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Surface Tension Effect on Hele-Shaw Immiscible Displacements.

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The interface between two immisicble displacing fluids in a Hele-Shaw cell is unstable when the displacing fluid is less viscous. The Saffman-Taylor growth constant is *unbounded* in terms of the wavenumbers k when the *surface tension on the interface is zero.* In this paper we consider a middle region between the displacing fluids and *zero surface tension on the interfaces.* We prove that the growth constants corresponding to a suitable *variable* viscosity in the middle region are *bounded* in terms of k and *decreasing* in terms of the middle region length. In the case of successive regions with *constant* viscosities between the initial displacing fluids (the multi-layer Hele-Shaw model), we prove that the corresponding growth constants are *unbounded* in terms of the wave number k if all involved surfaces tensions are zero.

Key words: Hele-Shaw immiscible displacement; Hydrodynamic stability; Zero surface tension.

1. Introduction. Saffman and Taylor (1958) studied the linear stability of the displacement of two immiscible fluids in a Hele-Shaw cell and proved the well know result: if the displacing fluid is less viscous, then the interface becomes unstable and the fingering phenomenon appears. This model is used to study the oil recovery from a porous medium by displacing it with a second fluid (water). The surface tension acting on the interface between the displacing fluids is giving us a cutoff behind the growth constants σ (in time) of perturbations become negative - see the formula (21). If the surface tension on the interface is zero, then the growth constant becomes unbounded in terms of k. Homsy (1987) and Saffman (1980) analyzed many aspects of the fingering phenomena appearing in porous media and Hele-Shaw cells in the presence of a surface tension on the interface between displacing fluids.

Experimental studies proved the useful effect on a forerunner polymer-solute with exponentially decaying viscosity in the secondary oil recovery process - see Slobod and Lestz (1960), Uzoigwe, Scanlon and Jewett (1974), Shah and Schecter (1977).

In the seminal paper of Gorell and Homsy (1983), an optimal policy for minimizing the Saffman - Taylor instability was introduced, based on an intermediate region between water and oil, where a polymer-solute with an unknown viscosity μ exists - a three-layer Hele-Shaw model. An appropriate choice of an increasing μ, between the water viscosity μ_W and oil viscosity μ_O, is giving an important improvement of the displacing process. Numerical experiments confirmed the experimental results mentioned above. The corresponding growth constants are bounded in terms of k, due to the surface tensions existing on the two interfaces: water-middle region and middle region-oil.

Daripa and Pasa (2004, 2005, 2006) and Daripa (2008a, 2008b) studied the multi-layer Hele-Shaw flows, considering a large number of successive regions with constant viscosities between the displacing fluid (water) and oil. They proved that a large enough number of intermediate regions can give an *arbitrary* small (positive) growth constant in time of perturbations. This result holds only when the involved surface tensions are not zero. The effect of variable viscosities in the successive regions between water and oil was studied by Daripa and Ding (2012), also considering some surface tensions on the interfaces; an important improvement of flow stability was obtained.

In this paper we study the displacement of immiscible fluids in a Hele-Shaw cell, *without surface tensions on the interfaces*. In this case the Saffman-Taylor growth constant σ is unbounded in terms of the wave numbers k - see the formula (21) with $T = 0$. We prove that the growth constants, corresponding to a suitable *variable* viscosity in a middle region between the initial displacing fluids, are *bounded* in terms of k. Moreover, σ is *decreasing* in terms of the considered middle region length.

In section 2 we study a three-layer Hele-Shaw flow. We have three fluids in a Hele-Shaw cell: the displacing fluid (water), an intermediate fluid and the displaced fluid (oil). The intermediate fluid is a polymer-solute with an *a priori* variable unknown viscosity, contained in the middle region $(-L, 0)$ - see explanations after formulas (4) - (5). The flow stability is governed by the equation (12) with boundary conditions (19) at the ends of the middle region. The growth constant is given by the ratio (20) with zero surface tensions in $x = -L, x = 0$. The numerator of (20) is containing the sum of the viscosity jumps in $x = -L, x = 0$:

$$k^2U\{[\mu^+(0) - \mu^-(0)]f(0)^2 + [\mu^+(-L) - \mu^-(-L)]f(-L)^2\},$$

where U is the water velocity far upstream and the superscripts $^+, ^-$ denote the right and left limit values - see also explanations after the formula (20). The main point is to manage the above viscosity jumps $(\mu^+ - \mu^-)$. If both this terms are negative or zero, we get the upper bound (27) of σ, *which is not depending on k, even if the surface tensions are zero in* $x = 0,\ x = -L$. In the Figure 1 we plot discontinuous viscosity profiles b), c), d), where the differences between right and left values at the middle region ends are *negative*. The *linear continuous* profile relating the water and oil viscosities, plotted in Figure 1 a), is giving the formula (28) of the *Remark 2* . This upper bound of σ is not depending on k and becomes arbitrary small for a large enough middle region. We compare the continuous linear profile with the continuous exponential and parabolic profiles. Moreover, for convex increasing viscosity profiles in the middle region, we prove that the corresponding upper bound of the growth constant is decreasing in terms of the used amount of polymer.

