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Vasile Drăgan
E-mail: Vasile.Dragan@imar.ro

Toader Morozan
E-mail: Toader.Morozan@imar.ro

H_2 optimal control for a class of discrete-time linear stochastic systems with periodic coefficients

Vasile Dragan and Toader Morozan

Institute of Mathematics "Simion Stoilow" of the Romanian Academy,

P.O.Box. 1-764, RO-014700, Bucharest, Romania

Vasile.Dragan@imar.ro, Toader.Morozan@imar.ro

Abstract

This paper presents an optimal solution of the H_2 state feedback control problem for discrete-time time-varying periodic stochastic linear systems simultaneously affected by jump Markov perturbations and multiplicative white noise. it is proved that the optimal solution is a static gain which is also optimal in the class of all higher order controllers. The gain matrix of the optimal controller is expressed in terms of the stabilizing solution of some generalized coupled discrete-time Riccati type equations. An iterative procedure which allows us to compute the stabilizing solution of the corresponding generalized discrete-time Riccati equation is proposed. The derivation of the state space representation of the optimal controller is essentially based on the formula of the value of the H_2-norms of the considered stochastic systems. As in the deterministic case, the value of H_2-norms are expressed in terms of periodic solutions of some linear matrix equations extending to this framework the well known equations of the observability Gramian and controlability Gramian. It is worth mentioning that the value of the H_2-norms of a discrete-time linear stochastic system with periodic coefficients affected by an homogenous Markov chain is strongly dependent on the existence of the limit of P^t for t tends to ∞ where P is a transition probability matrix of the Markov chain. We feel that this is an unexpected fact and it is specific to the discrete-time case.

Keywords: linear stochastic systems, Markovian jumping, H_2-norms, discrete-time Riccati equations, periodic coefficients.

I. INTRODUCTION.

In the last three decades a great amount of effort was dedicated to the solution of the H_2 and H_∞ control problems. Originated in two optimization problems stated in frequency domain consisting in the minimization either of the H_2-norm or of the H_∞-norm of some suitable transfer functions, these two control problems get later an equivalent formulation in the time domain. The state space interpretation of the H_2-norm and of the H_∞-norm of a linear time invariant system allowed important extensions of the H_2 and H_∞ control problems to deterministic time varying control systems as well as to both time invariant and time varying stochastic systems. For the readers convenience we refer to the monographs [3], [21], [24], [1], [27] and their references for the deterministic case and to [27], [6], [29], [15], [16], [19] for the stochastic case.

Traditionally, in the stochastic framework a special attention was paid to two classes of stochastic systems, namely Markov jump linear systems and systems subject to multiplicative white noise. When an important and unpredictible variation causes a discrete change in the plant characterization at isolated points in time, a Markov chain with a finite state space is a natural model for the plant parameter process. Some illustrative applications of these systems can be found for example in [2], [18], [25], [6], [26] and their references.

More recently, the H_2 control problem for continuous time Markov jump linear stationary systems has been studied in [4] for the state feedback case and in [10] for the output feedback case. The H_2 optimal control problem for discrete-time time-invariant Markov jump systems was solved in [5], [6].

The stochastic systems with multiplicative white noise naturally arise in control problems of linear uncertain systems with stochastic uncertainty (see [17], [23], [28], [30] and the references therein). Results concerning the H_2 control problem for this type of systems are derived for instance in [9], [12] for the continuous time case and [19], [20] for the discrete-time case.

Recently, there is an increasing interest in various control problems for stochastic systems simultaneously affected by multiplicative white noise perturbations and Markovian jumping (see e.g. [13], [15] for continuous time case and [7], [8], [16] for the discrete-time case).

Lately, there is an increasing interest to encounter periodic phenomena in many practical fields, such as spacecraft altitude control, industrial process control, communication systems and economics ([1], [31]). It is worth mentioning that in the case of controlled systems with periodic coefficients of period $\theta \geq 2$, the solutions of many control problems can be effectively computed and implemented as in the time invariant case, that is the periodic case of period $\theta = 1$.

In the present paper the H_2 optimal control problem for a class of discrete-time linear stochastic systems with periodic coefficients is considered. This paper may be viewed as the discrete-time counterpart of [14]. The consider controlled systems have state space representations described by linear systems with periodic coefficients simultaneously affected by an homogenous Markov chain and multiplicative and additive white noise perturbations. Related to such a class of systems we introduced two quadratic cost functionals which we call "H_2-norms". These cost functionals are defined via the Cesaro limit of the mean square of a suitable output. In the special case when the multiplicative white noise perturbations would vanish and the Markov chain would has only one state, the considered performance criteria reduce to the well known H_2-norm introduced for a discrete-time deterministic system. This justify the terminology used in connection with the control problem investigated in this paper.

The H_2 optimization problem considered in this paper is solved under the assumption that perfect measurements of the state vectors are available to fed the controller. Also, it is assumed that at each time instance the states of the Markov chain are known. The class of the admissible controllers consist of arbitrary dynamic systems with periodic coefficients with the same period θ. The dimension of the state space of the admissible controllers is not prefixed. We show that among all admissible controllers the best performance is achieved by a zero order controller, that is a stabilizing state feedback. The gain matrix of the optimal controller is design based on the stabilizing solution of a suitable system of coupled discrete-time Riccati equations. The derivation of the state space representation of the optimal controller is essentially based on the formula of the value of the H_2 norms of the considered stochastic systems. As in the deterministic case, the value of H_2 norms are expressed in terms of periodic solutions of some linear matrix equations extending to this framework the well known equations of the observability Gramian and controlability Gramian. It is worth mentioning that the value of the H_2-norms of a discrete-time linear stochastic system with periodic coefficients affected by an homogenous Markov chain is strongly dependent on the existence of the limit of P^t for t tends to ∞. We feel that this fact is an unexpected one and it is specific to the discrete-time case.

The rest of the paper is organized as follows: in the first part of Section 2 we introduced two H_2 type norms for a discrete-time linear stochastic systems with periodic coefficients, simultaneously affected by a Markov chain and multiplicative and additive white noise perturbations. In the second part of Section 2 the H_2 optimization problems we want to solve in this paper, are stated. Section 3 collects some known definitions and results which help us to state and prove the main results of this paper. The values of the H_2 norms is computed in Section 4 in terms of periodic solutions of some suitable discrete-time linear matrix equations. The solutions of the H_2 optimization problems stated in Section 2 are provided in Section 5.

II. Problem formulation.

Consider the system $\mathbf{G}$ having the state space representation:

$$\mathbf{G}\begin{cases} x(t+1) = [A_0(t,\eta_t) + \sum_{k=1}^{r} w_k(t)A_k(t,\eta_t)]x(t) + B(t,\eta_t)v(t), \\ z(t) = C(t,\eta_t)x(t), \end{cases} \tag{1}$$

$t \in \mathbf{Z}_+ = \{0,1,2,...\}$, where $x(t) \in \mathbf{R}^n$ are the state vectors, while $z(t) \in \mathbf{R}^{n_z}$ are the values of a preferential output of the system $(\mathbf{G})$.

In (1) $\{\eta_t\}_{t\geq 0}$ is an homogeneous Markov chain on a given probability space $(\Omega, \mathscr{F}, \mathscr{P})$ with the set of the states $\mathfrak{S} = \{1,2,3,...,N\}$ and the transition probability matrix $P = (p_{ij})$, (for details see [11]) $\{w(t)\}_{t\geq 0}$ $(w(t) = (w_1(t), w_2(t), ..., w_r(t))^T)$, $\{v(t)\}_{t\geq 0}$ are sequences of independent random vectors satisfying the assumptions:

$\mathbf{H}_1$) $E[w(t)] = 0$, $E[w(t)w^T(t)] = I_r$ for all $t \in \mathbf{Z}_+$; the stochastic process $\{w(t)\}_{t\geq 0}$ is independent of the stochastic process $\{\eta_t\}_{t\geq 0}$.

$\mathbf{H}_2$) $E[v(t)] = 0$, $E[v(t)v^T(t)] = I_{m_v}$ for all $t \in \mathbf{Z}_+$; the stochastic process $\{v(t)\}_{t\geq 0}$ is independent of $\{w(t), \eta_t\}_{t\geq 0}$.

As usual $E[\cdot]$ stands for the mathematical expectation and superscript T denotes the transposition of a matrix or a vector.

Regarding the coefficients of the system (1) we assume that the sequences $\{A_k(t,i)\}_{t\geq 0} \subset \mathbf{R}^{n\times n}$, $0 \leq k \leq r$, $\{B(t,i)\}_{t\geq 0} \subset \mathbf{R}^{n\times m_v}$, $\{C(t,i)\}_{t\geq 0} \subset \mathbf{R}^{n_z\times n}$ are periodic with a period $\theta \geq 1$.

Throughout this section we assume that the discrete-time linear stochastic system

$$x(t+1) = [A_0(t,\eta_t) + \sum_{k=1}^{r} w_k(t)A_k(t,\eta_t)]x(t) \tag{2}$$

is exponentially stable in mean square (ESMS).

For precise definition and other properties of the concept of (ESMS) we refer to [16].

Consider $z(t;t_0,x_0,v) = C(t,\eta_t)x(t;t_0,x_0,v)$ where $x(t;t_0,x_0,v)$ is the solution of the system (1) with the initial value $x(t_0;t_0,x_0,v) = x_0 \in \mathbf{R}^n$. Using the decomposition $x(t;t_0,x_0,v) = x(t;t_0,x_0,0) + x(t;t_0,x_0,v)$ one obtains the following splitting of the output

$$z(t;t_0,x_0,v) = z(t;t_0,x_0,0) + z(t;t_0,0,v).$$

So, we obtain a separation of the influence of the initial conditions x_0 and of the additive white noise perturbation $v(t)$ on the output of the considered system. In this work we are interested in introducing a measure of the influence of the additive white noise perturbation through a preferred output of the given system. Also, in the case when the system contains control parameters, our aim is to construct a control law which minimize the effect of the additive white noise perturbations. To measure the influence of the additive white noise perturbation through an output of the system (1) we introduce the so called H_2-norm. Thus we introduce

$$\|\mathbf{G}\|_2 = \{\lim_{\tau\to\infty} \frac{1}{\tau} \sum_{t=0}^{\tau} E[|z_0(t)|^2]\}^{1/2} \tag{3}$$

where $z_0(t) = C(t,\eta_t)x(t;0,0,v) = z(t;0,0,v)$.

Motivated by the dependence of the coefficients of the considered system upon the Markov chain, we may introduce also the following H_2-type norm: $|||\cdot|||_2$,

$$|||\mathbf{G}|||_2 = \{\lim_{\tau\to\infty} \frac{1}{\tau} \sum_{t=0}^{\tau} \sum_{i\in\mathfrak{S}_0} E[|z_0(t)|^2|\eta_0 = i]\}^{1/2} \tag{4}$$

where $\mathfrak{S}_0 = \{i \in \mathfrak{S} | \mathscr{P}(\eta_0) = i > 0\}$.

Throughout the paper $E[\cdot|\eta_0 = i]$ stands for the conditional expectation with respect to the event $\{\eta_0 = i\}$.

