z) \vee z) \wedge 0] \rightsquigarrow^L z \overset{distrib.}{=}$
$[[((y \rightsquigarrow x) \rightsquigarrow z) \wedge 0] \vee (z \wedge 0)] \rightsquigarrow^L z$
$([[((y \rightsquigarrow x) \rightsquigarrow z) \wedge 0] \vee z] \rightsquigarrow z) \wedge 0 \overset{(27)}{=}$
$([((y \rightsquigarrow x) \rightsquigarrow z) \wedge 0] \rightsquigarrow z) \wedge (z \rightsquigarrow z) \wedge 0 \overset{(28)}{=}$
$([((y \rightsquigarrow x) \rightsquigarrow z) \rightsquigarrow z] \vee (0 \rightsquigarrow z)) \wedge 0 = (D \vee z) \wedge 0.$

$B = (A \to^L w) \to^L w = ([(D \vee z) \wedge 0] \to w) \wedge 0] \to^L w \overset{(28)}{=}$
$[[((D \vee z) \to w) \vee (0 \to w)] \wedge 0] \to^L w =$
$([[((D \vee z) \to w) \vee w] \wedge 0] \to w) \wedge 0 \overset{(28)}{=}$
$(([((D \vee z) \to w) \vee w] \to w) \vee (0 \to w)) \wedge 0 \overset{(27)}{=}$
$([(((D \vee z) \to w) \to w) \wedge (w \to w)] \vee w) \wedge 0 \overset{distrib.}{=}$
$[[(((D \vee z) \to w) \to w) \vee w] \wedge (0 \vee w)] \wedge 0 =$
$[(((D \vee z) \to w) \to w) \vee w] \wedge 0 \overset{(27)}{=}$
$[([(D \to w) \wedge (z \to w)] \to w) \vee w] \wedge 0 \overset{(28)}{=}$
$[[((D \to w) \to w) \vee ((z \to w) \to w))] \vee w] \wedge 0 =$
$[E \vee ((z \to w) \to w) \vee w] \wedge 0.$

$C = (x \rightsquigarrow^L y) \vee B =$
$[(x \rightsquigarrow y) \wedge 0] \vee [(E \vee ((z \to w) \to w) \vee w) \wedge 0] \overset{distrib.}{=}$
$[(x \rightsquigarrow y) \vee E \vee ((z \to w) \to w) \vee w] \wedge 0 = 0,$
since $(x \rightsquigarrow y) \vee E \vee ((z \to w) \to w) \vee w \geq (x \rightsquigarrow y) \vee E \geq 0$, by (49), and hence $[(x \rightsquigarrow y) \vee E] \wedge 0 = 0$. Thus, $C = \mathbf{1}$.

(1') has a similar proof, using Theorem 4.1 (c), in the first proof, and Theorem 4.6 (c1), in the second proof. □

Finaly, we present some intermediary results and an open problem.

Theorem 4.10 *(see Theorem 2.9)*

Let $\mathcal{G} = (G, \vee, \wedge, \to, \rightsquigarrow, 0)$ be a representable l-implicative-group. Then,

(1) the reversed left-pseudo-BCK(pP) lattice $\mathcal{G}^L = (G^-, \wedge, \vee, \to^L, \rightsquigarrow^L, \mathbf{1} = 0)$ (with the pseudo-product $\odot = +$), verifying condition (pC), verifies also the following conditions: for all $a, b \in G^-$,
(i) $(a \vee b)^2 = a^2 \vee b^2$, i.e. $(a \vee b) \odot (a \vee b) = (a \odot a) \vee (b \odot b)$,
(ii) Condition (i) is equivalent with condition

$$[b \to^L (a \rightsquigarrow^L (a \odot a))] \vee [a \rightsquigarrow^L (b \to^L (b \odot b))] = \mathbf{1}. \tag{50}$$

(iii) $(b \to^L a) \vee (a \rightsquigarrow^L b) = \mathbf{1}$,
(iv) Condition (iii) implies condition (50).

(1') the reversed right-pseudo-BCK(pS) lattice $\mathcal{G}^R = (G^+, \vee, \wedge, \to^R, \rightsquigarrow^R, \mathbf{0} = 0)$ (with the pseudo-sum $\oplus = +$), verifying the dual condition (pC^d), verifies also the following conditions: for all $a, b \in G^+$,
(i') $2(a \wedge b) = 2a \wedge 2b$, i.e. $(a \wedge b) \oplus (a \wedge b) = (a \oplus a) \wedge (b \oplus b)$,
(ii') Condition (i') is equivalent with condition

$$[b \to^R (a \rightsquigarrow^R (a \oplus a))] \vee [a \rightsquigarrow^R (b \to^R (b \oplus b))] = \mathbf{0}. \tag{51}$$

(iii') $(b \to^R a) \wedge (a \rightsquigarrow^R b) = \mathbf{0}$,
(iv') Condition (iii') implies condition (51).

Proof. We prove (1). We denote $\rightarrow = \rightarrow^L$ and $\rightsquigarrow = \rightsquigarrow^L$.

(i): follows obviously by Theorem 4.7 (b^d), since $\mathcal{G}$ is representable.

