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# Embedding theorems for free actions on $\Lambda$-trees

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**Abstract**

Using suitable deformations of simplicial trees, we show that any free action on a median set can be extended to a free and transitive one.

**Keywords.** free action, transitive action, simplicial tree, median set, median group, folding, abelian $l$-group, $\Lambda$-tree, $\Lambda$-tree-group, Lyndon length function

## 1 Introduction

A natural generalization of simplicial trees (i.e. acyclic connected graphs) was introduced by Morgan and Shalen in the fundamental paper [20] under the name of $\Lambda$-*trees*. This notion is obtained from that of a simplicial tree, interpreted as a metric space with an integer-valued distance function, by replacing $\mathbb{Z}$ with any totally ordered abelian group $\Lambda$. A still more general notion is introduced and investigated in [6], [8], by taking $\Lambda$ an arbitrary abelian $l$-group.

**Definition 1.1.** Let $\Lambda$ be an abelian $l$-group. By a $\Lambda$-*tree* we understand a $\Lambda$-metric space $(X, d : X \times X \longrightarrow \Lambda)$ satisfying the following conditions, where we put

$$[x, y] := \{z \in X \mid d(x, z) + d(z, y) = d(x, y)\}$$

for $x, y \in X$.

(1) For all $x, y, z \in X$, the set $[x, y] \cap [y, z] \cap [z, x]$ is a singleton; denote its unique element $m(x, y, z)$, and call it the *median* of the triple $(x, y, z)$.

(2) For all $x, y \in X$, the map $\iota_{x,y} : [x, y] \longrightarrow [0, d(x, y)], z \mapsto d(x, z)$ is bijective.

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Any $\Lambda$-tree $(X,d)$ has an underlying structure $(X,m)$ of *median set (algebra)* [1], and $[x,y] = \{m(x,y,z) \mid z \in X\} = \{z \in X \mid m(x,y,z) = z\}$ for $x,y \in X$. Note that the median set $(X,m)$ is *locally linear* (cf. [12, Definition 1.4 (4), Lemma 2.7] provided $\Lambda$ is totally ordered.

A $\Lambda$-metric version for *median groups* (cf. [12, Definition 1.5]) is defined as follows.

**Definition 1.2.** By a $\Lambda$-*tree-group* we mean a group $G$ together with a map $d : G^2 \longrightarrow \Lambda$ satisfying the following conditions.

(1) $(G,d)$ is a $\Lambda$-tree.

(2) For all $g,h,u \in G$, $d(ug,uh) = d(g,h)$.

In the present work we use the embedding theorem for free actions on median sets [12, Theorem 1.6] to prove an analogous result for free actions on $\Lambda$-trees, where $\Lambda$ is an arbitrary abelian $l$-group, extending in this way [17, Theorem 5.4.] devoted to free and without inversions actions on $\Lambda$-trees, where $\Lambda$ is a totally ordered abelian group.

The main result reads as follows.

**Theorem** *Let $H$ be a group acting freely on a nonempty set $X$. Fix a basepoint $b_1 \in X$. For a given abelian $l$-group $\Lambda$, we denote by $\mathcal{T}_\Lambda(X)$ the set of those distance maps $d : X \times X \longrightarrow \Lambda$ for which $(X,d)$ is a $\Lambda$-tree, compatible with the free action of $H$, i.e. $d(hx,hy) = d(x,y)$ for all $x,y \in X, h \in H$. Then there exists a group $\widehat{H}$ containing $H$ as its subgroup, together with an embedding $\iota : X \longrightarrow \widehat{H}$ and a retract $\varphi : \widehat{H} \longrightarrow X$ such that the following hold.*

(1) *The maps $\iota$ and $\varphi$ are $H$-equivariant, i.e. $\iota(h \cdot x) = h\iota(x), \varphi(hu) = h \cdot \varphi(u)$ for all $h \in H, x \in X, u \in \widehat{H}$.*

(2) *$\iota(b_1) = 1$, so $\iota(Hb_1) = H$, and $\iota(X)$ generates the group $\widehat{H}$.*

(3) *For each $d \in \mathcal{T}_\Lambda(X)$ there exists uniquely a map $\widehat{d} : \widehat{H} \times \widehat{H} \longrightarrow \Lambda$ with the following properties.*

  (i) *$(\widehat{H}, \widehat{d})$ is a $\Lambda$-tree-group.*

  (ii) *$\widehat{d}$ extends $d$, i.e. $\widehat{d}(\iota(x), \iota(y)) = d(x,y)$ for all $x,y \in X$.*

[1] By a *median set* we mean a set $X$ together with a ternary operation $m : X^3 \longrightarrow X$, called *median*, satisfying the following equational axioms.

(1) $m(x,y,z) = m(y,x,z) = m(x,z,y)$.

(2) $m(x,y,x) = x$.

(3) $m(m(x,u,v), m(y,u,v), x) = m(x,u,v)$.

(iii) *The map* $\iota \circ \varphi : \widehat{H} \longrightarrow \widehat{H}$ *is a folding identifying* $X$ *with a retractible* $\Lambda$-*subtree of* $(\widehat{H}, \widehat{d})$, *i.e.* $\bigcap_{x \in X} [\ \iota(x), u\ ] = [\ \iota(\varphi(u)), u\ ]$ *for all* $u \in \widehat{H}$.

**Corollary** *Let* $H$ *be a group acting freely on a* $\Lambda$-*tree* $\mathbb{X}$, *where* $\Lambda$ *is an abelian* $l$-*group. Then there exists a group* $\widehat{H}$ *acting freely and transitively on a* $\Lambda$-*tree* $\widehat{\mathbb{X}}$, *together with a group embedding* $H \longrightarrow \widehat{H}$ *and a* $H$-*equivariant* $\Lambda$-*isometry* $\mathbb{X} \longrightarrow \widehat{\mathbb{X}}$.

The part I of the paper is organized in three sections. Section 1 introduces the reader to the basic notions and facts concerning median sets and median groups. In Section 2 we associate to an arbitrary free action $H \times X \longrightarrow X$ a group $\widehat{H}$ with an underlying tree structure, together with a natural embedding of the $H$-set $X$ into $\widehat{H}$. Section 3 is devoted to the proof of the main result stated above using a procedure for deformation of the underlying simplicial tree of $\widehat{H}$ introduced in Section 2 into a suitable median group operation $\widehat{m}$ which extends any given median operation $m$ on the $H$-set $X$.

The result above is extended in the part II of the paper to more general actions on median sets, in particular on more general $\Lambda$-trees, where $\Lambda$ is an arbitrary Abelian $l$-group. More precisely, we shall consider actions $G \times X \longrightarrow X$ satisfying the following two conditions

(i) for all $x, y \in X$, the stabilizers $G_x$ and $G_y$ are conjugate subgroups of the group $G$, and

(ii) for some (for all) $x \in X$, the stabilizer $G_x$ has a complement $H_x$ in $G$, i.e. $H_x \cap G_x = 1$ and $G = H_x \cdot G_x$.

# 2 $\Lambda$-trees and their subspaces

In the following we denote by $\Lambda$ an abelian $l$-group with the group operation $+$, the partial order $\leq$, and the (distributive) lattice operations $\wedge, \vee$. Set $\Lambda_+ := \{\lambda \in \Lambda \mid \lambda \geq 0\}$, $\lambda_+ := \lambda \vee 0, \lambda_- := (-\lambda)_+ = -(\lambda \wedge 0)$, $|\lambda| := \lambda \vee (-\lambda) = \lambda_+ + \lambda_-$ for $\lambda \in \Lambda$. The abelian $l$-group $\Lambda$ has a canonical subdirect representation into the product $\prod_{\mathfrak{p} \in \mathcal{P}(\Lambda)} \Lambda/\mathfrak{p}$ of its maximal totally ordered quotients, where $\mathfrak{p}$ ranges over the set $\mathcal{P}(\Lambda)$ of the minimal prime convex $l$-subgroups of $\Lambda$, in bijection with the set of the minimal prime convex submonoids of $\Lambda_+$ as well as with the set of the ultrafilters of the distributive lattice $(\Lambda_+; \wedge, \vee)$ with $0$ as the least element.

## 2.1 Remarkable classes of $\Lambda$-metric spaces

**Definition 2.1.** By a pre-$\Lambda$-*metric space* we understand a set $X$ together with a $\Lambda$-valued *distance map* $d : X^2 \longrightarrow \Lambda$ satisfying the following two conditions.

(1) For $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$.

(2) $d(x,y) = d(y,x)$ for all $x, y \in X$.

A pre-$\Lambda$-metric space $(X,d)$ is a $\Lambda$-*metric space* if the *triangle inequality*

(3) $d(x,y) \leq d(x,z) + d(z,y)$ for all $x, y, z \in X$

is satisfied.

Assuming that $(X,d)$ is a $\Lambda$-metric space, we obtain $0 = d(x,x) \leq d(x,y) + d(y,x) = 2d(x,y)$ for all $x, y \in X$, therefore the map $d$ takes values in $\Lambda_+$. Note that $\Lambda$ itself becomes a $\Lambda$-metric space with $d(\lambda, \lambda') := |\lambda - \lambda'|$ for $\lambda, \lambda' \in \Lambda$.

For $x, y \in X$, set $[x,y] := \{z \in X \mid d(x,z) + d(z,y) = d(x,y)\}$; thus, $x, y \in [x,y]$, and $d(x,z) \leq d(x,y)$ for all $z \in [x,y]$. We define the map $\iota_{x,y} : [x,y] \longrightarrow [0, d(x,y)]$ by $\iota_{x,y}(z) := d(x,z)$. Call *cell* of $(X,d)$ any subset of $X$ having the form $[x,y]$ with $x, y \in X$. For any cell $C \subseteq X$, its *boundary* is the nonempty subset $\partial C := \{x \in X \mid C = [x,y] \text{ for some } y \in X\}$.

The metric $d$ on a $\Lambda$-metric space $X$ induces the *betweenness relation*

$$xzy \text{ (read } z \text{ is between } x \text{ and } y) \Longleftrightarrow z \in [x,y].$$

Note that $[x,x] = \{x\}$, $[x,y] = [y,x]$, $[x,y] = [x,z] \Longrightarrow y = z$, and

$$z \in [x,y], u \in [x,z] \Longrightarrow z \in [u,y].$$

**Definition 2.2.** A $\Lambda$-metric space $(X,d)$ is called *median* if for all $x, y, z \in X$, the intersection $[x,y] \cap [y,z] \cap [z,x]$ consists of a single element.

**Remark 2.3.** (1) The name *median* is justified since, according to [2], [8, Proposition 3.1.], a $\Lambda$-metric space $(X,d)$ satisfying the condition above becomes a median set, where the median $m(x,y,z)$ of any triple $(x,y,z)$ of elements of $X$ is the unique element of the set $[x,y] \cap [y,z] \cap [z,x]$.

(2) According to [8, Lemma 6.1.], a necessary and sufficient condition for a $\Lambda$-metric space $(X,d)$ to be median is that for all $x, y, z \in X$ there exists $u \in [y,z]$ such that $[x,y] \cap [x,z] = [x,u]$.

(3) In a median $\Lambda$-metric space $(X,d)$, the map $\iota_{x,y} : [x,y] \longrightarrow [0, d(x,y)]$ is not necessarily injective for all $x, y \in X$. The simplest example of a median $\mathbb{Z}$-metric space, where the $\iota$'s are surjective but not all are injective, is the square $X = \mathbb{Z}/4$ with

$$d(n \bmod 4, n+1 \bmod 4) = 1,\ d(n \bmod 4, n+2 \bmod 4) = 2,$$

so $X = [0 \bmod 4, 2 \bmod 4] = [1 \bmod 4, 3 \bmod 4]$.

**Definition 2.4.** Let $\mathbb{X} = (X, d)$ be a median $\Lambda$-metric space, and let $m$ be the induced median operation.

(1) $\mathbb{X}$ is called a *pre-$\Lambda$-tree* if for all $x, y \in X$, the map $\iota_{x,y} : [x, y] \longrightarrow [0, d(x, y)]$ is a $\Lambda$-isometry, i.e., $d(u, v) = |d(x, u) - d(x, v)|$ for all $u, v \in [x, y]$; in particular, the map $\iota_{x,y}$ is injective for all $x, y \in X$.

(2) $\mathbb{X}$ is called a *$\Lambda$-tree* if it is *locally faithfully full* (cf. [6, 1.3.]), i.e., for all $x, y \in X$, the map $\iota_{x,y} : [x, y] \longrightarrow [0, d(x, y)]$ is bijective.

**Remark 2.5.** (1) The necessary and sufficient condition for a median $\Lambda$-metric space $\mathbb{X} = (X, d, m)$ to be a pre-$\Lambda$-tree is that $d(x, m(x, u, v)) = d(x, u) \wedge d(x, v)$ for all $x, y \in X, u, v \in [x, y]$, whence the map $\iota_{x,y} : [x, y] \longrightarrow [0, d(x, y)]$ is an injective morphism of bounded distributive lattices for all $x, y \in X$.

(2) If $\mathbb{X} = (X, d)$ is a $\Lambda$-tree then for all $x, y \in X$, the map $\iota_{x,y} : [x, y] \longrightarrow [0, d(x, y)]$ is an isomorphism of $\Lambda$-metric spaces, and also an isomorphism of bounded distributive lattices; in particular, any $\Lambda$-tree is a pre-$\Lambda$-tree.

(3) Assuming that $\Lambda$ is totally ordered, we obtain the Morgan-Shalen $\Lambda$-trees [20], [1].

(4) The abelian $l$-group $\Lambda$, with $d(\lambda, \lambda') = |\lambda - \lambda'|$ for $\lambda, \lambda' \in \Lambda$, is obviously a $\Lambda$-tree.

(5) Let $\mathbb{X} = (X, d)$ be a $\Lambda$-tree. The canonical subdirect product representation of the abelian $l$-group $\Lambda$, $\Lambda \longrightarrow \prod_{\mathfrak{p} \in \mathcal{P}(\Lambda)} \Lambda_{\mathfrak{p}}$, with $\Lambda_{\mathfrak{p}} := \Lambda/\mathfrak{p}$ totally ordered, induces a canonical subdirect product representation $\mathbb{X} \longrightarrow \prod_{\mathfrak{p} \in \mathcal{P}(\Lambda)} \mathbb{X}_{\mathfrak{p}}$, where the $\Lambda_{\mathfrak{p}}$-tree $\mathbb{X}_{\mathfrak{p}} := (X_{\mathfrak{p}}, d_{\mathfrak{p}} : X_{\mathfrak{p}}^2 \longrightarrow \Lambda_{\mathfrak{p}})$ is the quotient of $\mathbb{X}$ by the congruence $x \sim_{\mathfrak{p}} y \Longleftrightarrow d(x, y) \in \mathfrak{p}$ for $x, y \in X$, and $d_{\mathfrak{p}}(x_{\mathfrak{p}}, y_{\mathfrak{p}}) = d(x, y) \bmod \mathfrak{p}$ for $x_{\mathfrak{p}} = x \bmod \sim_{\mathfrak{p}}, y_{\mathfrak{p}} = y \bmod \sim_{\mathfrak{p}}$ in $X_{\mathfrak{p}} = X/\sim_{\mathfrak{p}}$. Though, in general, we prefer direct proofs, sometimes we shall use the canonical representation above to transfer known results for $\Lambda$-trees with $\Lambda$ totally ordered (the *local* case) to $\Lambda$-trees with $\Lambda$ an abelian $l$-group (the *global* case).

(6) $\Lambda$-trees, where $\Lambda$ is an abelian $l$-group, not necessarily totally ordered, occur in a natural way in various algebraic and geometric contexts. Thus, the residue structures induced by Prüfer extensions have underlying $\Lambda$-tree structures with suitable abelian $l$-groups $\Lambda$ (cf. [6, 11]).

(7) Any median $\Lambda$-metric space is a pre-$\Lambda$-tree provided the induced median operation is locally linear, though $\Lambda$ is not necessarily totally ordered.

(8) A simple example of a pre-$\Lambda$-tree which is not a $\Lambda$-tree is obtained as follows: let $\Lambda = \mathbb{Z}\times\mathbb{Z}$ with $\Lambda_+ = \mathbb{N}\times\mathbb{N}$, and $X = \{a, b, c\} \subseteq \Lambda$, with $a = (0, 0), b = (1, 0), c = (0, 1)$, and the induced $\Lambda$-metric $d : X^2 \longrightarrow \Lambda$. $(X, d)$ is a pre-$\Lambda$-tree, but not a $\Lambda$-tree, since the injective map $\iota_{b,c}$ is not surjective : $[b, c] = X$ is of cardinality $3$, while the cardinality of $[0, d(b, c)]$ is $4$. Adding the point $(1, 1)$ to $X$, we obtain the $\Lambda$-tree closure of the pre-$\Lambda$-tree $(X, d)$ (see Corollary 2.11).

## 2.2 An equivalent description of $\Lambda$-trees

The next lemma introduces a class of $\Lambda$-metric spaces which contains the pre-$\Lambda$-trees as a proper subclass.

**Lemma 2.6.** *Let $(X, d)$ be a nonempty pre-$\Lambda$-metric space. Set*

$$(x, y)_u := \frac{1}{2}(d(x, u) + d(u, y) - d(x, y)) \in \mathbb{Z}[\frac{1}{2}] \otimes_{\mathbb{Z}} \Lambda \text{ for } x, y, u \in X.$$

*Fix a base point $a \in X$, and assume that the following two conditions are satisfied.*

(S1)$_a$ $(x, y)_a \in \Lambda$ *for all* $x, y \in X$.

(S2)$_a$ $(x, y)_a \geq (x, z)_a \wedge (y, z)_a$ *for all* $x, y, z \in X$.

*Then the following assertions hold.*

(1) $(x, y)_u \in \Lambda_+$ *for all $x, y, u \in X$, in particular, $(S1)_u$ holds for all $u \in X$.*

(2) $(X, d)$ *is a $\Lambda$-metric space.*

(3) $(x, y)_u \geq (x, z)_u \wedge (y, z)_u$ *for all $x, y, z, u \in X$, i.e., $(S2)_u$ holds for all $u \in X$.*

(4) *For all $x, y \in X$, the map $\iota_{x,y} : [x, y] \longrightarrow [0, d(x, y)]$ is a $\Lambda$-isometry. In particular, for $x, y \in X$, $\iota_{x,y}$ is injective, and for $z, u \in [x, y]$, $z \in [x, u]$ if and only if $d(x, z) \leq d(x, u)$.*

(5) *For all $x, y, z \in X$, the set $[x, y] \cap [y, z] \cap [z, x]$ has at most one element.*

(6) *For all $x, y, u, v \in X$, $u, v \in [x, y]$ if and only if $[u, v] \subseteq [x, y]$.*

*Proof.* Taking $z = a$ in (S2)$_a$, we obtain $(x, y)_a \geq (x, a)_a \wedge (y, a)_a = 0$, therefore $(x, y)_a \in \Lambda_+$ for all $x, y \in X$ by (S1)$_a$. Further it follows by (S2)$_a$ that $d(x, a) = (x, x)_a \geq (x, y)_a$, and, similarly, $d(y, a) \geq (x, y)_a$, and hence

$$(x, y)_a \in [0, d(x, a) \wedge d(y, a)] \subseteq \Lambda_+ \text{ for all } x, y \in X.$$

(1) We obtain

$$\begin{aligned}(x,y)_u &= d(u,a)-(x,u)_a-(y,u)_a+(x,y)_a\\ &\geq d(u,a)-(x,u)_a-(y,u)_a+((x,u)_a\wedge(y,u)_a)\\ &=(d(u,a)-(x,u)_a)\wedge(d(u,a)-(y,u)_a)\geq 0,\end{aligned}$$

and hence $(x,y)_u\in\Lambda_+$ for all $x,y,u\in X$, as desired.

(2) is a consequence of (1).

Note that (3) is equivalent with the inequality $A+B\geq(A_1+B_1)\wedge(A_2+B_2)$, where

$$A:=(x,y)_a, B:=(z,u)_a, A_1:=(x,u)_a, B_1:=(y,z)_a, A_2:=(y,u)_a, B_2:=(x,z)_a.$$

By (S2)$_a$ we deduce that

$$A\geq(A_1\wedge A_2)\vee(B_1\wedge B_2)=(A_1\vee B_1)\wedge(A_1\vee B_2)\wedge(A_2\vee B_1)\wedge(A_2\vee B_2),$$

and

$$B\geq(A_1\wedge B_2)\vee(B_1\wedge A_2)=(A_1\vee B_1)\wedge(A_1\vee A_2)\wedge(B_1\vee B_2)\wedge(A_2\vee B_2).$$

Using the identity $(\bigwedge_{1\leq i\leq n}a_i)+(\bigwedge_{1\leq i\leq n}b_i)=\bigwedge_{1\leq i,j\leq n}(a_i+b_j)$, it follows that

$$A+B\geq(A_1+B_1)\wedge(A_2+B_2)\wedge((A_1\vee B_1)+(A_2\vee B_2))=(A_1+B_1)\wedge(A_2+B_2),$$

as required, since

$$(A_1\vee B_1)+(A_2\vee B_2)-((A_1+B_1)\wedge(A_2+B_2))=|(A_1\wedge B_1)-(A_2\wedge B_2)|\geq 0.$$

(4) Let $z,u\in[x,y]$, i.e., $(z,y)_x=d(x,z),(u,y)_x=d(x,u)$. By (3) we get $(z,u)_x\geq(z,y)_x\wedge(u,y)_x=d(x,z)\wedge d(x,u)$, therefore $d(z,u)\leq|d(x,z)-d(x,u)|$. On the other hand, $|d(x,z)-d(x,u)|\leq d(z,u)$ by the triangle inequality, since $(X,d)$ is a $\Lambda$-metric space by (2). Thus, the map $\iota_{x,y}$ is a $\Lambda$-isometry for all $x,y\in X$.

(5) Let $x,y,z,u,v\in X$ be such that $u,v\in[x,y]\cap[y,z]\cap[z,x]$. Then $d(x,u)=(y,z)_x=d(x,v)$, and hence $u=v$ by (4).

(6) Let $u,v\in[x,y], z\in[u,v]$, i.e. $(x,y)_u=(x,y)_v=(u,v)_z=0$. We have to show that $z\in[x,y]$, i.e., $(x,y)_z=0$. As $(u,v)_z=0$ by assumption, it follows by (3) that $(x,u)_z\wedge(x,v)_z=0$, therefore $d(x,z)=(d(x,u)-d(u,z))\vee(d(x,v)-d(v,z))$. Similarly, we obtain $d(y,z)=(d(y,u)-d(u,z))\vee(d(y,v)-d(v,z))$, and hence

$$0\leq(x,y)_z=(-d(u,z))\vee(-d(v,z))\vee(-(x,u)_v)\vee(-(x,v)_u)\leq 0$$

by (1), so $(x,y)_z=0$ as desired. □

**Lemma 2.7.** *Let $\mathbb{X} = (X, d, m)$ be a pre-$\Lambda$-tree. Then $\mathbb{X}$ satisfies $(S1)_a$ and $(S2)_a$ for any element $a \in X$.*

*Proof.* (S1)$_a$ is satisfied since for all $x, y \in X$, $(x, y)_a = d(a, m(a, x, y)) \in \Lambda$.