In Section 4 we shall show how we can express the values of the two H_2-norms introduced above in terms of global solution of some suitable linear equations which extend to this framework the well known equations of the controllability Gramian and observability Gramian from the deterministic case. Also, it will be clear that the value of the norms (3) and (4) respectively, does not change if the output $z_0(t)$ is replaced by $z(t;0,x_0,v)$ for arbitrary $x_0 \in \mathbf{R}^n$.

Let us consider the controlled system $(\mathbf{G})$ described by

$$\begin{aligned} x(t+1) &= [A_0(t,\eta_t)+\sum_{k=1}^{r} w_k(t)A_k(t,\eta_t)]x(t)+[B_0(t,\eta_t)+ \\ &\quad +\sum_{k=1}^{r} w_k(t)B_k(t,\eta_t)]u(t)+B_v(t,\eta_t)v(t) \\ y(t) &= x(t) \qquad (5) \\ z(t) &= C_z(t,\eta_t)x(t)+D_z(t,\eta_t)u(t) \end{aligned}$$

where $x(t)\in\mathbf{R}^n$ are the state vectors, $u(t)\in\mathbf{R}^n$ are the control parameters, $y(t)\in\mathbf{R}^n$ are the available measurements to fed the controller, and $z(t)$ is the output which must be controlled.

In this paper, we consider the case when perfect state measurements are possible. That is $y(t)=x(t)$. The case when an output $y(t)=[C_0(t,\eta_t)+\sum_{k=1}^{r} w_k(t)C_k(t,\eta_t)]x(t)+D(t,\eta_t)v(t)$ is available for measurements will be considered elsewhere.

In (5) the stochastic processes $\{\eta_t\}_{t\geq 0}$, $\{w(t)\}_{t\geq 0}$, $\{v(t)\}_{t\geq 0}$ are as in the system (1) and satisfy the assumptions $\mathbf{H}_1$) and $\mathbf{H}_2$). Regarding the coefficients of the system (5) we assume that the sequences $\{A_k(t,i)\}_{t\geq 0}\subset\mathbf{R}^{n\times n}$, $\{B_k(t,i)\}_{t\geq 0}\subset\mathbf{R}^{n\times m}$, $0\leq k\leq r$, $\{B_v(t,i)\}_{t\geq 0}\subset\mathbf{R}^{n\times m_v}$, $\{C_z(t,i)\}_{t\geq 0}\subset\mathbf{R}^{n_z\times n}$, $\{D_z(t,i)\}_{t\geq 0}\subset\mathbf{R}^{n_z\times m}$, are periodic of period $\theta\geq 1$.

To control the system (5) we consider dynamic controllers $\mathbf{G}_c$ having the state space representation of the form:

$$\begin{aligned} x_c(t+1) &= [A_{0c}(t,\eta_t)+\sum_{k=1}^{r} w_k(t)A_{kc}(t,\eta_t)]x_c(t)+B_c(t,\eta_t)u_c(t) \qquad (6) \\ y_c(t) &= C_c(t,\eta_t)x_c(t)+D_c(t,\eta_t)u_c(t) \end{aligned}$$

where $x_c(t)\in\mathbf{R}^{n_c}$ are the state parameters of the controller, $u_c(t)\in\mathbf{R}^n$ are the inputs and $y_c(t)\in\mathbf{R}^m$ are the outputs of the controller.

In (6) the dimension $n_c\geq 0$ of the state space of a controller $\mathbf{G}_c$ is not prefixed. In the special case $n_c=0$, a controller (6) reduces to a state feedback law of the form $u_c(t)=D_c(t,\eta_t)u_c(t)$.

In (6) $\{A_{kc}(t,i)\}_{t\geq 0}\subset\mathbf{R}^{n_c\times n_c}$, $0\leq k\leq r$, $\{B_c(t,i)\}_{t\geq 0}\subset\mathbf{R}^{n_c\times n}$, $\{C_c(t,i)\}_{t\geq 0}\subset\mathbf{R}^{m\times n_c}$, $\{D_c(t,i)\}_{t\geq 0}\subset\mathbf{R}^{m\times n}$ are arbitrary θ-periodic sequences.

When coupling a controller $\mathbf{G}_c$ of type (6) to the system (5) taking $u_c(t) = y(t)$ and $u(t) = y_c(t)$ one obtains the following closed-loop system

$$(\mathbf{G}_{cl}(\mathbf{G}_c)) : \begin{cases} x_{cl}(t+1) = [A_{0cl}(t,\eta_t) + \sum_{k=1}^{r} w_k(t) A_{kcl}(t,\eta_t)] x_{cl}(t) + B_{vcl}(t,\eta_t) v(t), \\ z_{cl}(t) = C_{cl}(t,\eta_t) x_{cl}(t) \end{cases} \quad (7)$$

where

$$\begin{aligned} x_{cl}(t) &= \begin{pmatrix} x^T(t) & x_c^T(t) \end{pmatrix}^T, \\ A_{0cl}(t,i) &= \begin{pmatrix} A_0(t,i) + B_0(t,i) D_c(t,i) & B_0(t,i) C_c(t,i) \\ B_c(t,i) & A_{0c}(t,i) \end{pmatrix}, \\ A_{kcl}(t,i) &= \begin{pmatrix} A_k(t,i) + B_k(t,i) D_c(t,i) & B_k(t,i) C_c(t,i) \\ 0 & A_{kc}(t,i) \end{pmatrix}, 1 \le k \le r \\ B_{vcl}(t,i) &= \begin{pmatrix} B_v(t,i) \\ 0 \end{pmatrix}, \\ C_{cl}(t,i) &= \begin{pmatrix} C_z(t,i) + D_z(t,i) D_c(t,i) & D_z(t,i) C_c(t,i) \end{pmatrix}. \end{aligned} \quad (8)$$

Often we shall write $\mathbf{G}_{cl}$ instead of $\mathbf{G}_{cl}(\mathbf{G}_c)$ when no confusions are possible.

In the sequel, we denote $\mathscr{K}_s(\mathbf{G})$ the set of all controllers $\mathbf{G}_c$ with the state space representation of type (6) with the property that the linear system

$$x_{cl}(t+1) = [A_{0cl}(t,\eta_t) + \sum_{k=1}^{r} w_k(t) A_{kcl}(t,\eta_t)] x_{cl}(t) \quad (9)$$

is ESMS. In the sequel the controllers from $\mathscr{K}_s(\mathbf{G})$ will be called *"stabilizing controllers"*.

One sees that the closed loop systems (7) are of the form (1) while (9) is of type (2). Hence, in the case of systems (7) we may defined the H_2-norms (3) or (4), respectively.

Now, we are in position to formulate the H_2-optimization problems for the system (5) which are solved in this paper.

$\mathbf{OP}_j$ Find a stabilizing controller $\tilde{\mathbf{G}}_{c,j}$ with the property that

$$||\mathbf{G}_{cl}(\tilde{\mathbf{G}}_{c,j})||_{2,j} \le ||\mathbf{G}_{cl}(\mathbf{G}_c)||_{2,j}, j = 1,2 \quad (10)$$

for all $\mathbf{G}_c \in \mathscr{K}_s(\mathbf{G})$.

In (10) $||\cdot||_{2,j}=||\cdot||_2$ defined as in (3) for $j=1$ and $||\cdot||_{2,j}=|||\cdot|||_2$ defined as in (4) for $j=2$. The solution of this two H_2-optimization problems will be given in Section 5. We shall see that among all stabilizing controllers of type (6) of arbitrary order n_c, the best performance is achieved by a zero order controller.

III. Some auxiliary results.

In this section we briefly recall several already known notions which will be helpful to state and proof the main results of this paper. Throughout this paper $\mathscr{S}_n \subset \mathbf{R}^{n\times n}$ stands for the linear subspace of the $n\times n$ symmetric real matrices and $\mathscr{S}_n^N=\mathscr{S}_n\oplus\mathscr{S}_n\ldots\oplus\mathscr{S}_n$. One sees easily that $\mathscr{S}_n^N$ is an ordered real Hilbert space with respect to the inner product

$$<\mathbf{X},\mathbf{Y}>=\sum_{i=1}^{N}Tr[X(i)Y(i)] \tag{11}$$

for all $\mathbf{X}=(X(1),X(2),\ldots,X(N))$, $\mathbf{Y}=(Y(1),Y(2),\ldots,Y(N))\in\mathscr{S}_n^N$, and the order relation induced by the closed, solid, convex cone $\mathscr{S}_n^{N+}=\{\mathbf{X}\in\mathscr{S}_n^N|\mathbf{X}=(X(1),\ldots,X(N)),\ X(i)\geq 0,\ 1\leq i\leq N\}$. Here, $X(i)\geq 0$ means that $X(i)$ is a positive semidefinite matrix.

Based on the coefficients of the system (2) we construct the following sequence of linear operators $\{\mathscr{L}(t)\}_{t\in\mathbf{Z}}$, $\mathscr{L}(t):\mathscr{S}_n^N\rightarrow\mathscr{S}_n^N$ by $\mathscr{L}(t)(\mathbf{X})=(\mathscr{L}_1(t)(\mathbf{X}),\ldots,\mathscr{L}_N(t)(\mathbf{X}))$ with

$$\mathscr{L}_i(t)(\mathbf{X})=\sum_{k=0}^{r}\sum_{j=1}^{N}p_{ji}A_k(t,j)X(j)A_k^T(t,j) \tag{12}$$

for all $\mathbf{X}\in\mathscr{S}_n^N$. By direct calculations one obtains that the adjoint operator of $\mathscr{L}(t)$ with respect to the inner product (3.1) is given by $\mathscr{L}^*(t)(\mathbf{X})=(\mathscr{L}_1^*(t)(\mathbf{X}),\ldots,\mathscr{L}_N^*(t)(\mathbf{X}))$ with

$$\mathscr{L}_i^*(t)(\mathbf{X})=\sum_{k=0}^{r}A_k^T(t,i)\mathscr{E}_i(\mathbf{X})A_k(t,i) \tag{13}$$

for all $\mathbf{X}\in\mathscr{S}_n^N$. Throughout this paper

$$\mathscr{E}_i(\mathbf{Y})=\sum_{j=1}^{N}p_{ij}Y(j),(\forall)\mathbf{Y}\in\mathscr{S}_n^N. \tag{14}$$

It is worth mentioning that even if the system (1), (5) and (6) are well defined for $t\in\mathbf{Z}_+$ all coefficients of these systems $\{A_k(t,i)\}_{t\geq 0}$ and so on, can be extended by periodicity to $t\in\mathbf{Z}$. So, $\mathscr{L}(t)$ may be defined for $t\in\mathbf{Z}$.

The sequences $\{\mathscr{L}(t)\}_{t\in\mathbf{Z}}$ and $\{\mathscr{L}^*(t)\}_{t\in\mathbf{Z}}$ play an important role in characterization of the exponential stability in mean square of the discrete-time linear stochastic system (2).

