Mappings satisfying some modular inequalities

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Abstract: We study classes of continuous, open, discrete mappings $f : D \subset \mathbf{R}^n \to \mathbf{R}^n$ satisfying a modular inequality of type $M_q(f(\Gamma)) \leq \gamma(m_\omega^p(\Gamma))$. We show that this thing ensures important geometric properties and we extend known results from the theory of quasiregular mappings like Picard, Hurwitz theorems or equicontinuity and eliminability results.

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1 Introduction.

If Γ is a path family from $\mathbf{R}^n$, we set $F(\Gamma) = \{\rho : \mathbf{R}^n \to [0,\infty]$ Borel maps$| \int_\gamma \rho ds \geq 1$ for every $\gamma \in \Gamma$ locally rectifiable$\}$ and if $\omega : \mathbf{R}^n \to [0,\infty]$ is measurable and finite a.e., $\omega > 0$ a.e. and $p > 1$, we define the weight p-modulus of weight ω by $M_\omega^p(\Gamma) = \inf_{\rho \in F(\Gamma)} \int_{\mathbf{R}^n} \omega(x)\rho^p(x)dx$ for every path family Γ from $\mathbf{R}^n$. If $\omega = 1$, we obtain the usual p-modulus $M_p(\Gamma) = \inf_{\rho \in F(\Gamma)} \int_{\mathbf{R}^n} \rho^p(x)dx$ for every path family Γ in $\mathbf{R}^n$, which is a basic tool in the study of quasiregular mappings. The systematic utilization of the arbitrary weight p-modulus in the mapping theory was initiated by Cabiria Andreian in [2].

If $D \subset \mathbf{R}^n$ is a domain, we say that a map $f : D \to \mathbf{R}^n$ is of finite distortion if $f \in W_{loc}^{1,1}(D, \mathbf{R}^n)$, $J_f \in L_{loc}^1(D)$ and there exists $K : D \to [0,\infty]$ measurable and finite a.e. so that $|f'(x)|^n \leq K(x)J_f(x)$ a.e., and if in addition $f \in W_{loc}^{1,n}(D, \mathbf{R}^n)$ and $K \in L^\infty(D)$, we obtain the known class of quasiregular mappings. For more information about the theory of quasiregular mappings we send the reader to [21-22] and [27-29].

If $x = (x_1, ..., x_n) \in \mathbf{R}^n$, we set $|x| = (\sum_{i=1}^n x_i^2)^{\frac{1}{2}}$ and if $A \in \mathcal{L}(\mathbf{R}^n, \mathbf{R}^n)$, $detA \neq 0$, $p > 0$, we set $|A| = \sup_{|h|=1} |A(h)|$, $l(A) = \inf_{|h|=1} |A(h)|$, $K_{0,p}(A) = |A|^p/|detA|$, $K_{I,p}(A) = |detA|/l(A)^p$.

If $D \subset \mathbf{R}^n$ is a domain and $f : D \to \mathbf{R}^n$ is a.e. differentiable and $J_f(x) \neq 0$ a.e., we can define a.e. the mappings $K_{0,p}(f) : D \to [0,\infty]$ by $K_{0,p}(f)(x) = K_{0,p}(f'(x))$ a.e. and $K_{I,p}(f) : D \to [0,\infty]$ by $K_{I,p}(f)(x) = K_{I,p}(f'(x))$ a.e. A quasiregular map is open and discrete, is a.e. differentiable and $J_f(x) \neq 0$ a.e. and satisfies the known modular inequality of Poleckii which says that $M_n(f(\Gamma)) \leq K_{I,n}(f)M_n(\Gamma)$ for every path family Γ from D. This modular inequality is the key for proving most of the important geometric properties of quasiregular mappings.

If f is a map of finite distortion and either $K_{I,n}(f) \in BMO(D)$, or $exp(\mathcal{A} \circ K_{0,n}(f)) \in L_{loc}^1(D)$ for some Orlicz map A, then, using some weight modular inequalities, in [5,6], [13-15], [20], [23] are established basic geometric properties in this class of mappings. In [7] and [8] is studied a class of continuous, open, discrete mappings having local ACL^n inverses for which a generalized Poleckii's modular inequality of type "$M_n f(\Gamma) \leq M^n_{K_{I,n}(f)}(\Gamma)$ for every path family Γ" holds. This thing, together with the condition "$\lim_{a\to 0} M^n_{K_{I,n}(f)}(\Gamma_{x,a,b}) = 0$ for every $x \in D$ and every $b > 0$ so that $\overline{B}(x,b) \subset D$" permits us to reconstruct most of the geometric properties of the quasiregular mappings. Also, using the modulus method, in [16], [17], [24-26] are established for ring homeomorphisms and for mappings of finite length distortion equicontinuity and boundary extension results.

If $D \subset \mathbf{R}^n$ is a domain, $E, F \subset \overline{D}$, we denote by $\Delta(E,F,D) = \{\gamma : [a,b] \to \overline{D}$ path $|\gamma(a) \in E, \gamma(b) \in F$ and $\gamma((a,b)) \subset D\}$ and if $x \in D$ and $0 < a < b$ we set $\Gamma_{x,a,b} = \Delta(\overline{B}(x,a), S(x,b), B(x,b) \setminus \overline{B}(x,a))$. We denote by $M(D) = \{u : D \to [0,\infty]$ measurable and finite a.e.$\}$ and by $A(D)$ the set of all path families from D.

If $D \subset \mathbf{R}^n$ is a domain, we say that $M : A(D) \to [0,\infty]$ is a modulus on D if:

a) $M(\phi) = 0$.

b) $M(\Gamma_1) \leq M(\Gamma_2)$ if $\Gamma_1 \subset \Gamma_2$, $\Gamma_1, \Gamma_2 \in A(D)$.

c) $M(\bigcup_{p=1}^{\infty} \Gamma_p) \leq \sum_{p=1}^{\infty} M(\Gamma_p)$ if $\Gamma_1, ..., \Gamma_p, ...$ are from $A(D)$.

In a recent paper [9] we investigate the geometric properties of the continuous, open, discrete mappings $f : D \subset \mathbf{R}^n \to \mathbf{R}^n$ satisfying the condition "$M_1(f(\Gamma)) \leq \gamma(M_2(\Gamma))$ for every $\Gamma \in A(D)$", where M_1 and M_2 are some modulus on $\mathbf{R}^n$, respectively D and $\gamma : [0,\infty) \to [0,\infty)$ is increasing so that $\lim_{t\to 0} \gamma(t) = 0$. Picard, Montel and Liouville type theorems, modulus of continuity, equicontinuity and eliminability results are established for very general kind of modulus M_1 and M_2, and the modulus M_1 is of the form $M_N(\Gamma) = \inf_{\rho \in F(\Gamma)} N(\rho)$, for $\Gamma \in A(\mathbf{R}^n)$, where $N : \mathcal{M}(\mathbf{R}^n) \to [0,\infty]$ is a general convex operator.

In the present paper we study the geometric properties of some continuous, open, discrete mappings $f : D \subset \mathbf{R}^n \to \mathbf{R}^n$ satisfying a modular inequality of type "$M_q(f(\Gamma)) \leq \gamma(M^p_\omega(\Gamma))$ for every $\Gamma \in A(D)$, some $n-1 < q$, $p > 1$, a weight $\omega \in L^1_{loc}(D)$ so that $\lim_{a\to 0} M^p_\omega(\Gamma_{x,a,b}) = 0$ for some $x \in D$ and every $b > 0$ so that $\overline{B}(x,b) \subset D$ and a function $\gamma : [0,\infty) \to [0,\infty)$ increasing with $\lim_{t\to 0} \gamma(t) = 0$". We continue in this way the researches from [9] and we construct in Proposition 2 a class of homeomorphisms $f : D \to D^{'}$ between two domains from $\mathbf{R}^n$ satisfying a modular inequality of type "$M_q(f(\Gamma)) \leq C M^p_\omega(\Gamma)^{\frac{q}{p}}$ for every $\Gamma \in A(D)$ and some $1 < q < p$, a class which is larger than the class of homeomorphisms with finite mean dilatations of A. Goldberg from [10], [11] (see also Chapter 12 from [19]).