In section 3 we study the multi-layer Hele-Shaw model with successive regions of constant viscosities between the displacing fluid and oil. The growth constant is given by the formula (38). An estimate of the integrals I_i appearing in the denominator is obtained in *Lemma 1*. Lower estimates of σ in terms of k are given in *Lemma 2* - see the formulas (48)-(49). We prove that the corresponding growth constant is unbounded in terms of the wavenumber k if all involved surfaces tensions are zero - see the formula (50). We conclude in section 4.

2. The three-layer Hele-Shaw cell

. The mathematic model of the Hele-Shaw flow with a variable viscosity in the middle region was described in the seminal paper of Gorell and Homsy (1983). We recall here the basic elements.

The Hele-Shaw plates are parallel with the fix horizontal plane x_1Oy. The displacing fluid (water) with constant viscosity μ_W is moving with the velocity U far upstream, and is displacing the oil with constant viscosity μ_O. Between water and oil we consider a polymer-solute with an *a priori* unknown viscosity μ. We suppose the total amount of polymer is given, then we have

$$\frac{Dc}{dt} = 0, \tag{1}$$

where c is the solute concentration and the "total" derivative appears. The general expression of μ in terms of c, given also in Flory (1953) and Gilje (2008), is

$$\mu(c) = a_0 + a_1 c + a_2 c^2 + ... \tag{2}$$

where a_i are some coefficients which can depend on x_1, y. In the case of a dilute polymer-solute, which is studied here, a linear dependence of μ in terms of c can be considered with a good approximation, therefore μ is invertible in terms of c. Then from (1) it follows

$$\mu_t + u\mu_{x_1} + v\mu_y = 0, \tag{3}$$

where (u, v) are the velocity components and lower indices are denoting the partial derivatives. This equation is the crucial point of the model introduced by Gorell and Homsy (1983), and holds only in the middle region where μ is variable.

The middle region is the interval $Ut - L < x_1 < Ut$, moving with the constant water velocity U far upstream.

In all three regions of the Hele-Shaw cell we have the Darcy law and the continuity equation of the velocity (we consider incompressible fluids):

$$p_{x_1} = -\mu_d u, \quad p_y = -\mu_d v, \quad u_{x_1} + v_y = 0, \tag{4}$$

where

$$\mu_d = \mu_W,\ x_1 < Ut - L; \quad \mu_d = \mu,\ Ut - L < x_1 < Ut; \quad \mu_d = \mu_O,\ x_1 > Ut.$$

The *basic* solution is corresponding to the velocity $u = U, v = 0$ with the straight interfaces $x_1 = Ut - L, \quad x_1 = Ut$. The basic pressure is denoted by P and is continuos, because the curvature of the basic interfaces (appearing in the Laplace law below) is zero. The basic (unknown) viscosity μ in the middle layer is verifying the equation

$$\mu_t + U\mu_{x_1} = 0. \tag{5}$$

Consider the moving reference $x = x_1 - Ut, \tau = t$. The above equation is giving $\mu_\tau = \mu_t = 0$, then our basic viscosity is an unknown function of x. This is a second crucial point of the Gorell and Homsy model. The middle region in the moving frame is the segment $-L < x < 0$.

On the interfaces we consider the *Laplace law:* the pressure jump is given by the surface tension multiplied with the surface curvature and the component u is continuous; moreover, we consider that the interfaces are materials. As $\tau = t$, we denote the time by t.

The perturbations of the basic velocity, pressure and viscosity are denoted by u', v', p', μ'; by inserting it in the equations (3), (4) we get

$$(P + p')_{x_1} = -(\mu + \mu')(U + u'), \quad (P + p')_y = -(\mu + \mu')(v'); \quad (U + u')_{x_1} + v'_y = 0; \tag{6}$$

$$(\mu + \mu')_t + (U + u')(\mu + \mu')_{x_1} + v'(\mu + \mu')_y = 0. \tag{7}$$

We consider a Fourier mode decomposition for the horizontal perturbation:

$$u'(x, y, t) = f(x) \exp(iky + \sigma t) \tag{8}$$

where σ is the growth constant and k are the wavenumbers. As the velocity along the axis Ox is continuous, the above amplitude $f(x)$ is continuous.

