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The conditions (C_i) in modular lattices, and applications

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Abstract

In this paper we introduce and investigate the latticial counterparts of the conditions (C_i), $i = 1, 2, 3, 11, 12$, for modules. In particular, we study the lattices satisfying the condition (C_1), we call CC lattices (for *Closed are Complements*), i.e., the lattices such that any closed element is a complement, that are the latticial counterparts of CS modules (for *Closed are Summands*). Applications of these results are given to Grothendieck categories and module categories equipped with a torsion theory.

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Introduction

In this paper we shall illustrate a general strategy which consists on putting a module-theoretical definition/result in a latticial frame, in order to translate that definition/result to Grothendieck categories and to module categories equipped with a torsion theory. Thus, we provide latticial counterparts of known results about modules satisfying the conditions

(C_i), $i = 1, 2, 3, 11, 12$. Our proofs are not always simple adaptations of the corresponding ones in the module case because not all the involved module-theoretical tools work in a latticial frame.

In Section 0 we list some definitions and results about lattices, especially from [4] and [12]. In Section 1 we define the conditions (C_i), $i = 1, 2, 3, 11, 12$, for lattices, and prove some of their basic properties. Section 2 is devoted to the investigation of inheritance properties of condition (C_{11}) under direct joins and complement intervals. The last two sections present some applications to Grothendieck categories and module categories equipped with a hereditary torsion theory.

0 Preliminaries

All lattices considered in this paper are assumed to have a least element denoted by 0 and a last element denoted by 1. Throughout this paper, $(L, \leqslant, \wedge, \vee, 0, 1)$, or more simply, just L, will always denote such a lattice. If the lattices L and L' are isomorphic, we denote this by $L \simeq L'$. We shall denote by $\mathcal{M}$ the class of all modular lattices with 0 and 1. We shall use $\mathbb{N}$ to denote the set $\{1, 2, \ldots\}$ of all positive integers.

For a lattice L and elements $a \leqslant b$ in L we write

$$b/a := [a, b] = \{\ x \in L \mid a \leqslant x \leqslant b\ \}.$$

A *subfactor* of L is any interval b/a of L with $a \leqslant b$.

An element $c \in L$ is a *complement in* L if there exists an element $a \in L$ such that $a \wedge c = 0$ and $a \vee c = 1$; we say in this case that c is a *complement* of a *in* L. One denotes by $D(L)$ the set of all complements of L. By a *complement interval* of L we mean any interval $d/0$ of L with $d \in D(L)$. The lattice L is called *indecomposable* if $L \neq \{0\}$ and $D(L) = \{0, 1\}$. The lattice L is said to be *complemented* if every element of L has a complement in L.

For a lattice L and $a, b, c \in L$, the notation $a = b \dot{\vee} c$ will mean that $a = b \vee c$ and $b \wedge c = 0$, and we say that a is a *direct join* of b and c. Also, for a non-empty subset S of L, we use the *direct join* notation $a = \dot{\bigvee}_{b \in S} b$ if S is an independent subset of L and $a = \bigvee_{b \in S} b$. Recall that a non-empty subset S of L is called *independent* if $0 \notin S$, and for every $x \in S$, positive integer n, and subset $T = \{t_1, \ldots, t_n\}$ of S with $x \notin T$, $x \wedge (t_1 \vee \cdots \vee t_n) = 0$. Clearly a subset S of L is independent if and only if every finite subset of S is independent. Alternatively, we say that a finite family $(x_i)_{i \in I}$ of elements of a lattice L is *independent* if $x_i \neq 0$ and $x_i \wedge (\bigvee_{j \in I \setminus \{i\}} x_j) = 0$ for every $i \in I$, and in that case, necessarily $x_p \neq x_q$ for each $p \neq q$ in I. Thus, the definitions of independence, using subsets or families of elements of L, are essentially the same.

An element $b \in L$ is a *pseudo-complement in* L if there exists an element $a \in L$ such that $a \wedge b = 0$ and b is maximal with this property; we say in this case that b is a *pseudo-complement* of a, and $P(a)$ will denote the set, possibly empty, of all pseudo-complements of a in L. One denotes also by $P(L)$ the set of all pseudo-complement elements of L.

As in [4], L is called *pseudo-complemented* if every element of L has a pseudo-complement, and *strongly pseudo-complemented* if for all $a, b \in L$ with $a \wedge b = 0$, there exists a pseudo-complement p of a in L such that $b \leqslant p$. Every upper continuous modular lattice L is strongly pseudo-complemented. Notice that the term of a pseudo-complemented lattice has in [12] the following stronger meaning: for every $a \leqslant b$ in L and for every $x \in b/a$, there exists a pseudo-complement of x in b/a.

An element $e \in L$ is *essential in* L if $e \wedge x \neq 0$ for every $x \neq 0$ in L. One denotes by $E(L)$ the set of all essential elements of L. The lattice L is called *uniform* if $L \neq \{0\}$ and $x \wedge y \neq 0$ for every non-zero elements $x, y \in L$. An element u of L is called *uniform* if the interval $u/0$ of L is a uniform lattice. As in [4], L is called *E-complemented* (“E” for essential) if for each $a \in L$ there exists $b \in L$ such that $a \wedge b = 0$ and $a \vee b \in E(L)$.

An element $c \in L$ is said to be *closed* if $c \notin E(a/0)$ for all $a \in L$ with $c < a$. One denotes by $C(L)$ the set of all closed elements of L. As in [4], the lattice L is called *essentially closed* if for all $a \in L$, the set $S_a = \{\, e \in L \mid a \in E(e/0) \,\}$ has a maximal element, or equivalently, for any $a \in L$ there exists $c \in C(L)$ with $a \in E(c/0)$.

An element $c \in L$ is *compact in* L if whenever $c \leqslant \bigvee_{x\in A} x$ for a subset A of L, there is a finite subset F of A such that $c \leqslant \bigvee_{x\in F} x$. The lattice L is said to be *compact* if 1 is a compact element in L, and *compactly generated* if it is complete and every element of L is a join of compact elements.

For all other undefined notation and terminology on lattices, the reader is referred to [4], [5], [6], and/or [12].

Throughout this paper R will denote an associative ring with non-zero identity element, and Mod-R the category of all unital right R-modules. The notation M_R will be used to designate a unital right R-module M, and $N \leqslant M$ will mean that N is a submodule of M. The lattice of all submodules of a module M_R will be denoted by $\mathcal{L}(M_R)$.

1 Conditions (C_i), $i = 1, 2, 3, 11, 12,$ in lattices

The purpose of this section is to define the conditions (C_i), $i = 1, 2, 3, 11, 12,$ in lattices, and to establish their basic properties. These are the latticial counterparts of the well-known corresponding conditions on modules (see [8], [10], [11]).

Recall that for a lattice L we use throughout this paper the following notation:

$P(L) =$ the set of all *pseudo-complement* elements of L (P for “*Pseudo*”),

$E(L)$ = the set of all *essential* elements of L (E for "*Essential*"),

$C(L)$ = the set of all *closed* elements of L (C for "*Closed*"),

$D(L)$ = the set of all *complement* elements of L (D for "*Direct summand*").

Also, recall that for any $a \in L$, we have denoted by $P(a)$ the set, possibly empty, of all pseudo-complements of a in L, so $P(L) = \bigcup_{a \in L} P(a)$.