We prove equicontinuity and eliminability results and we extend the theorems of Picard and Hurwitz in the new introduced class of mappings. A basic tool in our proof is a known result from the theory of quasiregular mappings, established in [27] for the M_n modulus in $\mathbf{R}^n$ and by P. Caraman in Theorem 4 from [3] for the p-modulus in $\mathbf{R}^n$. Caraman's result shows that if E, F are disjoint subsets from $\mathbf{R}^n$ so that $S(x,t) \cap E \neq 0$, $S(x,t) \cap F \neq 0$ for some $x \in \mathbf{R}^n$ and every $a < t < b$, then there exists a constant $C(n,p)$ depending only on n and p so that $M_p(\Delta(E,F,B(x,b) \setminus \overline{B}(x,a))) \geq C(n,p)(b^{n-p} - a^{n-p})$ if $p > n-1$, $p \neq n$, whether the classical result from Theorem 10.12, page 31 from [26] says that $M_n(\Delta(E,F,B(x,b) \setminus \overline{B}(x,a))) \geq C(n) \ln(\frac{b}{a})$.

If $p \geq 1$, we denote by $W^{1,p}_{loc}(D, \mathbf{R}^m)$ the Sobolev space of all functions $f : D \to \mathbf{R}^m$ which are locally in L^p together with their first order derivatives. We say that f is ACL if f is continuous and for every cube $Q \subset D$ with the sides parallel to coordinate axes and every face S of Q it results that $f|P_S^{-1}(y) \cap Q : P_S^{-1}(y) \cap Q \to \mathbf{R}^m$ is absolutely continuous for a.e. $y \in S$, where $P_S : \mathbf{R}^n \to S$ is the projection on S. An ACL map has a.e. partial derivatives and we say that f is ACL^p if f is ACL and the partial derivatives are locally in L^p. We see from Proposition 1.1.2, page 6 from [22] that if $p \geq 1$ and $f \in C(D, \mathbf{R}^m)$, then f is ACL^p if and only if $f \in W^{1,p}_{loc}(D, \mathbf{R}^m)$.

If $D \subset \mathbf{R}^n$ is a domain, $f : D \to \mathbf{R}^n$ is continuous, open, discrete, $x \in D$ and $r > 0$ is so that $\overline{B}(x, r) \subset D$, we set $L(x, f, r) = \sup\limits_{|y-x|=r} |f(y) - f(x)|$, $l(x, f, r) = \inf\limits_{|y-x|=r} |f(y) - f(x)|$, $H(x, f) = \limsup\limits_{r \to 0} L(x, f, r)/l(x, f, r)$. We know that a homeomorphism $f : D \to D^{\text{'}}$ between two domains from $\mathbf{R}^n$ is quasiconformal if and only if there exists $H \geq 1$ so that $H(x, f) \leq H$ for every $x \in D$, and we also know that if f is differentiable in a point $x \in D$ and $J_f(x) \neq 0$, then $H(x, f) = |f^{\text{'}}(x)|/l(f^{\text{'}}(x))$. If $D \subset \mathbf{R}^n$ is a domain, $p > 0$ and $f : D \to \mathbf{R}^n$ is a map, we define the map $H_{I,p}(f) : D \to \mathbf{R}$ by $H_{I,p}(f)(x) = K_{I,p}(f^{\text{'}}(x))$ if f is differentiable in x and $J_f(x) \neq 0$, $H_{I,p}(f)(x) = 0$ otherwise.

If $D \subset \mathbf{R}^n$ is a domain, $b \in \partial D$ and $f : D \to \mathbf{R}^n$ is a map, we set $C(f, b) = \{z \in \overline{\mathbf{R}}^n|$ there exists $b_p \in D, b_p \to b$ so that $f(b_p) \to z\}$ and if $A \subset D$, $y \in \mathbf{R}^n$, we put $N(y, f, A) = Card f^{-1}(y) \cap A$ and $N(f, A) = \sup\limits_{y \in \mathbf{R}^n} N(y, f, A)$. We also set $L(x, f) = \limsup\limits_{h \to 0} \frac{|f(x+h)-f(x)|}{|h|}$ if $x \in D$. If $p > 0$ and A is a Borel set from $\mathbf{R}^n$, we set $m_p(A)$ the p Hausdorff measure of A and we set $\mu_n(A)$ the Lebesgue measure of A. If $a, b \in \overline{\mathbf{R}}^n$, we denote by $q(a, b)$ the chordal distance between a and b and $q(a, b) = |a-b|/(1+|a|^2)^{1/2}(1+|b|^2)^{1/2}$ if $a, b \in \mathbf{R}^n$, $q(a, \infty) = 1/(1+|a|^2)^{1/2}$ if $a \in \mathbf{R}^n$ and if $A \subset \overline{\mathbf{R}}^n$, we set $q(A)$ the diameter of A considering the chordal metric on $\overline{\mathbf{R}}^n$. Also, if $A \subset \mathbf{R}^n$, we set $d(A)$ the diameter of A considering the euclidean metric on $\mathbf{R}^n$.

If E, F are Hausdorff spaces and $f : E \to F$ is a map, we say that f is open if f carries open sets into open sets and we say that f is discrete if $f^{-1}(y)$ is discrete or empty for every $y \in F$. If $p : [0, 1] \to F$ is a path and $x \in E$ is so that $f(x) = p(0)$, we say that the path $q : [0, 1] \to E$ is a lifting of p from x if $q(0) = x$ and $f \circ q = p$ and we say that $q : [0, a) \to E$ is a maximal lifting of p from x if q is a path, $q(0) = x$, $0 < a \leq 1$, $f \circ q = p|[0, a)$ and a is maximal with this property. If $D \subset \mathbf{R}^n$ is a domain, $f : D \to \mathbf{R}^n$ is continuous, open, discrete, $p : [0, 1] \to f(D)$ is a path, $x \in D$ is so that $f(x) = p(0)$, there exists allways a maximal lifting of p from x. If $q : [0, a) \to \mathbf{R}^n$ is a path, we say that a point $x \in \overline{\mathbf{R}}^n$ is a limit point of q if there exists $t_p \to a$ so that $q(t_p) \to x$.

If M is a modulus on $\mathbf{R}^n$ and $E \subset \overline{\mathbf{R}}^n$, we say that E is of zero M modulus (and we write $M(E) = 0$) if the M modulus of all paths having some limit point in E is zero. We say that $E = (A, C)$ is a condenser if $C \subset A \subset \mathbf{R}^n$, C is compact and A is open, and if $p > 1$, we define $cap_p(E) = \inf \int\limits_{\mathbf{R}^n} |\nabla u|^p(x)dx$ the p capacity of E, where the infimum is taken over all $u \in C_0^\infty(A)$ so that $u \geq 1$ on C. If $C \subset \mathbf{R}^n$ is compact we say that $cap_p C = 0$ if $cap_p(A, C) = 0$ for every open set A so that $C \subset A \subset \mathbf{R}^n$, and the definition does not depend on the choice of the open set A so that $C \subset A$. If $K \subset \mathbf{R}^n$, we say that $cap_p K = 0$ if $cap_p C = 0$ for every compact set $C \subset K$. We know from Prop. II.10.2, page 54 from [22] that if $C \subset \mathbf{R}^n$ is closed, then $cap_p C = 0$ if and only if $M_p(C) = 0$.

If X, Y are metric spaces and W is a family of mappings $f : X \to Y$, we say that the family W is equicontinuous at a point $x \in D$ if for every $\epsilon > 0$, there exists $\delta_\epsilon > 0$ so that $d(f(y), f(x)) \leq \epsilon$ if $d(x, y) \leq \delta_\epsilon$ for every $f \in W$.

