**INSTITUTUL DE MATEMATICA**
**“SIMION STOILOW”**
**AL ACADEMIEI ROMANE**

PREPRINT SERIES OF THE INSTITUTE OF MATHEMATICS
OF THE ROMANIAN ACADEMY

ISSN 0250 3638

# Some properties of open, discrete, generalized ring mappings

by

Mihai Cristea

Preprint nr. 1/2015

BUCURESTI

# Some properties of open, discrete, generalized ring mapping

by

Mihai Cristea

Preprint nr. 1/2015

January 2015

University of Bucharest Faculty of Mathematics and Computer Sciences, Str. Academiei 14, R-010014 Bucharest, Romania
**E-mail: mcristea@fmi.unibuc.ro**

# Some properties of open, discrete, generalized ring mappings

Mihai Cristea

**Abstract:** We study the properties of open, discrete ring mappings satisfying generalized modular inequalities, namely the equicontinuity, the distortion and the limit mapping of certain homeomorphisms from these classes. Such mappings generalize the known class of quasiregular mappings and their extensions known as mappings of finite distortion. We apply our results to open discrete ring mappings $f : D \subset \overline{\mathbb{R}^n} \to D_f \subset \overline{\mathbb{R}^n}$ satisfying condition $(N)$ and having local $ACL^q$ inverses, and we focus especially on the case $n-1 < q < n$. We show that such mappings cannot have essential singularities and also that Zoric's theorem can hold in this case and in some conditions even if $n = 2$. This is in contrast even with the known case of quasiregular mappings.

*Keywords: generalizations of quasiregular mappings, special classes of Sobolev mappings.*

*AMS 2000 Subject Classification: 30C65.*

## 1 Introduction.

In this paper we continue the research of the properties of mappings satisfying generalized modular inequalities from [7-14] and [31-44].

Given a domain $D \subset \overline{\mathbb{R}^n}$, we denote by $A(D)$ the set of all path families from $D$ and if $\Gamma \in A(D)$, we set $F^{`}(\Gamma) = \{\rho : \mathbb{R}^n \to [0,\infty]$ Borel maps $\mid \int_\gamma \rho ds \geq 1$ for every $\gamma \in \Gamma$ locally rectifiable$\}$. If $D \subset \overline{\mathbb{R}^n}$ is open, $M : A(D) \to [0,\infty]$ is a modulus if:

1) $M(\phi) = 0$.

2) $M(\Gamma_1) \leq M(\Gamma_2)$ if $\Gamma_1 > \Gamma_2$, $\Gamma_1, \Gamma_2 \in A(D)$.

3) $M(\bigcup_{i=1}^{\infty} \Gamma_i) \leq \sum_{i=1}^{\infty} M(\Gamma_i)$ if $\Gamma_1, ..., \Gamma_i, ... \in A(D)$.

Here, if $\Gamma_1, \Gamma_2 \in A(D)$, we say that $\Gamma_1 > \Gamma_2$ if every path $\gamma_1 \in \Gamma_1$ has a subpath in $\Gamma_2$.

We define for $p > 1$ and $\omega : D \to [0,\infty]$ measurable and finite a.e. the $p$-modulus of weight $\omega$

$$M_\omega^p(\Gamma) = \inf_{\rho \in F(\Gamma)} \int_{\mathbb{R}^n} \omega(x)\rho(x)^p dx \; for \; \Gamma \in A(D).$$

For $\omega = 1$ we have the classical $p$ modulus

$$M_p(\Gamma) = \inf_{\rho \in F(\Gamma)} \int_{\mathbb{R}^n} \rho(x)^p dx \; for \; \Gamma \in A(D).$$

A map $f : D \subset \mathbb{R}^n \to \mathbb{R}^n$ is of finite distortion if $f \in C(D, \mathbb{R}^n) \cap W^{1,1}_{loc}(D, \mathbb{R}^n)$, $J_f \in L^1_{loc}(D)$ and there exists $K : D \to [0,\infty]$ measurable and finite a.e. such that $|f^{`}(x)|^n \leq K(x)J_f(x)$

a.e. and if $f \in W^{1,n}_{loc}(D,\mathbb{R}^n)$, we say that $f$ is of finite dilatation. If $K \in L^\infty(D)$, we obtain the known class of quasiregular mappings, and we recommend the reader the books [29, 30, 45, 46] for basic facts concerning quasiregular mappings. The important modular inequality of Poleckii says that if $f$ is $K$-quasiregular, then $M_n(f(\Gamma)) \le KM_n(\Gamma)$ for every $\Gamma \in A(D)$, and this is the key for proving most of the geometric properties of this class of mappings. If $f : D \to G$ is a homeomorphism between two domains from $\mathbb{R}^n$, we say that $f$ is $K$-quasiregular if $\frac{M_n(\Gamma)}{K} \le M_n(f(\Gamma)) \le KM_n(\Gamma)$ for every $\Gamma \in A(D)$. This is equivalent to the fact that $f$ is $ACL^n$, a.e. differentiable and $\frac{|f^{`}(x)|^n}{K} \le |J_f(x)| \le Kl(f^{`}(x))^n$ a.e. and it results that if $f$ is quasiconformal, then $f$ and $f^{-1}$ are $ACL^n$ and satisfy condition $(N)$.

General classes of such mappings were intensively studied using the modulus method in the last 20 years. Important steps in this direction were done by Ryazanov and his students in [15-16], [21], [23-25], [31-44], by Koskela and his students in [18-20], [27-28] and by the author in [3-6]. They extended most of the geometric properties of the known class of quasiregular mappings to this class of mappings. Several conditions were imposed on the dilatation $K$ of the function $f$, like $K \in BMO(D)$, or such that there exists an Orlicz function $A$ such that $\exp(A \circ K) \in L^1_{loc}(D)$, or such that $f$ has locally $ACL^n$ inverses. For all of them the modular inequality "$M_n(f(\Gamma)) \le M^n_{K^{n-1}}(\Gamma)$" holds for every $\Gamma \in A(D)$ and this is the main instrument in studying these functions.

In some recent papers [7-14] we studied classes of continuous, open discrete mappings $f : D \to \mathbb{R}^n$ for which a modular inequality of type "$M_q(f(\Gamma)) \le \gamma(M^p_\omega(\Gamma))$" holds for every $\Gamma \in A(D)$, where $p > 1$, $q > n-1$, $\omega : D \to [0,\infty]$ is measurable and finite a.e. and $\gamma : [0,\infty) \to [0,\infty)$ is increasing with $\lim_{t\to 0} \gamma(t) = 0$. We gave Liouville, Montel, Picard type theorems, equicontinuity and boundary extension results and estimates of the modulus of continuity for such mappings. We extended in this way some of the geometric properties of quasiregular mappings and of their generalizations mentioned before from [3-6], [15-16], [18-21], [23-25], [27-29] and [31-44].

Let now $D \subset \overline{\mathbb{R}^n}$ be open and $M : A(D) \to [0,\infty]$ be a modulus of the form

$$M(\Gamma) = \inf_{\rho \in F(\Gamma)} T(\rho),\ \textit{where}\ T : \mathcal{B}(D) \to [0,\infty]\ \textit{is an operator} \qquad (\alpha)$$

Throughout this paper we shall work with a modulus of this type. We shall have in mind especially the modulus $M = M^p_\omega$, for which the operator $T : \mathcal{B}(D) \to [0,\infty]$ is given by $T(\rho) = \int_D \omega(x)\rho(x)^p dx$ for every $\rho \in \mathcal{B}(D)$, but we can consider more general operators $T$, like $T(\rho) = \int_D \omega(x)\Phi(\rho(x))dx$ for every $\rho \in \mathcal{B}(D)$, where $\Phi : [0,\infty) \to [0,\infty)$ is a homeomorphism, or like $T(\rho) = \int_D \omega(x)\rho(x)^{\Phi(x)}dx$ for every $\rho \in \mathcal{B}(D)$, where $\Phi : D \to [1,\infty]$ is Borel measurable (see also Theorem 1 in [9] for some more general cases).

If $D \subset \overline{\mathbb{R}^n}$ is open, $E, F \subset \overline{D}$, we set $\Delta(E,F,D) = \{\gamma : [0,1] \to \overline{D}$ path $|\gamma(0) \in E, \gamma(1) \in F$ and $\gamma((0,1)) \subset D\}$. We say that $\infty \in D$ if there exists $r > 0$ such that $\complement B(0,r) \subset D$.

A domain $A \subset \overline{\mathbb{R}^n}$ is a ring if $\complement A$ has exactly two components $Q_0$ and $Q_1$ and we denote this thing by $A = R(Q_0,Q_1)$. Then $\partial A$ has exactly two components $B_0 = Q_0 \cap \overline{A}$ and $B_1 = Q_1 \cap \overline{A}$ and we associate to the ring $A$ the path family $\Gamma_A = \Delta(B_0,B_1,A)$.

If $x \in \overline{\mathbb{R}^n}$ and $0 < a < b$, we set

$\Gamma_{x,a,b} = \Delta(\overline{B}(x,a), S(x,b), B(x,b) \setminus \overline{B}(x,a))$ if $x \in \mathbb{R}^n$.

$\Gamma_{\infty,a,b} = \Delta(\overline{B}(0,a), S(0,b), B(0,b) \setminus \overline{B}(0,a))$ if $x = \infty$.

$L_{x,a,b} = \{\rho : \mathbb{R}^n \to [0,\infty] |$ there exists $\eta : (a,b) \to [0,\infty]$ a Borel map such that $\int\limits_a^b \eta(t)dt \geq 1$, $\rho(z) = \eta(|z-x|)$ if $z \in B(x,b) \setminus \overline{B}(x,a)$, $\rho(z) = 0$ otherwise$\}$.

$L_{\infty,a,b} = \{\rho : \mathbb{R}^n \to [0,\infty]$ there exists $\eta : (a,b) \to [0,\infty]$ a Borel map such that $\int\limits_a^b \eta(t)dt \geq 1, \rho(z) = \eta(|z|)$ if $z \in B(0,b) \setminus \overline{B}(0,a), \rho(z) = 0$ otherwise$\}$.

If $M$ is a modulus as in $(\alpha)$, we set

$$\Delta_M(\Gamma_{x,a,b}) = \inf_{\rho \in L_{x,a,b}} T(\rho).$$

Since $L_{x,a,b} \subset F(\Gamma_{x,a,b})$, we see that $\Delta_M(\Gamma_{x,a,b}) \geq M(\Gamma_{x,a,b})$. If $M = M_\omega^p$, we set $\Delta_M = \Delta_\omega^p$ and if $M = M_p$, we set $\Delta_M = \Delta_p$. We say that $M(x) = 0$ if $M(\Gamma_x) = 0$, where $\Gamma_x = \{\gamma$ path $|x \in \overline{Im\gamma}\}$. We say that $\Delta_M(x) = 0$ if there exists $c > 0$ such that $B(x,c) \subset D$ and $\lim\limits_{a\to 0} \Delta_M(\Gamma_{x,a,b}) = 0$ for every $0 < a < b < c$ if $x \neq \infty$ and we say that $\Delta_M(\infty) = 0$ if there exists $c > 0$ such that $\complement B(0,c) \subset D$ and $\lim\limits_{a\to\infty} \Delta_M(\Gamma_{\infty,b,a}) = 0$ for every $0 < c < b < a$. If $\Delta_M(x) = 0$, then $M(x) = 0$ and we say that $\Delta_M(x) > 0$ if $\Delta_M(x) \neq 0$.

If $n \geq 2$, $q > 1$, $\gamma : [0,\infty) \to [0,\infty)$ is increasing with $\lim\limits_{t\to 0} \gamma(t) = 0$, $x \in \partial D$ is an isolated point of $\partial D$ and $f : D \to \overline{\mathbb{R}^n}$ is a map, we say that $f$ satisfies a ring $(q, M, \gamma)$ condition in $x$ if

$M_q(f(\Gamma_{x,a,b})) \leq \gamma(\Delta_M(\Gamma_{x,a,b}))$ for every $0 < a < b$ such that $\overline{B}(x,b) \subset D$ if $x \neq \infty$

$M_q(f(\Gamma_{\infty,a,b})) = \gamma(\Delta_M(\Gamma_{\infty,a,b}))$ for every $0 < a < b$ such that $\complement B(0,a) \subset D$ if $x = \infty$.

We say that $f$ satisfies a generalized ring $(q, M, \gamma)$ condition in $x$ if $M_q(f(\Gamma_A)) \leq \gamma(M(\Gamma_A))$ for every ring $A$ such that $x \notin \overline{A}$ and $\overline{A}$ is compact in $D \cap \mathbb{R}^n$.

We say that $f : D \subset \overline{\mathbb{R}^n} \to \overline{\mathbb{R}^n}$ satisfies a ring $(q, M, \gamma)$ condition if $f$ satisfies a ring $(q, M, \gamma)$ condition in every point $x \in D$, and we say that $f$ satisfies a generalized ring $(q, M, \gamma)$ condition if $f$ satisfies a generalized ring $(q, M, \gamma)$ condition in every point $x \in D$.

If $x = (x_1, ..., x_n) \in \mathbb{R}^n$, we set $|x| = (\sum\limits_{i=1}^n x_i^2)^{\frac{1}{2}}$ and if $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$, we set $|A| = \sup\limits_{|x|=1} |A(x)|$, $l(A) = \inf\limits_{|x|=1} |A(x)|$. We denote by $\mu_n$ the Lebesgue measure in $\mathbb{R}^n$ and if $p > 1$, we denote by $m_p$ the $p$-Hansdorff measure in $\mathbb{R}^n$.

If $D \subset \mathbb{R}^n$ is open and $f : D \to \mathbb{R}^n$ is a map, we say that $f$ satisfies condition $(N)$ if $\mu_n(f(A)) = 0$ whenever $A \subset D$ is such that $\mu_n(A) = 0$. We also denote for $x \in D$ by $L(x,f) = \limsup\limits_{h\to 0} \frac{|f(x+h)-f(x)|}{|h|}$ and we set $B_f = \{x \in D | f$ is not a local homeomorphism at $x\}$.

If $D \subset \mathbb{R}^n$ is open, we say that $\varphi : \mathcal{B}(D) \to [0,\infty]$ is a set function if $\varphi(A) < \infty$ for every compact $A \subset D$ and $\varphi(\bigcup\limits_{i=1}^\infty A_i) = \sum\limits_{i=1}^\infty \varphi(A_i)$, if $A_1, ..., A_i, ...$ are disjoint Borel sets in $D$. We say $\varphi$ has a derivative $\varphi^{'}(x)$ in a point $x \in D$ if there exists $\varphi^{'}(x) = \lim\limits_{r\to 0} \frac{\varphi(B(x,r))}{\mu_n(B(x,r))}$. A set function $\varphi$ has a.e. a derivative $\varphi^{'}(x)$ and the function $\varphi^{'}$ is Borel measurable.

If $D, G$ are domains in $\mathbb{R}^n$ and $f : D \to G$ is a homeomorphism, we set $\mu_f : \mathcal{B}(D) \to [0,\infty]$ by $\mu_f(A) = \mu_n(f(A))$ for every $A \in \mathcal{B}(D)$. Then $\mu_f$ is a set function and if $h : \mathcal{B}(\mathbb{R}^n) \to [0,\infty]$ is Borel measurable and $f$ satisfies condition $(N)$, then $\int\limits_A h(f(x))\mu_f^{'}(x)dx = \int\limits_{f(A)} h(y)dy$ for every $A \in \mathcal{B}(D)$ (see [45], page 81-83).

Let now $q > 1$, $D \subset \mathbb{R}^n$ open, $E \subset D$ closed in $D$ such that $\mu_n(E) = 0$ and $f : D \setminus E \to \mathbb{R}^n$ a local homeomorphism. We can define a.e. the function $K_{I,q}(f) : D \to [0,\infty]$ by $K_{I,q}(f)(x) = L(f(x), g_x)\mu_f^{'}(x)$. Here $g_x$ is a local inverse of $f$ around $x$ such that $g_x(f(x)) = x$. Then $K_{I,q}(f)$

is a Borel map and if $x \in D \setminus E$ is such that $f$ is differentiable in $x$ and $J_f(x) \neq 0$, then $K_{I,q}(f)(x) = \frac{|J_f(x)|}{l(f^{`}(x))^q}$.

The following Theorem is proved in Theorem 1 in [12].

**Theorem A.** Let $n \geq 2$, $1 < q < p$, $D \subset \mathbb{R}^n$ a domain, $f : D \to \mathbb{R}^n$ continuous and satisfying condition $(N)$ such that $m_1(B_f) = 0$ and having $ACL^q$ inverses on $f(D \setminus B_f)$. Then $M_q(f(\Gamma)) \leq M^q_{K_{I,q}(f)}(\Gamma)$ for every $\Gamma \in A(D)$ and $M_q(f(\Gamma)) \leq (\int\limits_D K_{I,q}(f)(x)^{p/(p-q)}dx)^{\frac{p-q}{p}} M_p(\Gamma)^{q/p}$ for every $\Gamma \in A(D)$.