2.1. *The equation governing the amplitude f* . From the relations (6), (7) in the moving reference $x = x_1 - Ut$, we obtain

$$p'_x = -\mu u' - \mu' U, \quad p'_y = \mu v', \quad u'_x + v'_y = 0, \tag{9}$$

$$\mu'_t + u'\mu_x = 0, \tag{10}$$

therefore we get

$$v' = (-1/ik) f_x \exp(iky + \sigma t), \quad p' = -(\mu/k^2) f_x \exp(iky + \sigma t), \quad \mu' = (-1/\sigma) f \mu_x. \tag{11}$$

The above expression of p' and the cross derivation of the relations $(9)_1, (9)_2$ are giving the equation which governs the amplitude f inside the middle region:

$$-(\mu f_x)_x + k^2 \mu f = \frac{1}{\sigma} U k^2 f \mu_x. \tag{12}$$

Out of the middle region, the above equation becomes $-f_{xx} + k^2 f = 0$ because the viscosity is constant. The perturbations must decay to zero in the far field, then we have

$$f(x) = \exp[k(x+L)],\ \forall x < -L; \quad f(x) = \exp(-kx),\ \forall x > 0. \tag{13}$$

2.2. *The Laplace law in a point with discontinuous viscosity.* Consider an arbitrary point x_0 where a jump of the viscosity exists. The amplitude f is continuous in this point, but we have a jump of f_x. For using the above mentioned Laplace law in x_0, we need to describe the perturbed interface near x_0, which is denoted by $\eta(y,t)$. The crucial relation is

$$\eta = u_t, \quad \text{then} \quad \eta = (1/\sigma) f, \tag{14}$$

which holds because the interface is a material one. We need to obtain the right and left limit values of the pressure in the point x_0, denoted by $p^+(x_0), \quad p^-(x_0)$. For this we use the basic pressure $P(x_0)$, the Taylor first order expansion of P near x_0 and the above expression $(11)_2$ of p':

$$p^+(x_0) = P^+(x_0) + P^+_x(x_0)\eta + p'^+(x_0) = \tag{15}$$

$$P^+(x_0) + [-\mu^+(x_0)U](1/\sigma) f(x_0) \exp(iky + \sigma t) + [-\mu^+(x_0)/k^2] f^+_x(x_0) \exp(iky + \sigma t),$$

$$p^-(x_0) = P^-(x_0) + P^-_x(x_0)\eta + p'^-(x_0) = \tag{16}$$

$$P^-(x_0) + [-\mu^-(x_0)U](1/\sigma) f(x_0) \exp(iky + \sigma t) + [-\mu^-(x_0)/k^2] f^+_x(x_0) \exp(iky + \sigma t),$$

where $\mu^+(x_0), \quad \mu^-(x_0)$ are the right and left limit viscosity values in x_0.

As we mentioned above, the basic pressure is continuous, because the basic interfaces are straight lines, therefore in the above two relations we have $P^+(x_0) = P^-(x_0)$. On the contrary, a jump of the gradient of the basic pressure exists, giving the terms $-\mu^+(x_0)U$, $-\mu^-(x_0)U$.

We use the following form of the Laplace's law in the point x_0:

$$p^+(x_0) - p^-(x_0) = T(x_0)\eta_{yy}, \tag{17}$$

where $T(x_0)$ is the surface tension acting in the point x_0 and the curvature of the perturbed interface is approximated by η_{yy}. From the relations (14) - (17) it follows

$$-\mu^+(x_0)[\frac{Uf(x_0)}{\sigma} + \frac{f_x^+(x_0)}{k^2}] + \mu^-(x_0)[\frac{Uf(x_0)}{\sigma} + \frac{f_x^-(x_0)}{k^2}] = -\frac{T(x_0)}{\sigma}f(x_0)k^2, \tag{18}$$

$$-\mu^+(x_0)f_x^+(x_0) + \mu^-(x_0)f_x^-(x_0) = \frac{k^2Uf(x_0)[\mu^+(x_0) - \mu^-(x_0)] - T(x_0)k^4}{\sigma}f(x_0). \tag{19}$$

2.3. *The growth constant expression.* We multiply with f in the amplitude equation (12), we integrate on $(-L, 0)$, we use the above boundary conditions (19) in the points $x_0 = -L, \quad x_0 = 0$ and get

$$\sigma = \frac{[k^2U(\mu^+ - \mu^-)_0 - T_0k^4]f_0^2 + [k^2U(\mu^+ - \mu^-)_N - T_Nk^4]f_N^2 + k^2U\int_{-L}^0 \mu_x f^2}{\mu_O k f_0^2 + \mu_W k f_N^2 + \int_{-L}^0 [\mu f_x^2 + k^2\mu f^2]}, \tag{20}$$

where $(\mu^+ - \mu^-)_0, \ (\mu^+ - \mu^-)_N, \ T_0, \ T_N, \ f_0, \ f_N$ are the viscosity jumps, the surface tensions and the amplitude values in the points $x = 0, \ x = -L$. The viscosity jumps in the above expression are given by the difference between the *right and left* limit values: $(\mu^+ - \mu^-)_0 = \mu_O - \mu, \quad (\mu^+ - \mu^-)_N = \mu - \mu_W$.

