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THE CHARACTERISTIC SYSTEM METHOD FOR LINEAR HIGHER-ORDER SPDES OF PARABOLIC TYPE

I. MOLNAR AND C. VARSAN

Linear higher-order PDEs and SPDEs and of parabolic type are solved relying on the corresponding solution of the characteristic system which involves a fundamental system of solutions in the kernel of a Kolmogorov parabolic equation.

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Key words: parabolic higher-order PDEs and SPDEs; characteristic system method.

1. Introduction

The analysis here is subjected to those higher-order PDEs and SPDEs which involve a parabolic operator $U(\varphi)(t,x) = \partial_t\varphi(t,x) - L(\varphi)(t,x)$ and its iterations U^k, $0 \leqslant k \leqslant m-1$. Here the linear second operator

$$L(\varphi)(t,x) = \langle \partial_x\varphi(t,x), X_0(x)\rangle + \frac{1}{2}\sum_{j=1}^{d}\langle \partial_x^2\varphi(t,x)X_j(x), X_j(x)\rangle$$

is generated by a finite set of smooth vector fields $\{X_0, X_1, \ldots, X_d\} \subset (\mathcal{C}_b^1 \cap \mathcal{C}^2)(\mathbb{R}^n;\mathbb{R}^n))$. A Cauchy problem solution is a composition of a smooth $\lambda = \psi \in \ker U$, $\psi \in \mathcal{C}^{1,2}((0,T] \times \mathbb{R}^n;\mathbb{R}^n)$ and the unique solution satisfying a linear system of ODEs or SDEs $\{y(t;\lambda) \in \mathbb{R}^m, t \in [0,T], \lambda \in \mathbb{R}^n\}$. The main results (see Theorem in section 3) is focused on the linear systems of parabolic SPDEs which lead us to an integral representation of the solution satisfying a higher-order SPDEs. In section 2 are analyzed higher-order PDEs of parabolic type. The first cited work (see [1], Th. 6.1, p.124) allow us to use a general parabolic operator as it is contained in the Kolmogorov equation. The characteristic system method involved here is dealing with linear systems for which the procedure used in [2] and [3] is not applicable.

2. Preliminaries

Consider a finite set of smooth complete vector fields $\{X_0, X_1, \dots, X_d\} \subset (\mathcal{C}_b^1 \cap \mathcal{C}^2)(\mathbb{R}^n; \mathbb{R}^n))$. Associate a linear second order operator $L : \mathcal{C}^2(\mathbb{R}^n; \mathbb{R}) \to \mathcal{C}(\mathbb{R}^n; \mathbb{R})$ where

$$L(\varphi)(x) = \langle \partial_x \varphi(x), X_0(x) \rangle + \frac{1}{2} \sum_{k=1}^{d} \langle \partial_x^2 \varphi(x) X_k(x), X_k(x) \rangle, \; x \in \mathbb{R}^n$$

and define $U : \mathcal{C}^{1,2}([0,T] \times \mathbb{R}^n; \mathbb{R}) \to \mathcal{C}([0,T] \times \mathbb{R}^n; \mathbb{R})$ by

$$U(\varphi)(t,x) = \partial_t \varphi(t,x) - L(\varphi(t,\cdot))(x), \; t \in [0,T], \; x \in \mathbb{R}^n.$$

The space $\mathcal{C}_b^1(\mathbb{R}^n; \mathbb{R}^n)$ consists of all first order continuously differentiable functions $h : \mathbb{R}^n \to \mathbb{R}^n$ for which the gradient $\partial_x h_i(x)$, $x \in \mathbb{R}^n$, is bounded for any $1 \leqslant i \leqslant n$ (see $h = (h_1, \dots, h_n)$). A simple m-th order deterministic parabolic equation has the following expression

$$U^m(\varphi)(t,x) = \sum_{k=0}^{m-1} a_k(t) U^k(\varphi)(t,x) + f(t), \; (t,x) \in (0,T] \times \mathbb{R}^n, \tag{1}$$

where $f, a_k \in \mathcal{C}_b(\mathbb{R}; \mathbb{R})$ and $U^k(\varphi)$ is the corresponding k-iteration of a linear operator, for $k = 0, 1, \dots, m-1$.