(ii): We shall prove that (i) $\Longleftrightarrow$ (50). Indeed,

(i) $\Longrightarrow$ (50):

(i) $(a \vee b) \odot (a \vee b) = (a \odot a) \vee (b \odot b) \Longleftrightarrow$
$[(a \vee b) \odot a] \vee [(a \vee b) \odot b] = (a \odot a) \vee (b \odot b) \Longleftrightarrow$
$a \odot a \vee b \odot a \vee a \odot b \vee b \odot b = a \odot a \vee b \odot b \Longleftrightarrow$

$$a \odot b \vee b \odot a \leq a \odot a \vee b \odot b. \quad (52)$$

And (52) $\Longrightarrow$
$a \odot b \leq a \odot a \vee b \odot b \Longrightarrow$
$b \rightarrow (a \odot b) \leq b \rightarrow (a \odot a \vee b \odot b) \Longrightarrow$

$$a \rightsquigarrow (b \rightarrow (a \odot b)) \leq a \rightsquigarrow (b \rightarrow (a \odot a \vee b \odot b)). \quad (53)$$

But $a \rightsquigarrow (b \rightarrow (a \odot b) = b \rightarrow (a \rightsquigarrow (a \odot b)) \leq b \rightarrow b = \mathbf{1}$, since $b \leq a \rightsquigarrow (a \odot b)$. Hence, (53) $\Longrightarrow$
$a \rightsquigarrow (b \rightarrow (a \odot a \vee b \odot b)) = \mathbf{1} \overset{(pprel)}{\Longleftrightarrow}$
$a \rightsquigarrow [(b \rightarrow a \odot a) \vee (b \rightarrow b \odot b)] = \mathbf{1} \overset{(pprel)}{\Longleftrightarrow}$
$[a \rightsquigarrow (b \rightarrow a \odot a)] \vee [a \rightsquigarrow (b \rightarrow b \odot b)] = \mathbf{1} \Longleftrightarrow$
$[b \rightarrow (a \rightsquigarrow (a \odot a))] \vee [a \rightsquigarrow (b \rightarrow (b \odot b))] = \mathbf{1}$, i.e.(50) holds.
Note we have used an equivalent condition with (pprel) denoted ($\text{pprel}_{\Rightarrow\vee}$) in [18], pag. 386:
($\text{pprel}_{\Rightarrow\vee}$) $x \rightarrow (y \vee z) = (x \rightarrow y) \vee (x \rightarrow z)$ and $x \rightsquigarrow (y \vee z) = (x \rightsquigarrow y) \vee (x \rightsquigarrow z)$.

50) $\Longrightarrow$ (i):

(50) $[b \rightarrow (a \rightsquigarrow (a \odot a))] \vee [a \rightsquigarrow (b \rightarrow (b \odot b))] = \mathbf{1} \Longleftrightarrow$
$[a \rightsquigarrow (b \rightarrow (a \odot a))] \vee [a \rightsquigarrow (b \rightarrow (b \odot b))] = \mathbf{1} \overset{(pprel)}{\Longleftrightarrow}$
$a \rightsquigarrow (b \rightarrow (a \odot a \vee b \odot b)) = \mathbf{1} \Longleftrightarrow$
$\mathbf{1} \leq a \rightsquigarrow (b \rightarrow (a \odot a \vee b \odot b)) \Longrightarrow$
$a = a \odot \mathbf{1} \leq a \odot [a \rightsquigarrow (b \rightarrow (a \odot a \vee b \odot b))] \overset{(pdiv)}{\Longleftrightarrow}$
$a \leq a \wedge (b \rightarrow (a \odot a \vee b \odot b)) \leq a \Longrightarrow$
$a = a \wedge (b \rightarrow (a \odot a \vee b \odot b)) \Longleftrightarrow$
$a \leq (b \rightarrow (a \odot a \vee b \odot b)) \Longrightarrow$
$a \odot b \leq (b \rightarrow (a \odot a \vee b \odot b)) \odot b \overset{(pdiv)}{\Longleftrightarrow}$
$a \odot b \leq b \wedge (a \odot a \vee b \odot b) \leq a \odot a \vee b \odot b \Longrightarrow$
$a \odot b \leq a \odot a \vee b \odot b$.
Similarly,
$b \odot a \leq b \odot b \vee a \odot a$,
i.e. $a \odot a \vee b \odot b$ is an upper bound of $a \odot b$ and $b \odot a$. It follows that
$a \odot b \vee b \odot a \leq a \odot a \vee b \odot b$, i.e. (52) holds. And we have seen above that (52) $\Longleftrightarrow$ (i).

(iii): $(b \rightarrow^L a) \vee (a \rightsquigarrow^L b) = [(b \rightarrow a) \wedge 0] \vee [(a \rightsquigarrow b) \wedge 0] =$
$[(b \rightarrow a) \vee (a \rightsquigarrow b)] \wedge 0 \geq (0 \vee [(b \rightsquigarrow a) \rightsquigarrow (b \rightarrow a)]) \wedge 0 = 0 = \mathbf{1}$, by Theorem 4.7 ((a) $\Longleftrightarrow$ ($b1^d$)).

(iv): Condition (iii) implies condition (50). Indeed,
since $a \leq a \rightsquigarrow^L (a \odot a)$ and $b \leq b \rightarrow^L (b \odot b)$ by [18], condition (10.3), it follows that
$b \rightarrow^L a \leq b \rightarrow^L [a \rightsquigarrow^L (a \odot a)]$ and $a \rightsquigarrow^L b \leq a \rightsquigarrow^L [b \rightarrow^L (b \odot b)]$, hence
$\mathbf{1} = (b \rightarrow^L a) \vee (a \rightsquigarrow^L b) \leq (b \rightarrow^L [a \rightsquigarrow^L (a \odot a)]) \vee (a \rightsquigarrow^L [b \rightarrow^L (b \odot b)])$, hence
$(b \rightarrow^L [a \rightsquigarrow^L (a \odot a)]) \vee (a \rightsquigarrow^L [b \rightarrow^L (b \odot b)]) = \mathbf{1}$.

(1') has a similar proof. □

Proposition 4.11 *(see Theorem 2.9) Let $\mathcal{G} = (G, \vee, \wedge, \rightarrow, \rightsquigarrow, 0)$ be an l-implicative-group.*

(1) If $\mathcal{G}$ verifies the condition ($b1^d$") from Remarks 4.4:
($b1^d$") for all $a, b \in G$, $(b \rightarrow a) \vee (a \rightsquigarrow b) \geq 0$,
then the reversed left-pseudo-BCK(pP) lattice $\mathcal{G}^L = (G^-, \wedge, \vee, \rightarrow^L, \rightsquigarrow^L, \mathbf{1} = 0)$ verifies the condition (iii)

from Theorem 4.10 (1):
(iii) for all $a, b \in G^-$, $(b \to^L a) \vee (a \rightsquigarrow^L b) = \mathbf{1} = 0$.
(1') If $\mathcal{G}$ *verifies the condition (b1") from Remarks 4.4:*
(b1") for all $a, b \in G$, $(b \to a) \wedge (a \rightsquigarrow b) \leq 0$,
then the reversed right-pseudo-BCK(pS) lattice $\mathcal{G}^R = (G^+, \vee, \wedge, \to^R, \rightsquigarrow^R, \mathbf{0} = 0)$ *verifies the condition (iii') from Theorem 4.10 (1'):*
(iii') for all $a, b \in G^+$, $(b \to^R a) \wedge (a \rightsquigarrow^R b) = \mathbf{0} = 0$.