Using this result, we can easily produce mappings which are not quasiregular and satisfying generalized modular inequalities used in this paper.

Indeed, let $D = (0,1)^n$, $\frac{n-1}{n} < \frac{q}{n} < \alpha < 1$, $f : (0,1)^2 \to (0, \frac{1}{\alpha})^2$ given by $f(x,y) = (\frac{x^\alpha}{\alpha}, \frac{y^\alpha}{\alpha})$ for $x, y \in (0,1)$ and let $F : D \to (0,1)^{n-2} \times (0, \frac{1}{\alpha})^2$ given by
$F(x_1, ..., x_n) = (x_1, ..., x_{n-2}, f(x_{n-1}, x_n))$ for $x = (x_1, ..., x_n) \in D$.

We see that $F$ is a homeomorphism, $l(f^{`}(x_{n-1}, x_n)) = \min\{x_{n-1}^{\alpha-1}, x_n^{\alpha-1}\}$ if $x_{n-1}, x_n \in (0,1)$ and $l(F^{`}(x)) = \min\{1, l(f^{`}(x_{n-1}, x_n))\} = 1$, $J_f(x) = x_{n-1}^{\alpha-1} x_n^{\alpha-1}$ if $x \in D$.

We find that $K_{I,n}(F)(x) = x_{n-1}^{\alpha-1} x_n^{\alpha-1} \to \infty$ if $x \to 0$, hence $F$ is not quasiconformal. Also, $K_{I,q}(F)(x) = x_{n-1}^{\alpha-1} x_n^{\alpha-1}$ for $x \in D$ and let $C = (\int\limits_D K_{I,q}(F)(x)^{n/(n-q)}dx)^{\frac{n-q}{n}}$. Then $C = (\int\limits_0^1 dx_1 ... \int\limits_0^1 dx_{n-2} \int\limits_0^1 x_{n-1}^{\frac{n(\alpha-1)}{n-q}} dx_{n-1} \int\limits_0^1 x_n^{\frac{n(\alpha-1)}{n-q}} dx_n)^{\frac{n-q}{n}} < \infty$.

Using Theorem A, we see that $M_q(f(\Gamma)) \leq C M_n(\Gamma)^{q/n}$ for every $\Gamma \in A(D)$, hence $F$ is a generalized ring $(q, M_n, \gamma)$ homeomorphism, where $\gamma(t) = Ct^{q/n}$ for $t > 0$.

Another interesting example of an open, discrete map $F : B(0,1) \to \mathbb{R}^n$ such that $B_F \neq \phi$ and $K_{I,q}(F)(x) = m$ for every $x \in B(0,1)$, $n-1 < q < n$ and for which the modular inequality "$M_q(f(\Gamma)) \leq m M_q(\Gamma)$" holds for every $\Gamma \in A(B(0,1))$ is given in [15].

One of the main result of this paper shows that in the case $n - 1 < q < n$, continuous, open discrete ring $(q, M, \gamma)$ mappings in an isolated point $x \in \partial D$ cannot have an essential singularity in $x$ if $\Delta_M(x) = 0$. This is in contrast with the case $q = n$ and even with the known case of analytic mapping.

**Theorem 1.** Let $n \geq 2$, $n-1 < q < n$, $D \subset \overline{\mathbb{R}^n}$ a domain, $x \in \partial D$ an isolated point of $\partial D$, let $f : D \to \overline{\mathbb{R}^n}$ be continuous, open, discrete and suppose that $f$ satisfies a ring $(q, M, \gamma)$ condition in $x$ and $\Delta_M(x) = 0$. Then there exists $\lim\limits_{z \to x} f(z) = l \in \mathbb{R}^n$.

**Corollary 1.** Let $n \geq 2$, $n-1 < q < p \leq n$, $D \subset \overline{\mathbb{R}^n}$ a domain, $x \in \partial D$ an isolated point of $\partial D$, let $f : D \to \overline{\mathbb{R}^n}$ be continuous, open discrete, satisfying condition $(N)$ such that $m_1(B_f) = 0$ and having local $ACL^q$ inverses on $f(D \setminus B_f)$. Suppose that one of the following conditions hold:

1) $\Delta^q_{K_{I,q}(f)}(x) = 0$.

2) $\int\limits_D K_{I,q}(f)(x)^{p/(p-q)}dx < \infty$ and $p = n$ if $x = \infty$.

Then there exists $\lim\limits_{z \to x} f(z) = l \in \mathbb{R}^n$.

The known of Zoric says that if $f : \mathbb{R}^n \to \mathbb{R}^n$ is quasiregular and a local homeomorphism and $n \geq 3$, then $f$ is a homeomorphism, but the result is false in the case $n = 2$, as $\exp : \mathbb{C} \to \mathbb{C}$ shows. We also see that $\Delta_n(\infty) = 0$.

In the case $n-1 < q < n$ we cannot have such a phenomenon in the case of ring $(q, M, \gamma)$ local homeomorphisms $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $\Delta_M(\infty) = 0$, because such mappings do not exist, as we can see from the following theorem:

**Theorem 2.** Let $n \geq 2$, $n-1 < q < n$. Then there exists no continuous, open, discrete ring $(q, M, \gamma)$ mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ at $\infty$ such that $\Delta_M(\infty) = 0$.

**Corollary 2.** Let $n \geq 2$, $n-1 < q < n$. Then there exists no continuous, open discrete mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ satisfying condition $(N)$, such that $m_1(B_f) = 0$, having local $ACL^q$ inverses on $f(D \setminus B_f)$ and such that either $\Delta^q_{K_{I,q}(f)}(\infty) = 0$, or $\int\limits_{\mathbb{R}^n} K_{I,q}(f)(x)^{n/(n-q)}dx < \infty$.

On the other side, if we permit the mapping $f : \mathbb{R}^n \to \overline{\mathbb{R}^n}$ to take the value $\infty$, Zoric's theorem is valid even in the case $n = 2$ for ring $(q, M, \gamma)$ local homeomorphisms such that $\Delta_M(\infty) = 0$ and $n - 1 < q < n$.

**Theorem 3.** Let $n \geq 2$, $n-1 < q < n$ and $f : \mathbb{R}^n \to \overline{\mathbb{R}^n}$ be a ring $(q, M, \gamma)$ local homeomorphism at $\infty$ such that $\Delta_M(\infty) = 0$. Then $f$ extends to a homeomorphism $F : \overline{\mathbb{R}^n} \to \overline{\mathbb{R}^n}$ such that $F(\infty) \in \mathbb{R}^n$.

**Corollary 3.** Let $n \geq 2$, $n-1 < q < n$, $f : \mathbb{R}^n \to \overline{\mathbb{R}^n}$ be a local homeomorphism satisfying condition $(N)$ and having local $ACL^q$ inverses and suppose that either $\Delta^q_{K_{I,q}(f)}(\infty) = 0$, or $\int\limits_{\mathbb{R}^n} K_{I,q}(f)(x)^{n/(n-q)}dx < \infty$. Then $f$ extends to a homeomorphism $F : \overline{\mathbb{R}^n} \to \overline{\mathbb{R}^n}$ such that $F(\infty) \in \mathbb{R}^n$ and if $x \in \mathbb{R}^n$ is such that $f(x) = \infty$, then $\Delta^q_{K_{I,q}(f)}(x) > 0$.

We see for instance that if $f : \overline{\mathbb{R}^n} \to \overline{\mathbb{R}^n}$ is given by $f(x) = x$ for $x \in \mathbb{R}^n$, then, if $n-1 < q < n$, we have that $K_{I,q}(f)(x) = 1$ for every $x \in \mathbb{R}^n$ and $\Delta^q_{K_{I,q}(f)}(\infty) = \infty$. We shall also show that if $f : \overline{\mathbb{R}^n} \to \overline{\mathbb{R}^n}$ is given by $f(x) = \frac{x}{|x|^2}$ if $x \neq 0$, $f(0) = \infty$, $f(\infty) = 0$, then $\Delta^q_{K_{I,q}(f)}(0) = \infty$ and $\Delta^q_{K_{I,q}(f)}(\infty) = 0$.

If $x$ is a boundary point of a domain in $D \neq \mathbb{R}^n$, we have:

**Theorem 4.** Let $n \geq 3$, $n-1 < q < n$, $D \subset \overline{\mathbb{R}^n}$ a domain, $x \in \partial D$ an isolated point of $\partial D$, and let $f : D \to \overline{\mathbb{R}^n}$ be a ring $(q, M, \gamma)$ local homeomorphism in $x$ such that $\Delta_M(x) = 0$. Then there exists $F : D \cup \{x\} \to \overline{\mathbb{R}^n}$ a local homeomorphism around $x$ such that $F|D = f$ and $F(\infty) \in \mathbb{R}^n$.

**Corollary 4.** Let $n \geq 3$, $n-1 < q < p \leq n$, $D \subset \overline{\mathbb{R}^n}$ a domain, $x \in \partial D$ an isolated point of $\partial D$, let $f : D \to \overline{\mathbb{R}^n}$ be a local homeomorphism satisfying condition $(N)$ and having local $ACL^q$ inverses and suppose that one of the following conditions hold:

1) $\Delta^q_{K_{I,q}(f)}(x) = 0$.

2) $(\int\limits_D K_{I,q}(f)(x)^{p/(p-q)}dx)^{\frac{p-q}{p}} = K < \infty$ and $p = n$ if $x = \infty$.

Then there exists $F : D \cup \{x\} \to \overline{\mathbb{R}^n}$ a local homeomorphism around $x$ such that $F|D = f$ and $F(\infty) \in \mathbb{R}^n$.

Using the ideas from Lemma 2.1 in [15] and [16], we prove an equicontinuity result for a family $W$ of continuous, open discrete ring $(q, M, \gamma)$ mappings $f : D \subset \overline{\mathbb{R}^n} \to \overline{\mathbb{R}^n}$, $n-1 < q < n$, extending the result proved in [15] and [16], for $M = M^q_\omega$ and $\gamma(t) = t$ for $t > 0$. We show that we don't need to impose to each mapping $f \in W$ to avoid a set of positive $q$ capacity, and this in contrast with the case $q = n$ and even with the known case of a family $W$ of $K$-quasiregular mappings.

**Theorem 5.** Let $n \geq 2$, $n-1 < q < n$, $D \subset \overline{\mathbb{R}^n}$ a domain, $x \in \partial D$ an isolated point of $\partial D$ and let $W$ be a family of continuous, open, discrete ring $(q, M, \gamma)$ mappings $f : D \to \overline{\mathbb{R}^n}$ in $x$ such that $\Delta_M(x) = 0$. Then each function $f \in W$ can be extended by continuity to a function $f_x : D \cup \{x\} \to \overline{\mathbb{R}^n}$ and the family $W_x = (f_x)_{f \in W}$ is equicontinuous at $x$, and we take on $D$ and on $\overline{\mathbb{R}^n}$ the chordal metric.

**Corollary 5.** Let $n \geq 2$, $n-1 < q < p \leq n$, $D \subset \overline{\mathbb{R}^n}$ a domain, $x \in \partial D$ an isolated point of $\partial D$ and let $W$ be a family of continuous, open discrete mappings $f : D \to D_f \subset \overline{\mathbb{R}^n}$ satisfying condition $(N)$ and having $ACL^q$ inverses on $f(D \setminus B_f)$ and such that $m_1(B_f) = 0$. Suppose that one of the following conditions hold:

1) $\Delta^q_{K_{I,q}(f)}(x) = 0$ for every $f \in W$ and there exists $\omega : D \to [0, \infty]$ measurable and finite a.e. such that $K_{I,q}(f) \le \omega$ for every $f \in W$.

2) There exists $0 < C < \infty$ such that $(\int\limits_D K_{I,q}(f)(x)^{p/(p-q)}dx)^{\frac{p-q}{p}} < C$ for every $f \in W$ and $p = n$ if $x = \infty$.

Then each function $f \in W$ can be extended by continuity to a function $f_x : D \cup \{x\} \to \overline{\mathbb{R}^n}$ and the family $W_x = (f_x)_{f \in W}$ is equicontinuous at $x$, and we take on $D$ and on $\overline{\mathbb{R}^n}$ the chordal metric.

A large number of results are devoted to find estimates of the modulus of continuity of certain classes of mappings of finite distortion (see for instance [1], [18] or [22]). We give here some estimates of the modulus of continuity of some continuous, open discrete ring $(q, M, \gamma)$ mappings, $n - 1 < q < n$, extending in Theorem 6 some results from [15] and [16] given for $M = M^q_\omega$ and $\gamma(t) = t$ for $t > 0$.

**Theorem 6.** Let $n \ge 2$, $n - 1 < q < n$, $p > 1$, $D \subset \mathbb{R}^n$ a domain, $x \in D$, $d = d(x, \partial D)$, $\omega : D \to [0, \infty]$ measurable and finite a.e., $\gamma : [0, \infty) \to [0, \infty)$ increasing with $\lim\limits_{t \to 0} \gamma(t) = 0$, let $f : D \to \mathbb{R}^n$ be continuous, open, discrete satisfying a ring $(q, M^p_\omega, \gamma)$ condition in $x$ such that $f(D) \subset B(0, r)$ and suppose that one of the following conditions is satisfied:

a) There exists $M > 0$ and $0 \le \alpha < p - 1$ such that $\int\limits_{B(x,\delta)} \omega(z)dz \le M\delta^p(\ln(\frac{de}{\delta}))^\alpha$ for every $0 < \delta < d$.

b) $\omega \in L^{n/(n-p)}(D)$.

Then, if condition 1) holds, it results that

$$|f(y) - f(x)| \le (\frac{(V_n)^{1-n+q}}{C_1})^{\frac{1}{q}} r^{\frac{n(1-n+q)}{q}} (\gamma(Me^p \sum_{k=1}^{\infty} \frac{1}{k^{p-\alpha}} /$$

$$/(\ln\ln(\frac{de}{|y-x|}))^p)^{\frac{n-1}{q}} \ for\ every\ 0 < |y - x| < d \tag{1}$$

and if $\rho > 0$ is small enough, then

$$|f(y) - f(x)| \le (\gamma(Me^p \sum_{k=1}^{\infty} \frac{1}{k^{p-\alpha}} /(\ln\ln(\frac{de}{|y-x|}))^p)^{\frac{n-1}{q}} \ if\ 0 < |y - x| < \rho \tag{2}$$

If condition b) holds, then

$$|f(y) - f(x)| \le (\frac{(V_n)^{1-n+q}}{C_1})^{\frac{1}{q}} r^{\frac{n(1-n+q)}{q}} (\gamma((\omega_{n-1})^{p/n}(\|\omega^{n/(n-p)}\|_{B(x,d)})^{\frac{n-p}{n}}$$

$$(\ln(\frac{d}{|y-x|}))^{\frac{p(1-n)}{n}})^{\frac{n-1}{q}} \ for\ every\ 0 < |y - x| < d \tag{3}$$

and for $\rho > 0$ small enough, we have

$$|f(y) - f(x)| \le (\gamma((\omega_{n-1})^{p/n}(\|\omega^{n/(n-p)}\|_{B(x,d)})^{\frac{n}{n-p}}(\ln(\frac{\rho}{|y-x|}))^{\frac{p(1-n)}{n}})^{\frac{n-1}{q}} \ if\ 0 < |y - x| < \rho \tag{4}$$

**Corollary 6.** Let $n \ge 2$, $n - 1 < q < n$, $D \subset \mathbb{R}^n$ a domain, $x \in D$, $d = d(x, \partial D)$, let $f : D \to \mathbb{R}^n$ be continuous, open discrete, satisfying condition $(N)$ such that $m_1(B_f) = 0$ and having local $ACL^q$ inverses on $f(D \setminus B_f)$ and $f(D) \subset B(0,r)$ and suppose that one of the following conditions is satisfied:

a) There exists $M > 0$ and $0 \le \alpha < q-1$ such that $\int\limits_{B(x,\delta)} K_{I,q}(f)(z)dz \le M\delta^q(\ln(\frac{de}{\delta}))^\alpha$ for every $0 < \delta < d$.

b) $K_{I,q}(f) \in L^{n/(n-q)}(D)$.