Remark 1. When the middle region tends to zero and we have a viscosity jump only in the point x_0, from the relations (13) and (19) we recover the Saffman-Taylor formula

$$\sigma_{ST} = \frac{[kU(\mu^+ - \mu^-)_0 - T_0k^3]}{\mu_0^+ + \mu_0^-}, \tag{21}$$

because $f_x^+(0) = -kf(0), \quad f_x^-(0) = kf(0)$. It is possible to see that

$$T_0 = 0, \quad (\mu^+ - \mu^-)_0 > 0 \quad \textit{and} \quad k \to \infty \Rightarrow \sigma_{ST} \to \infty \tag{22}$$

□

The main point of this paper is to prove that a suitable variable viscosity in the middle region is giving us *a bounded growth constant in terms of* k, even if the surface tensions acting at the ends of intermediate region are both zero. For this, we consider the formula (20) and the fourth viscosity profiles, plotted in the Figure 1 a) - c) on the last page. We plotted only *linear* profiles, but an arbitrary shape can be considered, subjected to *negative* jumps of viscosity at the middle region ends. In all fourth cases plotted in the Figure 1 we consider linear increasing viscosities such that

$$(\mu^+ - \mu^-)(x_q) \leq 0, \quad for \quad x_q \in \{-L, \quad 0\}. \tag{23}$$

Therefore we can neglect the viscosity jumps in the numerator. We neglect also the positive values $\mu_O k f_0^2,\ \mu_W k f_N^2,\ \int_{-L}^0 \mu f_x^2$ in the denominator of (20) and obtain the upper estimate (24) below.

$$\sigma \leq \frac{k^2 U \int_{-L}^0 \mu_x f^2}{\int_{-L}^0 [\mu f_x^2 + k^2 \mu f^2]} \leq \frac{k^2 U \int_{-L}^0 \mu_x f^2}{\int_{-L}^0 k^2 \mu f^2} = U \frac{\int_{-L}^0 \mu_x f^2}{\int_{-L}^0 \mu f^2}. \tag{24}$$

In some previous paper was used an approximation formula for the integrals appearing in the ratio of the right hand side of the above inequality. For example, if we use the rectangle rule for approximating integrals and the interior points $x_i = -iL/N, \quad i = 0, 1, .., N$ with N a large enough natural number, we obtain

$$\frac{\int_{-L}^0 \mu_x f^2}{\int_{-L}^0 \mu f^2} \approx \frac{\sum_{i-0}^{i=N} (\mu_x)_i f_i^2}{\sum_{i-0}^{i=N} \mu_i f_i^2}, \tag{25}$$

where $(\mu_x)_i = \mu_x(x_i), \quad \mu_i = \mu(x_i), f_i = f(x_i)$. We can use the inequality (53) proved in the section 3 and get

$$\sigma \leq U Max_i \{\frac{(\mu_x)_i}{\mu_i}\} \approx U Max_{x \in (-L,0)} \{\frac{\mu_x}{\mu}\}. \tag{26}$$

However, the error of the rectangle formula (and others approximation method) are depending on the derivative of the functions μ_x, f, μ in some points contained in the middle region. As we don't known the expressions of these functions, we can't give an accurate estimate of the error. Moreover, if the function f contains a combination of exponentials, the ratio of the derivatives of f can be unbounded in terms of k - see *Remark 4*. Therefore this method is not useful to obtain a bounded estimate estimate of σ in terms of k.

Consider μ_{inf} the smallest (positive) value of μ in the intermediate region, which can be less than μ_W - see Figure 1 c), d). As the viscosity profile is increasing we have $\mu_x > 0$, and from the above formula we get the folowing upper bound of the growth constant:

$$\sigma \leq U \frac{Max_x \{\mu_x\}}{\mu_{inf}}. \tag{27}$$

We can see that the above upper bound is not depending on the *maximum value* of the viscosity, but only on the *maximum value of his derivative* and on μ_{inf}.

Remark 2. The simplest case of a *linear* continuous viscosity profile relating μ_W and μ_O (see Figure 1 a) and the point i) in subsection 2.4 below) is giving the estimate

$$\sigma \leq U\frac{\mu_O - \mu_W}{L\mu_W}, \tag{28}$$

therefore a large middle region (that means a large quantity of polymer-solute) can give us an arbitrary small growth constant, even if the surface tensions in $x = -L, \quad x = 0$ are zero □

2.4. *Comparison between some continuous convex increasing viscosity profiles in the middle region.* In this case, the maximum value of the viscosity derivative is obtained in $x = 0$ and the fromula (27) is giving the estimate

$$\sigma \leq U\frac{\mu_x(0)}{\mu_W} \tag{29}$$

if the the smallest value of μ in the middle region is μ_W.