If X is a separable metric space and $(A_i)_{i \in N}$ is a collection of sets, we define the superior limit of the collection $(A_i)_{i \in N}$ to be $\limsup A_i = \{x \in X|$ every neighborhood of x contains points from infinitely many sets $A_i\}$ and we define the inferior limit of the collection $(A_i)_{i \in N}$ to be $\liminf A_i = \{x \in X|$ every neighborhood of x contains points of all but a finite member of sets $A_i\}$. If for a collection $(A_i)_{i\in N}$ we have $\limsup A_i = \liminf A_i$, we say that $A_i \to A$, where A is the common value of $\limsup A_i$ and $\liminf A_i$. We know from Theorem 7.1 [30], page 11 that every sequence of sets $(A_i)_{i\in N}$ contains a convergent subsequence and from [30] page 15 we see that if $(A_i)_{i\in N}$ is a convergent sequence of compact connected sets so that $\overline{\bigcup_{i\in N} A_i}$ is compact, then, if $A_i \to A$, it results that A is compact and connected.

If $D \subset \mathbf{R}^n$ is a domain, x is an isolated point of ∂D and $f : D \to \mathbf{R}^n$ is continuous, open, discrete, we say that x is an essential singularity of f if there exists not $\lim\limits_{z\to x} f(z) \in \overline{\mathbf{R}}^n$.

2 Examples of mappings satisfying generalized modular inequalities.

Proposition 1. Let $n \geq 2$, $D \subset \mathbf{R}^n$ a domain, $1 < q < p$, $f \in ACL^q(D, \mathbf{R}^n)$, f a.e. differentiable so that $J_f(x) \neq 0$ a.e., $\alpha, \beta \in [0,1]$ so that $\alpha + \beta = 1$, $K_{0,p}(f) \in L^{q\alpha/(p-q)}(D)$, $N(f,D) < \infty$, and let $C = N(f,D)^{q/p}(\int\limits_D K_{0,p}(f)(x)^{q\alpha/(p-q)}dx)^{(p-q)/p}$. Then $M_q(\Gamma) \leq C(M_\omega^p(f(\Gamma))^{q/p})$ for every $\Gamma \in A(D)$, where $\omega = H_{I,p}(f^{-1})^\beta$.

Proof: Let Γ be a path family from D and let $\rho^{`} \in F(f(\Gamma))$. Let $\rho : \mathbf{R}^n \to [0,\infty]$, $\rho(x) = \rho^{`}(f(x)) \cdot L(x,f)$ if $x \in D$, $\rho(x) = 0$ otherwise and let $\Gamma_0 = \{\gamma \in \Gamma | f \circ \gamma^0$ is absolutely continuous$\}$. Using Fuglede's theorem (see [27], Theorem 28.2, page 95), we have $M_q(\Gamma) = M_q(\Gamma_0)$ and from Theorem 5.3, page 12 from [27], we see that $\rho \in F(\Gamma_0)$. Using the change of variable formulae (3) from [12] and Hölder's inequality, we have

$$\int\limits_D \rho^q(x)dx = \int\limits_D \rho^{`q}(f(x))L(x,f)^q dx = \int\limits_D \rho^{`q}(f(x))|f^{`}(x)|^q dx \leq$$

$$\leq \int\limits_D \rho^{`q}(f(x))H_{I,p}(f^{-1})(f(x))^{\beta q/p}K_{0,p}(f)(x)^{\alpha q/p}|J_f(x)|^{q/p}dx \leq$$

$$\leq (\int\limits_D \rho^{`p}(f(x))H_{I,p}(f^{-1})(f(x))^\beta |J_f(x)|dx)^{q/p}(\int\limits_D K_{0,p}(f)(x)^{\alpha q/(p-q)}dx)^{(p-q)/p} \leq$$

$$\leq (\int\limits_{\mathbf{R}^n} \rho^{`p}(y)H_{I,p}(f^{-1})(y)^\beta N(y,f,D)dy)^{q/p}(\int\limits_D K_{0,p}(f)(x)^{\alpha q/(p-q)}dx)^{(p-q)/p} \leq$$

$$\leq C(\int\limits_{\mathbf{R}^n} \rho^{`p}(y)H_{I,p}(f^{-1})(y)^\beta dy)^{q/p}.$$

It results that $M_q(\Gamma) = M_q(\Gamma_0) \leq \int\limits_D \rho^q(x)dx \leq C(\int\limits_{\mathbf{R}^n} \rho^{`p}(y)\omega(y)dy)^{q/p}$ for every $\rho^{`} \in F(f(\Gamma))$, hence $M_q(\Gamma) \leq C(M_\omega^p(f(\Gamma))^{q/p})$.

We used here the fact that $H_{I,p}(f^{-1})(f(x)) = K_{0,p}(f)(x)$ in every point $x \in D$ so that f is differentiable in x and $J_f(x) \neq 0$.

Proposition 2. Let $n \geq 2$, $1 < q < p$, $D, D^{`}$ domains from $\mathbf{R}^n$, $h : D^{`} \to D$ a homeomorphism, $f = h^{-1}$ that $f \in ACL^q(D, D^{`})$, $f$ is a.e. differentiable and $J_f(x) \neq 0$ in $D$, $\alpha, \beta \in [0,1]$ with $\alpha + \beta = 1$, $\gamma = pq\alpha/(p - \beta q)$, $C = (\int\limits_{D^{`}} H_{I,\gamma}(h)(y)^{(p-\beta q)/(p-q)}dy)^{(p-q)/p}$ and let $\omega = H_{I,p}(h)^\beta$. Then $M_q(h(\Gamma^{`})) \leq C(M_\omega^p(\Gamma^{`}))^{q/p}$ for every $\Gamma^{`} \in A(D^{`})$.

Proof: Let $A = \{x \in D | f$ is differentiable in x and $J_f(x) \neq 0\}$ and $B = \{y \in D^{\text{`}} | h$ is differentiable in y and $J_h(y) \neq 0\}$. Then $f(A) \subset B$ and $\mu_n(CA) = 0$. Using the change of variable formulae (3) from [12] we see that

$$\int\limits_D K_{0,p}(f)(x)^{\alpha q/(p-q)} dx = \int\limits_A (|f^{\text{`}}(x)|^p / |J_f(x)|)^{\alpha q/(p-q)} = \int\limits_A |f^{\text{`}}(x)|^{pq\alpha/(p-q)} / |J_f(x)|^{\alpha q/(p-q)} dx =$$

$$= \int\limits_A (J_h(f(x))^{(p-\beta q)/(p-q)} / l(h^{\text{`}}(f(x))^{pq\alpha/(p-q)})) |J_f(x)| dx \leq$$

$$\leq \int\limits_{f(A)} |J_h(y)|^{(p-\beta q)/(p-q)} / l(h^{\text{`}}(y))^{pq\alpha/(p-q)} dy \leq \int\limits_B |J_h(y)|^{(p-\beta q)/(p-q)} / l(h^{\text{`}}(y))^{pq\alpha/(p-q)} dy =$$

$$= \int\limits_B (|J_h(y)| / l(h^{\text{`}}(y))^{\gamma})^{(p-\beta q)/(p-q)} dy = \int\limits_{D^{\text{`}}} H_{I,\gamma}(h)(y)^{(p-\beta q)/(p-q)} dy.$$

Let $\Gamma^{\text{`}} \in A(D^{\text{`}})$ and $\Gamma = h(\Gamma^{\text{`}})$. Then $\Gamma^{\text{`}} = f(\Gamma)$ and using Proposition 1, we see that $M_q(h(\Gamma^{\text{`}})) = M_q(\Gamma) \leq C(M_\omega^p(f(\Gamma)))^{q/p} = C(M_\omega^p(\Gamma^{\text{`}}))^{q/p}$.