To check (S2)$_a$, let $x, y, z \in X$. As $(X, m)$ is a median set, it follows that

$$\begin{aligned}[a, m(a, x, z)] \cap [a, m(a, y, z)] &= [a, m(a, m(a, x, z), m(a, y, z))] \\ &= [a, m(a, m(a, x, y), z)] \\ &= [a, z] \cap [a, m(a, x, y)].\end{aligned}$$

Since the map $\iota_{a,z}$ is a $\Lambda$-isometry, we deduce that

$$\begin{aligned}(x, y)_a = d(a, m(a, x, y)) &\geq d(a, m(a, m(a, x, z), m(a, y, z)) \\ &= d(a, m(a, x, z)) \wedge d(a, m(a, y, z)) = (x, z)_a \wedge (y, z)_a\,,\end{aligned}$$

as required. □

The next statement provides an equivalent description of $\Lambda$-trees. It is a slight extension of [6, Proposition 1.3.], [8, Proposition 6.2.]) and also a generalization of the characterization of $\Lambda$-trees, where $\Lambda$ is a totally ordered abelian group (cf. [1, Theorem 3.17.]

**Proposition 2.8.** *Let $(X, d)$ be a nonempty pre-$\Lambda$-metric space. For any $a \in X$, the following assertions are equivalent.*

(1) *$(X, d)$ is a $\Lambda$-tree.*

(2) *$(X, d)$ satisfies $(S1)_a, (S2)_a$, and*

  (T) *for all $x, y \in X$, the map $\iota_{x,y} : [x, y] \longrightarrow [0, d(x, y)]$ is onto.*

(3) *$(X, d)$ satisfies the conditions $(S1)_a, (S2)_a$,*

  (T)$_a$ *for all $x \in X$, the map $\iota_{a,x} : [a, x] \longrightarrow [0, d(a, x)]$ is onto, and*

  (T') *for all $x, y \in X, z \in [x, y]$, there is $u \in [x, y]$ such that $d(x, u) = d(y, z)$.*

*Proof.* (1) $\Longrightarrow$ (2) follows by Lemma 2.7, while (2) $\Longrightarrow$ (3) is obvious.

(3) $\Longrightarrow$ (1). Assume that the pre-$\Lambda$-metric space $(X, d)$ satisfies the assertion (3) of the statement. Thus, $(X, d)$ is a $\Lambda$-metric space by Lemma 2.6 (2), and for all $x \in X$, the map $\iota_{a,x}$ is bijective by (T)$_a$ and Lemma 2.6 (4). Note also that for all $x, y \in X$, there exists uniquely $u := m(a, x, y) \in [a, x] \cap [a, y] \cap [x, y]$. Indeed, there is uniquely $u \in [a, x]$ such that $d(a, u) = (x, y)_a$ since $\iota_{a,x}$ is bijective and $(x, y)_a \in [0, d(a, x)]$, so we have to show that $d(y, u) = (a, x)_y$. By Lemma 2.6 (1, 3), we obtain

$$d(y, u) - (a, x)_y = 2((a, y)_u \wedge (x, y)_u) \in [0, 2(a, x)_u] = \{0\},$$

therefore $d(y,u) = (a,x)_y$ as desired.

According to Lemma 2.6 (2, 4, 5), it remains to show that for all $x, y \in X$, the map $\iota_{x,y}$ is onto (and hence bijective), while the set $[x,y] \cap [y,z] \cap [z,x]$ is nonempty (and hence a singleton) for all $x, y, z \in X$.

Let $x, y \in X, \lambda \in [0, d(x,y)]$. We have to show that there exists $z \in [x,y]$ such that $d(x,z) = \lambda$. Let $w := m(a,x,y)$,

$$\begin{aligned}\mu &:= d(a,x) - (\lambda \wedge d(x,w)) \\ &= (d(a,x) - \lambda) \vee d(a,w) \in [d(a,w), d(a,x)],\end{aligned}$$

and

$$\begin{aligned}\zeta &:= d(a,y) - ((d(x,y) - \lambda) \wedge d(y,w)) \\ &= (d(a,y) + \lambda - d(x,y)) \vee d(a,w) \in [d(a,w), d(a,y)].\end{aligned}$$

As the maps $\iota_{a,x}$ and $\iota_{a,y}$ are bijective, there exist uniquely $u \in [a,x]$ , $v \in [a,y]$ such that $d(a,u) = \mu, d(a,v) = \zeta$. As $\mu, \zeta \geq d(a,w)$ and the maps $\iota_{a,x}, \iota_{a,y}$ are $\Lambda$-isometries, we deduce that $w \in [a,u] \cap [a,v]$, $u \in [w,x], v \in [w,y]$, therefore $u, v \in [x,y]$, and hence $[u,v] \subseteq [x,y]$ by Lemma 2.6 (6). On the other hand, since the map $\iota_{x,y}$ is a $\Lambda$-isometry by Lemma 2.6 (4), it follows that $w \in [u,v]$, and hence $w = m(a,u,v)$. According to (T') there is $z \in [u,v]$ such that $d(u,z) = d(v,w)$, therefore $d(v,z) = d(u,w)$. As $d(x,y) = d(x,w) + d(w,y) = d(x,u) + d(u,z) + d(z,v) + d(v,y)$ and $z \in [u,v] \subseteq [x,y]$, we deduce that $d(x,z) = d(x,u) + d(u,z) = d(a,x) - \mu + \zeta - d(a,w) = \lambda$ as desired.

Finally, we can use the same argument as in the beginning of the proof of the implication (3) $\Longrightarrow$ (1), with the point $a$ replaced by any element $z \in X$, to conclude that for all $x, y, z \in X$, the set $[x,y] \cap [y,z] \cap [z,x]$ is nonempty (and hence a singleton) as required. $\square$

**Remark 2.9.** (1) If $\Lambda$ is totally ordered then (T') is a consequence of the rest of hypotheses since

$$[x,y] = [x, m(a,x,y)] \cup [m(a,x,y), y] \subseteq [a,x] \cup [a,y]$$

and $[x, m(a,x,y)] \cap [m(a,x,y), y] = \{m(a,x,y)\}$. However, in general, the condition (T') cannot be omitted (see Remark 2.5 (5)).

(2) Let $(X,d)$ be a nonempty $\Lambda$-tree with the induced median operation $m : X^3 \longrightarrow X$. It follows from Proposition 2.8 that the following assertions are equivalent for a subset $S \subseteq X$.

(a) $S$ is a convex subset of the median set $(X,m)$.

(b) $(S, d|_{S^2})$ is a $\Lambda$-tree, so $S$ is a *sub-$\Lambda$-tree* of $(X,d)$.

In particular, the $\Lambda$-tree $(X,d)$ is spanned by a subset $S \subseteq X$ if and only if $X$ is the convex closure of $S$ in the median set $(X,m)$.

A nonempty sub-$\Lambda$-tree $S$ of the $\Lambda$-tree $(X,d)$ is said to be *closed* if the intersection of $S$ with any cell of $(X,d)$ is either empty or a cell. In particular, the cells are the simplest closed sub-$\Lambda$-trees of $(X,d)$. According to [3, Proposition 7.3.], the following assertions are equivalent for a nonempty subset $S \subseteq X$.

(i) $S$ is a closed sub-$\Lambda$-tree of $(X,d)$.

(ii) $S$ is a retractible convex subset of the median set $(X,m)$.

(iii) $S=\varphi(X)$, where $\varphi : X \longrightarrow X$ is a *folding* of the median set $(X,m)$, i.e., $\varphi(m(x,y,z)) = m(\varphi(x),\varphi(y),z)$ for all $x,y,z\in X$, in particular, $\varphi=\varphi\circ\varphi$ is an idempotent endomorphism of the median set $(X,m)$.

If $S$ is a closed sub-$\Lambda$-tree of $(X,d)$ then the folding $\varphi$ satisfying $S=\varphi(X)$ is unique, so we may call it the *folding associated to* $S$. For all $s\in S, x\in X$, $S\cap[s,x]=[s,\varphi(x)]$, and for any $x\in X$, $\bigcap_{s\in S}[x,s]=[x,\varphi(x)]$ is called the *bridge* from the point $x$ to the closed sub-$\Lambda$-tree $S$ of $(X,d)$. In particular, the distance from $x$ to $S$ is well defined :

$$d(x,S) := d(x,\varphi(x)) = \min\{d(x,s)\,|\,s\in S\}.$$

(3) For any cell $C$ of a $\Lambda$-tree $(X,d)$, put $\mathfrak{d}(C):=d(x,y)$ for some (for all) $x,y\in X$ such that $C=[x,y]$; call $\mathfrak{d}(C)\in\Lambda_+$ the *diameter* of the cell $C$. By a *midpoint* (or a *center*) of the cell $C$ we mean a point $c\in C$ such that $d(c,x)=d(c,y)$ for all $x,y\in\partial C$. The necessary and sufficient condition for the existence of a (*unique*) midpoint $c$ of the cell $C$ is that $\mathfrak{d}(C)\in 2\Lambda$, whence the *radius* $d(c,x)=\frac{\mathfrak{d}(C)}{2}\in\Lambda_+$ for all $x\in\partial C$. More generally, for any two cells $C$ and $C'$ such that $C\subseteq C'$, we say that $C$ is *centrally situated* in $C'$ if for some (for all) $x\in\partial C, x'\in\partial C'$, $d(x,x')=d(\neg x,\neg x')$, where $\neg x\in\partial C, \neg x'\in\partial C'$ are unique with the property $C=[x,\neg x], C'=[x',\neg x']$. It follows that $\mathfrak{d}(C')-\mathfrak{d}(C)=2d(x',m(x,x',\neg x))\in 2\Lambda_+$ for all $x\in\partial C, x'\in\partial C'$.

(4) Let $(X,d)$ be a $\Lambda$-tree, and $x,y,z,u\in X$ be such that $z\in[x,y]$. Then $v:=m(x,y,u)\in[u,z]$, and hence

$$\begin{aligned} d(u,z) &= d(u,v)+d(v,z)\\ &= (x,y)_u + |(u,y)_x - d(z,x)|\\ &= (d(u,x)-d(x,z))\vee(d(u,y)-d(y,z)). \end{aligned}$$

Similarly, assuming that $z\in[x_1,x_2]$, $u\in[y_1,y_2]$, we obtain

$$d(z,u) = \bigvee_{i,j\in\{1,2\}} (d(x_i,y_j)-d(x_i,z)-d(y_j,u)). \tag{2.1}$$

In particular, if $d(x,y)=2r\in 2\Lambda$ and $z$ is the midpoint of the cell $[x,y]$, it follows that $d(u,z)=(d(u,x)\vee d(u,y))-r$. Consequently, if $c,c'$ are the midpoints of the cells $[x,y]$ and $[x',y']$ respectively, with $d(x,y)=2r, d(x',y')=2r'$, then

$$d(c,c') = (d(x,x')\vee d(x,y')\vee d(y,x')\vee d(y,y'))-(r+r'). \tag{2.2}$$

(5) Let $(X,d)$ be a $\Lambda$-tree. We denote by $\mathrm{Dir}(X)$ the median set of all directions on the underlying median set $(X,m)$. According to [3, Definition 3.1.], a *direction* $D$ on $(X,m)$ is a semilattice operation $\vee_D$ on $X$ with the property that for every $a\in X$, the map $X\longrightarrow X, x\mapsto a\vee_D x$ is a folding of $(X,m)$, i.e., $a\vee_D m(x,y,z)=m(a\vee_D x, a\vee_D y, z)$ for all $x,y,z\in X$. By [3, Proposition 8.3.], the set $\mathrm{Dir}(X)$ becomes a median set with respect to the ternary operation $(D_1,D_2,D_3)\mapsto D$, where the direction $D$ is defined by

$$x\vee_D y := m(x\vee_{D_1} y, x\vee_{D_2} y, x\vee_{D_3} y)$$

for all $x,y\in X$. Moreover the map $X\longrightarrow \mathrm{Dir}(X), a\mapsto d_a$, with $x\vee_{d_a} y := m(a,x,y)$ for $x,y\in X$, is injective, identifying $(X,m)$ with a convex subset of the median set $\mathrm{Dir}(X)$. The directions $d_a$, for $a\in X$, are called *internal*, while the rest of directions are called *external*. If $X$ is the convex closure in $(X,m)$ of some finite subset of $X$ then $(X,m)\cong\mathrm{Dir}(X)$, i.e., $\mathrm{Dir}(X)$ consists entirely of internal directions. Note also that $\mathrm{Dir}(X)\cong\mathrm{Dir}(\mathrm{Dir}(X))$ according to [3, Proposition 8.7.].

For $D\in\mathrm{Dir}(X), a\in X$, the convex subset $[a,D]\cap X=\{x\in X\,|\,a\vee_D x = x\}$, called the *ray from* $a$ *in the direction* $D$, is a distributive lattice with the least element $a$, the meet operation $(x,y)\mapsto m(x,a,y)$, and the join operation $(x,y)\mapsto x\vee_D y$.

On the other hand, the family $(C_i)_{i\in I}$ of all nonempty finitely generated convex subsets of $(X, m)$ form an inverse system, where the connecting morphisms are the canonical retracts $\pi_{i,j} : C_j \longrightarrow C_i$ of the embeddings $C_i \subseteq C_j$ (for any $x \in C_j$, the bridge from $x$ to $C_i$ is the cell $[x, \pi_{i,j}(x) = \vee_{d_x} C_i]$). According to [3, Proposition 8.1.], the median set $\mathrm{Dir}(X)$ is identified with the inverse limit of the inverse system above via the isomorphism $D \mapsto (\vee_D C_i)_{i\in I}$, whose inverse sends any compatible family $(c_i)_{i\in I} \in \prod_{i\in I} C_i$ to the direction $D$ defined by $x \vee_D y := m(x, y, c_i)$ for some (for all) $i \in I$ satisfying $x, y \in C_i$.

Using the $\Lambda$-tree structure of $X$, we can define the following convex subset of $\mathrm{Dir}(X)$ lying over $X$. Fix a point $a \in X$, and consider the family of balls

$$B(\lambda) = B_a(\lambda) = \{x \in X \mid d(a, x) \leq \lambda\} \ (\lambda \in \Lambda_+).$$

One checks easily that for any $\lambda \in \Lambda_+$, the ball $B(\lambda)$ is a closed sub-$\Lambda$-tree of $(X, d)$, with the associated folding $\varphi_\lambda : X \longrightarrow X, x \mapsto$ the unique point $y \in [a, x]$ satisfying $d(a, y) = \lambda \wedge d(a, x)$, so the cell $[x, \varphi_\lambda(x)]$ is the bridge from $x$ to the ball $B(\lambda)$. The balls $B(\lambda)(\lambda \in \Lambda_+)$ form an inverse system of median sets with the connecting morphisms $\varphi_\lambda|_{B(\mu)} : B(\mu) \longrightarrow B(\lambda)$ for $\lambda \leq \mu$. We denote by $B = B_a$ the inverse limit of this inverse system of median sets. The injective map $X \longrightarrow B, x \mapsto (\varphi_\lambda(x))_{\lambda\in\Lambda_+}$ identifies $(X, m)$ with a convex subset of the median set $B$, and hence the canonical embedding $X \longrightarrow \mathrm{Dir}(X)$ extends uniquely to a morphism of median sets $B \longrightarrow \mathrm{Dir}(X), b \mapsto d_b$, where $x \vee_{d_b} y = m(x, y, b) = m(x, y, b_\lambda)$ for some (for all) $\lambda \in \Lambda_+$ satisfying $\lambda \geq d(a, x) \vee d(a, y)$ $(x, y \in X)$. Moreover the map $B \longrightarrow \mathrm{Dir}(X)$ is injective, identifying $B$ with a convex subset of $\mathrm{Dir}(X)$. The construction does not depend on the choice of the base point $a \in X$, and $\mathrm{Dir}(X) \cong \mathrm{Dir}(B)$. Consequently, $B \cong \mathrm{Dir}(X)$ provided the $\Lambda$-tree $B(\lambda)$ is spanned by a finite subset for all $\lambda \in \Lambda_+$. For instance, this happens for $X = \Lambda$, with $d(x, y) = |x - y|$ (see for details 2.1.1.). By contrast, for $\Lambda = \mathbb{R}, X = [0, 1) = \{x \in \mathbb{R} \mid 0 \leq x < 1\}$, we obtain $X = B \neq \mathrm{Dir}(X) \cong [0, 1]$.

## 2.3 Subspaces of $\Lambda$-trees and their $\Lambda$-tree closures

The next statement provides a characterization of the $\Lambda$-metric spaces which are isometric to subspaces of $\Lambda$-trees. It extends to arbitrary abelian $l$-groups [1, Theorem 3.17], where $\Lambda$ is a totally ordered abelian group.

**Theorem 2.10.** *Let $(X, d)$ be a nonempty pre-$\Lambda$-metric space. For any element $a \in X$, the following assertions are equivalent.*

(1) *$(X, d)$ is isometric to a subspace of a $\Lambda$-tree.*

(2) *$(X, d)$ satisfies $(S1)_a$ and $(S2)_a$.*

*Proof.* The implication (1) $\Longrightarrow$ (2) is immediate by Proposition 2.8. To prove the converse, assume that $(X, d)$ satisfies (S1)$_a$ and (S2)$_a$, in particular, $(X, d)$ is a $\Lambda$-metric space by Lemma 2.6 (2). We denote by $\widetilde{X}$ the subset of $\Lambda_+^X$ consisting of those maps $f : X \longrightarrow \Lambda_+$ satisfying the following three conditions

($\alpha$) $(x, y)_f := \frac{1}{2}(f(x) + f(y) - d(x, y)) \in \Lambda_+$ for all $x, y \in X$,

($\beta$) $(x, y)_f \geq (x, z)_f \wedge (y, z)_f$ for all $x, y, z \in X$, and

$(\gamma)_a$ there exists a nonempty finite subset $M_f \subseteq X$ such that $\bigwedge_{x\in M_f} (a, x)_f = 0$.

Note that the definition of $\widetilde{X}$ does not depend on the choice of a base point $a \in X$. Indeed, we may assume without loss that $a \in M_f$, and hence it follows by $(\alpha)$, $(\beta)$ and $(\gamma)_a$ that

$$0 \leq \bigwedge_{x \in M_f} (b, x)_f = \bigwedge_{x \in M_f} ((b, a)_f \wedge (b, x)_f) \leq \bigwedge_{x \in M_f} (a, x)_f = 0,$$

so $(\gamma)_b$ holds for any $b \in X$.

By Lemma 2.6 (1, 3), $X$ is identified with a subset of $\widetilde{X}$ via the injective map $u \mapsto \widetilde{u}$, where $\widetilde{u}(x) = d(u, x)$ for $x \in X$, with $M_{\widetilde{u}} = \{u\}$ for $u \in X$.

We define a map $\widetilde{d} : \widetilde{X} \times \widetilde{X} \longrightarrow \Lambda_+$ as follows. Let $f, g \in \widetilde{X}$. By assumption there is a nonempty finite subset $M \subseteq X$ such that $\bigwedge_{x,y \in M} (x, y)_f = \bigwedge_{x,y \in M} (x, y)_g = 0$. Setting

$$L := \bigvee_{x \in M} (f(x) - g(x)),\ R := \bigvee_{x \in M} (g(x) - f(x)),$$

it follows by $(\alpha)$ and $(\beta)$ that for all $x \in X$,

$$f(x) - g(x) - R = 2 \bigwedge_{y \in M} ((x, y)_f - (x, y)_g) \leq 2 \bigwedge_{y \in M} (x, y)_f \leq 2 \bigwedge_{y,z \in M} (y, z)_f = 0,$$

in particular, $L \leq R$. By symmetry, we obtain $L = R \geq |f(x) - g(x)|$ for all $x \in X$, and hence it makes sense to define

$$\widetilde{d}(f, g) = \widetilde{d}(g, f) := L = R = \bigvee_{x \in X} |f(x) - g(x)| \in \Lambda_+.$$

It follows that $(\widetilde{X}, \widetilde{d})$ is a $\Lambda$-metric space, and $\widetilde{d}(f, \widetilde{x}) = f(x)$ for all $f \in \widetilde{X}, x \in X$. In particular, $(X, d)$ is identified to a $\Lambda$-metric subspace of $(\widetilde{X}, \widetilde{d})$.

To conclude that $(\widetilde{X}, \widetilde{d})$ is a $\Lambda$-tree we have to show that the assertion (2) of Proposition 2.8 is satisfied.

(i) First, taking $\widetilde{a} = a \in X$ as a base point of $\widetilde{X}$, we have to show that

$$(f, g)_a := \frac{1}{2}(f(a) + g(a) - \widetilde{d}(f, g)) \in \Lambda \text{ for all } f, g \in \widetilde{X}.$$

Given $f, g \in \widetilde{X}$, choose a nonempty finite subset $M \subseteq X$ as in the definition of the distance $\widetilde{d}(f, g)$. Then

$$\begin{aligned} 2(f, g)_a &= f(a) + g(a) - \widetilde{d}(f, g) \\ &= \bigwedge_{x \in M} (f(a) + g(a) - f(x) + g(x)) \\ &= 2 \bigwedge_{x \in M} ((f, x)_a + (a, x)_g) \in 2\Lambda, \end{aligned}$$

so $(f, g)_a \in \Lambda$ as desired.

(ii) Next we have to check the inequality $(f,g)_a \geq (f,h)_a \wedge (g,h)_a$ for $f,g,h \in \widetilde{X}$. Since the abelian $l$-group $\Lambda$ is a subdirect product of the family of all its maximal totally ordered quotients, it suffices to show that the inequality above holds in each totally ordered quotient of $\Lambda$, so we may assume without loss that $\Lambda$ itself is totally ordered. Then it follows by $(\gamma)_a$ that there are $b,c,e \in X$ such that $(a,b)_f = (a,c)_g = (a,e)_h = 0$. As $(b,c)_a \geq (b,e)_a \wedge (c,e)_a$ by assumption, it suffices to show that $(f,g)_a = f(a) \wedge g(a) \wedge (b,c)_a$ provided $(a,b)_f = (a,c)_g = 0$. According to the definition of $\widetilde{d}$, we obtain

$$\widetilde{d}(f,g) = (f(a) - g(a)) \vee (f(b) - g(b)) \vee (f(c) - g(c)),$$

therefore

$$\begin{aligned}(f,g)_a &= g(a) \wedge ((f,b)_a + (a,b)_g) \wedge ((f,c)_a + (a,c)_g) \\ &= g(a) \wedge (f(a) + (a,b)_g) \wedge (f,c)_a.\end{aligned}$$

By symmetry, we get $(f,g)_a = f(a) \wedge (g(a) + (a,c)_f) \wedge (g,b)_a)$, and hence $(f,g)_a = (f,c)_a \wedge (g,b)_a$ since $(f,c)_a \leq f(a), (g,b)_a \leq g(a)$. Thus, it remains to show that $(f,c)_a = f(a) \wedge (b,c)_a$ since the equality $(g,b)_a = g(a) \wedge (b,c)_a$ follows by symmetry. As $(a,b)_f = 0$ by assumption, it follows by $(\alpha)$ and $(\beta)$ that $(a,c)_f \wedge (b,c)_f = 0$, therefore

$$(f,c)_a = ((f,c)_a + (a,c)_f) \wedge ((f,c)_a + (b,c)_f) = f(a) \wedge (b,c)_a,$$

as required.

