Γ-correlated processes. Some geometrical considerations.

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Abstract

Some geometrical aspects of the Γ-correlated processes are analyzed, starting from the properties of a Γ-orthogonal projection, which is not a proper one. Geometrical results are generalized to Γ-correlated case, especially the problem of the angle between the past and the future of some Γ-correlated processes. In the periodically Γ-correlated case it is proved that the positivity of the angle is preserved by its stationary dilation process. The generalized Friedrichs angle and other geometrical concepts are used in analysing some properties of periodically Γ-correlated processes.

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1 Preliminaries

A Γ-correlated processes is a sequence $(f_t)_{t\in G}$ in a right $\mathcal{L}(\mathcal{E})$-module $\mathcal{H}$ endowed with a correlation of the action of $\mathcal{L}(\mathcal{E})$. Here G is $\mathbb{Z}$, $\mathbb{R}$, or more generally a locally compact abelian group, and by $\mathcal{L}(\mathcal{E})$ is denoted the C^*-algebra of all linear bounded operators on a separable Hilbert space $\mathcal{E}$.

By an *action* of $\mathcal{L}(\mathcal{E})$ on $\mathcal{H}$ we mean the map $\mathcal{L}(\mathcal{E})\times\mathcal{H}$ into $\mathcal{H}$ given by $Ah := hA$ in the sense of the right $\mathcal{L}(\mathcal{E})$-module $\mathcal{H}$. We are writting Ah instead of hA to respect the classical notations from the scalar case. A *correlation* of the action of $\mathcal{L}(\mathcal{E})$ on $\mathcal{H}$ is a map Γ from $\mathcal{H}\times\mathcal{H}$ into $\mathcal{L}(\mathcal{E})$ having the properties:

(i) $\Gamma[h,h]\geq 0,\ \ \Gamma[h,h]=0$ implies $h=0$;

(ii) $\Gamma[g,h]=\Gamma[h,g]^*$;

(iii) $\Gamma[h,Ag]=\Gamma[h,g]A$.

In various calculations we will use the formula

$$\Gamma\Big[\sum_i A_ih_i,\sum_j B_jg_j\Big]=\sum_{i,j}A_i^*\Gamma[h_i,g_j]B_j$$

obtained by (ii) and (iii) for finite sums of actions of $\mathcal{L}(\mathcal{E})$ on $\mathcal{H}$.

A triplet $\{\mathcal{E},\mathcal{H},\Gamma\}$ defined as above was called [17] a *correlated action* of $\mathcal{L}(\mathcal{E})$ on $\mathcal{H}$.

By the fact that generally in $\mathcal{H}$ we have no topology, the prediction subsets, such as past and present, future, etc., can not be seen as closed subspaces, therefore the powerful tool of the orthogonal projection can not be directly used.

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An example of correlated action can be constructed as follows. Take as the right $\mathcal{L}(\mathcal{E})$-module $\mathcal{H} = \mathcal{L}(\mathcal{E}, \mathcal{K})$ – the space of the linear bounded operators from $\mathcal{E}$ into $\mathcal{K}$, where $\mathcal{E}$ and $\mathcal{K}$ are Hilbert spaces. An action of $\mathcal{L}(\mathcal{E})$ on $\mathcal{L}(\mathcal{E}, \mathcal{K})$ is given if we consider $AV := VA$ for each $A \in \mathcal{L}(\mathcal{E})$ and $V \in \mathcal{L}(\mathcal{E}, \mathcal{K})$. It is easy to see that $\Gamma[V_1, V_2] = V_1^* V_2$ is a correlation of the action of $\mathcal{L}(\mathcal{E})$ on $\mathcal{L}(\mathcal{E}, \mathcal{K})$, and the triplet $\{\mathcal{E}, \mathcal{L}(\mathcal{E}, \mathcal{K}), \Gamma\}$ is a correlated action (the *operatorial model*). It was proved [17] that any abstract correlated action $\{\mathcal{E}, \mathcal{H}, \Gamma\}$ can be embedded into the operatorial model. Namely, there exists an algebraic embedding $h \to V_h$ of $\mathcal{H}$ into $\mathcal{L}(\mathcal{E}, \mathcal{K})$, where $\mathcal{K}$ is obtained as the Aronsjain reproducing kernel Hilbert space given by a positive definite kernel obtained from the correlation Γ. The generators of $\mathcal{K}$ are elements of the form $\gamma_{(a,h)} : \mathcal{E} \times \mathcal{H} \to \mathbb{C}$, where $\gamma_{(a,h)}(b, g) = \langle \Gamma[g, h]a, b\rangle_{\mathcal{E}}$ and the embedding $h \to V_h$ is given by $V_h a = \gamma_{(a,b)}$.

Due to such an embedding of any correlated action $\{\mathcal{E}, \mathcal{H}, \Gamma\}$ into the operatorial model, prediction problems can be formulated and solved using operator techniques. In the particular case when the embedding $h \to V_h$ is onto, the correlated action $\{\mathcal{E}, \mathcal{H}, \Gamma\}$ is caled a *complete correlated action*. In this paper most of properties are analysed in the complete correlated case.