Definitions 1.1. *For a lattice L one may consider the following conditions:*

(C_1) *For every $x \in L$ there exists $d \in D(L)$ such that $x \in E(d/0)$.*

(C_2) *For every $x \in L$ such that $x/0 \simeq d/0$ for some $d \in D(L)$, one has $x \in D(L)$.*

(C_3) *For every $d_1, d_2 \in D(L)$ with $d_1 \wedge d_2 = 0$, one has $d_1 \vee d_2 \in D(L)$.*

(C_{11}) *For every $x \in L$ there exists a pseudo-complement p of x with $p \in D(L)$, in other words, $D(L) \cap P(x) \neq \varnothing$.*

(C_{12}) *For every $x \in L$ there exist $d \in D(L)$, $e \in E(d/0)$, and a lattice isomorphism $x/0 \simeq e/0$.* □

Definitions 1.2. *A lattice L is called* CC *or* extending *if it satisfies* (C_1), continuous *if it satisfies* (C_1) *and* (C_2), *and* quasi-continuous *if it satisfies* (C_1) *and* (C_3). □

First, we list below four results from [4] that will be used in our study of conditions (C_i), $i = 1, 2, 3, 11, 12$.

Lemma 1.3. ([4, Lemma 1.2.6]). *Let $L \in \mathcal{M}$, and let $a, b, c \in L$ be such that $a \wedge b = 0$ and $(a \vee b) \wedge c = 0$. Then $(a \vee c) \wedge b = 0$.* □

Lemma 1.4. ([4, Lemma 1.2.7]). *Let $L \in \mathcal{M}$, and let $a, b, c \in L$ be such that $a \wedge b = 0$ and $c \in E(b/0)$. Then $a \vee c \in E((a \vee b)/0)$.*

Lemma 1.5. ([4, Proposition 1.2.16 and Corollary 1.2.17]). *For any $L \in \mathcal{M}$, $P(L) \subseteq C(L)$, and $P(L) = C(L)$ if additionally L is E-complemented.* □

Lemma 1.6. ([4, Theorem 1.2.24]). *A modular lattice L is strongly pseudo-complemented if and only if L is E-complemented and essentially closed.* □

Proposition 1.7. *The following assertions hold for a lattice $L \in \mathcal{M}$.*

(1) *$D(L) \subseteq P(L) \subseteq C(L)$.*

(2) *$D(L) \cap (a/0) \subseteq D(a/0)$ for every $a \in L$.*

(3) *$D(L) \cap (d/0) = D(d/0)$ for every $d \in D(L)$.*

Proof. (1) Let $d \in D(L)$. Then there exists $c \in L$ with $c \vee d = 1$ and $c \wedge d = 1$. If $c' \in L$ is such that $c \leqslant c'$ and $d \wedge c' = 0$, then, by modularity, we have $c' = 1 \wedge c' = (c \vee d) \wedge c' = c \vee (d \wedge c') = c \vee 0 = c$, which proves that $d \in P(L)$, and so $D(L) \subseteq P(L)$.

The other inclusion $P(L) \subseteq C(L)$ follows from Lemma 1.5.

(2) Let $d \in D(L) \cap (a/0)$, and let $c \in L$ be a complement of d in L. Then $1 = c \vee d$ and $c \wedge d = 0$. It follows that $(c \wedge a) \wedge d = 0$ and $(c \wedge a) \vee d = (c \vee d) \wedge a = 1 \wedge a = a$, which shows that $c \wedge a$ is a complement of d in $a/0$, i.e., $d \in D(a/0)$.

(3) Let $d' \in D(d/0)$. Then there exists $d'' \in L$ such that $d' \wedge d'' = 0$ and $d' \vee d'' = d$. Also, $a \vee d = 1$ and $a \wedge d = 0$ for some $a \in L$. Thus $d' \vee (d'' \vee a) = 1$. Now, observe that $a \wedge (d' \vee d'') = a \wedge d = 0$, so we can apply Lemma 1.3 to obtain $d' \wedge (d'' \vee a) = 0$. This shows that $d'' \vee a$ is a complement of d' in L, i.e., $d' \in D(L)$. Since $d' \leqslant d$, we deduce that $d' \in D(L) \cap (d/0)$. So $D(d/0) \subseteq D(L) \cap (d/0)$. The other inclusion follows from (2). □

The next two results provide characterizations of conditions (C_{11}) and (C_{12}).

Proposition 1.8. *The following statements hold for a lattice $L \in \mathcal{M}$.*

(1) *L satisfies $(C_{11}) \Longleftrightarrow \forall\, x \in L, \exists\, d \in D(L)$ with $d \wedge x = 0$ and $d \vee x \in E(L)$.*

(2) *L is uniform $\Longrightarrow L$ satisfies (C_{11}).*

(3) *If L is indecomposable, then L satisfies $(C_{11}) \Longleftrightarrow L$ is uniform.*

Proof. (1) Assume that L satisfies (C_{11}). Then, it is well-known (see, e.g., [12, Chapter 3, Proposition 6.4]) that for every $p \in P(x)$ one has $p \vee x \in E(L)$.

Conversely, assume that L has the stated properties, and let $x \in L$ and $d \in D(L)$ with $d \wedge x = 0$ and $d \vee x \in E(L)$. We claim that d is a pseudo-complement of x in L. Suppose not. Then, there exists $q \in L$ with $d < q$ and $x \wedge q = 0$. But $d \in C(L)$ by Proposition 1.7(1), so $d \notin E(q/0)$. Then, there exists $0 \neq y \leqslant q$ such that $d \wedge y = 0$. Next, $(d \vee y) \wedge x = 0$ gives that $(d \vee x) \wedge y = 0$ by Lemma 1.3, and hence $y = 0$ because $d \vee x \in E(L)$, which is a contradiction. Thus d is a pseudo-complement of x in L, as desired.

(2) Let $x \in L$. If $x = 0$, then $1 \in D(L)$, $1 \wedge x = 0$, and $1 \vee x = 1 \in E(L)$. If $x \neq 0$, then $0 \in D(L)$, $0 \wedge x = 0$, and $0 \vee x = x \in E(L)$ because L is uniform. So, by (1), L satisfies (C_{11}).

(3) If L is uniform then it satisfies (C_{11}) by (2). Now assume that L satisfies (C_{11}), and let $0 \neq x \in L$. By (1), there exists $d \in D(L) = \{0, 1\}$ with $d \wedge x = 0$ and $d \vee x \in E(L)$, so necessarily $d = 0$, and then $x \in E(1/0) = E(L)$. Hence L is uniform. □

Proposition 1.9. *An essentially closed modular lattice L satisfies (C_{12}) if and only if for every $c \in C(L)$ there exist $d \in D(L)$, $e \in E(d/0)$, and a lattice isomorphism $c/0 \simeq e/0$.*

Proof. If L satisfies (C_{12}), then by definition, it clearly has the stated properties. Conversely, assume that L has the stated properties, and let $x \in L$. Then, there exists $c \in C(L)$ such that

$x \in E(c/0)$ because L is essentially closed. By assumption, there exist $d \in D(L)$, $e \in E(d/0)$, and a lattice isomorphism $\alpha : c/0 \longrightarrow e/0$. Then, $y := \alpha(x) \in E(e/0)$, so $y \in E(d/0)$, and by restriction of α to $x/0$ we obtain a lattice isomorphism $x/0 \simeq y/0$, which proves that L satisfies (C_{12}). □

The next result presents the connections between the conditions (C_i), $i = 1, 2, 3, 11, 12$, and characterizes essentially closed CC lattices in terms of closeness; in particular, it explains the term of CC, acronym for Closed elements are Complements.