We consider the following three profiles relating the water and oil viscosities.

i) The linear continuous profile:

$$\mu(x) = \frac{\mu_O - \mu_W}{L}x + \mu_O, \quad \mu(-L) = \mu_W, \quad \mu(0) = \mu_O, \tag{30}$$

$$\mu_x(0) = \frac{\mu_O - \mu_W}{L}, \; I(\mu) = \int_{-L}^{0} \mu(x) = \frac{(\mu_O + \mu_W)L}{2}.$$

ii) The parabolic continuous profile:

$$\zeta(x) = ax(x + L) + \frac{\mu_O - \mu_W}{L}x + \mu_O, \quad \zeta(-L) = \mu_W, \quad \zeta(0) = \mu_O, \tag{31}$$

$$\zeta_x(0) = aL + \frac{\mu_O - \mu_W}{L}, \quad I(\zeta) = -\frac{aL^3}{6} + \frac{(\mu_O + \mu_W)L}{2}; \quad 0 < a < \frac{3(\mu_O + \mu_W)}{L^2}.$$

iii) The exponential continuous profile:

$$\eta(x) = \exp[\frac{ln(\mu_O/\mu_W)}{L}x + \ln\mu_O], \quad \eta(-L) = \mu_W, \quad \eta(0) = \mu_O, \tag{32}$$

$$\eta_x(0) = \mu_O\frac{ln(\mu_O/\mu_W)}{L}, \; I(\eta) = \frac{L(\mu_O - \mu_W)}{ln(\mu_O/\mu_W)}.$$

From the above three relations it follows

$$I(\zeta) < I(\mu), \; \zeta_x(0) \geq \mu_x(0); \quad I(\eta) < I(\mu), \; \eta_x(0) \geq \mu_x(0); \tag{33}$$

that means the upper bound of σ decreases with increasing of total amount of polymer. The inequalities $(33)_3$, $(33)_4$, can be obtained as follows. We consider $\mu_O = (1+\alpha)\mu_W, \quad \alpha > 1$, then we have to prove

$$\frac{\alpha}{ln(1+\alpha)} < \frac{2+\alpha}{2}, \quad (1+\alpha)ln(1+\alpha) \geq \alpha. \tag{34}$$

For $(34)_1$ we prove the stronger inequality

$$\frac{\alpha}{ln(1+\alpha)} < \frac{1+\alpha}{2} \tag{35}$$

as follows. Consider $F(x) = ln(1+x) - 2x/(x+1)$, then $F'(x) \geq 0, \quad \forall x \geq 1$. As $F(0) = 0$ we get $(34)_1$. Then we obtained $(x+1)ln(x+1) > 2x$ which gives $(34)_2$.

We prove here a more general relation, *verified by positive functions with equal values at the ends of* $(-L, 0)$:

$$G(x) < H(x), \ \forall x \in (-L, 0) \Rightarrow G_x(0) \geq H_x(0). \tag{36}$$

For this, we consider $z < 0$ very close to 0. We have

$$G(z) < H(z), \quad G(z) - G(0) < H(z) - H(0), \quad \frac{G(z) - G(0)}{z} > \frac{H(z) - H(0)}{z}.$$

When $z \to 0$ we get the relation (36). The physical meaning of this relation is quite similar with the property mentioned after the relations (33). If G, H are denoting the viscosities of two kind of polymer-solute contained in the middle region, then the upper bound (27) is *decreasing* in terms of the total amount of polymer in the middle region.

From the relations (30), (31), (32) we obtain an important result: only the linear and exponential profiles are giving upper bounds (27) *decreasing* to zero for very large L.

3. The multi-layer Hele-Shaw cell. This model is based on a successive small intermediate regions between displacing and displaced fluids. A natural choice is to consider constant viscosities in each region. Daripa (2008a, 2008b), proved that if some surface tensions T_i are acting on the viscosity jumps points x_i, then a very large number of small intermediate regions can give us an arbitrary (positive) small growth constant σ. For this, T_i must verify some conditions.

As we mentioned above, in this paper we consider $T_i = 0, \ \forall i$. We divide the middle region in small intervals (a_i, b_i) of length (L/N). On each small interval we introduce the intermediate constant viscosities

$$\mu_i = \mu_W + (N-i)(\mu_O - \mu_W)/N = \mu_O - i(\mu_O - \mu_W)/N, \quad \mu_0 = \mu_O, \quad \mu_N = \mu_W. \tag{37}$$

We multiply the amplitude equation (12) on each interval (a_i, b_i) with f, we use the boundary conditions (19) in the jump points of the viscosity (with all surface tensions zero), we integrate and get the following formula of the growth constant (see also Daripa(2008a)):

$$\sigma = \frac{\sum_{i=0}^{i=N} k^2 U(\mu^+ - \mu^-)_i f_i^2}{k\mu_W f_N^2 + k\mu_O f_0^2 + \sum_{i=0}^{i=N} I_i}, \quad I_i = \int_{a_i}^{b_i} \mu_i (f_x^2 + k^2 f^2) \tag{38}$$

where $f_i = f(x_i), \quad x_i = -L + (N-i)L/N, \quad a_i = x_{i+1}, \quad b_i = x_i.$

In the following we prove that if all surface tensions are zero and *if the viscosity is increasing form the displacing fluid to the displaced one*, then the above multi-layer constant viscosities method is giving an unbounded growth constant in terms of the wavenumber k. For this, we consider the following integral

$$I(a,c) = \int_a^c \{f_x^2 + k^2 f^2\} \tag{39}$$

and analyze his magnitude in terms of $k, f^2(a), f^2(c)$.