2 Some geometrical aspects

A first geometrical aspect is the existence of a Γ-orthogonal projection "on" a right $\mathcal{L}(\mathcal{E})$-submodule $\mathcal{H}_1$ of $\mathcal{H}$.

PROPOSITION 2.1. *Let $\mathcal{H}_1$ be a submodule in the right $\mathcal{L}(\mathcal{E})$-module $\mathcal{H}$ and*

$$\mathcal{K}_1 = \bigvee_{x \in \mathcal{H}_1} V_x \mathcal{E} \subset \mathcal{K}. \tag{2.1}$$

For each $h \in \mathcal{H}$ there exists a unique element $h_1 \in \mathcal{H}$ such that for each $a \in \mathcal{E}$ we have

$$V_{h_1} a \in \mathcal{K}_1 \qquad \text{and} \qquad V_{h-h_1} a \in \mathcal{K}_1^{\perp}. \tag{2.2}$$

Moreover, we have

$$\Gamma[h - h_1, h - h_1] = \inf_{x \in \mathcal{H}_1} \Gamma[h - x, h - x], \tag{2.3}$$

where the infimum is taken in the set of all positive operators from $\mathcal{L}(\mathcal{E})$.

A complete proof can be found in [17]. This result assure that if we put

$$\mathcal{P}_{\mathcal{H}_1} h = h_1, \tag{2.4}$$

then we can interpret the endomorphism $\mathcal{P}_{\mathcal{H}_1}$ of $\mathcal{H}$ as a Γ-orthogonal projection "on" $\mathcal{H}_1$, since we have $\mathcal{P}_{\mathcal{H}_1}^2 = \mathcal{P}_{\mathcal{H}_1}$ and $\Gamma[\mathcal{P}_{\mathcal{H}_1} h, g] = \Gamma[h, \mathcal{P}_{\mathcal{H}_1} g]$.

Also as a geometrical aspect, let us remark that the unique element h_1 obtained by the Γ-orthogonal projection of $h \in \mathcal{H}$ can belongs not necessary to $\mathcal{H}_1$, but, due to (2.3) it is close enough to be considered as the best estimation.

The previous result can be generalized to an "orthogonal projection" from $\mathcal{H}^T$ - the cartesian product of T copies of $\mathcal{H}$ on a submodule $\mathcal{M}$ of $\mathcal{H}^T$, as follows. Firstly, the embedding of $\mathcal{H}^T$ into $\mathcal{L}(\mathcal{E}, \mathcal{K}^T)$ is defined by

$$W_X a = (V_{x_1} a, \ldots, V_{x_T} a) \tag{2.5}$$

for $a \in \mathcal{E}$ and $X = (x_1, \dots, x_T) \in \mathcal{H}^T$, and then the extended "orthogonal projection" $\mathcal{P}_{\mathcal{M}}X$ it follows with respect to an appropriate correlation [20], considering $\mathcal{K}_1^T = \bigvee_{X \in \mathcal{M}} W_X \mathcal{E}$ in $\mathcal{K}^T$.

The action of $\mathcal{L}(\mathcal{E})$ on $\mathcal{H}^T$ is given by acting on the components, which is a particular case of the matrix action of $\mathcal{L}(\mathcal{E})^{T\times T}$ on $\mathcal{H}^T$ in the sense of the right multiplication.

A Γ-correlated process $(f_t) \subset \mathcal{H}$ is *stationary* if $\Gamma[f_s, f_t]$ depends only on $t - s$ and not by s and t separately. For a Γ-correlated process (not necessary stationary) the *past-present* at the moment $t = n$ is the right $\mathcal{L}(\mathcal{E})$-submodule

(2.6) $$\mathcal{H}_n^f = \Big\{ \sum_k A_k f_k;\ A_k \in \mathcal{L}(\mathcal{E}),\ k \le n \Big\},$$

and the *future* is

(2.7) $$\widetilde{\mathcal{H}}_n^f = \Big\{ \sum_k A_k f_k;\ A_k \in \mathcal{L}(\mathcal{E}),\ k > n \Big\}.$$

By the embedding $h \to V_h$ of $\mathcal{H}$ into $\mathcal{L}(\mathcal{E}, \mathcal{K})$, the corresponding *past* and *future* will be the closed subspaces of $\mathcal{K}$ given by

(2.8) $$\mathcal{K}_n^f = \bigvee_{j \le n} V_{f_j} \mathcal{E}$$

and

(2.9) $$\widetilde{\mathcal{K}}_n^f = \bigvee_{j > n} V_{f_j} \mathcal{E}.$$

Similarly, various processes can be considered in the right $\mathcal{L}(\mathcal{E})$-module, or $\mathcal{L}(\mathcal{E})^{T\times T}$-module $\mathcal{H}^T$, and appropriate past and future constructed. Also, various correlations can be done. Between these, especially for the study of periodically correlated processes, the following correlations are of interest. For $X = (x_1, \dots, x_T)$ and $Y = (y_1, \dots, y_T)$ from $\mathcal{H}^T$, taking into account the right action of $\mathcal{L}(\mathcal{E})$, respectively of $\mathcal{L}(\mathcal{E})^{T\times T}$ on $\mathcal{H}^T$, it is simply to see that $\Gamma_1 : \mathcal{H}^T \times \mathcal{H}^T \to \mathcal{L}(\mathcal{E})$ and $\Gamma_T : \mathcal{H}^T \times \mathcal{H}^T \to \mathcal{L}(\mathcal{E})^{T\times T}$ defined, respectively, by

(2.10) $$\Gamma_1[X, Y] = \sum_{k=1}^{T} \Gamma[x_k, y_k]$$

and the matriceal one

(2.11) $$\Gamma_T[X, Y] = \Big(\Gamma[x_i, y_j] \Big)_{i,j \in \{1,2,\dots,T\}}$$

are correlations on $\mathcal{H}^T$.