Using standard notations, $\varphi = y_0$, $U^k(\varphi) = y_k$, $0 \leqslant k \leqslant m-1$, rewrite (1) as a system of parabolic equations

$$\begin{cases} U(y_0)(t,x) = y_1(t,x), \dots, U(y_{m-2})(t,x) = y_{m-1}(t,x) \\ U(y_{m-1})(t,x) = \sum_{k=0}^{m-1} a_k(t) y_k(t,x) + f(t), \; (t,x) \in (0,T] \times \mathbb{R}^n. \end{cases} \tag{2}$$

A classical solution of the system (2) means a function $y \in \mathcal{C}^{1,2}((0,T] \times \mathbb{R}^n; \mathbb{R}^m)$, $y(t,x) = (y_0(t,x), \dots, y_{m-1}(t,x)) \in \mathbb{R}^m$, $t \in [0,T]$, $x \in \mathbb{R}^n$, which is first order continuously differentiable of $t \in [0,T]$ and second order continuously differentiable of $x \in \mathbb{R}^n$ such that the system (2) is satisfied for any $t \in (0,T]$ and $x \in \mathbb{R}^n$. The first component $\varphi(t,x) = y_0(t,x)$ of a solution satisfying (2) stands for a classical solution of the higher-order parabolic equation (1). Each Cauchy problem associated with (2) (see $y(0,x) = y^0(x)$, $y^0 \in \mathcal{C}^1(\mathbb{R}^n; \mathbb{R}^n)$) has a solution which can be constructed using so called the characteristic system method. More precisely, using the vectorial notation $y(t,x) = (y_0(t,x),\ldots,y_{m-1}(t,x)) \in \mathbb{R}^m$, rewrite (2) as a linear system of parabolic equations

(3)

$$\begin{cases} U(y)(t,x) = A_0 y(t,x) + \sum_{i=0}^{m-1} a_i(t) B_i y(t,x) + b_m f(t),\ t \in (0,T], x \in \mathbb{R}^n, \\ y_0(0,x) = y^0(x),\ y^0 \in \mathcal{C}^1(\mathbb{R}^n;\mathbb{R}^n), b_m = \mathrm{col}(0,0,\ldots,1) \in \mathbb{R}^m, \end{cases}$$

where the $(m \times m)$ matrices $A_0, B_i,\ 0 \leqslant i \leqslant m-1$, satisfy the following properties

$$(4)\quad \begin{cases} A_0 = [\theta, e_0, \ldots, e_{m-2}],\ B_i = [\theta,\ldots,\theta,e_i,\theta,\ldots,\theta], \\ [B_i,B_j] := B_j B_i - B_i B_j = \Theta \text{ (null matrix)},\ i,j \in \{0,1,\ldots,m-1\}. \end{cases}$$

Here $\theta \in \mathbb{R}^m$ is the origin, $\{e_0, e_1, \ldots, e_{m-1}\} \subset \mathbb{R}^m$ is the canonical basis and notice that $e_{m-1} = b_m$. A solution of (3) can be found as a composition of two smooth mappings $y(t,x) = \widehat{y}(t;\psi(t,x)),\ t \in [0,T],\ x \in \mathbb{R}^n$, where $\{\widehat{y}(t,\lambda) \in \mathbb{R}^m, t \in [0,T], \lambda \in \mathbb{R}^n\}$ and $\{\lambda = \psi(t,x), t \in [0,T], x \in \mathbb{R}^n\}$ satisfy the following characteristic system

(5)

$$\frac{d\widehat{y}}{dt}(t;\lambda) = A_0 \widehat{y}(t;\lambda) + \sum_{i=0}^{m-1} a_i(t) B_i \widehat{y}(t;\lambda) + b_m f(t),\ \widehat{y}(0;\lambda) = y^0(\lambda), t \in [0,T],$$

$$(6)\quad \partial_t \psi(t,x) = L(\psi(t,\cdot))(x),\ t \in (0,T), x \in \mathbb{R}^n \text{ (see } \psi \in \ker U),\ \psi(0,x) = x.$$