Then, if condition a) holds, it results that

$$|f(y)-f(x)| \le (\frac{(V_n)^{1-n+q}}{C_1})^{\frac{1}{q}} r^{\frac{n(1-n+q)}{q}} (Me^q \sum_{k=1}^{\infty} \frac{1}{k^{q-\alpha}} /$$

$$/(\ln\ln(\frac{de}{|y-x|}))^q)^{\frac{n-1}{q}} \; for\ every\ 0 < |y-x| < d \qquad (1)$$

and if $\rho > 0$ is small enough, then

$$|f(y)-f(x)| \le 1/\ln\ln(\frac{\rho e}{|y-x|}))^{n-1} \; if\ 0 < |y-x| < \rho \qquad (2)$$

If condition b) holds, then

$$|f(y)-f(x)| \le (\frac{(V_n)^{1-n+q}}{C_1})^{\frac{1}{q}} r^{\frac{n(1-n+q)}{q}} ((\omega_{n-1})^{q/n} (\|\omega^{n/(n-q)}\|_{B(x,d)})^{\frac{n-q}{n}}$$

$$(\ln(\frac{d}{|y-x|}))^{\frac{q(1-n)}{n}})^{\frac{n-1}{q}} \; for\ every\ 0 < |y-x| < d \qquad (3)$$

and if $\rho > 0$ is small enough, then

$$|f(y)-f(x)| \le 1/(\ln(\frac{\rho}{|y-x|}))^{\frac{(n-1)^2}{n}} \; if\ 0 < |y-x| < \rho \qquad (4)$$

We also estimate the behaviour at infinite of some open, discrete ring $(q, M, \gamma)$ mappings, $n-1 < q < n$.

**Theorem 7.** Let $n \ge 2$, $n-1 < q < n$, $p > 1$, $\lambda > 0$, $D \subset \overline{\mathbb{R}^n}$ a domain such that $\complement B(0,\lambda) \subset D$, let $\omega : D \to [0,\infty]$ be measurable and finite a.e., $\gamma : [0,\infty) \to [0,\infty)$ increasing with $\lim\limits_{t\to 0}\gamma(t) = 0$, let $f : D \to \mathbb{R}^n$ be continuous, open discrete, satisfying a ring $(q, M^p_\omega, \gamma)$ condition at $\infty$ and $f(D) \subset B(0,r)$ and suppose that one of the following conditions hold:

a) There exists $M > 0$ and $0 \le \alpha < p-1$ such that $\int\limits_{B(0,\delta)\setminus B(0,\lambda)} \omega(z)dz \le M\delta^p(\ln\delta)^\alpha$ for every $\delta > \lambda$.

b) $\int\limits_{\complement B(0,\lambda)} \omega(z)^{n/(n-p)}dz < \infty$.

Then, if condition a) holds and $C = Me^p 2^\alpha((\ln\lambda)^\alpha \sum\limits_{k=1}^{\infty} \frac{1}{k^p} + \sum\limits_{k=1}^{\infty} \frac{1}{k^{p-\alpha}})$, it results that there exists $l \in \mathbb{R}^n$ such that

$$|f(y)-l| \le (\frac{(V_n)^{1-n+q}}{C_1})^{\frac{1}{q}} r^{\frac{n(1-n+q)}{q}} (\gamma(C/(\ln\ln(\frac{|y|e}{\lambda}))^p)^{\frac{n-1}{q}} \; if\ |y| > \lambda \qquad (1)$$

and if $\rho$ is great enough, then

$$|f(y)-l| \le \gamma(C/\ln\ln(\frac{|y|e}{\rho}))^p)^{\frac{n-1}{q}} \; if\ |y| > \rho \qquad (2)$$

If condition b) holds, there exists $l \in \mathbb{R}^n$ such that

$$|f(y)-l| \leq (\frac{(V_n)^{1-n+q}}{C_1})^{\frac{1}{q}} r^{\frac{n(1-n+q)}{q}} (\gamma((\omega_{n-1})^{p/n}$$

$$(\|\omega^{n/(n-p)}\|_{\complement B(0,\lambda)})^{\frac{n-p}{n}} (\ln(\frac{|y|}{\lambda}))^{\frac{p(1-n)}{n}})^{\frac{n-1}{q}} \; if \; |y| > \lambda \; (3)$$

and if $\rho$ is great enough, then

$$|f(y)-l) \leq (\gamma((\omega_{n-1})^{p/n}(\|\omega^{n/(n-p)}\|_{\complement B(0,\lambda)})^{\frac{n-p}{n}} (\ln(\frac{|y|}{\lambda}))^{\frac{p(1-n)}{n}})^{\frac{n-1}{q}} \; if \; |y| > \rho \qquad (4)$$

**Corollary 7.** Let $n \geq 2$, $n-1 < q < n$, $\lambda > 0$, $D \subset \overline{\mathbb{R}^n}$ a domain such that $\complement B(0,\lambda) \subset D$, let $f : D \to \mathbb{R}^n$ be continuous, open discrete, satisfying condition $(N)$ such that $m_1(B_f) = 0$, having local $ACL^q$ inverses on $f(D \backslash B_f)$ and $f(D) \subset B(0,r)$ and suppose that one the following conditions is satisfied:

a) There exists $M > 0$ and $0 \leq \alpha < q-1$ such that $\int\limits_{B(0,\delta)\backslash \overline{B(0,\lambda)}} K_{I,q}(f)(x)dx \leq M\delta^q(\ln\delta)^\alpha$ for every $\delta > \lambda$.

b) $\int\limits_{\complement B(0,\lambda)} K_{I,q}(f)(x)^{n/(n-q)}dx < \infty$.

Then, if condition a) holds and $C = Me^q 2^\alpha((\ln\lambda)^\alpha \sum\limits_{k=1}^{\infty} \frac{1}{k^q} + \sum\limits_{k=1}^{\infty} \frac{1}{k^{q-\alpha}})$, it results that there exists $l \in \mathbb{R}^n$ such that

$$|f(y)-l| \leq (\frac{(V_n)^{1-n+q}}{C_1})^{\frac{1}{q}} r^{\frac{n(1-n+q)}{q}} (C/(\ln\ln(\frac{|y|e}{\lambda}))^q)^{\frac{n-1}{q}} \; if \; |y| > \lambda \qquad (1)$$

and if $\rho$ is great enough, then

$$|f(y)-l| \leq 1/(\ln\ln(\frac{|y|e}{\rho}))^{n-1} \; if \; |y| > \rho \qquad (2)$$

If condition b) holds, there exists $l \in \mathbb{R}^n$ such that

$$|f(y)-l| \leq (\frac{(V_n)^{1-n+q}}{C_1})^{\frac{1}{q}} r^{\frac{n(1-n+q)}{q}} ((\omega_{n-1})^{q/n}(\|K_{I,q}(f)^{n/(n-q)}\|_{\complement B(0,\lambda)})^{\frac{n-q}{n}}$$

$$(\ln(\frac{|y|}{\lambda}))^{\frac{q(1-n)}{n}})^{\frac{n-1}{q}} \; if \; |y| > \lambda \qquad (3)$$

and if $\rho$ is great enough, then

$$|f(y)-l| \leq 1/(\ln(\frac{|y|}{\rho}))^{\frac{(n-1)^2}{n}} \; if \; |y| > \rho \qquad (4)$$

If the function $f$ is a ring homeomorphism, we have some better estimates and in this case we don't need the boundedness of $f$ in the calculus of the modulus of continuity of $f$.

We also don't need in the preceding theorems the boundedness of $f$ for the estimates of the modulus of continuity of $f$ if $0 < |y-x| < \rho$ is small enough.

**Theorem 8.** Let $n \geq 2$, $n-1 < q < n$, $p > 1$, $D \subset \mathbb{R}^n$ a domain, $x \in D$, $d = d(x, \partial D)$, $\omega : D \to [0,\infty]$ be measurable and finite a.e., $\gamma : [0,\infty) \to [0,\infty)$ increasing with $\lim\limits_{t\to 0}\gamma(t) = 0$, let $f : D \to D_f \subset \mathbb{R}^n$ be a ring $(q, M_\omega^p, \gamma)$ homeomorphism in $x$ and suppose that one of the following conditions hold:

a) There exists $M > 0$ and $0 \le \alpha < p-1$ such that $\int\limits_{B(x,\delta)} \omega(z)dz \le M\delta^p(\ln(\frac{de}{\delta}))^\alpha$ for every $0 < \delta < d$.

b) $\int\limits_D \omega(z)^{n/(n-p)}dz < \infty$.

Then, if condition a) holds, it results that

$$|f(y) - f(x)| \le (\frac{1}{C_0}\gamma(Me^p\sum_{k=1}^{\infty}\frac{1}{k^{p-\alpha}}/(\ln\ln(\frac{de}{|y-x|}))^p)^{\frac{1}{n-q}}\ if\ 0 < |y-x| < d \quad (1)$$

If condition b) holds, it results that

$$|f(y) - f(x)| \le (\frac{1}{C_0}\gamma((\omega_{n-1})^{p/n}(\|\omega^{n/(n-p)}\|_{B(x,d)})^{\frac{n-p}{n}}/$$

$$/(\ln(\frac{d}{|y-x|}))^{\frac{p(1-n)}{n}})^{\frac{1}{n-q}}\ if\ 0 < |y-x| < d\ (2)$$

**Corollary 8.** Let $n \ge 2$, $n-1 < q < n$, $D \subset \mathbb{R}^n$ a domain, $x \in D$, $d = d(x, \partial D)$ and let $f : D \to D_f \subset \mathbb{R}^n$ be a homeomorphism satisfying condition $(N)$ such that its inverse is $ACL^q$ and suppose that one of the following conditions hold:

a) There exists $M > 0$ and $0 \le \alpha < q-1$ such that $\int\limits_{B(x,\delta)} K_{I,q}(f)(z)dz \le M\delta^q(\ln(\frac{de}{\delta}))^\alpha$ for every $0 < \delta < d$.

b) $\int\limits_D K_{I,q}(f)(z)^{n/(n-q)}dz < \infty$.

Then, if condition a) holds, it results that

$$|f(y) - f(x)| \le (\frac{Me^q}{C_0}\sum_{k=1}^{\infty}\frac{1}{k^{q-\alpha}}/(\ln\ln(\frac{de}{|y-x|}))^q)^{\frac{1}{n-q}}\ if\ 0 < |y-x| < d \quad (1)$$

If condition b) holds, it results that

$$|f(y) - f(x)| \le (\frac{(\omega_{n-1})^{q/n}}{C_0}(\|K_{I,q}(f)^{n/(n-q)}\|_{B(x,d)})^{\frac{n-q}{n}}/(\ln(\frac{d}{|y-x|}))^{\frac{q(n-1)}{n}})^{\frac{1}{n-q}}$$

for every $0 < |y-x| < d$.

We also have some better estimates of the behaviour at infinite of ring $(q, M_\omega^p, \gamma)$ homeomorphisms at $\infty$, $n-1 < q < n$ and also in this case we don't need the boundedness of $f$.

**Theorem 9.** Let $n \ge 2$, $n-1 < q < n$, $p > 1$, $\lambda > 0$, $D \subset \overline{\mathbb{R}^n}$ a domain such that $\complement B(0,\lambda) \subset D$, let $\omega : D \to [0,\infty]$ be measurable and finite a.e, $\gamma : [0,\infty) \to [0,\infty)$ increasing with $\lim\limits_{t\to 0}\gamma(t) = 0$, let $f : D \to D_f \subset \mathbb{R}^n$ be a ring $(q, M_\omega^p, \gamma)$ homeomorphism at $\infty$ and suppose that one of the following conditions hold:

a) There exists $M > 0$ and $0 \le \alpha < p-1$ such that $\int\limits_{B(0,\delta)\setminus\overline{B}(0,\lambda)} \omega(z)dz \le M\delta^p(\ln\delta)^\alpha$ for every $\delta > \lambda$.

b) $\int\limits_{\complement B(0,\lambda)} \omega(z)^{n/(n-p)}dz < \infty$.

Then, if condition a) holds and $C = Me^p 2^\alpha ((\ln\lambda)^\alpha \sum_{k=1}^{\infty} \frac{1}{k^p} + \sum_{k=1}^{\infty} \frac{1}{k^{p-\alpha}})$, there exists $l \in \mathbb{R}^n$ such that

$$|f(y) - l| \le (\frac{1}{C_0}\gamma(C/(\ln\ln(\frac{|y|e}{\lambda}))^p)^{\frac{1}{n-q}} \; if \; |y| > \lambda \qquad (1)$$

If condition b) holds, there exists $l \in \mathbb{R}^n$ such that

$$|f(y) - l| \le (\frac{1}{C_0}\gamma((\omega_{n-1})^{p/n}(\|\omega^{n/(n-p)}\|_{\complement B(0,\lambda)})^{\frac{n-p}{n}}(\ln(\frac{|y|}{\lambda}))^{\frac{p(1-n)}{n}})^{\frac{1}{n-q}} \; if \; |y| > \lambda \qquad (2)$$

**Corollary 9.** Let $n \ge 2$, $n-1 < q < n$, $\lambda > 0$, $D \subset \overline{\mathbb{R}^n}$ a domain such that $\complement B(0,\lambda) \subset D$ and let $f : D \to D_f \subset \mathbb{R}^n$ be a homeomorphism satisfying condition $(N)$ such that its inverse is $ACL^q$ and suppose that one of the following conditions hold:

a) There exists $M > 0$ and $0 \le \alpha < q-1$ such that $\int\limits_{B(0,\delta)\setminus\overline{B}(0,\lambda)} \omega(x)dx \le M\delta^q(\ln\delta)^\alpha$ for every $\delta > \lambda$.

b) $\int\limits_{\complement B(0,\lambda)} K_{I,q}(f)(x)^{n/(n-q)}dx < \infty$.

Then, if condition a) holds and $C = Me^q 2^\alpha ((\ln\lambda)^\alpha \sum_{k=1}^{\infty} \frac{1}{k^q} + \sum_{k=1}^{\infty} \frac{1}{k^{q-\alpha}})$, there exists $l \in \mathbb{R}^n$ such that

$$|f(y) - l| \le (\frac{1}{C_0}(C/(\ln\ln(\frac{|y|e}{\lambda}))^q)^{\frac{1}{n-q}} \; if \; |y| > \lambda \qquad (1)$$

If condition b) holds, there exists $l \in \mathbb{R}^n$ such that

$$|f(y) - l| \le (\frac{(\omega_{n-1})^{q/n}}{C_0}(\|K_{I,q}(f)^{n/(n-q)}\|_{\complement B(0,\lambda)})^{\frac{n-q}{n}}(\ln(\frac{|y|}{\lambda}))^{\frac{q(1-n)}{n}})^{\frac{1}{n-q}} \; if \; |y| > \lambda \qquad (2)$$

A known theorem from the theory of quasiconformal mappings says that if $f_j : D \to D_j \subset \overline{\mathbb{R}^n}$ is a sequence of $K$-quasiconformal mappings such that $f_j \to f$, then either the limit mapping is a $K$-quasiconformal homeomorphism, or Card Im$f \le 2$, and the convergence may be not uniform on the compact subsets of $D$. Also, if $f_j \to f$ uniformly on the compact subsets of $D$, then either $f$ is a $K$-quasiconformal homeomorphism, or $f$ is constant on $D$ (see Theorem 21.1, page 69 in [45] or Corollary 37.3, page 125 in [45]). A version of the last result for generalized ring $(n, M^p_\omega, \gamma)$ homeomorphisms is proved in Theorem 21.9, page 73 in [25] and Corollary 37.3, page 125 in [25], in Theorem 9 in [8] and in Theorem 4.1 in [32].

Another main result of this paper shows that in the case $n-1 < q < n$, we have a stronger result. We prove that if $f_j : D \to D_j \subset \mathbb{R}^n$ is a sequence of generalized ring $(q, M^p_\omega, \gamma)$ homeomorphisms such that $f_j \to f$, $n-1 < q < n$, then either $f$ is constant on $D$, or $f$ is a generalized ring $(q, M^p_\omega, \gamma)$ homeomorphism and the convergence is uniform on the compact subsets of $D$.

**Theorem 10.** Let $n \ge 2$, $n-1 < q < n$, $p > 1$, $D \subset \overline{\mathbb{R}^n}$ a domain, $\omega \in L^1_{loc}(D)$ such that $\Delta^p_\omega(x) = 0$ for every $x \in D$, $\gamma : [0,\infty) \to [0,\infty)$ increasing with $\lim\limits_{t\to 0}\gamma(t) = 0$, let $f_j : D \to D_j \subset \mathbb{R}^n$ be generalized ring $(q, M^p_\omega, \gamma)$ homeomorphisms for every $j \in \mathbb{N}$ and suppose that $f_j \to f$. Then $f_j \to f$ uniformly on the compact subsets of $D$ and either $f$ is constant on $D$, or there exists a domain $G \subset \mathbb{R}^n$ such that $f : D \to G$ is a generalized ring $(q, M^p_\omega, \gamma)$

homeomorphism. In the last case, if $K \subset G$ is compact, there exists $j_0 \in \mathbb{N}$ such that $K \subset D_j$ for every $j \geq j_0$ and $f_j^{-1}|K \to f^{-1}|K$ uniformly on $K$.

**Corollary 10.** Let $n \geq 2$, $n-1 < q < n$, $D \subset \overline{\mathbb{R}^n}$ a domain, $\omega \in L^1_{loc}(D)$ and let $f_j : D \to D_j \subset \mathbb{R}^n$ be homeomorphisms satisfying condition $(N)$ and having $ACL^q$ inverses for every $j \in \mathbb{N}$ such that $f_j \to f$. Suppose that one of the following conditions hold:

1) $\Delta^q_\omega(x) = 0$ for every $x \in D$ and $K_{I,q}(f_j) \leq \omega$ for every $j \in \mathbb{N}$.