Remember that a process (f_t) is periodically Γ-correlated if there exists a positive T such that $\Gamma[f_{s+T}, f_{t+T}] = \Gamma[f_s, f_t]$.

For a Γ-correlated process (f_t), if we take sequences of consecutive T terms

(2.12) $$X_n = \big(f_n, f_{n+1}, \dots, f_{n+T-1} \big),$$

then (X_n) is a stationary Γ_1-correlated process in $\mathcal{H}^T$. Also, taking consecutive blocks of length T

$$X_n^T = \big(f_{nT}, f_{nT+1}, \dots, f_{nT+T-1}\big), \tag{2.13}$$

then (X_n^T) is a stationary Γ_T-correlated process in $\mathcal{H}^T$.

From prediction point of view and the study of periodically Γ-correlated processes, the following result [20] was proved.

PROPOSITION 2.2. *Let $(f_n)_{n\in\mathbb{Z}}$ be a Γ-correlated process in $\mathcal{H}$, $T \geq 2$, (X_n) and (X_n^T) defined by* (2.12) *and* (2.13). *The following are equivalent:*
(i) $\{f_n\}$ is periodically Γ-correlated in $\mathcal{H}$, with the period T;
(ii) $\{X_n\}$ is stationary Γ_1-correlated in $\mathcal{H}^T$;
(iii) $\{X_n^T\}$ is stationary Γ_T-correlated in $\mathcal{H}^T$.

Between other strong geometrical aspects, such as the Wold decomposition of a Γ-correlated process, the dilation of a nonstationary process to a stationary one is very useful for prediction resons. A nonstatioary Γ-correlated process (f_t) in $\mathcal{H}$ has a *stationary dilation* if there exists a larger right module H and a stationary process (g_t) in H such that $f_t = \mathcal{P}^H_{\mathcal{H}} g_t$. It is easy to see that each periodically Γ-correlated process $(f_t) \subset \mathcal{H}$ has a stationary Γ_1-correlated dilation $(X_n) \subset \mathcal{H}^T$.

The geometrical property of a process to have a stationary dilation permits us to use some stationary techniques in the study of some nonstationary processes. This is the case at least for the processes very close to the stationary processes, such as periodically, harmonizable, or uniformly bounded linearly stationary processes.

A nice geometrical aspect is the fact that in the discrete case ($G = \mathbb{Z}$) each periodically Γ-correlated process with the period T is Γ-harmonizable and its spectral distribution is an $\mathcal{L}(\mathcal{E})$-valued semispectral measure supported on $2T-1$ equidistant stright line segments parallel to the diagonal of the square $[0, 2\pi] \times [0, 2\pi]$. Unfortunately this nice property is not generally valid in the continuous case. The stationarity is characterized by the fact that the support is reduced to the diagonal.

3 The angle between past and future

One of the prediction problem is the study of the angle between the past and the future of a process. Starting with the study of Helson and Szegő [9], the results was generalized in various contexts, helping in the characterization of stationary and some nonstationary processes. Here a generalization in the stationary Γ-correlated case is obtained, and some results for periodically case are analysed.

Actually the notions of the angles between two subspaces of a Hilbert space arise in [6] and [5], starting from the general definition of the scalar product of two vectors into the form $\langle h, g\rangle = \|h\|\,\|g\| \cdot \cos\alpha$. The *angle* (sometimes called the Dixmier angle) between two subspaces $\mathcal{M}$ and $\mathcal{N}$ from a Hilbert space $\mathcal{K}$ is given by its cosinus

$$\rho(\mathcal{M}, \mathcal{N}) := \sup\big\{\, |\langle h, g\rangle|\,;\ h \in \mathcal{M} \cap B_{\mathcal{K}}, g \in \mathcal{N} \cap B_{\mathcal{K}}\big\}. \tag{3.1}$$

where $B_{\mathcal{K}}$ is the unit ball of $\mathcal{K}$.

In the context of a complete correlated action $\{\mathcal{E}, \mathcal{H}, \Gamma\}$ the cosinus between the submodules $\mathcal{M}$ and $\mathcal{N}$ of the right $\mathcal{L}(\mathcal{E})$-module $\mathcal{H}$ is given by

$$\rho(\mathcal{M}, \mathcal{N}) = \sup\left\{\, |\langle \Gamma[g,h]a, b\rangle| \,;\, \|\Gamma[h,h]a\| \leq 1, \|\Gamma[g,g]b\| \leq 1\right\},$$

where $h \in \mathcal{M}, g \in \mathcal{N},\ a, b \in \mathcal{E}$.