Definition 1. We say that a solution $y(t,x) \in \mathbb{R}^n$, $t \in [0,T]$, $x \in \mathbb{R}^n$, of the linear parabolic system (3) is constructed by characteristic system method if it can be represented by $y(t,x) = \widehat{y}(t;\psi(t,x))$, $t \in [0,T]$, $x \in \mathbb{R}^n$, where the smooth vector function $\{\widehat{y}(t;\lambda), t \in [0,T], \lambda \in \mathbb{R}^n\}$ and $\{\lambda = \psi(t,x) \in \mathbb{R}^n, t \in [0,T], x \in \mathbb{R}^n\}$, $\psi \in \mathcal{C}^{1,2}((0,T] \times \mathbb{R}^n;\mathbb{R}^n)$, satisfy the characteristic system (5).

This goal will be achieved under the following hypotheses:

(a) the Cauchy condition $y^0 \in \mathcal{C}^1(\mathbb{R}^n;\mathbb{R}^n)$ is linear, i.e. $y^0(x) = Cx + v$ for some $(n \times m)$ matrix C and $v \in \mathbb{R}^m$,

(b) $X_k \in (\mathcal{C}_b^1 \cap \mathcal{C}^2)(\mathbb{R}^n;\mathbb{R}^n)$, $0 \leqslant k \leqslant d$, satisfy a polynomial growth condition $|\partial^2_{x_i x_j} X_k(x)| \leqslant K(1+|x|^N)$, $x \in \mathbb{R}^n$, $i,j \in \{1,\dots,n\}$.

Remark 1. The first component $\{\widehat{y}(t;\lambda), t \in [0,T], \lambda \in \mathbb{R}^n\}$ satisfying (5) can be represented as follows

(7)

$$\widehat{y}(t;\lambda) = [M(\widehat{h}(t))]\widehat{v}(t;\lambda),\ t \in [0,T], \lambda \in \mathbb{R}^n, \widehat{h}_i(t) = \int_0^t a_i(s)ds, 0 \leqslant i \leqslant m-1,$$

where the $(m \times m)$ nonsingular matrix $M(p)$, $p \in \mathbb{R}^m$ is given by

$$M(p) = [\exp t_0 B_0] \dots [\exp t_{m-1} B_{m-1}],\ p = (t_0, \dots, t_{m-1} \in \mathbb{R}^m) \tag{8}$$

and $\widehat{v}(t;\lambda) \in \mathbb{R}^m$ satisfies the linear system of ODEs

$$\begin{cases} \dfrac{d\widehat{v}}{dt}(t;\lambda) = A(t)\widehat{v}(t;\lambda) + [M(-\widehat{h}(t))]b_m f(t),\ t \in [0,T], \lambda \in \mathbb{R}^n, \\ \widehat{v}(0;\lambda) = y^0(\lambda),\ A(t) := [M(-\widehat{h}(t))]A_0[M(\widehat{h}(t))]. \end{cases}$$

A direct implication of Remark 1 is

Lemma 1. *Consider the* $(m \times m)$ *matrices* A_0, B_i, $0 \leqslant i \leqslant m-1$, *such that* (4) *are satisfied. Then the smooth vector function* $\widehat{y}(t;\lambda) = [M(\widehat{y}(t))]\widehat{v}(t;\lambda) \in \mathbb{R}^m$ *defined in* (7) *is the unique solution of the characteristic system* (5)*. In addition, assuming the hypothesis* (a)*, the unique solution of* (5) *is a linear mapping of* $\lambda \in \mathbb{R}^n$.