Proposition 1.10. *The following statements hold for a lattice $L \in \mathcal{M}$.*

(1) *L is uniform $\Longrightarrow$ L is quasi-continuous $\Longrightarrow$ L is CC.*

(2) *If L is indecomposable, then L is CC $\Longleftrightarrow$ L is uniform.*

(3) *If additionally L is essentially closed, then*

$$L \text{ is CC} \Longleftrightarrow C(L) \subseteq D(L) \Longleftrightarrow C(L) = D(L).$$

(4) *If additionally L is strongly pseudo-complemented, then*

$$L \text{ is CC} \Longleftrightarrow C(L) \subseteq D(L) \Longleftrightarrow C(L) = D(L) \Longleftrightarrow P(L) \subseteq D(L) \Longleftrightarrow P(L) = D(L).$$

(5) *L satisfies $(C_2) \Longrightarrow L$ satisfies (C_3).*

(6) *L satisfies $(C_1) \Longrightarrow L$ satisfies (C_{11}).*

(7) *L satisfies $(C_{11}) \Longrightarrow L$ satisfies (C_{12}).*

Proof. (1) Assume that L is uniform. Then $D(L) = \{0, 1\}$. Let $x \in L$. If $x = 0$ then $0 \in D(L)$ and $0 \in E(0/0)$. If $x \neq 0$ then $1 \in D(L)$ and $x \in E(1/0) = E(L)$. So L satisfies (C_1). Let $d_1, d_2 \in D(L) = \{0, 1\}$. Then $d_1 \vee d_2 = 0$ or 1, and so $d_1 \vee d_2 \in D(L)$. Thus L satisfies (C_3).

(2) By (1), if L is uniform then it is CC. Now assume that L is CC and let $0 \neq x \in L$. By hypothesis, $x \in E(d/0)$ for some $d \in D(L) = \{0, 1\}$, so necessarily $d = 1$, and then $x \in E(1/0) = E(L)$. Hence L is uniform.

(3) Assume that L is CC, and let $x \in C(L)$. Then, there exists $d \in D(L)$ such that $x \in E(d/0)$, and hence $x = d \in D(L)$ because L is essentially closed. So $C(L) \subseteq D(L)$. Observe that, by Proposition 1.7(1), $C(L) \subseteq D(L) \Longleftrightarrow C(L) = D(L)$.

Finally assume that $C(L) \subseteq D(L)$, and let $x \in L$. There exists $c \in C(L)$ such that $x \in E(c/0)$. By assumption, $c \in D(L)$. It follows that L is CC.

(4) follows at once from (3) and Lemmas 1.5 and 1.6.

(5) Assume that L satisfies (C_2), and let $k, l \in D(L)$ with $k \wedge l = 0$. Then $1 = k \vee k'$ and $k \wedge k' = 0$ for some $k' \in L$.

Consider the element $u := (k \vee l) \wedge k'$. By modularity, we have

$$k \vee l = (k \vee l) \wedge (k \vee k') = ((k \vee l) \wedge k') \vee k = u \vee k.$$

Notice that $u \leqslant k'$ and $u \wedge k = 0$, so

$$l/0 = l/(k \wedge l) \simeq (k \vee l)/k = (u \vee k)/k \simeq u/(u \wedge k) = u/0.$$

By (C_2), $u \in D(L)$. Let m be a complement of u in L. Using again modularity we have

$$(u \vee k) \wedge k' = u \vee (k \wedge k') = u \vee 0 = u,$$

and consequently

$$(u \vee k) \wedge (k' \wedge m) = 0.$$

Also, by modularity, we obtain

$$u \vee (k' \wedge m) = (u \vee m) \wedge k' = 1 \wedge k' = k',$$

and so

$$(u \vee k) \vee (k' \wedge m) = k \vee k' = 1.$$

Thus $k \vee l = u \vee k \in D(L)$, as desired.

(6) Assume that L satisfies (C_1), and let $x \in L$. Then, there exists $d \in D(L)$ such that $x \in E(d/0)$. So, $d \wedge d' = 0$ and $d \vee d' = 1$ for some $d' \in L$. Now $d \wedge d' = 0$ implies that $x \vee d' \in E((d \vee d')/0) = E(1/0) = E(L)$ by Lemma 1.4. Since $x \wedge d' = 0$, by Proposition 1.8(1), we deduce that L satisfies (C_{11}).

(7) Assume that L satisfies (C_{11}), and let $x \in L$. Then, there exists $p \in P(x) \cap D(L)$, so $x \wedge p = 0$ and $x \vee p \in E(L)$. Also, there exists $p' \in L$ such that $p \wedge p' = 0$ and $p \vee p' = 1$.

Let $e := (x \vee p) \wedge p'$. Since $x \vee p \in E(L)$, it follows that $e \in E(p'/0)$ by well-known properties of essential elements. We have $1/p = (p \vee p')/p$ and $p'/0 = p'/(p \wedge p')$. By modularity, the map $\varphi : 1/p \longrightarrow p'/0$, $u \mapsto u \wedge p'$, is a lattice isomorphism. Since $\varphi(x \vee p) = e$, we deduce that

$$(x \vee p)/p \simeq e/0.$$

Using modularity again we have

$$x/0 = x/(x \wedge p) \simeq (x \vee p)/p,$$

hence

$$x/0 \simeq e/0 \quad \text{for } e \in E(p'/0) \text{ and } p' \in D(L).$$

Thus, L satisfies (C_{12}). □

Lemma 1.11. ([4, Corollary 1.2.14]). *Let $L \in \mathcal{M}$ be an E-complemented lattice, and let $c \leqslant d$ in L be such that $c \in C(d/0)$ and $d \in C(L)$. Then $c \in C(L)$.* □

Lemma 1.12. *Let $L \in \mathcal{M}$ be an E-complemented lattice, and let $c \leqslant d$ in L be such that $c \in C(d/0)$ and $d \in D(L)$. Then $c \in C(L)$.*

Proof. The result follows immediately from Lemma 1.11 because $D(L) \subseteq C(L)$. □

Lemma 1.13. *Let $L \in \mathcal{M}$, $d \in D(L)$, and $k \in D(d/0)$. Then $k \in D(L)$.*

Proof. There exist $k' \in d/0$ and $d' \in L$ such that $k \vee k' = d$, $k \wedge k' = 0$, $d \vee d' = 1$, and $d \wedge d' = 0$. Then $k \vee (k' \vee d') = 1$ and $k \wedge (k' \vee d') = 0$ by Lemma 1.3, and so, $k \in D(L)$. □

Lemma 1.14. ([4, Lemma 1.2.20]). *If L is a strongly pseudo-complemented lattice, then so is also its sublattice $a/0$ for any element $a \in L$.* □

Proposition 1.15. *Let $L \in \mathcal{M}$ be a strongly pseudo-complemented lattice (in particular, an upper continuous lattice), and let $d \in D(L)$. If L satisfies (C_i), $i = 1, 2, 3$, then $d/0$ also satisfies (C_i), $i = 1, 2, 3$, in other words, the conditions (C_i), $i = 1, 2, 3$, are inherited by complement intervals.*

Proof. First assume that L satisfies (C_1), and let $c \in C(d/0)$. Since $d \in D(L)$, it follows that $c \in C(L)$ by Lemma 1.12. But $C(L) = D(L)$ by Proposition 1.10(3), therefore $c \in D(L) \cap (d/0) = D(d/0)$ by Proposition 1.7(3). Thus $C(d/0) \subseteq D(d/0)$. Now, observe that $d/0$ is strongly pseudo-complemented by Lemma 1.14, so we can apply again Proposition 1.10(3) to deduce that $d/0$ satisfies (C_1).

Now assume that L satisfies (C_2). Let $x \in d/0$, $k \in D(d/0)$, and let $f : x/0 \longrightarrow k/0$ be a lattice isomorphism. Then $k \in D(L)$ by Lemma 1.13, and so $x \in D(L)$ because L satisfies (C_2). So, $x \in D(L) \cap (d/0) = D(d/0)$ by Proposition 1.7(3), i.e., $d/0$ satisfies (C_2).

