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Serban A. Basarab: Simion Stoilow Institute of Mathematics of the Romanian Academy, P.O. Box 1-764, RO-014700
Bucharest, ROMANIA
E-mail: serban.basarab@imar.ro

Embedding theorems for median-free groups

Şerban A. Basarab

To the memory of my parents, Ileana and Alexandru Basarab

Abstract

Using suitable deformations of simplicial trees and the duality theory for median sets, we show that every free action on a median set can be extended to a free and transitive one. We also prove that the category of median groups is a reflective full subcategory of the category of free actions on pointed median sets.

Keywords. free action, transitive action, transitive closure, simplicial tree, median set (algebra), median group, folding, free group, free product, spectral space, coherent map, distributive lattice with negation

1 Introduction

Let $\mathbf{K}$ be a nonempty class of mathematical structures closed under isomorphisms. We denote by $\mathbf{KFG}$ the class of $\mathbf{K}$-*free groups* consisting of the groups G for which there exists $\mathbb{X} \in \mathbf{K}$ such that G acts freely on $\mathbb{X}$. Let $\mathbf{KFTG}$ be the subclass of $\mathbf{KFG}$ consisting of the groups G for which there exists $\mathbb{X} \in \mathbf{K}$ such that G acts freely and transitively on $\mathbb{X}$. Consider also the class $\mathbf{KFTG}_\forall$ whose members are the groups which can be embedded in groups belonging to $\mathbf{KFTG}$. The inclusions

$$\mathbf{KFTG} \subseteq \mathbf{KFTG}_\forall \subseteq \mathbf{KFG}$$

are obvious. It is natural to look for classes $\mathbf{K}$ for which some of the inclusions above become equalities. Thus, it is a remarkable fact that, taking $\mathbf{K}$ the class of $\mathbb{Z}$-trees and assuming, in addition, that the involved free actions are without inversions, the three classes above coincide with the class of free groups. On the other hand, taking $\mathbf{K}$ the class of $\mathbb{R}$-trees, it follows from cardinality reasons that the inclusion $\mathbf{KFTG} \subseteq \mathbf{KFTG}_\forall$ is proper, while the equality $\mathbf{KFTG}_\forall = \mathbf{KFG}$ holds according to the following more general result due to I. Chiswell and T. Müller

Şerban A. Basarab: Simion Stoilow Institute of Mathematics of the Romanian Academy, P.O. Box 1-764, RO-014700 Bucharest, ROMANIA; e-mail: serban.basarab@imar.ro

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Theorem 1.1. ([19, Theorem 5.4.]) *Let G be a group acting freely and without inversions on a Λ-tree X, where Λ is a totally ordered abelian group. Then there exists a group $\widehat{G}$ acting freely, without inversions, and transitively on a Λ-tree $\widehat{X}$, together with a group embedding $G \to \widehat{G}$ and a G-equivariant Λ-isometry $X \to \widehat{X}$.*

In the proof of this result, the authors use the connection between Lyndon length functions and actions on Λ-trees, as well as string-rewriting systems techniques.

One of the main goals of the present work, a slightly revised version of [12], is to prove an analogue (see Theorem 1.6) of Theorem 1.1 for free actions on more general arboreal structures, namely median sets (algebras). The method of proof, different from that used in [19], is based on a procedure of deformation of ordinary trees into median sets and the duality theory for median sets developed in [4]. The results of the present work will be used in a forthcoming paper to extend Theorem 1.1 to free actions on Λ-trees (cf. [5, 1.3], [13, Definition 3.4]), where Λ is an arbitrary abelian l-group, not necessarily totally ordered.

In order to state the main results, we introduce (recall) some basic notions. More details will be given in Section 2 having a preliminary character.

Definition 1.2. By a *median set* (or *median algebra* [1]), we understand a set X together with a ternary operation $m : X^3 \longrightarrow X$ satisfying the following three equational axioms.

(M 1) *Symmetry*: $m(x, y, z) = m(y, x, z) = m(x, z, y)$

(M 2) *Absorptive law*: $m(x, y, x) = x$

(M 3) *Selfdistributive law*: $m(m(x, y, z), u, v) = m(m(x, u, v), m(y, u, v), z)$

The element $m(x, y, z)$ is called the *median of the triple* (x, y, z). Note that (M 3) can be replaced by

$$(\mathrm{M}\,3')\ m(m(x, u, v), m(y, u, v), x) = m(x, u, v).$$

In particular, taking $u = y, v = z$, we obtain $m(x, y, m(x, y, z)) = m(x, y, z)$ for all $x, y, z \in X$.

For $x, y \in X$, we denote

$$[x, y] := \{z \in X \mid m(x, y, z) = z\} = \{m(x, y, z) \mid z \in X\}.$$

Note that $[x, y] = [y, x]$, and $u, v \in [x, y] \Longrightarrow [u, v] \subseteq [x, y]$.

Remark 1.3. (1) Any distributive lattice $(L, \leq, \wedge, \vee)$, not necessarily bounded, has a canonical underlying structure of median set with the median operation

$$m(x, y, z) := (x \wedge y) \vee (y \wedge z) \vee (z \wedge x) = (x \vee y) \wedge (y \vee z) \wedge (z \vee x), \tag{1.1}$$

and $[x, y] = [x \wedge y, x \vee y] := \{z \in L \mid x \wedge y \leq z \leq y\}$ for $x, y \in L$.

(2) Any Λ-tree (cf. [5, 1.3], [13, Definition 3.4]) $(X, d : X \times X \longrightarrow \Lambda)$, where Λ is an abelian l-group, has an underlying structure of median set, and $[x, y] = \{z \in X \mid d(x, z) + d(z, y) = d(x, y)\}$ for $x, y \in X$.

Definition 1.4. Let $\mathbb{X} = (X, m)$ be a median set.

(1) By a *median subset* of $\mathbb{X}$ we mean a subset $A \subseteq X$ closed under median operation m.

(2) A subset $A \subseteq X$ is said to be *convex* if $[x, y] \subseteq A$ for all $x, y \in A$. In particular, a convex subset is median, and $[x, y]$ is convex for all $x, y \in X$.

(3) A nonempty convex subset $A \subseteq X$ is *retractible* if for all $x \in X$ there exists (uniquely) $\psi(x) \in A$ such that $[x, a] \cap A = [\psi(x), a]$ for all (for some) $a \in A$; call the map $\psi : X \longrightarrow X$, with $\psi(X) = A$, the *folding* associated to the retractible convex subset A.

(4) $\mathbb{X}$ is said to be *locally linear* if $[x, y] = [x, z] \cup [z, y]$ for all $x, y \in X, z \in [x, y]$.

(5) $\mathbb{X}$ is called *simplicial* (or *discrete*) if $[x, y]$ is finite for all $x, y \in X$.

Definition 1.5. By a *median group* [6, 2.2.] we understand a group G together with a ternary operation $m : G^3 \longrightarrow G$ making G a median set on which the group G acts freely and transitively from the left, i.e., $u\,m(x, y, z) = m(ux, uy, uz)$ for all $u, x, y, z \in G$.

The median group (G, m) is called *locally linear* (*simplicial*) if its underlying median set is locally linear (simplicial).

Remark. Though median sets could seem too general to be considered genuinely "treelike", they are, however, very well suited for the study of various natural arboreal and metric structures on groups and rings; see for instance [3], [5] - [10], [13], [14], [16], [18]. Moreover the median sets and the median groups form proper subclasses of the larger class of *connected median groupoids of groups* introduced in [8, 9] as a frame for an extension of the Bass-Serre theory [23] to actions on more general arboreal structures.

We denote by **AMS** the category of actions on median sets. The objects of **AMS** are systems $(G, \mathbb{X})$, where G is a group together with an action from the left $G \times \mathbb{X} \longrightarrow \mathbb{X}$, $(g, x) \mapsto g \cdot x$ on a nonempty median set $\mathbb{X} = (X, m)$, while we take as morphisms $(G, \mathbb{X}) \longrightarrow (G', \mathbb{X}')$ the pairs (ψ_0, ψ), where $\psi_0 : G \longrightarrow G'$ is a group morphism, and $\psi : \mathbb{X} \longrightarrow \mathbb{X}'$ is a morphism of median sets, compatible with the actions, i.e. $\psi(g \cdot x) = \psi_0(g) \cdot \psi(x)$ for all $g \in G, x \in X$. We denote by **FAMS** the full subacategory of **AMS** whose objects are the free actions on nonempty median sets, while by **FTAMS** we denote the full subcategory of **FAMS** consisting of the free and transitive actions on nonempty median sets.

We consider also the category **FAPMS** of free actions on pointed median sets having as objects the systems $(G, \mathbb{X}, x_0)$, where G is a group acting freely on a median set $\mathbb{X} = (X, m)$, while $x_0 \in X$ is a fixed base point. The morphisms in **FAPMS** are the morphisms in **FAMS** which preserve the base points. The category **MG** of median groups, with naturally defined morphisms, is equivalent with **FTAMS** and is identified with a full subcategory of **FAPMS**, by taking the neutral element 1 of a median group $\mathbb{G} = (G, m)$ as the base point of its underlying median set.

A strong connection between free actions on median sets and median groups is described by the first main result of the paper.

Theorem 1.6. *Let H be a group acting freely on a nonempty set X. Fix a basepoint $b_1 \in X$. We denote by $\mathcal{M}(X)$ the set of all median operations $m : X^3 \longrightarrow X$ which are compatible with the action of H, i.e., $m(hx, hy, hz) = h \cdot m(x, y, z)$ for all $h \in H$, $x, y, z \in X$. Let $\mathcal{M}_l(X)(\mathcal{M}_s(X))$ be the subset of $\mathcal{M}(X)$ consisting of those median operations which are locally linear (simplicial). Then there exists a group $\widehat{H}$ containing H as its subgroup, together with an embedding $\iota : X \longrightarrow \widehat{H}$ and a retract $\varphi : \widehat{H} \longrightarrow X$, such that the following assertions hold.*

(1) *The maps ι and φ are H-equivariant, i.e., $\iota(h \cdot x) = h\iota(x), \varphi(hu) = h \cdot \varphi(u)$ for all $h \in H, x \in X, u \in \widehat{H}$.*

(2) *$\iota(b_1) = 1$, so $\iota(Hb_1) = H$, and $\iota(X)$ generates the group $\widehat{H}$.*

(3) *Every $m \in \mathcal{M}(X)$ extends uniquely to a ternary operation $\widehat{m} : \widehat{H}^3 \longrightarrow \widehat{H}$ such that $(\widehat{H}, \widehat{m})$ is a median group and the map $\iota \circ \varphi : \widehat{H} \longrightarrow \widehat{H}$ is a folding identifying X with a retractible convex subset of the median set $(\widehat{H}, \widehat{m})$.*

(4) *The median group $(\widehat{H}, \widehat{m})$ is locally linear (simplicial) provided $m \in \mathcal{M}_l(X)$ ($m \in \mathcal{M}_s(X)$).*

Corollary 1.7. *Let G be a group. Then the following assertions are equivalent.*

(1) *G acts freely on some median set (locally linear median set, simplicial median set).*

(2) *There exist a median group (locally linear median group, simplicial median group) (H, m) and a monomorphism $\iota : G \longrightarrow H$.*

The other two main results of the paper are concerned with the construction of two functors on the category **FAMS** with suitable universal properties.

Theorem 1.8. *Let H be a group acting freely on a nonempty median set $\mathbb{X} = (X, m)$. Then there exists a group $\widehat{H}$ acting freely on a median set $\widehat{\mathbb{X}} = (\widehat{X}, \widehat{m})$ together with a* monomorphism *$\iota_0 : H \longrightarrow \widehat{H}$ and a H-equivariant* embedding *of median sets $\iota : \mathbb{X} \rightarrow \widehat{\mathbb{X}}$ such that $\iota(X) \subseteq \widehat{H}\iota(x)$ for some (for all) $x \in X$, and the following universal property is satisfied.*

(RTUP) *For every group $\widetilde{H}$ acting freely on a median set $\widetilde{\mathbb{X}} = (\widetilde{X}, \widetilde{m})$ and for every morphism $(\psi_0, \psi) : (H, \mathbb{X}) \longrightarrow (\widetilde{H}, \widetilde{\mathbb{X}})$ in* **FAMS** *such that $\psi(X) \subseteq \widetilde{H}\psi(x)$ for some* *(for all)* $x \in X$*, there exists uniquely a morphism* $(\widehat{\psi}_0, \widehat{\psi}) : (\widehat{H}, \widehat{\mathbb{X}}) \longrightarrow (\widetilde{H}, \widetilde{\mathbb{X}})$ *in* **FAMS** *such that* $\widehat{\psi}_0 \circ \iota_0 = \psi_0$ *and* $\widehat{\psi} \circ \iota = \psi$*.*

Call the (unique up to a unique isomorphism) free action $(\widehat{H}, \widehat{\mathbb{X}})$ extending $(H, \mathbb{X})$, with the property (RTUP), the *relatively-transitive closure* of the free action $(H, \mathbb{X})$.

By iterating the functorial construction above we obtain

Theorem 1.9. *Let* H *be a group acting freely on a nonempty median set* $\mathbb{X}$*. Then there exists a group* $\mathfrak{H}$ *acting* freely and transitively *on a median set* $\mathfrak{X}$*, together with an* embedding $(\iota_0, \iota) : (H, \mathbb{X}) \longrightarrow (\mathfrak{H}, \mathfrak{X})$ *in the category* **FAMS** *such that the following universal property is satisfied.*

(TUP) *For every group* $\widetilde{H}$ *acting* freely and transitively *on a median set* $\widetilde{\mathbb{X}}$ *and for every morphism* $(\psi_0, \psi) : (H, \mathbb{X}) \longrightarrow (\widetilde{H}, \widetilde{\mathbb{X}})$ *in the category* **FAMS***, there exists uniquely a morphism* $(\Psi_0, \Psi) : (\mathfrak{H}, \mathfrak{X}) \longrightarrow (\widetilde{H}, \widetilde{\mathbb{X}})$ *in* **FTAMS** *such that* $\Psi_0 \circ \iota_o = \psi_0$ *and* $\Psi \circ \iota = \psi$*.*

Call the (unique up to a unique isomorphism) free and transitive action $(\mathfrak{H}, \mathfrak{X})$ extending $(H, \mathbb{X})$, with the property (TUP), the *transitive closure* of the free action $(H, \mathbb{X})$.

In category and median group theoretic terms, Theorem 1.9 can be rephrased as follows.

Theorem 1.10. *The category* **MG** *of median groups is a reflective full subcategory of* **FAPMS***, i.e., the embedding functor* **MG** $\longrightarrow$ **FAPMS** *has a left adjoint which* embedds *every free action on a pointed median set into the median group* freely generated *by it.*

The present paper is organized as follows. Section 2 introduces the reader to the basic notions and facts concerning median sets, free actions on median sets, and median groups. In Section 3 we associate to an arbitrary free action of a group H on a nonempty set X a group $\widehat{H}$ with an underlying tree structure, together with a natural embedding of the H-set X into $\widehat{H}$. Section 4 is devoted to the proof of a more explicit version of Theorem 1.6 by providing a procedure for deformation of the underlying simplicial tree of $\widehat{H}$ introduced in Section 3 into suitable median group operations on $\widehat{H}$ which extend given median operations on the H-set X. Then, in Section 5 we use Theorem 1.6 and the duality theory for median sets to prove Theorem 1.8. Finally, by iteration of the functorial construction provided by Theorem 1.8, we prove Theorems 1.9 and 1.10 in Section 6.

2 Preliminaries on median sets, free actions on median sets, and median groups

2.1 Median sets

The notion of median set (algebra) (see Definition 1.2) appeared as a common generalization of trees and distributive lattices. We recall here some basic definitions and properties related to median sets. For proofs and further details we refer the reader to the papers [1], [2], [4], [8], [15], [21], [22], [24], [25].

The median sets form a category with naturally defined morphisms.

Let $\mathbb{X} = (X, m)$ be a nonempty median set. To any element $a \in X$ we associate the binary operation $(x, y) \mapsto x \underset{a}{\cap} y := m(x, a, y)$. With respect to this operation, X is a meet-semilattice with the induced partial order $x \underset{a}{\subset} y \iff m(x, a, y) = x$ and the least element a, while for arbitrary elements $x, y, z \in X$, $m(x, y, z)$ is the join with respect to the partial order $\underset{a}{\subset}$ of the triple $(x \underset{a}{\cap} y, y \underset{a}{\cap} z, z \underset{a}{\cap} x)$.

According to Definition 1.4 (2), the intersection of any family of convex subsets of the median set $\mathbb{X}$ is convex, hence we may speak on the *convex closure of* (or *the convex subset generated by*) a subset $A \subseteq X$, and denote it by $[A]$. Thus, $[\varnothing] = \varnothing, [\{a\}] = \{a\}$ for $a \in X$, and $[\{a, b\}] = [a, b] = \{m(a, b, x) \mid x \in X\} = \{x \in X \mid m(a, b, x) = x\}$ for $a, b \in X$.

The median sets have a rich geometry whose basic pieces are the cells, a natural generalization of the common closed intervals,

Definition 2.1. (1) By a *cell* of $\mathbb{X}$ we mean a convex subset of the form $[a, b]$ with $a, b \in X$.

(2) Given a cell C of $\mathbb{X}$, any element $a \in X$ for which there exists $b \in X$ such that $C = [a, b]$ is called an *end* of the cell C. The (nonempty) subset of all ends of a cell C is denoted by ∂C and is called the *boundary* of C.

The boundary ∂C of a cell C is a median subset of C, and there is a canonical automorphism $\neg$ of the median set ∂C such that $C = [a, \neg a]$ and $\neg\neg\, a = a$ for all $a \in \partial C$. For any element $a \in \partial C$, the cell C becomes a bounded distributive lattice with the partial order $\underset{a}{\subset}$, the meet $\underset{a}{\cap}$, the join $\underset{\neg a}{\cap}$, the least element a, and the last element $\neg\, a$. The boundary ∂C is identified with the boolean subalgebra of the bounded distributive lattice $(C, \underset{a}{\subset}, \underset{a}{\cap}, \underset{\neg a}{\cap})$ consisting of those elements which have (unique) complements.