Proof. (1): $(b \to^L a) \vee (a \rightsquigarrow^L b) = [(b \to a) \wedge 0] \vee [(a \rightsquigarrow b) \wedge 0] \stackrel{distrib.}{=}$
$[(b \to a) \vee (a \rightsquigarrow b)] \wedge 0 \stackrel{(b1^{d}")}{=} 0 = \mathbf{1}$.
(1'): $(b \to^R a) \wedge (a \rightsquigarrow^R b) = [(b \to a) \vee 0] \wedge [(a \rightsquigarrow b) \vee 0] =$
$[(b \to a) \wedge (a \rightsquigarrow b)] \vee 0 \stackrel{(b1")}{=} 0 = \mathbf{0}$. □

Open problems 4.12
(1) Find if there are connections between the representability of $\mathcal{G}^L = (G^-, \wedge, \vee, \to^L, \rightsquigarrow^L, \mathbf{1} = 0)$ (or of the left-pseudo-MV algebra $[u', 0]$) and the conditions (i) $\Longleftrightarrow$ (50), (iii).
(1') Find if there are connections between the representability of $\mathcal{G}^R = (G^+, \vee, \wedge, \to^R, \rightsquigarrow^R, \mathbf{0} = 0)$ (or of the right-pseudo-MV algebra $[0, u]$) and the conditions (i') $\Longleftrightarrow$ (51), (iii').

Open problem 4.13 Find connections between the representability at l-group (l-implicative-group) G level and the representability at $[u', 0] \subset G^-$, $[0, u] \subset G^+$ level and at $G^-_{-\infty}$, $G^+_{+\infty}$ level.

5 States

We study the additive-states on po-groups [15] and on l-groups and we introduce and study the implicative-states and the Bosbach-states. New properties needed will be first proved.

5.1 New properties

Following the ideas from [7], we define the following distances:

Definition 5.1 Let $(G, \vee, \wedge, +, -, 0)$ be an l-group. We define the following *distance functions*, by (33) and (34):

$$d_1^L(x, y) \stackrel{def.}{=} (x \vee y) \to (x \wedge y) \in G^-, \quad d_2^L(x, y) \stackrel{def.}{=} (x \vee y) \rightsquigarrow (x \wedge y) \in G^-,$$
$$d_1^R(x, y) \stackrel{def.}{=} (x \wedge y) \to (x \vee y) \in G^+, \quad d_2^R(x, y) \stackrel{def.}{=} (x \wedge y) \rightsquigarrow (x \vee y) \in G^+.$$

Proposition 5.2 $(G, \vee, \wedge, +, -, 0)$ *be an* l*-group. Then the above defined distance functions verify the following properties (see [7]): for all* $x, y, z \in G$,
(1) $d_1^L(x, y) = d_1^L(y, x)$, $\quad d_2^L(x, y) = d_2^L(y, x)$,
(1') $d_1^R(x, y) = d_1^R(y, x)$, $\quad d_2^R(x, y) = d_2^R(y, x)$,
(2) $d_1^L(x, y) = 0 \Longleftrightarrow x = y \Longleftrightarrow d_2^L(x, y) = 0$,
(2') $d_1^R(x, y) = 0 \Longleftrightarrow x = y \Longleftrightarrow d_2^R(x, y) = 0$,
(3) $d_1^L(x, 0) = d_2^L(x, 0) = \begin{cases} -x & , \text{ if } x \geq 0 \\ x & , \text{ if } x < 0, \end{cases}$
(3') $d_1^R(x, 0) = d_2^R(x, 0) = \begin{cases} -x & , \text{ if } x \leq 0 \\ x & , \text{ if } x > 0, \end{cases}$
(4) $d_1^L(x, y) = d_2^L(-x, -y)$, $\quad d_2^L(x, y) = d_1^L(-x, -y)$,
(4') $d_1^R(x, y) = d_2^R(-x, -y)$, $\quad d_2^R(x, y) = d_1^R(-x, -y)$,
(5) $d_2^L(x, y) + d_2^L(y, z) + d_2^L(x, y) \leq d_2^L(x, z)$, $\quad d_2^L(y, z) + d_2^L(x, y) + d_2^L(y, z) \leq d_2^L(x, z)$,
(6) $d_1^L(x, y) + d_1^L(y, z) + d_1^L(x, y) \leq d_1^L(x, z)$, $\quad d_1^L(y, z) + d_1^L(x, y) + d_1^L(y, z) \leq d_1^L(x, z)$,
(5') $d_2^R(x, y) + d_2^R(y, z) + d_2^R(x, y) \geq d_2^R(x, z)$, $\quad d_2^R(y, z) + d_2^R(x, y) + d_2^R(y, z) \geq d_2^L(x, z)$,
(6') $d_1^R(x, y) + d_1^R(y, z) + d_1^R(x, y) \geq d_1^R(x, z)$, $\quad d_1^R(y, z) + d_1^R(x, y) + d_1^R(y, z) \geq d_1^R(x, z)$.

Proof.

(1), (1'): Obvious, by (33) and (34).

(2): $d_1^L(x,y) = 0 \Longleftrightarrow (x \to y) \wedge (y \to x) \wedge 0 = 0 \Longleftrightarrow (x \to y) \wedge (y \to x) \geq 0 \Longleftrightarrow x \to y \geq 0,\ y \to x \geq 0 \Longleftrightarrow y - x \geq 0,\ x - y \geq 0 \Longleftrightarrow y \geq x,\ x \geq y \Longleftrightarrow x = y$. The other equivalence has a similar proof.