(iii) Thus, we have shown that $(\widetilde{X}, \widetilde{d})$ satisfies (S1)$_a$ and (S2)$_a$. To conclude, by Proposition 2.8, that $(\widetilde{X}, \widetilde{d})$ is a $\Lambda$-tree, it remains to prove that the map

$$\iota_{f,g} : [f,g] \longrightarrow [0, \widetilde{d}(f,g)], h \mapsto \widetilde{d}(f,h),$$

is onto for all $f,g \in \widetilde{X}$. Let $f,g \in \widetilde{X}, \lambda \in [0, \widetilde{d}(f,g)]$. Define the map $h : X \longrightarrow \Lambda_+$ by

$$h(x) := (f,g)_x + |(g,x)_f - \lambda| = f(x) + \lambda - 2(\lambda \wedge (g,x)_f).$$

It follows by symmetry that

$$h(x) = g(x) + \widetilde{d}(f,g) - \lambda - 2((\widetilde{d}(f,g) - \lambda) \wedge (f,x)_g).$$

We have to show that $h \in \widetilde{X}$, $\widetilde{d}(f,h) = \lambda$, and $\widetilde{d}(g,h) = \widetilde{d}(f,g) - \lambda$.

$(\alpha)$ Let $x,y \in X$. As $x,y,f,g \in \widetilde{X}$, it follows by Lemma 2.6 (3) that $(x,y)_f \geq (g,x)_f \wedge (g,y)_f$, and hence

$$\begin{aligned}(x,y)_h &= (x,y)_f + \lambda - \lambda \wedge (g,x)_f - \lambda \wedge (g,y)_f \\ &= ((x,y)_f - (\lambda \wedge (g,x)_f \wedge (g,y)_f)) + (\lambda - (\lambda \wedge ((g,x)_f \vee (g,y)_f))) \in \Lambda_+\end{aligned}$$

as a sum of two elements of $\Lambda_+$.

$(\beta)$ Let $x, y, z \in X$. The inequality $(x,y)_h \geq (x,z)_h \wedge (y,z)_h$ is equivalent with the inequality $A + B \geq (A_1 + B_1) \wedge (A_2 + B_2)$, where

$$A = (x,y)_f, B = \lambda \wedge (g,z)_f,$$
$$A_1 = (x,z)_f, B_1 = \lambda \wedge (g,y)_f,$$
$$A_2 = (y,z)_f, B_2 = \lambda \wedge (g,x)_f.$$

Since $x, y, z, f, g \in \widetilde{X}$, it follows by Lemma 2.6 (3) that

$$A \geq (A_1 \wedge A_2) \vee (B_1 \wedge B_2) \text{ and } B \geq (A_1 \wedge B_2) \vee (B_1 \wedge A_2),$$

therefore $A + B \geq (A_1 + B_1) \wedge (A_2 + B_2)$ as shown in the proof of the assertion (3) of Lemma 2.6.

$(\gamma)_a$ As $f, g \in \widetilde{X}$, there is a nonempty finite subset $M \subseteq X$ such that $\bigwedge_{x\in M}(a,x)_f = \bigwedge_{x\in M}(a,x)_g = 0$. Set $B := \bigwedge_{x\in M}(a,x)_h$. Note that $B \geq 0$ by $(\alpha)$. To conclude that $h \in \widetilde{X}$, it remains to show that $B \leq 0$. We obtain

$$B \leq (\bigwedge_{x\in M}(a,x)_f) + (\lambda - \lambda \wedge (g,a)_f) = (\lambda - (g,a)_f)_+,$$

and, by symmetry, $B \leq (\widetilde{d}(f,g) - \lambda - (f,a)_g)_+ = (\lambda - (g,a)_f)_-$, therefore $B \leq 0$ as desired.

Finally note that the equalities $\widetilde{d}(f,h) = \lambda, \widetilde{d}(g,h) = \widetilde{d}(f,g) - \lambda$ follow easily from the definitions of $h$ and $\widetilde{d}$. □

**Corollary 2.11.** *Let $(X,d)$ be a $\Lambda$-metric space satisfying $(S1)_a$ and $(S2)_a$ for some (for all) $a \in X$. Then there exists the* $\Lambda$-tree closure $(\widetilde{X},\widetilde{d})$ *of $(X,d)$, i.e., the following conditions hold.*

(1) *$(X,d)$ is a $\Lambda$-metric subspace of the $\Lambda$-tree $(\widetilde{X},\widetilde{d})$.*

(2) *Any $\Lambda$-isometry $\rho : (X,d) \longrightarrow (\widehat{X},\widehat{d})$ from $(X,d)$ into a $\Lambda$-tree $(\widehat{X},\widehat{d})$ extends uniquely to a $\Lambda$-isometry $\widetilde{\rho} : (\widetilde{X},\widetilde{d}) \longrightarrow (\widehat{X},\widehat{d})$.*

*In particular, the unique up to isomorphism $\Lambda$-tree $(\widetilde{X},\widetilde{d})$ is spanned by $(X,d)$, and $\widetilde{X}$ is the convex closure of $X$ into the underlying median set of $(\widetilde{X},\widetilde{d})$.*

*Proof.* Let $(\widetilde{X},\widetilde{d})$ be the $\Lambda$-tree extension of $(X,d)$, defined as in the proof of Theorem 2.10. Denote by $m : \widetilde{X}^3 \longrightarrow \widetilde{X}$ the induced median operation. First, let us show that $\widetilde{X}$ is the convex closure of $X$ in the median set $(\widetilde{X},m)$, equivalently, by Remark 2.9 (2), the $\Lambda$-tree $(\widetilde{X},\widetilde{d})$ is spanned by $X$. Let $X' \subseteq \widetilde{X}$ be such that $X \subseteq X'$ and $X'$ is a convex subset of $(\widetilde{X},m)$. We have to show that $X' = \widetilde{X}$. Fix a base point $a \in X$, and let $f \in \widetilde{X}$.

According to the definition of $\widetilde{X}$, there exists a finite subset $M \subseteq X$ such that $a \in M$ and $\bigwedge_{x\in M}(a,x)_f = 0$. As $a \in M$ and $(\widetilde{X}, m)$ is a median set, there is an unique element $g \in \widetilde{X}$ such that

$$\bigcap_{x\in M}[f,x] = \bigcap_{x\in M}[f, m(f,a,x)] = [f,g] \subseteq [f,a].$$

Since $M \subseteq X \subseteq X'$ and $X'$ is a convex subset of $(\widetilde{X}, m)$ by assumption, it follows that $g \in X'$. As the canonical map $\iota_{f,a} : [f,a] \longrightarrow [0, \widetilde{d}(f,a)]$ is an isomorphism of $\Lambda$-metric spaces, we deduce that $\widetilde{d}(f,g) = \bigwedge_{x\in M}\widetilde{d}(f, m(f,a,x)) = \bigwedge_{x\in M}(a,x)_f = 0$, therefore $f = g \in X'$ as desired.

To prove the universal property (2), let $\rho : (X,d) \longrightarrow (\widehat{X}, \widehat{d})$ be a $\Lambda$-isometry from $(X,d)$ into a $\Lambda$-tree $(\widehat{X}, \widehat{d})$. We have to show that $\rho$ extends uniquely to a $\Lambda$-isometry $\widetilde{\rho} : (\widetilde{X}, \widetilde{d}) \longrightarrow (\widehat{X}, \widehat{d})$. Since we have already proved that the $\Lambda$-tree $(\widetilde{X}, \widetilde{d})$ is spanned by $X$, we may assume without loss that $\rho$ is an inclusion and the $\Lambda$-tree $(\widehat{X}, \widehat{d})$ is spanned by $X$, equivalently $\widehat{X}$ is the convex closure of $X$ into the underlying median set $(\widehat{X}, \widehat{m})$. Thus we have to show that $(\widetilde{X}, \widetilde{d})$ and $(\widehat{X}, \widehat{d})$ are canonically isomorphic over $(X,d)$. Consider the map $F : \widehat{X} \longrightarrow \Lambda_+^X$, where $F(\widehat{x})(x) := \widehat{d}(\widehat{x}, x)$ for $\widehat{x} \in \widehat{X}, x \in X$. In particular, $F(x) = \widetilde{x}$ for $x \in X$, so the restriction $F|_X$ is identified with the identity map $1_X$. It remains only to show that $F(\widehat{X}) \subseteq \widetilde{X}$ and that $F$ is a $\Lambda$-isometry since the uniqueness is immediate from the definition of $(\widetilde{X}, \widetilde{d})$.

Let $\widehat{x} \in \widehat{X}$. Then $F(\widehat{x})$ satisfies obviously the conditions $(\alpha)$ and $(\beta)$ of the definition of $\widetilde{X}$. To check the condition $(\gamma_a)$, we use the assumption that $\widehat{X}$ is the convex closure of $X$ in the median set $(\widehat{X}, \widehat{m})$, and hence, according to [5, Proposition 3.1.(e, g)], there is a finite subset $M \subseteq X$ such that $a \in M$ and $\widehat{x}$ belongs to the convex closure $[\,M\,]_{\widehat{X}}$ of $M$ in $(\widehat{X}, \widehat{m})$, i.e.,

$$\bigcap_{y\in M}[\widehat{x}, y]_{\widehat{X}} = \bigcap_{y\in M}[\widehat{x}, \widehat{m}(\widehat{x}, a, y)]_{\widehat{X}} = \{\widehat{x}\}.$$

Since the canonical map $\widehat{\iota}_{\widehat{x},a} : [\widehat{x}, a]_{\widehat{X}} \longrightarrow [0, \widehat{d}(\widehat{x}, a)]$ is an isomorphism of $\Lambda$-metric spaces, it follows that

$$\bigwedge_{y\in M}(a,y)_{\widehat{x}} = \bigwedge_{y\in M}\widehat{d}(\widehat{x}, \widehat{m}(\widehat{x}, a, y)) = \widehat{d}(\widehat{x}, \widehat{x}) = 0$$

as required. Thus $F(\widehat{x}) \in \widetilde{X}$ for all $\widehat{x} \in \widehat{X}$.

To prove that $F : (\widehat{X}, \widehat{d}) \longrightarrow (\widetilde{X}, \widetilde{d})$ is a $\Lambda$-isometry, let $\widehat{x}, \widehat{y} \in \widehat{X}$. As shown above there exists a nonempty finite subset $M \subseteq X$ such that $\widehat{x}, \widehat{y} \in [\,M\,]_{\widehat{X}}$, therefore, by [5, Proposition 3.1.(f)], $[\widehat{x}, \widehat{y}]_{\widehat{X}} = [\{\widehat{m}(\widehat{x}, \widehat{y}, z) \mid z \in M\}]_{\widehat{X}}$.

Since the map $\widehat{\iota}_{\widehat{x},\widehat{y}} : [\widehat{x}, \widehat{y}]_{\widehat{X}} \longrightarrow [0, \widehat{d}(\widehat{x}, \widehat{y})]$ is an isomorphism of $\Lambda$-metric spaces, it follows that

$$\bigwedge_{z\in M}\widehat{d}(\widehat{x}, \widehat{m}(\widehat{x}, \widehat{y}, z)) = \bigwedge_{z\in M}\widehat{d}(\widehat{y}, \widehat{m}(\widehat{x}, \widehat{y}, z)) = 0,$$

therefore

$$\widehat{d}(\widehat{x},\widehat{y}) = \bigvee_{z\in M}(F(\widehat{x})(z) - F(\widehat{y})(z)) = \bigvee_{z\in M}(F(\widehat{y})(z) - F(\widehat{x})(z)) = \widetilde{d}(F(\widehat{x}), F(\widehat{y}))$$

as desired, according to the definitions of $F$ and $\widetilde{d}$. □

**Corollary 2.12.** *Let $(X,d)$ be a $\Lambda$-tree, and $S \subseteq X$ be such that the $\Lambda$-tree $(X,d)$ is spanned by $S$. Denote by $G$ the group consisting of the automorphisms $f$ of $(X,d)$ satisfying $f(S) = S$. Then the restriction map $G \longrightarrow \mathrm{Aut}(S, d|_{S\times S})$ is an isomorphism.*

**Remark 2.13.** Using Corollaries 2.11 and 2.12, we can construct a *base change* functor which extends to abelian $l$-groups the construction from [1, I, 4 Base change] concerning $\Lambda$-trees with $\Lambda$ totally ordered. We fix a morphism $h : \Lambda \longrightarrow \Lambda'$ of abelian $l$-groups, identifying the quotient $\Lambda/\mathrm{Ker}\,(h)$ with the $l$-subgroup $h(\Lambda)$ of $\Lambda'$. We construct a covariant functor $\mathbb{X} \mapsto \mathbb{X}\otimes_\Lambda \Lambda'$ from $\Lambda$-trees to $\Lambda'$-trees.

Let $\mathbb{X} = (X,d)$ be a $\Lambda$-tree with the induced median operation $m$. One checks easily that the binary relation $x \sim y \iff h(d(x,y)) = 0$ is a congruence on $\mathbb{X}$ and that the quotient $\overline{\mathbb{X}} := (\overline{X} := X/\sim, \overline{d})$, where the distance map $\overline{d} : \overline{X}\times\overline{X} \longrightarrow h(\Lambda)$ is induced by $d : X \times X \longrightarrow \Lambda$, is a $h(\Lambda)$-tree, with the median operation $\overline{m}$ induced by $m$. As $h(\Lambda)$ is an $l$-subgroup of $\Lambda'$, it follows that $\overline{X}$ is a pre-$\Lambda'$-tree. We define $\mathbb{X}\otimes_\Lambda \Lambda'$ as the $\Lambda'$-tree closure of the pre-$\Lambda'$-tree $\overline{\mathbb{X}}$. The desired functoriality follows easily by Corollary 2.11.

Moreover, for any $\Lambda$-tree $\mathbb{X}$, we obtain a group morphism $\mathrm{Aut}(\mathbb{X}) \longrightarrow \mathrm{Aut}(\mathbb{X}\otimes_\Lambda \Lambda'), \sigma \mapsto \sigma\otimes_\Lambda \Lambda'$, by composing the morphism $\mathrm{Aut}(\mathbb{X}) \longrightarrow \mathrm{Aut}(\overline{\mathbb{X}})$ (not necessarily surjective), induced by the projection $\mathbb{X} \longrightarrow \overline{\mathbb{X}}$, with the embedding $\mathrm{Aut}(\overline{\mathbb{X}}) \longrightarrow \mathrm{Aut}(\mathbb{X}\otimes_\Lambda \Lambda')$ provided by Corollary 2.12. Consequently, the morphism $\mathrm{Aut}(\mathbb{X}) \longrightarrow \mathrm{Aut}(\mathbb{X}\otimes_\Lambda \Lambda')$ is injective whenever the morphism $h : \Lambda \longrightarrow \Lambda'$ is injective.

In particular, if $\mathbb{X}$ is the $\Lambda$-tree with support $\Lambda$, and $\Lambda'$ is an abelian $l$-group containing $\Lambda$, then the $\Lambda'$-tree $\mathbb{X}\otimes_\Lambda \Lambda'$ is isomorphic over $\mathbb{X}$ with the underlying $\Lambda'$-tree of the convex closure $\bigcup_{\lambda\in\Lambda_+}\{\lambda' \in \Lambda' \mid |\lambda'| \le \lambda\}$ of $\Lambda$ in $\Lambda'$.

# 3 Automorphisms of $\Lambda$-trees

We extend in this section some basic results concerning the automorphisms of $\Lambda$-trees, where $\Lambda$ is a totally ordered abelian group (see [1, II, 6]), to the more general case of an abelian $l$-group $\Lambda$.

Let $(X,d)$ be a nonempty $\Lambda$-tree, where $\Lambda$ is an abelian $l$-group. Let $m : X^3 \longrightarrow X$ be the induced median operation on $X$. The automorphism group Aut$(X,d)$ of the $\Lambda$-metric space $(X,d)$ consists of all bijective maps $\sigma : X \longrightarrow X$ satisfying $d(\sigma x, \sigma y) = d(x,y)$ for all $x,y \in X$ (bijective isometries). Any $\sigma \in$ Aut$(X,d)$ is also an automorphism of the median set $(X,m)$, so Aut$(X,d)$ is a subgroup of the automorphism group Aut$(X,m)$. In general, Aut$(X,d)$ is a proper subgroup of Aut$(X,m)$. For instance, if $X = \mathbb{R}$, $d(x,y) = |x-y|$, and $\lambda \in \mathbb{R}\setminus\{-1,0,1\}$, then the map $x \mapsto \lambda x$ belongs to the complement of Aut$(X,d)$ in Aut$(X,m)$.

Usually the elements of Aut$(X,d)$ are classified as follows.

**Definition 3.1.** Let $\sigma \in \mathrm{Aut}(X,d)$.

(E) The automorphism $\sigma$ is called *elliptic* if $\mathrm{Fix}(\sigma) := \{x \in X \mid \sigma x = x\} \neq \varnothing$.

(I) $\sigma$ is said to be an *inversion* if $\mathrm{Fix}(\sigma) = \varnothing$ and $\mathrm{Fix}(\sigma^2) \neq \varnothing$.

(H) $\sigma$ is called *hyperbolic* if $\mathrm{Fix}(\sigma^2) = \varnothing$, in particular, $\mathrm{Fix}(\sigma) = \varnothing$.

Thus, $\mathrm{Aut}(X, d)$ is a disjoint union $\mathcal{E}(X, d) \sqcup \mathcal{I}(X, d) \sqcup \mathcal{H}(X, d)$, where $\mathcal{E}(X, d)$, $\mathcal{I}(X, d)$ and $\mathcal{H}(X, d)$ denote the subsets of elliptic automorphisms, inversions, and hyperbolic automorphisms respectively, which are closed under inversion ($\sigma \mapsto \sigma^{-1}$) and conjugation ($\sigma \mapsto \tau \circ \sigma \circ \tau^{-1}$ for $\tau \in \mathrm{Aut}(X, d)$). Note that $1_X$ is a priviliged element of $\mathcal{E}(X, d)$, while the sets $\mathcal{E}(X, d) \setminus \{1_X\}$, $\mathcal{I}(X, d)$ and $\mathcal{H}(X, d)$ may be empty.

To study the automorphisms of a $\Lambda$-tree $\mathbb{X} = (X, d)$, where $\Lambda$ is an abelian $l$-group, we may try to transfer the known results concerning the automorphisms of $\Lambda$-trees, where $\Lambda$ is a totally ordered abelian group, via the embedding

$$\mathrm{Aut}(\mathbb{X}) \longrightarrow \prod_{\mathfrak{p} \in \mathcal{P}(\Lambda)} \mathrm{Aut}(\mathbb{X}_{\mathfrak{p}}), \sigma \mapsto (\sigma_{\mathfrak{p}})_{\mathfrak{p} \in \mathcal{P}(\Lambda)},$$

induced by the canonical subdirect product representation $\mathbb{X} \longrightarrow \prod_{\mathfrak{p} \in \mathcal{P}(\Lambda)} \mathbb{X}_{\mathfrak{p}}$ from Remark 2.5 (3). Indeed, any automorphism $\sigma \in \mathrm{Aut}(\mathbb{X})$ induces an automorphism $\sigma_{\mathfrak{p}}$ of the $(\Lambda/\mathfrak{p})$-tree quotient $\mathbb{X}_{\mathfrak{p}}$ of $\mathbb{X}$, for all $\mathfrak{p} \in \mathcal{P}(\Lambda)$, and the map above identifies $\mathrm{Aut}(\mathbb{X})$ with a subgroup of the product $\prod_{\mathfrak{p} \in \mathcal{P}(\Lambda)} \mathrm{Aut}(\mathbb{X}_{\mathfrak{p}})$. However, the morphism $\mathrm{Aut}(\mathbb{X}) \longrightarrow \mathrm{Aut}(\mathbb{X}_{\mathfrak{p}})$ is not necessarily surjective for all $\mathfrak{p} \in \mathcal{P}(\Lambda)$, so, in general, the injective morphism above is not a subdirect product representation of $\mathrm{Aut}(\mathbb{X})$.

Though, in general, we prefer a direct global approach and find again the results in the local case, we will use in certain circumstances the transfer from the local case to the global one, as, for instance, in the next subsection devoted to the particular case of the isometry group of an abelian $l$-group $\Lambda$.

## 3.1 The isometry group of an abelian $l$-group

Let $\Lambda \neq \{0\}$ be an abelian $l$-group, and take $X = \Lambda$ with $d(x, y) = |x - y|$. Among the isometries of $\Lambda$ we distinguish the *translations* $t_a(x) = a + x$, and the *reflections* $r_a(x) = a - x \, (a \in \Lambda)$. Under composition, they form a group isomorphic with the semidirect product $T_\Lambda \ltimes \{t_0 = 1_\Lambda, r_0 = -1_\Lambda\}$ of the normal abelian subgroup $T_\Lambda \cong \Lambda$ of translations with the group $\{\pm 1_\Lambda\} \cong \mathbb{Z}/2\mathbb{Z}$.

If $\Lambda$ is totally ordered then the isometries of $\Lambda$ form a group $\mathrm{Aut}_{\mathrm{metric}}(\Lambda)$ consisting entirely of translations and reflections (see [1, Proposition 2.5.(a)]). Note that in this case $r_0 = -1_\Lambda$ is the unique isometry $\sigma \neq 1_\Lambda$ satisfying $\sigma(0) = 0$.

To see what happens in the more general case when $\Lambda$ is not necessarily totally ordered, we denote by $G_0$ the set of those isometries $\sigma$ satisfying $\sigma(0) = 0$, in particular, $\{1_\Lambda, r_0\} \subseteq G_0$. The set $G_0$ is closed under composition, $T_\Lambda \cap G_0 = \{1_\Lambda\}$, and any isometry $\sigma$ is uniquely written in the form $\sigma = t \circ \tau$ with $t \in T_\Lambda, \tau \in G_0$, where $t = t_{\sigma(0)}, \tau = t^{-1} \circ \sigma$.

[1, Proposition 2.5.(a)] is extended to a similar structure theorem for the isometries of an abelian $l$-group $\Lambda$ as follows.

**Proposition 3.2.** *Let $\Lambda$ be an abelian $l$-group. Then, with the notation above, the following assertions hold.*

(1) *$G_0$ is an abelian group of exponent 2. Consequently, all the isometries of $\Lambda$ are bijective, so they form a group $G := \mathrm{Aut}_{\mathrm{metric}}(\Lambda)$.*

(2) $G_0 = G \cap \mathrm{Aut}_{\mathrm{group}}(\Lambda) = \{\sigma \in \mathrm{Aut}_{\mathrm{group}}(\Lambda) \mid \forall x \in \Lambda, |\sigma(x)| = |x|\}$.