Definition 2.2. A median set $\mathbb{X}$ is said to be *locally boolean* if $C = \partial C$ for all cells C of $\mathbb{X}$.

In particular, any boolean algebra $(L; \leq, \wedge, \vee, \neg, 0, 1)$ is a locally boolean median set with respect to the canonical median operation m defined by (1.1). For $x, y, z \in L$, $[x, y] = [x \wedge y, x \vee y] = [m(x, y, z), m(x, y, \neg z)]$; in particular, $L = [0, 1] = [x, \neg x]$ for all $x \in L$.

Remark 2.3. Let $\mathbb{X} = (X, m)$ be a median set. Then the following assertions hold.

(1) For $a, b, c \in X, [a, b] \cap [a, c] = [a, b \underset{a}{\cap} c], [a, b] \cap [b, c] \cap [c, a] = \{m(a, b, c)\}$, and $c \in [a, b]$ if and only if $[a, c] \cap [b, c] = \{c\}$.

(2) For nonempty finite subsets $A, B \subseteq X$, $[A] \cap [B]$ is either empty or the finitely generated convex subset $[\{\underset{a}{\cap} B \mid a \in A\}] = [\{\underset{b}{\cap} A \mid b \in B\}]$, where $\underset{a}{\cap} B$ denotes the meet of the finite set B with respect to the partial order $\underset{a}{\subset}$. In particular, for $a, b, c, d \in X$, $[a, b] \cap [c, d]$ is either empty or the cell $[a \underset{c}{\cap} b, a \underset{d}{\cap} b] = [c \underset{a}{\cap} d, c \underset{b}{\cap} d]$.

Among the convex subsets of a median set, we distinguish the prime ones, defined as follows.

Definition 2.4. A convex subset P of a median set $\mathbb{X}$ is said to be *prime* if its complement $X \setminus P$ is also a convex subset of $\mathbb{X}$.

The set $Spec\,\mathbb{X}$ of prime convex subsets of $\mathbb{X}$ contains the empty set $\varnothing$ as well as the whole X, and is closed under the involution $P \mapsto \neg P := X \setminus P$. For $A \subseteq X$, we denote $U(A) := \{P \in Spec\,\mathbb{X} \mid P \cap A = \varnothing\}$, $V(A) := \{P \in Spec\,\mathbb{X} \mid A \subseteq P\}$. Thus, $U(A) = \bigcap_{a \in A} U(a), V(A) = \bigcap_{a \in A} V(a)$, where $U(a) := U(\{a\}), V(a) := V(\{a\})$ for $a \in A$.

The next result collects some basic properties of the space $Spec\,\mathbb{X}$. For a proof see [4, Theorems 5.2.1, 6.4.]

Theorem 2.5. (1) *For $A, B \subseteq X, V(A) \cap U(B) \neq \varnothing$ if and only if $[A] \cap [B] = \varnothing$; in particular, $[A] = \bigcap_{P \in V(A)} P$.*

(2) *With respect to the topology defined by the basic open sets $U(A)$ for A ranging over the finite subsets of X, $Spec\,\mathbb{X}$ is an irreducible spectral space* [1] *with the generic point $\varnothing$ and the unique closed point X.*

(3) *The proper quasicompact open subsets of $Spec\,\mathbb{X}$ form a distributive lattice*

$$\mathcal{L}(\mathbb{X}) := \{\bigcup_{i=1}^{n} U(A_i) \mid n \geq 1, \varnothing \neq A_i \subseteq X \text{ finite}, i = \overline{1, n}\},$$

closed under the negation *operator*

$$\mathcal{U} \mapsto \neg\mathcal{U} := \{P \in Spec\,\mathbb{X}) \mid \neg P \notin \mathcal{U}\},$$

while the embedding $X \longrightarrow \mathcal{L}(\mathbb{X}), x \mapsto U(x)$ identifies the median set $\mathbb{X}$ to the subset $\mathrm{Fix}_\neg(\mathcal{L}(\mathbb{X})) := \{\mathcal{U} \in \mathcal{L}(\mathbb{X}) \mid \neg\mathcal{U} = \mathcal{U}\}$, *with the canonical median operation*

$$\mathfrak{m}(\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3) := (\mathcal{U}_1 \cap \mathcal{U}_2) \cup (\mathcal{U}_2 \cap \mathcal{U}_3) \cup (\mathcal{U}_3 \cap \mathcal{U}_1) = (\mathcal{U}_1 \cup \mathcal{U}_2) \cap (\mathcal{U}_2 \cup \mathcal{U}_3) \cap (\mathcal{U}_3 \cup \mathcal{U}_1).$$

[1] A topological space S is said to be *spectral* (or *coherent*) if

i) S is *sober*, i.e., every irreducible nonempty closed subset of S is the closure of a unique point of S, and

ii) the family of all quasicompact open subsets of S is closed under finite intersection (in particular, S itself is quasicompact) and forms a base for the topology of S.

A map $f : S' \longrightarrow S$ between spectral spaces is called *coherent* if $f^{-1}(U) \subseteq S'$ is a quasicompact open set provided $U \subseteq S$ is a quasicompact open set. In particular, a coherent map is continuous.

(4) *The correspondence* $\mathbb{X} \mapsto \mathit{Spec}\,\mathbb{X}$ *yields a duality between the category of median sets and a category of irreducible spectral spaces with additional structure* [2].

(5) *The correspondence* $\mathbb{X} \mapsto \mathcal{L}(\mathbb{X})$ *yields an equivalence between the category of median sets and the category of distributive lattices with negation* $(L, \neg)$ *which are generated (as lattices) by their subsets* $\mathrm{Fix}_\neg(L) := \{a \in L \mid \neg\, a = a\}$.

As a corollary we obtain a description of the median set $\mathrm{fms}(A)$ *freely generated by* an arbitrary set A.

Corollary 2.6. (1) *The restriction map* $\mathit{Spec}\,\mathrm{fms}(A) \longrightarrow \mathcal{P}(A), P \mapsto P \cap A$ *is bijective, identifying the spectral space* $\mathit{Spec}\,\mathrm{fms}(A)$ *to the power set* $\mathcal{P}(A)$ *with the basic quasicompact open sets* $U(F) = \mathcal{P}(A \setminus F)$ *for* F *ranging over the finite subsets of* A.

(2) *The elements of the distributive lattice* $\mathcal{L}(\mathrm{fms}(A))$ *correspond bijectively to families* $(F_i)_{i=\overline{1,n}}$, $n \geq 1$, *where the* F_i*'s are incomparable nonempty finite subsets of* A*; such a family* $(F_i)_{i=\overline{1,n}}$ *corresponds to the proper quasicompact open set*

$$\bigcup_{i=1}^{n} U(F_i) = \bigcup_{i=1}^{n} \mathcal{P}(A \setminus F_i).$$

(3) *The negation operator* $\neg$ *sends a family* $(F_i)_{i=\overline{1,n}}$ *as above to the finite family of the subsets* $E \subseteq \bigcup_{i=1}^{n} F_i$ *which are minimal with the property* $E \cap F_i \neq \varnothing$ *for* $i = \overline{1,\, n}$.

(4) *The elements of the median set* $\mathrm{fms}(A)$ *freely generated by the set* A *correspond bijectively to families* $(F_i)_{i=\overline{1,n}}$, $n \geq 1$, *where the* F_i*'s are incomparable nonempty finite subsets of* A *satisfying*

(i) $F_i \cap F_j \neq \varnothing$ *for* $1 \leq i, j \leq n$, *and*

(ii) *for each subset* $E \subseteq \bigcup_{i=1}^{n} F_i$ *such that* $E \cap F_i \neq \varnothing$ *for* $i = \overline{1,\, n}$, *there is* $1 \leq j \leq n$ *such that* $F_j \subseteq E$.

In other words, the F_i*'s,* $i = \overline{1,\, n}$, *are the minimal elements with respect to inclusion of the finite set* $\{E \subseteq \bigcup_{i=1}^{n} F_i \mid E \cap F_i \neq \varnothing, i = \overline{1,\, n}\}$.

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[2] More precisely, according to [4], the objects of the dual category of the category of median sets are the systems $(S, 0, 1, \neg)$, where S is an irreducible spectral space with generic point 0, 1 is the unique closed point of S, and $\neg : S \longrightarrow S$ is an involution satisfying the following conditions.

(i) For every quasicompact open set $U \subseteq S$, the set $\neg U := \{s \in S \mid \neg s \notin U\}$ is quasicompact open.

(ii) The quasicompact open sets $U \subseteq S$ satisfying $\neg U = U$ generate the topology of S.

It follows that $\neg 0 = 1$.

The morphisms $f : (S, 0, 1, \neg) \longrightarrow (S', 0', 1', \neg')$ are the coherent maps $f : S \longrightarrow S'$ satisfying $f(0) = 0'$ and $f \circ \neg = \neg' \circ f$, whence $f(1) = 1'$.

In particular, the finitely generated median sets are finite, and any median set is a direct limit of finite median sets.

The next statement collects some useful equivalent descriptions for locally linear median sets (see Definition 1.4 (4)).

Lemma 2.7. *Let $\mathbb{X}$ be a median set. Then the following assertions are equivalent.*

(1) *$\mathbb{X}$ is locally linear.*

(2) *Every cell C of $\mathbb{X}$ has at most two ends, i.e., $|\partial C| \in \{1, 2\}$.*

(3) *For $a, b \in X$, the partial order $\underset{a}{\subset}$ restricted to the cell $[\![a, b]\!]$ is total (linear) with the least element a and the last element b.*

(4) *For $P, Q \in Spec\,\mathbb{X}$ such that $P \cap Q \neq \emptyset$ and $P \cup Q \neq X$, either $P \subseteq Q$ or $Q \subseteq P$.*

Remark 2.8. Locally linear median sets are strongly related with order-trees. Recall that an *order-tree* is a poset $\mathbb{T} = (T, \leq)$ satisfying

(i) For every pair (x, y) of elements in T, there exists the meet $x \wedge y$, and

(ii) $x \leq z$ and $y \leq z$ imply either $x \leq y$ or $y \leq x$, i.e., for all $z \in T$, the subset $z \downarrow := \{x \in X \mid x \leq z\}$ is totally ordered.

Since for any triple (x, y, z) of elements in an order-tree $\mathbb{T}$, the set $\{x \wedge y, y \wedge z, z \wedge x\}$ has at most two distinct elements, it follows that $\mathbb{T}$ has a natural structure of locally linear median set with

$$m(x, y, z) := (x \wedge y) \vee (y \wedge z) \vee (z \wedge x) \in \{x \wedge y, y \wedge z, z \wedge x\},$$

and $[\![x, y]\!] = [x \wedge y, x] \cup [x \wedge y, y]$.

More generally, we may consider posets $(T, \leq)$ satisfying the condition (i) above, while (ii) is replaced by the following two weaker conditions

(ii)' For all $z \in T$, the poset $z \downarrow$ is a distributive lattice with the last element z, and

(iii) For all $x, y, z \in T$, there exists the join $m(x, y, z) := (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$ of the triple $(x \wedge y, y \wedge z, z \wedge x)$.

It turns out that the ternary operation $m : T^3 \longrightarrow T$ provided by (iii) is a median operation on such structures, which can be seen as natural generalizations of the ordinary *flag complexes*, while the partial order $\leq$ is a *direction* compatible with the median operation m (see for details [2]).

The next notion will be very useful in the present paper.

Definition 2.9. By a *folding* of a median set $\mathbb{X} = (X, m)$ we mean a map $\varphi : X \longrightarrow X$ satisfying $\varphi(m(x, y, z)) = m(\varphi(x), y, \varphi(z))$ for all $x, y, z \in X$.

One checks easily that a map $\varphi : X \to X$ is a folding if and only if φ is an idempotent endomorphism of the median set $\mathbb{X}$ and the image $\varphi(X)$ is a convex subset of $\mathbb{X}$. In addition, according to [2, Proposition 7.3], the map $\varphi \mapsto \varphi(X)$ maps bijectively the foldings of $\mathbb{X}$ onto the nonempty convex subsets A of $\mathbb{X}$ satisfying the following equivalent conditions.

i) A is *retractible*, i.e., there is a (unique) median set retract $p : X \longrightarrow A$ to the median set embedding $A \longrightarrow X$;

ii) For some (for all) $a \in A, A \cap [a, x]$ is a cell for all $x \in X$;

iii) For all $x \in X$, the meet $\underset{x}{\cap} A$ with respect to the partial order $\underset{x}{\subset}$ exists and belongs to A.

In particular, to any nonempty finite subset A of a median set $\mathbb{X} = (X, m)$, we associate the folding φ_A defined by $\varphi_A(x) = \underset{x}{\cap} A$, the meet with respect to the partial order $\underset{x}{\subset}$ of the finite set A, whence $\bigcap_{a \in A} [x, a] = [x, \varphi_A(x)]$ for all $x \in X$, and $\varphi_A(X) = [A]$.

Finally, some words on simplicial median sets (see Definition 1.4 (5)). To a simplicial median set $\mathbb{X}$ one assigns an integer-valued "distance" function $d : X \times X \longrightarrow \mathbb{N}$, where for $x, y \in X, d(x, y)$ is the length of some (of all) maximal chain(s) in the finite distributive lattice $([x, y], \underset{x}{\subset})$. With respect to d, $\mathbb{X}$ becomes a $\mathbb{Z}$-metric space such that for all $x, y \in X, [x, y] = \{z \in X \mid d(x, z) + d(z, y) = d(x, y)\}$, and the map

$$[x, y] \longrightarrow [0, d(x, y)], z \mapsto d(x, z),$$

induced by d, is onto. In particular, $d(x, y) = d(u, v)$ whenever $[x, y] = [u, v]$, so we may speak on the "diameter" of any cell of $\mathbb{X}$. Note also that for $x, y, z \in X, d(x, y \underset{x}{\cap} z) = \frac{1}{2}(d(x, y) + d(x, z) - d(y, z))$.

An equivalent graph theoretic description of simplicial median sets is given by [8, Proposition 7.3]. In particular, the *simplicial trees*, i.e., acyclic connected graphs, are identified with those simplicial median sets $\mathbb{X}$ which are locally linear, i.e., for all $x, y \in X$, the map $[x, y] \longrightarrow [0, d(x, y)]$ induced by d is bijective.

Note also that in any simplicial median set, the nonempty convex subsets are retractible, and the finitely generated convex subsets are finite.

2.2 Groups acting freely on median sets

We denote by **MSFG** the class of *median-free groups* consisting of those groups G for which there exist a median set $\mathbb{X}$ and a free action from the left of G on $\mathbb{X}$.

Lemma 2.10. **MSFG** *is a quasivariety, i.e., a nonempty class of groups closed under isomorphisms, subgroups, and reduced products.*

Proof. We have only to show that the class **MSFG** is closed under reduced products. Let I be a nonempty set, $\mathcal{F}$ a filter on I, and $(G_i)_{i \in I}$ a family of median-free groups. For each $i \in I$, let $G_i \times X_i \longrightarrow X_i$ be a free action on a nonempty median set X_i. Using the filter $\mathcal{F}$, we define the normal subgroup N of the product $\prod_{i \in I} G_i$ and the congruence $\equiv$ on the median set product $\prod_{i \in I} X_i$ by

$$N := \{(g_i)_{i \in I} \in \prod_{i \in I} G_i \mid \{i \in I \mid g_i = 1_i\} \in \mathcal{F}\},$$

$$x \equiv y \Longleftrightarrow \{i \in I \mid x_i = y_i\} \in \mathcal{F}, \text{ for } x = (x_i)_{i\in I}, y = (y_i)_{i\in I} \in \prod_{i\in I} X_i.$$

One checks easily that the canonical free action $(\prod_{i\in I} G_i) \times (\prod_{i\in I} X_i) \longrightarrow \prod_{i\in I} X_i$ induces a free action of the reduced product $G := (\prod_{i\in I} G_i)/N$ on the quotient median set $X := (\prod_{i\in I} X_i)/\equiv$ as desired. □

Lemma 2.11. *As a full subcategory of the category* $\mathbf{G}$ *of groups,* $\mathbf{MSFG}$ *is reflective. The reflector, the left adjoint of the embedding* $\mathbf{MSFG} \longrightarrow \mathbf{G}$*, sends a group* G *to its quotient* G/N*, where* N *is the smallest normal subgroup such that* G/N *belongs to* $\mathbf{MSFG}$.

Proof. Given a group G, we denote by $\mathcal{N}$ the set of those normal subgroups U of G for which the quotient G/U belongs to $\mathbf{MSFG}$. The set $\mathcal{N}$ is nonempty since $G \in \mathcal{N}$. Set $N := \bigcap_{U\in\mathcal{N}} U$, the kernel of the canonical morphism $G \longrightarrow \prod_{U\in\mathcal{N}} G/U$. As $\mathbf{MSFG}$ is closed under products and subgroups, it follows that G/N belongs to $\mathbf{MSFG}$, whence N is the least member of the poset $\mathcal{N}$ with respect to inclusion. The required adjunction property is immediate. □

As a quasivariety, the class of median-free groups is axiomatized by quasi-identities (see [17, Theorem 2.25], [20, Theorem 9.4.7]). These quasi-identities turn out to be quite simple according to the next lemma.

Lemma 2.12. *Let* G *be a group. Then the following assertions are equivalent.*

(1) G *is a median-free group.*

(2) *The canonical action of* G *on the median set* $\mathrm{fms}(G)$ *freely generated by the underlying set of* G *is free.*

(3) *For every* $g \in G$*, either* g *is of infinite order or the order of* g *is a power of* 2.