Finally, suppose that L satisfies (C_3), and let $d_1, d_2 \in D(d/0)$ with $d_1 \wedge d_2 = 0$. Then $d_1 \vee d_2 \in D(L) \cap (d/0) = D(d/0)$ again by Proposition 1.7(3), and we are done. □

Lemma 1.16. *Let L be a strongly pseudo-complemented lattice, let $a \in L$, let $p \in P(a)$ and let $q \in P(p)$ with $a \leqslant q$. Then $p \in P(q)$, so, p and q are pseudo-complements of each other.*

Proof. Let $b \in L$ be such that $p \leqslant b$ and $b \wedge q = 0$. Then also $b \wedge a = 0$, so $b = p$ by the definition of p. Hence p is maximal with respect to $p \wedge q = 0$, and so, p is a pseudo-complement of q. □

Lemma 1.17. ([4, Corollary 2.2.2]). *Let S and T be non-empty finite subsets of a lattice $L \in \mathcal{M}$. Then $S \cup T$ is an independent subset of L if and only if S and T are both independent subsets of L and $\left(\bigvee S\right) \wedge \left(\bigvee T\right) = 0$.* □

Proposition 1.18. *The following statements are equivalent for a strongly pseudo-complemented lattice L.*

(1) *L is quasi-continuous.*

(2) *$1 = p_1 \dot{\vee} p_2$ for every $p_1, p_2 \in P(L)$ which are pseudo-complements of each other.*

(3) *For every $x_1, x_2 \in L$ with $x_1 \wedge x_2 = 0$, there exist $d_1, d_2 \in L$ with $1 = d_1 \dot{\vee} d_2$ and $x_1 \leqslant d_1$, $x_2 \leqslant d_2$.*

Proof. First note that, by Proposition 1.10(4), L is CC $\Longleftrightarrow P(L) \subseteq D(L)$.

(1) $\Longrightarrow$ (2) Assume that L is quasi-continuous, and let $p_1, p_2 \in P(L)$ which are pseudo-complements of each other. By condition (C_1), $p_1, p_2 \in D(L)$. Now, condition (C_3) yields that $p_1 \vee p_2 \in D(L)$. But $p_1 \vee p_2 \in E(L)$, and so $1 = p_1 \dot{\vee} p_2$.

(2) $\Longrightarrow$(1) We prove first that L is CC. By the remark that starts the proof, it suffices to show that $P(L) \subseteq D(L)$. To do this, let $p \in P(L)$. Then, there exists $a \in L$ such that $p \in P(a)$. Since L is strongly pseudo-complemented, there exists a $q \geqslant a$ such that $q \in P(p)$. By Lemma 1.16, p and q are pseudo-complements of each other, and by our hypothesis, we have $p \dot{\vee} q = 1$, and so $p \in D(L)$, as desired.

Now, we show that L satisfies (C_3). Let $d_1, d_2 \in D(L)$ be such that $d_1 \wedge d_2 = 0$ and prove that $d_1 \vee d_2 \in D(L)$. Let $p_1 \in P(d_2)$ with $d_1 \leqslant p_1$, and $p_2 \in P(p_1)$ with $d_2 \leqslant p_2$. By Lemma 1.16, p_1 and p_2 are pseudo-complements of each other, so, by assumption, $1 = p_1 \dot{\vee} p_2$. Therefore, $p_1, p_2 \in D(L)$, so by Proposition 1.7(3), we have $d_i \in D(L) \cap (p_i/0) = D(p_i/0)$ for $i = 1, 2$. Let $e_i \in D(p_i/0)$, be such that $p_i = d_i \dot{\vee} e_i$ for $i = 1, 2$. Now apply Lemma 1.17 for $S = \{d_1, e_1\}$ and $T = \{d_2, e_2\}$ to deduce that

$$(d_1 \vee d_2) \dot{\vee} (e_1 \vee e_2) = 1,$$

so $d_1 \vee d_2 \in D(L)$, as desired.

(2) $\Longrightarrow$(3) Let $x_1, x_2 \in L$ with $x_1 \wedge x_2 = 0$, and pick $d_1 \in P(x_2)$ with $x_1 \leqslant d_1$ and $d_2 \in P(d_1)$ with $x_2 \leqslant d_2$. By Lemma 1.16, d_1 and d_2 are pseudo-complements of each other, so $1 = d_1 \dot{\vee} d_2$ by assumption, and we are done.

(3) $\Longrightarrow$(2) Let $p_1, p_2 \in P(L)$ which are pseudo-complements of each other, i.e., $p_2 \in P(p_1)$ and $p_1 \in P(p_2)$. Then $p_1 \wedge p_2 = 0$, so, by assumption, there exist $d_1, d_2 \in L$ with $1 = d_1 \dot{\vee} d_2$ and $p_1 \leqslant d_1$, $p_2 \leqslant d_2$. Then $p_1 \wedge d_2 \leqslant d_1 \wedge d_2 = 0$ and $p_2 \wedge d_1 \leqslant d_1 \wedge d_2 = 0$, so necessarily $p_i = d_i$ because $p_i \in P(L)$, $i = 1, 2$. Thus $1 = p_1 \dot{\vee} p_2$, as desired. □

We end this section by stating a latticial counterpart involving CC lattices of the following renown result of Module Theory that provides sufficient conditions for a finitely generated (respectively, cyclic) module to be a finite direct sum of uniform submodules.

The Osofsky-Smith Theorem [9]. *A finitely generated* (*respectively, cyclic*) *right* R-*module such that all of its finitely generated* (*respectively, cyclic*) *subfactors are CS modules is a finite direct sum of uniform submodules.* □

Recall that a module M is said to be *CS* (or *extending*) if every submodule of M is essential in a direct summand of M, or, equivalently, if any complement submodule of M is a direct summand of M. The name CS is an acronym for Complements submodules are direct Summands. Recall that in Module Theory one says that a submodule N of M is a *complement* if there exists a submodule L of M such that $N \cap L = 0$ and N is maximal in the set of all submodules P of M such that $P \cap L = 0$, i.e., the element N of the lattice $\mathcal{L}(M)$ of all submodules of M is a pseudo-complement element in this lattice. Consequently, a module M is CS if and only if the lattice $\mathcal{L}(M)$ is CC.

Though the Osofsky-Smith Theorem is a module-theoretical result, our contention is that it is a result of a strong latticial nature. The following latticial version of this theorem was established in [1], and applications of it to Grothendieck categories and module categories equipped with a torsion theory were given in [2].

Theorem 1.19. (The Latticial Osofsky-Smith Theorem [1]). *Let* L *be a compact, compactly generated, modular lattice. Assume that all compact subfactors of* L *are CC. Then* 1 *is a finite direct join of uniform elements of* L. □

2 Inheritance of condition (C_{11}) under direct joins and complement intervals

The condition (C_1) is, in general, not inherited by direct joins, as this is well-known for modules (see, e.g., [7]), in contrast with the condition (C_{11}) by the theorem that will follow. But first, we need some preparatory results.

Lemma 2.1. *Let* $n \in \mathbb{N}$, $n \geqslant 2$. *A set* $\{a_1, \ldots, a_n\}$ *of non-zero elements of a lattice* $L \in \mathcal{M}$ *is independent if and only if* $a_{k+1} \wedge (a_1 \vee \cdots \vee a_k) = 0$ *for all* k, $1 \leqslant k \leqslant n-1$.

Proof. See, e.g., [4, Lemma 2.2.1]. □

Lemma 2.2. *Let* $L \in \mathcal{M}$, *and let* $a_1, a_2 \in L$ *be such that* $a_1 \wedge a_2 = 0$. *Suppose that for every* $i \in \{1, 2\}$, d_i' *is a complement of* d_i *in* $a_i/0$. *Then* $d_1' \vee d_2'$ *is a complement of* $d_1 \vee d_2$ *in* $(a_1 \vee a_2)/0$.