(3) *$G = T_\Lambda \ltimes G_0$ is the semidirect product of the normal abelian subgroup $T_\Lambda \cong \Lambda$ of translations with $G_0$.*

(4) *The embedding $G \longrightarrow \prod_{\mathfrak{p} \in \mathcal{P}(\Lambda)} G_\mathfrak{p}, \sigma \mapsto (\sigma_\mathfrak{p})_{\mathfrak{p} \in \mathcal{P}(\Lambda)}$, where $G_\mathfrak{p} := \mathrm{Aut}_{\mathrm{metric}}(\Lambda/\mathfrak{p})$, is a subdirect product representation of $G = \mathrm{Aut}_{\mathrm{metric}}(\Lambda)$. In particular, $G_0$ is a subdirect product of $\prod_{\mathfrak{p} \in \mathcal{P}(\Lambda)} \{\pm 1_{\Lambda/\mathfrak{p}}\} \cong (\mathbb{Z}/2\mathbb{Z})^{\mathcal{P}(\Lambda)}$.*

*Proof.* The proof is straightforward. It suffices to note that for any $\sigma \in G_0$, $\sigma_\mathfrak{p} = \pm 1_{\Lambda/\mathfrak{p}}$ for all $\mathfrak{p} \in \mathcal{P}(\Lambda)$, and hence $\sigma^2 = 1_\Lambda$, and $\sigma(x - y) = \sigma(x) - \sigma(y)$ for all $x, y \in \Lambda$. □

According to Definition 3.1, the isometries of an abelian $l$-group $\Lambda$ are easily classified as follows.

**Lemma 3.3.** *Let $\Lambda$ be an abelian $l$-group. We denote by $\mathcal{E}(\Lambda), \mathcal{I}(\Lambda)$ and $\mathcal{H}(\Lambda)$ the corresponding sets of elliptic isometries, inversions and hyperbolic isometries of $\Lambda$; in particular, $\mathcal{E}(\Lambda/\mathfrak{p}) = \{r_a \mid a \in 2(\Lambda/\mathfrak{p})\}$, $\mathcal{I}(\Lambda/\mathfrak{p}) = \{r_a \mid a \in (\Lambda/\mathfrak{p}) \setminus 2(\Lambda/\mathfrak{p})\}$, and $\mathcal{H}(\Lambda/\mathfrak{p}) = T_{\Lambda/\mathfrak{p}} \setminus \{1_{\Lambda/\mathfrak{p}}\}$ for any $\mathfrak{p} \in \mathcal{P}(\Lambda)$. Then the following assertions hold.*

(1) *$\mathcal{E}(\Lambda)$ is the conjugacy class of $G_0$, the stabilizer of $0$ in $G = \mathrm{Aut}_{\mathrm{metric}}(\Lambda)$, so*

$$\begin{aligned}\mathcal{E}(\Lambda) &= \bigcup_{a \in \Lambda} t_a G_0 t_{-a} \\ &= \{t_a \tau \mid a \in 2\Lambda, \tau \in G_0, \tau(a) = -a\} \\ &= \{\sigma \in G \mid \sigma^2 = 1_\Lambda, \sigma(0) \in 2\Lambda\}.\end{aligned}$$

*For any $\sigma \in G, \sigma \in \mathcal{E}(\Lambda)$ if and only if $\sigma_\mathfrak{p} \in \mathcal{E}(\Lambda/\mathfrak{p})$ for all $\mathfrak{p} \in \mathcal{P}(\Lambda)$. In particular, $\{r_a \mid a \in 2\Lambda\} \subseteq \mathcal{E}(\Lambda)$.*

(2) $\mathcal{I}(\Lambda) = \{t_a\tau \mid a \in \Lambda \setminus 2\Lambda, \tau \in G_0, \tau(a) = -a\} = \{\sigma \in G \mid \sigma^2 = 1_\Lambda, \sigma(0) \notin 2\Lambda\}$. *For any* $\sigma \in G, \sigma \in \mathcal{I}(\Lambda)$ *if and only if* $\{\mathfrak{p} \in \mathcal{P}(\Lambda) \mid \sigma_\mathfrak{p} \in \mathcal{I}(\Lambda/\mathfrak{p})\} \neq \varnothing$ *and* $\{\mathfrak{p} \in \mathcal{P}(\Lambda) \mid \sigma_\mathfrak{p} \in \mathcal{H}(\Lambda/\mathfrak{p})\} = \varnothing$. *In particular,* $\{r_a \mid a \in \Lambda \setminus 2\Lambda\} \subseteq \mathcal{I}(\Lambda)$.

(3) $\mathcal{H}(\Lambda) = \{\sigma \in G \mid \sigma^2 \neq 1_\Lambda\} = \{t_a\tau \mid a \in \Lambda, \tau \in G_0, \tau(a) \neq -a\}$, *in particular,* $T_\Lambda \setminus \{1_\Lambda\} \subseteq \mathcal{H}(\Lambda)$. *For any* $\sigma \in G, \sigma \in \mathcal{H}(\Lambda)$ *if and only if there exists* $\mathfrak{p} \in \mathcal{P}(\Lambda)$ *such that* $\sigma_\mathfrak{p} \in \mathcal{H}(\Lambda/\mathfrak{p})$.

### 3.1.1 The action of $\mathrm{Aut}_{\mathrm{metric}}(\Lambda)$ on the median set of directions of $\Lambda$

If $\Lambda$ is totally ordered then the median set $\mathrm{Dir}(\Lambda)$ is obtained by adding to the internal directions $d_\lambda (\lambda \in \Lambda)$, identified with the elements of $\Lambda$, two external directions $D_+$ and $D_-$ induced by the linear order $\leq$ and its opposite. Thus $\mathrm{Dir}(\Lambda)$ is a cell $[D_-, D_+]$ with the boundary $\partial[D_-, D_+] = \{D_-, D_+\}$. With respect to the direction $D_+$, $\mathrm{Dir}(\Lambda)$ is a bounded totally ordered set with the least element $D_-$ and the last element $D_+$. The action of $\mathrm{Aut}_{\mathrm{metric}}(\Lambda)$ on $\Lambda$ is extended to an action on $\mathrm{Dir}(\Lambda)$, with the normal subgroup $T_\Lambda \cong \Lambda$ as stabilizer of both external directions $D_+$ and $D_-$, inducing a free and transitive action of $G_0 = \{1_\Lambda, r_0 = -1_\Lambda\} \cong \mathbb{Z}/2\mathbb{Z}$ on the boundary $\{D_-, D_+\}$. Note also that the reflexion $r_0$ is a negation operator on $\mathrm{Dir}(\Lambda)$ with $0$ as its unique fixed point.

To extend this very simple situation to the general case of an abelian $l$-group $\Lambda$, we consider the family of closed balls $B(\lambda) := \{x \in \Lambda \mid |x| \leq \lambda\}$ $(\lambda \in \Lambda_+)$.

Any such ball $B(\lambda)$ is a cell with the boundary $\partial B(\lambda) = \{x \in \Lambda \mid |x| = \lambda\}$ and the midpoint $0$. With respect to the partial order $\leq$, $B(\lambda)$ is a bounded distributive lattice with the least element $-\lambda$ and the last element $\lambda$, while the restriction $r_0|_{B(\lambda)}, x \mapsto -x$, is a negation operator with $0$ as its unique fixed point. With respect to the lattice operations $\vee, \wedge$ and the negation operator, the boundary $\partial B(\lambda)$ is a boolean algebra with the least element $-\lambda$ and the last element $\lambda$, and $B(\lambda) = [-x, x]$ for all $x \in \partial B(\lambda)$. The isometries of $B(\lambda)$ form an abelian group $\mathrm{Aut}_{\mathrm{metric}}(B(\lambda))$ of exponent 2, identified, via the embedding $B(\lambda) \longrightarrow \prod_{\mathfrak{p} \in \mathcal{P}(\Lambda)} B(\lambda \bmod \mathfrak{p})$, with a subdirect product of the power $\{\pm 1\}^{U(\lambda)}$, where $U(\lambda) := \{\mathfrak{p} \in \mathcal{P}(\Lambda) \mid \lambda \bmod \mathfrak{p} > 0\}$. The isometry group $\mathrm{Aut}_{\mathrm{metric}}(B(\lambda))$ acts freely and transitively on the boundary $\partial B(\lambda)$, and the kernel of the restriction map $G_0 \longrightarrow \mathrm{Aut}_{\mathrm{metric}}(B(\lambda))$, not necessarily surjective, is $G_0 \cap G_\lambda = G_0 \cap t_\lambda G_0 t_{-\lambda}$.

The balls $B(\lambda)(\lambda \in \Lambda_+)$ form an inverse system of median sets, with the connecting surjective morphisms $\varphi_\lambda|_{B(\mu)} : B(\mu) \longrightarrow B(\lambda)$ $(\lambda \leq \mu)$, where

$$\varphi_\lambda(x) = m(-\lambda, x, \lambda) = (-\lambda) \vee (\lambda \wedge x) = \lambda \wedge ((-\lambda) \vee x)\ (x \in \Lambda)$$

is the folding of the median set $(\Lambda, m)$, with image $B(\lambda)$. Note that $\varphi_\lambda(x)_+ = \lambda \wedge x_+$, $\varphi_\lambda(x)_- = \lambda \wedge x_-$, $|\varphi_\lambda(x)| = \lambda \wedge |x|$.

We denote by $B$ the inverse limit of the inverse system of balls $B(\lambda)\,(\lambda \in \Lambda_+)$, so $B$ consists of the maps $\psi : \Lambda_+ \longrightarrow \Lambda$ satisfying $\psi(\lambda) = \varphi_\lambda(\psi(\mu))$ for all $\lambda, \mu \in \Lambda_+$, with $\lambda \leq \mu$, while the median operation on $B$ is defined component-wise. According to Remark 2.9 (5), the median set $B$ is identified with the median set $\mathrm{Dir}(\Lambda)$ of the directions of the median set $(\Lambda, m)$, via the isomorphism $\mathrm{Dir}(\Lambda) \longrightarrow B, D \mapsto \psi_D$, where $\psi_D(\lambda) := (-\lambda) \vee_D \lambda\,(\lambda \in \Lambda_+)$, with the inverse $B \longrightarrow \mathrm{Dir}(\Lambda), \psi \mapsto D_\psi$, where the direction $D_\psi$ is defined by $x \vee_{D_\psi} y := m(x, y, \psi(\lambda))$ for some (for all) $\lambda \geq |x| \vee |y|$. In particular, the internal directions $d_x\,(x \in \Lambda)$ correspond bijectively to the maps $\psi_x : \Lambda_+ \longrightarrow \Lambda, \lambda \mapsto \varphi_\lambda(x)\,(x \in \Lambda)$, while the external directions $D_+, D_-$, induced by the partial order $\leq$ and its opposite, correspond to the maps $\psi_+(\lambda) = \lambda, \psi_-(\lambda) = -\lambda\,(\lambda \in \Lambda_+)$ respectively. Note also that $\mathrm{Dir}(\Lambda) \cong B$ is a subdirect product of

$$\prod_{\mathfrak{p} \in \mathcal{P}(\Lambda)} \mathrm{Dir}(\Lambda/\mathfrak{p}) \cong \prod_{\mathfrak{p} \in \mathcal{P}(\Lambda)} (\{D_{-\,\mathfrak{p}}\} \cup (\Lambda/\mathfrak{p}) \cup \{D_{+\,\mathfrak{p}}\}),$$

where $D_{+\,\mathfrak{p}}$ and $D_{-\,\mathfrak{p}}$ are the external opposite directions of the nontrivial totally ordered factor $\Lambda/\mathfrak{p}$.

It follows that $\mathrm{Dir}(\Lambda) \cong B$ is a cell, and its boundary

$$\partial B = \{\psi \in B \mid \forall \lambda \in \Lambda_+, |\psi(\lambda)| = \lambda\}$$

is the inverse limit of the boundaries $\partial B(\lambda) = \{x \in \Lambda \mid |x| = \lambda\}$ of the balls $B(\lambda)\,(\lambda \in \Lambda_+)$, isomorphic to a subdirect product of $\prod_{\mathfrak{p} \in \mathcal{P}(\Lambda)} \{D_{-\,\mathfrak{p}}, D_{+\,\mathfrak{p}}\}$. Note that, though the projections $B(\mu) \longrightarrow B(\lambda)\,(\lambda \leq \mu)$ and $B \longrightarrow B(\lambda)\,(\lambda \in \Lambda_+)$ are onto, the induced maps $\partial B(\mu) \longrightarrow \partial B(\lambda)$ and $\partial B \longrightarrow \partial B(\lambda)$ are not necessarily onto.

With respect to the partial order $\psi \leq \psi' \Longleftrightarrow \forall \lambda \in \Lambda_+, \psi(\lambda) \leq \psi'(\lambda)$, induced by the direction $D_+$, $B$ is a bounded distributive lattice with the least element $\psi_-$ and the last element $\psi_+$, while the map $\psi \mapsto -\psi$ is a negation operator with $\psi_0 = 0$ as its unique fixed point. With respect to the lattice operations and the negation operator, the boundary $\partial B$ is the boolean subalgebra of $B$ consisting of those elements which have (unique) complements.

The isometry group $G = \mathrm{Aut}_{\mathrm{metric}}(\Lambda)$ acts canonically on the median set $\mathrm{Dir}(\Lambda)$ according to the rule

$$x \vee_{\sigma D} y := \sigma(\sigma^{-1}(x) \vee_D \sigma^{-1}(y)) \ (\sigma \in G, D \in \mathrm{Dir}(\Lambda), x, y \in \Lambda).$$

The action above induces an action of $G$ on $B$, given by $(\sigma\psi)(\lambda) := \varphi_\lambda(\sigma(\psi(\mu)))$ for some (for all) $\mu \in \Lambda_+$ satisfying $\mu \geq \lambda + |\sigma(0)|$. In particular, $(\sigma\psi)(\lambda) = \sigma(\psi(\lambda))$ for all $\lambda \in \Lambda_+$ whenever $\sigma \in G_0$. It follows that the kernel of the induced action of $G$ on the boundary $\partial B$ is the normal subgroup $T_\Lambda$, while the action on $\partial B$ of the group $G_0$, isomorphic to the inverse limit of the isometry groups $\mathrm{Aut}_{\mathrm{metric}}(B(\lambda)) \cong \mathrm{Aut}_{\mathrm{metric}}(\partial B(\lambda))\,(\lambda \in \Lambda_+)$, is free and transitive.

## 3.2 Invariants associated to automorphisms of $\Lambda$-trees

A key role in the study of the automorphism group Aut$(X, d)$ is played by the restriction to Aut$(X, d)$ of the map $\mathrm{Aut}(X, m) \longrightarrow X^X$ which assigns to any $\sigma \in \mathrm{Aut}(X, m)$ the map $\psi_\sigma : X \longrightarrow X$ defined by $\psi_\sigma(x) := m(\sigma^{-1}x, x, \sigma x)$.

Note that $\psi_{1_X} = 1_X, \psi_\sigma = \psi_{\sigma^{-1}}$, and $\psi_{\tau\circ\sigma\circ\tau^{-1}} = \tau \circ \psi_\sigma \circ \tau^{-1}$ for all $\sigma, \tau \in \mathrm{Aut}(X, m)$. In particular, $\psi_\sigma \circ \sigma = \sigma \circ \psi_\sigma$, whence any $\sigma \in \mathrm{Aut}(X, d)$ induces by restriction an automorphism of the $\Lambda$-metric subspace $\psi_\sigma(X)$. Note also that for every $\sigma \in \mathrm{Aut}(X, d)$ and for every point $x \in X$, the element $\sigma\psi_\sigma(x) = \psi_\sigma(\sigma x)$ is the midpoint of the cell $[x, \sigma^2 x]$, in particular, $d(x, \sigma^2 x) \in 2\Lambda_+$.

In the following we shall show that the maps $\psi_\sigma : X \longrightarrow X$, for $\sigma \in \mathrm{Aut}(X, d)$, have remarkable properties, in particular, they are endomorphisms of the median set $(X, m)$, and moreover $\psi_\sigma^2 := \psi_\sigma \circ \psi_\sigma$ is a folding of the median set $(X, m)$ for every $\sigma \in \mathrm{Aut}(X, d)$ (see Proposition 3.10) In addition, we shall extend to our more general context the basic notion of hyperbolic length of an automorphism.

We fix an automorphism $\sigma$ of the nonempty $\Lambda$-tree $(X, d)$.

**Lemma 3.4.** *Let $x \in X$, and put $\psi := \psi_\sigma, \mathfrak{D} := d(x, \sigma^2 x) - d(x, \sigma x)$. Then the following assertions hold.*

(1) *The cell $[\psi(x), \sigma\psi(x)]$ is centrally situated in the cell $[x, \sigma x]$.*

(2) $m(\psi(x), x, \sigma\psi(x)) = \psi^2(x)$, *and* $m(\psi(x), \sigma x, \sigma\psi(x)) = \sigma\psi^2(x)$.

(3) $[\psi(x), \sigma\psi(x)] = [\psi^2(x), \sigma\psi^2(x)]$, *whence* $d(\psi(x), \sigma\psi(x)) = d(\psi^2(x), \sigma\psi^2(x))$ *and* $d(\psi^2(x), \sigma\psi(x)) = d(\psi(x), \sigma\psi^2(x))$.

(4) $d(x, \sigma x) = d(\psi(x), \sigma\psi(x)) + 2d(x, \psi^2(x))$.

(5) $\mathfrak{D}_+ = d(\psi^2(x), \sigma\psi(x)), \mathfrak{D}_- = d(\psi(x), \psi^2(x))$, *and* $|\mathfrak{D}| = d(\psi(x), \sigma\psi(x))$.

*Proof.* (1) is immediate since $\psi(x), \sigma\psi(x) \in [x, \sigma x]$ and $d(x, \psi(x)) = d(\sigma x, \sigma\psi(x))$.

(2). Put $z := m(\psi(x), x, \sigma\psi(x)), z' := m(\psi(x), x, \sigma^{-1}\psi(x))$. We have to show that $z = z' = \psi^2(x)$. As $z, z' \in [x, \psi(x)]$, and

$$d(x, z) + d(z', \psi(x)) = (\psi(x), \sigma\psi(x))_x + (x, \sigma^{-1}\psi(x))_{\psi(x)} = d(x, \psi(x)),$$

it follows that

$$z = z' \in [\psi(x), \sigma\psi(x)] \cap [\psi(x), \sigma^{-1}\psi(x)] = [\psi(x), \psi^2(x)].$$

To conclude that $\psi^2(x) = z$, it remains to show that $z \in [\sigma^{-1}\psi(x), \sigma\psi(x)]$. Assuming the contrary, it follows by [12, Theorem 1.6.(1)] that there exists $P \in \mathit{Spec}(X, m)$ such that $\sigma^{-1}\psi(x), \sigma\psi(x) \notin P, z \in P$. Consequently, $x \in P$, $\psi(x) \in P \cap [\sigma x, \sigma^{-1}x]$, whence either $\sigma x \in P$ or $\sigma^{-1}x \in P$. Assuming that $\sigma x \in P$, we obtain $\sigma\psi(x) \in [x, \sigma x] \subseteq P$, a contradiction, while assuming that $\sigma^{-1}x \in P$, we obtain $\sigma^{-1}\psi(x) \in [x, \sigma^{-1}x] \subseteq P$, again a contradiction.

(3) and (4) are immediate consequences of (2).

(5). We obtain

$$\begin{aligned}\mathfrak{D} &= 2d(x, \sigma\psi(x)) - d(x, \sigma x)\\ &= d(x, \sigma\psi(x)) - d(\sigma x, \sigma\psi(x))\\ &= d(\psi^2(x), \sigma\psi(x)) - d(\psi^2(x), \psi(x)).\end{aligned}$$

Since the map $\iota_{\psi^2(x),\sigma\psi^2(x)} : [\psi^2(x), \sigma\psi^2(x)] \longrightarrow [0, d(\psi^2(x), \sigma\psi^2(x)]$, is an isomorphism of $\Lambda$-metric spaces, and $[\psi^2(x), \sigma\psi^2(x)] = [\psi(x), \sigma\psi(x)]$ by (3), it follows that $d(\psi^2(x), \psi(x)) \wedge d(\psi^2(x), \sigma\psi(x)) = 0$, whence the identities from (5). □

**Corollary 3.5.** $\psi^3 = \psi$, *whence* $\psi(X) = \psi^2(X) = \mathrm{Fix}(\psi^2) := \{x \in X \mid \psi^2(x) = x\}$, *and the restriction* $\psi|_{\psi(X)}$ *is an involution. Thus, for any point* $x \in X$, $x \in \psi(X)$ *if and only if* $x$ *is the midpoint of the cell* $[\sigma^{-1}\psi(x), \sigma\psi(x)]$.

*Proof.* Let $x \in X$. By Lemma 3.4 (4) applied to $\psi(x)$, we obtain $2d(\psi(x), \psi^3(x)) = d(\psi(x), \sigma\psi(x)) - d(\psi^2(x), \sigma\psi^2(x))$. Consequently, $d(\psi(x), \psi^3(x)) = 0$, i.e., $\psi^3(x) = \psi(x)$, since $d(\psi(x), \sigma\psi(x)) = d(\psi^2(x), \sigma\psi^2(x))$ by Lemma 3.4 (3). □

Using the map $\psi_{\sigma^2}$, we obtain another useful description of $\psi_\sigma(X)$.

**Lemma 3.6.** $\psi_\sigma(X) = \mathrm{Fix}(\psi_{\sigma^2}) \subseteq \psi_{\sigma^2}(X)$. *Thus, for any point* $x \in X$, $x \in \psi_\sigma(X)$ *if and only if* $x$ *is the midpoint of the cell* $[\sigma^{-2}x, \sigma^2 x]$.

*Proof.* Put $\psi := \psi_\sigma, \psi' := \psi_{\sigma^2}$. As $\psi(X) = \mathrm{Fix}(\psi^2)$ by Corollary 3.5, we have to show that $\mathrm{Fix}(\psi') = \mathrm{Fix}(\psi^2)$. First, let us show that $\mathrm{Fix}(\psi') \subseteq \mathrm{Fix}(\psi^2)$. Let $x \in \mathrm{Fix}(\psi')$, whence $x$ is the midpoint of the cell $[\sigma^{-2}x, \sigma^2 x]$. Since $\sigma^{-1}\psi(x) \in [\sigma^{-2}x, x]$, $\sigma\psi(x) \in [x, \sigma^2 x]$, and $d(x, \sigma\psi(x)) = d(x, \sigma^{-1}\psi(x))$, it follows that $x = \psi^2(x)$ is the midpoint of the cell $[\sigma^{-1}\psi(x), \sigma\psi(x)]$ as desired.