Proof. (1) $\Longrightarrow$ (3). Assume that G acts freely on the nonempty median set $\mathbb{X}$, and there is $g \in G$ of prime order $p \neq 2$, say $p = 2k+1, k \geq 1$. We have to get a contradiction. We may assume without loss that G is cyclic of order p, generated by g. Fix an element $x \in X$, so Gx, the G-orbit of x, has cardinality p. We denote by $\mathcal{F}$ the (finite) set of all subsets $F \subseteq Gx$ of cardinality $|F| = k+1$.

For any $F \in \mathcal{F}$, the set $U(F) = \{P \in Spec\,\mathbb{X} \mid P \cap F = \varnothing\}$ is a basic quasicompact open subset of the spectral space $Spec\,\mathbb{X}$ of prime convex subsets of $\mathbb{X}$ (cf. Theorem 2.5 (2)). Consequently, the union

$$\mathcal{U} := \bigcup_{F\in\mathcal{F}} U(F) = \{P \in Spec\,\mathbb{X} \mid |P \cap Gx| \leq k\} \in \mathcal{L}(\mathbb{X})$$

is a proper quasicompact open subset of $Spec\,\mathbb{X}$.

Moreover $\mathcal{U} = \neg\mathcal{U} = \{P \in Spec\,\mathbb{X} \,|\, X \setminus P \notin \mathcal{U}\}$, therefore, according to Theorem 2.5 (3), there exists uniquely $y \in X$ such that $\mathcal{U} = U(y) = \{P \in Spec\,\mathbb{X} \,|\, y \notin P\}$. Equivalently, by Theorem 2.5 (1), y is the unique common element of the convex closure $[F] = \bigcap_{F \subseteq P \in Spec\,\mathbb{X}} P$ of F in the median set $\mathbb{X}$ for F ranging over $\mathcal{F}$. Since g acts as a permutation on the finite set $\mathcal{F}$, we deduce that $gy = y$, contrary to the assumption that G acts freely on $\mathbb{X}$.

(3) $\Longrightarrow$ (2). Assume that G satisfies (3) and there is $g \in G \setminus \{1\}$ such that $gx = x$ for some $x \in \mathrm{fms}(G)$. We have to derive a contradiction. By Corollary 2.6 (4), the element $x = gx \in \mathrm{fms}(G)$ is uniquely determined by a suitable finite family $\mathcal{F} = (F_i)_{i=\overline{1,n}}$, $n \geq 1$, of incomparable nonempty finite subsets of G, whence $gF_i = \{gh \,|\, h \in F_i\} = F_{\sigma(i)}$, $i = \overline{1, n}$, for some permutation σ of the finite set $\{1, \ldots, n\}$. Consequently, g has finite order, so we may assume without loss that g has order 2. Set $F := \bigcup_{i=1}^{n} F_i$. Since $gF = \{gh \,|\, h \in F\} = F$, there is $E \subseteq F$ such that F is the disjoint union of its subsets E and gE, in particular, $|F| = 2k$ with $k = |E| = |gE| \geq 1$. We distinguish the following two cases.

(i) $E \cap F_i = \varnothing$ for some $1 \leq i \leq n$. Then $F_i \subseteq gE$, whence $F_{\sigma(i)} = gF_i \subseteq g^2E = E$, therefore $F_i \cap F_{\sigma(i)} = \varnothing$, contrary to the condition (i) of Corollary 2.6 (4) satisfied by the family $\mathcal{F}$.

(ii) $E \cap F_i \neq \varnothing$ for all $i = \overline{1, n}$. Then, by condition (ii) of Corollary 2.6 (4) satisfied by the family $\mathcal{F}$, there is $1 \leq j \leq n$ such that $F_j \subseteq E$, whence $F_{\sigma(j)} = gF_j \subseteq gE$, so $F_j \cap F_{\sigma(j)} = \varnothing$, again a contradiction.

Finally, note that the implication (2) $\Longrightarrow$ (1) is trivial. □

Corollary 2.13. *The quasivariety* **MSFG** *of median-free groups is axiomatized by the quasi-identities*

$$x^p = 1 \to x = 1,$$

for p ranging over the set of odd prime numbers. In particular, **MSFG** *is closed under free products.*

Remark 2.14. Given a median-free group G, let us consider the poset $\mathrm{Cong}_G(\mathrm{fms}(G))$ consisting of those congruences on the median set $\mathrm{fms}(G)$ which are compatible with the canonical action of G. By [2, Proposition 1.6.1.], $\mathrm{Cong}_G(\mathrm{fms}(G))$ is a Heyting algebra, in particular, a bounded distributive lattice. The quotients $\mathrm{fms}(G)/R$, $R \in \mathrm{Cong}_G(\mathrm{fms}(G))$, are up to isomorphism the pointed median G-sets (X, m, x_0) for which the median set (X, m) is generated by the G-orbit $Gx_0 \subseteq X$ (in particular, X is the smallest convex subset of the median set (X, m) containing the G-orbit Gx_0). Since G acts freely on $\mathrm{fms}(G)$ according to Lemma 2.12, we obtain a lower subset $\mathcal{C}(G) \subseteq \mathrm{Cong}_G(\mathrm{fms}(G))$ with equality as the least element, consisting of those congruences R for which the induced action of G on $\mathrm{fms}(G)/R$ is free. As the nonempty poset $\mathcal{C}(G)$ is inductively ordered, it follows that it is the lower subset of $\mathrm{Cong}_G(\mathrm{fms}(G))$ generated by the subset $\mathrm{Max}(\mathcal{C}(G))$ of its maximal elements. The congruences belonging to $\mathrm{Max}(\mathcal{C}(G))$ correspond up to isomorphism to the *minimal* pointed median sets on which the group G acts freely.

2.3 Median groups

We recall here some basic definitions and properties related to median groups. For proofs and further details we refer the reader to the papers [3], [6], [10].

Definition 2.15. (1) Let G be a group. By a *median group operation* on G we understand a ternary operation $m : G^3 \longrightarrow G$ satisfying

(i) (G, m) is a median set, and

(ii) $u\, m(x, y, z) = m(ux, uy, uz)$ for all $u, x, y, z \in G$.

(2) By a *median group* we understand a group G together with a median group operation m on G.

(3) By a *formally-median group* we mean a group G satisfying the following equivalent conditions.

(i) There exists a median group operation on G.

(ii) G acts freely and transitively on some nonempty median set.

(iii) There exists a G-equivariant retract $\varphi : \mathrm{fms}(G) \longrightarrow G$ to the canonical G-equivariant embedding of G into the median set $(\mathrm{fms}(G), m)$ freely generated by the set G such that $\varphi(m(x, y, z)) = \varphi(m(\varphi(x), y, z))$ for $x, y, z \in \mathrm{fms}(G)$.

(iv) There is a congruence $R \in \mathrm{Max}(\mathcal{C}(G))$ such that for all $x \in \mathrm{fms}(G)$ there exists (uniquely) $g \in G$ such that $(x, g) \in R$, i.e., the action of G on the quotient median set $\mathrm{fms}(G)/R$ is free and transitive.

(v) There exist a pointed median set (X, m, x_0), a free action of G on (X, m), and a G-equivariant retract $\varphi : X \longrightarrow G$ to the G-equivariant embedding $G \longrightarrow X$, $g \mapsto gx_0$ such that $\varphi(m(x, y, z)) = \varphi(m(\varphi(x)x_0, y, z))$ for all $x, y, z \in X$.

Remark 2.16. By contrast with the free actions, without inversions, on ordinary trees [23, I.3.3, Theorem 4], the existence of a free action on a median set is far from being enough to make the acting group a median group.

Let $\mathbb{G} = (G, m)$ be a median group. Taking the neutral element 1 as a basepoint of the underlying median set of $\mathbb{G}$, we get the meet-semilattice operation $x \cap y = m(x, 1, y)$ with the induced partial order $\subset$. Thus, $1 \subset x$ for all $x \in G$, and $x \subset y \Longleftrightarrow x \in [1, y]$. Note also that $z \in [x, y] \Longleftrightarrow x^{-1}z \subset x^{-1}y \Longleftrightarrow y^{-1}z \subset y^{-1}x$, and

$$m(x, y, z) = x(x^{-1}y \cap x^{-1}z) = y(y^{-1}x \cap y^{-1}z) = z(z^{-1}x \cap z^{-1}y)$$

for all $x, y, z \in G$. In particular, for $x, y \in G$, $x \cap y$ is the unique element $z \in G$ satisfying $z \subset x$, $z \subset y$ and $x^{-1}z \subset x^{-1}y$.

The next statement furnishes an useful order theoretic description of median groups.

Proposition 2.17. (see [6, Proposition 2.2.1.]) *Let G be a group. Then the map sending a ternary operation $m : G^3 \longrightarrow G$ to the binary operation $\cap$, defined by $x \cap y := m(x, 1, y)$, maps bijectively the median group operations on G onto the binary operations $\cap$ on G satisfying*

(1) $(G, \cap)$ *is a meet-semilattice; let $x \subset y \Longleftrightarrow x \cap y = x$ be the induced partial order,*

(2) $1 \subset x$ *for all $x \in G$,*

(3) $x \subset y, y \subset z \Longrightarrow z^{-1}y \subset z^{-1}x$, *and*

(4) $x^{-1}(x \cap y) \subset x^{-1}y$ *for all $x, y \in G$.*

In both the signatures $(1,^{-1}, \cdot, m)$ and $(1,^{-1}, \cdot, \cap)$ of type $(0, 1, 2, 3)$ and $(0, 1, 2, 2)$ respectively, the median groups form a variety. In particular, the class of median groups is closed under arbitrary products, with group and median operations defined componentwise.

Remark 2.18. The class of median groups is also closed under semidirect products. Indeed, let $\mathbb{G}$ and $\mathbb{A}$ be median groups, and $G \times \mathbb{A} \longrightarrow \mathbb{A}, (g, a) \mapsto {}^g a$, be an action of the underlying group G of $\mathbb{G}$ on the median group A. Let $\mathbb{A} \rtimes \mathbb{G}$ be the cartesian product $A \times G$ equipped with the group operation $(a, g)(b, h) := (a\, {}^g b, gh)$ and the semilattice operation $(a, g) \cap (b, h) := (a \cap b, g \cap h)$, with the induced order $(a, g) \subset (b, h) \Longleftrightarrow a \subset b$ and $g \subset h$. One checks easily that $\mathbb{A} \rtimes \mathbb{G}$ is a median group, $\mathbb{A}$ is identified with a convex normal subgroup of $\mathbb{A} \rtimes \mathbb{G}$, and $\mathbb{G}$ is identified with the quotient median group $(\mathbb{A} \rtimes \mathbb{G})/\mathbb{A}$.

Recall that a median group is *locally linear* (*simplicial*) if its underlying median set is locally linear (simplicial). Equivalent descriptions for locally linear and simplicial median groups are provided by the next two statements.

Corollary 2.19. (see [6, Corollary 2.2.2.]) *Let G be a group. Then the map sending a ternary operation $m : G^3 \longrightarrow G$ to the binary relation $x \subset y \Longleftrightarrow m(x, 1, y) = x$ maps bijectively the locally linear median group operations on G onto the partial orders $\subset$ on G satisfying*

(1) $1 \subset x$ *for all $x \in G$,*

(2) $x \subset y, y \subset z \Longrightarrow z^{-1}y \subset z^{-1}x$,

(3) *for all $x, y \in G$, there exists $z \in G$ such that $z \subset x, z \subset y$ and $x^{-1}z \subset x^{-1}y$, and*

(4) $x \subset z, y \subset z \Longrightarrow$ *either $x \subset y$ or $y \subset x$.*

Corollary 2.20. *The necessary and sufficient condition for a median group $\mathbb{G} = (G, m)$ to be simplicial is that for all $x \in G$, the cell $[1, x] = \{y \in G \mid y \subset x\}$ has finitely many elements.*

2.3.1 Examples of median groups

Certain classes of median groups are studied in [3, 6, 7, 10]. For convenience of the reader we mention here some relevant examples.

Example 2.21. Any l-group $(G, \cdot, \leq, \wedge, \vee)$, not necessarily commutative, has a canonical underlying structure of median group. Indeed, as the underlying lattice of G is distributive, G has a canonical structure of median set with the median operation $m : G^3 \longrightarrow G$ defined by (1.1), compatible with multiplication. Note that $x \subset y$ if and only if $x_+ \leq y_+$ and $x_- \leq y_-$, while $(x \cap y)_+ = x_+ \wedge y_+, (x \cap y)_- = x_- \wedge y_-$, where $x_+ = x \vee 1, x_- = (x^{-1})_+ = (x \wedge 1)^{-1}$.

Example 2.22. A *left-ordered group* is a group G together with a total order $\leq$ on G such that $u \leq v \Longrightarrow gu \leq gv$ for all $g, u, v \in G$; the group G is *left-orderable* if $(G, \leq)$ is left-ordered for some total order $\leq$ on G. If $(G, \leq)$ is a left-ordered group then the total order $\leq$ determines a locally linear median group operation m on G whose associated partial order $\subset$ is given by $u \subset v \Longleftrightarrow$ either $1 \leq u \leq v$ or $v \leq u \leq 1$. In other words, the median operation m is induced by the betweenness relation associated to the total order $\leq$. Note that there exist groups G together with total orders $\leq$ such that $(G, \leq)$ is not left-ordered but the median operation induced by the betweenness relation associated to the total order $\leq$ is compatible with the left multiplication (see Example 2.25). In any case, possible connections with the much more studied class of left-orderable groups could be fruitful.

Example 2.23. Let G be a group with $S \subseteq G \setminus \{1\}$ a set of generators. Define the partial order $\subset$ on G by $x \subset y \Longleftrightarrow l(x) + l(x^{-1}y) = l(y)$, where $l : G \longrightarrow \mathbb{N}$ is the standard length function on (G, S), with $l(s) = 1$ for $s \in S$. Denote by $x \cap y, x \cup y$ the meet and the join of any pair (x, y) respectively, whenever they exist. The elements $x, y \in G$ are called *orthogonal* (written $x \perp y$) if $x \cap y = 1$ and the join $x \cup y$ exists. According to [6, Theorem 2.4.1], [10, Corollary 3.7], (G, S) is a *right-angled Artin group* if and only if $S \cap S^{-1} = \varnothing$ and the partial order $\subset$ makes G a (simplicial) median group satisfying the condition $x \perp y \Longrightarrow x \cup y = xy$ (in particular, $xy = yx$).

Example 2.24. Though, according to Corollary 2.13, all cyclic groups of order $2^n, n \in \mathbb{N}$, are median-free, only three of them, with $n = 0, 1, 2$, are formally-median groups. The corresponding median group operations are uniquely determined: the point, the segment $[1, \sigma], \sigma^2 = 1$, and the square $[1, \sigma^2] = [\sigma, \sigma^3], \sigma^4 = 1$, respectively.

Example 2.25. The infinite cyclic group $(\mathbb{Z}, +)$ has a canonical structure of simplicial and locally linear median group with respect to the median group operation m_0 associated to the usual simplicial tree on $\mathbb{Z}$ induced by the natural order:

$$m_0(x, y, z) = y \Longleftrightarrow \text{either } x \leq y \leq z \text{ or } z \leq y \leq x.$$

However there are still two distinct median group operations m_1 and m_{-1} on $(\mathbb{Z},+)$, both locally linear but not simplicial, related to each other through the unique nonidentical automorphism $n \mapsto -n$ of the group $(\mathbb{Z}.+)$:

$$m_{-1}(x,y,z) = -m_1(-x,-y,-z) \text{ for } x,y,z \in \mathbb{Z}.$$

Thus, there are only two (up to isomorphism) median groups with the underlying group $(\mathbb{Z},+)$. To prove the assertion above and describe explicitely the median group operation m_1, introduced in [10, Remarks 3.2.(3)], we proceed as follows.

Let m be a median group operation on $(\mathbb{Z},+)$, with the associated meet-semilattice operation $\cap$ and partial order $\subset$ (cf. Proposition 2.17). First, let us show that the following implication holds

$$0 < x < y \Longrightarrow 0 \le x \cap y \le y. \tag{2.1}$$

Assuming the contrary, let $x, y \in \mathbb{Z}$ be such that $0 < x < y$, $z := m(0,x,y) = x \cap y$, and either $y < z$ or $z < 0$. In the case $y < z$, let M be the median subset of $(\mathbb{Z},m)$ generated by the finite set $\{0,1,\dots,z\}$; in particular, M is finite. Let u be the minimal element of $\mathbb{N} \setminus M$. Setting $w := u - z$, and using the fact that the translation $n \mapsto n+w$ is an automorphism of the median set $(\mathbb{Z},m)$, we obtain $u = z + w = m(0,x,y) + w = m(w, x+w, y+w)$. As $0 < w, x+w, y+w < u$, the natural numbers $w, x+w, y+w$ belong to the median subset M, therefore $u = m(w,x+w,y+w) \in M$, i.e., a contradiction. Similarly, assuming that $z < 0$, we get again a contradiction taking M the median subset of $(\mathbb{Z},m)$ generated by the finite set $\{z, z+1,\dots,y\}$ and $u \le y$ maximal with the property $u \notin M$, in particular, $u < z$. Consequently, the implication (2.1) is proved.

In particular, we get $1 \cap 2 = m(0,1,2) \in \{0,1,2\}$. We distinguish the following three cases.

(i) $1 \cap 2 = 1$, i.e., $1 \subset 2$. To obtain $m = m_0$, it suffices to show that $n \subset n+1$ for all $n \ge 1$. As $1 \subset 2$ by hypothesis, assuming by induction that $k \subset k+1$ for $1 \le k < n, n \ge 2$, we have to show that $s := m(0,n,n+1) = n$. By (2.1), $0 \le s \le n+1$. Assuming that $s \ne n$, there are three possibilities.

(a) $1 \le s \le n-1$, whence $s-1 \subset n-1 \subset n$ by the induction hypothesis. On the other hand, $s \in [n,n+1]$, therefore $s-1 \in [n-1,n]$, so $n-1 \subset s-1 \subset n$. Consequently, $s-1 = n-1$, contrary to the assumption $s \le n-1$.