Remark 1. Of course, the most natural case we can have in Proposition 2 is when $\beta = 0$, i.e. when the weight $\omega = 1$. In this case we obtain Proposition 2 from [9], which is in connection with the class of homeomorphisms with finite mean dilatations of A. Goldberg from [10] and [11]. However, we can find an example of a homeomorphism $h : D^{\text{`}} \to D$ and $1 < q < p$, $\alpha, \beta \in [0,1]$ so that $p/q < p\alpha + \beta < p$ and so that if $\gamma = pq\alpha/(p - \beta q)$, it results that $H_{I,\gamma}(h) \in L^{(p-\beta q)/(p-q)}(D^{\text{`}})$ and $H_{I,q}(h) \notin L^{p/(p-q)}(D^{\text{`}})$ and this shows that Proposition 2 extends Proposition 2 from [9]. It also shows that if $\omega = H_{I,p}(h)^\beta$, then $M_q(h(\Gamma^{\text{`}})) \leq C(M_\omega^p(\Gamma^{\text{`}}))^{q/p}$ for every $\Gamma^{\text{`}} \in A(D^{\text{`}})$, where $C = (\int\limits_{D^{\text{`}}} H_{I,\gamma}(h)(y)^{(p-\beta q)/(p-q)} dy)^{(p-q)/p}$.

Example 1. Let $D = (0,1)^n$, $1 < q < p$, $\alpha, \beta \in [0,1]$ so that $\alpha + \beta = 1$, $p/q < p\alpha + \beta < p$ and let $\gamma = pq\alpha/(p - \beta q)$. We see that $pq - p > 0$, $pq\alpha + \beta q - p > 0$, $(p-q)/(pq-p) < (p-q)/(pq\alpha + \beta q - p)$ and let $(p-q)/(pq-p) < c < (p-q)/(pq\alpha + \beta q - p)$ and $h : D \to \mathbf{R}^n$ be defined by $h(x_1, ..., x_n) = (x_1, ..., x_{n-1}, \frac{x_n^{1+c}}{1+c})$ for $x = (x_1, ..., x_n) \in D$. We see that h is a homeomorphism onto a domain $D^{\text{`}}$ from $\mathbf{R}^n$, that $h \in C^1(D, D^{\text{`}})$, $J_h(x) = x_n^c \neq 0$, $l(h^{\text{`}}(x)) = x_n^c$, $|h^{\text{`}}(x)| = 1$ for every $x \in D$, hence $H(x,h) = |f^{\text{`}}(x)| / l(f^{\text{`}}(x)) = x_n^{-c} \to \infty$ if $x \to 0$ and hence h is not quasiconformal. Also, $H_{I,q}(h)(x) = x_n^{c(1-q)}$ and let $J = \int\limits_D H_{I,q}(h)(x)^{p/(p-q)} dx$. Then $J = \int\limits_0^1 x_n^{pc(1-q)/(p-q)} dx_n$ and since $p - q < c(pq - p)$, we see that $\frac{pc(1-q)}{p-q} + 1 < 0$ and hence $J = \infty$.

Let now $I = \int\limits_D H_{I,\gamma}(h)(x)^{(p-\beta q)/(p-q)} dx$.

Then $I = \int\limits_0^1 x_n^{c(1-\gamma)(p-\beta q)/(p-q)} dx_n = \int\limits_0^1 x_n^{c(p-\beta q - pq\alpha)/(p-q)} dx_n$, and since $\frac{c(p-\beta q - pq\alpha)}{p-q} + 1 > 0$, we see that $I = (p-q)/(p-q-c(pq\alpha + \beta q - p)) < \infty$.

We denote by $W_{q,p,\omega,\gamma}(D) = \{f : D \to \mathbf{R}^n | D \subset \mathbf{R}^n$ is a domain, f is continuous, open, discrete, $1 \leq n - 1 < q$, $p > 1$, $\omega \in L^1_{loc}(D)$, $\gamma : [0, \infty) \to [0, \infty)$ is increasing with $\lim\limits_{t \to 0} \gamma(t) = 0$ and $M_q(f(\Gamma)) \leq \gamma(M_\omega^p(\Gamma))$ for every $\Gamma \in A(D)\}$. We also denote by $Q_{q,p,\omega,\gamma}(D) = \{f : D \to \mathbf{R}^n | D \subset \mathbf{R}^n$ is a domain, f is continuous, open, discrete, $n - 1 < q$, $p > 1$, $\omega \in L^1_{loc}(D)$, $\gamma : [0, \infty) \to [0, \infty)$ is increasing with $\lim\limits_{t \to \infty} \gamma(t) = \infty$ and $M_q(f(\Gamma)) \leq \gamma(M_\omega^p(\Gamma))$ for every

$\Gamma \in A(D)$ so that $\Gamma = \Delta(E, F, G)$ with G a domain so that $\overline{G} \subset D$ and $E, F \subset \overline{G}$ compact, connected$\}$.

We showed that the class $W_{q,p,\omega,\gamma}(D)$ is nonempty if $q < p$ and we found in Proposition 2 a map $f \in W_{q,p,\omega,\gamma}(D)$ which is not quasiconformal.

3 Geometric properties of the mappings satisfying generalized modular inequalities.

We shall study the geometric properties of the functions from the class $W_{q,p,\omega,\gamma}(D)$. The following lemma, together with Caraman's result from [3] will be an important instrument for proving this things.

Lemma 1. Let $1 \le n-1 < q$, $E \subset \mathbf{R}^n$ closed so that $cap_q(E) > 0$, $\delta > 0$ and let Q_m be compact, connected subsets from $\overline{\mathbf{R}}^n$ so that $Q_m \cap E = \phi$ and $q(Q_m) \ge \delta$ for every $m \in N$. Then there exists $\epsilon > 0$ so that $M_q(\Delta(Q_m, E, \mathbf{R}^n)) > \epsilon > 0$ for every $m \in N$.

Proof. If $q = n$, the theorem holds due to Lemma III.2.6, page 65 from [22].

Let now $q > n-1$, $q \ne n$ and suppose that the theorem is false. In this case we can find sets Q_{m_k}, $k \in N$ so that $M_q(\Delta(Q_{m_k}, E, \mathbf{R}^n)) \to 0$ and using Theorem 7.1, page 11 from [30], we can find $Q \subset \overline{\mathbf{R}}^n$ compact, connected in $\overline{\mathbf{R}}^n$ and a subsequence $(Q_{m_{k_p}})_{p \in N}$ so that $Q_{m_{k_p}} \to Q$. We can suppose that $Q_m \to Q$ and $M_q(\Delta(Q_m, E, \mathbf{R}^n)) \to 0$ and we show that we shall obtain a contradiction.

We suppose first that Q is compact in $\mathbf{R}^n$, and in this case we can suppose that we find $R > 0$ so that $Q_m \subset B(0, R)$ for every $m \in N$ and $cap_q(E \cap \overline{B}(0, R)) > 0$. Let $\Gamma_0 = \Delta(E \cap \overline{B}(0, R), S(0, 2R), B(0, 2R))$. Then there exists $\delta_1 > 0$ so that $M_q(\Gamma_0) > \delta_1$ and there exists $d > 0$ so that $d(Q_m) > d$ for every $m \in N$. Let $\delta_2 = C(n, q)d(n-q)(1/3R)^{q-n+1}$ and let $\delta_3 = C(n, q)(n-q)R(1/2R)^{q-n+1}$, where $C(n, q)$ is the constant from Caraman's result (see Theorem 4 from [3]). Then $\epsilon = \frac{1}{3^q} min\{\delta_1, \delta_2, \delta_3\} > 0$ does not depend on m. Let $\tilde{\Gamma}_m = \Delta(Q_m, S(0, 2R), B(0, 2R))$ and let $\Gamma_m = \Delta(E \cap \overline{B}(0, R), Q_m, B(0, 2R))$ for $m \in N$. We fix now $m \in N$. We choose now $a_m, b_m \in Q_m$ so that $|a_m - b_m| > d$, let d_m be the line determined by the points a_m and b_m and let $c_m \in S(0, 2R) \cap d_m$ be so that $b_m \in (a_m, c_m)$. We see that $S(c_m, t) \cap Q_m \ne \phi$, $S(c_m, t) \cap CB(0, 2R) \ne \phi$ for every $|b_m - c_m| < t < |a_m - c_m|$ and using Caraman's result from [3] we have that $M_q(\tilde{\Gamma}_m) \ge C(n, q)(|c_m - a_m|^{n-q} - |b_m - c_m|^{n-q}) \ge \delta_2 > 0$.