Conversely, let $x \in \mathrm{Fix}(\psi^2)$, so $x$ is the midpoint of the cell $[\sigma^{-1}\psi(x), \sigma\psi(x)]$. Consequently, $d(x, \sigma^2 x) = 2d(x, \sigma\psi(x)) = d(\psi(x), \sigma^2\psi(x)) = (d(x, \sigma^4 x) - d(x, \sigma^2 x))_+$, where the last equality is obtained by applying the formula (2.2) to $\psi(x)$ and $\sigma^2\psi(x)$ - the midpoints of the cells $[\sigma^{-1}x, \sigma x]$ and $[\sigma x, \sigma^3 x]$ respectively. From the equality above we deduce that $d(x, \sigma^4 x) = 2d(x, \sigma^2 x)$, therefore $x = \psi'(x)$ is the midpoint of the cell $[\sigma^{-2}x, \sigma^2 x]$ as required. □

**Corollary 3.7.** *For all* $x, y \in \psi_\sigma(X)$, $d(x, \sigma^2 x) = d(y, \sigma^2 y)$.

*Proof.* Let $x \in \psi_\sigma(X)$. Set $x_n := \sigma^n x$ for $n \in \mathbb{Z}$. By Lemma 3.6, $x = \psi_{\sigma^2}(x)$ is the midpoint of the cell $[x_{-2}, x_2]$. Applying again Lemma 3.6 to the automorphism $\sigma^2$, we get $\psi_{\sigma^2}(X) = \mathrm{Fix}(\psi_{\sigma^4})$. It follows that the cell $[x_{-2}, x_2]$ is centrally situated in the cell $[x_{-4}, x_4]$, and $x$ is the midpoint of the cell $[x_{-4}, x_4]$ too.

Setting $d := d(x, x_2), d' := d(y, y_2)$ for $x, y \in \psi_\sigma(X)$, we have to show that $d = d'$. Applying the formula (2.2) to $x$ and $y$ - the midpoints of the cells $[x_{-2}, x_2]$ and $[y_{-4}, y_4]$ respectively, we obtain $d(x, y) + d + 2d' = d''$, where

$$d'' := d(x_2, y_4) \vee d(x_2, y_{-4}) \vee d(x_{-2}, y_4) \vee d(x_{-2}, y_{-4}).$$

By symmetry, interchanging $x$ and $y$, we get $d(x, y) + 2d + d' = d''$, and hence $d = d'$ as desired. □

**Definition 3.8.** Let $l(\sigma) := \frac{d(x,\sigma^2 x)}{2} \in \Lambda_+$ for some (for all) $x \in \psi_\sigma(X)$. We call $l(\sigma)$ the *hyperbolic length* of the automorphism $\sigma$.

As a consequence of Lemma 3.4 and Corollaries 3.5 and 3.7, we obtain

**Corollary 3.9.** *Let $\sigma$ be an automorphism of a $\Lambda$-tree $(X, d)$. Then the following assertions hold.*

(1) $l(\sigma) = (d(x, \sigma^2 x) - d(x, \sigma x))_+ = d(\psi_\sigma^2(x), \sigma\psi_\sigma(x)) = d(\psi_\sigma(x), \sigma\psi_\sigma^2(x))$ *for all* $x \in X$.

(2) $l(\sigma) = \min\{\frac{d(x,\sigma^2 x)}{2} \mid x \in X\} = \min\{d(x, \sigma\psi_\sigma(x)) \mid x \in X\}$.

(3) $\psi_\sigma(X) = \{x \in X \mid d(x, \sigma^2 x) = 2l(\sigma)\} = \{x \in X \mid d(x, \sigma\psi_\sigma(x)) = l(\sigma)\}$.

(4) $l(\tau\sigma\tau^{-1}) = l(\sigma)$ *and* $\psi_{\tau\sigma\tau^{-1}}(X) = \tau\psi_\sigma(X)$ *for all* $\tau \in \mathrm{Aut}(X, d)$.

**Proposition 3.10.** *Let $\sigma$ be an automorphism of a $\Lambda$-tree $(X, d)$. Then the following assertions hold.*

(1) *$\psi_\sigma(X)$ is a closed sub-$\Lambda$-tree of $(X, d)$, and $\psi_\sigma^2$ is its associated folding.*

(2) *For any $x \in X$, the cell $[x, \psi_\sigma^2(x)]$ is the bridge from $x$ to $\psi_\sigma(X)$, and $d(x, \psi_\sigma(X)) = d(x, \psi_\sigma^2(x)) = \min\{d(x, y) \mid y \in \psi_\sigma(X)\}$.*

(3) *The map $\psi_\sigma$ is an endomorphism of the underlying median set $(X, m)$ of the $\Lambda$-tree $(X, d)$.*

*Proof.* First we show that $\psi_\sigma(X)$ is a convex subset of $(X, m)$, and hence a sub-$\Lambda$-tree of $(X, d)$. Let $x, y \in \psi_\sigma(X)$, $z \in [x, y]$. According to Corollary 3.9 (3), $d(x, \sigma^2 x) =$

$d(y, \sigma^2 y) = 2l(\sigma)$, and we have to show that $z \in \psi_\sigma(X)$, i.e., $d(z, \sigma^2 z) = 2l(\sigma)$. Applying the formula (2.1) to the points $z \in [x, y]$ and $\sigma^2 z \in [\sigma^2 x, \sigma^2 y]$, we obtain

$$2l(\sigma) - d(z, \sigma^2 z) = 2(d(x, z) \wedge d(y, z) \wedge (x, \sigma^2 y)_y \wedge (x, \sigma^{-2} y)_y) = 0,$$

as desired, since $0 \leq (x, \sigma^2 y)_y \wedge (x, \sigma^{-2} y)_y \leq (\sigma^2 y, \sigma^{-2} y)_y = 0$ as $y \in \psi_\sigma(X)$, and hence $y \in [\sigma^{-2} y, \sigma^2 y]$ by Lemma 3.6.

Next we show that the involution $\psi_\sigma|_{\psi_\sigma(X)}$ is an automorphism of the $\Lambda$-tree $\psi_\sigma(X)$, i.e., $d(\psi_\sigma(x), \psi_\sigma(y)) = d(x, y)$ for all $x, y \in \psi_\sigma(X)$. Applying the formula (2.2) to $\psi(x)$ and $\psi(y)$ - the midpoints of the cells $[\sigma^{-1} x, \sigma x]$ and $[\sigma^{-1} y, \sigma y]$ respectively, we obtain with the same argument as above that

$$d(x, y) - d(\psi_\sigma(x), \psi_\sigma(y)) = 2(l(\sigma) \wedge (x, \sigma^2 y)_y \wedge (x, \sigma^{-2} y)_y) = 0,$$

as required.

To conclude that $\psi_\sigma(X)$ is a closed sub-$\Lambda$-tree with associated folding $\psi_\sigma^2$, it remains to show that $\psi_\sigma^2(x) \in [x, y]$ for all $x \in X, y \in \psi_\sigma(X)$. As shown above, the restriction $\psi_\sigma|_{\psi_\sigma(X)}$ is an involutive automorphism of the sub-$\Lambda$-tree $\psi_\sigma(X)$, therefore $d(\psi_\sigma^2(x), y) = d(\psi_\sigma(x), \psi_\sigma(y))$ for $x \in X, y \in \psi_\sigma(X)$. Applying the formula (2.2) to $\psi_\sigma(x)$ and $\psi_\sigma(y)$ - the midpoints of the cells $[\sigma^{-1} x, \sigma x]$ and $[\sigma^{-1} y, \sigma y]$ respectively, and using the identities $d(\psi_\sigma(x), \sigma x) = d(x, \psi_\sigma^2(x)) + l(\sigma)$, $d(\psi_\sigma(y), \sigma y) = l(\sigma)$ (cf. Lemma 3.4 and Corollary 3.9), we obtain

$$d(x, y) - d(x, \psi_\sigma^2(x)) - d(\psi_\sigma^2(x), y) = 2(l(\sigma) \wedge (x, \sigma^2 y)_y \wedge (x, \sigma^{-2} y)_y) = 0,$$

as desired.

Finally note that the map $\psi_\sigma = \psi_\sigma^3$ is an endomorphism of the median set $(X, m)$ as a composition of the folding $\psi_\sigma^2 : X \longrightarrow \psi_\sigma(X)$ with the automorphism $\psi_\sigma|_{\psi_\sigma(X)}$ of $\psi_\sigma(X)$. □

## 3.3 Elliptic automorphisms and inversions

The next characterization of the automorphisms $\sigma$ with hyperbolic length $l(\sigma) = 0$ is an immediate consequence of Corollary 3.9.

**Proposition 3.11.** *Let $\sigma$ be an automorphism of a nonempty $\Lambda$-tree $(X, d)$. Then the following assertions are equivalent.*

(1) $l(\sigma) = 0$.

(2) *$\sigma^2$ is elliptic, i.e., either $\sigma$ is elliptic or $\sigma$ is an inversion.*

(3) $\mathrm{Fix}(\sigma^2) = \psi_\sigma(X)$.

(4) $\sigma|_{\psi_\sigma(X)} = \psi_\sigma|_{\psi_\sigma(X)}$.

(5) $\sigma|_{\psi_\sigma(X)}$ *is an involution.*

(6) *For all* $x \in X, d(x, \sigma^2 x) \leq d(x, \sigma x)$.

(7) *There exists* $x \in X$ *such that* $d(x, \sigma^2 x) \leq d(x, \sigma x)$.

As a consequence of Proposition 3.11, we obtain

**Corollary 3.12.** *Let* $\sigma \in \mathrm{Aut}(X, d)$ *with* $l(\sigma) = 0$. *Then the nonempty set* $\mathrm{Fix}(\sigma^2)$ *is a closed sub-*$\Lambda$*-tree of* $(X, d)$, *with the associated folding* $\psi_\sigma^2 = \sigma \circ \psi_\sigma = \psi_\sigma \circ \sigma$ *sending any point* $x \in X$ *to the midpoint of the cell* $[x, \sigma^2 x]$, *while* $d(x, \mathrm{Fix}(\sigma^2)) = \frac{d(x, \sigma^2 x)}{2}$.

The next statement provides a characterization of the elliptic automorphisms.

**Proposition 3.13.** *Let* $\sigma$ *be an automorphism of a nonempty* $\Lambda$*-tree* $(X, d)$. *Then the following assertions are equivalent.*

(1) *The automorphism* $\sigma$ *is elliptic, i.e.,* $\mathrm{Fix}(\sigma) \neq \emptyset$.

(2) $l(\sigma) = 0$ *and* $d(x, \sigma x) \in 2\Lambda$ *for all* $x \in X$.

(3) *For all* $x \in X, d(x, \sigma^2 x) \leq d(x, \sigma x)$ *and* $d(x, \sigma x) \in 2\Lambda$.

(4) *There exists* $x \in X$ *such that* $d(x, \sigma^2 x) \leq d(x, \sigma x)$ *and* $d(x, \sigma x) \in 2\Lambda$.

(5) *There exists* $x \in X$ *such that* $d(x, \sigma^2 x) = d(x, \sigma x)$.

*Proof.* The implications $(2) \iff (3), (3) \implies (4), (1) \implies (5)$, and $(5) \implies (4)$ are obvious.

$(1) \Longrightarrow (2)$. Let $x \in X$. We have only to show that $d(x, \sigma x) \in 2\Lambda$. Let $a \in \mathrm{Fix}(\sigma)$. Then $m(x, a, \sigma x)$ is the midpoint of the cell $[x, \sigma x]$, whence $d(x, \sigma x) \in 2\Lambda$ as desired.

$(4) \Longrightarrow (1)$. Let $x \in X$ be such that $d(x, \sigma^2 x) \leq d(x, \sigma x) \in 2\Lambda$. Let $y$ be the midpoint of the cell $[x, \sigma x]$. It follows by Proposition 3.11 that $\psi_\sigma(x) \in \mathrm{Fix}(\sigma^2)$, whence $y \in \mathrm{Fix}(\sigma)$ as a midpoint of the cell $[\sigma\psi_\sigma(x) = \psi_\sigma^2(x), \psi_\sigma(x) = \sigma\psi_\sigma^2(x)]$, centrally situated in the cell $[x, \sigma x]$. □

The next statement extends to abelian $l$-groups [1, Proposition 6.1.] concerning elliptic automorphisms of $\Lambda$-trees, where $\Lambda$ is a totally ordered abelian group.

**Corollary 3.14.** *Let* $\sigma$ *be an elliptic automorphism of the* $\Lambda$*-tree* $(X, d)$. *Then the following assertions hold.*

(1) *The nonempty set* $\mathrm{Fix}(\sigma) = \mathrm{Fix}(\psi_\sigma) \subseteq \mathrm{Fix}(\sigma^2) = \psi_\sigma(X)$ *is a closed sub-*$\Lambda$*-tree of* $(X, d)$.

(2) *The folding* $\theta := \theta_\sigma : X \longrightarrow X$, *with* $\theta(X) = \mathrm{Fix}(\sigma)$, *sends any point* $x \in X$ *to* $\theta(x) := m(x, a, \sigma x)$ *for some (for all)* $a \in \mathrm{Fix}(\sigma)$. $\theta(x)$ *is the midpoint of the cell* $[\psi_\sigma^2(x) = \sigma\psi_\sigma(x), \psi_\sigma(x) = \sigma\psi_\sigma^2(x)]$, *centrally situated in the cells* $[x, \sigma x]$ *and* $[x, \sigma^{-1}x]$, *while the cell* $[x, \theta(x)]$ *is the bridge from* $x$ *to* $\mathrm{Fix}(\sigma)$, $d(x, \mathrm{Fix}(\sigma)) = d(x, \theta(x)) = \frac{d(x,\sigma x)}{2}$, *and* $\psi_\sigma^2 \circ \theta = \theta \circ \psi_\sigma^2 = \theta$.

(3) $\mathrm{Fix}(\sigma)$ *is the smallest sub-*$\Lambda$*-tree which meets any* $\langle\sigma\rangle$*-invariant sub-*$\Lambda$*-tree of* $(X, d)$.

*Proof.* The proof is straightforward, and hence it is left to the reader. □

**Remark 3.15.** Let $(X, d)$ be a $\Lambda$-tree, $\sigma \in \mathrm{Aut}(X, d)$, and $M := \{x \in X \mid d(x, \sigma^2 x) = d(x, \sigma x)\}$. Then $\mathrm{Fix}(\sigma) = \psi_\sigma(M) \subseteq M$, while $\mathrm{Fix}(\sigma) = M$ if and only if $M \subseteq \mathrm{Fix}(\sigma^2)$. In particular, the equality $\mathrm{Fix}(\sigma) = M$ holds whenever $\sigma$ is an involution.The simplest example of an elliptic automorphism $\sigma$, with the nonempty set $\mathrm{Fix}(\sigma)$ properly contained in $M$, is obtained by taking $\Lambda = \mathbb{Z}$, $X := \{x_0, x_1, x_2, x_3\}$ , $d(x_0, x_i) = 1$ for $i \neq 0$, $m(x_1, x_2, x_3) = x_0$, while $\sigma$ is the cycle $(x_1, x_2, x_3)$ of length 3. Then $\mathrm{Fix}(\sigma) = \{x_0\} \neq M = X$.

**Corollary 3.16.** *Let* $\sigma$ *be an automorphism of a* $\Lambda$*-tree* $(X, d)$. *Then the following assertions are equivalent.*

(1) $\sigma$ *is elliptic.*

(2) $\sigma^n$ *is elliptic for some odd* $n \in \mathbb{Z}$.

(3) $\sigma^n$ *is elliptic for all* $n \in \mathbb{Z}$.

*Proof.* The implications $(1) \Longrightarrow (3)$ and $(3) \Longrightarrow (2)$ are obvious.

$(2) \Longrightarrow (1)$. Assume that $\sigma^n$ is elliptic for some odd $n \in \mathbb{Z} \setminus \{1, -1\}$, and let $p$ be a prime number such that $p \mid n$. It suffices to show that $\tau := \sigma^{\frac{n}{p}}$ is elliptic. By assumption, $A := \mathrm{Fix}(\sigma^n) = \mathrm{Fix}(\tau^p)$ is a nonempty sub-$\Lambda$-tree of $(X, d)$, stable under the action of $\tau$. Assuming that $\mathrm{Fix}(\tau) \subseteq A$ is empty, it follows that the action on the underlying median set of $A$ of the cyclic group of odd prime order $p$ generated by $\tau|_A$ is free, contrary to [12, Lemma 2.12]. Consequently, $\mathrm{Fix}(\tau) \neq \varnothing$ as desired. □

The next statement is a completion of Proposition 3.11.

**Corollary 3.17.** *Let* $\sigma$ *be an automorphism of a* $\Lambda$*-tree* $(X, d)$. *Then the following assertions are equivlent.*

(1) $\sigma^2$ *is elliptic.*

(2) *For all* $x \in X$, $d(x, \sigma^4 x) \leq d(x, \sigma^2 x)$.

(3) *There exists* $x \in X$ *such that* $d(x, \sigma^4 x) \leq d(x, \sigma^2 x)$.

(4) *There exists* $x \in X$ *such that* $d(x, \sigma^4 x) = d(x, \sigma^2 x)$.

(5) $\sigma^{2n}$ *is elliptic for some* $n \in \mathbb{Z} \setminus \{0\}$.

(6) $\sigma^{2n}$ *is elliptic for all* $n \in \mathbb{Z}$.

*Proof.* As $d(x, \sigma^2 x) \in 2\Lambda$ for all $x \in X$, the logical equivalence of the assertions $(1)-(4)$ follows by applying Proposition 3.13 to the automorphism $\sigma^2$, while the implications $(1) \Longrightarrow (6)$ and $(6) \Longrightarrow (5)$ are obvious.

$(5) \Longrightarrow (1)$. Assume that $\sigma^{2n}$ is elliptic for some $n \in \mathbb{Z}\setminus\{-1, 0, 1\}$. By Corollary 3.16, we may assume that $n = 2^m, m \geq 1$. Setting $\tau := \sigma^{\frac{n}{2}} = \sigma^{2^{m-1}}$, it suffices to show that $\tau^2$ is elliptic. As $\tau^4 = \sigma^{2n}$ is elliptic by assumption, there exists $x \in X$ such that $d(x, \tau^4 x) = 0 \leq d(x, \tau^2 x)$, and hence $\tau$ satisfies (3). As $(3) \Longrightarrow (1)$, we deduce that $\tau^2$ is elliptic as desired. □

The next two statements extend to abelian $l$-groups [1, Proposition 6.3.] concerning inversions of $\Lambda$-trees, where $\Lambda$ is a totally ordered abelian group.

**Proposition 3.18.** *The following assertions are equivalent.*

(1) *The automorphism $\sigma$ is an inversion, i.e.,* $\mathrm{Fix}(\sigma) = \varnothing$ *and* $\mathrm{Fix}(\sigma^2) \neq \varnothing$, *so* $l(\sigma) = 0$.

(2) *For all* $x \in X$, $d(x, \sigma^2 x) \leq d(x, \sigma x)$ *and* $d(x, \sigma x) \notin 2\Lambda$, *in particular,* $d(x, \sigma^2 x) < d(x, \sigma x)$.

(3) *There exists* $x \in X$ *such that* $d(x, \sigma^2 x) < d(x, \sigma x)$ *and* $d(x, \sigma x) \notin 2\Lambda$.

(4) *For all* $x \in X$, $\sigma\psi_\sigma(x) = \psi_\sigma^2(x) \neq \psi_\sigma(x)$, *so* $\sigma|_{\psi_\sigma(X)}$ *is an involution without fixed points.*

(5) *There exists* $x \in X$ *such that* $\sigma^2 x = x$ *and* $d(x, \sigma x) \notin 2\Lambda$.

(6) $\mathrm{Fix}(\sigma) = \varnothing$, *and if* $\Lambda'$ *is an abelian* $l$*-group containing* $\Lambda$ *such that* $d(x, \sigma x) \in 2\Lambda'$ *for some* $x \in X$ *then the automorphism* $\sigma \otimes_\Lambda \Lambda'$ *of the* $\Lambda'$*-tree* $(X, d) \otimes_\Lambda \Lambda'$ *is elliptic.*

(7) $\sigma^n$ *is an inversion for some odd* $n \in \mathbb{Z}$.

(8) $\sigma^n$ *is an inversion for all odd* $n \in \mathbb{Z}$.

*Proof.* The logical equivalence of the assertions $(1) - (5)$ follows by Propositions 3.11 and 3.13, while the logical equivalence of the assertions $(1), (7), (8)$ is a consequence of Corollary 3.16.

$(1) \Longrightarrow (6)$. By assumption $\sigma$ is an inversion, in particular, $\mathrm{Fix}(\sigma) = \varnothing$. Let $\Lambda'$ be an abelian $l$-group containing $\Lambda$, and $\mathbb{X}' := (X, d) \otimes_\Lambda \Lambda', \sigma' := \sigma \otimes_\Lambda \Lambda' \in \mathrm{Aut}(\mathbb{X}')$ be as defined in Remark 2.13. By assumption there exists $x \in X$ such that $d(x, \sigma x) \in \Lambda \cap 2\Lambda'$.

As $\sigma^2$ is elliptic, it follows by Proposition 3.11 (6) that $d(x, \sigma^2 x) \leq d(x, \sigma x)$, and hence $\mathrm{Fix}(\sigma') \neq \varnothing$ by Proposition 3.13 (4).