Combining Theorem 3.10 with Theorems 2.4 and 2.7 in [16] we obtain:

Proposition 3.1. *Under the assumption* $\mathbf{H}_1)$ *the following are equivalent:*

(i) the discrete-time linear stochastic system (2) is ESMS;

(ii) the discrete-time linear equation on $\mathscr{S}_n^N$, $\mathbf{X}(t+1) = \mathscr{L}(t)(\mathbf{X}(t))$ *is exponentially stable;*

(iii) the discrete-time backward affine equation on $\mathscr{S}_n^N$, $\mathbf{X}(t) = \mathscr{L}^*(t)(\mathbf{X}(t+1)) + I_n^N$ *has a* θ*-periodic solution* $\tilde{\mathbf{X}}(t) = (\tilde{X}(t,1),...,\tilde{X}(t,N))$ *with* $\tilde{X}(t,i) > 0$, $t \in \mathbf{Z}$, $1 \le i \le N$ *where* $I_n^N = \begin{pmatrix} I_n, & I_n, & \dots & ,I_n \end{pmatrix} \in \mathscr{S}_n^{N+}$;

(iv) for any θ*-periodic sequence* $\{\mathbf{H}(t)\}_{t\in\mathbf{Z}}$, $\mathbf{H}(t) = (H(t,1),...,H(t,N))$ *with* $H(t,i) > 0$, *for all* $(t,i) \in \mathbf{Z}\times\mathfrak{S}$, *the discrete-time backward affine equation*

$$\mathbf{X}(t) = \mathscr{L}^*(t)(\mathbf{X}(t+1)) + \mathbf{H}(t)$$

has a θ*-periodic solution* $\hat{\mathbf{X}}(t) = (\hat{X}(t,1),...,\hat{X}(t,N))$, $\hat{X}(t,i) > 0$, $(\forall)\ (t,i) \in \mathbf{Z}\times\mathfrak{S}$;

(v) the discrete-time forward equation

$$\mathbf{Y}(t+1) = \mathscr{L}(t)(\mathbf{Y}(t)) + I_n^N$$

has a θ*-periodic solution* $\tilde{\mathbf{Y}}(t) = (\tilde{Y}(t,1),...,\tilde{Y}(t,N))$ *with* $\tilde{Y}(t,i) > 0$, $(\forall)(t,i) \in \mathbf{Z}\times\mathfrak{S}$;

(vi) for any θ*-periodic sequence* $\{\mathbf{H}(t)\}_{t\in\mathbf{Z}} \subset \mathscr{S}_n^N$ *as in (iv) the discrete-time forward affine equation*

$$\mathbf{Y}(t+1) = \mathscr{L}(t)(\mathbf{Y}(t)) + \mathbf{H}(t)$$

has a θ*-periodic solution* $\hat{\mathbf{Y}}(t) = (\hat{Y}(t,1),...,\hat{Y}(t,N))$ *with* $\hat{Y}(t,i) > 0$, $(\forall)\ (t,i) \in \mathbf{Z}\times\mathfrak{S}$.

Further, combining Theorem 2.5 and Theorem 2.6 in [16] with Proposition 3.1 from above, we obtain:

Proposition 3.2. *Assume that assumption* $\mathbf{H}_1)$ *is fulfilled and the system (2) is ESMS. Then the following hold:*

(i) for any θ*-periodic sequence* $\{\mathbf{H}(t)\}_{t\in\mathbf{Z}} \subset \mathscr{S}_n^N$ *the discrete-time backward affine equation*

$$\mathbf{X}(t) = \mathscr{L}^*(t)(\mathbf{X}(t+1)) + \mathbf{H}(t) \tag{15}$$

has a unique θ*-periodic solution* $\tilde{\mathbf{X}}(t) = (\tilde{X}(t,1)...\tilde{X}(t,N))$. *Furthermore if* $\mathbf{H}(t) \in \mathscr{S}_n^{N+} \forall t \in \mathbf{Z}$ *then* $\tilde{\mathbf{X}}(t) \in \mathscr{S}_n^{N+}$, $\forall t \in \mathbf{Z}$;

(ii) for any θ-periodic sequence $\{\mathbf{H}(t)\}_{t\in\mathbf{Z}} \subset \mathscr{S}_n^N$ the discrete-time forward affine equation

$$\mathbf{Y}(t+1) = \mathscr{L}(t)(\mathbf{Y}(t)+\mathbf{H}(t)) \tag{16}$$

has a unique θ-periodic solution $\tilde{\mathbf{Y}}(t) = (\tilde{Y}(t,1),...,\tilde{Y}(t,N))$, $t \in \mathbf{Z}$. Moreover, if $\mathbf{H}(t) \in \mathscr{S}_n^{N_+}$, $(\forall)\ t \in \mathbf{Z}$, then $\tilde{\mathbf{Y}}(t) \in \mathscr{S}_n^{N_+}$, $\forall t \in \mathbf{Z}$.

Lemma 3.3. Under the assumptions $\mathbf{H}_1$) and $\mathbf{H}_2$) if $\mathbf{X}(t) = (X(t,1),...,X(t,N)) \in \mathscr{S}_n^N$, $t \geq t_0$, $X(t,i) = X^T(t,i)$ then the following equality holds:

$$\begin{aligned} E[x^T(t+1)X(t+1,\eta_{t+1})x(t+1)|\eta_{t_0}] = E[x^T(t)\mathscr{L}^*_{\eta_t}(t)(\mathbf{X}(t+1))x(t)|\eta_{t_0}] + \\ + \sum_{j=1}^{N} E[p_{\eta_t j} Tr[B^T(t,\eta_t)X(t+1,j)B(t,\eta_t)]|\eta_{t_0}] \end{aligned} \tag{17}$$

for all $t \geq t_0 \geq 0$ and for all trajectories $x(t) = x(t;t_0,x_0,v)$ of the system (1).

Proof. The equality in the statement follows immediately from Lemma 3.1 in [16] because the system (1) from above is a special case of the system (3.84) in the afore mentioned reference.

At the end of this section, let us recall several useful properties of a stochastic matrix. For details we refer to [11] (Chapter 5).

A matrix $\check{P} = (\check{p}_{ij}) \in \mathbf{R}^{N\times N}$ is named stochastic matrix if $\check{p}_{ij} \geq 0$, $(\forall)\ i,j$ and $\sum_{j=1}^{N} \check{p}_{ij} = 1$ $(\forall)\ 1 \leq i \leq N$.

Lemma 3.4 [11]. *If $\check{P}$ is a stochastic matrix then the Cesaro limit $\lim_{\tau\to\infty} \frac{1}{\tau} \sum_{t=0}^{\tau} \check{P}^t$ is well defined. If*

$$\check{Q} = \lim_{\tau\to\infty} \frac{1}{\tau} \sum_{t=0}^{\tau} \check{P}^t \tag{18}$$

then $\check{Q}$ is a stochastic matrix and satisfies:

$$\check{Q}\check{P} = \check{P}\check{Q} = \check{Q}.$$

It is known that if $\check{P}$ is a stochastic matrix, then the limit $\lim_{t\to\infty} \check{P}^t$ is not always well defined. However, if this limit exist, then we have:

$$\lim_{t\to\infty} \check{P}^t = \check{Q}$$

where $\check{Q}$ is the matrix defined in (18).

Let $\pi_t = (\pi_t(1),...,\pi_t(N))$ be the distribution of the random variable η_t. That is $\pi_t(i) = \mathscr{P}\{\eta_t = i\}$, $1 \le i \le N$. One obtains that $\pi_{t+1} = \pi_t P$, which leads to

$$\pi_t = \pi_0 P^t. \tag{19}$$

Applying Lemma 3.4 in the case $\check{P} = P$ we obtain that it is well defined π_∞ by

$$\pi_\infty = \lim_{\tau\to\infty} \frac{1}{\tau} \sum_{t=0}^{\tau} \pi_t. \tag{20}$$

We have

$$\pi_\infty = \pi_0 Q \tag{21}$$

where Q is a stochastic matrix defined by (18) for $\check{P} = P$.

IV. Computation of the H_2-norms.

In this section we shall compute the values of the H_2-norms introduced via (3) and (4). We shall see that in the case of discrete-time linear stochastic systems of type (1) with periodic coefficients of period $\theta \ge 2$ both the value of the norm (3) as well as (4) are strongly affected by the existence or non-existence of the limit $\lim_{t\to\infty} P^t$.

Theorem 4.1. *Assume that the system (2) is ESMS. Under the assumptions* $\mathbf{H}_1)$ *and* $\mathbf{H}_2)$ *the following equalities hold:*

$$\lim_{\tau\to\infty} \frac{1}{\tau} \sum_{t=0}^{\tau} E[|z(t;0,x_0,v)|^2/\eta_0 = i] = \sum_{l=1}^{N} \sum_{j=1}^{N} \frac{p_{jl}}{\theta} \sum_{s=0}^{\theta-1} \tilde{\mu}_{ij}(s) Tr[B^T(s,j)\tilde{X}(s+1,l)B(s,j)] \tag{22}$$

for all $i \in \mathfrak{S}$ *such that* $\mathscr{P}(\eta_0 = i) > 0$ *and for all* $x_0 \in \mathbf{R}^n$*, where* $\tilde{\mathbf{X}}(t) = (\tilde{X}(t,1),...,\tilde{X}(t,N))$ *is the unique* θ*-periodic solution of the discrete-time backward affine equation on the space* $\mathscr{S}_n^N$*:*

$$X(t,\iota) = \mathscr{L}_\iota^*(t)(\mathbf{X}(t+1)) + C^T(t,\iota)C(t,\iota), \quad 1 \le \iota \le N \tag{23}$$

and the scalars $\tilde{\mu}_{ij}(s)$ *are the elements of the matrix* $\tilde{M}(s)$ *defined as follows:*

$$\tilde{M}(s) = Q \tag{24}$$

if there exists $\lim_{t\to\infty} P^t$ *and*

$$\tilde{M}(s) = Q(\theta)P^s \tag{25}$$

if the limit $\lim_{t\to\infty} P^t$ *is not well defined; where* Q *is given by (18) when* $\check{P} = P$ *while* $Q(\theta)$ *is given by (18) for* $\check{P} = P^\theta$.

Proof. First, let us remark that based on Proposition 3.2 from above, the equation (23) has a unique θ-periodic solution $\tilde{\mathbf{X}}(t) = (\tilde{X}(t,1),...,\tilde{X}(t,N))$ $t \in \mathbf{Z}$.

Let $x(t) = x(t;0,x_0,v)$ be an arbitrary solution of the system (1) and $z(t) = C(t,\eta_t)x(t)$ be the corresponding output.

Using (23) we may write:

$$E[|z(t)|^2|\eta_0] = E[x^T(t)C^T(t,\eta_t)C(t,\eta_t)x(t)|\eta_0] =$$

$$= E[x^T(t)\tilde{X}(t,\eta_t)x(t)|\eta_0] - E[x^T(t)\mathscr{L}^*_{\eta_t}(t)(\tilde{\mathbf{X}}(t+1))x(t)|\eta_0]$$

$\forall t \geq 0$.