(2'): has a similar proof.

(3): By (33), $d_1^L(x,0) = (x \to 0) \wedge (0 \to x) \wedge 0 = (-x) \wedge x \wedge 0$:
- if $x \geq 0$, then $-x \leq 0$, hence $d_1^L(x,0) = 0 \wedge (-x) = -x$;
- if $x \leq 0$, then $-x \geq 0$, hence $d_1^L(x,0) = x \wedge 0 = x$.

$d_2^L(x,0) = (x \rightsquigarrow 0) \wedge (0 \rightsquigarrow x) \wedge 0 = (-x) \wedge x \wedge 0 = d_1^L(x,0)$.

(3'): has a similar proof.

(4): $d_1^L(x,y) = (x \to y) \wedge (y \to x) \wedge 0 \overset{(17)}{=} [(-y) \rightsquigarrow (-x)] \wedge [(-x) \rightsquigarrow (-y)] \wedge 0 = d_2^L(-x,-y)$.

$d_2^L(x,y) = (x \rightsquigarrow y) \wedge (y \rightsquigarrow x) \wedge 0 \overset{(17)}{=} [(-y) \to (-x)] \wedge [(-x) \to (-y)] \wedge 0 = d_1^L(-x,-y)$.

(4'): has a similar proof.

(5): by (G9) and Proposition 2.2 (a), we obtain:

$d_2^L(x,y) + d_2^L(y,z) + d_2^L(x,y) =$
$[(x \rightsquigarrow y) \wedge (y \rightsquigarrow x) \wedge 0] \wedge [(y \rightsquigarrow z) \wedge (z \rightsquigarrow y) \wedge 0] \wedge [(x \rightsquigarrow y) \wedge (y \rightsquigarrow x) \wedge 0] =$
$a \wedge b \wedge c \wedge d \wedge e \wedge f \wedge m \wedge n \wedge p \wedge$
$a' \wedge b' \wedge c' \wedge d' \wedge e' \wedge f' \wedge m' \wedge n' \wedge p' \wedge$
$a'' \wedge b'' \wedge c'' \wedge d'' \wedge e'' \wedge f'' \wedge m'' \wedge n'' \wedge p''$,

where:

$a = (x \rightsquigarrow y) + (y \rightsquigarrow z) + (x \rightsquigarrow y)$, $b = (x \rightsquigarrow y) + (y \rightsquigarrow z) + (y \rightsquigarrow x)$, $\mathbf{c} = (x \rightsquigarrow y) + (y \rightsquigarrow z) + 0 = x \rightsquigarrow z$,
$d = (x \rightsquigarrow y) + (z \rightsquigarrow y) + (x \rightsquigarrow y)$, $e = (x \rightsquigarrow y) + (z \rightsquigarrow y) + (y \rightsquigarrow x)$, $f = (x \rightsquigarrow y) + (z \rightsquigarrow y) + 0$,
$m = (x \rightsquigarrow y) + 0 + (x \rightsquigarrow y)$, $\mathbf{n} = (x \rightsquigarrow y) + 0 + (y \rightsquigarrow x) = \mathbf{0}$, $p = (x \rightsquigarrow y) + 0 + 0$,

$a' = (y \rightsquigarrow x) + (y \rightsquigarrow z) + (x \rightsquigarrow y)$, $b' = (y \rightsquigarrow x) + (y \rightsquigarrow z) + (y \rightsquigarrow x)$, $c' = (y \rightsquigarrow x) + (y \rightsquigarrow z) + 0$,
$d' = (y \rightsquigarrow x) + (z \rightsquigarrow y) + (x \rightsquigarrow y)$, $e' = (y \rightsquigarrow x) + (z \rightsquigarrow y) + (y \rightsquigarrow x)$, $f' = (y \rightsquigarrow x) + (z \rightsquigarrow y) + 0$,
$\mathbf{m'} = (y \rightsquigarrow x) + 0 + (x \rightsquigarrow y) = \mathbf{0}$, $n' = (y \rightsquigarrow x) + 0 + (y \rightsquigarrow x)$, $p' = (y \rightsquigarrow x) + 0 + 0$,

$a'' = 0 + (y \rightsquigarrow z) + (x \rightsquigarrow y)$, $b'' = 0 + (y \rightsquigarrow z) + (y \rightsquigarrow x)$, $c'' = 0 + (y \rightsquigarrow z) + 0$,
$d'' = 0 + (z \rightsquigarrow y) + (x \rightsquigarrow y)$, $\mathbf{e''} = 0 + (z \rightsquigarrow y) + (y \rightsquigarrow x) = z \rightsquigarrow x$, $f'' = 0 + (z \rightsquigarrow y) + 0$,
$\mathbf{m''} = 0 + 0 + (x \rightsquigarrow y)$, $n'' = 0 + 0 + (y \rightsquigarrow x)$, $\mathbf{p''} = 0 + 0 + 0 = \mathbf{0}$.

But, $d_2^L(x,y) + d_2^L(y,z) + d_2^L(x,y) \leq \mathbf{c} \wedge \mathbf{e''} \wedge \mathbf{p''} = (x \rightsquigarrow z) \wedge (z \rightsquigarrow x) \wedge 0 = d_2^L(x,z)$.

The second inequality of (5) has a similar proof.

(6), (5'), (6') have similar proofs. □

5.2 Additive-states and implicative-states

We generalize to arbitrary po-groups with strong unit the definition of states for the abelian po-groups from [15].

Definition 5.3 (see also [9]) Let $\mathcal{G} = (G, \leq, +, -, 0)$ be a po-group with strong unit u (i.e. $u \geq 0$ and for every $x \in G$, there exists some positive integer n such that $x \leq \underbrace{u + u + \ldots + u}_{n \text{ times}}$) and let $\mathbf{R} = (\mathbf{R}, \leq, +, -, 0))$ be the additive abelian po-group of real numbers with strong unit 1.