We say that $\mathcal{M}$ and $\mathcal{N}$ have a *positive angle*, if $\rho(\mathcal{M}, \mathcal{N}) < 1$, or equivalently, if there exists $\rho < 1$ such that for any $h \in \mathcal{M}, g \in \mathcal{N},\ a, b \in \mathcal{E}$

$$|\langle \Gamma[g,h]a, b\rangle_{\mathcal{E}}| \leq \rho \, \|V_h a\| \, \|V_g b\| \, . \tag{3.2}$$

In the study of prediction problems we are interested in the case when the angle between past and future is positive, i.e., when $\rho(n) = \rho(\mathcal{H}_n^f, \widetilde{\mathcal{H}}_n^f) < 1$.

As a remark, if (f_n) is a stationary Γ-correlated process in $\mathcal{H}$, then the angle between the past and future is constant, i.e. does not depends on the choosing of the present time $t = n$. Indeed, for $a, b \in \mathcal{E}$ we have

$$\begin{aligned}
\rho(n) &= \sup\left\{\, |\langle \Gamma[g,h]a, b\rangle| \,;\, h \in \mathcal{H}_n^f, g \in \widetilde{\mathcal{H}}_n^f\right\} = \\
&= \sup\left\{\, \left|\left\langle \Gamma[\sum_{k\leq n} A_k f_k, \sum_{p>n} A_p f_p]a, b\right\rangle\right| ;\ A_j \in \mathcal{L}(\mathcal{E})\right\} = \\
&= \sup\left\{\, \left|\sum_{k\leq n}\sum_{p>n} \left\langle A_p^* \Gamma[f_p, f_k] A_k a, b\right\rangle\right| \right\} = \\
&= \sup\left\{\, \left|\sum_{k\leq n}\sum_{p>n} \left\langle A_p^* \Gamma[f_{p+m}, f_{k+m}] A_k a, b\right\rangle\right| \right\} = \rho(n+m)
\end{aligned}$$

for any $m \in \mathbb{Z}$.

Generalizing to stationary Γ-correlated case a result of [9] we have

PROPOSITION 3.1. *Let (f_n) be a stationary Γ-correlated process in $\mathcal{H}$. The angle between past and future of (f_n) is positive if and only if there exists a finite constant C which depends only by (f_n) such that for each element of the form $\sum V_{f_n} a_n$ from the time domain $\mathcal{K}_\infty^f$ and for each $-\infty \leq n_1 \leq n_2 < \infty$ we have*

$$\left\| \sum_{k=n_1}^{n_2} V_{f_k} a_k \right\| \leq C \left\| \sum V_{f_k} a_k \right\| , \tag{3.3}$$

where in the second term the sum has finitely many non-zero elements.

Proof. It is known [9] that for two subspaces $\mathcal{M}$ and $\mathcal{N}$ from a Hilbert space we have $\rho(\mathcal{M}, \mathcal{N}) < 1$ if and only if there exists a finite constant C such that $\|x\| \leq C \|x + y\|$ for x and y generators in $\mathcal{M}$ and $\mathcal{N}$, respectively. Therefore for any sum of the form $\sum V_{f_n} a_n$ from the time domain $\mathcal{K}_\infty^f$, taking into account that $\rho(\mathcal{H}_n^f, \widetilde{\mathcal{H}}_n^f) < 1$, we have

$$\left\| \sum_{k\leq n} V_{f_k} a_k \right\| \leq C \left\| \sum_{k\leq n} V_{f_k} a_k + \sum_{k>n} V_{f_k} a_k \right\| = C \left\| \sum V_{f_k} a_k \right\| ,$$

where $\sum V_{f_k} a_k$ has finitely many non-zero elements. Since (f_n) is stationary Γ-correlated, for any $m \in \mathbb{Z}$ we have

$$\left\| \sum_{k\leq m} V_{f_k} a_k \right\|_{\mathcal{K}}^2 = \left\langle \sum_{k\leq m} V_{f_k} a_k, \sum_{p\leq m} V_{f_p} a_p \right\rangle = \sum_{k,p\leq m} \left\langle V_{f_p}^* V_{f_k} a_k, a_p \right\rangle_{\mathcal{E}} =$$

$$= \sum_{k,p\leq m} \langle \Gamma[f_p, f_k]a_k, a_p\rangle = \sum_{k,p\leq m} \left\langle \Gamma[f_{p-(m-n)}, f_{k-(m-n)}]a_k, a_p\right\rangle =$$

$$= \sum_{i,j\leq n} \langle \Gamma[f_j, f_i]a_i, a_j\rangle = \left\| \sum_{k\leq n} V_{f_k}a_k \right\|_{\mathcal{K}}^2 \leq C^2 \left\|\sum V_{f_k}a_k\right\|_{\mathcal{K}}^2.$$

Therefore

$$\left\| \sum_{k=n_1}^{n_2} V_{f_k}a_k \right\| = \left\| \sum_{k\leq n_2} V_{f_k}a_k - \sum_{k<n_1} V_{f_k}a_k \right\| \leq$$

$$\leq \left\| \sum_{k\leq n_2} V_{f_k}a_k \right\| + \left\| \sum_{k\leq n_1} V_{f_k}a_k \right\| \leq 2C \left\|\sum V_{f_k}a_k\right\|$$

and (3.3) is proved. □

Also the property of representing the elements from the time domain as a series (Schauder basis [13]) can be obtained for Γ-correlated processes.