(b) $s = 0$, i.e., $0 \in [n,n+1]$, and hence $n-1 \in [0,n] \subseteq [n,n+1]$. As $1 \in [0,2]$ implies $n \in [n-1,n+1]$, we deduce that $n-1 = n$, i.e., a contradiction.

(c) $s = n+1$, i.e. $n+1 \in [0,n]$, therefore $[n-1,n+1] \subseteq [0,n]$. Setting $t := m(0,n-1,n+1)$, we obtain by Remark 2.3 (2)

$$[n-1,n+1] = [n-1,n+1] \cap [0,n] = [t, m(n,n-1,n+1)] = [t,n],$$

whence $t \notin \{n-1,n,n+1\}$, and hence $0 \le t \le n-2$ by (2.1). Assuming that $t \ne 0$, it follows by the induction hypothesis that $t-1 \subset n-2 \subset n-1 \subset n$. Since $[n-1,n+1] = [t,n]$ implies $[n-2,n] = [t-1,n-1]$ (by translation with -1), we deduce that $n-1 = n$, i.e., a contradiction. Thus, $t = 0$, therefore $1 \in [0,n] = [n-1,n+1]$, and hence $0 \in [n-2,n]$ (by translation with -1). As $n-2 \subset n$, it follows that $n-2 = 0$, i.e., $[0,2] = [1,3]$, whence $[0,2] = [2,4]$ (by translation with 1), therefore $0 = 4$, i.e., again a contradiction.

Since in all three possible cases (a), (b), (c) we obtain a contradiction, we deduce that $n \subset n+1$ for all $n \in \mathbb{N}$, and hence $m = m_0$ as desired.

(ii) $1 \cap 2 = 2$, i.e., $2 \in [0,1]$, and hence $1 \in [0,-1]$ (by translation with -1), whence $2 \subset 1 \subset -1$. Applying succesively the translation $k \mapsto k+2$, it follows that $2n \subset 2n+2 \subset 2n+1 \subset 2n-1$ for $n \in \mathbb{N}$. Consequently, the median group operation $m := m_1$ is uniquely determined by the betweenness relation induced by the total order $\prec$ (or its opposite) on $\mathbb{Z}$, defined by $x \prec y$ if and only if one of the following assertions holds

(a) x, y are even and $x \leq y$;
(b) x, y are odd and $y \leq x$;
(c) x is even and y is odd.

(iii) $1 \cap 2 = 0$, i.e., $0 \in [1, 2]$, whence $-1 \in [0, 1]$ (by translation with -1), i.e., $-1 \subset 1$. It follows that the median group operation $m := m_{-1}$ is the conjugate of m_1 by the group automorphism $n \mapsto -n$, and hence it is uniquely determined by the betweenness relation induced by the total order $\prec'$ (or its opposite) on $\mathbb{Z}$ obtained from $\prec$ by replacing (c) with
(c') x is odd and y is even.

Note that, though the total orders $\prec$ and $\prec'$ are not compatible with the group operation on $\mathbb{Z}$, the induced median operations m_1 and m_{-1} are so. Note also that $2\mathbb{Z}$ is a median subgroup of $(\mathbb{Z}, m_i), i = 0, \pm 1$, and $m_i|_{2\mathbb{Z}} = m_0|_{2\mathbb{Z}}, i = \pm 1$. However, by contrast with the median group $(\mathbb{Z}, m_0)$ which has no proper convex subgroups, and hence no proper quotients, $2\mathbb{Z}$ is the unique proper convex subgroup of $(\mathbb{Z}, m_1) \cong (\mathbb{Z}, m_{-1})$, inducing the surjective morphism of *median groups* $(\mathbb{Z}, m_1) \cong (\mathbb{Z}, m_{-1}) \longrightarrow \mathbb{Z}/2$, whose kernel is isomorphic to the median group $(\mathbb{Z}, m_0)$. In other words, the isomorphic (non-simplicial) locally linear median groups $(\mathbb{Z}, m_1)$ and $(\mathbb{Z}, m_{-1})$ are extensions of the (simplicial) locally linear median group $\mathbb{Z}/2$ by the (simplicial) locally linear median group $(\mathbb{Z}, m_0)$. Note that the other two (up to isomorphism) extensions of the median group $\mathbb{Z}/2\mathbb{Z}$ by the median group $(\mathbb{Z}, m_0)$, namely the direct product $(\mathbb{Z}, m_0) \times \mathbb{Z}/2$ and the semidirect product (the infinite dihedral group) $(\mathbb{Z}, m_0) \rtimes \mathbb{Z}/2$ are both simplicial but not locally linear. Note also that the direct product $(\mathbb{Z}, m_1) \times \mathbb{Z}/2$ is the unique (up to isomorphism) median group extension of $\mathbb{Z}/2$ by $(\mathbb{Z}, m_1)$.

Remark 2.26. The construction from Example 2.25 has a nice interpretation in terms of the nonstandard arithmetic. Let ${}^*\mathbb{Z}$ be an *enlargement* of $\mathbb{Z}$. For our purposes it suffices to take ${}^*\mathbb{Z}$ an ultrapower of $\mathbb{Z}$ relative to a nonprincipal ultrafilter on $\mathbb{N}$. We denote by ${}^*m_i$ the median group operation on ${}^*\mathbb{Z}$ which extends the median group operation $m_i, i = 0, 1, -1$. Let $T_i := \{t \in {}^*\mathbb{Z} \,|\, \forall x, y \in \mathbb{Z},\ {}^*m_i(t, x, y) \in \mathbb{Z}\}$. T_i is the maximal median subset of $({}^*\mathbb{Z},\ {}^*m_i)$ lying over $\mathbb{Z}$ with the property that $\mathbb{Z}$ is convex in $(T_i,\ {}^*m_i|_{T_i})$. Since $(\mathbb{Z}, m_0)$ is simplicial, it follows that $T_0 = {}^*\mathbb{Z}$, while

$$T_1 = \{-2t \,|\, t \in {}^*\mathbb{N} \setminus \mathbb{N}\} \bigsqcup \mathbb{Z} \bigsqcup \{-2t + 1 \,|\, t \in {}^*\mathbb{N} \setminus \mathbb{N}\},$$
$$T_{-1} = \{2t + 1 \,|\, t \in {}^*\mathbb{N} \setminus \mathbb{N}\} \bigsqcup \mathbb{Z} \bigsqcup \{2t \,|\, t \in {}^*\mathbb{N} \setminus \mathbb{N}\}$$

are proper *median submonoids* of the median group $({}^*\mathbb{Z}, +,\ {}^*m_i)$ for $i = 1, -1$ respectively, containing $\mathbb{Z}$ as the maximal (convex) subgroup.

Let us consider the congruence $\equiv_i$ on the median set $(T_i,\ {}^*m_i|_{T_i})$, defined by

$$t \equiv_i t' \iff \forall x, y \in \mathbb{Z},\ {}^*m_i(t, x, y) = {}^*m_i(t', x, y).$$

According to [2, A.1.Proposition], the factor median set $T_i/ \equiv_i$ is isomorphic to the median set $\mathrm{Dir}(\mathbb{Z}, m_i)$ of the *directions* on the median set $(\mathbb{Z}, m_i)$, containing $\mathbb{Z}$ as the convex subset of *internal directions*. It follows that $\mathrm{Dir}(\mathbb{Z}, m_i) = [D_i, D_i'] = \{D_i\} \bigsqcup \mathbb{Z} \bigsqcup \{D_i'\}$, where the *external directions* D_i, D_i' are the equivalence classes $\{-t \,|\, t \in {}^*\mathbb{N}\setminus\mathbb{N}\}, \{t \,|\, t \in {}^*\mathbb{N}\setminus\mathbb{N}\}$ for $i = 0$, $\{-2t \,|\, t \in {}^*\mathbb{N}\setminus\mathbb{N}\}, \{-2t+1 \,|\, t \in {}^*\mathbb{N}\setminus\mathbb{N}\}$ for $i = 1$, and $\{2t + 1 \,|\, t \in {}^*\mathbb{N} \setminus \mathbb{N}\}, \{2t \,|\, t \in {}^*\mathbb{N} \setminus \mathbb{N}\}$ for $i = -1$ respectively. The induced total order on $\mathbb{Z}$ from D_i to D_i' is $\leq$ for $i = 0$, $\prec$ for $i = 1$, and $\prec'$ for $i = -1$. The canonical free action of $\mathbb{Z}$ on T_i, $(n, t) \in \mathbb{Z} \times T_i \mapsto n + t \in T_i$ induces an action on $\mathrm{Dir}(\mathbb{Z}, m_i) \cong T_i/ \equiv_i$ which is obviously free and transitive on the set $\mathbb{Z}$ of internal directions, identical on the external directions D_0, D_0', and acting as $\mathbb{Z}/2$ on the pair of external directions (D_i, D_i') for $i = 1, -1$.

2.3.2 An useful lemma

We end this preliminary section with a lemma relating free actions on median sets and median groups.

Lemma 2.27. *Let G be a group, H a subgroup of G, and $X \subseteq G$ a set of generators of G such that $H \subseteq X$, and $HX = X$, whence $H \times X \longrightarrow X, (h, x) \mapsto hx$ is a free action of H on the nonempty set X, and the embedding $\iota : X \longrightarrow G$ is H-equivariant. Let $\varphi : G \longrightarrow X$ be a H-equivariant retract of ι. We denote by $\mathcal{M}(X)$ the set of the median operations $m : X^3 \longrightarrow X$ which are compatible with the action of H, i.e., $m(hx, hy, hz) = hm(x, y, z)$ for $h \in H, x, y, z \in X$. On the other hand, we denote by $\mathcal{M}(G, \varphi)$ the set of those median group operations $\widehat{m} : G^3 \longrightarrow G$ for which the map φ is a folding, so X is a retractible convex subset of $(G, \widehat{m})$ with associated folding φ.*

Then the restriction map $\mathcal{M}(G, \varphi) \longrightarrow \mathcal{M}(X)$ is injective.

Proof. Let $I \subseteq X$ be a system of representatives for the H-orbits of X. Assume that $1 \in I$, and set $I' := I \setminus \{1\}$; thus the disjoint union $(H \setminus \{1\}) \sqcup I'$ generates the group G. Let $\widehat{m} \in \mathcal{M}(G, \varphi)$, and $m \in \mathcal{M}(X)$ be its restriction. To prove that $\widehat{m}$ is the unique prolongation of m, it suffices to show by duality (cf. Theorem 2.5 (4)) that $Spec(G, \widehat{m})$ is uniquely determined by $Spec(X, m)$ and φ.

The H-equivariant morphisms of median sets

$$\iota : (X, m) \longrightarrow (G, \widehat{m}), \varphi : (G, \widehat{m}) \longrightarrow (X, m)$$

satisfying $\varphi \circ \iota = 1_X$ induce by duality the morphisms of spectral spaces

$$Spec(G, \widehat{m}) \longrightarrow Spec(X, m), P \mapsto P \cap X,$$
$$Spec(X, m) \longrightarrow Spec(G, \widehat{m}), \mathfrak{p} \mapsto \varphi^{-1}(\mathfrak{p})$$

such that $\varphi^{-1}(\mathfrak{p}) \cap X = \mathfrak{p}$ for all $\mathfrak{p} \in Spec(X, m)$. As $\widehat{m}$ is a median group operation, G acts from the right on $Spec(G, \widehat{m}), (P, g) \mapsto P^g := g^{-1}P$, while H acts from the right on $Spec(X, m), (\mathfrak{p}, h) \mapsto \mathfrak{p}^h := h^{-1}\mathfrak{p}$, and the morphisms of spectral spaces above are H-equivariant.

Let $\mathcal{S} := \{\varphi^{-1}(\mathfrak{p})^g = g^{-1}\varphi^{-1}(\mathfrak{p}) \mid \mathfrak{p} \in Spec(X, m), g \in G\}$. Note that $\varnothing, G \in \mathcal{S}$, and $G \setminus P \in \mathcal{S}$ provided $P \in \mathcal{S}$. As the inclusion $\mathcal{S} \subseteq Spec(G, \widehat{m})$ is obvious, it remains to prove the opposite inclusion. Let $P \in Spec(G, \widehat{m}) \setminus \{\varnothing, G\}$. We distinguish the following three cases.

(1) $\mathfrak{p} := P \cap X \neq \varnothing, X$. Then $P = \varphi^{-1}(\mathfrak{p}) \in \mathcal{S}$. Indeed, assuming the contrary, say $P \not\subseteq \varphi^{-1}(\mathfrak{p})$, let $g \in P$ be such that $\varphi(g) \notin \mathfrak{p}$. As $\mathfrak{p} \neq \varnothing$ by assumption, choose some $x \in \mathfrak{p} \subseteq P$. Since φ is a folding of the median set $(G, \widehat{m})$ with $\varphi(G) = X$, and P is convex, we get $\varphi(g) \in [x, g] \cap X \subseteq P \cap X = \mathfrak{p}$, i.e., a contradiction. The case $\varphi^{-1}(\mathfrak{p}) \not\subseteq P$ follows similarly by replacing P with $G \setminus P$.

(2) $X \subseteq P$. As $P \neq G$, choose an element $g \in G \setminus P$ of minimal length $l(g)$ over the alphabet $J := (H \setminus \{1\}) \sqcup I'^{\pm 1}$. In particular $g \neq 1$, i.e., $l(g) \geq 1$, since $1 \in X \subseteq P$ by assumption. Let $g = g't$ be a reduced word representing g, with $t \in J$. Note that $g' = gt^{-1} \in P$ since $l(g') < l(g)$. There are two possibilities.

(i) $t \in (H \setminus \{1\}) \sqcup I'^{-1}$. Then $t^{-1} \in g^{-1}P \cap X$, while $1 \in X \setminus g^{-1}P$. Consequently, $\mathfrak{p} := g^{-1}P \cap X \in Spec(X, m) \setminus \{\varnothing, X\}$, therefore $P = g\varphi^{-1}(\mathfrak{p}) \in \mathcal{S}$ by (1).

(ii) $t \in I'$. Then $t = g'^{-1}g \in X \setminus g'^{-1}P$ and $1 = g'^{-1}g' \in X \cap g'^{-1}P$, whence $\mathfrak{p} := X \cap g'^{-1}P \in Spec(X, m) \setminus \{\varnothing, X\}$, therefore $P = g'\varphi^{-1}(\mathfrak{p}) \in \mathcal{S}$ by (1) again.

(3) $P \cap X = \varnothing$. Then $G \setminus P \in \mathcal{S}$ by (2), and hence $P \in \mathcal{S}$ as desired. □

In particular, taking $H = 1$, we obtain

Corollary 2.28. *Let G be a group, $1 \in X \subseteq G$ a set of generators, and $\varphi : G \longrightarrow X$ a surjective map such that $\varphi(x) = x$ for all $x \in X$. We denote by $\mathcal{M}(X)$ the set of median operations on X, and by $\mathcal{M}(G, \varphi)$ the set of median group operations on G for which X is a retractible conves subset with associated folding φ.*

Then the restriction map $\mathcal{M}(G, \varphi) \longrightarrow \mathcal{M}(X)$ is injective.

With the notation from Corollary 2.28, call the surjective map $\varphi : G \longrightarrow X$ *admissible* if $\mathcal{M}(G, \varphi) \neq \varnothing$. We give in the following some simple examples of admissible maps.

Example 2.29. For $G = \langle g \,|\, g^4 = 1 \rangle \cong \mathbb{Z}/4, X = \{1, g\}$, the map $\varphi : G \longrightarrow X$, with $\varphi(g^2) = \varphi(g) = g$, $\varphi(g^3) = \varphi(1) = 1$, is the unique admissible map, and $\mathcal{M}(G, \varphi)$ consists of the unique median group operation on G - the square with the pairs of opposite vertices $(1, g^2)$ and (g, g^3) -, so the restriction map $\mathcal{M}(G, \varphi) \longrightarrow \mathcal{M}(X)$ is obviously bijective.

Example 2.30. For $G = (\mathbb{Z}, +), X = \{0, 1\}$, the surjective map $\varphi : G \longrightarrow X$ with $\varphi^{-1}(1) = \mathbb{Z}_{\geq 1}$ is the unique admissible map, and $\mathcal{M}(G, \varphi) = \{m_0\}$, where m_0 is the canonical median group operation on $\mathbb{Z}$, corresponding to the natural simplicial tree on $\mathbb{Z}$, so the restriction map $\mathcal{M}(G, \varphi) \longrightarrow \mathcal{M}(X)$ is obviously bijective.

Example 2.31. A more interesting case is $G = (\mathbb{Z}, +), X = \mathbb{N}$, where we have to find those median group operations m on $\mathbb{Z}$ satisfying the strong condition that the submonoid $(\mathbb{N}, +)$ is a retractible convex subset of $(\mathbb{Z}, m)$, whence, by translation with elements $n \in \mathbb{Z}$, $\mathbb{Z}_{\geq n}$ is also retractible convex in $(\mathbb{Z}, m)$. This task is easy since we already know, according to Example 2.25, that there are only three distinct median group operations m_0, m_1, m_{-1} on $(\mathbb{Z}, +)$. We see that only two of them, namely m_0 and m_1, satisfy the requirement, providing the admissible maps $\varphi_i : \mathbb{Z} \longrightarrow \mathbb{N}, i = 0, 1$, defined by

$$\varphi_0(n) = \begin{cases} n & \text{if } n \geq 0 \\ 0 & \text{if } n < 0, \end{cases}$$

with $\mathcal{M}(\mathbb{Z}, \varphi_0) = \{m_0\}$, and

$$\varphi_1(n) = \begin{cases} n & \text{if } n \geq 0 \\ 0 & \text{if } n < 0 \text{ and } n \in 2\mathbb{Z} \\ 1 & \text{if } n < 0 \text{ and } n \in 2\mathbb{Z} + 1, \end{cases}$$

with $\mathcal{M}(\mathbb{Z},\varphi_1)=\{m_1\}$.