An *additive-state* or a *state* for short on $\mathcal{G}$ is any positive (or equivalently order preserving) group homomorphism $s : G \longrightarrow \mathbf{R}$ verifying $s(u) = 1$, i.e. s is a state iff the following properties hold: for all $x, y \in G$,
(s1) $s(x + y) = s(x) + s(y)$,
(s2) $x \geq 0$ implies $s(x) \geq 0$,
(s3) $s(u) = 1$.

Proposition 5.4 *Let s be a state on (G,u). Then, the following properties hold: for all $x, y \in G$*
(s4) $s(0) = 0$,
(s5) $s(-x) = -s(x)$,
(s6) $x \leq y$ *implies* $s(x) \leq s(y)$,
(s7) $s(x \to y) = s(x) \to s(y) = s(x \rightsquigarrow y)$.

Proof.
(s4): $s(x) = s(x+0) = s(x) + s(0)$ implies $s(0) = 0$.
(s5): $0 = s(0) = s(x + (-x)) = s(x) + s(-x)$ implies $s(-x) = -s(x)$.
(s6): $x \leq y$ implies $y - x \geq 0$ implies $s(y-x) \geq s(0)$, i.e. $s(y) + s(-x) \geq 0$, hence $s(y) - s(x) \geq 0$ hence $s(x) \leq s(y)$.
(S7): $s(x \to y) = s(y - x) = s(y) + s(-x) = s(y) - s(x) = s(x) \to s(y)$ and $s(x \rightsquigarrow y) = s(-x+y) = s(-x) + s(y) = -s(x) + s(y) = s(x) \rightsquigarrow s(y) = s(x) \to s(y)$. □

Definition 5.5 Let $\mathcal{G} = (G, \leq, \to, \rightsquigarrow, 0)$ be a po-implicative-group with strong unit u and let $(\mathbf{R}, \leq, \to, 0)$ be the abelian po-implicative-group with strong unit 1 of real numbers.
An *implicative-state* on (G,u) is any map $s : G \longrightarrow \mathbf{R}$ verifying: for all $x, y \in G$,
(is-1) $s(x \to y) = s(x) \to s(y) = s(x \rightsquigarrow y)$,
(is-2) $x \geq 0$ implies $s(x) \geq 0$,
(is-3) $s(u) = 1$.

Proposition 5.6 *Let s be an implicative-state on the po-implicative-group $\mathcal{G} = (G, \leq, \to, \rightsquigarrow, 0)$ with strong unit u. Then, the following property holds: for all $x, y \in G$*
(is-4) $s(0) = 0$,
(is-5) $s(-x) = -s(x)$.

Proof.
(is-4): $s(0) \overset{(I9)}{=} s(x \to x) = s(x) \to s(x) \overset{(I9)}{=} 0$,
(is-5): $s(-x) = s(x \to 0) = s(x) \to 0 = -s(x)$. □

Theorem 5.7 *The states on the po-group $\mathcal{G}_g = (G, \leq, +, -, 0)$ with strong unit u coincide with the implicative-states on the term equivalent po-implicative-group $\mathcal{G}_{ig} = (G, \leq, \to, \rightsquigarrow, 0)$.*

Proof.
• Let s be a state on $\mathcal{G}_g$. To prove that s is an implicative-state on $\mathcal{G}_{ig}$ it is sufficient to prove that (is-1) holds for all $x, y \in G$. Indeed,
$s(x \to y) = s(y - x) = s(y) + s(-x) = s(y) + (-s(x)) = s(x) \to s(y)$ and $s(x \rightsquigarrow y) = s(-x+y) = s(-x) + s(y) = -s(x) + s(y) = s(x) \rightsquigarrow s(y) = s(x) \to s(y)$.
• Let s be an implicative-state on $\mathcal{G}_{ig}$. To prove that s is a state on $\mathcal{G}_g$ it is sufficient to prove that (s1) holds for all $x, y \in G$. Indeed,
$s(x+y) = s(-(x \to (-y)) = -s(x \to (-y)) = -(s(x) \to s(-y)) = -(s(x) \to (-s(y))) = s(x) + s(y)$. □

5.3 State morphisms and implicative-state morphisms

Definition 5.8 Let $\mathcal{G} = (G, \vee, \wedge, +, -, 0)$ be an l-group with strong unit u and let $\mathbf{R} = (\mathbf{R}, \max, \min, +, -, 0)$ be the additive abelian l-group of real numbers with strong unit 1.
A *state morphism* on $\mathcal{G}$ is a state s on $\mathcal{G}$ verifying the following property: for all $x, y \in G$,
(s0) $s(x \wedge y) = s(x) \wedge s(y) = \min(s(x), s(y))$.

Note that (s0) can be replaced by the weaker condition

$$s(x) \wedge s(y) \leq s(x \wedge y),$$

since $s(x \wedge y) \leq s(x) \wedge s(y)$ always holds (indeed, $x \wedge y \leq x,\ y$ implies $s(x \wedge y) \leq s(x),\ s(y)$, i.e. $s(x \wedge y)$ is a lower bound of $s(x),\ s(y)$; hence, $s(x \wedge y) \leq s(x) \wedge s(y)$).

Proposition 5.9 *Let s be a state-morphism on the l-group with strong unit $\mathcal{G} = (G, \vee, \wedge, +, -, 0)$. Then, for all $x, y \in G$:*
(s0') $s(x \vee y) = s(x) \vee s(y) = \max(s(x), s(y))$.

Proof. $s(x\vee y) \overset{(G12)}{=} s(x-(x\wedge y)+y) = s(x)-s(x\wedge y) = s(y) = s(x)-s(x)\wedge s(y)+s(y) \overset{(G12)}{=} s(x)\vee s(y)$. $\square$

Note that an equivalent definition of a state-morphism on an l-group $\mathcal{G}$ with strong unit would be, by Proposition 5.9 and by (G12), the following:
A *state-morphism* on $\mathcal{G}$ is any state s on $\mathcal{G}$ verifying the following property: for all $x, y \in G$,
(s0') $s(x \vee y) = s(x) \vee s(y) = \max(s(x), s(y))$,
and then (s0) will follow.