$(6) \Longrightarrow (2)$. Let $x \in X$. We have to show that $d(x, \sigma^2 x) \leq d(x, \sigma x)$ and $d(x, \sigma x) \notin 2\Lambda$. Let $\Lambda' := \mathbb{Z}[\frac{1}{2}] \otimes_{\mathbb{Z}} \Lambda, \mathbb{X}' := (X, d) \otimes_\Lambda \Lambda', \sigma' := \sigma \otimes_\Lambda \Lambda'$. As $\Lambda \subseteq \Lambda' = 2\Lambda'$, it follows by assumption that $\sigma'$ is elliptic, therefore $d(x, \sigma^2 x) \leq d(x, \sigma x)$ by Proposition 3.13 (3). Since $\mathrm{Fix}(\sigma) = \varnothing$ by assumption, we deduce by Proposition 3.13 (4) that $d(x, \sigma x) \notin 2\Lambda$ as desired. □

**Proposition 3.19.** *Let $\mathbb{X} := (X, d)$ be a nonempty $\Lambda$-tree, and $\sigma \in \mathrm{Aut}(\mathbb{X})$ be an inversion. Let $\Lambda'$ be an abelian $l$-group containing $\Lambda$ such that $\sigma' := \sigma \otimes_\Lambda \Lambda' \in \mathrm{Aut}(\mathbb{X} \otimes_\Lambda \Lambda')$ is elliptic. Let $\theta' : X' \longrightarrow X'$ be the folding of the underlying median set $(X', m')$ of the $\Lambda'$-tree $\mathbb{X}' = \mathbb{X} \otimes_\Lambda \Lambda' = (X', d')$ sending any point $x' \in X'$ to the midpoint of the cell $[x', \sigma' x'] \subseteq X'$. Then the following assertions hold.*

(1) *The closed sub-$\Lambda'$-tree $\mathrm{Fix}(\sigma') = \theta'(X')$ of the $\Lambda'$-tree $\mathbb{X}'$ is the $\Lambda'$-tree closure of its nonempty subspace $\theta'(X) = \theta'(\psi_\sigma(X)) = \theta'(\mathrm{Fix}(\sigma^2))$.*

(2) *For all $x', y' \in \theta'(X), d'(x', y') \in \Lambda$, $\theta'(X)$ is a $\Lambda$-tree, and $\mathrm{Fix}(\sigma') \cong \theta'(X) \otimes_\Lambda \Lambda'$.*

(3) *The restriction map $\theta'|_X$ induces a bijection $\mathrm{Fix}(\sigma^2)/\sim\, \cong X/\sim\, \longrightarrow \theta'(X)$, where the congruence $\sim$ on the underlying median set $(X, m)$ is defined by*

$$x \sim y \Longleftrightarrow [x, \sigma x] \cap [y, \sigma y] \neq \varnothing.$$

*The congruence $\sim$ as well as the induced $\Lambda$-tree structure on the quotient $X/\sim$ do not depend on the extension $\Lambda'$ of $\Lambda$ satisfying $\mathrm{Fix}(\sigma \otimes_\Lambda \Lambda') \neq \varnothing$, and for any such $\Lambda'$, $\mathrm{Fix}(\sigma \otimes_\Lambda \Lambda') \cong (X/\sim) \otimes_\Lambda \Lambda'$.*

(4) *If $\Lambda$ is totally ordered then the automorphism $\sigma' = \sigma \otimes_\Lambda \Lambda'$ has a unique fixed point.*

*Proof.* (1) Since $\mathrm{Fix}(\sigma^2) = \psi_\sigma(X) = \psi_\sigma^2(X)$ by Proposition 3.11, while $\theta' \circ \psi_{\sigma'}^2 = \theta'$ by Corollary 3.14, it follows that $\theta'(X) = \theta'(\psi_\sigma(X)) = \theta'(\mathrm{Fix}(\sigma^2))$. As $X'$ is the convex closure of $X$ in the median set $(X', m')$ and $\theta'$ is an endomorphism of the median set $(X', m')$, we deduce that $\theta'(X') = \mathrm{Fix}(\sigma')$ is the convex closure of $\theta'(X)$, and hence the closed sub-$\Lambda'$-tree $\mathrm{Fix}(\sigma')$ is spanned by its subspace $\theta'(X)$. Consequently, by Corollary 2.11, $\mathrm{Fix}(\sigma')$ is the $\Lambda'$-tree closure of its subspace $\theta'(X)$.

(2) First we show that for all $x, y \in \mathrm{Fix}(\sigma^2)$, there exist $p, q \in \mathrm{Fix}(\sigma^2)$ such that $p, q \in [x, y], \theta'(p) = \theta'(x), \theta'(q) = \theta'(y)$, and $d(p, q) = d'(\theta'(p), \theta'(q))$, in particular, $\theta'(X) = \theta'(\mathrm{Fix}(\sigma^2))$ is a $\Lambda$-metric space, and for all $x', y' \in \theta'(X)$,

$$d'(x', y') = \min\{d(x, y) \mid x, y \in \mathrm{Fix}(\sigma^2), \theta'(x) = x', \theta'(y) = y'\}.$$

For $x, y \in \mathrm{Fix}(\sigma^2)$, set $p := m(x, y, \sigma x), q := m(y, p, \sigma y)$. Since $\theta'(x) = \theta'(\sigma x), \theta'(y) = \theta'(\sigma y)$, it follows that $\theta'(p) = \theta'(x), \theta'(q) = \theta'(y)$, and hence the cells $[p, \sigma p]$ and $[q, \sigma q]$ are centrally situated in the cells $[x, \sigma x]$ and $[y, \sigma y]$ respectively. On the other hand, as $p \in [x, y]$ and $q \in [p, y]$, we deduce that $q \in [x, y]$ too.

Moreover we claim that $[p, \sigma q] = [\sigma p, q]$, in particular, $d(p, \sigma p) = d(q, \sigma q)$.

Indeed, since $\sigma q \in [y, \sigma y]$ and $q = m(y, p, \sigma y)$, it follows that $q \in [p, \sigma q]$, and hence, by acting with $\sigma$, $\sigma q \in [\sigma p, q]$. To show that $p \in [\sigma p, q]$, whence $\sigma p \in [p, \sigma q]$, let $P \in \mathrm{Spec}(X, m)$ be a prime convex subset of the median set $(X, m)$ such that $\sigma p, q \in P$. By [12, Theorem 2.5 (1)], we have to show that $p \in P$. Assuming the contrary, it follows that $y, \sigma y \in P$, while $x, \sigma x \in X \setminus P$, and hence $\sigma p \in [x, \sigma x] \subseteq X \setminus P$, i.e., a contradiction. Consequently, $[p, \sigma q] = [\sigma p, q]$ as claimed.

As $\theta'(x) = \theta'(p)$ and $\theta'(y) = \theta'(q)$ are the midpoints of the cells $[p, \sigma p] \otimes_\Lambda \Lambda'$ and $[q, \sigma q] \otimes_\Lambda \Lambda'$ respectively, and $[p, \sigma q] = [\sigma p, q]$, it follows by (2.2) that $d'(\theta'(x), \theta'(y)) = d(p, q) \le d(x, y)$ as desired.

To conclude that $\theta'(X)$ is a $\Lambda$-tree, whence $\mathrm{Fix}(\sigma') \cong \theta'(X) \otimes_\Lambda \Lambda'$, we have to show by Proposition 2.8 that for all $x', y' \in \theta'(X)$ and for all $\lambda \in \Lambda_+$ such that $\lambda \le d'(x', y')$, the unique element $z'$ of the cell $[x', y'] \subseteq \mathrm{Fix}(\sigma')$ satisfying $d'(x', z') = \lambda$ belongs to $\theta'(X)$. Given $x', y' \in \theta'(X)$, we may choose as above $x, y \in \mathrm{Fix}(\sigma^2)$ such that $\theta'(x) = x', \theta'(y) = y'$, and $[x, \sigma y] = [\sigma x, y]$, whence $d'(x', y') = d(x, y)$. Since $\mathrm{Fix}(\sigma^2)$ is a sub-$\Lambda$-tree of $(X, d)$, it follows that for any $\lambda \in \Lambda_+$ such that $\lambda \le d'(x', y')$, there exists uniquely $z \in [x, y] \subseteq \mathrm{Fix}(\sigma^2)$ such that $d(x, z) = d(\sigma x, \sigma z) = \lambda$. It follows easily that $[x, \sigma z] = [\sigma x, z]$, and hence $\theta'(z) \in [x', y']$, with $d'(x', \theta'(z)) = d(x, z) = \lambda$ as desired.

(3) Let $x, y \in \mathrm{Fix}(\sigma^2)$ be such that $\theta'(x) = \theta'(y)$. Setting $p := m(x, y, \sigma x)$, $q := m(y, p, \sigma y)$, it follows by (2) that $p = q \in [x, \sigma x] \cap [y, \sigma y] \neq \varnothing$. Conversely, assuming that $C := [x, \sigma x] \cap [y, \sigma y] \neq \varnothing$, we obtain $C = [p, \sigma p] = [r, \sigma r]$, where $r := m(y, x, \sigma y)$. Consequently, $\theta'(x) = \theta'(p) = \theta'(r) = \theta'(y)$. Thus, the restriction map $\theta'|_{\mathrm{Fix}(\sigma^2)}$ induces an isomorphism of median sets $\mathrm{Fix}(\sigma^2)/\sim \longrightarrow \theta'(\mathrm{Fix}(\sigma^2)) = \theta'(X)$, where the congruence $x \sim y \iff [x, \sigma x] \cap [y, \sigma y] \neq \varnothing$ does not depend on the choice of the extension $\Lambda'$ of $\Lambda$ satisfying $\mathrm{Fix}(\sigma \otimes_\Lambda \Lambda') \neq \varnothing$. As $\mathrm{Fix}(\sigma^2) = \psi_\sigma(X)$, and for all $x \in X$, the cell $[\psi_\sigma(x), \sigma\psi_\sigma(x) = \psi_\sigma^2(x)]$ is centrally situated in the cell $[x, \sigma x]$, the binary relation $\sim$ is a congruence on $(X, m)$ too, and the inclusion $\mathrm{Fix}(\sigma^2) \subseteq X$ induces an isomorphism of median sets $\mathrm{Fix}(\sigma^2)/\sim \longrightarrow X/\sim$.

Moreover, as we have seen above, the induced $\Lambda$-tree structure on the quotient $X/\sim$ does not depend on $\Lambda'$, and $\mathrm{Fix}(\sigma \otimes_\Lambda \Lambda') \cong (X/\sim) \otimes_\Lambda \Lambda'$ for any suitable extension $\Lambda'$ of $\Lambda$.

(4) Assuming that $\Lambda$ is totally ordered, it suffices to show by (1) that $\theta'(X)$ is a singleton. Let $x', y' \in \theta'(X)$. By (2) there exists $x, y \in \mathrm{Fix}(\sigma^2)$ such that $d'(x', y') = d(x, y)$ and $[x, \sigma y] = [\sigma x, y]$. As $\Lambda$ is totally ordered by assumption, the median set $(X, m)$ is

locally linear, and hence either $x = \sigma x$ or $x = y$. Since $\mathrm{Fix}(\sigma) = \varnothing$ by assumption, we deduce that $x = y$, whence $x' = y'$ as desired. □

**Remark 3.20.** (1) If $\Lambda = 2\Lambda$ then inversions don't exist.

(2) The converse of (4) from Proposition 3.19 does not hold. For instance, let $\Lambda' = \mathbb{Z} \times \mathbb{Z}$, $\Lambda = 2\Lambda'$, $X = \{0, 2\} \times \{0, 2\}$ be the square with the induced $\Lambda$-tree structure, and $\sigma$ be the involution $((0, 0)(2, 2))((2, 0)(0, 2))$. Then $X \otimes_\Lambda \Lambda' = \{0, 1, 2\} \times \{0, 1, 2\}$, and $\mathrm{Fix}(\sigma \otimes_\Lambda \Lambda') = \{(1, 1)\}$. By contrast, taking $\tau$ the involution $((0, 0)(2, 0))((0, 2)(2, 2))$, we obtain $\theta'_\tau(X) = \{(1, 0), (1, 2)\} \subsetneq \mathrm{Fix}(\tau \otimes_\Lambda \Lambda') = \{(1, 0), (1, 1), (1, 2)\} = [(1, 0), (1, 2)]$.

(3) If $\sigma$ is an automorphism of a $\Lambda$-tree then $\sigma^2$ is not an inversion. Indeed, assuming that $\sigma^2$ is an inversion, it follows that $\sigma^4$ is elliptic, and hence $\sigma^2$ is also elliptic by Corollary 3.17 ($(5) \Longrightarrow (1)$), i.e., a contradiction.

(4) Let $G$ be a group acting freely on a $\Lambda$-tree $(X, d)$, and $\sigma \in G \setminus \{1\}$. Then either $\sigma$ is hyperbolic of infinite order or $\sigma$ is an inversion of order 2; note that the latter possibility does not occur whenever $\Lambda = 2\Lambda$. Indeed, if $\mathrm{Fix}(\sigma^2) \neq \varnothing$ then $\sigma^2 = 1$ and $\sigma$ is an inversion. On the other hand, if $\sigma$ is of finite order, say $n \geq 2$, then $\mathrm{Fix}(\sigma^{2n}) = X \neq \varnothing$, and hence $\mathrm{Fix}(\sigma^2) \neq \varnothing$ by Corollary 3.17 ($(5) \Longrightarrow (1)$).

### 3.4 Hyperbolic automorphisms

## 4 Actions on $\Lambda$-trees, length functions, and $\Lambda$-tree-groups

The connection between Lyndon length functions and actions on $\Lambda$-trees, where $\Lambda$ is a totally ordered abelian group (cf. [1, Theorem 5.4.]), can be extended to the more general case where $\Lambda$ is an arbitrary abelian $l$-group.

We fix an abelian $l$-group $\Lambda$.

Let $G$ be a group with neutral element $1$.

**Definition 4.1.** By a ($\Lambda$-valued) *length function* on $G$ we understand a map $L : G \longrightarrow \Lambda$ satifying the following conditions, where we put

$$(g, h)_L := \frac{1}{2}(L(g) + L(h) - L(g^{-1}h)) \in \Lambda \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{2}]$$

for $g, h \in G$.

(L0) $L(1) = 0$.

(L1) $L(g) = L(g^{-1})$ for all $g \in G$.

(L2) $(g, h)_L \geq 0$ for all $g, h \in G$.

**Remark 4.2.** Assuming (L0) and (L1), (L2) is equivalent with the folowing conditions.

(i) $L(gh) \leq L(g) + L(h)$ for all $g, h \in G$.

(ii) $(g, h)_L \leq L(g) \wedge L(h)$ for all $g, h \in G$.

Note also that the conditions (L0)-(L2) imply $L(g) = (g, g)_L \geq 0$ for all $g \in G$ and $(g, h)_L = (h, g)_L$ for all $g, h \in G$.

The next characterization of the length functions is immediate.

**Lemma 4.3.** *Let $L : G \longrightarrow \Lambda$ be a map defined on the group $G$ with values in the abelian $l$-group $\Lambda$. Set $K := L^{-1}(0)$. Then, the necessary and sufficient condition for the map $L$ to be a length function is that the following assertions hold.*

(1) *$K$ is a subgroup of $G$; set $\mathbb{G} := G/K = \{gK \mid g \in G\}$, $\mathbf{1} := 1 \cdot K$.*

(2) *The map $L : G \longrightarrow \Lambda$ factorizes through $K \setminus G/K := \{KgK \mid g \in G\}$; set $d(gK, hK) := L(g^{-1}h)$ for $g, h \in G$.*

(3) *$(\mathbb{G}, d)$ is a $\Lambda$-metric space on which the group $G$ acts transitively from the left by $\Lambda$-isometries.*

**Corollary 4.4.** *The map $L : G \longrightarrow \Lambda$ is a length function with the property $L(gh) = L(hg)$ for all $g, h \in G$ if and only if $K = L^{-1}(0)$ is a normal subgroup of $G$ and the quotient group $\mathbb{G} = G/K$ becomes a $\Lambda$-metric space with respect to the* biinvariant metric *defined by $d(gK, hK) = L(g^{-1}h)$.*

A remarkable class of length functions is defined as follows.

**Definition 4.5.** By a *($\Lambda$-valued) Lyndon length function* on $G$ we understand a length function $L : G \longrightarrow \Lambda$ satisfying the following conditions.

(L2)' $(g, h)_L \geq (g, u)_L \wedge (u, h)_L$ for all $g, h, u \in G$.

(L3) $(g, h)_L \in \Lambda$ for all $g, h \in G$.

**Remark 4.6.** (1) (L2) is a consequence of (L0), (L1) and (L2)': take $u = 1$ in (L2)'.

(2) Suppose that $G$ acts by $\Lambda$-isometries on a $\Lambda$-tree $(X, d)$ and $x \in X$ is any point. Then the displacement map $L = L_x : G \longrightarrow \Lambda$, defined by $L(g) = d(x, gx)$, is a Lyndon length function. Indeed properties (L0) and (L1) are evident, while $(g, h)_L = (gx, hx)_x = d(x, m(x, gx, hx))$, where $m : X^3 \longrightarrow X$ is the induced median operation on the $\Lambda$-tree $(X, d)$, whence properties (L2)' and (L3), by Proposition 2.8.

(3) The condition (L2)' is quite restrictive. A remarkable class of length functions which do not satisfy (L2)' is obtained by taking $G = S_n$, the symmetric group, $n \geq 3$, $\Lambda = \mathbb{R}$, and setting $L(\sigma) := \frac{\#\{i \mid \sigma(i) \neq i\}}{n}$. As $L(\sigma\tau) = L(\tau\sigma)$ for all $\sigma, \tau \in G$, the induced (*Hamming*) metric on $G$ is biinvariant. Letting $\sigma = (1, 2), \tau = (2, 3), \rho = (1, 2, 3)$, we obtain $(\sigma, \tau)_L = \frac{1}{2n}, (\sigma, \rho)_L = (\rho, \tau)_L = \frac{3}{2n}$, and hence (L2)' is not satisfied.

With the notation from Lemma 4.3, we obtain the following characterization of the Lyndon length functions.

**Corollary 4.7.** *The necessary and sufficient condition for a map $L : G \longrightarrow \Lambda$ to be a Lyndon length function is that the assertions $(1) - (3)$ from* Lemma 4.3 *hold and, in addition, the $\Lambda$-metric space $(\mathbb{G}, d)$ satisfies the conditions $(S1)_{\mathbf{1}}$ and $(S2)_{\mathbf{1}}$ from* Lemma 2.7.

*Proof.* Note that $(gK, hK)_{\mathbf{1}} = (g, h)_L$ for $g, h \in G$. □

The next statement is a converse of Remark 4.6 (2).

**Theorem 4.8.** *Let $L : G \longrightarrow \Lambda$ be a Lyndon length function. Then there exists a $\Lambda$-tree $\mathbb{T}$ with base point $\mathbf{1}$, and an action of $G$ on $\mathbb{T}$ (by $\Lambda$-isometries), with the following properties.*

(1) *For all $g \in G$, $L(g) = d(\mathbf{1}, g\mathbf{1})$, i.e., $L$ is the length function $L_{\mathbf{1}}$ arising from the action of $G$ on $\mathbb{T}$.*

(2) *Suppose that $G$ acts on a $\Lambda$-tree $\mathbb{X}$ and $L = L_x$ for some $x \in X$. Then there exists a unique $G$-equivariant $\Lambda$-isometry $\psi : \mathbb{T} \longrightarrow \mathbb{X}$ with $\psi(\mathbf{1}) = x$. The image of $\psi$ is the sub-$\Lambda$-tree of $\mathbb{X}$ spanned by the orbit $Gx$.*

*Proof.* (1) According to Corollary 4.7, $K := L^{-1}(0)$ is a subgroup of $G$, and the factor set $\mathbb{G} := G/K$ becomes a $\Lambda$-metric space with the metric $d : \mathbb{G} \times \mathbb{G} \longrightarrow \Lambda$ given by $d(gK, hK) := L(g^{-1}h)$. In addition $(\mathbb{G}, d)$ satisfies $(S1)_{\mathbf{1}}$ and $(S2)_{\mathbf{1}}$, where $\mathbf{1} := 1 \cdot K$, and the group $G$ acts transitively (by $\Lambda$-isometries) on $(\mathbb{G}, d)$. Applying Corollary 2.11, we define the required $\Lambda$-tree $\mathbb{T}$ as the $\Lambda$-tree closure $\widetilde{\mathbb{G}}$ of $\mathbb{G}$. By Corollary 2.12, the action of $G$ on $\mathbb{G}$ extends uniquely to an action on $\mathbb{T}$ with $L_{\mathbf{1}} = L$ as desired.

(2) Suppose that $G$ acts on a $\Lambda$-tree $\mathbb{X}$ such that $L = L_x$ for some $x \in X$. Then the map $G \longrightarrow X, g \mapsto gx$, induces a $G$-equivariant $\Lambda$-isometry $\mathbb{G} \longrightarrow \mathbb{X}$, which extends uniquely to a $G$-equivariant $\Lambda$-isometry $\psi : \mathbb{T} \longrightarrow \mathbb{X}$ by Corollary 2.11. Since $\psi(\mathbf{1}) = x$ and the $\Lambda$-tree $\mathbb{T} = \widetilde{\mathbb{G}}$ is spanned by $\mathbb{G} = G\mathbf{1}$, we deduce that the image of $\psi$ is the sub-$\Lambda$-tree of $\mathbb{X}$ spanned by $Gx$ as desired. □

We denote by $\mathbb{T}(L)$ the (unique up to isomorphism) $G$-$\Lambda$-tree with base point $\mathbf{1}$, associated as in Theorem 4.8 to a Lyndon length function $L : G \longrightarrow \Lambda$. Denote by $\nu : G \longrightarrow \mathbb{T}(L), g \mapsto g\mathbf{1}$, the canonical $G$-equivariant map. Recall that $\mathbb{T}(L)$ is the $\Lambda$-tree closure of its $\Lambda$-metric subspace $\nu(G)$.

**Examples 4.9.** (3) Let $G = \mathbb{Z}/4, \widehat{i} := i \bmod 4$ for $i \in \mathbb{Z}$. A map $L : G \longrightarrow \Lambda_+$, with $L(\widehat{0}) = 0$, is a Lyndon length function if and only if $\lambda := L(\widehat{1}) = L(\widehat{3}) \geq 2\mu := L(\widehat{2})$. Indeed, assuming that $L$ is a Lyndon length function. it follows that $\lambda := L(\widehat{1}) = L(\widehat{3}) \geq 0$ and $(\widehat{1}, \widehat{2})_L = (\widehat{2}, \widehat{3})_L = \frac{L(\widehat{2})}{2} \in \Lambda_+$, whence $L(\widehat{2}) = 2\mu$ for some $\mu \in \Lambda_+$ and $\lambda - \mu = (\widehat{1}, \widehat{3})_L \geq (\widehat{1}, \widehat{2})_L \wedge (\widehat{2}, \widehat{3})_L = \mu$, so $\lambda \geq 2\mu$ as desired. The converse is immediate. Consequently, $K := L^{-1}(0) = \{\widehat{0}\}$ if and only if $\mu > 0$. Assuming $\mu > 0$, $G$ becomes a $\Lambda$-metric space with $d(\widehat{i}, \sigma\widehat{i}) = \lambda$, $d(\widehat{i}, \sigma^2\widehat{i}) = 2\mu$, where $\sigma\widehat{i} := \widehat{i+1}$. The associated $G$-$\Lambda$-tree $\mathbb{T}(L)$ is the $\Lambda$-tree closure of $(G, d)$ whose points are identified with the maps $f : G \longrightarrow \Lambda_+$ satisfying the conditions $(\alpha), (\beta)$ from the proof of Theorem 2.10, with $X = G$, together with the identity $(\gamma)_{\widehat{0}}$ : $(\widehat{0}, \widehat{1})_f \wedge (\widehat{0}, \widehat{2})_f \wedge (\widehat{0}, \widehat{3})_f = 0$, where $(\widehat{0}, \widehat{i})_f = \frac{f(\widehat{0}) + f(\widehat{i}) - L(\widehat{i})}{2}$. The free action of $G$ on the $\Lambda$-metric space

$(G,d)$ extends canonically to a faithful action on the $\Lambda$-tree $\mathbb{T}(L)$ according to the rule $(\sigma f)(\widehat{i}) = f(\widehat{i-1})$. It follows that $\mathrm{Fix}(\sigma^2)$ is the cell $[c_0, c_1]$, where $c_0(\widehat{0}) = c_0(\widehat{2}) = \mu, c_0(\widehat{1}) = c_0(\widehat{3}) = \lambda - \mu$ and $c_1 = \sigma c_0$ are the midpoints of the diagonal cells $[\widehat{0},\widehat{2}]$ and $[\widehat{1},\widehat{3}]$ respectively; note that $c_0 = m(\widehat{0},\widehat{1},\widehat{2}) = m(\widehat{0},\widehat{2},\widehat{3})$, $c_1 = m(\widehat{0},\widehat{1},\widehat{3}) = m(\widehat{1},\widehat{2},\widehat{3})$, and $[c_0,c_1] = [\widehat{0},\widehat{1}] \cap [\widehat{2},\widehat{3}] = [\widehat{0},\widehat{3}] \cap [\widehat{1},\widehat{2}]$ is the intersection of opposite side cells as well as the bridge between the diagonal cells $[\widehat{0},\widehat{2}]$ and $[\widehat{1},\widehat{3}]$. On the other hand, $\mathrm{Fix}(\sigma) \neq \varnothing$ if and only if $\lambda \in 2\Lambda$, and in this case, $\mathrm{Fix}(\sigma) = \{c\}$ is a singleton, where $c(\widehat{i}) = \frac{\lambda}{2}$ for all $\widehat{i} \in G$ is the common midpoint of the four side cells $[\widehat{i}, \sigma\widehat{i}]$ of diameter $\lambda$ and of the cell $[c_0, c_1]$ of diameter $\lambda - 2\mu$; in particular, $\mathrm{Fix}(\sigma) = \mathrm{Fix}(\sigma^2) = \{c\}$, the common midpoint of side and diagonal cells of the same diameter $\lambda$, if and only if $\lambda = 2\mu$. Thus, the automorphism $\sigma$ is either elliptic (for $\lambda \in 2\Lambda$) or an inversion (for $\lambda \notin 2\Lambda$). If the abelian group $\Lambda$ is totally ordered then the $\Lambda$-tree $\mathbb{T}(L)$ consists of the cells $[\widehat{0},\widehat{2}]$ and $[\widehat{1},\widehat{3}]$ connected by the bridge $[c_0, c_1]$.