Note that φ_1 is the folding $n\mapsto m_1(0,n,1)$ associated to the linear cell $[0,1]_{m_1}=\mathbb{N}$. Thus, with the exception of $m_0|_{\mathbb{N}}$ and $m_1|_{\mathbb{N}}$, the infinitely many median operations on the countable set $\mathbb{N}$ do not extend to median group operations on $(\mathbb{Z},+)$. To extend them to suitable median group operations we are forced to forget the *monoid* structure of $\mathbb{N}$ and look for larger groups containing the countable *set* $\mathbb{N}$ (see Corollary 4.7).

3 Simplicial trees induced by free actions on sets

The main goal of this section is to explore the underlying simplicial tree of a free action on an arbitrary nonempty set, as well as its extension to a simplicial tree on a group naturally associated to the given free action, in order to use it further for obtaining by suitable deformations more general arboreal structures.

Let H be a group acting freely on a nonempty set X. Let $B=\{b_i\,|\,i\in I\}\subseteq X$ be a set of representatives for the H-orbits. The bijection $H\times I\longrightarrow X,(h,i)\mapsto hb_i$ identifies up to isomorphism the H-set X with the cartesian product $H\times I$, with the canonical free action of the group H, $H\times(H\times I)\longrightarrow H\times I,(h_1,(h_2,i))\mapsto(h_1h_2,i)$.

We assume that $I\cap H=\{1\}$, and we shall take $b_1=(1,1)$ as *basepoint* in $X\cong H\times I$. Set $I':=I\setminus\{1\}, X':=\bigsqcup_{i\in I'}Hb_i\cong H\times I'$.

3.1 The underlying tree of the H-set X

The set $X\cong H\times I$ has a natural structure of simplicial tree with the elements of X as vertices, and the ordered pairs $(b_1,hb_1),h\in H\setminus\{1\}$ and $(hb_1,hb_i),h\in H,i\in I'$, as oriented edges. Taking b_1 as root, we obtain a rooted order-tree $(X,b_1,\le)$ with the partial order $\le$ given by

$$x<y\Longleftrightarrow \text{ either } x=b_1,y\ne b_1 \text{ or } \exists h\in H\setminus\{1\},i\in I',x=hb_1,y=hb_i,$$

the induced meet-semilattice operation $\wedge$, and the locally linear median operation

$$Y(x,y,z)=(x\wedge y)\vee(y\wedge z)\vee(z\wedge x)\in\{x\wedge y,y\wedge z,z\wedge x\}.$$

Remark 3.1. (1) With respect to the partial order $\le$, b_1 is the least element of X, while X' is the set of all maximal elements of X. In particular, for $x,y\in X,x\wedge y\in X'$ if and only if $x=y\in X'$, and hence $x\wedge y\in Hb_1$ provided $x\ne y$, whence $Y(x,y,z)\in Hb_1$ whenever $x\ne y,y\ne z,z\ne x$.

(2) Hb_1 is a retractible convex subset of the median set (X,Y), with associated folding $\theta:X\longrightarrow X$, $hb_i\mapsto hb_1$, compatible with the action of H, i.e., $\theta(hx)=h\theta(x)$ for all $h\in H,x\in X$.

(3) The median operation Y is *almost compatible* with the action of H on X in the following sense: $Y(hx,hy,hz)\in H\,Y(x,y,z)$ for all $h\in H,x,y,z\in X$, i.e., the map $Y:X^3\longrightarrow X$ induces a map $H\setminus X^3\longrightarrow H\setminus X$. Indeed, for $h\in H,x,y,z\in X$, we obtain

$$Y(hx,hy,hz)=\begin{cases}hY(x,y,z) & \text{if } |\{\theta(x),\theta(y),\theta(z)\}|\in\{1,2\},\\ Y(x,y,z)=b_1 & \text{if } |\{\theta(x),\theta(y),\theta(z)\}|=3.\end{cases}$$

Consequently, the necessary and sufficient condition for the median operation Y to be compatible with the action of H on X, i.e., $hY(x,y,z)=Y(hx,hy,hz)$ for all $h\in H,x,y,z\in X$, is that either $H=1$ or $H\cong\mathbb{Z}/2$.

3.2 The group $\widehat{H}$ and its underlying tree

We denote by F the free group with base I', and by $\widehat{H} := H * F$ the free product of the groups H and F. The group H is canonically identified with a subgroup of $\widehat{H}$, while the injective map $\iota : X \longrightarrow \widehat{H}, hb_i \mapsto hi$, identifies the H-set $X \cong H \times I$ with the disjoint union $H \bigsqcup (\bigsqcup_{i\in I'} Hi) \subseteq \widehat{H}$ on which H acts freely by left multiplication.

Using the natural tree structure of the free group F with base I', we extend as follows the underlying tree of the H-set X as defined in 3.1 to a simplicial tree on the underlying set of the free product $\widehat{H} = H * F$.

Let $l : \widehat{H} \longrightarrow \mathbb{N}$ denote the length function associated to the system of generators $J = J^{-1} := (H \setminus \{1\}) \bigsqcup I'^{\pm 1}$, so $l(w)$ is the minimum length of any expression $w = w_1 \cdots w_n$ with $w_k \in J$. In particular, $l(w) = 0 \iff w = 1$, $l(w^{-1}) = l(w)$ for all $w \in \widehat{H}$, and $l(uv) \leq l(u) + l(v)$ for all $u, v \in \widehat{H}$. Since $\widehat{H} = H * F$ and F is free with base I', it follows that the expression of minimal length above is unique for any $w \in \widehat{H}$; call it the *reduced normal form* of w, and set $\mathrm{o}(w) := w_1$, $\mathrm{t}(w) := w_n$ provided $l(w) = n \geq 1$. For $u, v \in \widehat{H}$, put $u \leq v \iff l(v) = l(u) + l(u^{-1}v)$, and write $v = u \bullet (u^{-1}v)$ provided $u \leq v$. The binary relation $\leq$ is a partial order extending the partial order $\leq$ on X as defined in 3.1. Moreover the partial order $\leq$ makes $\widehat{H}$ a *rooted order-tree* with 1 as distinguished base point, the *root*, while the corresponding meet-semilattice operation $\wedge$ and the locally linear median operation Y are extensions of the operations $\wedge$ and Y on X respectively.

For all $u, v \in \widehat{H}$, the cell $[u, v] := \{Y(u, v, w) \,|\, w \in \widehat{H}\}$ is the union of the closed intervals $[u \wedge v, u]$ and $[u \wedge v, v]$. Since for all $u, v \in \widehat{H}$, the cell $[u, v]$ has finitely many elements, $\widehat{H}$ is a $\mathbb{Z}$-tree with the distance function $d : \widehat{H} \times \widehat{H} \longrightarrow \mathbb{Z}$ defined by $d(u, v) := |[u, v]| - 1 = l(w^{-1}u) + l(w^{-1}v)$, where $w = u \wedge v$. Thus $d(u, v) = l(u^{-1}v)$ provided $u \leq v$, in particular, $d(u, 1) = l(u)$ for all $u \in \widehat{H}$, and hence

$$d(u, v) = d(u, u \wedge v) + d(v, u \wedge v) \geq l(u^{-1}v) \text{ for all } u, v \in \widehat{H}.$$

Consequently, $d(u, v) = l(u^{-1}v)$ if and onl if $u^{-1}v = (u^{-1}(u \wedge v)) \bullet ((u \wedge v)^{-1}v)$, while $d(u, v) = l(u^{-1}v) + 1$ otherwise. The latter situation holds whenever $u \neq u \wedge v \neq v$ and $\mathrm{o}((u\wedge v)^{-1}u), \mathrm{o}((u\wedge v)^{-1}v) \in H \setminus \{1\}$. In graph theoretic terms, the underlying tree of $\widehat{H}$ has the elements of $\widehat{H}$ as vertices, and the ordered pairs (u, v), with $u \leq v, l(u^{-1}v) = 1$, as oriented edges.

X is a retractible convex subset of the median set $(\widehat{H}, Y)$, with the canonical retract $\varphi : \widehat{H} \longrightarrow X$ defined by $\varphi(w) :=$ the greatest element $x \in X$ for which $x \leq w$, i.e., $w = x \bullet (x^{-1}w)$. Thus $\varphi(w) = h \in H \iff w = h$ or $hi^{-1} \leq w$ for some $i \in I'$, while $\varphi(w) = hi$ with $h \in H, i \in I' \iff hi \leq w$. In particular, $H = Hb_1$ is a retractible convex subset of the median set $(\widehat{H}, Y)$ with the retract $\widehat{\theta} := \theta \circ \varphi : \widehat{H} \longrightarrow H$. Thus $\widehat{\theta}(w) = 1 \iff$ either $w = 1$ or $\mathrm{o}(w) \in I'^{\pm 1}$, and $\widehat{\theta}(w) = h \in H \setminus \{1\} \iff \mathrm{o}(w) = h$.

Remark 3.2. (1) The maps φ and $\widehat{\theta}$ are H-equivariant.

(2) Let $u, v \in \widehat{H}$ be such that $u^{-1} \wedge v = 1$. Then $\varphi(uv) = \varphi(u)$ if and only if either $u \notin H$ or $\varphi(v) = 1$.
(3) $\varphi(ux) = \varphi(u)$ for all $u \in \widehat{H} \setminus H, x \in X$.

Some useful facts concerning the relation between the group $\widehat{H}$ and its underlying tree are collected in the next lemma.

Lemma 3.3. *The following assertions hold.*

(1) $u \leq v$ *and* $v \leq w \Longrightarrow w^{-1}v \leq w^{-1}u$.

(2) *For* $u, v \in \widehat{H}$, *let* $a := u^{-1} \wedge v, u' := ua, v' := a^{-1}v, u'' := u'\widehat{\theta}(u'^{-1}), v'' := \widehat{\theta}(v')^{-1}v', h := \widehat{\theta}(u'^{-1})^{-1}\widehat{\theta}(v') \in H$, *with* $h = 1 \Longleftrightarrow \widehat{\theta}(u'^{-1}) = \widehat{\theta}(v') = 1$. *Then* $uv = u''hv'' = u'' \bullet h \bullet v''$, *i.e.,* $l(uv) = l(u'') + l(h) + l(v'')$.
In particular, $u \leq v \Longleftrightarrow u^{-1} \wedge (u^{-1}v) = 1$ *and either* $\widehat{\theta}(u^{-1}) = 1$ *or* $\widehat{\theta}(u^{-1}v) = 1$.

(3) *The necessary and sufficient condition for* $\widehat{H}$ *together with the median operation* Y *to be a median group is that either* $H = 1$ *or* $H \cong \mathbb{Z}/2$.

(4) *For all* $s, u, v \in \widehat{H}, su \wedge sv \leq sY(u, v, s^{-1})$, *with* $(su \wedge sv)^{-1}sY(u, v, s^{-1}) \in H$.

(5) *For all* $u, v, w \in \widehat{H}, Y(t^{-1}u, t^{-1}v, t^{-1}w) = 1$, *where* $t := Y(u, v, w)$.

Proof. The proof of the assertions (1) and (2) is straightforward.

(3) It follows by Proposition 2.17 that the necessary and sufficient condition for $(\widehat{H}, Y)$ to be a median group is that $u^{-1}(u \wedge v) \leq u^{-1}v$ for all $u, v \in \widehat{H}$. According to (2), the latter condition holds if and only if for all $u, v \in \widehat{H}$, either $\widehat{\theta}((u \wedge v)^{-1}u) = 1$ or $\widehat{\theta}((u \wedge v)^{-1}v) = 1$. One checks easily that the last sentence is equivalent with $|H| \leq 2$.

(4) Let $s, u, v \in \widehat{H}$. Setting $a := u \wedge v \wedge s^{-1}, b := Y(u, v, s^{-1})$, we have to show that $sb = (su \wedge sv) \bullet p$ with $p \in H$. We distinguish the following three cases.

(4.1) $a = u \wedge s^{-1} = v \wedge s^{-1} \leq b = u \wedge v$: We may assume without loss that $a = 1$ since $s = s' \bullet a^{-1}, u = a \bullet u', v = a \bullet v'$ with $a' := u' \wedge s'^{-1} = v' \wedge s'^{-1} = 1$, therefore $su = s'u', sv = s'v'$, and

$$sY(u, v, s^{-1}) = s(u \wedge v) = s'(u' \wedge v') = s'Y(u', v', s'^{-1}),$$

so we may replace the elements u, v, s by u', v', s' respectively. As $u \wedge s^{-1} = 1$, it follows that either $su = s \bullet u$ or $su = (s' \bullet g)(h \bullet u') = s' \bullet (gh) \bullet u'$ where $g, h \in H \setminus \{1\}, gh \neq 1$. A similar alternative holds for the pair (s, v). Thus we have the following possible situations.

(4.1.1) $su = s \bullet u, sv = s \bullet v$: Then $su \wedge sv = s \bullet b$, so $p = 1$.

(4.1.2) $su \neq s \bullet u, sv = s \bullet v$: Then $s = s' \bullet g, u = h \bullet u'$ with $g, h \in H \setminus \{1\}, gh \neq 1$, therefore $su = s' \bullet (gh) \bullet u'$, and either $v = 1$ or $\mathrm{o}(v) \in I'^{\pm 1}$. Consequently, $b = 1, su \wedge sv = s', sb = s = s' \bullet p$ with $p = g \in H \setminus \{1\}$.

(4.1.3) $su = s \bullet u, sv \neq s \bullet v$: We proceed as in case (4.1.2).

(4.1.4) $su \neq s \bullet u, sv \neq s \bullet v$: Then $s = s' \bullet g, u = h_1 \bullet u', v = h_2 \bullet v'$ with $g, h_j \in H \setminus \{1\}$, $gh_j \neq 1, j = 1, 2$. We have the alternative: either $h := h_1 = h_2$ or $h_1 \neq h_2$. In the first case we obtain $b = h \bullet (u' \wedge v'), sb = su \wedge sv = s' \bullet (gh) \bullet (u' \wedge v')$, so $p = 1$, while in the second case we get $b = 1, su \wedge sv = s', sb = s = s' \bullet p$ with $p = g \in H \setminus \{1\}$ as desired.

(4.2) $a = u \wedge s^{-1} = u \wedge v < b = v \wedge s^{-1}$: As in case (4.1.), we may assume that $a = 1$. We get $s = s' \bullet b^{-1}, v = b \bullet v'$ with $u \wedge b = v' \wedge s'^{-1} = 1$. We have to show that $s' = (s'b^{-1}u \wedge s'v') \bullet p$ with $p \in H$. We have the following possible situations.

(4.2.1) $b^{-1}u = b^{-1} \bullet u, s'v' = s' \bullet v'$: Then $s'b^{-1}u \wedge s'v' = s'$, so $p = 1$.

(4.2.2) $b^{-1}u = b^{-1} \bullet u, s'v' \neq s' \bullet v'$: Then $s' = s'' \bullet g, v' = h \bullet v''$ with $g, h \in H \setminus \{1\}, gh \neq 1$. The desired result follows with $p = g \in H \setminus \{1\}$.

(4.2.3) $b^{-1}u \neq b^{-1} \bullet u, s'v' = s' \bullet v'$: Then $b = g \bullet b', u = h \bullet u'$ with $g, h \in H \setminus \{1\}, g \neq h$. First let us assume that $b' \neq 1$. Since $s = s' \bullet b^{-1}$, it follows that $s'b^{-1}u = s' \bullet b'^{-1} \bullet (g^{-1}h) \bullet u'$. As $v = b \bullet v' = g \bullet b' \bullet v'$, we deduce that $v' \wedge b'^{-1} = 1$. Since $s'v' = s' \bullet v'$ by assumption, the required result follows with $p = 1$.

Next let us assume that $b' = 1$, i.e. $b = g \in H \setminus \{1\}$. Then $b^{-1}u = (g^{-1}h) \bullet u'$, $s = s' \bullet b^{-1} = s' \bullet g^{-1}$, therefore either $s' = 1$ or $\mathrm{t}(s') \in I'^{\pm 1}$. Consequently, $s'b^{-1}u = s' \bullet (g^{-1}h) \bullet u'$. On the other hand, since $v = b \bullet v' = g^{-1} \bullet v'$, it follows that either $v' = 1$ or $\mathrm{o}(v') \in I'^{\pm 1}$, and hence $v' \wedge (g^{-1}h) = 1$. As $s'v' = s' \bullet v'$, we get as above the desired result with $p = 1$.

(4.2.4) $b^{-1}u \neq b^{-1} \bullet u, s'v' \neq s' \bullet v'$: According to (4.2.3) we get $s' \leq s'b^{-1}u$. On the other hand, it follows by assumption that $s' = s'' \bullet g', v' = h' \bullet v''$ with $g', h' \in H \setminus \{1\}, g'h' \neq 1$. Thus

$$s'v' \wedge s' = (s'' \bullet (g'h') \bullet v'') \wedge (s'' \bullet g') = s'' < s',$$

and hence the required result with $p = g' \in H \setminus \{1\}$.

(4.3) $a = v \wedge s^{-1} = u \wedge v < b = u \wedge s^{-1}$: We proceed as in case (4.2).

(5) Let $u, v, w \in \widehat{H}, t := Y(u, v, w)$. Since $Y(u, v, t) = t$, it follows by (4) that $t^{-1}u \wedge t^{-1}v \leq t^{-1}Y(u, v, t) = 1$, therefore $t^{-1}u \wedge t^{-1}v = 1$, and similarly, $t^{-1}v \wedge t^{-1}w = t^{-1}w \wedge t^{-1}u = 1$, and hence $Y(t^{-1}u, t^{-1}v, t^{-1}w) = 1$ as desired. □

4 Deformation of the underlying tree of $\widehat{H}$ into median group operations

Let H be a group acting freely on a nonempty set X. As shown in Section 3, the free action $H \times X \longrightarrow X$ is extended via the H-equivariant embedding $\iota : X \longrightarrow \widehat{H}$, with the H-equivariant retract $\varphi : \widehat{H} \longrightarrow X$, to the transitive and free action by left multiplication of the group $\widehat{H} = H * F$ on its underlying set. Thus the conditions (1) and (2) of Theorem 1.6 are obviously satisfied.