Remark 2. A smooth solution $\psi \in \ker U$ satisfying the characteristic system (6) can be obtained via a fundamental system of solutions satisfying the corresponding backward Kolmogorov equation (see [1], Th.6.1, p.124). Under the hypothesis (b) there exist n solutions $\{\varphi_i(t,x), t \in (0,T], x \in \mathbb{R}^n\}$, $0 \leqslant i \leqslant n$, satisfying a backward parabolic equation

$$\begin{cases} \partial_t \varphi_i(t,x) + L(\varphi_i(t,\cdot))(x) = 0,\ t \in [0,T], x \in \mathbb{R}^n, \\ \varphi_i(T,x) = x_i,\ 1 \leqslant i \leqslant n. \end{cases} \tag{9}$$

Denote $\psi(t,x) = \varphi(T-t,x)$, $1 \leqslant i \leqslant n$, $\psi(t,x) = (\psi_1(t,x),\dots,\psi_n(t,x))$ and using (9) we get $\psi \in \ker U$, $\psi(0,x) = x$, satisfying the characteristic system (6). The content of Remark 2 will be restated as

Lemma 2. *Assume that* $\{X_0, X_1, \ldots, X_d\} \subset (\mathcal{C}_b^1 \cap \mathcal{C}^2)(\mathbb{R}^n; \mathbb{R}^n)$ *satisfy the polynomial growth condition in the hypothesis (b). Then the conditions of Theorem 6.1 in [1] are satisfied and there exists a smooth solution* $\psi \in \mathcal{C}^{1,2}((0,T] \times \mathbb{R}^n); \mathbb{R}^n$ *verifying forward parabolic equations* (6).

Remark 3. Under the hypotheses (a) and (b), consider the smooth mappings $\{\widehat{y}(t;\lambda) \in \mathbb{R}^n, t \in [0,T], \lambda \in \mathbb{R}^n\}$ defined in Lemma 1 and $\{\lambda = \psi(t,x) \in \mathbb{R}^n, t \in [0,T], x \in \mathbb{R}^n\}$ constructed in Lemma 2. Define a smooth mapping

$$y(t,x) = \widehat{y}(t;\psi(t,x)),\ t \in [0,T], x \in \mathbb{R}^n.$$

By a direct computation, we get that $\{y(t,x) \in \mathbb{R}^m, t \in [0,T], x \in \mathbb{R}^n\}$ verifies the linear system of parabolic equations (3), where the linear initial condition $y^0(x) = Cx + v$ is used. More precisely, by definition, $y(0,x) = \widehat{y}(0;x) = Cx + v = y^0(x)$, $x \in \mathbb{R}^n$ and partial derivative $\partial_t y(t,x)$ satisfies

$$(10)\qquad \begin{aligned} \partial_t y(t,x) &= \frac{d\widehat{y}}{dt}(t;\widehat{\lambda}) + \partial_\lambda \widehat{y}(t;\widehat{\lambda})\partial_t \psi(t,x) = A_0 y(t,x) + \\ &+ \sum_{i=0}^{m-1} a_i(t) B_i y(t,x) + b_m f(t) + \partial_\lambda \widehat{y}(t;\widehat{\lambda}) L(\psi(t,\cdot))(x), \end{aligned}$$

where $\widehat{\lambda} = \psi(t,x)$. On the other hand, $\widehat{y}(t;\lambda)$ is a linear mapping of $\lambda \in \mathbb{R}^n$ and the last term in (10) coincides with $L(y(t,\cdot))(x)$ which allow us to rewrite (10) as the linear system (3), i.e.

$$\begin{cases} U(y)(t,x) := \partial_t y(t,x) - L(y(t,\cdot))(x) \\ \qquad\qquad = A_0 y(t,x) + \displaystyle\sum_{i=0}^{m-1} a_i(t) B_i y(t,x) + b_m f(t), \\ y(0,x) = Cx + v,\ t \in (0,T], x \in \mathbb{R}^n. \end{cases}$$

We restate Remark 3 as

Lemma 3. *Assume that* $y^0 \in \mathcal{C}^1(\mathbb{R}^n; \mathbb{R}^n)$ *and* $\{X_0, X_1, \ldots, X_d\} \subset (\mathcal{C}_b^1 \cap \mathcal{C}^2)(\mathbb{R}^n; \mathbb{R}^n)$ *are given such that the hypotheses (a) and (b) are verified. Then there exists a smooth solution* $\{y(t,x) \in \mathbb{R}^m, t \in [0,T], x \in \mathbb{R}^n\}$ *of the linear* *parabolic system* (3) *constructed by the characteristic system method. In addition, the first component* $\varphi = y_0$, *of* $y(t,x) = (y_0(t,x),\dots,y_{m-1}(t,x))$ *verifies the liniar higher-order deterministic parabolic equation* (1).