(4) Let $G = S_3 \cong \mathbb{D}_6 = \langle \sigma, \tau \,|\, \sigma^2 = \tau^3 = (\sigma\tau)^2 = 1 \rangle$. Let $L : G \longrightarrow \Lambda$ be a length function inducing a biinvariant metric, so $L(\sigma) = L(\tau\sigma) = L(\tau^2\sigma)$ and $L(\tau) = L(\tau^2)$. Since $(\sigma,\tau)_L = (\tau,\tau\sigma)_L = \frac{L(\tau)}{2}$ and $(\sigma,\tau\sigma)_L = L(\sigma) - \frac{L(\tau)}{2}$, $L$ is a Lyndon length function if and only if $L(\sigma) = \lambda, L(\tau) = 2\mu$ for some $\lambda, \mu \in \Lambda_+$ such that $\lambda \geq 2\mu$. Assuming $\lambda \geq 2\mu > 0$, the transitive free action of $G$ on the $\Lambda$-metric space $(G,d)$ is extended to a faithful action of $G$ on the $\Lambda$-tree closure $\mathbb{T}(L)$ of $(G,d)$, and $\mathrm{Fix}(\tau) = \mathrm{Fix}(\tau^2)$ is the cell $[c_0, \sigma c_0]$ of diameter $\lambda - 2\mu$, where $c_0 = m(1,\tau,\tau^2)$ is the common midpoint of the cells $[1,\tau], [1,\tau^2]$ and $[\tau,\tau^2]$ of diameter $2\mu$, while $\sigma c_0 = m(\sigma, \tau\sigma, \tau^2\sigma)$ is the common midpoint of the cells $[\sigma,\tau\sigma], [\sigma,\tau^2\sigma]$ and $[\tau\sigma,\tau^2\sigma]$ also of diameter $2\mu$. On the other hand, $\mathrm{Fix}(\sigma) = \mathrm{Fix}(\tau\sigma) = \mathrm{Fix}(\tau^2\sigma) \neq \varnothing$ if and only if $\lambda \in 2\Lambda$; in this case, $\mathrm{Fix}(\sigma) = \{c\}$, where $c$ is the common midpoint of the cells $[\tau^i, \tau^j\sigma]$ $(i,j \in \mathbb{Z}/3)$ of diameter $\lambda \in 2\Lambda_+$ and of the cell $[c_0, \sigma c_0]$ of diameter $2(\frac{\lambda}{2} - \mu)$. In particular, $\mathrm{Fix}(\tau) = \mathrm{Fix}(\sigma) = \{c\}$ if and only if $\lambda = 2\mu$. Thus, $\tau$ and $\tau^2$ are elliptic, while the involutions $\sigma, \tau\sigma$ and $\tau^2\sigma$ are either elliptic (for $\lambda \in 2\Lambda$) or inversions (for $\lambda \notin 2\Lambda$).

The next statement extends [17, Lemma 5.2.] concerning strongly regular Lyndon length functions with values in totally ordered abelian groups.

**Lemma 4.10.** *Let $L : G \longrightarrow \Lambda$ be a Lyndon length function.*

(1) *The following assertions are equivalent.*

*(i) $\nu(G)$ is a pre-$\Lambda$-tree on which $G$ acts transitively by $\Lambda$-isometries.*

*(ii) The length function $L$ is* regular, *i.e., for any two elements $g, h \in G$, there exists $u \in G$ such that $(u, u^{-1}g)_L = (u, u^{-1}h)_L = 0$ and $L(u) = (g,h)_L$.*

(2) *The following assertions are equivalent.*

*(i) $\nu$ is onto, i.e., the action of $G$ on the $\Lambda$-tree $\mathbb{T}(L)$ is transitive.*

*(ii) The length function $L$ is* strongly regular, *i.e., for all $g \in G, \lambda \in [0, L(g)]$, there exists $h \in G$ such that $L(h) = \lambda$, and $L(h^{-1}g) = L(g) - \lambda$.*

*Proof.* By Corollary 4.7, the Lyndon length function $L$ makes $\nu(G) = (G/K, d)$ a $\Lambda$-metric space satisfying $(S1)_{\mathbf{1}}$ and $(S2)_{\mathbf{1}}$, where $K = L^{-1}(0)$, $\mathbf{1} = 1{\cdot}K$, and $d(gK, hK) = L(g^{-1}h)$ for $g, h \in G$. According to Lemma 2.6 (4, 5), the map

$$\iota_{gK,hK} : [gK, hK] \longrightarrow [0, d(gK,hK) = L(g^{-1}h)]$$

is a $\Lambda$-isometry for all $g, h \in G$, and the set

$$M_{g,h,u} := [gK, hK] \cap [hK, uK] \cap [uK, gK]$$

has at most one element for all $g, h, u \in G$. Consequently, $\nu(G)$ is a pre-$\Lambda$-tree (cf. Definition 2.4 (1)) if and only if the set $M_{g,h,u}$ is nonempty for all $g, h, u \in G$. Since $G$ acts transitively by $\Lambda$-isometries on $\nu(G)$, the condition above is equivalent with the fact that the set $M_{g,h,1}$ is nonempty for all $g, h \in G$, i.e., the Lyndon length function $L$ is regular. The assertion (1) of the corollary is thus proved.

On the other hand, since $G$ acts transitively by $\Lambda$-isometries on $\nu(G)$, it follows by Proposition 2.8 that $\nu(G)$ is a $\Lambda$-tree (equivalently, $\nu$ is onto, by Theorem 4.8) if and only if the map $\iota_{\mathbf{1},gK} : [\mathbf{1}, gK] \longrightarrow [0, d(\mathbf{1}, gK) = L(g)]$ is onto for all $g \in G$, i.e., the Lyndon length function $L$ is strongly regular. Thus the assertion (2) of the corollary is also proved. □

In particular, if the map $\nu : G \longrightarrow \mathbb{T}(L)$ is injective, i.e., $L^{-1}(0) = \{1\}$, we obtain the following classes of arboreal groups having underlying structures of median groups.

**Definition 4.11.** Let $G$ be a group, and $L : G \longrightarrow \Lambda$ be a map with values in an abelian $l$-group $\Lambda$. Assume that $L^{-1}(0) = \{1\}$, and let $d : G \times G \longrightarrow \Lambda$ be the map defined by $d(g, h) := L(g^{-1}h)$; thus, $d(1, g) = L(g)$ for all $g \in G$, and $d(ug, uh) = d(g, h)$ for all $u, g, h \in G$.

(1) $(G, L, d)$ is called a *pre-$\Lambda$-tree-group* if the following equivalent conditions are satisfied.

   (i) $L : G \longrightarrow \Lambda$ is a regular Lyndon length function.

   (ii) $(G, d)$ is a pre-$\Lambda$-tree.

(2) $(G, L, d)$ is called a *$\Lambda$-tree-group* if the following equivalent conditions are satisfied.

   (i) $L : G \longrightarrow \Lambda$ is a strongly regular Lyndon length function.

   (ii) $(G, d)$ is a $\Lambda$-tree.

The arboreal groups defined above form categories which are equivalent with categories of free and transitive actions on pointed pre-$\Lambda$-trees and pointed $\Lambda$-trees respectively.

We end this section with a lemma which will be used in the next section to prove the main result of the paper.

**Lemma 4.12.** *Let $(G, m)$ be a median group. We denote by $\cap$ the meet-semilattice operation defined by $g \cap h := m(g, h, 1)$, and by $\subset$ the associated partial order. Let $L : G \longrightarrow \Lambda_+$ be a map, and define $d : G \times G \longrightarrow \Lambda_+$ by $d(g, h) := L(g^{-1}h)$; thus, $L(g) = d(1, g)$ for all $g \in G$, and $d(ug, uh) = d(g, h)$ for all $u, g, h \in G$. Then the following assertions hold.*

(1) $(G, d)$ *is a median* $\Lambda$*-metric space with the induced median operation* $m$ *if and only if the following condition is satisfied.*

$(i)$ *For all* $g, h \in G, h \subset g \Longleftrightarrow L(g) = L(h) + L(g^{-1}h)$.

(2) *The necessary and sufficient condition for* $(G, L, d)$ *to be a pre-*$\Lambda$*-tree-group with the induced median operation* $m$ *is that* $(i)$ *and the following condition are satisfied.*

$(ii)$ *For all* $g, u, v \in G,\ u \subset g, v \subset g \Longrightarrow L(u^{-1}v) = |L(u) - L(v)|$.

(3) *The necessary and sufficient condition for* $(G, L, d)$ *to be a* $\Lambda$*-tree-group with the induced median operation* $m$ *is that* $(i)$ *and the following condition are satisfied.*

$(iii)$ *For all* $g \in G, \lambda \in [0, L(g)]$*, there exists uniquely* $h \in G$ *such that* $L(h) = \lambda$ *and* $L(h^{-1}g) = L(g) - \lambda$.

*Proof.* (1) Assuming that $(G, d)$ is a median $\Lambda$-metric space with the induced median operation $m$ and partial order $\subset$, it follows that

$$h \subset g \Longleftrightarrow h \in [1, g] \Longleftrightarrow d(1, g) = d(1, h) + d(g, h) \Longleftrightarrow L(g) = L(h) + L(g^{-1}h)$$

for all $g, h \in G$, therefore $(i)$ is satisfied.

Conversely, assuming that $(i)$ is satisfied, it follows that $L^{-1}(0) = \{1\}$ and $L(g) = L(g^{-1})$ since $1 \subset g$ for all $g \in G$, and $g \subset 1 \Longrightarrow g = 1$. Next, using the assumption $L(G) \subseteq \Lambda_+$ and applying the implication $\Longrightarrow$ from $(i)$ to the relations $g \cap h \subset g, g \cap h \subset h$ and $g^{-1}(g \cap h) \subset g^{-1}h$, we deduce that $(g, h)_L = L(g \cap h) \in \Lambda_+$ for all $g, h \in G$, so (L2) and (L3) are satisfied. Consequently, $(G, d)$ is a median $\Lambda$-metric space with the induced median operation $m$. Note that the inequality (L2)' is not necessarily satisfied, so $L$ is not necessarily a Lyndon length function: take $G = \mathbb{Z}/4, \widehat{0} \subset \widehat{1}, \widehat{3} \subset \widehat{2}$, and $L : G \longrightarrow \mathbb{Z}$ with $L(\widehat{0}) = 0, L(\widehat{1}) = L(\widehat{3}) = 1, L(\widehat{2}) = 2$; $(i)$ is satisfied, but (L2)' fails since $(\widehat{1}, \widehat{2})_L = (\widehat{2}, \widehat{3})_L = 1 > 0 = (\widehat{1}, \widehat{3})_L$.

(2), (3) For any $g \in G$, consider the map $\iota_{1,g} : [1, g] \longrightarrow [0, d(1, g) = L(g)]$. Then $(ii)$ means that $\iota_{1,g}$ is a $\Lambda$-isometry, while $(iii)$ means that $\iota_{1,g}$ is bijective. Consequently, in the both assertions an implication is obvious. Conversely, it suffices to show that (L2)' is satisfied, so $L$ is a Lyndon length function and $L^{-1}(0) = \{1\}$. Indeed, in this case, $(i)$ implies that $L$ is regular, while $(iii)$ implies that $L$ is strongly regular as desired.

To check (L2)', let $g, h, u \in G$. Since $L(G) \subseteq \Lambda_+$ and $g \cap h \cap u \subset g \cap h$, it follows by $(i)$ that $(g, h)_L = L(g \cap h) \geq L(g \cap h \cap u)$, so it remains to note that the identity

$$L(g \cap h \cap u) = L(g \cap u) \wedge L(h \cap u) \tag{4.1}$$

holds since $g \cap u, h \cap u \subset u$ and the map $\iota_{1,u}; [1, u] \longrightarrow [0, L(u)]$ is a $\Lambda$-isometry by $(ii)$, while $(i)$ and $(iii) \Longrightarrow (ii)$. Note that $(i) \Longrightarrow (ii)$ provided the median operation $m$ is locally linear. □

# 5 Embedding free actions on $\Lambda$-trees into $\Lambda$-tree-groups

In this section we use the embedding theorem for free actions on median sets [12, Theorem 1, Theorem 3.1.] to prove an analogous result for free actions on $\Lambda$-trees, where $\Lambda$ is an arbitrary abelian $l$-group. In particular, we recover [17, Theorem 5.4] for a totally ordered abelian group $\Lambda$.

Let $H$ be a group acting freely on a nonempty set $X$. Let $B = \{b_i \mid i \in I\} \subseteq X$ be a set of representatives for the $H$-orbits. The bijection $H \times I \longrightarrow X, (h,i) \mapsto hb_i$ identifies up to isomorphism the $H$-set $X$ to the cartesian product $H \times I$ with the canonical free action of the group $H$, $H \times (H \times I) \longrightarrow H \times I, (h_1,(h_2,i)) \mapsto (h_1h_2,i)$.

We assume that $I \cap H = \{1\}$, and we shall take $b_1 = (1,1)$ as *basepoint* in $X \cong H \times I$. Setting $I' := I \setminus \{1\}$, we denote by $F$ the free group with free base $I'$, and by $\widehat{H} := H * F$ the free product of the groups $H$ and $F$. The group $H$ is canonically identified with a subgroup of $\widehat{H}$, while the injective map $\iota : X \longrightarrow \widehat{H}, hb_i \mapsto hi$, identifies the $H$-set $X \cong H \times I$ with the disjoint union $H \bigsqcup (\bigsqcup_{i \in I'} Hi) \subseteq \widehat{H}$ on which $H$ acts freely by left multiplication. In particular, the base point $b_1$ of $X$ is identified with the neutral element $1 \in H \subseteq \widehat{H}$. As shown in [12, 2.2.], $\widehat{H}$ is endowed with a simplicial tree structure induced by the length function $l : \widehat{H} \longrightarrow \mathbb{N}$ associated to the set of generators $J = J^{-1} := (H \setminus \{1\}) \sqcup I'^{\pm 1}$. With respect to the partial order $u \leq v \Longleftrightarrow l(v) = l(u) + l(u^{-1}v)$, $\widehat{H}$ is an order-tree with the least element 1; write $v = u \bullet (u^{-1}v)$ provided $u \leq v$. Denote by $\wedge$ and $Y$ the corresponding meet-semilattice and median operations. $X$ is a retractible convex subset of the locally linear median set $(\widehat{H}, Y)$, with the canonical $H$-equivariant retract $\varphi : \widehat{H} \longrightarrow X$ defined by $\varphi(w) :=$ the greatest element $x \in X$ for which $x \leq w$, i.e., $w = x \bullet (x^{-1}w)$.

For a fixed abelian $l$-group $\Lambda$, let us denote by $\mathcal{T}_\Lambda(X)$ the set consisting of those maps $d : X \times X \longrightarrow \Lambda$ for which $(X,d)$ is a $\Lambda$-tree and $d(hx,hy) = d(x,y)$ for all $h \in H$, $x,y \in X$. On the other hand, we denote by $\mathcal{PT}_\Lambda(\widehat{H}, \varphi)$ the set consisting of those maps $\widehat{d} : \widehat{H} \times \widehat{H} \longrightarrow \Lambda$ for which $(\widehat{H}, \widehat{d})$ is a pre-$\Lambda$-tree-group and the map $u \mapsto \varphi(u)$ is a folding of the underlying median set of $(\widehat{H}, \widehat{d})$. We have to show that the restriction map res : $\mathcal{PT}_\Lambda(\widehat{H}, \varphi) \longrightarrow \mathcal{T}_\Lambda(X), \widehat{d} \mapsto \widehat{d}|_{X \times X}$ is bijective.

We assume that $\mathcal{T}_\Lambda(X) \neq \varnothing$ since otherwise we have nothing to prove. Let $d \in \mathcal{T}_\Lambda(X)$. We denote by $m : X^3 \longrightarrow X$ the induced median operation. Recall that for $x,y,z \in X$, $m(x,y,z)$ is the unique element of the set $[x,y] \cap [y,z] \cap [z,x]$, where

$$[x,y] := \{t \in X \mid d(x,t) + d(t,y) = d(x,y)\}$$

for $x,y \in X$. Let $\cap$ be the meet-semilattice operation defined by $x \cap y = m(x,y,1)$ for $x,y \in X$, with the induced partial order (with the least element 1) denoted by $\subset$. Define the map $L : X \longrightarrow \Lambda_+$ by $L(x) = d(1,x)$, the prolongation of the Lyndon length function $H \longrightarrow \Lambda_+, h \mapsto L(h) = d(1, h)$, and note that for all $h \in H, x, y \in X$,

$$d(hx, hy) = d(x, y) = L(x) + L(y) - 2L(x \cap y). \tag{5.1}$$

Note also that, according to Remark 3.20 (4), for any $h \in H - \{1\}$, either $h$ is of infinite order or $h^2 = 1$. In the first case, $h$ is hyperbolic, equivalently, by Proposition 3.11, with the condition $L(h^2) \not\leq L(h)$, i.e., $(L(h^2) - L(h))_+ > 0$. In the latter case, $h$ is an inversion, equivalently, by Proposition 3.19, with the condition $L(h) \notin 2\Lambda$, in particular, $\Lambda \neq 2\Lambda$. Consequently, since $\widehat{H} = H * F$ and $F$ is free, the elements of finite order of $\widehat{H} - \{1\}$ are conjugate with the elements of order 2 of $H$ (if these ones exist).

**Proposition 5.1.** *Let $d \in \mathcal{T}_\Lambda(X)$ with the induced median operation $m : X^3 \longrightarrow X$ and the function $L : X \longrightarrow \Lambda_+, x \mapsto d(1, x)$. Then there exists uniquely $\widehat{d} \in \mathcal{PT}_\Lambda(\widehat{H}, \varphi)$ such that $\widehat{d}|_{X \times X} = d$.*

*Proof.* By [12, Theorem 3.1], the median operation $m : X^3 \longrightarrow X$ extends uniquely to a median group operation $\widehat{m} : \widehat{H}^3 \longrightarrow \widehat{H}$ such that the map $\varphi$ is a folding identifying $X = \varphi(\widehat{H})$ with a retractible convex subset of the median set $(\widehat{H}, \widehat{m})$. Recall that

$$\widehat{m}(u, v, w) = tm(\varphi(t^{-1}u), \varphi(t^{-1}v), \varphi(t^{-1}w)) \tag{5.2}$$

for $u, v, w \in \widehat{H}$, where $t := Y(u, v, w)$. It follows that

$$u \cap v := \widehat{m}(u, v, 1) = am(\varphi(a^{-1}), \varphi(b), \varphi(c)) \tag{5.3}$$

for $u, v \in \widehat{H}$, where $a := u \wedge v, b := a^{-1}u, c := a^{-1}v$, i.e., $u = a \bullet b, v = a \bullet c, b \wedge c = 1$. Consequently,

$$u \subset v \iff u = u \cap v \iff b = \varphi(b) \in [\varphi(a^{-1}), \varphi(c)] \subseteq X, \tag{5.4}$$

and $u \cap v = \varphi(u \cap v) = \varphi(u) \cap \varphi(v) \in X$ provided $u \wedge v \in H$.

We have to show that there exists uniquely a distance map $\widehat{d} : \widehat{H} \times \widehat{H} \longrightarrow \Lambda$ extending $d : X \times X \longrightarrow \Lambda$ such that $(\widehat{H}, \widehat{d})$ is a pre-$\Lambda$-tree-group whose induced median group operation is $\widehat{m}$.

Equivalently, according to Lemma 4.12, we have to show that there exists uniquely a map $\widehat{L} : \widehat{H} \longrightarrow \Lambda_+$ such that $\widehat{L}|_X = L$, with the following properties.

(1) For all $u, v \in \widehat{H}, u \subset v \iff \widehat{L}(v) = \widehat{L}(u) + \widehat{L}(v^{-1}u)$.

(2) For all $u, u', v \in \widehat{H}$, $u, u' \subset v \implies \widehat{L}(u^{-1}u') = |\widehat{L}(u) - \widehat{L}(u')|$.

In particular, $\widehat{L}$ is a Lyndon length function, $\widehat{L}(u \cap v) = (u, v)_{\widehat{L}}$ for $u, v \in \widehat{H}$, and the distance map $\widehat{d} : \widehat{H} \times \widehat{H} \longrightarrow \Lambda$ is given by $\widehat{d}(u, v) = \widehat{L}(u^{-1}v)$.