Definition 5.10 Let $\mathcal{G} = (G, \vee, \wedge, \to, \rightsquigarrow, 0)$ be an l-implicative-group with strong unit u and let $\mathbf{R} = (\mathbf{R}, \max, \min, \to, 0)$ be the abelian l-implicative-group of real numbers with strong unit 1.
An *implicative-state morphism* on $\mathcal{G}$ is an implicative-state s on $\mathcal{G}$ verifying (s0).

By Theorem 5.7, we immediately obtain that **the states morphisms on the l-group $\mathcal{G}_g = (G, \vee, \wedge, +, -, 0)$ with strong unit u coincide with the implicative-states morphisms on the term equivalent l-implicative-group $\mathcal{G}_{ig} = (G, \vee, \wedge, \to, \rightsquigarrow, 0)$.**

5.4 Bosbach-states

In this subsection, $\mathcal{G} = (G, \vee, \wedge, +, -, 0)$ is an l-group with strong unit u and $\mathbf{R} = (\mathbf{R}, \vee, \wedge, +, -, 0)$ is the additive l-group of real numbers with the strong unit 1; we denote them (G, u) and $(\mathbf{R}, 1)$ respectively.

Proposition 5.11 *(See [12], Proposition 2.1)*
Let $s : G \longrightarrow \mathbf{R}$ such that $s(0) = 0$. Then, the following are equivalent:
(i) $s(x \vee y) + s(d_1^L(x, y)) = s(x \wedge y)$,
(ii) $s(y) + s((y \to x) \wedge 0) = s(x \wedge y)$,
(iii) $s(x) + s((x \to y) \wedge 0) = s(y) + s((y \to x) \wedge 0)$.

Proof.
(i) $\Longleftrightarrow$ (ii): Let us consider $a \le b$ in G; then $a \wedge b = a$ and $a \vee b = b$, hence $d_1^L(a, b) = b \to a$. It follows, by (i), that

$$s(b) + s(b \to a) = s(a). \tag{54}$$

Let us take $a = x \wedge y$ and $b = y$. By (54), we obtain:
$s(y) + s((y \to x) \wedge 0) = s(y) + s(y \to (x \wedge y)) = s(x \wedge y)$, i.e. (ii) holds.
(ii) $\Longleftrightarrow$ (iii): $s(x) = s((x \to y) \wedge 0) \overset{(ii)}{=} s(y \wedge x) = s(x \wedge y) = s(y) + s((y \to x) \wedge 0)$, i.e. (iii) holds.
(iii) $\Longleftrightarrow$ (i): $s(x \vee y) + s(d_1^L(x, y)) = s(x \vee y) + s((x \vee y) \to (x \wedge y)) \overset{(33)}{=}$
$s(x \vee y) + s([(x \vee y) \to (x \wedge y)] \wedge 0) \overset{(iii)}{=} s(x \wedge y) + s([(x \wedge y) \to (x \vee y)] \wedge 0) = s(x \wedge y) + s(0) = s(x \wedge y)$, since
$(x \vee y) \to (x \wedge y) \le 0$ and hence $(x \vee y) \to (x \wedge y) = [(x \vee y) \to (x \wedge y)] \wedge 0$ and since
$x \wedge y \le x \vee y$ and hence $[(x \wedge y) \to (x \vee y)] \overset{Corollary\ 2.3}{\ge} 0$. $\square$
The following proposition has a similar proof:

Proposition 5.12 *(See [12], Proposition 2.2)*
Let $s : G \longrightarrow \mathbf{R}$ such that $s(0) = 0$. Then, the following are equivalent:
(i) $s(x \vee y) + s(d_2^L(x, y)) = s(x \wedge y)$,
(ii) $s(y) + s((y \rightsquigarrow x) \wedge 0) = s(x \wedge y)$,
(iii) $s(x) + s((x \rightsquigarrow y) \wedge 0) = s(y) + s((y \rightsquigarrow x) \wedge 0)$.

Dually, we have:

Proposition 5.13

Let $s : G \longrightarrow \mathbf{R}$ such that $s(0) = 0$. Then, the following are equivalent:

(i') $s(x \wedge y) + s(d_1^R(x, y)) = s(x \vee y)$,

(ii') $s(y) + s((y \to x) \vee 0) = s(x \vee y)$,

(iii') $s(x) + s((x \to y) \vee 0) = s(y) + s((y \to x) \vee 0)$.

Proof.

(i') $\Longleftrightarrow$ (ii'): Let us consider $a \leq b$ in G; then $a \wedge b = a$ and $a \vee b = b$, hence $d_1^L(a, b) = b \to a$. It follows, by (i'), that

$$s(a) + s(a \to b) = s(b). \tag{55}$$

Let us take $a = y$ and $b = x \vee y$. By (55), we obtain:
$s(y) + s((y \to x) \vee 0) = s(y) + s(y \to (x \vee y)) = s(x \vee y)$, i.e. (ii') holds.

(ii') $\Longleftrightarrow$ (iii'): $s(x) = s((x \to y) \vee 0) \stackrel{(ii')}{=} s(y \vee x) = s(x \vee y) = s(y) + s((y \to x) \vee 0)$, i.e. (iii') holds.

(iii') $\Longleftrightarrow$ (i'): $s(x \wedge y) + s(d_1^R(x, y)) = s(x \wedge y) + s((x \wedge y) \to (x \vee y)) \stackrel{(33)}{=}$
$s(x \wedge y) + s([(x \wedge y) \to (x \vee y)] \vee 0) \stackrel{(iii')}{=} s(x \vee y) + s([(x \vee y) \to (x \wedge y)] \vee 0) = s(x \vee y) + s(0) = s(x \vee y)$, since
$x \wedge y \leq x \vee y$ and hence $[(x \wedge y) \to (x \vee y)] \stackrel{Corollary\ 2.3}{\geq} 0$. □

The following proposition has a similar proof:

Proposition 5.14

Let $s : G \longrightarrow \mathbf{R}$ such that $s(0) = 0$. Then, the following are equivalent:

(i') $s(x \wedge y) + s(d_2^R(x, y)) = s(x \vee y)$,

(ii') $s(y) + s((y \rightsquigarrow x) \vee 0) = s(x \vee y)$,

(iii') $s(x) + s((x \rightsquigarrow y) \vee 0) = s(y) + s((y \rightsquigarrow x) \vee 0)$.