Let $\rho \in F(\Gamma_m)$ and suppose that $3\rho \notin F(\Gamma_0)$ and $3\rho \notin F(\tilde{\Gamma}_m)$. Then there exists paths α_m joining $E \cap \overline{B}(0, R)$ with $S(0, 2R)$ in $B(0, 2R)$ and β_m joining Q_m with $S(0, 2R)$ in $B(0, 2R)$ so that $\int\limits_{\alpha_m} \rho ds < \frac{1}{3}$ and $\int\limits_{\beta_m} \rho ds < \frac{1}{3}$. Let $\Delta_m = \Delta(Im\alpha_m, Im\beta_m, B(0, 2R) \setminus \overline{B}(0, R))$. Using again Caraman's result from [3], we see that $M_q(\Delta_m) \ge C(n, q)((2R)^{n-q} - R^{n-q}) \ge \delta_3 > 0$. Let $\varphi_m \in \Delta_m$. We can find a subpath $\tilde{\alpha}_m$ of α_m, a subpath $\tilde{\beta}_m$ of β_m and a subpath $\tilde{\varphi}_m$ of φ_m so that the path $\Psi_m = \tilde{\alpha}_m \bigvee \tilde{\varphi}_m \bigvee \tilde{\beta}_m \in \Gamma_m$. We see that $1 \le \int\limits_{\Psi_m} \rho ds = \int\limits_{\tilde{\alpha}_m} \rho ds + \int\limits_{\tilde{\varphi}_m} \rho ds + \int\limits_{\tilde{\beta}_m} \rho ds \le \int\limits_{\alpha_m} \rho ds + \int\limits_{\varphi_m} \rho ds + \int\limits_{\beta_m} \rho ds < \frac{2}{3} + \int\limits_{\varphi_m} \rho ds$, hence $1 \le 3 \int\limits_{\varphi_m} \rho ds$ for every $\rho_m \in \Delta_m$.

It results that $3\rho \in F(\Delta_m)$ and we proved that $M_q(\Gamma_m) = \inf\limits_{\rho \in F(\Gamma_m)} \int\limits_{\mathbf{R}^n} \rho^q(x) dx \ge \frac{1}{3^q} \min\{M_q(\Gamma_0), M_q(\tilde{\Gamma}_m), M_q(\Delta_m)\} \ge \epsilon > 0$ for every $m \in N$. We reached a contradiction, since we supposed that $M_q(\Delta(Q_m, E, \mathbf{R}^n)) \to 0$, and on the other side we see that $M_q(\Delta(Q_m, E, \mathbf{R}^n)) \ge M_q(\Gamma_m) > \epsilon > 0$ for every $m \in N$.

Suppose now that Q is not compact in $\mathbf{R}^n$. We can suppose in this case that there exists $R > 0$ so that $cap_q(E \cap \overline{B}(0, 2R)) > 0$ and $Q_m \cap S(0, t) \neq \phi$ for $R < t < 2R$ for every $m \in N$. We see that $d(Q_m \cap (B(0, 2R) \setminus \overline{B}(0, R))) \geq R > 0$ for every $m \in N$ and $cap_q(E \cap \overline{B}(0, 2R)) > 0$ and using the preceding argument, we can find $\epsilon > 0$ so that $M_q(\Delta(E \cap \overline{B}(0, 2R), Q_m, B(0, 4R) \setminus \overline{B}(0, 2R))) > \epsilon > 0$ for every $m \in N$. We reached again a contradiction and the theorem is now proved.

Theorem 1. (Generalization of Picard's theorem). Let $1 \leq n - 1 < q$, $E \subset \mathbf{R}^n$ closed, $f : \mathbf{R}^n \setminus E \to \mathbf{R}^n$ continuous, open, discrete, M a modulus on $\mathbf{R}^n$ so that $M(E \cup \{\infty\}) = 0$ and $M_q(f(\Gamma)) = 0$ for every $\Gamma \in A(\mathbf{R}^n \setminus E)$ with $M(\Gamma) = 0$. Then $cap_q\overline{C(f(\mathbf{R}^n \setminus E))} = 0$.

Proof. Let $K \subset \mathbf{R}^n \setminus E$ be compact, connected so that $Card K > 1$. Since f is continuous, open, discrete, we see that $f(K)$ is compact, connected so that $Card f(K) > 1$, and we also see that $f(K) \cap \overline{Cf(\mathbf{R}^n \setminus E)} = \phi$. Let $\Gamma^{‘} = \Delta(f(K), \overline{Cf(\mathbf{R}^n \setminus E)}, \mathbf{R}^n)$ and let Γ be the family of all maximal lifting of some paths from $\Gamma^{‘}$ starting from some points of K. Then $\Gamma^{‘} > f(\Gamma)$ and since every path from Γ has at least a limit point in $E \cup \{\infty\}$, we see that $M(\Gamma) = 0$. Using the hypothesis, we see that $M_q(f(\Gamma)) = 0$, hence $M_q(\Gamma^{‘}) = 0$. On the other side, if $cap_q\overline{Cf(\mathbf{R}^n \setminus E)} > 0$, we see from Lemma 1 that $M_q(\Gamma^{‘}) > 0$ and we reached a contradiction. We therefore proved that $cap_q\overline{Cf(\mathbf{R}^n \setminus E)} = 0$.

The following equicontinuity result extends Corollary 2.7, page 66 from [22]. It also extends Theorem 14 from [9] in the case $M_N = M_q$ with $q > n - 1$.

Theorem 2. Let $1 \leq n - 1 < q$, $Q \subset \mathbf{R}^n$ with $cap_q\overline{Q} > 0$, $D \subset \mathbf{R}^n$ a domain, $x \in D$, M a modulus on D so that $\lim_{a \to 0} M(\Gamma_{x,a,b}) = 0$ for every $b > 0$ so that $\overline{B}(x, b) \subset D$, $\gamma : [0, \infty) \to [0, \infty)$ increasing with $\lim_{t \to 0} \gamma(t) = 0$, let W be a family of continuous, open, discrete mappings $f : D \to \mathbf{R}^n \setminus Q$ so that $M_q(f(\Gamma)) \leq \gamma(M(\Gamma))$ for every $\Gamma \in A(D)$ and every $f \in W$. Then the family W is equicontinuous at x, and we take on D the euclidean metric and we take on $\overline{\mathbf{R}}^n$ the chordal metric.

Proof. Let $\epsilon > 0$ be so that $\overline{B}(x, \epsilon) \subset D$. Suppose that the family W is not equicontinuous at x. Then there exists $\delta > 0$, $r_m \to 0$ and $f_m \in W$ so that $q(f_m(\overline{B}(x, r_m))) \geq \delta$ for every $m \in N$ and let $G_m = f_m(\overline{B}(x, r_m))$ for $m \in N$. Since $Imf_m \cap Q = \phi$ and Imf_m are open sets, we see that $Imf_m \cap \overline{Q} = \phi$, hence $G_m \cap \overline{Q} = \phi$ for $m \in N$. Let $\Gamma^{‘}_m = \Delta(G_m, \overline{Q}, \mathbf{R}^n)$ for $m \in N$ and let Γ_m be the family of all maximal lifting of some paths from $\Gamma^{‘}_m$ starting from some points from $\overline{B}(x, r_m)$ for every $m \in N$. Since every path from Γ_m has at least a limit point outside $B(x, \epsilon)$, we see that $\Gamma_m > \Gamma_{x,r_m,\epsilon}$. We also see that $\Gamma^{‘}_m > f(\Gamma_m)$ for $m \in N$ and from Lemma 1 we can find $\alpha > 0$ so that $M_q(\Gamma^{‘}_m) > \alpha$ for every $m \in N$. We obtain that $M_q(\Gamma^{‘}_m) \leq M_q(f(\Gamma_m)) \leq \gamma(M(\Gamma_m)) \leq \gamma(M(\Gamma_{x,r_m,\epsilon})) \to 0$ if $m \to \infty$, and we reached a contradiction. It results that the family W is equicontinuous at x.