Further, applying Lemma 3.3 for $\mathbf{X}(t) = \tilde{\mathbf{X}}(t)$ and summing from $t = 0$ to $t = \tau$ one obtains:

$$\sum_{t=0}^{\tau} E[|z(t)|^2|\eta_0] = \sum_{t=0}^{\tau}\sum_{l=1}^{N} E[p_{\eta_t l} Tr[B^T(t,\eta_t)\tilde{X}(t+1,l)B(t,\eta_t)]|\eta_0] + \Psi(x_0,\tau), \quad \tau \geq 1, \quad (26)$$

where we denoted $\Psi(x_0,\tau) = x_0^T\tilde{X}(0,\eta_0)x_0 - E[x^T(\tau+1)\tilde{X}(\tau+1,\eta_{\tau+1})x(\tau+1)|\eta_0]$.

Let $i \in \{1,2,...,N\}$ with the property that $\mathscr{P}\{\eta_0 = i\} > 0$. In this case (26) yields

$$\sum_{t=0}^{\tau} E[|z(t)|^2|\eta_0 = i] = \sum_{t=0}^{\tau}\sum_{l=1}^{N} E[p_{\eta_t l} Tr[B^T(t,\eta_t)\tilde{X}(t+1,l)B(t,\eta_t)]|\eta_0 = i] + E[\Psi(x_0,\tau)|\eta_0 = i]. \quad (27)$$

We have

$$E[\Psi(x_0,\tau)|\eta_0 = i] = x_0^T\tilde{X}(0,i)x_0 - E[x^T(\tau+1)\tilde{X}(\tau+1,\eta_{\tau+1})x(\tau+1)|\eta_0 = i]. \quad (28)$$

Applying Corollary 3.9 (ii) in [16] we deduce that $\sup_{t\geq 0} E[|x(t)|^2|\eta_0 = i] \leq c(x_0) < \infty$.

So we may deduce from (28) that $\lim_{\tau\to\infty} \frac{1}{\tau}E[\Psi(x_0,\tau)|\eta_0 = i] = 0, \quad (\forall) x_0 \in \mathbf{R}^n$.

This allows us to conclude via (27) that

$$\lim_{\tau\to\infty}\frac{1}{\tau}\sum_{t=0}^{\tau} E[|z(t)|^2|\eta_0 = i] = \lim_{\tau\to\infty}\frac{1}{\tau}\sum_{t=0}^{\tau}\sum_{l=1}^{N} E[p_{\eta_t l} Tr[B^T(t,\eta_t)\tilde{X}(t+1,l)B(t,\eta_t)]|\eta_0 = i] \quad (29)$$

for all $x_0 \in \mathbf{R}^n$.

Let $\nu(\tau) = [\frac{\tau}{\theta}]$ be the largest integer less or equal than the rational number $\frac{\tau}{\theta}$. It follows that $\lim_{\tau\to\infty} \nu(\tau) = \infty$ and $\nu(\tau)\theta \leq \tau \leq \nu(\tau)\theta + \theta - 1$.

Hence, $\lim_{\tau\to\infty}\frac{1}{\tau}\sum_{t=\nu(\tau)\theta}^{\tau}\sum_{l=1}^{N}E[p_{\eta_t l}Tr[B^T(t,\eta_t)\tilde{X}(t+1,l)B(t,\eta_t)]|\eta_0=i]=0.$

In this way (29) reduces to

$$\lim_{\tau\to\infty}\frac{1}{\tau}\sum_{t=0}^{\tau}E[|z(t)|^2|\eta_0=i]=\lim_{\tau\to\infty}\frac{1}{\tau}\sum_{t=0}^{\nu(\tau)\theta-1}\sum_{l=1}^{N}E[p_{\eta_t l}Tr[B^T(t,\eta_t)\tilde{X}(t+1,l)B(t,\eta_t)]|\eta_0=i].\quad(30)$$

Further we write:

$$E[p_{\eta_t l}Tr[B^T(t,\eta_t)\tilde{X}(t+1,l)B(t,\eta_t)]|\eta_0=i]=\sum_{j=1}^{N}p_{jl}Tr[B^T(t,j)\tilde{X}(t+1,l)B(t,j)]\mathscr{P}\{\eta_t=j|\eta_0=i\}.$$

Let e_k, $1\le k\le N$ be the vectors of the canonical base of $\mathbf{R}^N$. If we take into account that $\mathscr{P}\{\eta_t=j|\eta_0=i\}=e_i^TP^te_j$ we obtain

$$\sum_{t=0}^{\nu(\tau)\theta-1}\sum_{l=1}^{N}E[p_{\eta_t l}Tr[B^T(t,\eta_t)\tilde{X}(t+1,l)B(t,\eta_t)]|\eta_0=i]=\sum_{l=1}^{N}\sum_{j=1}^{N}p_{jl}\sum_{t=0}^{\nu(\tau)\theta-1}\rho(t,l,j)e_i^TP^te_j\quad(31)$$

where we denoted $\rho(t,l,j)=Tr[B^T(t,j)\tilde{X}(t+1,l)B(t,j)]$. It is obvious that the sequences $\{\rho(t,l,j)\}_{t\in\mathbf{Z}}$ are θ-periodic sequences.

Exploiting the periodicity property of these sequences we may obtain

$$\sum_{t=0}^{\nu(\tau)\theta-1}\rho(t,l,j)e_i^TP^te_j=\sum_{k=0}^{\nu(\tau)-1}\sum_{s=0}^{\theta-1}\rho(k\theta+s,l,j)e_i^TP^{k\theta+s}e_j=\sum_{s=0}^{\theta-1}\rho(s,l,j)e_i^T\sum_{k=0}^{\nu(\tau)-1}P^{k\theta}P^se_j.$$

This allows us to conclude that

$$\begin{aligned}\lim_{\tau\to\infty}\frac{1}{\nu(\tau)}\sum_{t=0}^{\nu(\tau)\theta-1}\rho(t,l,j)e_i^TP^te_j &= \sum_{s=0}^{\theta-1}\rho(s,l,j)e_i^T[\lim_{\tau\to\infty}\frac{1}{\nu(\tau)}\sum_{k=0}^{\nu(\tau)-1}(P^\theta)^k]P^se_j=\\ &=\sum_{s=0}^{\theta-1}\rho(s,l,j)e_i^TQ(\theta)P^se_j.\end{aligned}\quad(32)$$

Finally (30), (31), (32) together with the equality $\lim_{\tau\to\infty}\frac{\nu(\tau)}{\tau}=\frac{1}{\theta}$ lead to

$$\lim_{\tau\to\infty}\frac{1}{\tau}\sum_{t=0}^{\tau}E[|z(t)|^2|\eta_0=i]=\sum_{l=1}^{N}\sum_{j=1}^{N}\frac{p_{jl}}{\theta}\sum_{s=0}^{\theta-1}Tr[B^T(s,j)\tilde{X}(s+1,l)B(s,j)]e_i^TQ(\theta)P^se_j.\quad(33)$$

If we take into account that $e_i^TQ(\theta)P^se_j=\tilde{\mu}_{ij}(s)$ we have that (33) is just (22) with $\tilde{\mu}_{ij}(s)$ constructed via (25). To end the proof it remains to show that (25) reduces to (24) if the limit $\lim_{t\to\infty}P^t$ is well defined. Indeed if $\lim_{t\to\infty}P^t$ exists, then, it is equal with Q, where Q is the stochastic matrix defined by (18) for $\check{P}=P$.

Thus, one gets $Q(\theta)=\lim_{\nu\to\infty}\frac{1}{\nu}\sum_{k=0}^{\nu}P^{k\theta}=\lim_{k\to\infty}P^{k\theta}=\lim_{t\to\infty}P^t=Q$. So, we obtain that $Q(\theta)=Q$. Since $QP=Q$ we deduce that $\tilde{M}(s)=Q$, $0\le s\le\theta-1$. Thus the proof is complete.

Remark 4.1. The matrices $\tilde{M}(s)$ were defined for $0 \le s \le \theta - 1$ by (24) and (25), respectively. It is easy to see that both (24) and (25), respectively, make sense for every $s \in \mathbf{Z}_+$. If $\tilde{M}(\cdot)$ is constructed via (25) we have $\tilde{M}(s+\theta) = Q(\theta)P^{s+\theta} = Q(\theta)P^s = \tilde{M}(s)$ for all $s \in \mathbf{Z}_+$. Hence, the sequence of matrices $\{\tilde{M}(s)\}_{s\in\mathbf{Z}_+}$ is periodic with period θ.

Now, we are in a position to prove the main results of this section:

Theorem 4.2. *Under the assumptions of Theorem 4.1 the value of the H_2-norm defined by (3) for the system* $\mathbf{G}$ *is given by:*

$$||\mathbf{G}||_2^2 = \sum_{l=1}^{N}\sum_{j=1}^{N}\frac{p_{jl}}{\theta}\sum_{s=0}^{\theta-1}\mu_j^{\pi_0}(s)Tr[B^T(s,j)\tilde{X}(s+1,l)B(s,j)] =$$
$$= \frac{1}{\theta}\sum_{s=0}^{\theta-1}\sum_{i=1}^{N}Tr[C(s,i)Y^{\pi_0}(s,i)C^T(s,i)] = \lim_{\tau\to\infty}\frac{1}{\tau}\sum_{t=0}^{\tau}E[|z(t;0,x_0,v)|^2] \tag{34}$$

for all $x_0 \in \mathbf{R}^n$, where $\tilde{\mathbf{X}}(t) = (\tilde{X}(t,1),...,\tilde{X}(t,N))$ is the unique θ-periodic solution of the discrete-time backward affine equation (23) while $\mathbf{Y}^{\pi_0}(t) = (Y^{\pi_0}(t,1),...,Y^{\pi_0}(t,N))$ is the unique θ-periodic solution of the discrete-time forward affine equation

$$Y(t+1,i) = \mathscr{L}_i(t)(\mathbf{Y}(t)) + \sum_{j=1}^{N}p_{ji}\mu_j^{\pi_0}(t)B(t,j)B^T(t,j) \tag{35}$$

$1 \le i \le N$. The scalars $\mu_j^{\pi_0}(t)$ are defined by

$$\mu_j^{\pi_0}(t) = \sum_{l=1}^{N}\pi_0(l)\tilde{\mu}_{lj}(t), \quad 1 \le j \le N,\ t \in \mathbf{Z}_+, \tag{36}$$

$\pi_0 = (\pi_0(1),...,\pi_0(N))$ being the initial distribution of the Markov chain.