Proof. We have $d_1 \wedge d_1' = 0$ and $(d_1 \vee d_1') \wedge d_2 = 0$ since $(d_1 \vee d_1') \wedge d_2 = a_1 \wedge d_2 \leqslant a_1 \wedge a_2 = 0$, and also $(d_2 \vee d_2') \wedge a_1 = a_2 \wedge a_1 = 0$. By Lemma 1.3, it follows that $(d_2 \vee a_1) \wedge d_2' = 0$, so

$$(d_1 \vee d_1' \vee d_2) \wedge d_2' = 0\,.$$

By Lemmas 1.17 and 2.1, we deduce that the family $\{d_1, d_1', d_2, d_2'\}$ is independent, hence

$$(d_1 \vee d_2) \wedge (d_1' \vee d_2') = 0\,.$$

To end the proof, observe that $(d_1 \vee d_2) \vee (d_1' \vee d_2') = a_1 \vee a_2\,.$ □

Lemma 2.3. ([4, Corollary 1.2.8]). *Let* $L \in \mathcal{M}$*, and let* $a_i,\, b_i \in L$ *be such that* $a_i \in E(b_i/0)$ $(i = 1, 2)$ *and* $a_1 \wedge a_2 = 0$*. Then* $a_1 \vee a_2 \in E((b_1 \vee b_2)/0)$*.* □

Lemma 2.4. *Let* $L \in \mathcal{M}$*, and let* $x,\, a_1,\, a_2 \in L$ *be such that* $a_1 \wedge a_2 = 0$ *and* $x \leqslant a_1 \vee a_2$*. Suppose that there exist an element* $d_1 \in D(a_1/0)$ *such that*

$$(x \wedge a_1) \wedge d_1 = 0,\ (x \wedge a_1) \vee d_1 \in E(a_1/0),$$

and an element $d_2 \in D(a_2/0)$ *such that*

$$((x \vee d_1) \wedge a_2) \wedge d_2 = 0,\ ((x \vee d_1) \wedge a_2) \vee d_2 \in E(a_2/0)\,.$$

Then

$$d_1 \vee d_2 \in D((a_1 \vee a_2)/0),\ x \wedge (d_1 \vee d_2) = 0 \ \text{ and } \ x \vee (d_1 \vee d_2) \in E((a_1 \vee a_2)/0).$$

Proof. If we set $d := d_1 \vee d_2$, then $d \in D((a_1 \vee a_2)/0)$ by Lemma 2.2. Since $x \wedge d_1 \leqslant a_1$ and $(x \wedge d_1) \wedge a_1 = 0$, it follows that $x \wedge d_1 = 0$. By modularity, we have

$$(x \vee d_1) \wedge a_1 = (x \wedge a_1) \vee d_1 \in E(a_1/0).$$

Similarly, $(x \vee d_1) \wedge d_2 = 0$ and

$$(x \vee d) \wedge a_2 = ((x \vee d_1) \vee d_2) \wedge a_2 = ((x \vee d_1) \wedge a_2) \vee d_2 \in E(a_2/0).$$

Since $x \wedge d_1 = 0$ and $(x \vee d_1) \wedge d_2 = 0$, by Lemma 1.3, we have $x \wedge d = x \wedge (d_1 \vee d_2) = 0$. To conclude, we show that $x \vee d \in E((a_1 \vee a_2)/0)$. Indeed, since $(x \vee d_1) \wedge a_1 \in E(a_1/0)$, it follows that $(x \vee d) \wedge a_1 \in E(a_1/0)$. We also have $(x \vee d) \wedge a_2 \in E(a_2/0)$. Using now Lemma 2.3, we deduce that $((x \vee d) \wedge a_1) \vee ((x \vee d) \wedge a_2) \in E((a_1 \vee a_2)/0)$. But $((x \vee d) \wedge a_1) \vee ((x \vee d) \wedge a_2) \leqslant x \vee d$, thus $x \vee d \in E((a_1 \vee a_2)/0)$, and we are done. □

Now, we are in a position to prove the main result of this section. First, we prove it for any finite independent family of elements of an arbitrary modular lattice L. Then, we prove it also for infinite independent families of elements of L, where the additional condition that L is upper continuous is required in order to use Zorn's Lemma.

Proposition 2.5. *Let* $L \in \mathcal{M}$*, and let* $(a_i)_{1 \leqslant i \leqslant n}$ *be a finite independent family of elements of* L *such that* $1 = \dot{\bigvee}_{1 \leqslant i \leqslant n} a_i$ *and* $a_i/0$ *satisfies* (C_{11}) *for all* $1 \leqslant i \leqslant n$*. Then* L *satisfies* (C_{11})*.*

Proof. We proceed by induction on n. The result is clear for $n = 1$. Now, let $1 < i < n$ and suppose that the result is true for i and prove it for $i + 1$.

For every $1 \leqslant i \leqslant n$ set $b_i := \bigvee_{1 \leqslant j \leqslant i} a_i$. We have $b_{i+1} = b_i \dot{\vee} a_{i+1}$. Now, $b_i/0$ satisfies the condition (C_{11}) by the inductive hypothesis, and $a_{i+1}/0$ satisfies the condition (C_{11}) by hypothesis. So, it is sufficient to prove the result only for $n = 2$. By Proposition 1.8(1) and Lemma 2.4, $(a_1 \vee a_2)/0$ satisfies (C_{11}), and we are done. □

Theorem 2.6. *Let L be an upper continuous modular lattice, and let $(a_i)_{i\in I}$ be an independent family of elements of L such that $1 = \dot{\bigvee}_{i\in I} a_i$ and $a_i/0$ satisfies (C_{11}) for all $i \in I$. Then L satisfies (C_{11}).*

Proof. Let $x \in L$ be a fixed element. For each $\varnothing \neq J \subseteq I$, set $a_J := \bigvee_{i\in J} a_i$. Consider the set $\mathcal{H}$, depending on x, of all triplets (J, d, d') such that

$$\varnothing \neq J \subseteq I,\ d,\, d' \in L,\ d \wedge d' = 0,\ d \vee d' = a_J,\ (x \wedge a_J) \wedge d = 0,\ (x \wedge a_J) \vee d \in E(a_J/0)\,,$$

which becomes a partially ordered set by the componentwise order $\preceq$ defined by

$$(J_1, d_1, d_1') \preceq (J_2, d_2, d_2') \iff J_1 \subseteq J_2,\ d_1 \leqslant d_2, \text{ and } d_1' \leqslant d_2'\,.$$

Since $a_i/0$ satisfies (C_{11}), for every $i \in I$ there exist $d, d' \in a_i/0$ such that $(\{i\}, d, d') \in \mathcal{H}$. Thus $\mathcal{H} \neq \varnothing$.

We are now going to show that $\mathcal{H}$ is an inductive set, so that, we can apply Zorn's Lemma to find a maximal element of it. To do this, consider a chain $\mathcal{C}$ in $\mathcal{H}$, and set

$$\mathcal{J} := \{\, J \subseteq I \mid \exists\, d_J,\, d_J' \in D(a_J/0) \text{ with } (J, d_J, d_J') \in \mathcal{C}\,\}.$$

Since $\mathcal{C}$ is a chain in $\mathcal{H}$, it follows that $\mathcal{J}$ is a chain of subsets of I.

Now, notice that, for $(J, d_J, d_J') \in \mathcal{C}$ and $(K, d_K, d_K') \in \mathcal{C}$ such that $J \subseteq K$ we have $d_J \leqslant d_K$ and $d_J' \leqslant d_K'$. Indeed, because $\mathcal{C}$ is a chain, we have $(J, d_J, d_J') \preceq (K, d_K, d_K')$ or $(K, d_K, d_K') \preceq (J, d_J, d_J')$, so $J \subseteq K, d_J \leqslant d_K, d_J' \leqslant d_K'$ or $K \subseteq J, d_K \leqslant d_J, d_K' \leqslant d_J'$. In the second case, it follows that $J = K$, and then, observe that $d_K \wedge d_J' \leqslant d_J \wedge d_J' = 0$ and $a_J = a_K = d_K \vee d_K' \leqslant d_K \vee d_J' \leqslant d_J \vee d_J' = a_J = a_K$, so d_J' is a complement of d_K in $a_K/0$. Because d_K' is also a complement of d_K in the modular lattice $a_K/0$ and $d_K' \leqslant d_J'$, then necessarily $d_K' = d_J'$. Similarly, in this case, we have $d_K = d_J$.