Definition 5.15 A *Bosbach-state* on (G, u) is a function $s : G \longrightarrow \mathbf{R}$ such that: for all $x, y \in G$,

(S1) $s(x) + s((x \to y) \wedge 0) = s(y) + s((y \to x) \wedge 0)$,

(S2) $s(x) + s((x \rightsquigarrow y) \wedge 0) = s(y) + s((y \rightsquigarrow x) \wedge 0)$,

(S1') $s(x) + s((x \to y) \vee 0) = s(y) + s((y \to x) \vee 0)$,

(S2') $s(x) + s((x \rightsquigarrow y) \vee 0) = s(y) + s((y \rightsquigarrow x) \vee 0)$,

(S3) $s(0) = 0$, $s(u) = 1$,

(S3') $x \geq 0$ implies $s(x) \geq 0$.

Note that Propositions 5.11, 5.12 (5.13, 5.14) give equivalent conditions to (S1), (S2) ((S1'), (S2') respectively).

Proposition 5.16 *(See [12], Proposition 2.7)*

Let s be a Bosbach-state on $\mathcal{G}$. Then, for all $x, y \in G$:

(S4) $s((x \to y) \wedge 0) = s((x \rightsquigarrow y) \wedge 0)$,

(S4') $s((x \to y) \vee 0) = s((x \rightsquigarrow y) \vee 0)$,

(S5) $s(d_1^L(x, y)) = s(d_2^L(x, y))$,

(S5') $s(d_1^R(x, y)) = s(d_2^R(x, y))$,

(S6) $s(x) + s(-x \wedge 0) = s(x \wedge 0)$,

(S6') $s(x) + s(-x \vee 0) = s(x \vee 0)$,

(S7) $s(-x) = -s(x)$,

(S8) $x \leq y$ implies $s(y \to x) = s(y \rightsquigarrow x) = s(y) \to s(x)$,

(S8') $x \leq y$ implies $s(x \to y) = s(x \rightsquigarrow y) = s(x) \to s(y)$,

(S9) $x \leq y$ implies $s(x) \leq s(y)$.

Proof.

(S4): $(s(x) + s((x \to y) \wedge 0) = s(y \wedge x) = s(x) + s((x \rightsquigarrow y) \wedge 0)$, by Propositions 5.11 (ii) and 5.12 (ii). Hence, $s((x \to y) \wedge 0) = s((x \rightsquigarrow y) \wedge 0)$.

(S4'): $(s(x) + s((x \to y) \vee 0) = s(y \vee x) = s(x) + s((x \rightsquigarrow y) \vee 0)$, by Propositions 5.13 (ii') and 5.14 (ii'). Hence, $s((x \to y) \vee 0) = s((x \rightsquigarrow y) \vee 0)$.

(S5): $s(x \vee y) + s(d_1^L(x,y)) = s(x \wedge y) = s(x \vee y) + s(d_2^L(x,y))$, by Propositions 5.11 (i) and 5.12 (i). Hence, $s(d_1^L(x,y)) = s(d_2^L(x,y))$.

(S5'): Similarly, by Propositions 5.13 (i') and 5.14 (i').

(S6): $s(x) + s(-x \wedge 0) \overset{(5)}{=} s(x) + s((x \to 0) \wedge 0) \overset{(S1)}{=} s(0) + s((0 \to x) \wedge 0) \overset{(9)}{=} 0 + s(x \wedge 0) = s(x \wedge 0)$.

(S6'): $s(x) + s(-x \vee 0) \overset{(5)}{=} s(x) + s((x \to 0) \vee 0) \overset{(S1')}{=} s(0) + s((0 \to x) \vee 0) \overset{(9)}{=} 0 + s(x \vee 0) = s(x \vee 0)$.

(S7): First, we prove that:

$$s(-x) + s(x \wedge 0) = s(-x \wedge 0). \tag{56}$$

Indeed, $s(-x) + s(x \wedge 0) = s(-x) + s((-(-x)) \wedge 0) \overset{(S6)}{=} s(-x \wedge 0)$.

Now, $s(x \wedge 0) \overset{(S6)}{=} s(x) + s(-x \wedge 0) \overset{(56)}{=} s(x) + s(-x) + s(x \wedge 0)$. It follows that $s(x) + s(-x) = 0$, i.e. $s(-x) = -s(x)$.

(S8): Let $x \leq y$; then $x \to y \geq 0$, $x \rightsquigarrow y \geq 0$ and $y \to x \leq 0$, $y \rightsquigarrow x \leq 0$, by Corollary 2.3. Then:
$s(y) + s(y \to x) = s(y) + s((y \to x) \wedge 0) \overset{(S1)}{=} s(x) + s((x \to y) \wedge 0) = s(x) + s(0) = s(x)$; hence, $s(y \to x) = s(x) - s(y) = s(y) \to s(x)$. Similarly,
$s(y) + s(y \rightsquigarrow x) = s(y) + s((y \rightsquigarrow x) \wedge 0) \overset{(S2)}{=} s(x) + s((x \rightsquigarrow y) \wedge 0) = s(x) + s(0) = s(x)$; hence, $s(y \rightsquigarrow x) = s(x) - s(y) = s(y) \to s(x)$.