Assuming that $\widehat{L} : \widehat{H} \longrightarrow \Lambda_+$ extends $L : X \longrightarrow \Lambda_+$ and satisfies (1) and (2), it follows that $\widehat{L}$ is unique with these properties, being defined by induction on the combinatorial length $l(u)$ for $u \in \widehat{H}$ as follows: $\widehat{L}(u) = 0$ if $l(u) = 0$, i.e., $u = 1$, while for $l(u) \geq 1$, say $u = x \bullet u'$ with $l(x) = 1$,

$$\widehat{L}(u) = \widehat{L}(x) + \widehat{L}(u') - 2L(x^{-1} \cap u'), \tag{5.5}$$

where

$$\widehat{L}(x) = \begin{cases} L(x) & \text{if} \quad x \in (H - \{1\}) \sqcup I', \\ L(x^{-1}) & \text{if} \quad x \in I'^{-1}. \end{cases} \tag{5.6}$$

Indeed, $2\widehat{L}(x^{-1} \cap u') = 2(x^{-1}, u')_{\widehat{L}} = \widehat{L}(x^{-1}) + \widehat{L}(u') - \widehat{L}(xu') = \widehat{L}(x) + \widehat{L}(u') - \widehat{L}(u)$, while $\widehat{L}(x^{-1} \cap u') = L(x^{-1} \cap u')$ since $\widehat{L}|_X = L$ and $x^{-1} \wedge u' = 1$ (by assumption) implies $x^{-1} \cap u' = \varphi(x^{-1} \cap u') = \varphi(x^{-1}) \cap \varphi(u') \in X$. Moreover, the last equality implies that

$$\widehat{L}(u) = \begin{cases} \widehat{L}(u') - L(\varphi(u')) + L(\varphi(u)) & \text{if} \quad x \in H \setminus \{1\}, \\ \widehat{L}(u') + L(x) = \widehat{L}(u') + L(\varphi(u)) & \text{if} \quad x \in I', \\ \widehat{L}(u') - L(\varphi(u')) + d(x^{-1}, \varphi(u')) & \text{if} \quad x \in I'^{-1}. \end{cases} \tag{5.7}$$

Thus, it remains only to show that the map $\widehat{L}$ defined inductively as above extends the map $L : X \longrightarrow \Lambda_+$ and satisfies (1) and (2).

The equality $\widehat{L}|_X = L$ is immediate, while, by induction on $l(u)$, it follows that $\widehat{L}(u) \geq L(\varphi(u)) \geq 0$ for all $u \in \widehat{H}$. Note also that $\widehat{L}^{-1}(0) = \{1\}$. Indeed, let $u \in \widehat{H} \setminus \{1\}$. As $\widehat{L}(u) \geq L(\varphi(u)) \geq 0$, we may assume that $\varphi(u) = 1$, whence $u = i^{-1} \bullet u'$ for some $i \in I', u' \in \widehat{H}$. We deduce that $\widehat{L}(u) = \widehat{L}(u') - L(\varphi(u')) + d(i, \varphi(u')) \geq d(i, \varphi(u')) > 0$ since $u = i^{-1} \bullet u'$ implies $\varphi(u') \neq i$.

Before doing the verification of the conditions (1) and (2), let us show by induction that $\widehat{L}$ satisfies (L1), i.e., $\widehat{L}(u) = \widehat{L}(u^{-1})$ for all $u \in \widehat{H}$, and

$$\widehat{L}(uv) = \widehat{L}(u) + \widehat{L}(v) - 2L(u^{-1} \cap v) \quad \text{provided} \quad u^{-1} \wedge v = 1. \tag{5.8}$$

To check (L1), we have to consider the case $l(u) \geq 2$, say $u = x \bullet u' \bullet y$ with $l(x) = l(y) = 1$. By the induction hypothesis, $\widehat{L}(x \bullet u') = \widehat{L}(u'^{-1} \bullet x^{-1})$ and $\widehat{L}(u' \bullet y) = \widehat{L}(y^{-1} \bullet u'^{-1})$. Consequently,

$$\widehat{L}(u) = \widehat{L}(x) + \widehat{L}(y) + \widehat{L}(u') - 2L(y \cap u'^{-1}) - 2L(x^{-1} \cap (u' \bullet y)),$$
$$\widehat{L}(u^{-1}) = \widehat{L}(y) + \widehat{L}(x) + \widehat{L}(u') - 2L(x^{-1} \cap u') - 2L(y \cap (u'^{-1} \bullet x^{-1})),$$

while the identity

$$L(y \cap u'^{-1}) + L(x^{-1} \cap (u' \bullet y)) = L(x^{-1} \cap u') + L(y \cap (u'^{-1} \bullet x^{-1}))$$

follows by (5.1) and [12, Remark 2.2]

To check (5.8), we may assume that $u \neq 1, v \neq 1$, say $u = u' \bullet x, v = y \bullet v'$ with $l(x) = l(y) = 1$. By assumption $u^{-1} \wedge v = 1$, therefore $xy \neq 1$. Consequently, $u'^{-1} \wedge xv = 1$, and either $xv = x \bullet v$ or $xv = (xy) \bullet v'$ with $x, y \in H \setminus \{1\}, v' \in \widehat{H}$. As $l(u') < l(u)$, it follows by the induction hypothesis applied to the pair $(u', xv)$ that

$$\widehat{L}(uv) = \widehat{L}(u') + \widehat{L}(xv) - 2L(u'^{-1} \cap (xv)),$$

whence the desired identity (5.8) by straightforward computation using (5.1) and [12, Remark 2.2].

Now, let us verify (1). First assume that $u \subset v$, i.e., $u = a \bullet b, v = a \bullet c, b \wedge c = 1$, and $b \in [\varphi(a^{-1}), \varphi(c)] \subseteq X$. According to (5.8), applied to the pairs $(a, b), (a, c)$ and $(c^{-1}, b)$, together with (L1) and (5.1), we obtain

$$\widehat{L}(v) - \widehat{L}(u) - \widehat{L}(v^{-1}u) = d(\varphi(a^{-1}), \varphi(c)) - d(\varphi(a^{-1}), b) - d(b, \varphi(c)) = 0$$

as desired. Conversely, assume that $\widehat{L}(v) = \widehat{L}(u) + \widehat{L}(v^{-1}u)$. Setting $u = a \bullet b, v = a \bullet c$ with $b \wedge c = 1$, it follows by (5.8), (L1) and (5.1) that

$$0 \leq \widehat{L}(b) - L(\varphi(b)) = d(\varphi(a^{-1}), \varphi(c)) - d(\varphi(a^{-1}), \varphi(b)) - d(\varphi(b), \varphi(c)) \leq 0,$$

therefore $b = \varphi(b) \in [\varphi(a^{-1}), \varphi(c)]$, i.e., $u \subset v$ as required.

Thus, according to Lemma 4.12, we have shown that $(\widehat{H}, \widehat{d})$ is a median $\Lambda$-metric group, where $\widehat{d}(u, v) := \widehat{L}(u^{-1}v)$ for $u, v \in \widehat{H}$.

Finally, it remains to check (2). Let $u, u', v \in \widehat{H}$ be such that $u \subset v, u' \subset v$. If $u$ and $u'$ are comparable we have nothing to prove by (1), so we may assume that $u \not\subset u'$ and $u' \not\subset u$. According to [12, Corollary 3.3], there exists $w \leq v$ such that

$$w\varphi(w^{-1}) \subset u, u' \subset w\varphi(w^{-1}v) \subset v,$$

whence $w^{-1}u, w^{-1}u' \in [\varphi(w^{-1}), \varphi(w^{-1}v)] \subseteq X$. It follows by (1) that

$$\widehat{L}(u) = \widehat{L}(w\varphi(w^{-1})) + \widehat{d}(w\varphi(w^{-1}), u) = \widehat{L}(w\varphi(w^{-1})) + d(\varphi(w^{-1}), w^{-1}u),$$

and, similarly, $\widehat{L}(u') = \widehat{L}(w\varphi(w^{-1})) + d(\varphi(w^{-1}), w^{-1}u')$. Consequently,

$$|\widehat{L}(u) - \widehat{L}(u')| = |d(\varphi(w^{-1}), w^{-1}u) - d(\varphi(w^{-1}), w^{-1}u')| = d(w^{-1}u, w^{-1}u') = \widehat{L}(u^{-1}u')$$

as desired, since $w^{-1}u$ and $w^{-1}u'$ belong to the cell $[\varphi(w^{-1}), \varphi(w^{-1}v)]$ of the $\Lambda$-tree $(X, d)$. According to Lemma 4.12, $(\widehat{H}, \widehat{L}, \widehat{d})$ is a pre-$\Lambda$-tree group with the induced median operation $\widehat{m}$, and the statement is proved. □

Using [12, Corrolary 3.3], we obtain an explicit version of the inductive definition (5.5)-(5.7) as folows.

**Corollary 5.2.** *Let* $d \in \mathcal{T}_\Lambda(X)$ *with the induced median operation* $m : X^3 \longrightarrow X$ *and the function* $L : X \longrightarrow \Lambda_+, x \mapsto d(1,x)$. *Let* $\widehat{d} \in \mathcal{PT}_\Lambda(\widehat{H}, \varphi)$ *be the unique extension of* $d$. *Then, for any* $v \in \widehat{H} - \{1\}$,

$$\widehat{L}(v) := \widehat{d}(1,v) = \sum_{w \in C_v} d(\varphi(w^{-1}), \varphi(w^{-1}v)), \tag{5.9}$$

*where the finite set* $C_v$ *consists of those* $w \le v$ *satisfying*

$$\varphi(w^{-1}) \in I, \text{ and } \varphi(w^{-1}) = 1 \Longrightarrow \varphi(w^{-1}v) \neq 1.$$

*Proof.* Set $C_v = \{w_i \,|\, i = \overline{1,n}\}$, $n \ge 1$, with $w_i < w_{i+1}$, and let $\zeta_i := w_i\varphi(w_i^{-1})$ for $i = \overline{1,n}$, $\zeta_{n+1} := v$. It follows that $\zeta_1 = 1, \zeta_i \le w_i \le \zeta_{i+1} = w_i \bullet \varphi(w_i^{-1}v)$ for $i = \overline{1,n}$, whence the totally ordered finite set $([1,v], \le)$ is the union of $n$ adjacent proper closed intervals $[\zeta_i, \zeta_{i+1}]$, $i = \overline{1,n}$, called in [12, Section 3], the *combinatorial configuration* associated to the element $v \in \widehat{H} - \{1\}$. According to [12, Corollary 3.3], the cell $[1,v]$ of the median group $(\widehat{H}, \widehat{m})$ is a *deformation* of the combinatorial configuration above induced by the median operation $m$ on $X$, being the union of the adjacent cells

$$[\zeta_i, \zeta_{i+1}] = w_i[\varphi(w_i^{-1}), \varphi(w_i^{-1}v)] \subseteq w_iX,\ i = \overline{1,n}\,,$$

with $\zeta_i \subsetneq \zeta_{i+1}$ for $i = \overline{1,n}$, $\zeta_1 = 1$, and $\zeta_{n+1} = v$, in particular, $w_n^{-1}v = \varphi(w_n^{-1}v) \in X$. Consequently, we obtain

$$\widehat{L}(v) = \widehat{d}(1,v) = \sum_{i=1}^{n} \widehat{d}(\zeta_i, \zeta_{i+1}) = \sum_{i=1}^{n} d(\varphi(w_i^{-1}), \varphi(w_i^{-1}v))\,,$$

as desired. □

With notation above, let us denote by $\mathcal{T}_\Lambda(\widehat{H}, \varphi)$ the subset of $\mathcal{PT}_\Lambda(\widehat{H}, \varphi)$ consisting of the maps $\widehat{d} : \widehat{H} \times \widehat{H} \longrightarrow \Lambda$ for which $(\widehat{H}, \widehat{d})$ is a $\Lambda$-tree group and the map $u \mapsto \varphi(u)$ is a folding of the underlying median set of $(\widehat{H}, \widehat{d})$. Though, in general, $\mathcal{T}_\Lambda(\widehat{H}, \varphi) \neq \mathcal{PT}_\Lambda(\widehat{H}, \varphi)$, assuming that $\Lambda$ is totally ordered we obtain

**Corollary 5.3.** *Assume that the abelian group* $\Lambda$ *is totally ordered. Then* $\mathcal{T}_\Lambda(\widehat{H}, \varphi) = \mathcal{PT}_\Lambda(\widehat{H}, \varphi)$, *and the restriction map* $\mathcal{T}_\Lambda(\widehat{H}, \varphi) \longrightarrow \mathcal{T}_\Lambda(X), \widehat{d} \mapsto \widehat{d}|_{X \times X}$ *is bijective.*

*Proof.* Let $\widehat{d} \in \mathcal{PT}_\Lambda(\widehat{H}, \varphi)$ with $d = \widehat{d}|_{X\times X}, \widehat{L}(v) = \widehat{d}(1,v)$ for $v \in \widehat{H}$. We have only to show that for all $v \in \widehat{H}, \lambda \in [0, \widehat{L}(v)]$, there exists (uniquely) $u \in [1,v]$ such that $\widehat{L}(u) = \lambda$. Since $\Lambda$ is totally ordered by assumption, it follows by Corollary 5.2 that there exists uniquely $w \in C_v$ such that $\widehat{L}(w\varphi(w^{-1})) \le \lambda \le \widehat{L}(w\varphi(w^{-1}v))$, and hence

$$\lambda - \widehat{L}(w\varphi(w^{-1})) \in [0, d(\varphi(w^{-1}), \varphi(w^{-1}v))].$$

As $(X, d)$ is a $\Lambda$-tree, the map

$$\iota_{\varphi(w^{-1}),\varphi(w^{-1}v)} : [\varphi(w^{-1}), \varphi(w^{-1}v)] \longrightarrow [0, d(\varphi(w^{-1}), \varphi(w^{-1}v))]$$

is bijective, therefore there exists uniquely $x \in [\varphi(w^{-1}), \varphi(w^{-1}v)]$ such that

$$d(\varphi(w^{-1}), x) = \lambda - \widehat{L}(w\varphi(w^{-1})).$$

Consequently, $wx \in w[\varphi(w^{-1}), \varphi(w^{-1}v)] \subseteq [1, v]$, so $w\varphi(w^{-1}) \subset wx \subset v$ and

$$\widehat{L}(wx) = \widehat{L}(w\varphi(w^{-1})) + \widehat{d}(w\varphi(w^{-1}), wx) = \widehat{L}(w\varphi(w^{-1})) + d(\varphi(w^{-1}), x) = \lambda$$

as desired. □

Thus, we have provided a different proof of [17, Theorem 5.4], where $\Lambda$ is a totally ordered abelian group. To extend this result to arbitrary abelian $l$-groups $\Lambda$, it remains to iterate the construction furnished by Proposition 5.1 as follows.

## 5.1 Proof of the main result

Let $\mathbb{X} = (X, d : X^2 \longrightarrow \Lambda_+)$ be a $\Lambda$-tree, and $m : X^3 \longrightarrow X$ be the induced median operation. Assuming that the group $H$ acts freely by $\Lambda$-isometries on $\mathbb{X}$, we identify $H$ with a subset of $X$ via the $H$-equivariant embedding $H \longrightarrow X, h \mapsto hb_1$, where $b_1$ is a fixed base point of $X$. Extend the Lyndon length function $L : H \longrightarrow \Lambda_+, h \mapsto d(b_1, hb_1)$ to the map $L : X \longrightarrow \Lambda_+, x \mapsto d(b_1, x)$. Let $\widehat{H} = H * F$ and the $H$-equivariant embedding $\iota : X \longrightarrow \widehat{H}$ with its $H$-equivariant retract $\varphi : \widehat{H} \longrightarrow X$ be as defined at the beggining of Section 5.

According to Proposition 5.1, the map $L : X \longrightarrow \Lambda_+$ extends uniquely to a regular Lyndon length function $\widehat{L} : \widehat{H} \longrightarrow \Lambda_+$ such that $(\widehat{H}, \widehat{d} : \widehat{H}^2 \longrightarrow \Lambda_+)$, with $\widehat{d}(u, v) = \widehat{L}(u^{-1}v)$, is a pre-$\Lambda$-tree group with induced median operation $\widehat{m} : \widehat{H}^3 \longrightarrow \widehat{H}$, while the map $\varphi$ is a folding, identifying $(X, m)$ with a retractible convex subset of $(\widehat{H}, \widehat{m})$.

Let $\mathbb{X}_1 = (X_1, d_1 : X_1^2 \longrightarrow \Lambda_+) := \mathbb{T}(\widehat{L})$ be the $\Lambda$-tree closure of the pre-$\Lambda$-tree $(\widehat{H}, \widehat{d})$. We may assume that $X_1 \neq \widehat{H}$, so $\Lambda$ is not totally ordered. Thus, $(\widehat{H}, \widehat{d})$ is identified with a $\Lambda$-metric subspace of $\mathbb{X}_1$, the $\Lambda$-tree $\mathbb{X}_1$ is spanned by $\widehat{H}$, and $X_1$ is the convex closure of $\widehat{H}$ into the underlying median set of $\mathbb{X}_1$. The action by left multiplication of the group $\widehat{H}$ on itself is naturally extended to a faithful action by $\Lambda$-isometries on the $\Lambda$-tree $\mathbb{X}_1$ according to the rule $(u \cdot f)(v) = f(u^{-1}v)$ for $u, v \in \widehat{H}, (f : \widehat{H} \longrightarrow \Lambda_+) \in X_1$ (see Theorem 2.10, Corollary 2.12 and Theorem 4.8). Moreover we obtain

**Lemma 5.4.** *The action of $\widehat{H}$ on the $\Lambda$-tree $\mathbb{X}_1$ is free. For any $u \in \widehat{H} - \{1\}$, the induced automorphism $\Phi_u$ on $\mathbb{X}_1$ is hyperbolic if and only if $u$ is of infinite order, while $\Phi_u$ is an inversion if and only if $u$ is conjugate to some element $h \in H$ of order $2$.*

*Proof.* Let $1 \neq u \in \widehat{H} = H * F$. Then we distinguish the following two cases.

Case 1: $u$ is of infinite order. First, let us show that $u = w \bullet v \bullet w^{-1}$, where $w = u \wedge u^{-1}, v = w^{-1}uw$, and $v \wedge v^{-1} = 1$. We have $u = w \bullet u'$ with $u' = w^{-1}u \neq 1$ since otherwise $u = w \leq u^{-1}$, and hence $u = u^{-1}$ as $l(u) = l(u^{-1})$; thus, $u^2 = 1$, contrary to our assumption. Further we obtain $w \leq u^{-1} = u'^{-1} \bullet w^{-1}$, whence either $w < u'^{-1}$ or $u'^{-1} < w$. The latter case cannot occur since assuming that $w = u'^{-1} \bullet w'$ with $w' \neq 1$, we get $u = u'^{-1} \bullet w' \bullet u'$ and $u^{-1} = u'^{-1} \bullet w'^{-1} \bullet u'$, therefore $w' = w'^{-1}$, and hence $u^2 = 1$, again a contradiction. Consequently, we have $u'^{-1} = w \bullet v^{-1}$ with $v \neq 1$, whence $u = w \bullet v \bullet w^{-1}$ with $v \wedge v^{-1} = 1$ as desired.

Since we get the equality of hyperbolic lengths $\mathcal{L}(\Phi_u) = \mathcal{L}(\Phi_v)$ by Corollary 3.9 (4), it suffices to show that $\mathcal{L}(\Phi_v) > 0$ to conclude that the automorphism $\Phi_u$ is hyperbolic. According to Proposition 3.11, (1) $\Longrightarrow$ (6), we have to check that $\widehat{L}(v^2) \not\leq \widehat{L}(v)$, i. e., $E_+ > 0$, where $E := \widehat{L}(v^2) - \widehat{L}(v)$. As $v \wedge v^{-1} = 1$ by assumption, we have $v \cap v^{-1} = \varphi(v \cap v^{-1}) = \varphi(v) \cap \varphi(v^{-1}) \in X$, and hence

$$\widehat{L}(v^2) = 2(\widehat{L}(v) - L(v \cap v^{-1})) = 2(\widehat{L}(v) - L(\varphi(v) \cap \varphi(v^{-1})),$$

by (5.8). Consequently, $E = \widehat{L}(v) - 2L(\varphi(v) \cap \varphi(v^{-1}))$. As $E = \widehat{L}(v) > 0$ provided either $\varphi(v) = 1$ or $\varphi(v^{-1}) = 1$, we may assume that $\varphi(v) \neq 1$ and $\varphi(v^{-1}) \neq 1$, whence $\varphi(v) \neq \varphi(v^{-1})$ since $\varphi(v) \wedge \varphi(v^{-1}) \leq v \wedge v^{-1} = 1$.

Assuming that $v = \varphi(v) \in X$, it follows that $v \in H - \{1\}$ since otherwise $\varphi(v^{-1}) = 1$. As $v = \varphi(v) \neq \varphi(v^{-1}) = v^{-1}$, i. e., $v^2 \neq 1$, and $H$ acts freely on the $\Lambda$-tree $\mathbb{X}$ by assumption, it follows that $v$ acts as a hyperbolic automorphism on $\mathbb{X}$, and hence $E_+ > 0$ as desired.

It remains to consider the case $\varphi(v) < v$, whence $\varphi(v^{-1}) < v^{-1}$. Setting $v = \varphi(v) \bullet s$ and $v^{-1} = \varphi(v^{-1}) \bullet t^{-1}$ with $s \neq 1, t \neq 1$, we get $v = t \bullet \varphi(v^{-1})^{-1}$, therefore either $t < \varphi(v)$ or $\varphi(v) \leq t$. Assuming that $t < \varphi(v)$ it follows that $t \in H - \{1\}$ and $\varphi(v) = t \bullet i$ with $i \in I'$. Consequently, $\varphi(v^{-1}) = s^{-1} \bullet i^{-1} \notin X$, which is a contradiction. Thus, setting $t = \varphi(v) \bullet w^{-1}$, we get $v = \varphi(v) \bullet w^{-1} \bullet \varphi(v^{-1})^{-1}$. Since $\varphi(v) \subset v$, we have $\widehat{L}(v) = L(\varphi(v)) + \widehat{L}(\varphi(v^{-1})w)$. On the other hand, $\varphi(v^{-1}) \leq \varphi(v^{-1})w \leq v^{-1}$ implies $\varphi(v^{-1}) = \varphi(\varphi(v^{-1})w) \subset \varphi(v^{-1})w$, therefore $\widehat{L}(\varphi(v^{-1})w) = L(\varphi(v^{-1})) + \widehat{L}(w)$. Since $2L(\varphi(v) \cap \varphi(v^{-1})) = L(\varphi(v)) + L(\varphi(v^{-1})) - d(\varphi(v), \varphi(v^{-1}))$ and $\varphi(v) \neq \varphi(v^{-1})$, we deduce that $E_+ = E = \widehat{L}(w) + d(\varphi(v), \varphi(v^{-1})) \geq d(\varphi(v), \varphi(v^{-1})) > 0$ as desired.

Case 2: $u$ has finite order, and hence $u = s \bullet h \bullet s^{-1}$ with $s \in \widehat{H}$ and $h \in H - \{1\}$ of order 2, so $L(h) \notin 2\Lambda$ since $h$ acts as an inversion on the $\Lambda$-tree $\mathbb{X}$. According to Proposition 3.19, (3) $\Longrightarrow$ (1), we have to show that $\widehat{L}(u) \notin 2\Lambda$ to conclude that $\Phi_u$ is an inversion. As $s^{-1} \wedge (hs^{-1}) = h^{-1} \wedge s^{-1} = 1$, it follows by (5.8) and (L1) that $\widehat{L}(u) = L(h) + 2(\widehat{L}(s) - L(s^{-1} \cap (hs^{-1})) - L(h^{-1} \cap s^{-1})) \equiv L(h) \bmod 2\Lambda$, and hence $\widehat{L}(u) \notin 2\Lambda$ as desired. □

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