Remark 4. One may wonder about the stochastic partial differential equation (SPDE) satisfied by a solution $y(t,x) = \widehat{y}(t;\psi(t,x)) \in \mathbb{R}^m$, $t \in [0,T]$, $x \in \mathbb{R}^n$, constructed by the characteristic system method (see(5) and (6)) when $\{\widehat{y}(t;\lambda) \in \mathbb{R}^m, t \in [0,T], \lambda \in \mathbb{R}^n\}$ of a linear system of stochastic differential equations

$$(11)\qquad \begin{cases} d_t\widehat{y}(t;\lambda) = A_0(t)\widehat{y}(t;\lambda)dt + \sum\limits_{i=0}^{m-1} B_i\widehat{y}(t;\lambda)a_i(t)\circ dw_i(t) + b(t)dt, \\ \widehat{y}(0;\lambda) = C\lambda + v,\ \lambda \in \mathbb{R}^n, t \in [0,T], \end{cases}$$

replacing the linear system of ODEs in (5). Here the continuous matrix function $A_0(t) : [0,T] \to M_{m,m}$ and the constant $(m \times m)$-matrices B_i, $0 \leqslant i \leqslant m-1$, do not satisfy the special properties given in (4), and $b(t) : [0,T] \to \mathbb{R}^m$ is a continuous function. In addition, $\{w(t) = (w_0(t),\dots,w_{m-1}(t)) \in \mathbb{R}^m : t \in [0,T]\}$ is a standard m-dimensional Wiener process over the complete filtered probability space $\{\Omega, \mathcal{F} \supset \{\mathcal{F}_t\}?, P\}$ and a Stratonovich integral "$\circ$" is computed by

$$h_i(t,y)\circ dw_i(t) = h_i(t,y)\cdot dw_i(t) + \frac{1}{2}\partial_y h_i(t,y)h_i(t,y)dt$$

using Ito integral "$\cdot$", $0 \leqslant i \leqslant m-1$. The unique solution of SDEs (11) has a simple integral representation similar to $\widehat{y}(t;\lambda) = [M(\widehat{h}(t))]\widehat{v}(t;\lambda)$, $t \in [0,T]$, $\lambda \in \mathbb{R}^n$, used in Lemma 1, provided we assume

(c) the $(m \times m)$ constant matrices B_i, $0 \leqslant i \leqslant m-1$, mutually commute, i.e. $[B_i, B_j] := B_jB_i - B_iB_j = \Theta$ (null matrix) for any $i,j \in \{0,1,\dots,m-1\}$.

Denote $\widehat{w}(t) = (\widehat{w}_0(t),\dots,\widehat{w}_{m-1}(t))$, $t \in [0,T]$, $\widehat{w}_i(t) := \int\limits_0^t a_i(s)\cdot dw_i(s)$, $0 \leqslant i \leqslant m-1$. Under the hypothesis (c), by a direct application of the standard rule of stochastic derivation, we get that the unique solution of SDEs (11) is represented by

$$\widehat{y}(t;\lambda) = [M(\widehat{w}(t))]\widehat{v}(t;\lambda),\ t \in [0,T], \lambda \in \mathbb{R}^n,$$

where the nonsingular matrix $M(p)$, $p \in \mathbb{R}^m$, $p = (t_0, \ldots, t_{m-1})$ is defined in Lemma 1 (see (8)) and $\widehat{v}(t;\lambda) \in \mathbb{R}^m$ satisfies a linear system of ODEs with random parameter

$$(12)\qquad \begin{cases} \dfrac{d\widehat{v}(t;\lambda)}{dt} = A(t)\widehat{v}(t;\lambda) + [M(-\widehat{w}(t))]b(t),\ t \in [0,T], \lambda \in \mathbb{R}^n, \\ \widehat{v}(0;\lambda) = y^0(\lambda) = C\lambda + v. \end{cases}$$