(S8'): Let $x \leq y$; then $x \to y \geq 0$, $x \rightsquigarrow y \geq 0$ and $y \to x \leq 0$, $y \rightsquigarrow x \leq 0$, by Corollary 2.3. Then:
$s(y) = s(y) + 0 = s(y) + s(0) = s(y) + s((y \to x) \vee 0) \overset{(S1')}{=} s(x) + s((x \to y) \vee 0) = s(x) + s(x \to y)$; hence, $s(x \to y) = s(y) - s(x) = s(x) \to s(y)$. Similarly, $s(x \rightsquigarrow y) = s(y) - s(x) = s(x) \to s(y)$.

(S9): Let $x \leq y$; hence $x \to y \geq 0$. By (S3'), $s(x \to y) \geq 0$; but, by (S8'), $s(x \to y) = s(y) - s(x)$; it follows that $s(y) - s(x) \geq 0$, hence $s(x) \leq s(y)$. □

Theorem 5.17 *Any state is a Bosbach-state.*

Proof. By definitions of states and Bosbach-states, it remains to prove (S1)-(S2'). Indeed,

(S1): $s(x) + s((x \to y) \wedge 0) = s((x \to y) \wedge 0) + s(x) \overset{s\ state}{=} s([(x \to y) \wedge 0] + x) = s([(x \to y) + x] \wedge x) = s(y \wedge x) = s(x \wedge y) = \ldots = s(y) + s((y \to x) \wedge 0)$.

(S2): $s(x) + s((x \rightsquigarrow y) \wedge 0) \overset{s\ state}{=} s(x + [(x \rightsquigarrow y) \wedge 0]) = s([x + (x \rightsquigarrow y)] \wedge x) = s(y \wedge x) = s(x \wedge y) = \ldots = s(y) + s((y \rightsquigarrow x) \wedge 0)$.

(S1'): $s(x) + s((x \to y) \vee 0) = s((x \to y) \vee 0) + s(x) \overset{s\ state}{=} s([(x \to y) \vee 0] + x) = s([(x \to y) + x] \vee x) = s(y \vee x) = s(x \vee y) = \ldots = s(y) + s((y \to x) \vee 0)$.

(S2'): $s(x) + s((x \rightsquigarrow y) \vee 0) \overset{s\ state}{=} s(x + [(x \rightsquigarrow y) \vee 0]) = s([x + (x \rightsquigarrow y)] \vee x) = s(y \vee x) = s(x \vee y) = \ldots = s(y) + s((y \rightsquigarrow x) \vee 0)$. □

Open problem 5.18 Study the restrictions of the various kinds of states from the l-group level to the G^-, G^+ level, the $[u', 0]$, $[0, u]$ level and the $G^-_{-\infty}$, $G^+_{+\infty}$ level.

References

[1] M. Andersen, T. Feil, Lattice-Ordered Groups - An Introduction, D. Reidel Publishing Company, 1988.

[2] C.J. van Alten, Representable biresiduated lattices, *J. Algebra* 247, 2002, 672-691.

[3] R. Ceterchi, On Algebras with Implications, categorically equivalent to Pseudo-MV Algebras, *The Proceedings of the Fourth International Symposium on Economic Informatics*, INFOREC Printing House, Bucharest, Romania, May 1999, 912-916.

[4] R. Ceterchi, Pseudo-Wajsberg Algebras, *Mult. Val. Logic (A special issue dedicated to the memory of Gr. C. Moisil)*, 2001, Vol. 6, No. 1-2, 67-88.

[5] C. C. Chang, Algebraic analysis of many valued logics, *Trans. Amer. Math. Soc.* 88, 1958, 467-490.

[6] P.F. Conrad, Lattice-ordered Groups, Tulane Lecture Notes, New Orleans, 1970.

[7] A. Di Nola, G. Georgescu, A. Iorgulescu, Pseudo-BL algebras: Part I, *Multi. Val. Logic*, 2002, Vol. 8 (5-6), 673-714.

[8] A. Dvurečenskij, Pseudo MV-algebras are intervals in l-groups, *Journal of Australian Mathematical Society*, 72, 2002, 427-445.

[9] A. Dvurečenskij, States on Unital Partially-Ordered Groups, submitted.

[10] J.M. Font, A. J. Rodriguez, A. Torrens, Wajsberg algebras, *Stochastica* Vol. VIII, No. 1, 1984, 5-31.

[11] N. Galatos, P. Jipsen, T. Kowalski, H. Ono, Residuated lattices: an algebraic glimps at substrucural logic, *Studies in Logic and the Foundation of Mathematics*, Vol. 151, Elsevier, 2007.

[12] G. Georgescu, Bosbach states on fuzzy structures, *Soft Computing* no. 8, 2004, 217-230.

[13] G. Georgescu, A. Iorgulescu, Pseudo-MV Algebras: a Noncommutative Extension of MV Algebras, *The Proceedings of the Fourth International Symposium on Economic Informatics*, INFOREC Printing House, Bucharest, Romania, May 1999, 961-968.

[14] G. Georgescu, A. Iorgulescu, Pseudo-MV algebras, *Mult. Val. Logic (A special issue dedicated to the memory of Gr. C. Moisil)*, Vol. 6, Nr. 1-2, 2001, 95-135.

[15] K.R. Goodearl, Partially ordered abelian groups with interpolation, Mathematical Surveys and Monographs, no. 20, A.M.S, Providence, Rhode Island, 1986.

[16] A. Iorgulescu, On pseudo-BCK algebras and porims, Scientiae Mathematicae Japonicae, Vol. 10, Nr. 16, 2004, 293-305.

[17] A. Iorgulescu, Classes of pseudo-BCK algebras - Part I, *J. of Multiple-Valued Logic and Soft Computing*, Vol. 12, No. 1-2, 2006, 71-130.

[18] A. Iorgulescu, Algebras of logic as BCK algebras, Academy of Economic Studies Press, Bucharest, 2008.

[19] A. Iorgulescu, The implicative-group - a term equivalent definition of the group coming from algebras of logic - Part I, submitted.

[20] J. Kühr, Pseudo-BCK-algebras and related structures, Universita Palackého v Olomouci, 2007.

[21] D. Mundici, MV-algebras are categorically equivalent to bounded commutative BCK-algebras, *Math. Japonica* 31, No. 6, 1986, 889-894.