2) There exists $q < p \leq n$ and $K > 0$ such that $(\int\limits_D K_{I,q}(f_j)(x)^{p/(p-q)}dx)^{\frac{p-q}{q}} \leq K < \infty$ for every $j \in \mathbb{N}$ and $\infty \notin D$ if $p < n$.

Then $f_j \to f$ uniformly on the compact subsets of $D$ and either $f$ is constant on $D$, or there exists a domain $G \subset \mathbb{R}^n$ such that $f : D \to G$ is a generalized ring homeomorphism and if $F \subset G$ is compact in $G$, $f_j^{-1}|F \to f^{-1}|F$ uniformly on $F$.

We shall also give analogues of some known results from the theory of quasiconformal mappings concerning the limit function of a sequence of $K$-quasiconformal mappings, which are valid for sequences of generalized ring $(q, M^p_\omega, \gamma)$ homeomorphisms whose inverses are generalized ring $(r, M^t_\eta, \lambda)$ homeomorphisms, $q, r \in (n-1, n]$.

# 2 Preliminaries.

If $a, b \in \overline{\mathbb{R}^n}$, we denote by $q(a,b)$ the chordal distance between $a$ and $b$ and $q(a,b) = |a-b|(1+|a|^2)^{-\frac{1}{2}}(1+|b|^2)^{-\frac{1}{2}}$ if $a, b \in \mathbb{R}^n$, $q(a,\infty) = (1+|a|^2)^{-\frac{1}{2}}$ if $a \in \mathbb{R}^n$. If $A \subset \overline{\mathbb{R}^n}$, we set $q(A)$ the chordal diameter of $A$ and if $A \subset \mathbb{R}^n$ we set $d(A)$ the diameter of $A$ in the euclidian metric. We denote by $B_q(x,r)$ the ball of center $x$ and radius $r$ in the chordal metric and by $S_q(x,r)$ the sphere of center $x$ and radius $r$ in the chordal metric. We set $B(x,r) = \{z \in \mathbb{R}^n | |z-x| < r\}$, $S(x,r) = \{z \in \mathbb{R}^n | |z-x| = r\}$ if $x \in \mathbb{R}^n$ and $r > 0$.

Given $0 < r \leq 1$ and $q > 1$, we set $\psi_{n,q}(r)$ the set of all rings $A = R(Q_0, Q_1)$ such that $q(Q_0) \geq r$, $q(Q_1) \geq r$ and $Q_1$ is unbounded and we set $\lambda_{n,q}(r) = \inf\limits_{A \in \psi_{n,q}(r)} M_q(\Gamma_A)$.

We see from Theorem 10.2 in [2] that if $q > n-1$, then $\lambda_{n,q}(r) > 0$ for every $0 < r \leq 1$, the function $\lambda_{n,q} : (0,1] \to (0,\infty)$ is increasing and $\lim\limits_{r \to 0} \lambda_{n,q}(r) = 0$.

If $0 \leq t \leq 1$, let $\phi_{n,q}(r,t)$ be the set of all rings $A = R(Q_0, Q_1)$ such that $q(Q_0) \geq r$, $q(Q_1) \geq r$, $q(Q_0, Q_1) \leq t$ and $Q_1$ is unbounded. We set $\overline{\lambda}_{n,q}(r,t) = \inf\limits_{A \in \phi_{n,q}(r,t)} M_q(\Gamma_A)$. We see from Theorem 10.2 in [2] that if $q > n-1$, the function $\overline{\lambda}_{n,q}$ is increasing in $r$ and decreasing in $t$, $\overline{\lambda}_{n,q}(r,t) \geq \lambda_{n,q}(r)$ for every $0 < r \leq 1$ and $0 \leq t \leq 1$ and that $\lim\limits_{t \to 0} \overline{\lambda}_{n,q}(r,t) = \infty$ for every fixed $0 < r \leq 1$. Also, if $A = R(Q_0, Q_1)$, $a, c \in Q_0$, $d, \infty \in Q_1$, we see from Theorem 9 in [2] that there exists a constant $C_0$ depending only on $n$ and $q$ such that $C_0|a-c|^{n-q} \leq \lambda_{n,q}(\frac{|d-c|}{|a-c})(|a-c|)^{n-q} \leq M_q(\Gamma_A)$. Throughout this paper we shall denote by $C_0$ this constant.

Let $E_1, ..., E_j, ...$ be a sequence of sets in $\overline{\mathbb{R}^n}$. The Kernel $Ker_{j \to \infty} E_j$ of this sequence of sets is the set $\{x \in \overline{\mathbb{R}^n} |$ there exists $U_x \in \mathcal{V}(x)$ and $j_x \in \mathbb{N}$ such that $U_x \subset E_j$ for every $j \geq j_x\}$.

If $D \subset \mathbb{R}^n$ is open and $f : D \to \mathbb{R}^n$ is a map, we say that $f$ is $ACL$ if $f$ is continuous and for every cube $Q \subset\subset D$ with the sides parallel to coordinate axes and for every face $S$ of $Q$ it results that $f|P_S^{-1}(y) \cap Q : P_s^{-1}(y) \cap Q \to \mathbb{R}^n$ is absolutely continuous for a.e. $y \in S$, where $P_S : \mathbb{R}^n \to S$ is the projection on $S$. An $ACL$ map has a.e. first partial derivatives and if $q > 1$, we say that $f$ is $ACL^q$ if $f$ is $ACL$ and the first partial derivatives are locally in $L^q$.

If $q > 1$, we denote by $W^{1,q}_{loc}(D, \mathbb{R}^n)$ the Sobolev space of all functions $f : D \to \mathbb{R}^n$ which are locally in $L^q$ together with the first order distributional derivatives. We see from Proposition 1.2, page 6 in [30] that if $f \in C(D, \mathbb{R}^n)$, then $f$ is $ACL^q$ if and only if $f \in W^{1,q}_{loc}(D, \mathbb{R}^n)$.

If $D \subset \overline{\mathbb{R}^n}$ is open, $b \in \partial D$, and $f : D \to \overline{\mathbb{R}^n}$ is a map, we set $C(f, b) = \{z \in \overline{\mathbb{R}^n}|$ there exists $b_p \in D$ such that $b_p \to b$ and $f(b_p) \to z\}$.

If $D \subset \overline{\mathbb{R}^n}$ is open and $f : D \to \overline{\mathbb{R}^n}$ is a map, we say that $f$ is open if $f$ carries open sets into open sets, we say that $f$ is discrete if $f^{-1}(y)$ is empty or discrete for every $y \in \mathbb{R}^n$ and we say that $f$ is light if for every $x \in D$ and every $U \in \mathcal{V}(x)$, $U \subset D$, there exists $Q \in \mathcal{V}(x)$ such that $\overline{Q} \subset U$ and $f(x) \notin f(\partial Q)$. We say that $f$ lifts the paths if for every path $p : [0, 1] \to f(D)$ and every $x \in D$ such that $f(x) = p(0)$, there exists $q : [0, 1] \to D$ a path such that $q(0) = x$ and $f \circ q = p$. We say that $q : [0, a) \to D$ is a maximal lifting of $p$ from $x$ if $0 < a \leq 1$, $q$ is a path, $q(0) = x$, $f \circ q = p|[0, a)$ and $q$ is maximal with this property. If $f : D \to \mathbb{R}^n$ is continuous, open discrete, $p : [0, 1] \to \mathbb{R}^n$ is a path and $x \in D$ is such that $f(x) = p(0)$, then there exists a maximal lifting of $p$ from $x$.

We denote by $V_n$ the volume of the unit ball from $\mathbb{R}^n$ and by $\omega_{n-1}$ the area of the unit sphere from $\mathbb{R}^n$.

If $\gamma : [a, b] \to \mathbb{R}^n$ is rectifiable, we set $s_\gamma(t) = l(\gamma|[a, t])$ for $t \in [a, b]$ and we define the reparametrization $\gamma^0 : [0, l(\gamma)] \to \mathbb{R}^n$ of $\gamma$ by setting $\gamma(t) = \gamma^0(s_\gamma(t))$ for $t \in [a, b]$.

If $D \subset \mathbb{R}^n$ is open, $f : D \to \mathbb{R}^n$ is a map, we set $\fint\limits_A f(x)dx = \int\limits_A f(x)dx/\mu_n(A)$ for $A \subset D$ measurable and if $x \in D$ and $\overline{B}(x, \epsilon) \subset D$, we set $f_\epsilon = \fint\limits_{B(x,\epsilon)} f(z)dz$. We say that $f$ is of finite mean oscillation in $x$ if $\limsup\limits_{\epsilon \to 0} \fint\limits_{B(x,\epsilon)} |f(z) - f_\epsilon|dz < \infty$, and we write $f \in FMO(x)$.

If $D, G$ are domains in $\overline{\mathbb{R}^n}$, $q > 1$, $f : D \to G$ is a homeomorphism and $g$ is its inverse which satisfies condition $(N)$, we can define a.e. the $q$ outer dilatation of $f$ in $x$, $K_{0,q}(f)(x) = L(x, f)^q \mu'_g(f(x))$, and if $f$ is differentiable in $x$ and $J_f(x) \neq 0$, then $K_{0,q}(f)(x) = \frac{|f'(x)|^q}{|J_f(x)|}$.

We say that $E = (A, C)$ is a condenser if $C \subset A \subset \mathbb{R}^n$, $C$ is compact and $A$ is open. If $p > 1$, we define $cap_p(E) = \inf \int\limits_{\mathbb{R}^n} |\nabla u(x)|^p dx$, the $p$ capacity of $E$, where the infimum is taken over all $u \in C_0^\infty(A)$ such that $u \geq 1$ on $C$. We also define $\Gamma_E = \Delta(C, \partial A, A)$ and we see from Proposition II.10.2, page 54 in [30] that $M_p(\Gamma_E) = cap_p(E)$.

We shall use the following capacity inequalities (see [17] and Proposition 6 in [26]):

$$cap_p(E) \geq \left(C_1 \frac{d(C)^p}{\mu_n(A)^{1-n+p}}\right)^{\frac{1}{n-1}} \ for \ p > n - 1 \qquad (I)$$

$$cap_p(E) \geq C_2 \mu_n(C)^{\frac{n-p}{n}} \ for \ 1 < p < n \qquad (II)$$

Here $C_1$ and $C_2$ are constants depending only on $n$ and $p$. Throughout this paper we denote by $C_1$ and $C_2$ the constants from the formulae (I) and (II).

The following is a result similar to Theorem A. We give its proof for the sake of completeness.

**Theorem B.** Let $n \geq 2$, $1 < q < p$, let $D, G$ be domains in $\mathbb{R}^n$, $f : D \to G$ be an $ACL^q$ homeomorphism, let $g$ be its inverse and suppose that $g$ satisfies condition $(N)$. Then $K_{0,q}(f) \circ g \in L^1_{loc}(G)$ and $M_q(\Gamma) \leq M^q_{K_{0,q}(f)\circ g}(f(\Gamma))$ for every $\Gamma \in A(D)$, and if $C = (\int\limits_G K_{0,q}(f)(g(y))^{p/(p-q)}dy)^{\frac{p-q}{p}}$, it results that $M_q(\Gamma) \leq CM_p(f(\Gamma))^{q/p}$ for every $\Gamma \in A(D)$.

**Proof.** Since $f$ is an $ACL^q$ homeomorphism, we can find a constant $C(n, q)$ depending only on $n$ and $q$ such that $L(x, f) \leq C(n, q)|f'(x)|$ a.e in $D$. Let $Q \subset\subset D$ be open. Then

$\int\limits_Q K_{0,q}(f)(g(y))dy = \int\limits_Q L(g(y),f)^q \mu_g^{'}(y)dy \leq \int\limits_{g(Q)} L(x,f)^q dx \leq C(n,q)^q \int\limits_{g(Q)} |f^{'}(x)|^q dx < \infty$, hence $K_{0,q}(f) \circ g \in L^1_{loc}(G)$.

Let $\Gamma \in A(D)$, $\Gamma_0 = \{\gamma \in \Gamma | \gamma$ is locally rectifiable and $f \circ \gamma^\circ$ is absolutely continuous$\}$, let $\eta \in F(f(\Gamma_0))$ and let $\rho : \mathbb{R}^n \to [0,\infty]$ be defined by $\rho(x) = \eta(f(x))L(x,f)$ if $x \in D$, $\rho(x) = 0$ otherwise. We see from Theorem 5.7, page 15 in [45] that $\rho \in F(\Gamma_0)$ and using Fuglede's theorem (see Theorem 28.2, page 95 in [45]), we have:

$$M_q(\Gamma) = M_q(\Gamma_0) \leq \int\limits_D \rho(x)^q dx = \int\limits_D \eta(f(x))^q L(x,f)^q dx =$$

$$= \int\limits_G \eta(f(g(y)))^q L(g(y),f)^q \mu_g^{'}(y)dy = \int\limits_G \eta(y)^q K_{0,q}(f)(g(y))dy.$$

Since $\eta \in F(f(\Gamma_0))$ was arbitrarily chosen, we see that

$$M_q(\Gamma) \leq M^q_{K_{0,q}(f)\circ g}(f(\Gamma_0)) \leq M^q_{K_{0,q}(f)\circ g}(f(\Gamma)).$$

Also, using Hölder's inequality, we see that

$$M_q(\Gamma) = \int\limits_G \eta(y)^q K_{0,q}(f)(g(y))dy \leq C(\int\limits_G \eta(y)^p dy)^{q/p}$$

and since $\eta \in F(f(\Gamma_0))$ was arbitrarily chosen, we find that

$$M_q(\Gamma) \leq CM_p(f(\Gamma_0))^{q/p} \leq CM_p(f(\Gamma))^{q/p}.$$

**Remark 1.** If in Theorem A $g$ is a.e. differentiable on $G$ and $J_g(y) \neq 0$ a.e. in $G$, then $f$ satisfies condition $(N)$. Also, if in Theorem B $f$ is a.e. differentiable on $D$ and $J_f(x) \neq 0$ a.e. in $D$, then $g$ satisfies condition $(N)$.

We denote $C_{x,a,b} = \{z \in \mathbb{R}^n | a < |z-x| < b\}$ for $x \in \mathbb{R}^n$ and $0 < a < b$.

Let $n > q > 1$, $D \subset \overline{\mathbb{R}^n}$ be open, $\omega : D \to [0,\infty]$ be measurable and finite a.e., $x \in D$ and $0 < a < b$.

We find some estimates for $\Delta^q_\omega(\Gamma_{x,a,b})$.

Let $\eta : (a,b) \to [0,\infty]$ be a Borel map and let $I_{a,b} = \int\limits_a^b \eta(t)dt$. Then, if $0 < I_{a,b} < \infty$, $\Delta^q_\omega(\Gamma_{x,a,b}) \leq \int\limits_{C_{x,a,b}} \omega(z)\eta(|z-x|)^q dz/(I_{a,b})^q$.

Letting $\eta(t) = \frac{1}{t}$ for $t \in (a,b)$ and using Hölder's inequality, we have

$$\Delta^q_\omega(\Gamma_{x,a,b}) \leq \int\limits_{C_{x,a,b}} \frac{\omega(z)}{|z-x|^q} dz/(\int\limits_a^b \frac{dt}{t})^q \leq (\int\limits_{C_{x,a,b}} \frac{dz}{|z-x|^n})^{q/n} (\int\limits_{C_{x,a,b}} \omega(z)^{n/(n-q)} dz)^{\frac{n-q}{n}} / (\ln(\frac{b}{a}))^q \leq$$

$$\leq (\|\omega^{n/(n-q)}\|_{C_{x,a,b}})^{\frac{n-q}{n}} (\omega_{n-1} \int\limits_a^b \frac{dt}{t})^{q/n} / (\ln(\frac{b}{a}))^q \leq (\omega_{n-1})^{q/n} (\|\omega^{n/(n-q)}\|_{C_{x,a,b}})^{\frac{n-q}{n}} (\ln(\frac{b}{a}))^{\frac{q(1-n)}{n}}.$$

It results that

$$\Delta^q_\omega(\Gamma_{x,a,b}) \leq (\omega_{n-1})^{q/n}(\|\omega^{n/(n-q)}\|_{C_{x,a,b}})^{\frac{n-q}{n}}(\ln(\frac{b}{a}))^{\frac{q(1-n)}{n}} \qquad (III)$$

It results that if $x \in \mathbb{R}^n$, $\overline{B}(x,b) \subset D$ and $\int\limits_{B(x,b)} \omega(z)^{n/(n-q)}dz < \infty$, then $\Delta^q_\omega(x) = 0$, and if $\complement B(0,b) \subset D$ and $\int\limits_{\complement B(0,b)} \omega(z)^{n/(n-q)}dz < \infty$, then $\Delta^q_\omega(\infty) = 0$.