Here the continuous random matrix $A(t)$ is defined by

$$A(t) := [M(-\widehat{w}(t))]A_0(t)[M(\widehat{w}(t))],\ t \in [0,T].$$

We restate Remark 4 as

Lemma 4. *Consider the linear system of SDEs in* (11) *where the* $(m\times m)$ *matrices* $\{B_0, \ldots, B_{m-1}\}$ *mutually commute (see (c)). Then the unique solution of* (11) *can be represented by*

$$(13)\qquad \widehat{y}(t;\lambda) = [M(\widehat{w}(t))]\widehat{v}(t;\lambda),\ t \in [0,T], \lambda \in \mathbb{R}^n,$$

where the nonsingular $(m \times m)$ *matrix* $M(p)$ *is given by*

$$M(p) := [\exp t_0 B_0] \ldots [\exp t_{m-1} B_{m-1}],\ p = (t_0, \ldots, t_{m-1}) \in \mathbb{R}^m$$

(see (8)*) and the smooth vector* $\widehat{v}(t;\lambda) \in \mathbb{R}^m$ *satisfies the linear ODEs* (12). *In addition,* $\widehat{y}(t;\lambda)$ *is a linear mapping with respect to* $\lambda \in \mathbb{R}^n$.

Problem A. One may wonder about the linear SPDEs of parabolic type satisfied by the continuous process

$$y(t,x) := \widehat{y}(t;\psi(t,x)),\ t \in [0,T], x \in \mathbb{R}^n,$$

where $\widehat{y}(t;\lambda) \in \mathbb{R}^m$ is represented in Lemma 4 (see (13)) and $\lambda = \psi(t,x) \in \mathbb{R}^n$, $\psi \in \mathcal{C}^{1,2}((0,T] \times \mathbb{R}^n;\mathbb{R}^n)$, satisfies forward parabolic equations (6) (see

Lemma 2). Prove that, under the hypotheses of Lemmas 2 and (4), the following linear system of parabolic SPDEs is satisfied,

$$\begin{cases} d_t y(t,x) = L(y(t,\cdot))(x)dt + A_0(t)y(t,x)dt + \\ \qquad\qquad + \sum_{i=0}^{m-1} B_i y(t,x) a_i(t) \circ dw_i(t) + b(t), \\ y(0,x) = y^0(x) = Cx + v,\ t \in (0,T], x \in \mathbb{R}^n. \end{cases}$$

In particular, when the $(m \times m)$-matrices A_0, B_i, $0 \leqslant i \leqslant m-1$, satisfy the special conditions in (4) and $b(t) = b_m f(t)$ (see (3)), then the first component $y_0 = \varphi$ of $y(t,x) = (y_0(t,x), \ldots, y_{m-1}(t,x))$ satisfies a linear higher-order SPDE of parabolic type

$$\begin{cases} d_t y_0(t,x) = L(y_0(t,\cdot))(x)dt + y_1(t,x)dt, \\ \cdots\cdots\cdots \\ d_t y_{m-2}(t,x) = L(y_{m-2}(t,\cdot))(x)dt + y_{m-1}(t,x)dt, \\ d_t y_{m-1}(t,x) = L(y_{m-1}(t,\cdot))(x)dt + \sum_{i=0}^{m-1} a_i(t) y_i(t,x) \circ dw_i(t) \\ \qquad\qquad + f(t)dt,\ t \in [0,T], x \in \mathbb{R}^n, \end{cases}$$

where the Stratonovich integral "$\circ$" is computed by

$$h_i(t,y) \circ dw_i(t) = h_i(t,y) \cdot dw_i(t) + \frac{1}{2}\partial_y h_i(t,y) h_i(t,y) dt$$

using Ito integral "$\cdot$".

3. Main result (solution for Problem A)

Under the same notations as before, we recall that a finite set of smooth vector fields are given such that

(d) $\{X_0, \ldots, X_d\} \subset (\mathcal{C}_b^1 \cap \mathcal{C}^2)(\mathbb{R}^n; \mathbb{R}^n)$ satisfy a polynomial growth condition $|\partial_{x_i x_j} X_k(x)| \leqslant K(1 + |x|^N)$, $x \in \mathbb{R}^n, i,j \in \{1, \ldots, n\}, k \in \{0, \ldots, d\}$.