Proof. With the convention $E[|z(t;0,x_0,v)|^2|\eta_0 = i] = 0$ if $\mathscr{P}\{\eta_0 = i\} = 0$, we obtain, based on (22) the equality:

$$\lim_{\tau\to\infty}\frac{1}{\tau}\sum_{t=0}^{\tau}E[|z(t;0,x_0,v)|^2] = \lim_{\tau\to\infty}\frac{1}{\tau}\sum_{t=0}^{\tau}\sum_{i=1}^{N}\pi_0(i)E[|z(t;0,x_0,v)|^2\eta_0 = i] =$$
$$= \sum_{i=1}^{N}\pi_0(i)\lim_{\tau\to\infty}\frac{1}{\tau}\sum_{t=0}^{\tau}E[|z(t;0,x_0,v)|^2|\eta_0 = i] = \tag{37}$$
$$= \sum_{j=1}^{N}\sum_{l=1}^{N}\frac{p_{jl}}{\theta}\sum_{s=0}^{\theta-1}\mu_j^{\pi_0}(s)Tr[B^T(s,j)\tilde{X}(s+1,l)B(s,j)]$$

$(\forall)x_0 \in \mathbf{R}^n$. Taking $x_0 = 0$ in (37) one obtains (via (3)) the first equality in (34). Furthermore (37) shows that the value of the H_2-norm defined by (3) is not influenced by the initial condition x_0 of the trajectory $x(t;0,x_0,v)$ involved in the definition of that norm.

To prove the second equality from (34) let us introduce the notation:

$$\mathscr{B}^{\pi_0}(t) = (B^{\pi_0}(t,1),...,B^{\pi_0}(t,N)) \tag{38}$$

$$B^{\pi_0}(t,l) = \sum_{j=1}^{N} p_{jl}\mu_j^{\pi_0}(t)B(t,j)B^T(t,j), \quad t \in \mathbf{Z}_+, \quad 1 \leq l \leq N.$$

Using (38) and (11), we write:

$$\sum_{l=1}^{N}\sum_{j=1}^{N} p_{jl}\mu_j^{\pi_0}(s)Tr[B^T(s,j)\tilde{X}(s+1,l)B(s,j)] = \tag{39}$$

$$= \sum_{l=1}^{N} Tr[\tilde{X}(s+1,l)B^{\pi_0}(s,l)] = \langle \tilde{\mathbf{X}}(s+1), \mathscr{B}^{\pi_0}(s)\rangle.$$

On the other hand, the linear equation (35) may be written in a compact form:

$$Y^{\pi_0}(t+1) = \mathscr{L}(t)(Y^{\pi_0}(t)) + \mathscr{B}^{\pi_0}(t) \tag{40}$$

and the linear equation (23) can be written in the compact form:

$$\tilde{\mathbf{X}}(t) = \mathscr{L}^*(t)(\tilde{\mathbf{X}}(t+1) + \mathscr{C}(t)) \tag{41}$$

where $\mathscr{C}(t) = (\mathscr{C}(t,1),...,\mathscr{C}(t,N))$, $\mathscr{C}(t,i) = C^T(t,i)C(t,i)$, $1 \leq i \leq N$, $t \in \mathbf{Z}$.

Based on (40) and (41) we may write:

$$\langle \tilde{\mathbf{X}}(s+1), \mathscr{B}^{\pi_0}(s)\rangle = \langle \tilde{\mathbf{X}}(s+1), \mathbf{Y}^{\pi_0}(s+1)\rangle - \langle \tilde{\mathbf{X}}(s+1), \mathscr{L}(s)(\mathbf{Y}^{\pi_0}(s))\rangle =$$

$$= \langle \tilde{\mathbf{X}}(s+1), \mathbf{Y}^{\pi_0}(s+1)\rangle - \langle \mathscr{L}^*(s)(\tilde{\mathbf{X}}(s+1)), \mathbf{Y}^{\pi_0}(s)\rangle =$$

$$= \langle \tilde{\mathbf{X}}(s+1), \mathbf{Y}^{\pi_0}(s+1)\rangle - \langle \tilde{\mathbf{X}}(s), \mathbf{Y}^{\pi_0}(s)\rangle + \langle \mathscr{C}(s), \mathbf{Y}^{\pi_0}(s)\rangle.$$

This allows us to obtain via (39):

$$\sum_{l=1}^{N}\sum_{j=1}^{N} p_{jl}\mu_j^{\pi_0}(s)Tr[B^T(s,j)\tilde{\mathbf{X}}(s+1,l)B(s,j)] =$$

$$= \langle \tilde{\mathbf{X}}(s+1), \mathbf{Y}^{\pi_0}(s+1)\rangle - \langle \tilde{\mathbf{X}}(s), \mathbf{Y}^{\pi_0}(s)\rangle + \langle \mathscr{C}(s), \mathbf{Y}^{\pi_0}(s)\rangle.$$

This leads to

$$\sum_{l=1}^{N}\sum_{j=1}^{N} \frac{p_{jl}}{\theta}\sum_{s=0}^{\theta-1} \mu_j^{\pi_0}(s)Tr[B^T(s,j)\tilde{\mathbf{X}}(s+1,j)B(s,j)] = \frac{1}{\theta}\sum_{s=0}^{\theta-1}\langle \mathscr{C}(s), \mathbf{Y}^{\pi_0}(s)\rangle. \tag{42}$$

To obtain (42) we have used the periodicity property of the solutions $\mathbf{Y}^{\pi_0}(\cdot)$ and $\tilde{\mathbf{X}}(\cdot)$. Invoking again (11) we obtain that (42) coincides with the second equality from (34).

The last equality from (34) follows obviously from (37) and (42). Thus the proof is complete.

Theorem 4.3. *Assume:*

a) the assumptions $\mathbf{H}_1)$, $\mathbf{H}_2)$ *are fulfilled and the system (2) is ESMS.*

b) $\mathscr{P}\{\eta_0 = i\} > 0,\ 1 \leq i \leq N$.

Under these conditions, the value of the H_2*-norm of the system* $\mathbf{G}$ *introduced via (4) is given by*

$$|||\mathbf{G}|||_2^2 = \sum_{l=1}^{N}\sum_{j=1}^{N}\frac{p_{jl}}{\theta}\sum_{s=0}^{\theta-1}\tilde{\mu}_j(s)Tr[B^T(s,j)\tilde{X}(s+1,l)B(s,j)] =$$

$$= \frac{1}{\theta}\sum_{s=0}^{\theta-1}\sum_{i=1}^{N}Tr[C(s,i)\tilde{Y}(s,i)C^T(s,i)] = \lim_{\tau\to\infty}\frac{1}{\tau}\sum_{t=0}^{\tau}\sum_{i=1}^{N}E[|z(t;0,x_0,v)|^2|\eta_0 = i] \tag{43}$$

$(\forall)\ x_0 \in \mathbf{R}^n$, *where* $\tilde{\mathbf{X}}(t) = (\tilde{X}(t,1),...,\tilde{X}(t,N))$ *is the unique* θ*-periodic solution of the discrete-time backward affine equation (23) and* $\tilde{\mathbf{Y}}(t) = (\tilde{Y}(t,1),...,\tilde{Y}(t,N))$ *is the unique* θ*-periodic solution of the discrete-time forward affine equation on* $\mathscr{S}_n^N$,

$$Y(t+1,i) = \mathscr{L}_i(t)(\mathbf{Y}(t)) + \sum_{j=1}^{N}p_{ji}\tilde{\mu}_j(t)B(t,j)B^T(t,j) \tag{44}$$

$1 \leq i \leq N$. *The scalars* $\tilde{\mu}_j(t)$ *are defined by*

$$\tilde{\mu}_j(t) = \sum_{i=1}^{N}\tilde{\mu}_{ij}(t) \tag{45}$$

$1 \leq j \leq N, t \in \mathbf{Z}_+$.

Proof. Under the considered assumptions, the equalities (22) take place for any $i \in \mathfrak{S}$. The first equality from (43) is obtained directly by summing (22) for $i \in \{1,...,N\}$. The other two equalities from (43) are obtained in a similar way as in the proof of Theorem 4.2. The details are omitted.

The last equality from (43) shows that the value of the norm (4) does not change if in (4) the output $z(t;0,x_0,v)$ would be involved.

Remark 4.2. a) In the absence of the assumption b) in the statement of Theorem 4.3 the scalars $\tilde{\mu}_i(t)$ defined via (45) are replaced in (43) by

$$\mu_j^{\mathfrak{S}_0}(t) = \sum_{i\in\mathfrak{S}_0}\tilde{\mu}_{ij}(t), \quad t \in \mathbf{Z}_+. \tag{46}$$

b) Under the assumptions of Theorem 4.3, the value of the norm (4) does not depend upon the initial distribution π_0 of the Markov chain. From (34) and (43) one obtains that $||\mathbf{G}||_2 \leq |||\mathbf{G}|||_2$.

c) If to the assumptions of Theorem 4.1 we add the additional assumption that $\lim_{t\to\infty} P^t$ exists and P is a nondegenerate stochastic matrix we may introduce a new H_2-norm as follows:

$$\tilde{|||}\mathbf{G}\tilde{|||}_2 = \lim_{\tau\to\infty} \sum_{t=\tau}^{\tau+\theta-1} E[|z_0(t)|^2].$$

Using the same technique as in the proofs of Lemma 7.4 and Lemma 7.5 from [16] one obtains that

$$\tilde{|||}\mathbf{G}\tilde{|||}_2 = \|\mathbf{G}\|_2.$$

However, the norm $\tilde{|||}\mathbf{G}\tilde{|||}_2$ can be computed under more restrictive assumptions and that is why we do not use it in our approach.

Let $\mathbf{T}(t,\tau)$ be the linear evolution operator on $\mathcal{S}_n^N$ defined by the discrete-time linear equation

$$\mathbf{X}(t+1) = \mathcal{L}(t)(\mathbf{X}(t)). \tag{47}$$

This means that $\mathbf{T}(t,\tau) = \mathcal{L}(t-1)\mathcal{L}(t-2)\ldots\mathcal{L}(\tau)$ if $t > \tau$ and $\mathbf{T}(t,\tau) = I_{\mathcal{S}_n^N}$ if $t = \tau$, $I_{\mathcal{S}_n^N}$ being the identity operator on the linear space $\mathcal{S}_n^N$. The operator $\mathbf{T}(\theta,0) : \mathcal{S}_n^N \to \mathcal{S}_n^N$ is named the monodromy operator associated to the equation (47).

The eigenvalues of $\mathbf{T}(\theta,0)$ are involved in characterization of the asymptotic stability of the zero state equilibrium of (47). For details, we refer to [1] or [22]. The solution of backward affine equation (23) or equivalently (41) admits the representation $\tilde{\mathbf{X}}(t) = \mathbf{T}^*(\theta,t)(\tilde{\mathbf{X}}(\theta)) + \sum_{s=t}^{\theta-1} \mathbf{T}^*(s,t)\mathcal{C}(s)$, $t < \theta$.

The periodicity condition $\tilde{\mathbf{X}}(0) = \tilde{\mathbf{X}}(\theta)$ is equivalent to the solvability of the linear equation

$$(I_{\mathcal{S}_n^N} - \mathbf{T}^*(\theta,0))(\tilde{\mathbf{X}}(\theta)) = \sum_{s=0}^{\theta-1} \mathbf{T}^*(s,0)(\mathcal{C}(s)). \tag{48}$$

If we take into account that under the assumptions $\mathbf{H}_1$) and $\mathbf{H}_2$) the system (2) is ESMS if and only if

$$\rho[\mathbf{T}(\theta,0)] < 1 \tag{49}$$

we deduce that under the assumption $\mathbf{H}_1$), if the system (2) is ESMS then the operator $I_{\mathcal{S}_n^N} - \mathbf{T}^*(\theta,0)$ is invertible. Hence the equation (48) has a unique solution $\tilde{\mathbf{X}}(\theta)$. The equation

(48) can be viewed as a system on $\hat{n}$ scalar linear equations with $\hat{n}$ scalar unknowns, where $\hat{n} = \frac{Nn(n+1)}{2}$. So, to solve (48) one may use any existing method for systems of algebraic linear equations.