PROPOSITION 3.2. *The angle between past and future of a stationary Γ-correlated process (f_n) is positive if and only if each element k from the time domain $\mathcal{K}_\infty^f$ admits a unique representation of the form $k = \sum\limits_{n=-\infty}^{\infty} k_n$ with $k_n \in \overline{V_{f_n}\mathcal{E}}$.*

Proof. Using the previous Proposition, if we take $P_n(\sum V_{f_k}a_k) = V_{f_n}a_n$, then (f_n) is of positive angle if and only if P_n is a linear operator on $\mathcal{K}_\infty^f$, for each $n \in \mathbb{Z}$, and $\sum\limits_{n_1}^{n_2} P_i$ are uniformly bounded operators and

$$k = \lim_{n_1,n_2} \sum_{n_1}^{n_2} P_i k = \sum_{-\infty}^{\infty} P_i k = \sum_{-\infty}^{\infty} k_n$$

To prove the unicity, if $k = \sum\limits_{-\infty}^{\infty} k'_n$ with $k'_n \in \overline{V_{f_n}\mathcal{E}}$, then by the fact that for $i \neq n$ we have $P_i k = 0$, and it follows that for $n \in \mathbb{Z}$

$$k'_n = P_n(\sum_n k'_n) = P_n k = P_n(\sum_n k_n) = k_n.$$

Conversely, if each $k \in \mathcal{K}_\infty^f$ admits a unique representation of the form $k = \sum\limits_{-\infty}^{\infty} k_n$ with $k_n \in \overline{V_{f_n}\mathcal{E}}$, then the operators $T_n : \mathcal{K}_\infty^f \to \overline{V_{f_n}\mathcal{E}}$ defined by $T_n k = k_n$ are well-defined, bounded and the family of elements of the form $\left\| \sum\limits_{n=k}^{p} T_n \right\|$ is uniformly bounded, and by Proposition 3.1 it follows that the angle between past and future of (f_n) is positive. □

We have seen that a periodically Γ-correlated process $(f_n)_{n\in\mathbb{Z}}$ from $\mathcal{H}$ has a stationary Γ-correlated dilation (X_n) in $\mathcal{H}^T$. In [20] an explicit stationary dilation is constructed which help in obtaining the Wiener filter for prediction and the prediction-error operator function for a periodically Γ-correlated process, in terms of the operator coefficients of its attached maximal function. Here we prove the following result concerning the angle of the stationary dilation of a periodically Γ-correlated process.

Proposition 3.3. *If (f_n) from $\mathcal{H}$ is a periodically Γ-correlated process with a positive angle between its past and future, then the angle between the past and the future of its stationary Γ_1-correlated dilation (X_n) from $\mathcal{H}^T$ it is also positive.*

Proof. Analogously as in (2.6) and (2.7), in $\mathcal{H}^T$ the past H_n^X and the future $\tilde{H}_n^X$ for a process $(X_n) \subset \mathcal{H}^T$ is constructed as linear combinations of finite actions of $\mathcal{L}(\mathcal{E})$ on $(X_n) \subset \mathcal{H}^T$. If (f_n) from $\mathcal{H}$ is a periodically Γ-correlated process having a positive angle between its past and future, then at each time $t = n$ there exists $\rho(n) < 1$ such that

$$|\langle \Gamma[g,h]a,b\rangle_{\mathcal{E}}| \leq \rho(n)\,\|V_h a\|\,\|V_g b\|$$

for each $h \in \mathcal{H}_n^f$ and $g \in \tilde{\mathcal{H}}_n^f$. For each element $X = \sum\limits_{k\leq n} A_k X_k$ from the past H_n^X and $Y = \sum\limits_{p>n} B_p X_p$ from the future $\tilde{H}_n^X$ of the process (X_n) given by $X_n = (f_n, f_{n+1}, \ldots, f_{n+T-1})$, and for any $a, b \in \mathcal{E}$ we have