We denote by $\mathcal{M}(X)$ the set of all median operations m on X which are compatible with the action of H, while by $\mathcal{M}(\widehat{H}, \varphi)$ we denote the set of those median group operations $\widehat{m}$ on $\widehat{H}$ for which the retract φ is a folding identifying X with a retractible convex subset of the median set $(\widehat{H}, \widehat{m})$. We denote by $\mathcal{M}_l(X), \mathcal{M}_l(\widehat{H}, \varphi)$ ($\mathcal{M}_s(X), \mathcal{M}_s(\widehat{H}, \varphi)$) the subsets of $\mathcal{M}(X)$ and $\mathcal{M}(\widehat{H}, \varphi)$ respectively consisting of those median operations which are locally linear (simplicial).

According to Lemma 2.27, the restriction map $\mathrm{res} : \mathcal{M}(\widehat{H}, \varphi) \longrightarrow \mathcal{M}(X)$ is injective, so the induced maps $\mathrm{res}_l : \mathcal{M}_l(\widehat{H}, \varphi) \longrightarrow \mathcal{M}_l(X)$ and $\mathrm{res}_s : \mathcal{M}_s(\widehat{H}, \varphi) \longrightarrow \mathcal{M}_s(X)$ are injective too.

The present section is devoted to the proof of a more explicit version of Theorem 1.6. With the notation above we obtain

Theorem 4.1. *The map* $\mathrm{res} : \mathcal{M}(\widehat{H}, \varphi) \longrightarrow \mathcal{M}(X)$ *is bijective. Let* $m \in \mathcal{M}(X)$*. Then the following assertions hold.*

(1) *The unique median group operation* $\widehat{m} \in \mathcal{M}(\widehat{H}, \varphi)$ *extending* m *is a deformation of the underlying simplicial tree of* $\widehat{H}$ *induced by the median operation* m *and the retract* φ*, defined by*

$$\widehat{m}(u, v, w) = t\, m(\varphi(t^{-1}u), \varphi(t^{-1}v), \varphi(t^{-1}w))$$

for $u, v, w \in \widehat{H}$*, where* $t = Y(u, v, w)$*.*

(2) *The induced meet-semilattice operation* $u \cap v := \widehat{m}(u, 1, v)$ *is a deformation of the meet-semilattice operation* $\wedge$*, defined by*

$$u \cap v = (u \wedge v) m(\varphi((u \wedge v)^{-1} u), \varphi((u \wedge v)^{-1}), \varphi((u \wedge v)^{-1} v))$$

for $u, v \in \widehat{H}$.

(3) *The induced partial order* $\subset$ *is defined by*

$$u \subset v \Longleftrightarrow u \cap v = u \Longleftrightarrow (u \wedge v)^{-1} u \in [\varphi((u \wedge v)^{-1}), \varphi((u \wedge v)^{-1} v)] \subseteq X.$$

In particular, $u \subset v$ *whenever* $u \leq v$ *and* $1 \in [\varphi(u^{-1}), \varphi(u^{-1} v)]$.

Proof. Let $m \in \mathcal{M}(X)$. We have to define a median group operation $\widehat{m} \in \mathcal{M}(\widehat{H}, \varphi)$ such that $\widehat{m}(x, y, z) = m(x, y, z)$ for all $x, y, z \in X$. Since m is compatible with the action of H, the group H acts from the right on $Spec(X, m)$ according to the rule

$$\mathfrak{p}^h := h^{-1}\mathfrak{p} = \{h^{-1}x \mid x \in \mathfrak{p}\} \text{ for } \mathfrak{p} \in Spec(X, m), h \in H.$$

On the other hand, we consider the natural action from the right of the group $\widehat{H}$ on the power set $\mathcal{P}(\widehat{H})$, $(P, u) \mapsto P^u := u^{-1}P = \{u^{-1}v \mid v \in P\}$. Let

$$\mathcal{S} := \{\varphi^{-1}(\mathfrak{p})^u = u^{-1}\varphi^{-1}(\mathfrak{p}) \mid \mathfrak{p} \in Spec(X, m), u \in \widehat{H}\}$$

be the $\widehat{H}$-orbit of the subset $\varphi^{-1}(Spec(X, m))$.

The set $\mathcal{S}$ is closed under the involution $P \mapsto \widehat{H} \setminus P$ which is compatible with the action of $\widehat{H}$, i.e., $\widehat{H} \setminus (P^u) = (\widehat{H} \setminus P)^u$ for all $P \in \mathcal{S}, u \in \widehat{H}$.

Let $P = \varphi^{-1}(\mathfrak{p})^u \in \mathcal{S} \setminus \{\varnothing, \widehat{H}\}$, so $\mathfrak{p} \neq \varnothing, X$. Then $P \cap X = \{x \in X \mid \varphi(ux) \in \mathfrak{p}\}$. We distinguish the following three cases.

(i) $u \in H$: Then $P \cap X = \mathfrak{p}^u \in Spec(X, m) \setminus \{\varnothing, X\}$, and $P = \varphi^{-1}(\mathfrak{p}^u) \in \varphi^{-1}(Spec(X, m) \setminus \{\varnothing, X\})$.

(ii) $u \notin H$ and $\varphi(u) \in \mathfrak{p}$: Then $\varphi(ux) = \varphi(u) \in \mathfrak{p}$ for all $x \in X$, by Remark 3.2 (3), whence $X \subseteq P$.

(iii) $u \notin H$ and $\varphi(u) \in X \setminus \mathfrak{p}$: Then $X \subseteq \widehat{H} \setminus P$ by (ii), and hence $P \cap X = \varnothing$.

Consequently, the H-equivariant retract $\varphi : \widehat{H} \longrightarrow X$ to the H-equivariant embedding $X \longrightarrow \widehat{H}$ induces a H-equivariant embedding $Spec(X, m) \longrightarrow \mathcal{S}, \mathfrak{p} \mapsto \varphi^{-1}(\mathfrak{p})$, with the H-equivariant retract $\mathcal{S} \longrightarrow Spec(X, m), P \mapsto P \cap X$, and

$$\varphi^{-1}(Spec(X, m)) = \{\varnothing, \widehat{H}\} \cup \{P \in \mathcal{S} \mid P \cap X \notin \{\varnothing, X\}\}.$$

The group $\widehat{H}$ acts from the left on the power set $\mathcal{P}(\mathcal{S})$ according to the rule

$$u\mathcal{U} := \{P^{u^{-1}} = uP \mid P \in \mathcal{U}\} \text{ for } u \in \widehat{H}, \mathcal{U} \subseteq \mathcal{S}.$$

Define the negation operator on $\mathcal{P}(\mathcal{S}), \mathcal{U} \mapsto \neg\mathcal{U} := \{P \in \mathcal{S} \mid \widehat{H} \setminus P \notin \mathcal{U}\}$. It follows that $\neg\mathcal{U} \subseteq \neg\mathcal{V} \Longleftrightarrow \mathcal{V} \subseteq \mathcal{U}, \neg(\mathcal{U} \cup \mathcal{V}) = (\neg\mathcal{U}) \cap (\neg\mathcal{V}), \neg(\mathcal{U} \cap \mathcal{V}) = (\neg\mathcal{U}) \cup (\neg\mathcal{V})$, and $\neg(\neg\mathcal{U}) = \mathcal{U}$ for $\mathcal{U}, \mathcal{V} \subseteq \mathcal{S}$. In addition, the operator $\neg$ is compatible with the action of $\widehat{H}$, i.e., $\neg(u\mathcal{U}) = u(\neg\mathcal{U})$ for all $u \in \widehat{H}, \mathcal{U} \subseteq \mathcal{S}$.

The group $\widehat{H}$ and the power set $\mathcal{P}(\mathcal{S})$ are also related through the map

$$\widehat{H} \longrightarrow \mathcal{P}(\mathcal{S}), u \mapsto U(u) := \{P \in \mathcal{S} \mid u \notin P\}.$$

Note that $\neg U(u) = U(u)$, and $uU(v) = U(uv)$ for $u, v \in \widehat{H}$, so $\widehat{H}$ acts transitively on the set $\{U(u) \mid u \in \widehat{H}\}$ fixed by $\neg$. To show that the action is free, it suffices to show that $U(u) \nsubseteq U(1)$ for all $u \in \widehat{H} \setminus \{1\}$. We distinguish the following two cases.

(i) $\varphi(u) \neq 1$: Then, by Theorem 2.5 (1), there is $\mathfrak{p} \in Spec(X, m)$ such that $1 \in \mathfrak{p}$, $\varphi(u) \notin \mathfrak{p}$, and hence $\varphi^{-1}(\mathfrak{p}) \in U(u) \setminus U(1)$.

(ii) $\varphi(u) = 1$: Then, since $u \neq 1$, it follows that $u = i^{-1} \bullet v$ for some $i \in I', v \in \widehat{H}$, with $\varphi(v) \neq i$. Consequently, by Theorem 2.5 (1) again, there is $\mathfrak{p} \in Spec(X, m)$ such that $i \in \mathfrak{p}, \varphi(v) \notin \mathfrak{p}$, therefore $\varphi^{-1}(\mathfrak{p})^i \in U(u) \setminus U(1)$ as required.

To obtain the desired median group operation $\widehat{m}$ on $\widehat{H}$, identified via the injective map $u \mapsto U(u)$ with a $\widehat{H}$-subset of $\mathcal{P}(\mathcal{S})$, we have to show that the subset $\{U(u) \mid u \in \widehat{H}\}$ is closed under the canonical median operation $\mathfrak{m}$ on $\mathcal{P}(\mathcal{S})$

$$\mathfrak{m}(\mathcal{U}, \mathcal{V}, \mathcal{W}) := (\mathcal{U} \cap \mathcal{V}) \cup (\mathcal{V} \cap \mathcal{W}) \cup (\mathcal{W} \cap \mathcal{U}) = (\mathcal{U} \cup \mathcal{V}) \cap (\mathcal{V} \cup \mathcal{W}) \cap (\mathcal{W} \cup \mathcal{U}),$$

which is compatible with the action of $\widehat{H}$, i.e., $u\,\mathfrak{m}(\mathcal{U}, \mathcal{V}, \mathcal{W}) = \mathfrak{m}(u\mathcal{U}, u\mathcal{V}, u\mathcal{W})$ for $u \in \widehat{H}, \mathcal{U}, \mathcal{V}, \mathcal{W} \subseteq \mathcal{S}$.

Since the $U(u)$'s are fixed by the negation operator $\neg$, it suffices to show that for arbitrary elements $u, v, w \in \widehat{H}$, $\mathfrak{m}(U(u), U(v), U(w)) \subseteq U(tx)$, where $t := Y(u, v, w)$ and $x := m(\varphi(t^{-1}u), \varphi(t^{-1}v), \varphi(t^{-1}w))$. Using the compatibility of the median operation $\mathfrak{m}$ with the action of $\widehat{H}$ and the fact that $Y(t^{-1}u, t^{-1}v, t^{-1}w) = 1$ by Lemma 3.3 (5), it remains to show that $\mathfrak{m}(U(u), U(v), U(w)) \subseteq U(x)$, where $x := m(\varphi(u), \varphi(v), \varphi(w))$, for $u, v, w \in \widehat{H}$ with $Y(u, v, w) = 1$. Thus, it suffices to show that $U(u) \cap U(v) \subseteq U(x)$ provided $u \wedge v = 1$ and x belongs to the cell $[\varphi(u), \varphi(v)]$ of the median set (X, m). Assuming the contrary, let $P \in \mathcal{S}$ be such that $x \in P$, $u, v \notin P$, in particular, $P \neq \widehat{H}$ and $P \cap X \neq \varnothing$. We distinguish the following two cases.

(i) $\mathfrak{p} := P \cap X \neq X$, whence $P = \varphi^{-1}(\mathfrak{p})$. As $x \in [\varphi(u), \varphi(v)] \cap \mathfrak{p}$, it follows that either $\varphi(u) \in \mathfrak{p}$ or $\varphi(v) \in \mathfrak{p}$, whence either $u \in P$ or $v \in P$, and hence a contradiction.

(ii) $X \subseteq P$, whence $P = \varphi^{-1}(\mathfrak{q})^s$ for some $\mathfrak{q} \in Spec(X, m) \setminus \{\varnothing, X\}, s \in \widehat{H} \setminus H$, with $\varphi(s) \in \mathfrak{q}$. Since $u \wedge v = 1$ by assumption, it follows that either $s^{-1} \wedge u = 1$ or $s^{-1} \wedge v = 1$, therefore either $\varphi(su) = \varphi(s) \in \mathfrak{q}$ or $\varphi(sv) = \varphi(s) \in \mathfrak{q}$ according to Remark 3.2 (2). Consequently, either $u \in P$ or $v \in P$, again a contradiction.

Thus we have obtained the desired median group operation $\widehat{m} : \widehat{H}^3 \longrightarrow \widehat{H}$, inducing the meet-semilattice operation $\cap$ and the partial order $\subset$ as defined in the statements

(2) and (3) of the theorem. One checks easily that $\operatorname{res}(\widehat{m}) = m$ and φ is a folding of the median set $(\widehat{H}, \widehat{m})$ as required. According to Lemma 2.27, $\widehat{m}$ is unique with the properties above, and $Spec(\widehat{H}, \widehat{m}) = \mathcal{S}$. This completes the proof. □

Remark 4.2. An alternative more technical proof of Theorem 4.1 is given in [11, Theorem 3.1].

The next lemma provides equivalent descriptions for the partial order $\subset$; for other descriptions of the partial order $\subset$ see [11, Lemma 3.5].

Lemma 4.3. *Let $m \in \mathcal{M}(X)$ and $\subset$ be the partial order induced by the unique median group operation $\widehat{m} \in \mathcal{M}(\widehat{H}, \varphi)$ extending m. Then the following assertions are equivalent for $u, v \in \widehat{H}$.*

(1) $u \subset v$.

(2) $U(1) \cap U(v) \subseteq U(u)$, *where* $U(w) := \{P \in \mathcal{S} \,|\, w \notin P\}$ *for* $w \in \widehat{H}$, $\mathcal{S} := Spec(\widehat{H}, \widehat{m})$.

(3) *There exists* $w \in \widehat{H}$ *such that* $w \leq v$ *and* $w^{-1}u \in [\varphi(w^{-1}), \varphi(w^{-1}v)] \subseteq X$.

(4) *Either* $u = v = 1$ *or there is* $w \in \widehat{H}$ *such that* $w \leq v, \varphi(w^{-1}) \in I, \varphi(w^{-1}v) \neq 1$ *provided* $\varphi(w^{-1}) = 1$, *and* $w^{-1}u \in [\varphi(w^{-1}), \varphi(w^{-1}v)] \subseteq X$, *whence*

$$w\varphi(w^{-1}) \subset u \subset w\varphi(w^{-1}v) \subset v,$$

while $w\varphi(w^{-1}) < w \bullet \varphi(w^{-1}v) \leq v$.

Proof. (1) $\Longleftrightarrow$ (2) follows by Theorem 2.5 (1), while (4) $\Longrightarrow$ (3) is obvious.

(1) $\Longrightarrow$ (4). Let $u, v \in \widehat{H}$ be such that $u \subset v$. If $v = 1$ then $u = 1$, so let us assume $v \neq 1$. Setting $a := u \wedge v, u = a \bullet b, v = a \bullet c$ with $b \wedge c = 1$, we have by assumption $b \in [\varphi(a^{-1}), \varphi(c)]$. We distinguish the following three cases.

(i) $\varphi(a^{-1}) \in I$, with $\varphi(c) \neq 1$ provided $\varphi(a^{-1}) = 1$: Then $w := a$ satisfies the requirements.

(ii) $\varphi(a^{-1}) = \varphi(c) = 1$: Then $b = 1, u = a \leq v = u \bullet c$. As $v \neq 1$ it follows that either $u \neq 1$ or $c \neq 1$.

First assume that $u \neq 1$, whence $u = u' \bullet i$ with $i \in I'$ since $\varphi(u^{-1}) = 1, u \neq 1$ by assumption. Then $w := u'$ satisfies the required conditions provided $\varphi(u'^{-1}) \in I$. Assuming that $\varphi(u'^{-1}) \notin I$, we obtain $u = u'' \bullet h \bullet i$ with $h \in H \setminus \{1\}, \varphi(u''^{-1}) \in I$, therefore $w := u''$ satisfies the requirements.

Next assume that $c \neq 1$, whence $c = i^{-1} \bullet c'$ with $i \in I'$ since $\varphi(c) = 1, c \neq 1$. Then $w := u \bullet i^{-1}$ satisfies the required conditions.

(iii) $\varphi(a^{-1}) \notin I$, whence $a = w \bullet h$ with $h \in H \setminus \{1\}, \varphi(w^{-1}) \in I$: Then $\varphi(w^{-1}v) = \varphi(h \bullet c) = h \bullet \varphi(c) \neq 1$, and $w^{-1}u = h \bullet b \in h[\varphi(a^{-1}), \varphi(c)] = [\varphi(w^{-1}), \varphi(w^{-1}v)]$ as desired.

One checks easily that in all three cases above, $w\varphi(w^{-1}) < w\varphi(w^{-1}v) \leq v$ and $w\varphi(w^{-1}) \subset u \subset w\varphi(w^{-1}v) \subset v$.

(3) $\Longrightarrow$ (2). Let $u, v \in \widehat{H}$ be such that $w^{-1}u \in [\varphi(w^{-1}), \varphi(w^{-1}v)] \subseteq X$ for some $w \leq v$. Assuming that $U(1) \cap U(v) \nsubseteq U(u)$, let $P \in \mathcal{S}$ be such that $1, v \notin P, u \in P$, whence $w^{-1}, w^{-1}v \notin Q, w^{-1}u \in Q$, where $Q := P^w$. As $w^{-1}u \in X$ by assumption, it follows that $Q \cap X \neq \varnothing$, and hence there are the following two possibilities.