After the computation of $\tilde{\mathbf{X}}(\theta)$ the values $\tilde{\mathbf{X}}(s)$, $1 \le s \le \theta - 1$ are obtained directly from (23).

For the computation of the θ-periodic solution of (35) one proceeds in a similar way.

Example 4.1. Let us consider the linear stochastic system

$$x(t+1) = A_0(t,\eta_t)x(t) + B(t,\eta_t)v(t)$$
$$z(t) = C(t,\eta_t)x(t) \tag{50}$$

where $\{\eta_t\}_{t\ge 0}$ is a homogeneous Markov chain with two states and the transition probability matrix $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. $\{A_0(t,i)\}_{t\in\mathbf{Z}}$, $i = 1,2$ are θ-periodic sequences with $\theta = 2$. Hence, the system (50) is the special case of (1) with $A_k(t,i) = 0$, $1 \le k \le r$, $t,i \in \mathbf{Z} \times \mathfrak{S}$, $N = 2$, $\theta = 2$.

In this case the Lyapunov type operator (12) is defined by $\mathscr{L}(t)(\mathbf{X}) = (\mathscr{L}_1(t)(\mathbf{X}), \mathscr{L}_2(t)(\mathbf{X}))$ with

$$\mathscr{L}_1(t)(\mathbf{X}) = A_0(t,2)X(2)A_0^T(t,2)$$

$$\mathscr{L}_2(t)(\mathbf{X}) = A_0(t,1)X(1)A_0^T(t,1)$$

for all $\mathbf{X} = (X(1), X(2)) \in \mathscr{S}_n^2$.

The corresponding adjoint operator is $\mathscr{L}^*(t)(\mathbf{Y}) = (\mathscr{L}_1^*(t)(\mathbf{Y}), \mathscr{L}_2^*(t)(\mathbf{Y}))$ with

$$\mathscr{L}_1^*(t)(\mathbf{Y}) = A_0^T(t,1)Y(2)A_0(t,1)$$

$$\mathscr{L}_2^*(t)(\mathbf{Y}) = A_0^T(t,2)Y(1)A_0(t,2)$$

for all $\mathbf{Y} = (Y(1), Y(2)) \in \mathscr{S}_n^2$.

The monodromy operator $\mathbf{T}(2,0) = \mathscr{L}(1)\mathscr{L}(0)$ is given by $\mathbf{T}(2,0)(\mathbf{X}) = (\mathbf{T}_1(2,0)(\mathbf{X}), \mathbf{T}_2(2,0)(\mathbf{X}))$ where

$$\mathbf{T}_1(2,0)(\mathbf{X}) = A_0(1,2)A_0(0,1)X(1)A_0^T(0,1)A_0^T(1,2),$$

$$\mathbf{T}_2(2,0)(\mathbf{X}) = A_0(1,1)A_0(0,2)X(2)A_0^T(0,2)A_0^T(1,1).$$

The corresponding adjoint operator is $\mathbf{T}^*(2,0)(\mathbf{X}) = (\mathbf{T}_1^*(2,0)(\mathbf{X}), \mathbf{T}_2^*(2,0)(\mathbf{X}))$ where

$$\mathbf{T}_1^*(2,0)(\mathbf{X}) = A_0^T(0,1)A_0^T(1,2)X(1)A_0(1,2)A_0(0,1)$$

$$\mathbf{T}_2^*(2,0)(\mathbf{X}) = A_0^T(0,2)A_0^T(1,1)X(2)A_0(1,1)A_0(0,2)$$

for all $\mathbf{X} = (X(1), X(2)) \in \mathscr{S}_n^2$. In this case (49) becomes:

$$\rho[\mathbf{T}(2,0)] < 1. \tag{51}$$

Since $\mathbf{T}(2,0)$ is a positive linear operator, (51) is equivalent to the fact that the linear equation

$$\mathbf{X} = \mathbf{T}^*(2,0)(\mathbf{X}) + I_n^2 \tag{52}$$

has a solution $\tilde{\mathbf{X}} = (\tilde{X}(1), \tilde{X}(2))$ with $\tilde{X}(i) > 0$, $i = 1,2$.

Detailing (52) one obtains the system:

$$\tilde{X}(1) = [A_0(1,2)A_0(0,1)]^T \tilde{X}(1)[A_0(1,2)A_0(0,1)] + I_n, \quad \tilde{X}(1) > 0 \tag{53}$$

$$\tilde{X}(2) = [A_0(1,1)A_0(0,2)]^T \tilde{X}(2)[A_0(1,1)A_0(0,2)] + I_n, \quad \tilde{X}(2) > 0. \tag{54}$$

From (51), (53) and (54) we deduce that the linear stochastic system

$$x(t+1) = A_0(t,\eta_t)x(t)$$

is ESMS if and only if

$$max\{\rho[A_0(1,2)A_0(0,1)], \rho[A_0(1,1)A_0(0,2)]\} < 1. \tag{55}$$

The equation (23) corresponding to the system (50) is

$$\begin{aligned} X(t,1) &= A_0^T(t,1)X(t+1,2)A_0(t,1) + C^T(t,1)C(t,1) \\ X(t,2) &= A_0^T(t,2)X(t+1,1)A_0(t,2) + C^T(t,2)C(t,2). \end{aligned} \tag{56}$$

To compute the value $\tilde{\mathbf{X}}(2)$ of the periodic solution of period $\theta = 2$ of the system (56) we need to solve the equations:

$$\begin{aligned} \tilde{X}(2,1) &= [A_0(1,2)A_0(0,1)]^T \tilde{X}(2,1)[A_0(1,2)A_0(0,1)] + \\ &\quad + A_0^T(0,1)C^T(1,2)C(1,2)A_0(0,1) + C^T(0,1)C(0,1) \\ \tilde{X}(2,2) &= [A_0(1,1)A_0(0,2)]^T \tilde{X}(2,2)[A_0(1,1)A_0(0,2)] + \\ &\quad + A_0^T(0,2)C^T(1,1)C(1,1)A_0(0,2) + C^T(0,2)C(0,2). \end{aligned} \tag{57}$$

This is a system of two uncoupled Stein equations. If (55) is fulfilled then (57) has a unique solution $\tilde{\mathbf{X}}(2) = (\tilde{X}(2,1), \tilde{X}(2,2))$. Further, the values of $\tilde{\mathbf{X}}(1) = (\tilde{X}(1,1), \tilde{X}(1,2))$ are obtained directly from (56).

In order to compute the H_2-norms of the system (50) we need to determine the matrices $\tilde{\mathscr{M}}(s)$, $s = 0,1$. To this end, let us remark that $P^{2k} = I_2$ and $P^{2k+1} = P$, $k \in \mathbf{Z}_+$. This shows that $\lim_{t\to\infty} P^t$ does not exist. On the other hand, one obtains that the matrix $Q = \lim_{\tau\to\infty} \frac{1}{\tau} \sum_{t=0}^{\tau} P^t$ is $Q = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$. The matrices $\tilde{\mathscr{M}}(s)$, $s = 0,1$ are computed as in (25) for $\theta = 2$. The matrix $Q(2)$ defined by $Q(2) = \lim_{\tau\to\infty} \frac{1}{\tau} \sum_{k=0}^{\tau} P^{2k}$ is $Q(2) = I_2$. Hence, $\tilde{\mathscr{M}}(0) = I_2$, $\tilde{\mathscr{M}}(1) = P$. This allows us to compute the H_2-norms of the system (50) applying formulae given in Theorem 4.2 and Theorem 4.3, respectively.

V. H_2 OPTIMAL CONTROLLER FOR DISCRETE-TIME LINEAR STOCHASTIC SYSTEMS WITH PERIODIC COEFFICIENTS AND PERFECT STATE MEASUREMENTS.

For each stabilizing controller $\mathbf{G}_c$ we introduce the Lyapunov type operators $\mathscr{L}_{cl}(t) : \mathscr{S}^N_{n+n_c} \to \mathscr{S}^N_{n+n_c}$ obtained via (12) replacing the matrices $A_k(t,i)$ by $A_{kcl}(t,i)$ defined in (8). In the sequel, $\mathscr{L}^*_{cl}(t) : \mathscr{S}^N_{n+n_c} \to \mathscr{S}^N_{n+n_c}$ stands for the adjoint operator of $\mathscr{L}_{cl}(t)$, $t \in \mathbf{Z}$. Let $\tilde{\mathbf{X}}_{cl}(t) = (\tilde{X}_{cl}(t,1), ..., \tilde{X}_{cl}(t,N))$ be the unique θ-periodic solution of the discrete-time backward equation:

$$\mathbf{X}_{cl}(t) = \mathscr{L}^*_{cl}(t)(\mathbf{X}_{cl}(t+1)) + \mathscr{C}_{cl}(t) \tag{58}$$

where $\mathscr{C}_{cl}(t) = (C^T_{cl}(t,1)C_{cl}(t,1), ..., C^T_{cl}(t,N)C_{cl}(t,N)) \in \mathscr{S}^N_{n+n_c}$. Let us introduce the following cost functional $J : \mathscr{K}_s(\mathbf{G}) \to \mathbf{R}_+$, by

$$J(\mathbf{G}_c) = \sum_{l=1}^{N} \sum_{j=1}^{N} \frac{p_{jl}}{\theta} \sum_{s=0}^{\theta-1} \varepsilon_j(s) Tr[B^T_{vcl}(s,j)\tilde{X}_{cl}(s+1,l)B_{vcl}(s,j)] \tag{59}$$

where $\varepsilon_j(s)$, $0 \le s \le \theta-1$, $1 \le j \le N$ are nonnegative given scalars. Let us remark that under the assumptions of Theorem 4.2, we have

$$J(\mathbf{G}_c) = ||\mathbf{G}_{cl}(\mathbf{G}_c)||_2^2 \tag{60}$$

for all $\mathbf{G}_c \in \mathscr{K}_s(\mathbf{G})$, if

$$\varepsilon_j(s) = \mu_j^{\pi_0}(s) \quad 0 \le s \le \theta-1, \quad 1 \le j \le N. \tag{61}$$

If the assumptions of Theorem 4.3 are fulfilled, then

$$J(\mathbf{G}_c) = |||\mathbf{G}_{cl}(\mathbf{G}_c)|||_2^2 \tag{62}$$

for all $\mathbf{G}_c \in \mathscr{K}_s(\mathbf{G})$, if

$$\varepsilon_j(s) = \tilde{\mu}_j(s), \quad 0 \le s \le \theta - 1, \quad 1 \le j \le N. \tag{63}$$

Therefore, (60) and (62) allow us to find the controllers $\tilde{\mathbf{G}}_c$ minimizing one of H_2-norms of the closed-loop system by designing the controller which minimizes the cost (59).