Notice also that the sets $A = \{\, a_J \mid J \in \mathcal{J}\,\}$, $D = \{\, d_J \mid J \in \mathcal{J}\,\}$, and $D' = \{\, d_J' \mid J \in \mathcal{J}\,\}$ are chains in L. Set

$$\overline{J} := \bigcup_{J\in\mathcal{J}} J\,,\ \overline{d} := \bigvee D = \bigvee_{J\in\mathcal{J}} d_J\,, \text{ and } \overline{d'} := \bigvee D' = \bigvee_{J\in\mathcal{J}} d_J'\,.$$

Next, we prove that $(\overline{J}, \overline{d}, \overline{d'})$ is an upper bound for $\mathcal{C}$ in $\mathcal{H}$. For this, it is sufficient to show that $(\overline{J}, \overline{d}, \overline{d'}) \in \mathcal{H}$. We have

$$\overline{d} \vee \overline{d'} = \bigvee_{J\in\mathcal{J}} (d_J \vee d_J') = \bigvee_{J\in\mathcal{J}} a_J = \bigvee_{J\in\mathcal{J}} \big(\bigvee_{i\in J} a_i\big) = \bigvee_{i\in\overline{J}} a_i = a_{\overline{J}} = \bigvee A.$$

Using upper continuity, we have

$$\overline{d} \wedge \overline{d'} = \big(\bigvee_{J\in\mathcal{J}} d_J\big) \wedge \big(\bigvee_{K\in\mathcal{J}} d_K'\big) = \bigvee_{J\in\mathcal{J}} \big(d_J \wedge \big(\bigvee_{K\in\mathcal{J}} d_K'\big)\big) = \bigvee_{J\in\mathcal{J}} \big(\bigvee_{K\in\mathcal{J}} (d_J \wedge d_K')\big)\,.$$

For $J, K \in \mathcal{J}$, we have either $J \subseteq K$ or $K \subseteq J$. In the first case, we have $d_J \leqslant d_K$, so $d_J \wedge d_K' \leqslant d_K \wedge d_K' = 0$. In the second case, we have $d_K' \leqslant d_J'$, so $d_J \wedge d_K' \leqslant d_J \wedge d_J' = 0$.

Therefore, in both cases we obtain $d_J \wedge d'_K = 0$. So $\overline{d} \wedge \overline{d'} = 0$, and hence $\overline{d'}$ is a complement of $\overline{d}$ in $a_{\overline{J}}/0$.

Now, again by upper continuity, we have

$$(x \wedge a_{\overline{J}}) \wedge \overline{d} = x \wedge \overline{d} = x \wedge \big(\bigvee D\big) = \bigvee_{J \in \mathcal{J}} (x \wedge d_J) = \bigvee_{J \in \mathcal{J}} ((x \wedge a_J) \wedge d_J) = 0\,.$$

Next, we claim that $(x \wedge a_{\overline{J}}) \vee \overline{d} \in E(a_{\overline{J}}/0)$. To do this, let $y \in a_{\overline{J}}/0$ be such that $((x \wedge a_{\overline{J}}) \vee \overline{d}\,) \wedge y = 0$. Using several times modularity and upper continuity, we have

$$\begin{aligned} 0 &= ((x \wedge a_{\overline{J}}) \vee \overline{d}\,) \wedge y = ((x \vee \overline{d}\,) \wedge a_{\overline{J}}) \wedge y = \big(x \vee \big(\bigvee D\big)\big) \wedge (a_{\overline{J}} \wedge y) \\ &= \big(\bigvee_{J \in \mathcal{J}} (x \vee d_J)\big) \wedge (a_{\overline{J}} \wedge y) = \bigvee_{J \in \mathcal{J}} \big((x \vee d_J) \wedge (a_{\overline{J}} \wedge y)\big)\,. \end{aligned}$$

Thus, for each $J \in \mathcal{J}$, we have $(x \vee d_J) \wedge (a_{\overline{J}} \wedge y) = 0$. But $a_J \leqslant a_{\overline{J}}$, so $(x \vee d_J) \wedge (a_J \wedge y) = 0$ and consequently, by modularity, we obtain

$$((x \wedge a_J) \vee d_J) \wedge (a_J \wedge y) = ((x \vee d_J) \wedge a_J) \wedge (a_J \wedge y) = (x \vee d_J) \wedge (a_J \wedge y) = 0.$$

By the definition of the set $\mathcal{H}$, we have $(x \wedge a_J) \vee d_J \in E(a_J/0)$, hence $a_J \wedge y = 0$ for each $J \in \mathcal{J}$. Using again the upper continuity, it follows that

$$0 = \bigvee_{J \in \mathcal{J}} (a_J \wedge y) = \big(\bigvee_{J \in \mathcal{J}} a_J\big) \wedge y = a_{\overline{J}} \wedge y = y,$$

which proves our claim.

For now, we have proved that $(\overline{J}, \overline{d}, \overline{d'}\,) \in \mathcal{H}$. As we stated before, it follows that $\mathcal{C}$ has an upper bound in $\mathcal{H}$, and consequently, $\mathcal{H}$ is an inductive set. Using Zorn's Lemma, there exists an (H, d_H, d'_H) that is maximal in $\mathcal{H}$ with respect to $\preceq$.

To end the proof of this theorem, it suffices to show that H equals I. Suppose not. Pick $i \in I \setminus H$. Since $a_i/0$ satisfies (C_{11}), there exist $d_i, d'_i \in a_i/0$ such that d'_i is the complement of d_i in $a_i/0$ and, moreover $((x \vee d_H) \wedge a_i) \wedge d_i = 0$ and $((x \vee d_H) \wedge a_i) \vee d_i \in E(a_i/0)$. Consider the set $G := H \cup \{i\}$. Set $d_G := d_H \vee d_i$ and $d'_G := d'_H \vee d'_i$. By Lemma 2.2, d'_G is a complement of d_G in $a_G/0 = (a_H \vee a_i)/0$. We have $(x \wedge a_H) \wedge d_H = 0$, $(x \wedge a_H) \vee d_H \in E(a_H/0)$ because $(H, d_H, d'_H) \in \mathcal{H}$. Using now Lemma 2.4 with a_H instead of a_1, a_i instead of a_2, d_H instead of d_1, and d_i instead of d_2 in its statement, we deduce that

$$(x \wedge a_G) \wedge d_G = (x \wedge (a_H \vee a_i)) \wedge (d_H \vee d_i) = 0$$

and

$$(x \wedge a_G) \vee d_G = (x \wedge (a_H \vee a_i)) \vee (d_H \vee d_i) \in E((a_H \vee a_i)/0) = E(a_G/0).$$

Hence $(G, d_G, d'_G\,) \in \mathcal{H}$. But $i \in G \setminus H$, so $G \supsetneqq H$. Therefore $(G, d_G, d'_G\,)$ is strictly greater than the maximal element $(H, d_H, d'_H\,)$ of $\mathcal{H}$, which is a contradiction. It follows that $H = I$, so, $(I, d_I, d'_I\,) \in \mathcal{H}$. Having in mind that $a_I = 1$, we obtain for the given element $x \in L$, we started the proof with, an element $d_I \in D(L)$ such that $x \wedge d_I = 0$ and $x \vee d_I \in E(L)$, so by Proposition 1.8(1) we conclude that L satisfies the condition (C_{11}). □

Remark 2.7. Theorem 2.6 is the latticial counterpart of [10, Theorem 2.5] showing that any direct sum $\bigoplus_{i\in I} M_i$ of right R-modules M_i, all satisfying condition (C_{11}), also satisfies (C_{11}). Notice that its original proof in [10] is incomplete, because, without involving Zorn's Lemma, it does not work at all for an *infinite* family $(M_i)_{i\in I}$ of modules. □

Corollary 2.8. *Let L be an upper continuous modular lattice, and let $(a_i)_{i\in I}$ be an independent family of elements of L such that $a_i/0$ satisfies (C_1) for all $i \in I$. Then $(\bigvee_{i\in I} a_i)/0$ satisfies (C_{11}).*

Proof. By Proposition 1.10(6), any lattice satisfying (C_1) also satisfies (C_{11}), so, the result follows at once from Theorem 2.6. □

Corollary 2.9. *Let L be an upper continuous modular lattice, and let $(a_i)_{i\in I}$ be an independent family of uniform elements of L. Then $(\bigvee_{i\in I} a_i)/0$ satisfies (C_{11}).*

Proof. For every $i \in I$, $a_i/0$ is a uniform lattice, so it satisfies (C_{11}) by Proposition 1.8(2). Apply now Theorem 2.6. □

In contrast to conditions (C_i), i $= 1, 2, 3$, the condition (C_{11}) is not inherited by complement intervals, as this follows in module case from [11, Example 4]. Next, we obtain some positive results in this trend.