Define a linear second order operator

$$L(\varphi(t,\cdot))(x) = \langle \partial_x \varphi(t,x), X_0(x) \rangle + \frac{1}{2} \sum_{j=1}^{d} \langle \partial_x^2 \varphi(t,x) X_j(x), X_j(x) \rangle,\ x \in \mathbb{R}^n$$

for $\varphi \in \mathcal{C}^{1,2}([0,T] \times \mathbb{R}^n, \mathbb{R})$. A standard m-dimensional Wiener process $w(t) = (w_0(t), \ldots, w_{m-1}(t)) \in \mathbb{R}^m$, $t \in [0,T]$, over the complete filtered probability space $\{\Omega, \mathcal{F} \supset \{\mathcal{F}_t\}?, P\}$ is given and for some continuous scalars functions $a_i \in \mathcal{C}([0,T]; \mathbb{R})$, $0 \leqslant i \leqslant m-1$, define $\widehat{w}_i(t) = \int\limits_0^t a_i(s) dw_i(s)$, $0 \leqslant i \leqslant m-1$ and $\widehat{w}(t) = (\widehat{w}_0(t), \ldots, \widehat{w}_{m-1}(t))$, $t \in [0,T]$.

Define a linear system of parabolic SPDEs

$$(14) \qquad \begin{cases} d_t y(t,x) = L(y(t,\cdot))(x)dt + A_0(t)y(t,x)dt \\ \qquad\qquad + \sum\limits_{i=0}^{m-1} a_i(t) B_i y(t,x) \circ dw_i(t) + b(t)dt, \\ y(0,x) = y^0(x) = Cx + v, \; y \in \mathbb{R}^m, x \in \mathbb{R}^n, t \in [0,T] \end{cases}$$

Here

(e) the $m \times m$-matrix $A_0(t)$, $t \in [0,T]$, and the vector $b(t) \in \mathbb{R}^m$, $t \in [0,T]$, are continuous functions; the $(m \times m)$-matrices B_i, $0 \leqslant i \leqslant m-1$, mutually commute (see the hypothesis (c)).

Theorem 1. *Assume that the hypotheses (d) and (e) are satisfied. Let* $\{\widehat{y}(t;\lambda) \in \mathbb{R}^m, t \in [0,T], \lambda \in \mathbb{R}^n\}$ *be the continuous process defined in* (13) *(see Lemma 4) and consider the smooth mapping* $\{\lambda = \psi(t,x), t \in [0,T], x \in \mathbb{R}^n\}$ *given in Lemma 2 and satisfying forward parabolic equation* (11)*. Define a continuous process using the characteristic system method*

$$y(t,x) = \widehat{y}(t; \psi(t,x)), \; t \in [0,T], x \in \mathbb{R}^n.$$

Then $\{y(t,x) \in \mathbb{R}^m, t \in [0,T], x \in \mathbb{R}^n\}$ *is a solution of the linear system of parabolic type SPDEs* (14)*. In particular, when the* $(m \times m)$*-matrices* A_0, B_i*,* $0 \leqslant i \leqslant m-1$*, satisfy the conditions* (4) *and* $b(t) = b_m f(t)$ *(see* (3)*), then*

(f) the first component $y_0 = \varphi$ *of* $y(t,x) = (y_0(t,x), \ldots, y_{m-1}(t,x))$ *verifies the linear higher-order parabolic SPDE IN* (2)*.*

Proof. The main ingredients used for proving that the smooth mapping $\{y(t,x) \in \mathbb{R}^m, t \in [0,T], x \in \mathbb{R}^n\}$, constructed by the characteristic system method in

Lemma 3, satisfy the linear system of parabolic equations (3), are equally significant here. What is new consist of using stochastic rule of derivation for proving Lemma 4 (see $\{\widehat{y}(t;\lambda) \in \mathbb{R}^m\}$ as a continuous process verifying (11)). These lead us directly to the conclusion (1). The last conclusion (see (f)) is a rewritting of the system (14) when the special properties (4) are assumed. The proof is complete. □

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