In the developments in this section a crucial role is played by the following system of discrete-time Riccati equations (SDTREs):

$$\begin{aligned} X(t,i) = \sum_{k=0}^{r} A_k^T(t,i)\mathscr{E}_i(\mathbf{X}(t+1))A_k(t,i) - \\ [\sum_{k=0}^{r} A_k^T(t,i)\mathscr{E}_i(\mathbf{X}(t+1))B_k(t,i) + C_z^T(t,i)D_z(t,i)] \cdot \\ [\sum_{k=0}^{r} B_k^T(t,i)\mathscr{E}_i(\mathbf{X}(t+1))B_k(t,i) + D_z^T(t,i)D_z(t,i)]^{-1} \cdot \\ [\sum_{k=0}^{r} B_k^T(t,i)\mathscr{E}_i(\mathbf{X}(t+1))A_k(t,i) + D_z^T(t,i)C_z(t,i)] + C_z^T(t,i)C_z(t,i), \ 1 \le i \le N \end{aligned} \tag{64}$$

with the unknown $\mathbf{X}(t) = (X(t,1),...,X(t,N)) \in \mathscr{S}_n^N$, $t \in \mathbf{Z}$, $\mathscr{E}_i(\cdot)$ being defined as in (14). We recall that a global solution $\{\mathbf{X}_s(t)\}_{t\in\mathbf{Z}}$, $\mathbf{X}_s(t) = (X_s(t,1),...,X_s(t,N))$, of SDTRE (64) is named "stabilizing solution" if the following closed-loop system

$$x(t+1) = [A_0(t,\eta_t) + B_0(t,\eta_t)F_s(t,\eta_t) + \sum_{k=1}^{r} w_k(t)(A_k(t,\eta_t) + B_k(t,\eta_t)F_s(t,\eta_t))]x(t) \tag{65}$$

is ESMS, where

$$\begin{aligned} F_s(t,i) = -[\sum_{k=0}^{r} B_k^T(t,i)\mathscr{E}_i(\mathbf{X}_s(t+1))B_k(t,i) + D_z^T(t,i)D_z(t,i)]^{-1} \\ [\sum_{k=0}^{r} B_k^T(t,i)\mathscr{E}_i(\mathbf{X}_s(t+1))A_k(t,i) + D_z^T(t,i)C_z(t,i)] \end{aligned} \tag{66}$$

$t,i \in \mathbf{Z} \times \mathfrak{S}$. A set of necessary and sufficient conditions for the existence of the bounded and stabilizing solution $\mathbf{X}_s(\cdot)$ of SDTRE (64) satisfying the following sign condition

$$\sum_{k=0}^{r} B_k^T(t,i)\mathscr{E}_i(\mathbf{X}_s(t+1))B_k(t,i) + D_z^T(t,i)D_z(t,i) > 0 \tag{67}$$

for all $t,i \in \mathbf{Z} \times \mathfrak{S}$ may be found in the Appendix.

It is worth mentioning that the bounded stabilizing solution of (64) if it exists is a θ-periodic sequence (see e.g. Theorem 5.5 in [16]).

Let $\mathcal{K}_s^0(\mathbf{G}) \subset \mathcal{K}_s(\mathbf{G})$ be the family of the zero order admissible controllers.

The state space representation of such a controller reduces to the stabilizing feedback law

$$u(t) = D_c(t,\eta_t)x(t) \tag{68}$$

where $\{D_c(t,i)\}_{t\in\mathbf{Z}}$, $1 \le i \le N$ are θ-periodic sequences such that the closed-loop system

$$x(t+1) = [A_0(t,\eta_t) + B_0(t,\eta_t)D_c(t,\eta_t) + \sum_{k=1}^{r} w_k(t)(A_k(t,\eta_t) + B_k(t,\eta_t)D_c(t,\eta_t))]x(t) \tag{69}$$

is ESMS. The corresponding discrete-time backward equation (58) reduces to

$$X(t,i) = \sum_{k=0}^{r}[A_k(t,i) + B_k(t,i)D_c(t,i)]^T \mathcal{E}_i(\mathbf{X}(t+1))[A_k(t,i) + B_k(t,i)D_c(t,i)] + [C_z(t,i) + D_z(t,i)D_c(t,i)]^T [C_z(t,i) + D_z(t,i)D_c(t,i)] \tag{70}$$

$1 \le i \le N$. An example of zero order controller is $\tilde{\mathbf{G}}_c$ having the state space representation given by

$$u(t) = F_s(t,\eta_t)x(t) \tag{71}$$

where $F_s(t,i)$ are introduced via (66). In this case (69) coincides with (65).

The value of the performance (59) achieved by a zero order stabilizing controller $\mathbf{G}_c$ is

$$J(\mathbf{G}_c) = \sum_{l=1}^{N}\sum_{j=1}^{N}\frac{p_{jl}}{\theta}\sum_{s=0}^{\theta-1}\varepsilon_j(s)Tr[B_v^T(s,j)\tilde{X}(s+1,l)B_v(s,j)] \tag{72}$$

where $\tilde{\mathbf{X}}(s) = (\tilde{X}(1,1),...,\tilde{X}(s,N))$ is a unique θ-periodic solution of (70).

This allows us to prove:

Proposition 5.1. *Assume that the SDTRE (64) has a stabilizing solution $\{\mathbf{X}_s(t)\}_{t\in\mathbf{Z}}$ which is θ-periodic and satisfies the sign condition (67). Then the zero order stabilizing controller (71) is well defined and the corresponding performance (59) is given by*

$$J(\tilde{\mathbf{G}}_c) = \sum_{l=1}^{N}\sum_{j=1}^{N}\frac{p_{jl}}{\theta}\sum_{t=0}^{\theta-1}\varepsilon_j(t)Tr[B_v^T(t,j)X_s(t+1,l)B_v(t,j)]. \tag{73}$$

Proof. By direct calculations the SDTRE (64) verified by $\mathbf{X}_s(t)$ may be rewritten in the following form:

$$X_s(t,i)=\sum_{k=0}^{r}[A_k(t,i)+B_k(t,i)F_s(t,i)]^T\mathscr{E}_i(\mathbf{X}_s(t+1))[A_k(t,i)+B_k(t,i)F_s(t,i)]+ \\ +[C_z(t,i)+D_z(t,i)F_s(t,i)]^T[C_z(t,i)+D_z(t,i)F_s(t,i)] \quad (74)$$

$1\leq i\leq N$. Comparing (74) with (70) we conclude that (73) is the special case of (72) for $D_c(t,i)=F_s(t,i)$. Thus the proof is complete.

The main result of this section is:

Theorem 5.2. *Assume: a) the assumptions* $\mathbf{H}_1$) *and* $\mathbf{H}_2$) *are fulfilled;*

b) the SDTRE (64) has a stabilizing solution $\{\mathbf{X}_s(t)\}_{t\in\mathbf{Z}}$ *which is* θ*-periodic and satisfies the sign condition (67).*

Under these conditions we have

$$\min_{\mathbf{G}_c\in\mathscr{K}_s(\mathbf{G})}J(\mathbf{G}_c)=J(\tilde{\mathbf{G}}_c) \quad (75)$$

where $\tilde{\mathbf{G}}_c$ *is the zero order stabilizing controller having the state space representation given by (71).*

Proof. Let $\mathbf{G}_c\in\mathscr{K}_s(\mathbf{G})$ be arbitrary but fixed and let $\tilde{\mathbf{X}}_{cl}(t)=(\tilde{X}_{cl}(t,1),...,\tilde{X}_{cl}(t,N))$ be the unique θ-periodic solution of the corresponding affine equation (58).

Let $\begin{pmatrix}\tilde{X}_{11}(t,i) & \tilde{X}_{12}(t,i)\\ \tilde{X}_{12}^T(t,i) & \tilde{X}_{22}(t,i)\end{pmatrix}$ be the partition of the matrix $\tilde{X}_{cl}(t,i)$ compatible with the coefficients of the closed-loop system (7) given in (8). This means that $\tilde{X}_{11}(t,i)\in\mathscr{S}_n$ and $\tilde{X}_{22}(t,i)\in\mathscr{S}_{n_c}$. On the other hand, the SDTRE (64) verified by the stabilizing solution may be rewritten as

$$X_s(t,i)=\sum_{k=0}^{r}[A_k(t,i)+B_k(t,i)D_c(t,i)]^T\mathscr{E}_i(\mathbf{X}_s(t+1))[A_k(t,i)+B_k(t,i)D_c(t,i)]+ \\ +[C_z(t,i)+D_z(t,i)D_c(t,i)]^T[C_z(t,i)+D_z(t,i)D_c(t,i)]-[F_s(t,i)-D_c(t,i)]^T\cdot \\ \cdot[\sum_{k=0}^{r}B_k^T(t,i)\mathscr{E}_i(\mathbf{X}_s(t+1))B_k(t,i)+D_z^T(t,i)D_z(t,i)][F_s(t,i)-D_c(t,i)] \quad (76)$$

$1\leq i\leq N$, where $D_c(t,i)$ arises in the state space representation (6) of the considered controller

$\mathbf{G}_c$. Set $U(t,i) = \begin{pmatrix} \tilde{X}_{11}(t,i) - X_s(t,i) & \tilde{X}_{12}(t,i) \\ \tilde{X}_{12}^T(t,i) & \tilde{X}_{22}(t,i) \end{pmatrix}$. By direct calculations involving (58) and (76), we conclude that $\mathbf{U}(t) = (U(t,1),...,U(t,N))$ is the θ-periodic solution of the discrete-time backward equation:

$$\mathbf{U}(t) = \mathscr{L}_{cl}^*(t)(\mathbf{U}(t+1)) + \mathbf{H}(t) \tag{77}$$

where $\mathbf{H}(t) = (H(t,1),...,H(t,N))$, $H(t,i) = \Theta^T(t,i)\mathscr{V}(t,i)\Theta(t,i) \in \mathscr{S}_{n+n_c}$,
$\Theta(t,i) = \begin{pmatrix} F_s(t,i) - D_c(t,i) & -C_c(t,i) \end{pmatrix}$, $\mathscr{V}(t,i) = \sum_{k=0}^{r} B_k^T(t,i)\mathscr{E}_i(\mathbf{X}_s(t+1))B_k(t,i) + D_z^T(t,i)D_z(t,i)$.