Lemma 1.

$$k\frac{e^{k(c-a)} - 1}{e^{k(c-a)} + 1}[f^2(a) + f^2(c)] \leq I(a,c) \leq \quad k\frac{e^{k(c-a)} + 1}{e^{k(c-a)} - 1}[f^2(a) + f^2(c)]. \tag{40}$$

Proof. If all intermediate viscosities are constant, the amplitude equation (12) becomes

$$-f_{xx} + k^2 f = 0 \tag{41}$$

in all intermediate region, except the jump points of the viscosity. Therefore on an small interval (a,c) contained in $(-L,0)$, the amplitude expression is

$$f(x) = Ae^{kx} + Be^{-kx},$$

where A, B are constant in terms of x. We multiply (41) with f and integrate on $(-L,0)$, then we have

$$I = \int_a^c (f_x f)_x dx = (f_x f)(c) - (f_x f)(a) = \tag{42}$$

$$k(Ae^{k(c)} - Be^{-k(c)})(Ae^{k(c)} + Be^{-k(c)}) - k(Ae^{ka} - Be^{-ka})(Ae^{ka} + Be^{-ka}) =$$

$$k[A^2e^{2kc} - B^2e^{-2kc} - A^2e^{2ka} + B^2e^{-2ka}] = k[A^2(e^{2kc} - e^{2ka}) + B^2(e^{-2ka} - e^{-2kc}] =$$

$$k\frac{e^{2kc} - e^{2ka}}{e^{2k(a+c)}}[A^2e^{2k(a+c)} + B^2].$$

We use the notation $f_0 = f(a), \quad f_1 = f(c)$ and get

$$A = \frac{f_0 e^{ka} - f_1 e^{kc}}{e^{2ka} - e^{2kc}}, \quad B = \frac{-f_0 e^{kc} + f_1 e^{ka}}{e^{2ka} - e^{2kc}} e^{k(a+c)},$$

therefore

$$D = A^2 e^{2k(a+c)} + B^2 =$$

$$[\frac{1}{e^{2ka} - e^{2kc}}]^2 e^{2k(a+c)} [f_0^2(e^{2ka} + e^{2kc}) + f_1^2(e^{2ka} + e^{2kc}) - 4 f_0 f_1 e^{k(a+c)}]. \tag{43}$$

In the above expression, we add and substract $2(f_0^2 + f_1^2)e^{k(a+c)}$, then it follows

$$D = [\frac{1}{e^{2ka} - e^{2kc}}]^2 e^{2k(a+c)} [(f_0^2 + f_1^2)(e^{ka} - e^{kc})^2 + 2(f_0^2 + f_1^2 - 2f_0 f_1) e^{k(a+c)}],$$

$$D \geq [\frac{1}{e^{2ka} - e^{2kc}}]^2 e^{2k(a+c)} (f_0^2 + f_1^2)(e^{ka} - e^{kc})^2. \tag{44}$$

On the other hand we have

$$-4 f_0 f_1 e^{k(a+c)} \leq 2(f_0^2 + f_1^2) e^{k(a+c)}$$

and by using this inequality in (43) we get

$$D \leq [\frac{1}{e^{2ka} - e^{2kc}}]^2 e^{2k(a+c)} (f_0^2 + f_1^2)(e^{ka} + e^{kc})^2. \tag{45}$$

The last part of the formula (30) and the expressions (44), (45) are giving

$$\frac{e^{2kc} - e^{2ka}}{e^{2k(a+c)}} [\frac{1}{e^{2ka} - e^{2kc}}]^2 e^{2k(a+c)} (e^{ka} - e^{kc})^2 = \frac{e^{kc} - e^{ka}}{e^{kc} + e^{ka}},$$

$$\frac{e^{2kc} - e^{2ka}}{e^{2k(a+c)}} [\frac{1}{e^{2ka} - e^{2kc}}]^2 e^{2k(a+c)} (e^{ka} + e^{kc})^2 = \frac{e^{kc} + e^{ka}}{e^{kc} - e^{ka}},$$

and we obtain the inequality (40).