Proposition 2.10. *Let L be a lattice which satisfies (C_{11}) and (C_3). Then any complement interval of L satisfies (C_{11}).*

Proof. Let $d \in D(L)$. We have to show that $d/0$ satisfies (C_{11}). There exists $d' \in L$ such that $d \wedge d' = 0$ and $d \vee d' = 1$. Let $k \in d/0$. By (C_{11}) for L, according to Proposition 1.8(1), there exists an $l \in D(L)$ such that $(k \vee d') \wedge l = 0$ and $(k \vee d') \vee l \in E(L)$. It follows that $d' \wedge l = 0$. Since d' and l are both complements in L, using (C_3) we deduce that $p = d' \vee l \in D(L)$.

Since $k \leqslant d$ and $d \wedge d' = 0$, it follows that $k \wedge d' = 0$. We also have $(k \vee d') \wedge l = 0$. By Lemma 1.3, we obtain that $k \wedge p = k \wedge (d' \vee l) = 0$. Notice that $k \vee p = k \vee (d' \vee l) = (k \vee d') \vee l \in E(L)$. We have $d/0 = d/(d \wedge d')$ and $1/d' = (d \vee d')/d'$. By modularity, the map

$$\varphi : 1/d' \longrightarrow d/0,\ \varphi(u) = u \wedge d,$$

is a lattice isomorphism. We have

$$k \wedge \varphi(p) = (k \wedge p) \wedge d = 0.$$

Moreover, using modularity, we also have $k \vee \varphi(p) = k \vee (d \wedge p) = d \wedge (k \vee p)$ and since $k \vee p \in E(L)$ we deduce that

$$k \vee \varphi(p) \in E(d/0)$$

by well-known properties of essential elements.

To conclude the proof, we show that $\varphi(p) \in D(d/0)$. Indeed, since $p \in D(L)$, it follows that p has a complement p' in L. But $d' \leqslant p$, and, by modularity we deduce that $d' \vee p'$ is a complement of p in $1/d'$. Thus $p \in D(1/d')$, and, because $\varphi : 1/d' \longrightarrow d/0$ is a lattice isomorphism, we deduce that $\varphi(p) \in D(d/0)$, as desired. □

Proposition 2.11. *The following statements are equivalent for a lattice L and a direct join decomposition $1 = m_1 \dot{\vee} m_2$ in L.*

(1) *$m_1/0$ satisfies (C_{11}).*

(2) *$\forall x \in m_1/0, \exists d \in D(L)$ such that $m_2 < d$, $d \wedge x = 0$, and $d \vee x \in E(L)$.*

Proof. (1) $\Longrightarrow$ (2) Assume that $m_1/0$ satisfies (C_{11}), and let $x \in m_1/0$. By Proposition 1.8(1), there exists $l \in D(m_1/0)$ such that $x \wedge l = 0$ and $x \vee l \in E(m_1/0)$. Then

$$(x \vee l) \wedge m_2 \leqslant m_1 \wedge m_2 = 0,$$

so $(l \vee m_2) \wedge x = 0$ by Lemma 1.3. If $d := l \vee m_2$, then $d \wedge x = 0$ and

$$d \vee x = (l \vee m_2) \vee x = (x \vee l) \vee m_2 \in E((m_1 \vee m_2)/0) = E(1/0) = E(L)$$

by Lemma 2.3. Now, by Lemma 1.13, we deduce that $d \in D(L)$ because $l \in D(m_1/0)$ and $m_1 \in D(L)$.

(2) $\Longrightarrow$ (1) Let $y \in m_1/0$. By assumption, there exists $k \in D(L)$ such that $m_2 < k$, $k \wedge y = 0$, and $k \vee y \in E(L)$. Now, by modularity, we have

$$k = k \wedge 1 = k \wedge (m_1 \vee m_2) = (k \wedge m_1) \vee m_2,$$

so that $k \wedge m_1 \in D(k/0) \subseteq D(L)$, and hence $k \wedge m_1 \in D(L) \cap (m_1/0) = D(m_1/0)$ by Lemma 1.7(3). Moreover, $y \wedge (k \wedge m_1) = 0$ and $y \vee (k \wedge m_1) = m_1 \wedge (y \vee k) \in E(m_1/0)$. So, by Proposition 1.8(1), $m_1/0$ satisfies (C_{11}). □

Proposition 2.12. *Let $L \in \mathcal{M}$ be a lattice satisfying (C_{11}) and having a direct join decomposition $1 = m_1 \dot{\vee} m_2$. Suppose that $k \vee m_2 \in D(L)$ for every $k \in D(L)$ with $k \wedge m_2 = 0$. Then $m_1/0$ satisfies (C_{11}).*

Proof. Let $x \in m_1/0$. By Proposition 1.8(1), there exists $k \in D(L)$ such that $(x \vee m_2) \wedge k = 0$ and $x \vee m_2 \vee k \in E(L)$. Moreover, by hypothesis, $k \vee m_2 \in D(L)$.

Now, observe that $x \wedge m_2 \leqslant m_1 \wedge m_2 = 0$, so, by Lemma 1.3, we have $(k \vee m_2) \wedge x = 0$. By Proposition 1.8(1), it follows that $m_1/0$ satisfies (C_{11}), as desired. □

3 Applications to Grothendieck categories

In this section we apply the lattice-theoretical results established in the previous sections to Grothendieck categories.

Throughout this section $\mathcal{G}$ will denote *Grothendieck category*, i.e.,an Abelian category with exact direct limits and with a generator, and for any object X of $\mathcal{G}$, $\mathcal{L}(X)$ will denote the lattice of all subobjects of X. It is well-known that $\mathcal{L}(X)$ is an upper continuous modular lattice (see, e.g., [12, Chapter 4, Proposition 5.3, and Chapter 5, Section 1].

For all undefined notation and terminology on Abelian categories the reader is referred to [3] and [12].

Recall that an object X of $\mathcal{G}$ is said to be *Noetherian* (respectively, *Artinian*) if the lattice $\mathcal{L}(X)$ is Noetherian (respectively, Artinian). More generally, if $\mathbb{P}$ is any property on lattices, we say that an object $X \in \mathcal{G}$ is/has $\mathbb{P}$ if the lattice $\mathcal{L}(X)$ is/has $\mathbb{P}$. Similarly, a subobject Y of an object $X \in \mathcal{G}$ is/has $\mathbb{P}$ if the element Y of the lattice $\mathcal{L}(X)$ is/has $\mathbb{P}$. Thus, we obtain the concepts of an *uniform* object, *compact* object, (C_i), $i = 1, 2, 3, 11, 12$, condition for an object, *CC* object, *quasi-continuous* object, *continuous* object, *pseudo-complement* subobject of an object, *essential* subobject of an object, *closed* subobject of an object, *complement* subobject of an object, etc. For a complement (respectively, compact) subobject of an object $X \in \mathcal{G}$ one uses the well-established term of a *direct summand* (respectively, *finitely generated* subobject) of X, and for this reason, instead of saying that X is a CC object we shall say that X is a CS object (acronym for *C*losed subobjects are direct *S*ummands).