Remark 2. If in the preceding theorem we take $M = M^p_\omega$ with $p \geq 2$ and $\omega \in L^1_{loc}(D)$ and the family W is a family of homeomorphisms, we can use Lemma 7 from [8] and Theorem 16 from [9] to extend a known result from the theory of quasiconformal mappings from [27], Theorem 19.2, page 65.

Theorem 3. Let $1 \leq n - 1 < q$, $p \geq 2$, $D \subset \mathbf{R}^n$ a domain, $x \in D$, $\omega \in L^1_{loc}(D)$ so that $M^p_\omega(\{x\}) = 0$, let $\gamma : [0, \infty) \to [0, \infty)$ be increasing with $\lim_{t \to 0} \gamma(t) = 0$, and let W be a family of homeomorphisms $f : D \to D_f \subset \mathbf{R}^n$ so that there exists $\delta > 0$ so that for every $f \in W$ there exists points $a_f, b_f \notin Imf$ with $q(a_f, b_f) \geq \delta$ and suppose that $M_q(f(\Gamma)) \leq \gamma(M^p_\omega(\Gamma))$ for every $\Gamma \in A(D)$. Then the family W is equicontinuous at x, and we take on D the euclidean metric, and we take on $\overline{\mathbf{R}}^n$ the chordal metric.

The following eliminability result extends partially Theorem 2.9, page 66 from [22]. It also extends Theorem 17 from [9] in the case $M_N = M_q$ with $q > n-1$.

Theorem 4. Let $1 \leq n-1 < q$, $D \subset \mathbf{R}^n$ a domain, $x \in D$, M a modulus on D so that $\lim_{a \to 0} M(\Gamma_{x,a,b}) = 0$ for every $b > 0$ so that $\overline{B}(x,b) \subset D$, let $\gamma : [0,\infty) \to [0,\infty)$ be increasing so that $\lim_{t \to 0} \gamma(t) = 0$ and let $E \subset D$ be closed in D and nowhere disconnecting so that $x \in E$ and $M(E) = 0$. Let $f : D \setminus E \to \mathbf{R}^n$ be continuous, open, discrete so that $M_q(f(\Gamma)) \leq \gamma(M(\Gamma))$ for every $\Gamma \in A(D \setminus E)$ and suppose that there exists $r_x > 0$ so that $\overline{B}(x,r_x) \subset D$ and $cap_q \overline{Cf(B(x,r_x) \setminus E)} > 0$. Then there exists $\lim_{z \to x} f(z) \in \overline{\mathbf{R}}^n$.

Proof: Suppose that $CardC(f,x) > 1$ and let $b_1, b_2 \in C(f,x)$, $b_1 \neq b_2$ and let $x_j, y_j \in B(x,r_x) \setminus E$ be so that $x_j \neq y_j$ for $j \in N$ and $f(x_j) \to b_1$, $f(y_j) \to b_2$. Let $r_j > 0$ be so that $0 < r_j < r_x$ and $x_j, y_j \in B(x,r_j)$ and let $C_j \subset B(x,r_j) \setminus E$ be compact, connected so that $x_j, y_j \in C_j$ for $j \in N$. We can find $\delta > 0$ so that $q(f(C_j)) \geq \delta$ for every $j \in N$ and we see that $f(C_j) \cap \overline{Cf(B(x,r_x) \setminus E)} = \phi$ for every $j \in N$. Let $\Gamma_j^{'} = \Delta(f(C_j), \overline{Cf(B(x,r_x) \setminus E)}, \mathbf{R}^n)$ for $j \in N$ and let Γ_j be the family of all maximal lifting of some paths from $\Gamma_j^{'}$ starting from some points from C_j for $j \in N$. Let $\Gamma_{1j} = \{\varphi \in \Gamma_j | \varphi$ has at least a limit point in $E\}$ and let $\Gamma_{2j} = \{\varphi \in \Gamma_j | \varphi$ has at least a limit point outside $B(x,r_x)\}$ for $j \in N$. We see that $M(\Gamma_{1j}) = 0$, that $\Gamma_j = \Gamma_{1j} \cup \Gamma_{2j}$, $\Gamma_{2j} > \Gamma_{x,r_j,r_x}$ and $\Gamma_j^{'} > f(\Gamma_j)$ for every $j \in N$. Using Lemma 1, we can find $\epsilon > 0$ so that $M_q(\Gamma_j^{'}) > \epsilon$ for every $j \in N$. It results that $0 < \epsilon < M_q(\Gamma_j^{'}) \leq M_q(f(\Gamma_j)) \leq \gamma(M(\Gamma_j)) = \gamma(M(\Gamma_{1j} \cup \Gamma_{2j})) \leq \gamma(M(\Gamma_{1j}) + M(\Gamma_{2j})) = \gamma(M(\Gamma_{2j})) \leq \gamma(M(\Gamma_{x,r_j,r_x})) \to 0$ if $j \to \infty$ and we reached a contradiction.

We obtained that $CardC(f,x) = 1$ and hence that there exists $\lim_{z \to x} f(z) \in \overline{\mathbf{R}}^n$.

Using the preceding theorem, we have the following extension of Theorem 18 from [9] in the case $M_N = M_q$ with $q > n-1$.

Theorem 5. Let $1 \leq n-1 < q$, $D \subset \mathbf{R}^n$ a domain, x an isolated point of ∂D, M a modulus on D so that $\lim_{a \to 0} M(\Gamma_{x,a,b}) = 0$ for every $b > 0$ so that $\overline{B}(x,b) \setminus \{x\} \subset D$, $\gamma : [0,\infty) \to [0,\infty)$ increasing with $\lim_{t \to 0} \gamma(t) = 0$, let $f : D \to \mathbf{R}^n$ be continuous, open, discrete such that x is an essential singularity of f and suppose that $M_q(f(\Gamma)) \leq \gamma(M(\Gamma))$ for every $\Gamma \in A(D)$. Then $cap_q \overline{Cf(B(x,b) \setminus \{x\})} = 0$ for every $b > 0$ so that $\overline{B}(x,b) \setminus \{x\} \subset D$.

If f is of finite multiplicity, we can use Theorem 19 from [9] to prove the following eliminability result:

Theorem 6. Let $1 \leq n-1 < q$, $p \geq 2$, $D \subset \mathbf{R}^n$ a domain, $\omega \in L^1_{loc}(D)$, $x \in D$, $\gamma : [0,\infty) \to [0,\infty)$ be increasing so that $\lim_{t \to 0} \gamma(t) = 0$, $E \subset D$ closed in D and nowhere disconnecting so that $x \in E$ and $M^p_\omega(E) = 0$. Let $f : D \setminus E \to \mathbf{R}^n$ be continuous, open, discrete so that there exists $U_x \in \mathcal{V}(x)$ and $n_x \in N$ so that $N(f, U_x \cap (D \setminus E)) \leq n_x$ and $M_q(f(\Gamma)) \leq \gamma(M^p_\omega(\Gamma))$ for every $\Gamma \in A(D \setminus E)$. Then there exists $\lim_{z \to x} f(z) = \overline{\mathbf{R}}^n$.

Remark 3. If the modulus M from Theorem 2,4,5 is of the form $M = M^p_\omega$ with $\omega \in L^1_{loc}(D)$ and $p \geq 2$, we see from Lemma 7 from [8] that the condition "$\lim_{a \to 0} M^p_\omega(\Gamma_{x,a,b}) = 0$ for every fixed $b > 0$ so that $\overline{B}(x,b) \subset D$" is equivalent to "$M^p_\omega(\{x\}) = 0$", and if $\omega = 1$ and $p \leq n$, then $M_p(\{x\}) = 0$. It results that if in Theorem 4 we have $M = M^p_\omega$ with $p \geq 2$ and $\omega \in L^1_{loc}(D)$, the condition "$M^p_\omega(E) = 0$" also implies the condition "$\lim_{a \to 0} M^p_\omega(\Gamma_{x,a,b}) = 0$ for every fixed $b > 0$ so that $\overline{B}(x,b) \subset D$" for a point $x \in E$. Of course, the most important case we have in mind for the singular set E from Theorem 4 and Theorem 6 is when $E = \{x\}$, i.e. when x is an isolated singularity of f. We see from Theorem 2 from [7] that $M^p_\omega(\{x\}) = 0$ if there exists $a > 0$ so that $\overline{B}(x,a) \subset D$ and either $\int_{B(x,a)} exp(A \circ \omega)(z)dz < \infty$ for some Orlicz map A, or there exists

$M > 0$ and $0 < \alpha < n-1$ so that $\fint_{B(x,r)} \omega(z)dz \le M(ln(ae/r))^\alpha$ for $0 < r < a$.