Let now $\omega_x(t) = \fint\limits_{S(x,t)} \omega(z)d_{S(x,t)}$ for $x \in D$ such that $\overline{B}(x,b) \subset D$ and $a < t < b$. Taken $\eta_0 : (a,b) \to [0,\infty]$ defined by $\eta_0(t) = (1/t^{\frac{n-1}{q-1}}\omega_x(t)^{\frac{1}{q-1}})$ for $a < t < b$ and $I_{a,b} = \int\limits_a^b \eta_0(t)dt$, we see from Remark 3 in [12] that

$$\Delta^q_\omega(\Gamma_{x,a,b}) = \frac{\omega_{n-1}}{(I_{a,b})^{q-1}} \qquad (IV)$$

It results that if $x \in \mathbb{R}^n$, $B(x,b) \subset D$, $\int\limits_a^b \eta_0(t)dt < \infty$ for $0 < a < b$ and $\int\limits_0^b \eta_0(t)dt = \infty$, then $\Delta^q_\omega(x) = 0$, and if $\complement B(0,a) \subset D$, $\int\limits_a^b \eta_0(t)dt < \infty$ for $0 < a < b$ and $\int\limits_a^\infty \eta_0(t)dt = \infty$, then also $\Delta^q_\omega(\infty) = 0$.

We also see from Corollary 6.3, Ch. 6 in [25] that if $n-1 < q < n$ and $f \in FMO(x)$, then $\Delta^q_\omega(x) = 0$.

In Proposition 3 in [12] is proved the following:

**Lemma A.** Let $n \geq 2$, $0 \leq \alpha < q-1$, $M > 0$, $D \subset \mathbb{R}^n$ a domain, $x \in D$, $b > 0$ such that $\overline{B}(x,b) \subset D$, let $\omega : D \to [0,\infty]$ be measurable and finite a.e. such that $\int\limits_{B(x,\delta)} \omega(z)dz \leq M\delta^q(\ln(\frac{be}{\delta}))^\alpha$ for every $0 < \delta < b$ and let $C = Me^q \sum\limits_{k=1}^{\infty} \frac{1}{k^{q-\alpha}}$. Then $\Delta^q_\omega(\Gamma_{x,a,b}) \leq C/(\ln\ln(\frac{be}{a}))^q$ for every $0 < a < b$.

Using the proof from Lemma A, we obtain a similar result for $x = \infty$.

**Lemma B.** Let $n \geq 2$, $0 \leq \alpha < q-1$, $M > 0$, $D \subset \overline{\mathbb{R}^n}$ a domain such that there exists $a > 0$ such that $\complement B(0,a) \subset D$, let $\omega : D \to [0,\infty]$ be measurable and finite a.e. such that $\int\limits_{C_{o,a,b}} \omega(z)dz \leq M\delta^q(\ln\delta)^\alpha$ for every $0 < a < b$ and let $C = Me^q 2^\alpha((\ln a)^\alpha \sum\limits_{k=1}^{\infty} \frac{1}{k^q} + \sum\limits_{k=1}^{\infty} \frac{1}{k^{q-\alpha}})$. Then

$$\Delta^q_\omega(\Gamma_{\infty,a,b}) \leq C/(\ln\ln(\frac{be}{a}))^q \; for\, every\, 0 < a < b. \qquad (V)$$

**Proof.** Let $0 < a < b$ and let $A_k = B(0,ae^{k+1}) \setminus \overline{B}(0,ae^k)$ for $k \in \mathbb{N}$. Then $\frac{1}{|x|} \leq \frac{e^{-k}}{a}$ and $\frac{1}{\ln(\frac{|x|e}{a})} \leq \frac{1}{k+1}$ for every $x \in A_k$ and every $k \in \mathbb{N}$. Let $\rho : \mathbb{R}^n \to [0,\infty]$ be defined by $\rho(x) = \frac{1}{\ln\ln(\frac{be}{a})}\frac{1}{|x|\ln(\frac{|x|e}{a})}$ if $x \in C_{0,a,b}$, $\rho(x) = 0$ otherwise. We see from Theorem 5.7, page 15 in [45] that $\rho \in L_{\infty,a,b}$ and let $k_0 \in \mathbb{N}$ be such that $k+1 < \ln a$ for $k < k_0$ and $\ln a \leq k+1$ for $k \geq k_0$. We have

$$\Delta^q_\omega(\Gamma_{\infty,a,b}) \leq \int\limits_{\mathbb{R}^n} \omega(z)\rho(z)^q dz \leq \int\limits_{\mathbb{R}^n} \omega(x)\frac{dx}{|x|^q(\ln(\frac{|x|e}{a}))^q}\frac{1}{(\ln\ln(\frac{be}{a}))^q} \leq$$

$$\leq \frac{1}{(\ln\ln(\frac{be}{a}))^q}\sum_{k=0}^{\infty}\int\limits_{A_k}\frac{\omega(x)dx}{|x|^q(\ln(\frac{|x|e}{a}))^q} \leq \frac{1}{(\ln\ln(\frac{be}{a}))^q}\sum_{n=0}^{\infty}\frac{1}{(ae^k)^q(k+1)^q}\int\limits_{A_k}\omega(z)dz \leq$$

$$\leq \frac{M}{(\ln\ln(\frac{be}{a}))^q}\sum_{k=0}^{\infty}\frac{(ae^{k+1})^q(\ln(ae^{k+1}))^{\alpha}}{(ae^k)^q(k+1)^q} = \frac{Me^q}{(\ln\ln(\frac{be}{a}))^q}\sum_{k=0}^{\infty}\frac{(\ln a+k+1)^{\alpha}}{(k+1)^q} =$$

$$= \frac{Me^q}{(\ln\ln(\frac{be}{a}))^q}(\sum_{k=0}^{k_0}\frac{(\ln a+k+1)^{\alpha}}{(k+1)^q} + \sum_{k=k_0+1}^{\infty}\frac{(\ln a+k+1)}{(k+1)^q})^{\alpha} \leq C/(\ln\ln(\frac{be}{a}))^q.$$

# 3 Open, discrete ring mappings. Proof of the results.

**Proof of Theorem 1.** Suppose that $x \in \mathbb{R}^n$ and let $b > 0$ be such that $\overline{B}(x,b) \subset D$ and let $\delta = \min\{b, C_2\}$. Let $0 < a < b$ be such that $\gamma(\Delta_M(\Gamma_{x,a,b})) \leq \frac{\delta}{2}$ and let $0 < \epsilon < a$. We can find $0 < c < \epsilon$ such that $\gamma(\Delta_M(\Gamma_{x,c,\epsilon})) < \frac{\delta}{2}$. Let $E = (B(x,b) \setminus \overline{B}(x,c), \overline{B}(x,a) \setminus B(x,\epsilon))$. Then $f(E)$ is a capacitor $(f(B(x,b) \setminus \overline{B}(x,c)),\ f(\overline{B}(x,a) \setminus B(x,\epsilon)))$ and let $\Gamma^*$ be the family of all maximal liftings of some paths from $\Gamma_{f(E)}$ starting from some points of $\overline{B}(x,a) \setminus B(x,\epsilon)$. Then $\Gamma_{f(E)} > f(\Gamma^*)$ and since $f$ is open, we see that $\Gamma^* \subset \Gamma_E$. Let $\Gamma_1 = \Gamma_{x,a,b}$, $\Gamma_2 = \Gamma_{x,c,\epsilon}$. Then $\Gamma_E > \Gamma_1 \cup \Gamma_2$ and $f(\Gamma^*) \subset f(\Gamma_E)$. Also $f(\Gamma_E) > f(\Gamma_1 \cup \Gamma_2) = f(\Gamma_1) \cup f(\Gamma_2)$. Using (II), we see that

$$C_2\mu_n(f(\overline{B}(x,a) \setminus B(x,\epsilon))^{\frac{n}{n-q}} \leq cap_q(f(E)) = M_q(\Gamma_{f(E)}) \leq M_q(f(\Gamma^*)) \leq$$

$$\leq M_q(f(\Gamma_E)) \leq M_q(f(\Gamma_1) \cup f(\Gamma_2)) \leq M_q(f(\Gamma_1)) + M_q(f(\Gamma_2)) \leq$$

$$\leq \gamma(\Delta_M(\Gamma_1)) + \gamma(\Delta_M(\Gamma_2)) \leq \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

We see that $\mu_n(f(\overline{B}(x,a) \setminus B(x,\epsilon)) \leq (\frac{\delta}{C_2})^{\frac{n-q}{n}} < 1$ for every $0 < \epsilon < a$ and letting $\epsilon \to 0$, we find that

$$\mu_n(f(B(x,a) \setminus \{x\})) < 1 \qquad (1)$$

Let $\rho > 0$ be fixed and let $0 < \beta < a$ be such that $\gamma(\Delta_M(\Gamma_{x,\beta,a})) < \frac{\rho}{2}$. Let $0 < \delta < \alpha < \beta$ be such that $\gamma(\Delta_M(\Gamma_{x,\delta,\alpha})) < \frac{\rho}{2}$ and let $\Gamma_1 = \Gamma_{x,\beta,a}$, $\Gamma_2 = \Gamma_{x,\delta,\alpha}$. Let $E = (B(x,a) \setminus \overline{B}(x,\delta), \overline{B}(x,\beta) \setminus B(x,\alpha))$. Then $f(E)$ is a capacitor $(f(B(x,a) \setminus \overline{B}(x,\delta)), f(\overline{B}(x.\beta) \setminus B(x,\alpha)),\ \Gamma_E > \Gamma_1 \cup \Gamma_2$, $M_q(\Gamma_{f(E)} \leq M_q(f(\Gamma_E))$ and using (I), we have:

$$(C_1 d(f(\overline{B}(x,\beta) \setminus B(x,\alpha)))^q)^{\frac{1}{n-1}} \leq (C_1\frac{d(f(\overline{B}(x,\beta) \setminus B(x,\alpha)))^q}{\mu_n(f(B(x,a) \setminus \overline{B}(x,\delta)))^{1-n+q}})^{\frac{1}{n-1}} \leq cap_q(f(E)) =$$

$$= M_q(\Gamma_{f(E)} \leq M_q(f(\Gamma_E)) \leq M_q(f(\Gamma_1) \cup f(\Gamma_2)) \leq \gamma(\Delta_M(\Gamma_1)) + \gamma(\Delta_M(\Gamma_2)) \leq \frac{\rho}{2} + \frac{\rho}{2} = \rho.$$

Letting $\alpha \to 0$, we find that

$$d(f(\overline{B}(x,\beta) \setminus \{x\}))^q \leq \frac{q^{n-1}}{C_1}\ \textit{for some}\ 0 < \beta < a < b \qquad (2)$$

It results from (2) that $C(f,x)$ is compact in $\mathbb{R}^n$ and that there exists $\lim\limits_{z\to x} f(z) = l \in \mathbb{R}^n$.

We use the same arguments if $x = \infty$.

**Proof of Corollary 1.** Suppose that condition 1) holds. We see from Theorem A that $M_q(f(\Gamma)) \leq M^q_{K_{I,q}(f)}(\Gamma)$ for every $\Gamma \in A(D)$ and we apply Theorem 1 with $M = M^q_{K_{I,q}(f)}$ and

$\gamma(t) = t$ for $t > 0$. Suppose now that condition 2) holds and let $C = (\int_D K_{I,q}(f)(x)^{p/(p-q)} dx)^{\frac{p-q}{p}}$. We see from Theorem A that $M_q(f(\Gamma)) \le C M_p(\Gamma)^{q/p}$ for every $\Gamma \in A(D)$ and we also see that $M_p(x) = 0$ if $0 < p \le n$ and $x \in \mathbb{R}^n$ and $M_n(\infty) = 0$. We apply Theorem 1 with $M = M_p$ and $\gamma(t) = C t^{q/p}$ for $t > 0$.

**Proof of Theorem 2.** Suppose that there exists $f : \mathbb{R}^n \to \mathbb{R}^n$ continuous, open and discrete such that $f$ is a ring $(q, M, \gamma)$ mapping at $\infty$ and $\Delta_M(\infty) = 0$. Using Theorem 1, we can find a function $F : \overline{\mathbb{R}^n} \to \mathbb{R}^n$ such that $F|\mathbb{R}^n = f$. Since $F$ is open and discrete on $\mathbb{R}^n$, we see that $F$ is a light map and we can easily see that taken on $\overline{\mathbb{R}^n}$ the chordal metric, $F$ is also an open map on $\overline{\mathbb{R}^n}$. Then $F(\overline{\mathbb{R}^n})$ is open and compact in $\overline{\mathbb{R}^n}$, hence $F(\overline{\mathbb{R}^n}) = \overline{\mathbb{R}^n}$ and we reached a contradiction.

**Proof of Corollary 2.** Suppose that there exists $f : \mathbb{R}^n \to \mathbb{R}^n$ continuous, open, discrete, satisfying condition $(N)$ such that $m_1(B_f) = 0$, having local $ACL^q$ inverses on $f(D \setminus B_f)$ and either $\Delta^q_{K_{I,q}(f)}(\infty) = 0$, or $\int_{\mathbb{R}^n} K_{I,q}(f)(x)^{n/(n-q)} dx < \infty$. Using Theorem A, we see that $M_q(f(\Gamma)) \le M^q_{K_{I,q}(f)}(\Gamma)$ for every $\Gamma \in A(D)$ and we also see from (III) that $\Delta^q_{K_{I,q}(f)}(\infty) = 0$. We apply now Theorem 2.

**Proof of Theorem 3.** Using Theorem 1, we can find $F : \overline{\mathbb{R}^n} \to \overline{\mathbb{R}^n}$ continuous such that $F|\mathbb{R}^n = f$ and $l = F(\infty) \in \mathbb{R}^n$. Let $g : \overline{\mathbb{R}^n} \to \overline{\mathbb{R}^n}$ be a homeomorphism such that $g(l) = \infty$ and $g(x) \in \mathbb{R}^n$ if $x \in \overline{\mathbb{R}^n} \setminus \{l\}$. Let $P = f^{-1}(l)$. Then $P$ is a discrete set in $\mathbb{R}^n$ and let $h : \mathbb{R}^n \setminus P \to \mathbb{R}^n$, $h = g \circ f$. We see that $\mathbb{R}^n \setminus P$ is pathwise connected, $\mathbb{R}^n$ is simply connected and $h$ lifts the paths, hence $h$ is a homeomorphism and we can easily see that $P = \phi$ and $\lim_{x \to \infty} h(x) = \infty$. We can find now a homeomorphism $H : \overline{\mathbb{R}^n} \to \overline{\mathbb{R}^n}$ such that $H|\mathbb{R}^n = h$ and $H(\infty) = \infty$. We see that $g \circ F = H$ and taking $F = g^{-1} \circ H$, we see that $F : \overline{\mathbb{R}^n} \to \overline{\mathbb{R}^n}$ is a homeomorphism and $F|\mathbb{R}^n = f$.

**Proof of Corollary 3.** We see from Theorem A that $M_q(f(\Gamma)) \le M^q_{K_{I,q}(f)}(\Gamma)$ for every $\Gamma \in A(D)$ and we also see that $\Delta^q_{K_{I,q}(f)}(\infty) = 0$. We apply now Theorem A.

**Example 1.** Let $n - 1 < q < n$ and $f : \overline{\mathbb{R}^n} \to \overline{\mathbb{R}^n}$, $f(x) = \frac{x}{|x|^2}$ if $x \ne 0$, $f(0) = \infty$, $f(\infty) = 0$. Then $f$ is a homeomorphism and using Example 16.2, page 49 in [45], we see that $l(f'(x)) = |f'(x)| = \frac{1}{|x|^2}$, $J_f(x) = |x|^{-2n}$ if $x \ne 0$. Then $K_{I,q}(f)(x) = |x|^{-2(n-q)}$ if $x \ne 0$ and using Lemma A, we see that $M_q(f(\Gamma)) \le M^q_{K_{I,q}(f)}(\Gamma)$ for every $\Gamma \in A(\mathbb{R}^n \setminus \{0\})$.

Let $Q = (0, \pi)^{n-2} \times (0, 2\pi)$ and let $\theta : Q \times (0, \infty) \to \mathbb{R}^n$ be the polar coordinates in $\mathbb{R}^n$ and let $\omega : \mathbb{R}^n \setminus \{0\} \to [0, \infty]$, $\omega(x) = K_{I,q}(f)(x)$ for $x \ne 0$. Then $\omega_x(t) = \fint_{S(x,t)} \omega(z) d_{S(x,t)} = \frac{1}{\omega_{n-1} t^{n-1}} \int_Q \omega(\theta(t, y)) |J_\theta(t, y)| dy = \frac{1}{\omega_{n-1} t^{n-1}} \int_Q \frac{t^{n-1} dt}{|\theta(t,y)|^{2(n-q)}} = t^{-2(n-q)}$ for $t > 0$ and $x \in \mathbb{R}^n$.