(i) $\mathfrak{q} := Q \cap X \neq X$, whence $Q = \varphi^{-1}(\mathfrak{q})$: As $w^{-1}, w^{-1}v \notin Q$, we obtain $\varphi(w^{-1})$, $\varphi(w^{-1}v) \in X \setminus \mathfrak{q}$, therefore, by Theorem 2.5 (1), $\mathfrak{q} \cap [\varphi(w^{-1}), \varphi(w^{-1}v)] = \varnothing$, contrary to the assumption that $w^{-1}u \in \mathfrak{q} \cap [\varphi(w^{-1}, \varphi(w^{-1}v)]$.

(ii) $X \subseteq Q$, whence $Q = \varphi^{-1}(\mathfrak{q})^s$ for some $\mathfrak{q} \in Spec(X, m) \setminus \{\varnothing, X\}, s \in \widehat{H} \setminus H$, with $\varphi(s) \in \mathfrak{q}$: As $w^{-1}, w^{-1}v \notin Q$, we obtain $\varphi(sw^{-1}), \varphi(sw^{-1}v) \in X \setminus \mathfrak{q}$. On the other hand, since $w \leq v$, it follows that $w^{-1} \wedge w^{-1}v = 1$, and hence either $s^{-1} \wedge w^{-1} = 1$ or $s^{-1} \wedge w^{-1}v = 1$. Consequently, by Remark 3.2 (2), we deduce that either $\varphi(sw^{-1}) = \varphi(s) \in \mathfrak{q}$ or $\varphi(sw^{-1}v) = \varphi(s) \in \mathfrak{q}$, i.e., a contradiction. This finishes the proof. □

To any $v \in \widehat{H} \setminus \{1\}$ we associate the following two sets

$$C_v := \{w \in \widehat{H} \,|\, w \leq v, \varphi(w^{-1}) \in I, \text{ and } \varphi(w^{-1}) = 1 \Longrightarrow \varphi(w^{-1}v) \neq 1\}$$

and

$$O_v := \{w\varphi(w^{-1}) \,|\, w \in C_v\},$$

together with the map $\zeta : C_v \longrightarrow O_v, w \mapsto w\varphi(w^{-1})$. It follows that $\zeta(w) \leq w$ for $w \in C_v$. C_v and O_v are nonempty finite sets, totally ordered with respect to $\leq$, and the map ζ is an isomorphism of totally ordered sets. Setting $C_v = \{w_i \,|\, i = \overline{1,\, n}\}$ with $n \geq 1, w_i < w_{i+1}$, it follows that $\zeta(w_1) = 1, w_i \leq \zeta(w_{i+1}) = w_i \bullet \varphi(w_i^{-1}v)$ for $i < n$, and $\zeta(w_n) < v = w_n \bullet \varphi(w_n^{-1}v)$. Thus the totally ordered finite set $([1, v], \leq)$ is the union of n adjacent proper closed intervals $J_i := [\zeta(w_i), \zeta(w_{l+1}], i = \overline{1,\, n-1}, J_n := [\zeta(w_n), v]$ with $w_i \in J_i, i = \overline{1,\, n}$. Call this decomposition in adjacent closed intervals the *combinatorial configuration associated to the element* $v \in \widehat{H} \setminus \{1\}$.

For instance, taking $v = i^{-2}hj^3g$ with $h, g \in H \setminus \{1\}, i, j \in I', l(v) = 7$, we obtain the totally ordered sets

$$C_v = \{i^{-1}, i^{-2}, i^{-2}hj, i^{-2}hj^2, i^{-2}hj^3\},\ O_v = \{1, i^{-1}, i^{-2}hj, i^{-2}hj^2, i^{-2}hj^3\}$$

of cardinality $n = 5$, and the adjacent closed intervals

$$J_1 = [1, i^{-1}], J_2 = [i^{-1}, i^{-2}hj], J_3 = [i^{-2}hj, i^{-2}hj^2], J_4 = [i^{-2}hj^2, i^{-2}hj^3], J_5 = [i^{-2}hj^3, v]$$

of cardinality $2, 4, 2, 2, 2$ respectively.

As a consequence of Lemma 4.3 and Corollaries 2.19, 2.20, we obtain

Corollary 4.4. *Let $m \in \mathcal{M}(X)$, and $\subset$ be the partial order induced by the unique median group operation $\widehat{m} \in \mathcal{M}(\widehat{H}, \varphi)$ extending m. Then, for any $v \in \widehat{H} \setminus \{1\}$, the cell $[1, v]$ of the median group $(\widehat{H}, \widehat{m})$, a bounded distributive lattice with respect to the partial* *order $\subset$, is a deformation of the combinatorial configuration associated to the element v, induced by the median operation m on X, in the following sense:*

Let $C_v = \{w_i \mid i = \overline{1,\, n}\}, O_v = \{\zeta(w_i) \mid i = \overline{1,\, n}\}$, and the adjacent closed intervals $J_i, i = \overline{1,\, n}, n \geq 1$, as defined above. Then the following assertions hold.

(1) $1 = \zeta(w_1) \subset \zeta(w_2) \subset \cdots \subset \zeta(w_n) \subset v$.

(2) *The cell $[1, v]$ of the median group $(\widehat{H}, \widehat{m})$ is the union of the adjacent cells*

$$\widehat{J_i} := [\zeta(w_i), \zeta(w_{i+1})] = w_i[\varphi(w_i^{-1}), \varphi(w_i^{-1}v)] \subseteq w_i X \textit{ for } i < n,$$

and $\widehat{J_n} := [\zeta(w_n), v] = w_n[\varphi(w_n^{-1}), \varphi(w_n^{-1}v)] \subseteq w_n X$, with the partial order $\subset$ given by

$$u \subset u' \iff \textit{either } i < k \textit{ or } i = k \textit{ and } w_i^{-1}u \in [\varphi(w_i^{-1}), w_i^{-1}u'] \subseteq X,$$

for $u \in \widehat{J_i}, u' \in \widehat{J_k}$.

Consequently, $(\widehat{H}, \widehat{m})$ is locally linear (simplicial) provided the median set (X, m) is locally linear (simplicial), so the restriction maps $\mathrm{res}_l : \mathcal{M}_l(\widehat{H}, \varphi) \longrightarrow \mathcal{M}_l(X)$ and $\mathrm{res}_s : \mathcal{M}_s(\widehat{H}, \varphi) \longrightarrow \mathcal{M}_s(X)$ are both bijective.

Thanks to Theorem 4.1 and Corollary 4.4, we can complete Lemma 2.12 as follows.

Corollary 4.5. *Let G be a group. Then the following assertions are equivalent.*

(1) *G acts freely on some nonempty median set (locally linear median set, simplicial median set).*

(2) *G is embeddable into the underlying group of some median group (locally linear median group, simplicial median group).*

Remark 4.6. According to Lemma 2.12, the groups acting freely on median sets form a quasivariety **MSFG** axiomatized by very simple quasi-identities. On the other hand, according to Corollary 4.5, the class of groups acting freely on locally linear median sets is axiomatized by the set of all universal sentences in the first order language of groups which are true in every locally linear median group. It would be of some interest to find a more concrete axiomatization, as well as a characterization of the finitely generated members of the class above. Concerning free actions on simplicial median sets, similar questions arise : characterize the (finitely generated) groups acting freely on simplicial median sets, as well as the models of the theory in the first order language of groups consisting of all universal sentences which are true in every simplicial median group.

In particular, taking $H = 1$ in the statements above, we obtain

Corollary 4.7. *Let $\mathbb{X} = (X, m)$ be a nonempty median set. Then there exist a median group $\mathbb{F} = (F, \widehat{m})$ and an embedding of median sets $\iota : \mathbb{X} \longrightarrow \mathbb{F}$ such that $1 \in \iota(X)$, $\iota(X) \setminus \{1\}$ freely generates the group F, and $\iota(X)$ is a retractible convex subset of $\mathbb{F}$. In addition, the median group $\mathbb{F}$ is locally linear (simplicial) provided the median set $\mathbb{X}$ is so.*

Example 4.8. Let F be the free group of rank 3 with generators $x_i, i = 1, 2, 3$, and let $X = \{1, x_1, x_2, x_3\}$. Define the retract $\varphi : F \longrightarrow X$ to the embedding $X \longrightarrow F$ by

$$\varphi(w) = \begin{cases} x_i & \text{if } x_i \le w, \\ 1 & \text{if either } w = 1 \text{ or } x_i^{-1} \le w \text{ for some } i \in \{1, 2, 3\}. \end{cases}$$

The finite set $\mathcal{M}(X)$ of median operations on X consists of 12 segments, 4 tripods and 3 squares. If m is the median operation of a segment or a tripod then the unique median group operation $\widehat{m} \in \mathcal{M}(F, \varphi)$ with res$(\widehat{m}) = m$ is the median operation of a simplicial tree on F determined by a suitable base. Thus, assuming that (X, m) is a segment, we distinguish the following two cases.

(S1) 1 is not an end point, say $X = [x_1, 1, x_2, x_3]$. Then the corresponding simplicial tree on F is determined by the base $\{x_1, x_2, x_2^{-1}x_3\}$. For the other 5 segments of type (S1), we proceed similarly via the action of the symmetric group S_3 on the set $X \setminus \{1\}$.

(S2) 1 is an end point, say $X = [1, x_1, x_2, x_3]$. Then the corresponding simplicial tree on F is determined by the base $\{x_1, x_1^{-1}x_2, x_2^{-1}x_3\}$, and for the other 5 segments of type (S2) we use as in case (S1) the action of S_3 on $X \setminus \{1\}$.

Assuming that (X, m) is a tripod, we also distinguish two cases.

(T1) 1 is not an end point, i.e., $m(x_1, x_2, x_3) = 1$, and hence the corresponding simplicial tree on F is determined by the base $\{x_1, x_2, x_3\}$.

(T2) 1 is an end point, say $m(1, x_1, x_2) = x_1 \cap x_2 = x_3$. Then the corresponding simplicial tree on F is determined by the base $\{x_3^{-1}x_1, x_3^{-1}x_2, x_3\}$, while for the other two tripods of type (T2) we proceed similarly.

On the other hand, if m is the median operation of the square with the pairs $(1, x_2), (x_1, x_3)$ of opposite vertices, then the corresponding median group $(F, \widehat{m})$ is simplicial, but not locally linear, with the set $\{w \in F \setminus \{1\} \,|\, [1, w] = \{1, w\}\} = \{x_1^{\pm 1}, x_3^{\pm 1}, (x_1^{-1}x_2)^{\pm 1}, (x_3^{-1}x_2)^{\pm 1}\}$ of cardinality 8, and the set $\{w \in F \,|\, [1, w] = \Box\} = \{x_2^{\pm 1}, (x_1^{-1}x_3)^{\pm 1}\}$ of cardinality 4. For any $v \in F \setminus \{1\}$, the cell $[1, v]$ is a finite union of adjacent segments and/or squares. For instance, taking $v = x_1^{-2}x_2x_3^2x_2$, the cell $[1, v]$ is the union of the 4 adjacent segments $[1, x_1^{-1}] = x_1^{-1}[x_1, 1], [x_1^{-1}, x_1^{-1}x_2] = x_1^{-2}[x_1, x_2], [x_1^{-2}x_2, x_1^{-2}x_2x_3] = x_1^{-2}x_2[1, x_3], [x_1^{-2}x_2x_3, x_1^{-2}x_2x_3^2] = x_1^{-2}x_2x_3[1, x_3]$ and of the square $[x_1^{-2}x_2x_3^2, v] = x_1^{-2}x_2x_3^2[1, x_2]$. The cases of the other two squares on X, with the pairs of opposite vertices $(1, x_1), (x_2, x_3)$ and $(1, x_3)$, (x_1, x_2) respectively, are similar.

5 The relatively-transitive closure of a free action on a median set

In this section we extend a given free action on a median set to a larger one with a suitable universal property. By iterating this construction, we shall obtain in the next section the *transitive closure* of any given free action on a median set.

Let H be a group acting freely on a median set $\mathbb{X} = (X, m)$. Fix as in the previous sections a set $B = \{b_i \,|\, i \in I\}$ of representatives of the H-orbits, with $1 \in I, I' := I \setminus \{1\}$. Let $\widehat{H} = H * F$ be the free product of H and the free group F with base I', and identify X to the H-subset $H \coprod (\coprod_{i \in I'} Hi) \subseteq \widehat{H}$, with the H-equivariant retract $\varphi : \widehat{H} \longrightarrow X$. With the notation above, the main result of this section (Theorem 1.8 from Introduction) reads as follows.

Theorem 5.1. *There exist a median set $\widehat{\mathbb{X}} = (\widehat{X}, \widehat{\mathfrak{m}})$ and a free action of $\widehat{H}$ on $\widehat{\mathbb{X}}$ such that, identifying $\widehat{H}$ with the $\widehat{H}$-orbit of a base point of $\widehat{\mathbb{X}}$, the composition of the maps*

$X \longrightarrow \widehat{H}$, $\widehat{H} \longrightarrow \widehat{X}$ *is a* H*-equivariant embedding of median sets* $\mathbb{X} \longrightarrow \widehat{\mathbb{X}}$ *satisfying the following universal property.*

(RTUP) *For every group* $\widetilde{H}$ *acting freely on a median set* $\widetilde{\mathbb{X}} = (\widetilde{X}, \widetilde{m})$*, every morphism* $(\psi_0, \psi) : (H, \mathbb{X}) \longrightarrow (\widetilde{H}, \widetilde{\mathbb{X}})$ *in the category* **FAMS** *of free actions on median sets, satisfying* $\psi(X) \subseteq \widetilde{H}\psi(1)$*, extends uniquely to a morphism* $(\widehat{\psi_0}, \widehat{\psi}) : (\widehat{H}, \widehat{\mathbb{X}}) \longrightarrow (\widetilde{H}, \widetilde{\mathbb{X}})$ *in* **FAMS***.*

In particular, the free action $(\widehat{H}, \widehat{\mathbb{X}})$ *satisfying* (RTUP)*, called the* relatively-transitive closure *of the free action* $(H, \mathbb{X})$*, is unique up to a unique isomorphism.*

Proof. To construct the median set $\widehat{\mathbb{X}}$, we consider the natural action from the right of $\widehat{H}$ on the power set $\mathcal{P}(\widehat{H})$, $P^u := u^{-1}P = \{u^{-1}v \,|\, v \in P\}$ for $u \in \widehat{H}, P \subseteq \widehat{H}$, and let

$$\mathfrak{S} := \{P \subseteq \widehat{H} \,|\, \forall u \in \widehat{H},\ P^u \cap X \in Spec\,\mathbb{X}\}.$$

The set $\mathfrak{S}$ is closed under the action of $\widehat{H}$ and the involution $P \mapsto \widehat{H} \setminus P$, and contains the $\widehat{H}$-set $\mathcal{S} := \{\varphi^{-1}(\mathfrak{p})^u \,|\, \mathfrak{p} \in Spec\,\mathbb{X}, u \in \widehat{H}\}$. Recall that, according to Theorem 4.1, $\mathcal{S} = Spec\,(\widehat{H}, \widehat{m})$, where $\widehat{m} : \widehat{H}^3 \longrightarrow \widehat{H}$ is the unique median group operation for which $\mathbb{X} = (X, m)$ is a retractible convex median subset of $(\widehat{H}, \widehat{m})$ with associated folding φ. Thus we obtain the H-equivariant embedding $Spec\,\mathbb{X} \longrightarrow \mathfrak{S}, \mathfrak{p} \mapsto \varphi^{-1}(\mathfrak{p})$ with the H-equivariant retract $\mathfrak{S} \longrightarrow Spec\,\mathbb{X}, P \mapsto P \cap X$.

The group $\widehat{H}$ acts canonically on the power set $\mathcal{P}(\mathfrak{S}) : u\,\mathfrak{M} := \{P^{u^{-1}} = uP \,|\, P \in \mathfrak{M}\}$ for $u \in \widehat{H}, \mathfrak{M} \subseteq \mathfrak{S}$, and the action is compatible with the negation operator $\mathfrak{M} \mapsto \neg\mathfrak{M} := \{P \in \mathfrak{S} \,|\, \widehat{H} \setminus P \notin \mathfrak{M}\}$. Consider the map $\widehat{H} \longrightarrow \mathcal{P}(\mathfrak{S}), u \mapsto \mathfrak{U}(u) := \{P \in \mathfrak{S} \,|\, u \notin P\}$ which maps $\widehat{H}$ onto the $\widehat{H}$-orbit of $\mathfrak{U}(1)$. Composing the map above with the restriction map $\mathcal{P}(\mathfrak{S}) \longrightarrow \mathcal{P}(\mathcal{S}), \mathfrak{M} \mapsto \mathfrak{M} \cap \mathcal{S}$, we obtain the injective map $\widehat{H} \longrightarrow \mathcal{P}(\mathcal{S})$ (see the proof of Theorem 4.1). Consequently, the map $\widehat{H} \longrightarrow \mathcal{P}(\mathfrak{S})$ is injective too, whence the transitive action of $\widehat{H}$ on the set $\{\mathfrak{U}(u) \,|\, u \in \widehat{H}\}$ is free. Note also that $\neg\mathfrak{U}(u) = \mathfrak{U}(u)$ for all $u \in \widehat{H}$.