Also, using Theorem 3 from [9] we see that $M^p_\omega(\{x\}) = 0$ if $\omega \in L^1_{loc}(D)$, $p > 1$ and there exists $b > 0$ so that $\overline{B}(x,b) \subset D$ and $p < l$ so that $\int_{B(x,r)} \omega(z)dz \le C(b)r^l$ if $0 < r < b$.

We also see from Theorem VII.1.15, page 166 from [22] that if $n-1 < q \le n$ and $A \subset \mathbf{R}^n$ is so that $cap_q A = 0$, then $m_1(A) = 0$ and hence A is totally disconnecting. It results that Theorem 1 generalizes Theorem 13 from [9], Theorem 2 extends Theorem 14 from [9], Theorem 4 generalizes Theorem 17 from [9] and Theorem 5 generalizes Theorem 18 from [9] in the case $M_N = M_q$ with $n-1 < q \le n$. Also, if in Theorem 4 and Theorem 6 we take $M = M_p$ with $p > 1$, then we also see from Theorem VII.1.15 from [22] that $m_{n-1}(E) = 0$ and hence that E is nowhere disconnecting. In this case, Theorem 4 has the following form:

Theorem 7. Let $1 \le n-1 < q$, $1 < p \le n$, $D \subset \mathbf{R}^n$ a domain, $E \subset D$ closed in D with $M_p(E) = 0$, $x \in E$, $\gamma : [0,\infty) \to [0,\infty)$ increasing with $\lim_{t \to 0} \gamma(t) = 0$, let $f : D \setminus E \to \mathbf{R}^n$ be continuous, open, discrete, so that $M_q(f(\Gamma)) \le \gamma(M_p(\Gamma))$ for every $\Gamma \in A(D \setminus E)$ and suppose that there exists $r_x > 0$ so that $\overline{B}(x,r_x) \subset D$ and $cap_q \overline{Cf(B(x,r_x) \setminus E)} > 0$. Then there exists $\lim_{z \to x} f(z) \in \overline{\mathbf{R}}^n$.

We showed in Theorem 12 from [9] that if $f \in W_{q,p,\omega,\gamma}(D)$ and there exists $\varphi : [0,\infty) \to [0,\infty)$ continuous, increasing with $\lim_{t \to 0} \varphi(t) = 0$ and so that $M^p_\omega(\Gamma_{x,a,b}) \le \varphi(a/b)$ for every $x \in D$ and $b > 0$ so that $\overline{B}(x,b) \subset D$, then there exists a constant $C(b)$ so that $|f(y) - f(x)| \le C(b)\gamma \circ \varphi(|y-x|)$ if $x \in D$ and $0 < |y-x| < b$, $\overline{B}(x,b) \subset D$. We found in this way an estimation of the modulus of continuity of the mappings from the class $W_{q,p,\omega,\gamma}(D)$ and we can use again Theorem 2 from [7] or Theorem 3 and Theorem 4 from [9] to establish some conditions of type "$M^p_\omega(\Gamma_{x,a,b}) \le \varphi(a/b)$, with $\varphi : (0,\infty) \to (0,\infty)$ continuous, increasing with $\lim_{t \to 0} \varphi(t) = 0$".

We give now a lower estimate of the distortion of a homeomorphism from the class $Q_{n,p,\omega,\gamma}(D)$, extending in this way Theorem 4.4, page 89 from [19] and Lemma 7.7, page 142 from [19].

Theorem 8. Let $f \in Q_{n,p,\omega,\gamma}(D)$ be injective and let $x \in D$ and $r > 0$ be so that $\overline{B}(x,r) \subset D$. Then there exists $\Psi : (0, \frac{r}{2}) \to \mathbf{R}$ increasing with $\lim_{t \to 0} \Psi(t) = 0$ and $|f(y) - f(x)| \ge \Psi(|y-x|)$ for every $y \in B(x, \frac{r}{2})$ and $\Psi(t) = L(x,f,\frac{r}{2}) exp(-\gamma(\int_{B(x,r)} \omega(z)dz \frac{1}{t^p}) \frac{1}{C(n)})$ for $t \in (0, \frac{r}{2})$, where $C(n)$ is a constant depending only on n.

Proof: Using Brouwer's theorem, we see that f is homeomorphism onto a domain $D^{`}$ from $\mathbf{R}^n$. Let $z \in S(x, \frac{r}{2})$ be so that $L(x,f,\frac{r}{2}) = |f(z) - f(x)|$ and let $S \in S(x,r)$ be so that $|f(x) - f(S)| = L(x,f,r)$, and let $y \in B(x, \frac{r}{2})$. Let P be the plane determined by the points x, y, z and let C be the circle under the intersection of P and $S(y, |y-x|)$. Let B be the tangency point to C of a ray emerging from z, let C_1 be the shortest arc from C joining x and B and let $E_1 = C_1 \cup [B, z]$. Then E_1 is compact, connected, $E_1 \setminus \{z\} \subset B(x, \frac{r}{2})$, $f(E_1)$ is compact and connected, joins $f(x)$ with $f(z)$ and $f(E_1) \setminus \{f(z)\} \subset B(f(x), L(x,f,\frac{r}{2}))$. We take a line d perpendicular on the plane P at the point y and let $M \in d \cap S(x,r)$ and $E_2 = [y, M] \cup S(x,r)$. Then E_2 is compact, connected, $E_2 \subset \overline{B}(x,r)$ and $f(E_2)$ is compact, connected and joins $f(y)$ with a point $f(S) \in f(S(x,r))$ and $d(E_1, E_2) \ge |y-x| > 0$.

Suppose that $f(y) \in B(f(x), L(x,f,\frac{r}{2}))$. Since $\overline{B}(f(x), L(x,f,\frac{r}{2})) \subset B(f(x), L(x,f,r))$, we see that $|f(x) - f(z)| < |f(x) - f(S)|$, hence $S(f(x),t) \cap f(E_1) \ne \phi, S(f(x),t) \cap f(E_2) \ne \phi$ for $|f(y) - f(x)| < t < L(x,f,\frac{r}{2})$. Let $\Gamma^{`} = \Delta(f(E_1), f(E_2), B(f(x), L(x,f,\frac{r}{2}) \setminus \overline{B}(f(x), |f(y) - f(x)|)))$. Let $\Gamma$ be the family of all maximal lifting of some paths from $\Gamma^{`}$ starting from some points of E_1.

Let $\varphi : [0,1] \to \mathbf{R}^n$, $\varphi \in \Gamma^{`}$. Then $\varphi([0,1)) \cap f(E_2) = \phi$ and let $\varphi_0 : [0,a) \to \mathbf{R}^n$ be a maximal lifting of φ from a point $x_0 \in E_1$ with $0 < a < 1$. Then $f(\varphi_0([0,a))) \cap f(E_2) = \phi$ and since $Im\varphi_0$ is connected and $\varphi_0(0) = x_0 \in E_1 \subset B(x,r)$, it results that $\varphi_0([0,a)) \subset B(x,r)$. Indeed, otherwise we can find $0 < c < a$ so that $\varphi_0(c) \in S(x,r)$, hence $f(\varphi_0(c)) \in f(S(x,r)) \subset f(E_2)$, and this contradicts the fact that $f(\varphi_0([0,a))) \cap f(E_2) = \phi$. Let b be a limit point of $\varphi_0 : [0,a) \to B(x,r)$. If $b \in S(x,r)$, then $f(b) = \varphi(a) \in \varphi([0,1)) \cap f(E_2)$ and this contradicts the hypothesis that $\varphi([0,1)) \cap f(E_2) = \phi$, and if $b \in B(x,r)$, this contradicts the maximality of the path $\varphi_0 : [0,a) \to \mathbf{R}^n$. It results that $a = 1$ and if b is a limit point of $\varphi_0 : [0,1) \to B(x,r)$, then $f(b) = \varphi(1) \in f(E_2)$ and since $f : D \to D^{`}$ is a homeomorphism, it results that $b \in f^{-1}(f(E_2)) = E_2$. We proved that if $\varphi_0 \in \Gamma$, $\varphi_0 : [0,1] \to D$, then φ_0 joins a point from E_1 with a point from E_2, hence, if φ_0 is locally rectifiable, it results that $l(\varphi_0) \geq |y-x|$.