Let $\eta_0 : (0, \infty) \to (0, \infty)$, $\eta_o(t) = 1/(t^{\frac{n-1}{q-1}} \omega_x(t)^{\frac{1}{q-1}})$ for $t > 0$. Then $\eta_0(t) = t^{\frac{n+1-2q}{q-1}}$ for $t > 0$.

Let $0 < a < b$ and $I_{a,b} = \int_a^b \eta_0(t) dt = \frac{q-1}{n-q}(b^{\frac{n-q}{q-1}} - a^{\frac{n-q}{q-1}})$ and using (IV), we find that $\Delta^q_\omega(\Gamma_{x,a,b}) = \frac{\omega_{n-1}}{(I_{a,b})^{q-1}}$. This implies that $\liminf_{a \to 0} \Delta^q_\omega(\Gamma_{0,a,b}) = \omega_{n-1}(\frac{n-q}{q-1})^{q-1} b^{q-n} > 0$, hence $\Delta^q_\omega(0) > 0$. We see that $\Delta^q_\omega(\infty) = 0$.

Let now $f : \overline{\mathbb{R}^n} \to \overline{\mathbb{R}^n}$, $f(x) = x$ for $x \in \overline{\mathbb{R}^n}$. Then $K_{I,q}(f)(x) = 1$ for every $x \in \mathbb{R}^n$ and $M_q(f(\Gamma)) = M_q(\Gamma)$ for every $\Gamma \in A(\mathbb{R}^n)$. Let $\omega : \mathbb{R}^n \to [0, \infty]$, $\omega(x) = K_{I,q}(f)(x)$ for $x \in \mathbb{R}^n$ and $\omega_x(t) = \fint_{S(x,t)} \omega(z) d_{S(x,t)}$ for $t > 0$ and $x \in \mathbb{R}^n$. Let $\eta_0 : (0, \infty) \to (0, \infty)$, $\eta_0(t) = 1/(t^{\frac{n-1}{q-1}} \omega_x(t)^{\frac{1}{q-1}})$ for $t > 0$. Then $\eta_0(t) = t^{\frac{1-n}{q-1}}$ for $t > 0$. Let $0 < a < b$ and

$I_{a,b} = \int\limits_a^b \eta_0(t)dt = \frac{q-1}{n-q}(a^{\frac{q-n}{q-1}} - b^{\frac{q-n}{q-1}})$. Then $\Delta^q_\omega(\infty) = (\frac{n-q}{q-1})^{q-1} a^{n-q} > 0$ and $\Delta^q_\omega(0) = \infty$.

Both homeomorphisms are conformal, hence $\Delta^n_{K_{I,n}(f)}(x) = 0$ for every $x \in \overline{\mathbb{R}^n}$.

**Proof of Theorem 4.** We see from Theorem 1 that there exists $F : D \cup \{x\} \to \overline{\mathbb{R}^n}$ continuous such that $F|D = f$ and we can easily see that $F$ is open and light and $F(x) \in \mathbb{R}^n$.

We suppose first that $x \neq \infty$ and we can also suppose that $F(D) \subset \mathbb{R}^n$. Let $r > 0$ be such that $\overline{B}(x,r) \subset D$ and $F(x) \notin F(S(x,r))$ and let $\rho = d(F(x), F(S(x,r))) > 0$. Let $0 < \epsilon < \rho$ and let $U$ be the component of $F^{-1}(B(F(x),\epsilon))$ containing $x$. Then $\overline{U} \subset B(x,r)$, $F(U) = B(F(x),\epsilon)$, $F(\partial U) = S(F(x),\epsilon)$ and let $E = F^{-1}(F(x)) \cap U$. Then $E$ is at most countable, since $F$ is a local homeomorphism on $U \setminus \{x\}$ and let $h = F|U \setminus E : U \setminus E \to B(F(x),\epsilon) \setminus \{F(x)\}$. Then $h$ is a local homeomorphism which lifts the paths, $U \setminus E$ is connected and $B(F(x),\epsilon) \setminus \{F(x)\}$ is simply connected and this implies that $h$ is a homeomorphism. We can easily see that $F$ is injective on $U$ and that $F$ is a local homeomorphism. If $x = \infty$ we apply a similar argument.

**Proof of Corollary 4.** If condition 1) holds, we see from Theorem A that $M_q(f(\Gamma)) \leq M^q_{K_{I,q}(f)}(\Gamma)$ for every $\Gamma \in A(D)$ and that $\Delta^q_{K_{I,q}(f)}(\infty) = 0$. If condition 2) holds, then $M_q(f(\Gamma)) \leq K M_p(\Gamma)^{q/p}$ for every $\Gamma \in A(D)$ and $M_p(x) = 0$ if $1 < p \leq n$ and $x \in \mathbb{R}^n$, $M_n(\infty) = 0$. We apply now Theorem 4.

**Proof of Theorem 5.** Suppose that $x \neq \infty$ and let $b > 0$ be such that $\overline{B}(x,b) \subset D$. We can prove relation (1) from Theorem 1 independently of the function $f \in W$, hence we can find $0 < a < b$ such that $\mu_n(f(\overline{B}(x,a))) < 1$ for every $f \in W$.

Let $\epsilon > 0$ and $0 < \delta < a$ be such that $\gamma(\Delta_M(\Gamma_{x,\delta,a})) < (\epsilon^q C_1)^{\frac{1}{n-1}}$. Let $E = (B(x,a), \overline{B}(x,\delta))$ and let $f \in W$. Then $f(E)$ is a capacitor $(f(B(x,a)), f(\overline{B}(x,\delta)))$ and let $\Gamma^*$ be the family of all maximal liftings of some paths from $\Gamma_{f(E)}$ starting from some points of $f(\overline{B}(x,\delta))$. Since $f$ is open, we have that $\Gamma_{f(E)} > f(\Gamma^*)$ and $\Gamma^* \subset \Gamma_E$. We have

$$(C_1 d(f(\overline{B}(x,\delta)))^q)^{\frac{1}{n-1}} \leq (\frac{C_1 d(f(\overline{B}(x,\delta)))^q}{\mu_n(f(B(x,a)))^{1-n+q}})^{\frac{1}{n-1}} \leq cap_q(f(E)) = M_q(\Gamma_{f(E)}) \leq$$

$$\leq M_q(f(\Gamma^*)) \leq M_q(f(\Gamma_E)) \leq \gamma(\Delta_M(\Gamma_{x,\delta,a})) \leq (\epsilon^q C_1)^{\frac{1}{n-1}}$$

for every $f \in W$. It results that $d(f(\overline{B}(x,\delta))) < \epsilon$ for every $f \in W$.

We proved that the family $W$ is equicontinuous at $x$. The proof is similar if $x = \infty$.

**Proof of Corollary 5.** Suppose that condition 1) holds and let $f \in W$. Using Theorem A, we see that $M_q(f(\Gamma)) \leq M^q_{K_{I,q}(f)}(\Gamma) \leq M^q_\omega(\Gamma)$ for every $\Gamma \in A(D)$, hence $f$ is a ring $(q, M^q_\omega, \gamma)$ mapping, with $\gamma(t) = t$ for $t > 0$.

Suppose now that condition 2) holds and let $f \in W$. Using again Theorem A, we see that $M_q(f(\Gamma)) \leq C M_p(\Gamma)^{q/p}$ for every $\Gamma \in A(D)$, hence $f$ is a ring $(q, M_p, \gamma)$ mapping, with $\gamma(t) = Ct^{q/p}$ for $t > 0$. We apply now Theorem 5.

**Proof of Theorem 6.** Let $0 < |y-x| < d$ and let $E = (B(x,d), \overline{B}(x,|y-x|))$. Then $f(E)$ is a capacitor and we have

$$(\frac{C_1 d(f(\overline{B}(x,|y-x|)))^q}{(V_n r^n)^{1-n+q}})^{\frac{1}{n-1}} \leq (\frac{C_1 d(f(\overline{B}(x,|y-x|)))^q}{\mu_n(f(B(x,d)))^{1-n+q}})^{\frac{1}{n-1}} \leq cap_q(f(E)) =$$

$$= M_q(\Gamma_{f(E)}) \leq M_q(f(\Gamma_E)) \leq \gamma(\Delta^p_\omega(\Gamma_{x,|y-x|,d})).$$

If condition a) holds, we see from Lemma A that $\Delta^p_\omega(\Gamma_{x,|y-x|,d}) \le Me^p \sum_{k=1}^{\infty} \frac{1}{k^{p-\alpha}} / (\ln\ln(\frac{de}{|y-x|}))^p$ and relation (1) is proved. If condition b) holds, we see from relation (III) that $\Delta^p_\omega(\Gamma_{x,|y-x|,d}) \le (\omega_{n-1})^{p/n}(\|\omega^{n/(n-p)}\|_{B(x,d)})^{\frac{n-p}{n}}(\ln(\frac{d}{|y-x|}))^{\frac{p(1-n)}{n}}$ and (3) is proved.

We see from the proof of Theorem 1 that if $0 < \rho < d$ is small enough, then $\mu_n(f(B(x,\rho)))$ is arbitrarily small and replacing in the preceding argument $d$ by $\rho$, we can prove relations (2) and (4).

**Proof of Corollary 6.** We see from Theorem A that $M_q(f(\Gamma)) \le M^q_{K_{I,q}(f)}(\Gamma)$ for every $\Gamma \in A(D)$ and we apply Theorem 6 with $\omega = K_{I,q}(f)$, $p = q$ and $\gamma(t) = t$ for $t > 0$.

**Proof of Theorem 7.** We see from Lemma B or from relation (III) that $\Delta^p_\omega(\infty) = 0$ and using Theorem 1, we see that in both cases there exists $\lim_{z\to x} f(z) = l \in \mathbb{R}^n$.

Let $0 < \lambda < \alpha < \beta$ and $k > 1$ and let $E = (\complement\overline{B}(0,\lambda) \setminus \complement B(0,k\beta), \complement B(0,\alpha) \setminus \complement\overline{B}(0,\beta))$. Then $E$ is a capacitor, $f(E)$ is a capacitor and if $\Gamma_1 = \Gamma_{\infty,\lambda,\alpha}$, $\Gamma_2 = \Gamma_{\infty,\beta,k\beta}$, we see as before that $\Gamma_E > \Gamma_1 \cup \Gamma_2$, $M_q(f(\Gamma_E)) \ge M_q(\Gamma_{f(E)})$, $f(\Gamma_E) > f(\Gamma_1 \cup \Gamma_2) = f(\Gamma_1) \cup f(\Gamma_2)$. We have

$$(\frac{C_1 d(f(\complement B(0,\alpha) \setminus \complement\overline{B}(0,\beta)))^q}{(V_n r^n)^{1-n+q}})^{\frac{1}{n-1}} \le (\frac{C_1 d(f(\complement B(0,\alpha) \setminus \complement\overline{B}(0,\beta)))^q}{\mu_n(f(\complement\overline{B}(0,\lambda) \setminus \complement B(0,k\beta)))^{1-n+q}})^{\frac{1}{n-1}} \le$$

$$\le cap_q(f(E)) = M_q(\Gamma_{f(E)}) \le M_q(f(\Gamma_E)) \le M_q(f(\Gamma_1 \cup \Gamma_2)) \le$$

$$\le M_q(f(\Gamma_1)) + M_q(f(\Gamma_2)) \le \gamma(\Delta^p_\omega(\Gamma_{\infty,\lambda,\alpha})) + \gamma(\Delta^p_\omega(\Gamma_{\infty,\beta,k\beta})).$$

Letting $k \to \infty$ and using the fact that $\Delta^p_\omega(\infty) = 0$, it results that $\lim_{k\to\infty} \Delta^p_\omega(\Gamma_{\infty,\beta,k\beta}) = 0$ and we find that

$$d(f(\complement B(0,\alpha) \setminus \complement\overline{B}(0,\beta))) \le (\frac{(V_n)^{1-n+q}}{C_1})^{\frac{1}{q}} r^{\frac{n(1-n+q)}{q}} \gamma(\Delta^p_\omega(\Gamma_{\infty,\lambda,\alpha}))^{\frac{n-1}{q}}.$$

Letting now $\beta \to \infty$, we find that

$$d(f(\complement B(0,\alpha))) \le (\frac{(V_n)^{1-n+q}}{C_1})^{\frac{1}{q}} r^{\frac{n(1-n+q)}{q}} \gamma(\Delta^p_\omega(\Gamma_{\infty,\lambda,\alpha}))^{\frac{n-1}{q}}.$$

Let now $y \in \complement\overline{B}(0,\lambda)$ and take $\alpha = |y|$. Then $f(y) \in f(\complement B(0,\alpha))$, $f(\complement B(0,\alpha)) \cup \{l\}$ is compact in $\mathbb{R}^n$ and $l \in Int(f(\complement B(0,\alpha)) \cup \{l\})$, hence $|f(y) - l| \le d(f(\complement B(0,\alpha)))$. We have

$$|f(y) - l| \le (\frac{(V_n)^{1-n+q}}{C_1})^{\frac{1}{q}} r^{\frac{n(1-n+q)}{q}} \gamma(\Delta^p_\omega(\Gamma_{\infty,\lambda,|y|}))^{\frac{n-1}{q}}.$$

Using Lemma B and relation (III), we prove relations (1) and (3). We can prove as in Theorem 1 that if $\rho$ is great enough, then $\mu_n(f(\complement B(0,\rho)))$ is small enough and replacing in the preceding arguments $\lambda$ by $\rho$, we prove relations (2) and (4).

**Proof of Corollary 7.** We see from Theorem A that $M_q(f(\Gamma)) \le M^q_{K_{I,q}(f)}(\Gamma)$ for every $\Gamma \in A(D)$ and we apply Theorem 7 with $\omega = K_{I,q}(f)$, $p = q$, $\gamma(t) = t$ for $t > 0$.

**Proof of Theorem 8.** Let $y \in B(x,d)$, $A = R(\overline{B}(x,|y-x|), \complement B(x,d))$. Then $f(A)$ is a ring $R(Q_0, Q_1)$, where $Q_0 = f(\overline{B}(x,|y-x|)$ is compact and $Q_1 = \complement f(B(x,d))$ is unbounded and let $b \in Q_1$. Using Theorem 9 in [2], we have:

$$C_0|f(y) - f(x)|^{n-q} \le |f(y) - f(x)|^{n-q} \lambda_{n,q}(\frac{|b - f(x)|}{|f(y) - f(x)|}) \le M_q(f(\Gamma_A)) \le \gamma(\Delta^p_\omega(\Gamma_{x,|y-x|,d})),$$

and it results that

$$|f(y)-f(x)| \leq (\frac{1}{C_0}\gamma(\Delta_\omega^p(\Gamma_{x,|y-x|,d})))^{\frac{1}{n-q}}.$$

Using relations (III) and (V), the theorem is proved.

**Proof of Corollary 8.** We see from Theorem A that $M_q(f(\Gamma)) \leq M^q_{K_{I,q}(f)}(\Gamma)$ for every $\Gamma \in A(D)$ and we apply Theorem 8 with $\omega = K_{I,q}(f)$, $p = q$ and $\gamma(t) = t$ for $t > 0$.

**Remark 2.** We can give an alternate proof of Corollary 8 in the case when condition b) is satisfied. Let $y \in B(x,d)$, $A = R(\overline{B}(x,|y-x|,\complement B(x,d)))$ and let $b \in \complement f(B(x,d))$. We see from Lemma A that $M_q(f(\Gamma)) \leq (\|K_{I,q}(f)^{n/(n-q)}\|_{B(x,d)})^{\frac{n-q}{n}} M_n(\Gamma)^{q/n}$ for every $\Gamma \in A(D)$, hence

$$C_0|f(y)-f(x)|^{n-q} \leq |f(y)-f(x)|^{n-q}\lambda_{n,q}(\frac{|b-f(x)|}{|f(y)-f(x)|}) \leq M_q(f(\Gamma_A)) \leq$$

$$\leq (\|K_{I,q}(f)^{n/(n-q)}\|_{B(x,d)})^{\frac{n-q}{n}} M_n(\Gamma_{x,|y-x|,d})^{q/n} =$$

$$= (\|K_{I,q}(f)^{n/(n-q)}\|_{B(x,d)})^{\frac{n-q}{n}} (\omega_{n-1}\ln(\frac{d}{|y-x|})^{1-n})^{q/n}$$

and we obtain the same result as in Corollary 8.

**Proof of Theorem 9.** We see from Lemma B or from relation (III) that $\Delta_\omega^p(\infty) = 0$ and using Theorem 1, we find that there exists $\lim_{x\to\infty} f(x) = l \in \mathbb{R}^n$.