Of course, all the notions and results of Sections 1 and 2 have categorical versions obtained by specializing them from an arbitrary modular lattice L to the upper continuous modular lattice $\mathcal{L}(X)$ of any object X of a Grothendieck category $\mathcal{G}$. No further proofs are required. We shall present below only two results, and leave the others to the reader.

Proposition 3.1. *An object X of a Grothendieck category $\mathcal{G}$ is quasi-continuous if and only if for any subobjects X_1 and X_2 of X with $X_1 \cap X_2 = 0$, there exist subobjects D_1 and D_2 of X with $X = D_1 \oplus D_2$ and $X_1 \subseteq D_1$, $X_2 \subseteq D_2$.* □

Theorem 3.2. *Any direct sum of objects satisfying the condition (C_{11}) of a Grothendieck category $\mathcal{G}$ also satisfies the condition (C_{11}).* □

4 Applications to module categories equipped with a hereditary torsion theory

In this section, we present relative versions with respect to a hereditary torsion theory on Mod-R of some module-theoretical results related to conditions (C_i). Their proofs are immediate applications of the lattice-theoretical results obtained in Sections 1 and 2.

Throughout this section R denotes a ring with non-zero identity, Mod-R the category of all unital right R-modules, $\tau = (\mathcal{T}, \mathcal{F})$ a fixed hereditary torsion theory on Mod-R, and $\tau(M)$ the τ-torsion submodule of a right R-module M. We shall use the notation M_R to emphasize that M is a right R-module. For any M_R we shall denote

$$\mathrm{Sat}_\tau(M) \ := \ \{ \ N \mid N \leqslant M \text{ and } M/N \in \mathcal{F} \ \},$$

and for any $N \leqslant M$ we shall denote the τ-*saturation* of N (in M) by

$$\overline{N} := \bigcap \{ \, C \mid N \leqslant C \leqslant M, \ M/C \in \mathcal{F} \, \}.$$

The submodule N is called τ-*saturated* if $N = \overline{N}$. Note that $\overline{N}/N = \tau(M/N)$ and

$$\mathrm{Sat}_\tau(M) = \{ \, N \mid N \leqslant M, \ N = \overline{N} \, \},$$

so $\mathrm{Sat}_\tau(M)$ is the set of all τ-*saturated* submodules of M, which explains the notation. It is known that $\mathrm{Sat}_\tau(M)$ is an upper continuous modular lattice for any M_R (see [12, Chapter 9, Proposition 4.1]).

For all undefined notation and terminology on torsion theories the reader is referred to [3] and [12].

A module M_R is said to be τ-CC if the lattice $\mathrm{Sat}_\tau(M)$ is CC. More generally if $\mathbb{P}$ is any property on lattices, we say that a module M_R is/has τ-$\mathbb{P}$ if the lattice $\mathrm{Sat}_\tau(M)$ is/has $\mathbb{P}$. Since the lattices $\mathrm{Sat}_\tau(M)$ and $\mathrm{Sat}_\tau(M/\tau(M))$ are canonically isomorphic, we deduce that M_R is τ-$\mathbb{P}$ if and only if $M/\tau(M)$ is τ-$\mathbb{P}$. Thus, we obtain the concepts of a τ-*Artinian* module, τ-*Noetherian* module, τ-*uniform* module, τ-*compact* module, τ-*compactly generated* module, condition τ-(C_i), τ-*quasi-continuous* module, τ-*continuous* module, etc. We say that a submodule N of M_R is/has τ-$\mathbb{P}$ if its τ-saturation $\overline{N}$, which is an element of $\mathrm{Sat}_\tau(M)$, is/has $\mathbb{P}$. Thus, we obtain the concepts of a τ-*pseudo-complement* submodule of a module, τ-*complement* submodule of a module, τ-*essential* submodule of a module, τ-*closed* submodule of a module, τ-*independent* set/family of submodules of a module, etc. Since $\overline{N} = \overline{\overline{N}}$, it follows that N is/has τ-$\mathbb{P}$ if and only if $\overline{N}$ is/has τ-$\mathbb{P}$. In the sequel we shall use the well-established term of a τ-*direct summand* of a module instead of that of a τ-*complement* submodule of a module and of a τ-*CS* module instead of that of a τ-*CC* module.

We present now intrinsic characterizations, that is, without explicitly referring to the lattice $\mathrm{Sat}_\tau(M)$, of the relative module-theoretical concepts involved in the conditions (C_i).

Proposition 4.1. ([2, Proposition 5.3]). *The following assertions hold for a module M_R and a submodule $N \leqslant M$.*

(1) *N is τ-essential in $M \Longleftrightarrow (\forall \, P \leqslant M, \ P \cap N \in \mathcal{T} \Longrightarrow P \in \mathcal{T})$.*

(2) *M is τ-uniform $\Longleftrightarrow (\forall \, P, \, K \leqslant M, \ P \cap K \in \mathcal{T} \Longrightarrow P \in \mathcal{T}$ or $K \in \mathcal{T})$.*

(3) *N is a τ-pseudo-complement in $M \iff \exists\, P \leqslant M$ such that $N \cap P \in \mathcal{T}$ and N is maximal among the submodules of M having this property; in this case $N \in \mathrm{Sat}_\tau(M)$ and $N \cap \overline{P} = \tau(M)$.*

(4) *N is τ-closed in $M \iff$ for any $P \leqslant M$ such that $N \subseteq P$ and N is a τ-essential submodule of P one has $P/N \in \mathcal{T}$. If additionally $N \in \mathrm{Sat}_\tau(M)$, then N is τ-closed in $M \iff N$ has no proper τ-essential extension in M.*

(5) *N is a τ-direct summand in $M \iff \exists\, P \leqslant M$ such that $M/(N+P) \in \mathcal{T}$ & $N \cap P \in \mathcal{T}$.*

(6) *M is τ-complemented $\iff \forall\, N \leqslant M, \exists\, P \leqslant M$ such that $M/(N+P) \in \mathcal{T}$ & $N \cap P \in \mathcal{T}$.*

(7) *A family $(N_i)_{1 \leqslant i \leqslant n}$ of submodules of M is τ-independent $\iff N_i \notin \mathcal{T}, \forall\, i,\ 1 \leqslant i \leqslant n$, and $N_{k+1} \cap \sum_{1 \leqslant j \leqslant k} N_j \subseteq \tau(M), \forall\, k,\ 1 \leqslant k \leqslant n-1$.* □

All the notions and results presented in Sections 1 and 2 for an arbitrary modular lattice L can now be easily specialized for the particular case when $L = \mathrm{Sat}_\tau(M_R)$ using the description from Proposition 4.1 of the relative concepts that intervene in their statements. We present below only three results, and leave the others to the reader.

Proposition 4.2. *A module M_R satisfies the condition τ-(C_{11}) if and only if for every $N \leqslant M$ there exist $D \leqslant M$ and $K \leqslant M$ such that $M/(D+K) \in \mathcal{T}$, $D \cap K \in \mathcal{T}$, $N \cap D \in \mathcal{T}$, and $(N+D) \cap X \notin \mathcal{T}$ for every $X \leqslant M$ with $X \notin \mathcal{T}$.* □

Theorem 4.3. *Let $(N_i)_{i \in I}$ be a τ-independent family of submodules of a module M_R such that all N_i satisfy the condition τ-(C_{11}). Then $\sum_{i \in I} N_i$ satisfies the condition τ-(C_{11}).* □

Proposition 4.4. *If M_R is a module satisfying the condition τ-(C_i), $i = 1, 2, 3$, then any τ-direct summand of M also satisfies the condition τ-(C_i). In particular, any τ-direct summand of a τ-CS module is also τ-CS.* □

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