Let $\rho = (1/|y-x|)\chi_{B(x,r)}$. Then $\rho \in F(\Gamma)$ and since $f(\Gamma) > \Gamma^{`}$, we have $C(n)\ln(L(x,f,\frac{r}{2})/$

$|f(y)-f(x)|) \leq M_n(\Gamma^{`}) \leq M_n(f(\Gamma)) \leq \gamma(M^p_\omega(\Gamma)) \leq \gamma(\int\limits_{\mathbf{R}^n} \omega(z)\rho^p(z)dz) \leq \gamma(\int\limits_{B(x,r)} \omega(z)dz\frac{1}{|y-x|^p})$.

This implies that $|f(y)-f(x)| \geq L(x,f,\frac{r}{2})\exp(-\gamma(\int\limits_{B(x,r)} \omega(z)dz\frac{1}{|y-x|^p})\frac{1}{C(n)})$ if $|y-x| < \frac{r}{2}$.

The preceding inequality holds also if $L(x,f,\frac{r}{2}) \leq |f(y)-f(x)|$, hence we can take now $\Psi : (0,\frac{r}{2}) \to \mathbf{R}$, $\Psi(t) = L(x,f,\frac{r}{2})\exp(-\gamma(\int\limits_{B(x,r)} \omega(z)dz\frac{1}{t^p})\frac{1}{C(n)})$ for $t \in (0,\frac{r}{2})$ and the theorem is now proved.

We use now the preceding theorem to prove a Hurwitz type theorem in class $Q_{n,p,\omega,\gamma}(D)$, extending in this way Theorem 21.9, page 73 from [27] and Corollary 37.3, page 125 from [27].

Theorem 9. (Hurwitz's theorem in the class $Q_{n,p,\omega,\gamma}(D)$). Let $f_m \in Q_{n,p,\omega,\gamma}(D)$ be homeomorphisms so that $f_m \to f$ uniformly on the compact subsets of D. Then f is either a constant map on D, or $f : D \to f(D)$ is a homeomorphism in the class $Q_{n,p,\omega,\gamma}(D)$.

Proof: Suppose that f is not discrete at a point $x \in D$ and let $r > 0$ be so that $\overline{B}(x,r) \subset D$. There exists points $x_k \in B(x,\frac{r}{2})$ so that $x_k \neq x$ and $f(x_k) = f(x)$ for every $k \in N$ and suppose that there exists $y \in B(x,\frac{r}{2})$ so that $f(y) \neq f(x)$. Let $\delta = |f(y)-f(x)| > 0$. We can suppose that $|f_k(y)-f_k(x)| \geq \frac{\delta}{2}$ for every $k \in N$ and let $\alpha = |y-x|$. Then $L(x,f_k,\frac{r}{2}) \geq L(x,f_k,\alpha) \geq |f_k(y)-f_k(x)| \geq \frac{\delta}{2}$ for every $k \in N$ and let $\Psi_k : (0,\frac{r}{2}) \to \mathbf{R}$ be given by $\Psi_k(t) = L(x,f_k,\frac{r}{2})\exp(-\gamma(\int\limits_{B(x,r)} \omega(z)dz\frac{1}{t^p})\frac{1}{C(n)})$ for $t \in (0,\frac{r}{2})$. We see from Theorem 8 that $|f_k(z)-f_k(x)| \geq \Psi_k(|z-x|)$ for $z \in B(x,\frac{r}{2})$ and every $k \in N$, hence
$|f_k(z)-f_k(x)| \geq \frac{\delta}{2}\exp(-\gamma(\int\limits_{B(x,r)} \omega(u)du\frac{1}{|z-x|^p})\frac{1}{C(n)})$ for every $z \in B(x,\frac{r}{2})$ and every $k \in N$.

Letting $z = x_m$ in the preceding inequality, we find that

$$|f_k(x_m)-f_k(x)| \geq \frac{\delta}{2}\exp(-\gamma(\int\limits_{B(x,r)} \omega(u)du\frac{1}{|x_m-x|^p})\frac{1}{C(n)}),\ for\ k,m \in N \qquad (1)$$

We fix now m in (1) and letting $k \to \infty$ we find that

$$0 = |f(x_m)-f(x)| \geq \frac{\delta}{2}\exp(-\gamma(\int\limits_{B(x,r)} \omega(u)du\frac{1}{|x_m-x|^p})\frac{1}{C(n)}) > 0$$

, and we reached a contradiction.

We proved that if $\overline{B}(x,r) \subset D$ and $x_k \to x$, $x_k \neq x$, $x_k \in B(x,r)$ and $f(x_k) = f(x)$ for every $k \in N$, then it results that $f(y) = f(x)$ for every $y \in B(x,\frac{r}{2})$. We therefore proved that if $x \in D$, then either f is discrete at x, or f is constant on a neighborhood of x.

Suppose that f is constant on a neighborhood of a point $x \in D$ and let $Q = \{y \in D|$ there exists $U_y \in \mathcal{V}(y)$ so that $f(z) = f(x)$ for every $z \in U_y\}$. Then Q is open in D and suppose that $\partial_D Q \neq \phi$, and let $y \in \partial_D Q$. Then there exists $x_k \in Q$, $x_k \to y$ and from what we have proved before we can find $U_y \in \mathcal{V}(y)$ so that $U_y \subset D$ and $f(z) = f(x)$ for every $z \in U_y$, and we reached a contradiction, since we supposed that $y \in \partial_D Q$. It results that $\partial_D Q = \phi$ and since D is a domain, we find that $D = Q$, i.e. f is constant on D. We therefore proved that either f is a constant map on D, or f is a discrete map on D.

Suppose that f is not constant on D. Then f is a discrete map and since $f_m \to f$ uniformly on the compact subsets of D and every map f_m is injective, we see from [4] that $f : D \to f(D)$ is a homeomorphism.

Let now G a domain so that $\overline{G} \subset D$ and E, F compact, connected so that $E, F \subset \overline{G}$ and let $\Gamma = \Delta(E, F, G)$. Since $f_m \to f$ uniformly on $\overline{G}$, we see that there exists $m_0 \in N$ so that $f(G) \subset f_m(G)$ for every $m \geq m_0$ (see for instance the first part of the proof of Theorem 21.9, page 73 from [27]). Then $M_n(\Delta(f_m(E), f_m(F), f(G))) \leq M_n(\Delta(f_m(E), f_m(F), f_m(G))) = M_n(f_m(\Delta(E, F, G))) \leq \gamma(M^p_\omega(\Delta(E, F, G))) = \gamma(M^p_\omega(\Gamma))$ for every $m \geq m_0$.

Using Lemma 6 from [5] or Lemma 6 from [8] we obtain that $M_n(f(\Gamma)) = M_n(f(\Delta(E, F, G))) = M_n(\Delta(f(E), f(F), f(G))) = \lim\limits_{m\to\infty} M_n(\Delta(f_m(E), f_m(F), f(G))) \leq \gamma(M^p_\omega(\Gamma))$, hence $f \in Q_{n,p,\omega,\gamma}(D)$.

Remark 4. Natural extensions of our results can be established on arbitrary metric measure spaces. Indeed, the important modular inequality of Caraman from Theorem 4 from [3] used in our Lemma 1 on $\mathbf{R}^n$, holds also on Ahlfors regular metric spaces (see Proposition 4.7 from [1]).

In a future paper we shall study the boundary behavior of the mappings from the class $W_{q,p,\omega,\gamma}(D)$.

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