Based on (67) one obtains that $H(t,i) \geq 0$, $t,i \in \mathbf{Z} \times \mathfrak{S}$. Since, the zero state equilibrium of the linear system (9) is ESMS one obtains via Theorem 3.10 and Theorem 2.5 in [16] that the θ-periodic solution of (77) satisfies

$$U(t,i) \geq 0, \quad \forall t,i \in \mathbf{Z} \times \mathfrak{S}. \tag{78}$$

Further, we write (59) as:

$$J(\mathbf{G}_c) = \sum_{l=1}^{N}\sum_{j=1}^{N}\frac{p_{jl}}{\theta}\sum_{t=0}^{\theta-1}\varepsilon_j(t)Tr[B_v^T(t,j)X_s(t+1,l)B_v(t,j)] + \\ + \sum_{l=1}^{N}\sum_{j=1}^{N}\frac{p_{jl}}{\theta}\sum_{t=0}^{\theta-1}\varepsilon_j(t)Tr[B_{vcl}^T(t,j)U(t+1,l)B_{vcl}(t,j)]. \tag{79}$$

Invoking (78) and 73), (79) yields:

$$J(\mathbf{G}_c) \geq \sum_{l=1}^{N}\sum_{j=1}^{N}\frac{p_{jl}}{\theta}\sum_{t=0}^{\theta-1}\varepsilon_j(t)Tr[B_v^T(t,j)X_s(t+1,l)B_v(t,j)] = J(\tilde{\mathbf{G}}_c). \tag{80}$$

Since $\mathbf{G}_c$ is arbitrary in $\mathscr{K}_s(\mathbf{G})$, (80) confirms the validity of (75). Thus the proof is complete.

Remark 5.1. The controller $\tilde{\mathbf{G}}_c$ which achieves the minimal value of the cost functional (59) does not depend upon the choice of the scalars $\varepsilon_j(t)$. From (60)-(63) and (75) we deduce that the solution of the two H_2-type optimization problems stated in Section 2 is given by the zero order controller (71).

VI. APPENDIX

A. Stabilizing solution of SDTRE (64).

In this section we briefly recall several aspects concerning necessary and sufficient conditions for the existence of the stabilizing solution of SDTRE (64) satisfying the sign condition (67). Details may be found in Chapter 5 in [16].

Firstly, let us consider the controlled system:

$$x(t+1) = [A_0(t,\eta_t) + \sum_{k=1}^{r} w_k(t)A_k(t,\eta_t)]x(t) + [B_0(t,\eta_t) + \sum_{k=1}^{r} w_k(t)B_k(t,\eta_t)]u(t) \quad (81)$$

obtained from (5) for $B_v(t,i) = 0$, $(\forall) \quad t,i \in \mathbf{Z}\times\mathfrak{S}$.

We say that the system (81) is stochastic stabilizable if there exist a θ-periodic matrix valued sequences $\{F(t,i)\}_{t\in\mathbf{Z}}$, $1 \le i \le N$, such that the zero state equilibrium of the closed-loop system

$$x(t+1) = [A_0(t,\eta_t) + B_0(t,\eta_t)F(t,\eta_t) + \sum_{k=1}^{r} w_k(t)(A_k(t,\eta_t) + B_k(t,\eta_t)F(t,\eta_t))]x(t) \quad (82)$$

is ESMS. It is worth mentioning that the concept of stochastic stabilizability introduced above is not restricted asking for θ-periodicity of the stabilizing feedback gains. For details one may view Theorem 5.7 in [16].

Applying Theorem 3.11 in [16] to the system (82) and using the Schur complement technique, one obtains the following criterion for stochastic stabilizability for discrete-time linear stochastic systems of type (81).

Proposition A1. *Under the assumption* $\mathbf{H}_1)$ *if* $\theta \ge 2$ *the following are equivalent:*

(i) the system (81) is stochastic stabilizable;

(ii) there exist matrices $Y(t,i) \in \mathscr{S}_n$, $W(t,i) \in \mathbf{R}^{m\times n}$, $Y(t,i) > 0$, $0 \le t \le \theta - 1$, $1 \le i \le N$ *satisfying the following system of LMIs:*

$$\begin{pmatrix} -Y(t+1,i) & \Psi_0(t,\mathbf{Y}(t),\mathbf{W}(t),i) & \dots & \Psi_r(t,\mathbf{Y}(t),\mathbf{W}(t),i) \\ \Psi_0^T(t,\mathbf{Y}(t),\mathbf{W}(t),i) & -\mathfrak{Y}(t) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ \Psi_r^T(t,\mathbf{Y}(t),\mathbf{W}(t),i) & 0 & \dots & -\mathfrak{Y}(t) \end{pmatrix} < 0 \quad (83)$$

for $0 \le t \le \theta - 1$ *with* $Y(\theta,i) = Y(0,i)$, *where* $\Psi_k(t,\mathbf{Y}(t),\mathbf{W}(t),i) =$

$$\begin{pmatrix} \sqrt{p_{1i}}(A_k(t,1)Y(t,1) + B_k(t,1)W(t,1)) & \dots & \sqrt{p_{Ni}}(A_k(t,N)Y(t,N) + B_k(t,N)(W(t,N))) \end{pmatrix} \in \mathbf{R}^{n\times\tilde{n}}$$

$0 \le t \le \theta - 1$, $0 \le k \le r$, $1 \le i \le N$, $\mathfrak{Y}(t) = diag(Y(t,1),\dots,Y(t,N) \in \mathscr{S}_{\tilde{n}}$, $0 \le t \le \theta - 1$, $\tilde{n} = nN$.

Furthermore, if $\mathbf{Y}(t) = (Y(t,1),\dots,Y(t,N))$, $\mathbf{W}(t) = (W(t,1),\dots,W(t,N))$, $0 \le t \le \theta - 1$ *is a solution of the system of LMIs (83), then*

$$F(t,i) = W(t,i)Y^{-1}(t,i), \quad 0 \le t \le \theta - 1,\ 1 \le i \le N \quad (84)$$

provides a stabilizing feedback gain.

The next result provides a set of necessary and sufficient conditions for the existence of the bounded and stabilizing solution of SDTRE (64) satisfying the sign condition (67). It follows immediately applying the general theory developed in Theorem 5.6 in [16].

Theorem A2. *Under the assumption* $\mathbf{H}_1$) *if* $\theta \geq 2$ *the following are equivalent:*

(i) the SDTRE (64) has a stabilizing solution $\{\mathbf{X}_s(t)\}_{t\in\mathbf{Z}}$ *which is* θ*-periodic and satisfies the sign condition (67).*

(ii) α*) the system (81) is stochastic stabilizable;*

β*) there exist the matrices* $Z(t,i) \in \mathscr{S}_n$, $0 \leq t \leq \theta - 1$, $1 \leq i \leq N$, *solving the following system of LMIs:*

$$\sum_{k=0}^{r} \begin{pmatrix} A_k(t,i) & B_k(t,i) \end{pmatrix}^T \mathscr{E}_i(\mathbf{Z}(t+1)) \begin{pmatrix} A_k(t,i) & B_k(t,i) \end{pmatrix} + \\ + \begin{pmatrix} C_z(t,i) & D_z(t,i) \end{pmatrix}^T \begin{pmatrix} C_z(t,i) & D_z(t,i) \end{pmatrix} - diag(Z(t,i),0) > 0 \tag{85}$$

$0 \leq t \leq \theta - 1$ *with* $\mathbf{Z}(\theta) = \mathbf{Z}(0)$, $1 \leq i \leq N$, $\mathbf{Z}(t+1) = (Z(t+1,1),...,Z(t+1,N))$.

For the numerical computation of the stabilizing and θ-periodic solution $\mathbf{X}_s(t)$ it is sufficient to compute its values $\mathbf{X}_s(t)$ for $0 \leq t \leq \theta - 1$.

To this end, one may use the following algorithm:

Step 0**.** One designs a stabilizing feedback gain $F^0(t) = (F^0(t,1),...,F^0(t,N))$, $0 \leq t \leq \theta - 1$. This stabilizing feedback gain may be computed, via (84), based on a solution of the system of LMIs (83).

Step 1**.** One computes $\mathbf{X}^1(t) = (X^1(t,1),...,X^1(t,N))$, $0 \leq t \leq \theta - 1$ as a solution of the following system of LMIs:

$$\sum_{k=0}^{r} [A_k(t,i) + B_k(t,i)F^0(t,i)]^T \mathscr{E}_i(\mathbf{X}^1(t+1))[A_k(t,i) + B_k(t,i)F^0(t,i)] - \\ -X^1(t,i) + [C_z(t,i) + D_z(t,i)F^0(t,i)]^T \cdot \\ \cdot [C_z(t,i) + D_z(t,i)F^0(t,i)] + \varepsilon I_n \leq 0, \quad 0 \leq t \leq \theta - 2; \tag{86}$$

$$\sum_{k=0}^{r}[A_k(\theta-1,i)+B_k(\theta-1,i)F^0(\theta-1,i)]^T\mathscr{E}_i(\mathbf{X}^1(0))[A_k(\theta-1,i)$$
$$+B_k(\theta-1,i)F^0(\theta-1,i)]-X^1(\theta-1,i)+[C_z(\theta-1,i)+D_z(\theta-1,i)F^0(\theta-1,i)]^T\cdot \tag{87}$$
$$\cdot[C_z(\theta-1,i)+D_z(\theta-1,i)F^0(\theta-1,i)]+\varepsilon I_n\leq 0,\quad 1\leq i\leq N.$$

Compute $F^1(t,i)=-[\sum_{k=0}^{r}B_k^T(t,i)\mathscr{E}_i(\mathbf{X}^1(t+1))B_k(t,i)+D_z^T(t,i)D_z(t,i)]^{-1}[\sum_{k=0}^{r}B_k^T(t,i)\mathscr{E}_i(\mathbf{X}^1(t+1))A_k(t,i)+D_z^T(t,i)C_z(t,i)]$, $0\leq t\leq\theta-1$, $1\leq i\leq N$, with $X^1(\theta,i)=X^1(0,i)$, $1\leq i\leq N$.

Step p. $p\geq 2$. Compute $\mathbf{X}^p(t)=(X^p(t,1),...,X^p(t,N))$, $F^P(t)=(F^P(t,1),...,F^P(t,N))$, $0\leq t\leq\theta-1$ by

$$X^p(t,i)=\sum_{k=0}^{r}[A_k(t,i)+B_k(t,i)F^{p-1}(t,i)]^T\mathscr{E}_i(\mathbf{X}^{p-1}(t+1))\cdot$$
$$\cdot[A_k(t,i)+B_k(t,i)F^{p-1}(t,i)]+[C_z(t,i)+D_z(t,i)F^{p-1}(t,i)]^T\cdot \tag{88}$$
$$\cdot[C_z(t,i)+D_z(t,i)F^{p-1}(t,i)]+\frac{\varepsilon}{p}I_n$$

$0\leq t\leq\theta-1$, with $X^{p-1}(\theta,i)=X^{p-1}(0,i)$, $1\leq i\leq N$,

$$F^p(t,i)=-[\sum_{k=0}^{r}B_k^T(t,i)\mathscr{E}_i(\mathbf{X}^p(t+1))B_k(t,i)+D_z^T(t,i)D_z(t,i)]^{-1}\cdot \tag{89}$$
$$\cdot[\sum_{k=0}^{r}B_k^T(t,i)\mathscr{E}_i(\mathbf{X}^p(t+1))A_k(t,i)+D_z^T(t,i)C_z(t,i)]$$

$0\leq t\leq\theta-1$, with $X^p(\theta,i)=X^p(0,i)$, $1\leq i\leq N$.

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