$$\begin{aligned}
|\langle \Gamma_1[X,Y]a,b\rangle_{\mathcal{E}}| &= \left|\left\langle \Gamma_1\Big[\sum_{p>n} B_p X_p, \sum_{k\leq n} A_k X_k\Big]a, b\right\rangle_{\mathcal{E}}\right| = \\
&= \left|\sum_{p>n}\sum_{k\leq n} \langle \Gamma_1[B_p X_p, A_k X_k]a,b\rangle_{\mathcal{E}}\right| = \\
&= \left|\sum_{p>n}\sum_{k\leq n}\sum_{i=0}^{T-1} \langle \Gamma[B_p f_{p+i}, A_k f_{k+i}]a,b\rangle_{\mathcal{E}}\right| = \\
&= \left|\sum_{p>n}\sum_{k\leq n}\sum_{i=0}^{T-1} \big\langle B_p^*\Gamma[f_{p+i}, f_{k+i}]A_k a,b\big\rangle_{\mathcal{E}}\right| = \\
&= \left|\sum_{i=0}^{T-1} \left\langle \Gamma\Big[\sum_{p>n} B_p f_{p+i}, \sum_{k\leq n} A_k f_{k+i}\Big]a,b\right\rangle_{\mathcal{E}}\right| \leq \\
&\leq \sum_{i=0}^{T-1}\rho_i(n)\left\|\sum_{k\leq n} A_k f_{k+i} a\right\|\left\|\sum_{p>n} B_p f_{p+i} b\right\| \leq \\
&\leq \rho(n)\sum_{i=0}^{T-1}\left\|\sum_{k\leq n} A_k f_{k+i} a\right\|\left\|\sum_{p>n} B_p f_{p+i} b\right\| \leq \\
\leq \rho(n)\Big(\sum_{i=0}^{T-1}\left\|\sum_{k\leq n} A_k f_{k+i} a\right\|^2\Big)^{1/2}&\Big(\sum_{i=0}^{T-1}\left\|\sum_{p>n} B_p f_{p+i} b\right\|^2\Big)^{1/2} = \\
= \rho\left\|\sum_{k\leq n} A_k W_{X_k} a\right\|&\left\|\sum_{p>n} B_p W_{X_p} b\right\| = \rho\,\|W_X a\|\,\|W_Y b\|,
\end{aligned}$$

where $\rho(n)$ is the maximum of $\rho_i(n)$; $i = 0, 1, \ldots, T-1$, and we used the embedding $X \to W_X$ of $\mathcal{H}^T$ into $\mathcal{L}(\mathcal{E}, \mathcal{K}^T)$ and the fact that $\rho(n) = \rho$ for stationary Γ_1-correlated proces (X_n). Therefore $|\langle \Gamma_1[X,Y]a,b\rangle_{\mathcal{E}}| \leq \rho\,\|W_X a\|\,\|W_Y b\|$ for each $X \in H_n^X$, $Y \in \tilde{H}_n^X$, and the angle between the past and the future of the stationary Γ_1-correlated dilation (X_n) is positive. □

A measure of the positive angle between the past and future is given by the operator $B \in \mathcal{L}(\mathcal{K})$ defined by [7]

$$B = P^- P^+ P^-, \tag{3.4}$$

where P^- is the projection on the past and P^+ is the projection on the future of a given process. More or less explicitly, in various situations this operator was used [11, 8, 2, 16, 19, 15, 4].

Another angle between two subspaces M_1 and M_2 of a Hilbert space $\mathcal{K}$ is the *Friedrichs angle* [6] defined to be the angle in $[0, \pi/2]$ whose cosine is given by

$$c(M_1, M_2) := \sup\{|\langle k_1, k_2\rangle| \,;\, k_i \in M_i \cap M^\perp \cap B_{\mathcal{K}}, \}, \tag{3.5}$$

where $M = M_1 \cap M_2$ and $B_{\mathcal{K}}$ is the unit ball of $\mathcal{K}$.

By (3.1) and (3.5) it follows that $c(M_1, M_2) \le \rho(M_1, M_2)$. Obviously we have $c(M_1, M_2) = \rho(M_1 \cap M^\perp, M_2 \cap M^\perp)$, and of course $c(M_1, M_2) = c(M_1^\perp, M_2^\perp)$. Various properties of the angles between subspaces in a Hilbert space can be found in [4]. Here some properties of the Friedrichs angle and the generalized Friedrichs angle [3] are used in the case of processes in a complete correlated action $\{\mathcal{E}, \mathcal{H}, \Gamma\}$.

If we take $(X_n) \subset \mathcal{H}^T$ the stationary Γ_1-correlated dilation of a periodically Γ-correlated process $(f_n) \subset \mathcal{H}$, then the Friedrichs angle between the past and the future of (X_n) is given by

$$c(K_n^X, \tilde{K}_n^X) = \sup\{|\langle X, Y\rangle| \,;\, X \in K_n^X \cap M^\perp \cap B_1, Y \in \tilde{K}_n^X \cap M^\perp \cap B_1\},$$

where $M = K_n^X \cap \tilde{K}_n^X$, B_1 is the unit ball in $\mathcal{K}^T$, and K_n^X and $\tilde{K}_n^X$ are the images of the past, respectively of the future from $\mathcal{K}^T$ by the embedding $X \to W_X$ of $\mathcal{H}^T$ into $\mathcal{L}(\mathcal{E}, \mathcal{K}^T)$

$$K_n^X = \bigvee_{k \le n} W_{X_k} \mathcal{E}, \qquad \tilde{K}_n^X = \bigvee_{j > n} W_{X_j} \mathcal{E}. \tag{3.6}$$

Even the angle between the past and the future of the stationary process $(X_n) \subset \mathcal{H}^T$ is constant, the angles between various pasts of the components of $X_n = (f_n, f_{n+1}, \ldots, f_{n+T-1})$ are variable and can be characterized by the generalized Friedrichs angle between several subspaces [3]. To do this, let us first remember the following characterization of the Friedrichs angle for two subspaces [3].