□

Lemma 2. For large values of k

$$\exists H = H(N, \ \mu_O, \ \mu_W) \quad s.t. \qquad \sigma \geq kH. \tag{46}$$

Proof. From *Lemma 1* we get

$$\int_{a_i}^{b_i} (f_x^2 + k^2 f^2) \leq k\Theta(k)[f_{i+1}^2 + f_i^2], \quad \Theta(k) = \frac{e^{kL/N} + 1}{e^{kL/N} - 1}, \tag{47}$$

therefore the above formula (38) is giving

$$\sigma \geq \frac{\sum_{i=0}^{i=N} k^2 U([\mu_O - \mu_W]/N) f_i^2}{k[\mu_O + \Theta(k)] f_0^2 + k \sum_{i=1}^{i=N-1} \Theta(k)[\mu_{i+1} + \mu_i] f_i^2 + k[\mu_W + \Theta(k)] f_N^2}. \tag{48}$$

We use the above lower bound, the inequality

$$B_i, x_i > 0 \Rightarrow \frac{\sum A_i x_i}{\sum B_i x_i} \geq Min_i \frac{A_i}{B_i}$$

and get

$$\sigma \geq k \frac{\mu_O - \mu_W}{N} Min_i \{ \frac{1}{\mu_O + \mu_1 \Theta(k)}, \quad \frac{1}{\Theta(k)[\mu_{i+1} + \mu_i]}, \quad \frac{1}{\mu_{N-1}\Theta(k) + \mu_W} \} = \tag{49}$$

$$k \frac{\mu_O - \mu_W}{N} \times \frac{1}{\mu_O + \mu_1 \Theta(k)}.$$

For *large values* of k we have $\Theta(k) \rightarrow 1$, therefore we finnally obtain

$$k \rightarrow \infty \Rightarrow \sigma \geq k H(N, \; \mu_O, \; \mu_W), \quad H(N, \; \mu_O, \; \mu_W) = \frac{\mu_O - \mu_W}{(2N-1)\mu_O + \mu_W}. \tag{50}$$

□

Remark 3. Recall the multi-layer Hele-Shaw model with constant successive intermediate viscosities and surface tensions $T_i \neq 0$ on each interface. A suitable choice of the intermediate viscosities and a very large number of intermediate regions can give us an arbitrary small growth constant *without increasing the total length L of the region between water and oil.* Indeed, in this case we have the formula

$$\sigma = \frac{\sum_{i=0}^{i=N} [k^2 U(\mu^+ - \mu^-)_i - k^4 T_i] f_i^2}{k \mu_W f_N^2 + k \mu_O f_0^2 + \sum_{i=0}^{i=N} \int_{a_i}^{b_i} \mu_i (f_x^2 + k^2 f^2)}, \tag{51}$$

given in Daripa (2008b).

From *Lemma 1* we see that $\sigma \rightarrow 0$ for very small wavenumbres k.

In the following we study the case of large k. We consider the N intermediate layers of the lenght L/N given in the relation (37). We use *Lemma 1* and for large k we get

$$\sigma \leq \frac{\sum_{i=0}^{i=N} [k U(\mu^+ - \mu^-)_i - k^3 T_i] f_i^2}{\mu_W f_N^2 + \mu_O f_0^2 + \sum_{i=1}^{i=N} \Lambda(k)(f_{i-1}^2 + f_i^2)}, \quad \Lambda(k) = \frac{e^{kL/N} - 1}{e^{kL/N} + 1}. \tag{52}$$

Consider $(\mu^+ - \mu^-)_i = (\mu_O - \mu_W)/N$. The function $F(k) = [kU(\mu_O - \mu_W)/N - k^3 T_i]$ has the maximum value

$$Max_k F = \frac{2}{3\sqrt{3 T_{max}}} [\frac{U(\mu_O - \mu_W)}{N}]^{3/2}.$$

The following inequality holds

$$B_i, x_i > 0 \Rightarrow \frac{\sum A_i x_i}{\sum B_i x_i} \leq Max_i \frac{A_i}{B_i}. \quad (53)$$

As $\Lambda(k) \to 1$ for large k, from the last three above relations it follows

$$lim_{k\to\infty}\sigma \leq \frac{2}{3\sqrt{3T_{max}}}[\frac{U(\mu_O - \mu_W)}{N}]^{3/2} \times Max_i\{\frac{1}{\mu_W + \mu_{N-1}}, \frac{1}{\mu_{i+1} + \mu i}, \frac{1}{\mu_1 + \mu_O}\}, \quad (54)$$

$$\mu_i = \mu_O - i\frac{\mu_0 - \mu_W}{N}.$$

Therefore for large k we have

$$\sigma(k \to \infty) \leq O(\frac{1}{\sqrt{NT_{max}}})$$

and a large enough N is giving an arbitrary small σ, *independent of the lenght* L □

Remark 4. A relation similar with (28) could be obtained for the multi-layer equal constant viscosities Hele-Shaw model, but not useful for our purposes. We have

$$\sum_{i=0}^{i=N}(\mu^+ - \mu^-)_i f_i^2 = \sum_{i=0}^{i=N}\frac{\mu_O - \mu_W}{N}f_i^2 = \frac{\mu_O - \mu_W}{L}\sum_{i=0}^{i=N}\frac{L}{N}f_i^2. \quad (55)$$