Let $0 < \lambda < \alpha$ and let $A = R(\overline{B}(0,\lambda), \complement B(0,\alpha))$. Then $f(A)$ is a ring $R(Q_0,Q_1)$, where $Q_0 = f(\complement B(0,\alpha)) \cup \{l\}$ is compact in $\mathbb{R}^n$ and $Q_1 = \complement f(\complement B(0,\lambda))$ is unbounded. Let $y \in \complement B(0,\lambda)$ and let $\alpha = |y|$. Then, $f(y)$, $l \in Q_0$ and if $b \in Q_1$, we see that

$$C_0|f(y)-l|^{n-q} \leq |f(y)-l|^{n-q}\lambda_{n,q}(\frac{|b-l|}{|f(y)-l}) \leq M_q(f(\Gamma_A)) \leq \gamma(\Delta_\omega^p(\Gamma_{\infty,\lambda,|y|})).$$

We apply now Lemma B and relation (III) and the theorem is proved.

**Proof of Corollary 9.** We see from Lemma A that $M_q(f(\Gamma)) \leq M^q_{K_{I,q}(f)}(\Gamma)$ for every $\Gamma \in A(D)$ and we apply Theorem 9 with $\omega = K_{I,q}(f)$, $p = q$ and $\gamma(t) = t$ for $t > 0$.

# 4 The limit mapping of generalized ring $(q, M, \gamma)$ homeomorphisms.

**Lemma 1.** Let $D, D_j$ be domains in $\overline{\mathbb{R}^n}$, $f_j : D \to D_j$ be homeomorphisms such that $f_j \to f$ uniformly on the compact subsets of $D$ and $f$ is a light map. Then $f(D) \subset Ker_{j\to\infty} D_j$.

**Proof:** Let $y \in f(D)$ and $x \in D$ be such that $y = f(x)$ and let $\delta > 0$ be such that $\overline{B}_q(x,\delta) \subset D$ and $y \notin f(S_q(x,\delta))$. Let $U = B_q(x,\delta)$ and $r = q(y, f(\partial U)) > 0$. Let $j_0 \in \mathbb{N}$ be such that $f_j(z) \in B_q(f(x),\frac{r}{2})$ for every $z \in \overline{U}$ and every $j \geq j_0$. Then $\overline{B}_q(y,\frac{r}{2}) \cap f_j(\partial U) = \phi$ for every $j \geq j_0$ and $B_q(y,\frac{r}{2}) = (B_q(y,\frac{r}{2}) \cap f_j(U)) \cup (B_q(y,\frac{r}{2}) \cap f_j(\complement\overline{U}))$ for $j \geq j_0$. We see that $B_q(y,\frac{r}{2})$ is connected, the sets $B_q(y,\frac{r}{2}) \cap f_j(U)$ and $B_q(y,\frac{r}{2}) \cap f_j(\complement\overline{U})$ are open sets and $B_q(y,\frac{r}{2}) \cap f_j(U) \neq \phi$ for $j \geq j_0$. This implies that $B_q(y,\frac{r}{2}) \subset f_j(U) \subset D_j$ for every $j \geq j_0$ and it results that $y \in Ker_{j\to\infty} D_j$.

**Proof of Theorem 10.** We see from Theorem 5 that the family $W = (f_j)_{j\in\mathbb{N}}$ is equicontinuous on $D$ and using Theorem 20.3 in [45], we see that $f_j \to f$ uniformly on the compact subsets of $D$. Let $Q = D \setminus \{\infty\}$ and take $x \in Q$. Since the family $W$ is equicontinuous at

$x$, we can find $r > 0$ and $j_0 \in \mathbb{N}$ such that $q(f(\overline{B}(x,r)) < \frac{1}{4}$ and $q(f_j(z), f(z)) < \frac{1}{4}$ for every $z \in \overline{B}(x,r)$ and every $j \geq j_0$, and we can take $j_0 = 1$. Let $U = B(x,r)$.

We show that either $f$ is constant on $U$, or $f$ is injective on $U$. If this thing is false, we can pick distinct points $a_1, a_2, a_3 \in U$ such that $f(a_1) \neq f(a_2) = f(a_3)$. We can join $a_1$ and $a_2$ by an arc $J$ in $U$ and let $I$ be another arc joining $a_3$ with a point $a_4 \in \partial U$ such that $I \cap J = \phi$. Let $A = R(J, \complement(U \setminus I))$. Then $f_j(A)$ is a ring $R(C_{0j}, C_{1j})$, $C_{0j} = f_j(J)$ is compact, $f_j(\overline{U})$ is compact in $\mathbb{R}^n$ and $C_{1j} = \complement(f_j(U \setminus I)) \supset \complement f_j(U)$ is unbounded, since $q(\complement f_j(U)) = 1$ for $j \in \mathbb{N}$. Let $r_j = q(f_j(a_1), f_j(a_2))$ and $t_j = q(f_j(a_2), f_j(a_3))$ for $j \in \mathbb{N}$ and let $\delta = q(f(a_1), f(a_2)) > 0$. Since $r_j \to \delta$, we can suppose that $r_j > \frac{\delta}{2}$ for every $j \in \mathbb{N}$. We see that $q(C_{0j}) \geq r_j$, $q(C_{1j}) \geq 1$, $q(C_{0j}, C_{1j}) \leq t_j$ for every $j \in \mathbb{N}$ and $t_j \to 0$. Let $h = d(J, \complement(U \setminus I)) > 0$ and let $\rho = \frac{1}{h}\mathcal{X}_U$. Then $\rho \in F(\Gamma_A)$ and $\overline{\lambda}_{n,q}(\frac{\delta}{2}, t_j) \leq \overline{\lambda}_{n,q}(r_j, t_j) \leq M_q(f_j(\Gamma_A)) \leq \gamma(M^p_\omega(\Gamma_A)) \leq \gamma(\int\limits_{\mathbb{R}^n} \frac{\omega(z)}{h^p}\mathcal{X}_U(z)dz) \leq \gamma(\frac{1}{h^p}\int\limits_U \omega(z)dz) < \infty$.

On the other side, we see from Theorem 2 in [2] that $\overline{\lambda}_{n,q}(\frac{\delta}{2}, t_j) \to \infty$ and we reached a contradiction. We proved that for every $x \in Q$ there exists $U_x \in \mathcal{V}(x)$, $U_x \subset Q$ such that either $f$ is injective on $U_x$, or $f$ is constant on $U_x$.

Suppose that there exists $z_1, z_2 \in Q$, $z_1 \neq z_2$ such that $f(z_1) = f(z_2)$ and let $\epsilon > 0$ be such that $z_2 \notin \overline{B}(z_1, \epsilon)$. Since $S(z_1, \epsilon)$ separates the points $z_1$ and $z_2$ and $f_j$ is a homeomorphism, we see that also $f_j(S(z_1, \epsilon))$ separates the points $f_j(z_1)$ and $f_j(z_2)$ for every $j \in \mathbb{N}$. Let $x_{j\epsilon} \in S(z_1, \epsilon)$ be such that

$$q(f_j(x_{j\epsilon}), f_j(z_1)) \leq q(f_j(z_1), f_j(z_2))\ for\ every\ j \in \mathbb{N} \tag{1}$$

Passing to a subsequence, we may assume that there exists $x_\epsilon \in S(z_1, \epsilon)$ such that $x_{j\epsilon} \to x_\epsilon$. Using the equicontinuity of the family $W$ in $x_\epsilon$, we see that $q(f_j(x_{j\epsilon}), f(x_\epsilon)) \leq q(f_j(x_{j\epsilon}), f_j(x_\epsilon)) + q(f_j(x_\epsilon), f(x_\epsilon)) \to 0$, hence $f_j(x_{j\epsilon}) \to f(x_\epsilon)$. Letting $j \to \infty$ in (1), we find that $f(x_\epsilon) = f(z_1)$.

We proved that if $z_2 \notin \overline{B}(z_1, \epsilon)$, then $f$ is not injective on $\overline{B}(z_1, \epsilon)$.

Let $Q_1 = \{x \in Q |$ there exists $U \in \mathcal{V}(x)$ such that $U \subset Q$ and $f$ is injective on $U\}$ and let $Q_2 = \{x \in Q |$ there exists $U \in \mathcal{V}(x)$ such that $U \subset Q$ and $f$ is constant on $U\}$. Then $Q_1$ and $Q_2$ are open and disjoint, $Q = Q_1 \cup Q_2$, $Q$ is connected and since $z_1 \notin Q_1$, we see that $z_1 \in Q_2$ and hence $Q_2 \neq \phi$. It results that $Q = Q_2$, hence $f$ is constant on $Q$ and since $f$ is continuous on $D$, we see that $f$ is constant on $D$.

Suppose that $f$ is not constant on $D$. We proved that in this case $f$ is injective on $Q$ and using Lemma 1, we see that $f(Q) \subset Ker_{j\to\infty} f_j(Q)$. Let $A = R(Q_0, Q_1)$ be a ring such that $\overline{A}$ is compact in $Q$ and let $B_0 = Q_0 \cap \overline{A}$, $B_1 = Q_1 \cap \overline{A}$. Since $\overline{A}$ is compact and $f(Q) \subset Ker_{j\to\infty} f_j(Q)$, we can find $j_0 \in \mathbb{N}$ such that $f(A) \subset f_j(A)$ for every $j \geq j_0$. Using Lemma 6 in [6], we see that $M_q(f_j(B_0), f_j(B_1), f(A)) \leq M_q(f_j(B_0), f_j(B_1), f_j(A)) = M_q(f_j(\Gamma_A)) \leq \gamma(M^p_\omega(\Gamma_A))$ for every $j \geq j_0$ and letting $j \to \infty$, we find that $M_q(f(\Gamma_A)) \leq \gamma(M^p_\omega(\Gamma_A))$, i.e. $f$ is a generalized ring $(q, M^p_\omega, \gamma)$ homeomorphism on $Q$ and hence $f$ is a generalized ring $(q, M^p_\omega, \gamma)$ homeomorphism on $D$.

We proved that if $f$ is not constant on $D$, there exists $G \subset \mathbb{R}^n$ a domain such that $f : D \to G$ is a generalized ring $(q, M^p_\omega, \gamma)$ homeomorphism on $D$.

Suppose that $f : D \to G$ is a generalized $(q, M^p_\omega, \gamma)$ homeomorphism and let $K \subset G$ be compact in $G$. Let $U \subset\subset D$ be a domain such that $K \subset f(U)$. Let $G_j = f_j(U)$, $h_j = f_j|U : U \to f_j(U)$ for $j \in \mathbb{N}$. Then $f(U) \subset Ker_{j\to\infty} f_j(U)$ and if $y \in K$, we can find $j(y) \in \mathbb{N}$ and $U_y \in \mathcal{V}(y)$ such that $U_y \subset G_j$ for $j \geq j(y)$. Since $K \subset \bigcup\limits_{y\in K} U_y$ and $K$ is compact, we can find

$y_1, ..., y_m \in K$ such that $K \subset \bigcup_{i=1}^{m} U_{y_i}$ and taking $j_0 = \max\{j(y_1), ..., j(y_m)\}$, we see that $K \subset G_j$ for every $j \geq j_0$. It results that $g_j = h_j^{-1} : G_j \to U$ is well defined for every $j \geq j_0$ and let $g : f(U) \to U$, $g = (f|U)^{-1}$. We show that $g_j \to g$ uniformly on $K$.

Suppose that this thing if false. Taking if necessarily a subsequence, we can find $\epsilon > 0$ and $y, y_j \in K$ such that $|g_j(y_j) - g(y_j)| > \epsilon$ for every $j \in \mathbb{N}$ and $y_j \to y$. Let $x_j = g(y_j)$ and $z_j = g_j(y_j)$ for $j \in \mathbb{N}$. Taking if necessarily a subsequence, we can suppose that there exists $x, z \in \overline{U}$ such that $x_j \to x$, $z_j \to z$. Due to equicontinuity of the family $W$ in $z$, we see that $f_j(z_j) \to f(z)$ and we also see that $f_j(z_j) = y_j = f(x_j) \to f(x)$ and we obtain that $f(x) = f(z)$. On the other side, $|x_j - z_j| \geq \epsilon$ for every $j \in \mathbb{N}$ and letting $j \to \infty$, we have that $|x - z| \geq \epsilon$, hence $x \neq z$ and $f(x) = f(z)$. We reached a contradiction, since we supposed that $f$ is injective on $D$.

We proved that $g_j \to g$ uniformly on $K$.

**Proof of Corollary 10.** If condition 1) holds, we see from Theorem A that $M_q(f_j(\Gamma)) \leq M^q_{K_{I,q}(f_j)}(\Gamma) \leq M^q_\omega(\Gamma)$ for every $\Gamma \in A(D)$ and every $j \in \mathbb{N}$. We apply now Theorem 10 and we see that if $f : D \to G$ is a homeomorphism, then $f$ is a generalized ring $(q, M^q_\omega, \gamma)$ homeomorphism, where $\gamma(t) = t$ for $t > 0$.

If condition 2) holds, we see from Theorem A that $M_q(f_j(\Gamma)) \leq K M_p(\Gamma)^{q/p}$ for every $\Gamma \in A(D)$ and every $j \in \mathbb{N}$ and we see from Theorem 19 in [2] that $M_p(x) = 0$ if $x \in \mathbb{R}^n$ and $M_n(\infty) = 0$. We apply Theorem 10 and we find that if $f : D \to G$ is a homeomorphism, then it is a generalized ring $(q, M_p, \gamma)$ homeomorphism, where $\gamma(t) = K t^{q/p}$ for $t > 0$.

We continue the researches concerning the properties of the limit mapping of a sequence of generalized ring $(q, M, \gamma)$ homeomorphisms, $n - 1 < q \leq n$, and we also study the case $q = n$.

We give first an equicontinuity result concerning families of ring $(q, M, \gamma)$ homeomorphisms $f : D \subset \overline{\mathbb{R}^n} \to D_f \subset \overline{\mathbb{R}^n}$, $q > n-1$ extending a known result from the theory of quasiconformal mappings (see Theorem 19.2, page 65 in [45]). The following result was given in Theorem 16 in [9] for families $W$ of ring $(q, M, \gamma)$ homeomorphisms, $q > n - 1$, establishing the equicontinuity of such a family in points $x \neq \infty$. We give here an alternate proof, valid also for the point $x = \infty$.

**Proposition 1.** Let $q > n - 1$, $D \subset \overline{\mathbb{R}^n}$ a domain, $x \in D$, $W$ be a family of ring $(q, M, \gamma)$ homeomorphisms $f : D \to D_f \subset \overline{\mathbb{R}^n}$ in $x$ and suppose that there exists $r > 0$ such that each $f \in W$ omits some points $a_f, b_f \notin \mathrm{Im} f$ with $q(a_f, b_f) \geq r$. Then the family $W$ is equicontinuous at $x$ and we take on $D$ and on $\overline{\mathbb{R}^n}$ the chordal metric.

**Proof:** Suppose that $x \neq \infty$, let $b > 0$ be such that $\overline{B}(x, b) \subset D$ and let $0 < \epsilon < r$. Let $0 < a_\epsilon < b$ be such that $\gamma(\Delta_M(\Gamma_{x,a_\epsilon,b})) < \lambda_{n,q}(\epsilon)$ and let $f \in W$. Then $f(\Gamma_{x,a_\epsilon,b})$ is a ring $A = R(C_{0f}, C_{1f})$. We see that if $f(x) \neq \infty$, then $C_{0f} = f(\overline{B}(x, a_\epsilon))$ is bounded and $C_{1f} = \complement f(B(x, b))$ is unbounded, and if $f(x) = \infty$, then $C_{0f}$ is unbounded and $C_{1f}$ is bounded. Also, $a_f, b_f \in C_{1f}$, hence $r \leq q(a_f, b_f) \leq q(C_{1f})$. Let $y \in B(x, a_\epsilon)$ and let $t = \min\{r, q(f(x), f(y))\}$. Then $q(C_{0f}) \geq q(f(x), f(y))$ and $q(C_{0f}) \geq t$, $q(C_{1f}) \geq t$, hence $\lambda_{n,q}(t) \leq M_q(\Gamma_A) = M_q(f(\Gamma_{x,a_\epsilon,b})) \leq \gamma(\Delta_M(\Gamma_{x,a_\epsilon,b})) < \lambda_{n,q}(\epsilon)$.

Since the function $\lambda_{n,q}$ is increasing and $\epsilon < r$, we see that $q(f(x), f(y)) \leq \epsilon$ for every $y \in B(x, a_\epsilon)$ and every $f \in W$. It results that the family $W$ is equicontinuous at $x$. The proof is similar if $x = \infty$.