Proposition 3.4. *If M_1 and M_2 are closed subspaces of $\mathcal{K}$, then the angle between M_1 and M_2 is given by*

$$\rho(M_1, M_2) = \sup\left\{\frac{2Re\,\langle m_1, m_2\rangle}{\|m_1\|^2 + \|m_2\|^2} \;;\; m_j \in M_j, (m_1, m_2) \ne (0,0)\right\}$$

and the Friedrichs angle is

$$c(M_1, M_2) = \sup\left\{\frac{2Re\,\langle m_1, m_2\rangle}{\|m_1\|^2 + \|m_2\|^2} \;;\; m_j \in M_j \cap M^\perp, (m_1, m_2) \ne (0,0)\right\}.$$

Then the Friedrichs angle to several subspaces $(M_1, M_2, \ldots, M_T)$ is defined [3] by

$$c(M_1, \ldots, M_T) = \sup\left\{\frac{2}{T-1}\frac{\sum_{j<k} Re\,\langle m_j, m_k\rangle}{\sum_{i=1}^T \|m_i\|^2}\right\} \tag{3.7}$$

for $m_j \in M_j \cap M^\perp$, $\sum_{i=1}^{T} \|m_i\|^2 \neq 0$.

In the case of a periodically Γ-correlated process (f_n), since $M = \bigcap_{i=0}^{T-1} \mathcal{K}_{n+i} = \mathcal{K}_n^f$, we have the Friedrichs angle associated to $(\mathcal{K}_n^f, \mathcal{K}_{n+1}^f, \dots, \mathcal{K}_{n+T-1}^f)$, corresponding to $X_n = (f_n, f_{n+1}, \dots, f_{n+T-1})$, defined by its cosinus (or *Friedrichs number*):

$$c(\mathcal{K}_n^f, \mathcal{K}_{n+1}^f, \dots, \mathcal{K}_{n+T-1}^f) = \sup \left\{ \frac{2}{T-1} \frac{\sum_{j<p} Re \langle k_j, k_p \rangle}{\sum_{i=0}^{T-1} \|k_i\|^2} \right\} \tag{3.8}$$

for $k_i \in \mathcal{K}_{n+i}^f \cap (\mathcal{K}_n^f)^\perp$, $\sum_{i=0}^{T-1} \|k_i\|^2 \neq 0$.

Analoguously, generalizing the angle ρ between two subspaces to T subspaces, a so called *Dixmier number* is obtained

$$\rho(\mathcal{K}_n^f, \mathcal{K}_{n+1}^f, \dots, \mathcal{K}_{n+T-1}^f) = \sup \left\{ \frac{2}{T-1} \frac{\sum_{j<p} Re \langle k_j, k_p \rangle}{\sum_{i=0}^{T-1} \|k_i\|^2} \right\}, \tag{3.9}$$

for $k_i \in \mathcal{K}_{n+i}^f$, $\sum_{i=0}^{T-1} \|k_i\|^2 \neq 0$.

Other definitions [3] of apparently geometric concepts which can help in the study of the geometry of some nonstationary processes are the following.

The *configurant constant*:

$$\kappa(\mathcal{K}_n^f, \mathcal{K}_{n+1}^f, \dots, \mathcal{K}_{n+T-1}^f) = \sup \left\{ \frac{1}{T} \frac{\left\| \sum_{j=0}^{T-1} k_j \right\|^2}{\sum_{i=0}^{T-1} \|k_i\|^2} \right\}, \tag{3.10}$$

for $k_i \in \mathcal{K}_{n+i}^f \cap (\mathcal{K}_n^f)^\perp$, $\sum_{i=0}^{T-1} \|k_i\|^2 \neq 0$.

The *non-reduced configurant constant*:

$$\kappa_0(\mathcal{K}_n^f, \mathcal{K}_{n+1}^f, \dots, \mathcal{K}_{n+T-1}^f) = \sup \left\{ \frac{1}{T} \frac{\left\| \sum_{j=0}^{T-1} k_j \right\|^2}{\sum_{i=0}^{T-1} \|k_i\|^2} \right\}, \tag{3.11}$$

for $k_i \in \mathcal{K}_{n+i}^f$, $\sum_{i=0}^{T-1} \|k_i\|^2 \neq 0$.

The *inclination* of $\mathcal{K}_n^f, \mathcal{K}_{n+1}^f, \dots, \mathcal{K}_{n+T-1}^f$:

$$l(\mathcal{K}_n^f, \mathcal{K}_{n+1}^f, \dots, \mathcal{K}_{n+T-1}^f) = \inf \left\{ \frac{\max_{0 \leq j \leq T-1} \operatorname{dist}(k, \mathcal{K}_{n+j}^f)}{\operatorname{dist}(k, \mathcal{K}_n^f)} \right\}, \tag{3.12}$$

for $k \neq \mathcal{K}_n^f$.