Let us now consider the sublattice $\mathfrak{L}$ of the boolean algebra $\mathcal{P}(\mathfrak{S})$ generated by the subsets $\mathfrak{U}(u) = u\,\mathfrak{U}(1), u \in \widehat{H}$. As $\varnothing \in \mathfrak{U}(u), \widehat{H} \notin \mathfrak{U}(u)$ for all $u \in \widehat{H}$, it follows that $\mathfrak{L} \subseteq \mathcal{P}(\mathfrak{S}) \setminus \{\varnothing, \mathfrak{S}\}$. More precisely, any element of $\mathfrak{L}$ has the form $\bigcup_{i=1}^{n} \mathfrak{U}(F_i),\ n \geq 1$, where the F_i's are nonempty finite subsets of $\widehat{H}$, and

$$\mathfrak{U}(F_i) := \bigcap_{u \in F_i} \mathfrak{U}(u) = \{P \in \mathfrak{S} \,|\, P \cap F_i = \varnothing\},\ i = \overline{1,\, n}.$$

Consequently, $\mathfrak{L}$ is closed under the negation operator $\neg$, so $\mathfrak{L}$ is an unbounded distributive lattice with negation on which $\widehat{H}$ acts canonically, $\widehat{X} := \{\mathfrak{M} \in \mathfrak{L} \,|\, \neg\,\mathfrak{M} = \mathfrak{M}\}$ is closed under the underlying median operation on the distributive lattice $\mathfrak{L}$

$$\begin{aligned}\mathfrak{m}(\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{M}_3) &= (\mathfrak{M}_1 \cap \mathfrak{M}_2) \cup (\mathfrak{M}_2 \cap \mathfrak{M}_3) \cup (\mathfrak{M}_3 \cap \mathfrak{M}_1) \\ &= (\mathfrak{M}_1 \cup \mathfrak{M}_2) \cap (\mathfrak{M}_2 \cup \mathfrak{M}_3) \cap (\mathfrak{M}_3 \cup \mathfrak{M}_1),\end{aligned}$$

and the action of $\widehat{H}$ on $\mathfrak{L}$ induces an action on the median set $\widehat{\mathbb{X}} = (\widehat{X}, \widehat{\mathfrak{m}})$, with $\widehat{\mathfrak{m}} := \widehat{m}\,|_{\widehat{X}}$.

The injective map $X \longrightarrow \widehat{X}, x \mapsto \mathfrak{U}(x)$ identifies the median set $\mathbb{X} = (X, m)$ with a median (not necessarily convex) subset of $\widehat{\mathbb{X}} = (\widehat{X}, \widehat{\mathfrak{m}})$, i.e., $\widehat{\mathfrak{m}}(\mathfrak{U}(x), \mathfrak{U}(y), \mathfrak{U}(z)) = \mathfrak{U}(m(x, y, z))$ for $x, y, z \in X$.

According to Theorem 2.5, the restriction map $Spec\,\mathfrak{L} \longrightarrow Spec\,\widehat{\mathbb{X}}, \mathfrak{P} \mapsto \mathfrak{P} \cap \widehat{\mathbb{X}}$ maps homeomorphically the spectral space $Spec\,\mathfrak{L}$ of prime ideals of the distributive lattice with negation $\mathfrak{L}$ onto the spectral space $Spec\,\widehat{\mathbb{X}}$, the dual of the median set $\widehat{\mathbb{X}}$. On the other hand, let us consider the $\widehat{H}$-equivariant restriction map

$$Spec\,\mathfrak{L} \longrightarrow \mathcal{P}(\widehat{H}), \mathfrak{P} \mapsto \mathfrak{P} \cap \widehat{H} := \{u \in \widehat{H} \,|\, \mathfrak{U}(u) \in \mathfrak{P}\}.$$

First let us show that $P := \mathfrak{P} \cap \widehat{H} \in \mathfrak{S}$ for all $\mathfrak{P} \in Spec\,\mathfrak{L}$, i.e., the map above takes values in $\mathfrak{S}$. We have to show that for all $u \in \widehat{H}$, $\mathfrak{q} := P^u \cap X \in Spec\,\mathbb{X}$. Let $x, y \in \mathfrak{q}$, i.e., $x, y \in X, ux, uy \in P$, so $\mathfrak{U}(x), \mathfrak{U}(y) \in \mathfrak{P}^u := \{u^{-1}\mathfrak{M} \,|\, \mathfrak{M} \in \mathfrak{P}\} \in Spec\,\mathfrak{L}$. Since $\mathfrak{U}(z) \subseteq \mathfrak{U}(x) \cup \mathfrak{U}(y)$ for every element z contained in the cell $[x, y]_{\mathbb{X}}$ of the median set $\mathbb{X}$, and $\mathfrak{P}^u$ is an ideal of $\mathfrak{L}$, we deduce that $\mathfrak{U}(z) \in \mathfrak{P}^u$ for all $z \in [x, y]_{\mathbb{X}}$, whence $[x, y]_{\mathbb{X}} \subseteq \mathfrak{q}$ for all $x, y \in \mathfrak{q}$. On the other hand, let $x, y \in X \setminus \mathfrak{q}$, whence $\mathfrak{U}(x), \mathfrak{U}(y) \notin \mathfrak{P}^u$. Assuming that $z \in [x, y]_{\mathbb{X}} \cap \mathfrak{q}$, we obtain $\mathfrak{U}(x) \cap \mathfrak{U}(y) \subseteq \mathfrak{U}(z) \in \mathfrak{P}^u$. As $\mathfrak{P}^u$ is a prime ideal of $\mathfrak{L}$, it follows that either $\mathfrak{U}(x) \in \mathfrak{P}^u$ or $\mathfrak{U}(y) \in \mathfrak{P}^u$, i.e., a contradiction.

Next let us show that the well defined restriction map $Spec\,\mathfrak{L} \longrightarrow \mathfrak{S}$ is bijective. For any $P \in \mathfrak{S}$, we denote by $\mathrm{Id}(P)$ the ideal of $\mathfrak{L}$ generated by the subset $\{\mathfrak{U}(u) \,|\, u \in P\}$. Let us show that $\mathrm{Id}(P)$ is prime. Let $\mathfrak{M}_1, \mathfrak{M}_2 \in \mathfrak{L}$ be such that $\mathfrak{M}_1 \cap \mathfrak{M}_2 \in \mathrm{Id}(P)$, i.e., $\mathfrak{M}_1 \cap \mathfrak{M}_2 \subseteq \bigcup_{i=1}^{n} \mathfrak{U}(u_i)$ for some $u_i \in P, i = \overline{1, n}, n \geq 1$. Obviously, $P \notin \mathfrak{M}_1 \cap \mathfrak{M}_2$, say $P \notin \mathfrak{M}_1$. As an element of $\mathfrak{L}$, $\mathfrak{M}_1 = \bigcup_{j=1}^{m} \mathfrak{U}(F_j)$ for some nonempty finite subsets $F_j \subseteq \widehat{H}, j = \overline{1, m}$, $m \geq 1$. Consequently, there exist $v_j \in F_j \cap P, j = \overline{1, m}$, therefore $\mathfrak{M}_1 \subseteq \bigcup_{j=1}^{m} \mathfrak{U}(v_j) \in \mathrm{Id}(P)$, whence $\mathfrak{M}_1 \in \mathrm{Id}(P)$, i.e., $\mathrm{Id}(P)$ is prime as desired. To conclude that the map $Spec\,\mathfrak{L} \longrightarrow \mathfrak{S}$ is bijective, it remains to show that $P = \mathrm{Id}(P) \cap \widehat{H}$ for $P \in \mathfrak{S}$, and $\mathrm{Id}(\mathfrak{P} \cap \widehat{H}) = \mathfrak{P}$ for $\mathfrak{P} \in Spec\,\mathfrak{L}$. The inclusions $\subseteq$ are obvious, while the opposite inclusions follow by straightforward verifications.

The bijection above identifies $\mathfrak{S}$ with $Spec\,\widehat{\mathbb{X}} \cong Spec\,\mathfrak{L}$, the dual of the median set $\widehat{\mathbb{X}}$, generated (as median set) by its subset $\{\mathfrak{U}(u) \,|\, u \in \widehat{H}\}$, the $\widehat{H}$-orbit of $\mathfrak{U}(1)$. The $\widehat{H}$-equivariant embedding $\mathcal{S} = Spec\,(\widehat{H}, \widehat{m}) \longrightarrow \mathfrak{S} \cong Spec\,\widehat{\mathbb{X}}$ induces a $\widehat{H}$-equivariant surjective morphism of median sets $\pi : \widehat{\mathbb{X}} = (\widehat{X}, \widehat{\mathfrak{m}}) \longrightarrow (\widehat{H}, \widehat{m})$, with $\pi|_X = 1_X$; in particular, the action of $\widehat{H}$ on the median set $\widehat{\mathbb{X}}$ is free. Note also that we obtain a canonical H-equivariant surjective morphism of median sets $p : \widehat{\mathbb{X}} \longrightarrow \mathbb{X}$, a retract to the H-equivariant embbeding $\mathbb{X} \longrightarrow \widehat{\mathbb{X}}$, by composing $\pi : \widehat{X} \longrightarrow \widehat{H}$ with the folding $\varphi : \widehat{H} \longrightarrow X$.

Finally, it remains to show that the free action $(\widehat{H}, \widehat{\mathbb{X}})$ extending $(H, \mathbb{X})$ satisfies the universal property (RTUP). Let $(\psi_0, \psi) : (H, \mathbb{X}) \longrightarrow (\widetilde{H}, \widetilde{\mathbb{X}})$ be a morphism in **FAMS** such that $\psi(X) \subseteq \widetilde{H}\psi(1)$, so we can identify $\widetilde{H}$ with the $\widetilde{H}$-orbit $\widetilde{H}\psi(1) \subseteq \widetilde{X}$ and $\psi(1) \in \widetilde{X}$ with the neutral element of the group $\widetilde{H}$, whence the map $\psi : X \longrightarrow \widetilde{X}$ factorizes through $\widetilde{H}$ and $\psi_0 = \psi|_H$. As $X \subseteq \widehat{H}$ is stable under the left multiplication with elements of H and $\widehat{H} = H * F$ is the free product of $H \subseteq X$ and the free group F with base $I' \subseteq X$, the map ψ extends uniquely to the group morphism $\widehat{\psi_0} : \widehat{H} \longrightarrow \widetilde{H}$ defined by $\widehat{\psi_0}(h) = \psi(h)$ for $h \in H$, $\widehat{\psi_0}(i) = \psi(i)$ for $i \in I'$. To extend the group morphism $\widehat{\psi_0} : \widehat{H} \longrightarrow \widetilde{H}$ to the desired morphism $(\widehat{\psi_0}, \widehat{\psi}) : (\widehat{H}, \widehat{\mathbb{X}}) \longrightarrow (\widetilde{H}, \widetilde{\mathbb{X}})$ in **FAMS** it suffices, by duality cf. Theorem 2.5, to define the morphism $Spec(\widehat{\psi}) : Spec\,\widetilde{\mathbb{X}} \longrightarrow Spec\,\widehat{\mathbb{X}} \cong \mathfrak{S}$ over $Spec\,\mathbb{X}$ in the unique possible way : $\widetilde{P} \in Spec\,\widetilde{\mathbb{X}} \mapsto \widehat{\psi_0}^{-1}(\widetilde{P} \cap \widetilde{H})$. The latter map is well defined, i.e., $\widehat{\psi_0}^{-1}(\widetilde{P} \cap \widetilde{H})^u \cap X = \psi^{-1}(\widetilde{P}^{\widehat{\psi_0}(u)}) = \psi^{-1}(\widehat{\psi_0}(u)^{-1}\widetilde{P}) \in Spec\,\mathbb{X}$ for $u \in \widehat{H}, \widetilde{P} \in Spec\,\widetilde{\mathbb{X}}$, since for all $x \in X$, $u \in \widehat{H}, \widetilde{P} \in Spec\,\widetilde{\mathbb{X}}$, we obtain

$$\begin{aligned} x \in \widehat{\psi_0}^{-1}(\widetilde{P} \cap \widetilde{H})^u &\Longleftrightarrow \widehat{\psi_0}(ux) = \widehat{\psi_0}(u)\psi(x) \in \widetilde{P} \\ &\Longleftrightarrow \psi(x) \in \widetilde{P}^{\widehat{\psi_0}(u)} \in Spec\,\widetilde{\mathbb{X}} \\ &\Longleftrightarrow x \in \psi^{-1}(\widetilde{P}^{\widehat{\psi_0}(u)}) \in Spec\,\mathbb{X}. \end{aligned}$$

Note also that the map $Spec(\widehat{\psi}) : Spec\,\widetilde{\mathbb{X}} \longrightarrow Spec\,\widehat{\mathbb{X}}$ above is coherent, as required, since for every finite subset $F \subseteq \widehat{H}$, the inverse image of the basic quasicompact open set $\mathfrak{U}(F) = \{P \in \mathfrak{S} \mid P \cap F = \varnothing\}$ of the spectral space $\mathfrak{S} \cong Spec\,\widehat{\mathbb{X}}$ is the quasicompact open set $\{\widetilde{P} \in Spec\,\widetilde{\mathbb{X}} \mid \widetilde{P} \cap \widehat{\psi_0}(F) = \varnothing\}$ of $Spec\,\widetilde{\mathbb{X}}$. This finishes the proof. □

One checks easily that the correspondence $(H, \mathbb{X}) \mapsto (\widehat{H}, \widehat{\mathbb{X}})$ provided by the statement above extends to an endofunctor RTC : **FAMS** $\longrightarrow$ **FAMS** (the *relatively-transitive closure*), together with a natural monomorphism rtc : $1_{\mathbf{FAMS}} \longrightarrow$ RTC.

6 The transitive closure of a free action on a median set

We are now in position to prove Theorem 1.9 and its median group theoretic version Theorem 1.10 (see Section 1).

Starting from the endofunctor RTC : **FAMS** $\longrightarrow$ **FAMS** (the *relatively-transitive closure*) and the natural monomorphism rtc : $1_{\mathbf{FAMS}} \longrightarrow$ RTC as defined in Section 5, we consider the direct system $\mathbb{RTC} := (\mathrm{RTC}_n)_{n\in\mathbb{N}}$ of endofunctors of **FAMS**, defined inductively by $\mathrm{RTC}_0 = 1_{\mathbf{FAMS}}, \mathrm{RTC}_{n+1} = \mathrm{RTC} \circ \mathrm{RTC}_n$, with the connecting natural *monomorphisms* $\mathrm{rtc}_n : \mathrm{RTC}_n \longrightarrow \mathrm{RTC}_{n+1}$, defined by $\mathrm{rtc}_n = \mathrm{RTC}_n \cdot \mathrm{rtc}$ for $n \in \mathbb{N}$.

Theorem 6.1. *The category* **FTAMS** *of free and transitive actions on median sets is a reflective full subcategory of* **FAMS***, i.e., the embedding* ι : **FTAMS** $\longrightarrow$ **FAMS** *has a left adjoint* TC : **FAMS** $\longrightarrow$ **FTAMS** *(the* transitive closure*). More precisely, the following assertions hold.*

(1) *The endofunctor* $\iota \circ \mathrm{TC}$ *of* **FAMS** *is the* direct limit *of the direct system* $\mathbb{RTC}$ *of endofunctors of* **FAMS**.

(2) *The natural transformation* $\mathrm{TC} \circ \iota \longrightarrow 1_{\mathbf{FTAMS}}$*, the* counit *of the ajunction, is a natural* isomorphism.

(3) *The natural transformation* $\mathrm{tc} : 1_{\mathbf{FAMS}} \longrightarrow \iota \circ \mathrm{TC}$*, the* unit *of the adjunction, is a natural* monomorphism.

Proof. Let $(H, \mathbb{X})$ be an object of **FAMS**, so we may identify H with the H-orbit of a base point of the median set $\mathbb{X}$ and the latter with the neutral element 1 of H. Applying step by step the endofunctor RTC : **FAMS** $\longrightarrow$ **FAMS**, we obtain a chain of suitable embeddings

$$H_0 \longrightarrow X_0 \longrightarrow \cdots \longrightarrow H_n \longrightarrow X_n \longrightarrow H_{n+1} \longrightarrow X_{n+1} \longrightarrow \cdots ,$$

with $(H_0, \mathbb{X}_0) = (H, \mathbb{X}), (H_{n+1}, \mathbb{X}_{n+1}) = \mathrm{RTC}(H_n, \mathbb{X}_n)$ for $n \in \mathbb{N}$. It follows easily that the union $\mathrm{TC}(H, \mathbb{X}) := \bigcup_{n \in \mathbb{N}} H_n = \bigcup_{n \in \mathbb{N}} X_n$ becomes a median group, and hence an object of **FTAMS** extending $(H, \mathbb{X})$ with the desired universal property (TUP) cf. Theorem 1.9 from Section 1. The rest of assertions concerning the adjunction between the categories **FAMS** and **FTAMS** follow by straightforward verifications. $\square$

The next example ilustrates the complexity of the functorial construction above in the simplest nontrivial case of the free median group with one generator.

Example 6.2. Let $\mathbb{G} = (G, m)$ be a nontrivial median group, and let $g \in G \setminus \{1\}$. The underlying group $\widetilde{G}$ of the median subgroup $\widetilde{\mathbb{G}}$ of $\mathbb{G}$ generated by the element g is the union of the ascending chain $(G_n)_{n \in \mathbb{N}}$ of subgroups of G, as well as the union of the ascending chain $(\mathbb{X}_n)_{n \in \mathbb{N}}$ of median subsets of (G, m), defined inductively by $G_0 = 1$, $X_0 = \{1, g\}$, G_{n+1} is the subgroup generated by $X_n \subseteq G$, while $\mathbb{X}_n$ is the median subset generated by $G_n \subseteq G$ for $n \geq 1$. In other words, the free and transitive action of the at most countable group $\widetilde{G}$ on its underlying median set is the *transitive closure of* the trivial action of $G_0 = 1$ on $X_0 = \{1, g\}$ *inside* the free and transitive action of G on its underlying median set. If $\mathbb{G} = \widetilde{\mathbb{G}}$ is the free median group with one generator g, i.e., $\mathbb{G} \cong \mathrm{TC}(1, X_0)$, then $G_1 \cong \mathbb{Z}$, G_n is a proper free factor of the free group G_{n+1} for $n \geq 1$, so G is a free group of countable rank, and $\mathbb{X}_{n+1}$ is the median set *freely generated by* $G_{n+1} \subseteq X_{n+1}$ *over* the median set $\mathbb{X}_n \subseteq G_{n+1}$, for $n \in \mathbb{N}$; in particular, $\mathbb{X}_1 \cong \mathrm{fms}(\mathbb{Z})$ is the free median set of countable rank.

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