If N is large enough, we can consider the expression $\sum_{i=0}^{i=N}\frac{L}{N}f_i^2$ as an approximation of $\int_{-L}^{0} f^2$ by the rectangle rule. It is well known that

$$\exists\theta \in (-L, 0), \quad \int_{-L}^{0} f^2 = \sum_{i=0}^{i=N}\frac{L}{N}f_i^2 - Lf'(\theta),$$

therefore the formulas (38) and (55)are giving

$$\sigma \leq U\{\frac{\mu_O - \mu_W}{L}\}\frac{\int_{-L}^{0} f^2 + Lf'(\theta)}{\mu_W \int_{-L}^{0} f^2} = U\{\frac{\mu_O - \mu_W}{\mu_W L}\}\{1 + \frac{Lf'(\theta)}{\int_{-L}^{0} f^2}\} \quad (56)$$

The above estimate is not useful *for large* k, because f is containing the exponential in terms of k, therefore the ratio $f'(\theta)/\int_{-L}^{0} f^2$ appearing in the right hand side of (56) is not bounded in terms of k. Then the rectangle rule for approximate $\int_{-L}^{0} f^2$ can't give a bounded upper estimate of σ in terms of k, even if the formulas (28) and (56) are similar.

4. Conclusions. In the seminal paper of Saffman and Taylor (1958) was studied the linear stability of the displacement of two immiscible fluids in a Hele-Shaw cell and the fingering phenomenon was proved: the interface between fluids is unstable if the displacing fluid is less viscous. A surface tension on the interface gives a cutoff behind all the growth constants (in time) of perturbations are negative. If the surface tension is zero on the interface, then the Saffman-Taylor growth constant in unbounded in terms of the wave numbers k - see the formula (21).

Gorell and Homsy (1983) introduced an intermediate region between displacing and displaced fluids, where an unknown viscosity is considered as a parameter, for minimizing the Saffman - Taylor instability. An optimal numerical viscosity profile in the middle region was obtained, in accord with previous experimental results. This model was generalized by Daripa and Pasa (2004, 2005, 2006) and Daripa (2008a, 2008b), who introduced the multi-layer model, based on a successive constant or variable viscosities between the initial displacing fluids. When a surface tension exists on the interfaces, this model is giving an important stability improvement.

We study here the displacements of immiscible fluids in horizontal three-layer and multi-layer Hele-Shaw cells *with zero surface tension on all interfaces.* The main results are:

1) A suitable *variable* viscosity μ in the middle region of a three-layer Hele-Shaw cell is giving us a *bounded* growth constant in terms of k - see the formula (27) in *Remark 2*. The main point is to manage the jump viscosities at the ends of the middle region $(-L, 0)$ appearing in the formula (20). We consider *negative viscosity jumps*: $\mu_O - \mu(0) \leq 0,\ \mu(-L) - \mu_W \leq 0$, where $\mu(0),\ \mu(-L)$ are the limit viscosity values in $0, -L$ and $\mu_W,\ \mu_O$ are the water and oil viscosities. The terms containing negative jumps become zero when $k = 0$, then we can't obtain a negative growth constant by using it. In section 2.4, three continuous *convex* increasing profiles relating the water and oil viscosities are analyzed: linear, parabolic and exponential. Only the linear and exponential profiles are giving an arbitrary small growth constant *when middle region length L is very large* - see the formulas (30), (31), (32). We prove also an important conclusion: an increasing amount of polymer is giving a decreasing upper bound (27) of the growth constant - see explanations for the relation (36).

2) In the case of the multi-layer Hele-Shaw model with constant viscosities, we prove that the corresponding growth constant is unbounded for large wave numbers. For this we estimate the integrals appearing in the denominator of the formula (38) - see *Lemma 1* - and get the lower estimates (48)- (49) (see *Lemma 2*).

In the *Remark 3* we briefly recall that if some surface tensions are acting on all intermediate interfaces, then the obtained growth constant can be arbitrary small when the number of intermediate regions is large enough. Moreover, this stability improvement is obtained *independent* of the intermediate regions length.

In conclusion, we proved that in the three-layer case, a large enough middle region (that means a large enough amount of polymer) and a suitable *variable* viscosity profile is giving an arbitrary small growth constant, *independent* of the wave number k, *even if the surfaces tensions on both interfaces are zero.* On the contrary, the constant multi-layer Hele-Shaw model is giving an unbounded growth constant in terms of k, if all involved surface tensions are zero.

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FIGURES

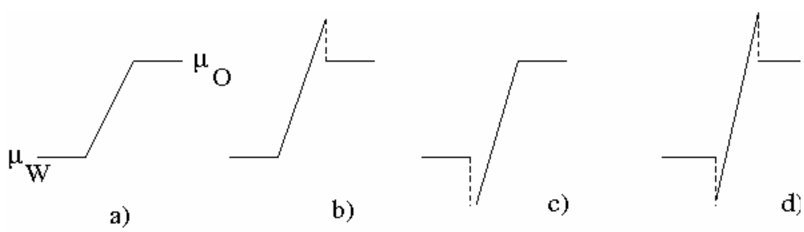

Figure 1: a) continuous linear viscosity between water and oil; b), c), d): discontinuous linear viscosities with *negative* jumps in $x = 0$ or (and) $x = -L$.