Proposition 3.5. *For a periodically Γ-correlated process $(f_n)_{n \in \mathbb{Z}}$ from $\mathcal{H}$, the configuration constant κ of the past spaces associated to its stationary Γ_1-correlated dilation (X_n) from $\mathcal{H}^T$ is given by*

$$\kappa(\mathcal{K}_n^f, \mathcal{K}_{n+1}^f, \dots, \mathcal{K}_{n+T-1}^f) = \sup \left\{ \frac{1}{T} \|G(k_0, \dots, k_{T-1})\| \right\}, \tag{3.13}$$

for $k_j \in \mathcal{K}_{n+j} \cap (\mathcal{K}_n^f)^\perp$, $\|k_j\| = 1$, $j = 0, 1, \dots, T-1$, where the matrix G is given by

$$G(k_0, \dots, k_{T-1}) = \Big(\langle k_i, k_j \rangle_{\mathcal{K}} \Big)_{i,j=0}^{T-1}$$

and

$$\langle k_i, k_j \rangle_{\mathcal{K}} = \langle \Gamma_T[Y_i, Y_j]\boldsymbol{b}, \boldsymbol{c} \rangle_{\mathcal{E}^T}, \tag{3.14}$$

$Y_i = (0, f_{n+1}, \ldots, f_{n+i}, 0, \ldots, 0) \subset \mathcal{H}^T$, *while* $\boldsymbol{b}$ *and* $\boldsymbol{c}$ *are vectors from* $\mathcal{E}^T$.

Proof. The characterization (3.13) of the configurant constant κ it follows by Proposition3.4 from [3], taking into account that $\bigcap_i \mathcal{K}^f_{n+i} = \mathcal{K}^f_n$. To prove (3.14), let us consider the generators k_j from $\mathcal{K}_{n+j} \cap (\mathcal{K}^f_n)^\perp$, $k_j = \sum_{k=0}^{j} \sum_r A_r V_{f_{n+k}} a_r$, $j = 0, 1, \ldots, T-1$, where $A_r \in \mathcal{L}(\mathcal{E})$, $a_r \in \mathcal{E}$, and the sums $\sum_r$ have finite non-zero terms. Then, taking into account the action of $\mathcal{L}(\mathcal{E})$ on $\mathcal{H}$ and the definition (2.11) of the Γ_T-correlation on $\mathcal{H}^T$, we have

$$\langle k_i, k_j \rangle_{\mathcal{K}} = \left\langle \sum_{k=0}^{i} \sum_r A_r V_{f_{n+k}} a_r, \sum_{p=0}^{j} \sum_s A_s V_{f_{n+p}} a_s \right\rangle =$$

$$= \left\langle \sum_{k=0}^{i} \sum_r V_{f_{n+k}} A_r a_r, \sum_{p=0}^{j} \sum_s V_{f_{n+p}} A_s a_s \right\rangle =$$

$$= \sum_{k=0}^{i} \sum_{p=0}^{j} \left\langle V^*_{f_{n+p}} V_{f_{n+k}} \sum_r A_r a_r, \sum_s A_s a_s \right\rangle =$$

$$= \sum_{k=0}^{i} \sum_{p=0}^{j} \langle \Gamma[f_{n+p}, f_{n+k}] b_k, c_p \rangle =$$

$$= \sum_{k=0}^{T} \sum_{p=0}^{T} \langle \Gamma[f_{n+p}, f_{n+k}] b_k, c_p \rangle = \langle \Gamma_T[Y_p, Y_k]\mathbf{b}, \mathbf{c} \rangle_{\mathcal{E}^T}$$

□

Considering $\mathbf{C}$ the cartesian product of $\mathcal{K}^f_n, \mathcal{K}^f_{n+1}, \ldots, \mathcal{K}^f_{n+T-1}$ and $\mathbf{D}$ the diagonal of $\mathcal{K}^T$, $\mathbf{D} = \{(k, \ldots, k); k \in \mathcal{K}\}$, in a similar way as in [3] can be proved the following characterization of the configurant constant and of the inclination of the past subspaces $\mathcal{K}^f_n, \mathcal{K}^f_{n+1}, \ldots, \mathcal{K}^f_{n+T-1}$ associated to the Γ_1-correlated dilation (X_n) of a periodically Γ-correlated process (f_n).

Proposition 3.6. *For* $T \geq 2$ *we have*

(i) $\rho(\mathcal{K}^f_n, \mathcal{K}^f_{n+1}, \ldots, \mathcal{K}^f_{n+T-1}) = \rho(\boldsymbol{C}, \boldsymbol{D})^2$,

(ii) $c(\mathcal{K}^f_n, \mathcal{K}^f_{n+1}, \ldots, \mathcal{K}^f_{n+T-1}) = c(\boldsymbol{C}, \boldsymbol{D})^2$,

(iii) $1 - l(\mathcal{K}^f_n, \ldots, \mathcal{K}^f_{n+T-1}) \leq c(\boldsymbol{C}, \boldsymbol{D}) \leq 1 - \frac{1}{2T} l(\mathcal{K}^f_n, \ldots, \mathcal{K}^f_{n+T-1})^2$.

As a remark, the inclination of the sequence of attached pasts subspaces $\mathcal{K}^f_n, \mathcal{K}^f_{n+1}, \ldots, \mathcal{K}^f_{n+T-1}$ is zero if and only if its Friedrichs angle is 1.

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