**Theorem 11.** Let $n \geq 2$, $p > 1$, $D \subset \overline{\mathbb{R}^n}$ a domain, $\omega \in L^1_{loc}(D)$ such that $\Delta^p_\omega(x) = 0$ for every $x \in D$, $\gamma : [0, \infty) \to [0, \infty)$ increasing with $\lim_{t \to 0} \gamma(t) = 0$ and let $f_j : D \to D_j \subset \overline{\mathbb{R}^n}$ be generalized ring $(n, M^p_\omega, \gamma)$ homeomorphisms such that $f_j \to f$. Then there are three possibilities:

1) $f : D \to G$ is a generalized ring $(n, M^p_\omega, \gamma)$ homeomorphism onto a domain $G \subset Ker\_j \to \infty D_j$, $f_j \to f$ uniformly on the compact subsets of $D$ and if $K \subset G$ is compact, there exists $j_0 \in \mathbb{N}$ such that $K \subset D_j$ for every $j \geq j_0$ and $f_j^{-1}|K \to f^{-1}|K$ uniformly on $K$.

2) Im$f = \{b_1, b_2\}$, $b_1 \neq b_2$ and there exists $a_1 \in D$ such that $f(a_1) = b_1$ and $f(x) = b_2$ for every $x \in D \setminus \{a_1\}$ and the convergence is not uniform on the compact subsets of $D$.

3) $f$ is constant. The convergence may be uniform on the compact subsets of $D$ or not.

**Proof:** If Im$f = \{b_1, b_2\}$ with $b_1 \neq b_2$ and $f(a_1) = b_1$, $f(a_2) = b_2$, we can suppose that there exists $r > 0$ such that $q(f_j(a_1), f_j(a_2)) > r > 0$ for every $j \in \mathbb{N}$ and using Proposition 1, we see that the family $W = (f_j)_{j\in\mathbb{N}}$ is equicontinuous on $D \setminus \{a_1, a_2\}$ and since $f$ is continuous on $D$, we see that either $f(x) = b_1$ for every $x \in D \setminus \{a_1, a_2\}$, or $f(x) = b_2$ for every $x \in D \setminus \{a_1, a_2\}$.

Suppose now that there exist three different points $b_i = f(a_i)$, $i = 1, 2, 3$. We can suppose that there exists $r > 0$ such that $q(f_j(a_i), f_j(a_k)) > r$ for $j \in \mathbb{N}$, $i, k = 1, 2, 3$, $i \neq k$. Using Proposition 1, we see that the family $W = (f_j)_{j\in\mathbb{N}}$ is equicontinuous on each sets $D \setminus \{a_1, a_2\}$, $D \setminus \{a_2, a_3\}$, $D \setminus \{a_1, a_3\}$ and this implies that the family $W$ is equicontinuous on $D$. Using Theorem 20.3 in [45], we see that $f_j \to f$ uniformly on the compact subsets of $D$.

If there exists $j_0 \in \mathbb{N}$ such that $\infty \notin$Im$f_j$ for every $j \geq j_0$, we take $Q = D \setminus \{\infty\}$. If this thing is not true, taking if necessarily a subsequence, we may suppose that there exists $x_j \in D$ such that $f_j(x_j) = \infty$ for every $j \in \mathbb{N}$ and we may also suppose that there exists $x_0 \in \overline{D}$ such that $x_j \to x_0$. We take in this case $Q = D \setminus \{x_0, \infty\}$.

Take $x \in Q$. Since $f$ is continuous in $x$, we take $r > 0$ small enough such that $d(x_0, x) > 2r$ and $q(f(\overline{B}(x, r)) < \frac{1}{4}$. Since $x_j \to x_0$, we can also suppose that $x_j \in B(x_0, r)$ for $j \in \mathbb{N}$ and since $f_j \to f$ uniformly on $\overline{B}(x, r)$, there exists $j_0 \in \mathbb{N}$ such that $q(f_j(z), f(z)) < \frac{1}{4}$ for every $z \in \overline{B}(x, r)$ and every $j \geq j_0$. We take $U = B(x, r)$ and we see that $x_j \notin \overline{U}$, $f_j(U) \subset B_q(f(x), \frac{1}{2})$ and hence $q(\complement f_j(U)) \geq 1$ for every $j \geq j_0$. We can take $j_0 = 1$. Since every $f_j(\overline{U})$ is compact in $\mathbb{R}^n$, we use the argument from Theorem 10 to see that either $f$ is constant on $U$, or $f$ is injective on $U$. We can prove as in Theorem 10 that either $f$ is constant on $Q$ (and hence on $D$), or there exists $G \subset Ker_{j\to\infty} D_j$ such that $f : D \to G$ is a generalized ring $(n, M^p_\omega, \gamma)$ homeomorphism. Also, in the last case, if $K \subset G$ is compact, there exists $j_0 \in \mathbb{N}$ such that $K \subset D_j$ for every $j \geq j_0$ and $f_j^{-1}|K \to f^{-1}|K$ uniformly on $K$.

**Theorem 12.** Let $n \geq 2$, $q, r \in (n-1, n]$, $p, t > 1$, $D, D_j$ be domains in $\overline{\mathbb{R}^n}$ for $j \in \mathbb{N}$, let $\omega \in L^1_{loc}(D)$ such that $\Delta^p_\omega(x) = 0$ for every $x \in D$, $A = Ker_{j\to\infty} D_j$, $\eta \in L^1_{loc}(A)$ such that $\Delta^t_\eta(y) = 0$ for every $y \in A$, $\gamma, \lambda : [0, \infty) \to [0, \infty)$ increasing with $\lim\limits_{s\to 0} \gamma(s) = 0$, $\lim\limits_{s\to 0} \lambda(s) = 0$, let $f_j : D \to D_j$ be generalized ring $(q, M^p_\omega, \gamma)$ homeomorphisms such that $g_j = f_j^{-1}|A : A \to f_j^{-1}(A)$ is a generalized ring $(r, M^t_\eta, \lambda)$ homeomorphism for every $j \in \mathbb{N}$ and $f_j \to f$. Suppose that if $r = n$, then Card$\partial D \geq 2$ and if $q = n$, then either $f_j \to f$ uniformly on the compact subsets of $D$, or there exists $r_0 > 0$ and $a_j, b_j \notin D_j$ such that $q(a_j, b_j) \geq r_0$ for every $j \in \mathbb{N}$. Then either $f(x) = c$ for every $x \in D$ and $c \in \complement(A \cup Ker_{j\to\infty} \complement D_j)$, or there exists a component $G$ of $A$ such that $f : D \to G$ is a generalized ring $(q, M^p_\omega, \gamma)$ homeomorphism, $f^{-1} : G \to D$ is a generalized ring $(r, M^t_\eta, \lambda)$ homeomorphism and $f_j^{-1} \to f^{-1}$ uniformly on the compact subsets of $G$.

**Proof:** If $q = n$ and there exists $r_0 > 0$ and $a_j, b_j \notin D_j$ such that $q(a_j, b_j) \geq r_0$ for every $j \in \mathbb{N}$, we see from Proposition 1 that the family $W = (f_j)_{j\in\mathbb{N}}$ is equicontinuous on $D$ and since $f_j \to f$, we see from Theorem 20.3, page 68 in [45] that $f_j \to f$ uniformly on the compact subsets of $D$. If $n - q < q < n$, we see from Theorem 5 that the family $W$ is equicontinuous. We proved in all cases that $f_j \to f$ uniformly on the compact subsets of $D$.

We also see from Theorem 10 and Theorem 11 that either $f(x) = c$ for every $x \in D$, or there exists a domain $G \subset A$ such that $f : D \to G$ is a generalized ring $(q, M^p_\omega, \gamma)$ homeomorphism and $f_j^{-1} \to f^{-1}$ uniformly on the compact subsets of $G$. Suppose that $f : D \to G$ is a homeomorphism and let $Q$ be the component of $A$ containing $G$. If there exists a point $b \in \partial G \cap Q$, let $V \in \mathcal{V}(b)$ be such that there exists $j_0 \in \mathbb{N}$ such that $V \subset D_j \cap Q$ for every $j \geq j_0$. Let $h_j = f_j^{-1}|V : V \to f_j^{-1}(V) \subset D_j$ for $j \geq j_0$. Using Theorem 5 or Proposition 1, we see that the family $(h_j)_{j \geq j_0}$ is equicontinuous on $V$ and using Ascoli's theorem, we can suppose that there exists $h : V \to \overline{\mathbb{R}^n}$ such that $h_j \to h$ uniformly on the compact subsets of $V$. Using Theorem 10 or Theorem 11, we see that either $h$ is constant on $V$, or $h$ is injective on $V$. The first case cannot hold, since we also see from Theorem 10 or Theorem 11 that $h|V \cap G = f^{-1}|V \cap G$. It results that $h : V \to h(V)$ is a homeomorphism.

We see from Lemma 1 that $h(V) \subset Ker_{j\to\infty} h_j(V) \subset D$, hence $h(b) \in D$. Let now $b_j \in G \cap V$ be such that $b_j \to b$. Then $h(b_j) \to h(b)$ and $b_j = f(h(b_j)) \to f(h(b))$ and this implies that $b = f(h(b)) \in f(D) = G$. We reached a contradiction, since we chose $b \in \partial G \cap Q$. It results that $G = Q$ and we see from Theorem 10 and Theorem 11 that $f^{-1}$ is a generalized ring $(r, M^t_\eta, \lambda)$ homeomorphism.

Suppose now that $f(x) = c$ for every $x \in D$. It is clear that $c \in \complement(Ker_{j\to\infty} \complement D_j)$ and suppose that $c \in A$. Let $j_0 \in \mathbb{N}$ and let $V \in \mathcal{V}(c)$ be such that $V \subset D_j$ for $j \geq j_0$ and let $h_j = f_j^{-1}|V : V \to f_j^{-1}(V) \subset D$ for $j \geq j_0$. Let $x \in D$. Then $f_j(x) \to c$ and using the equicontinuity of the family $(h_j)_{j\geq j_0}$ at the point $c$, we see that $q(x, h_j(c)) = q(h_j(f_j(x)), h_j(c) \to 0$. We obtain that $h_j(c) \to x$ for every $x \in D$ and hence $h_j(c) = x$ for every $x \in D$. We reached a contradiction, and we proved that $c \in \complement A$.

We immediately obtain:

**Theorem 13.** Let $n \geq 2$, $q, r \in (n-1, n]$, $p, t > 1$, $D, G$ be domains in $\overline{\mathbb{R}^n}$, $\omega \in L^1_{loc}(D)$ such that $\Delta^p_\omega(x) = 0$ for every $x \in D$, $\eta \in L^1_{loc}(G)$ such that $\Delta^t_\eta(y) = 0$ for every $y \in G$, let $\gamma, \lambda : [0, \infty) \to [0, \infty)$ increasing such that $\lim\limits_{s\to 0} \gamma(s) = 0$, $\lim\limits_{s\to 0} \lambda(s) = 0$ and suppose that if $r = n$, then Card$\partial D \geq 2$ and if $q = n$, then Card$\partial G \geq 2$. Let $f_j : D \to G$ be generalized ring $(q, M^p_\omega, \gamma)$ homeomorphisms such that their inverses are generalized ring $(r, M^t_\eta, \lambda)$ homeomorphisms such that $f_j \to f$. Then either $f(x) = c$ for every $x \in D$ and $c \in \partial D$, or $f : D \to G$ is a generalized ring $(q, M^p_\omega, \gamma)$ homeomorphism and $f^{-1} : G \to D$ is a generalized ring $(r, M^t_\eta, \lambda)$ homeomorphism and $f_j^{-1} \to f^{-1}$ uniformly on the compact subsets of $D$.

**Corollary 12.** Let $n \geq 2$, $q, r \in (n-1, n]$, $D, D_j$ be domains in $\overline{\mathbb{R}^n}$ for $j \in \mathbb{N}$, $A = Ker_{j\to\infty} D_j$, $\omega \in L^1_{loc}(D)$, $\eta \in L^1_{loc}(A)$, let $f_j : D \to D_j$ be $ACL^r$ homeomorphisms satisfying condition $(N)$ such that their inverses are $ACL^q$ and satisfy condition $(N)$ for every $j \in \mathbb{N}$ and $f_j \to f$. Suppose that if $r = n$, then Card$\partial D \geq 2$ and if $q = n$, then either $f_j \to f$ uniformly on the compact subsets of $D$, or there exists $r_0 > 0$ and $a_j, b_j \notin D_j$ such that $q(a_j, b_j) \geq r_0$ for every $j \in \mathbb{N}$. Suppose that one of the following conditions holds:

1) $K_{I,q}(f_j) \leq \omega$ for every $j \in \mathbb{N}$ and $\Delta^q_\omega(x) = 0$ for every $x \in D$.

2) There exists $q < p \leq n$ and $K > 0$ such that $(\int\limits_D K_{I,q}(f_j)(x)^{p/(p-q)} dx)^{\frac{p-q}{p}} < K$ for every $j \in \mathbb{N}$ and if $p < n$, then $\infty \notin D$.

Let $g_j : A \to f_j^{-1}(A)$ be the inverse of $f_j|f_j^{-1}(A) : f_j^{-1}(A) \to A$ for $j \in \mathbb{N}$. Suppose that also one of the following conditions holds:

3) $K_{0,r}(f_j) \circ g_j \leq \eta$ for every $j \in \mathbb{N}$ and $\Delta^r_\eta(y) = 0$ for every $y \in A$.

4) There exists $r < m \leq n$ and $T > 0$ such that $(\int\limits_{D_j} K_{0,r}(f_j)(g_j(y))^{m/(m-r)} dy)^{\frac{m-r}{m}} < T$ for every $j \in \mathbb{N}$ and if $m < n$, then $\infty \notin A$.

Then either $f(x) = c$ for every $x \in D$ and $c \in \complement(A \cup Ker_{j\to\infty}\complement D_j)$, or there exists a component $G$ of $A$ such that $f : D \to G$ is a generalized ring homeomorphism and its inverse is a generalized ring homeomorphism.

**Proof:** If condition 1) holds, we see from Theorem A that $M_q(f_j(\Gamma)) \le M^q_{K_{I,q}(f_j)}(\Gamma) \le M^q_\omega(\Gamma)$ for every $\Gamma \in A(D)$ and every $j \in \mathbb{N}$ and if condition 2) holds, then $M_q(f_j(\Gamma)) \le KM_p(\Gamma)^{q/p}$ for every $\Gamma \in A(D)$ and every $j \in \mathbb{N}$.

If condition 3) holds, we see from Theorem B that $M_r(\Gamma) \le M^r_{K_{0,r}(f_j)\circ g_j}(f_j(\Gamma)) \le M^r_\eta(f_j(\Gamma))$ for every $\Gamma \in A(D)$ and every $j \in \mathbb{N}$, hence $M_r(g_j(\Gamma)) \le M^r_\eta(\Gamma)$ for every $\Gamma \in A(A)$ and every $j \in \mathbb{N}$. If condition 4) holds, we see from Theorem B that $M_r(\Gamma) \le TM_m(f_j(\Gamma))^{r/m}$ for every $\Gamma \in A(D)$ and every $j \in \mathbb{N}$, hence $M_r(g_j(\Gamma)) \le TM_m(\Gamma)^{r/m}$ for every $\Gamma \in A(A)$ and every $j \in \mathbb{N}$.

We see from Theorem 12 that either $f(x) = c$ for every $x \in D$ and $c \in \complement(A \cup Ker_{j\to\infty}\complement D_j)$, or there exists a component $G$ of $A$ such that $f : D \to G$ is a homeomorphism. If condition 1) holds, then $f$ is a generalized ring $(q, M^q_\omega, \gamma)$ homeomorphism, where $\gamma(t) = t$ for $t > 0$ and if condition 2) holds, then $f$ is a generalized ring $(q, M_p, \gamma)$ homeomorphism, where $\gamma(t) = Kt^{q/p}$ for $t > 0$. Also, if condition 3) holds, then $f^{-1}$ is a generalized ring $(r, M^r_\eta, \gamma)$ homeomorphism, where $\gamma(t) = t$ for $t > 0$, and if condition 4) holds, then $f^{-1}$ is a generalized ring $(r, M_m, \gamma)$ homeomorphism, where $\gamma(t) = Tt^{r/m}$ for $t > 0$.

We immediately obtain:

**Corollary 13.** Let $n \ge 2$, $q, r \in (n-1, n]$, $D, G$ domains in $\overline{\mathbb{R}^n}$ such that $\mathrm{Card}\partial G \ge 2$ if $q = n$, $\mathrm{Card}\partial D \ge 2$ if $r = n$, let $\omega \in L^1_{loc}(D)$, $\eta \in L^1_{loc}(G)$, let $f_j : D \to G$ be $ACL^r$ homeomorphisms satisfying condition $(N)$ and their inverses $g_j : G \to D$ are $ACL^q$ and satisfy condition $(N)$ for every $j \in \mathbb{N}$ and $f_j \to f$. Suppose that one of the following conditions holds:

1) $K_{I,q}(f_j) \le \omega$ for every $j \in \mathbb{N}$ and $\Delta^q_\omega(x) = 0$ for every $x \in D$.

2) There exists $q < p \le n$ and $K > 0$ such that $(\int\limits_D K_{I,q}(f_j)(x)^{p/(p-q)}dx)^{\frac{p-q}{p}} < K$ for every $j \in \mathbb{N}$ and if $p < n$, then $\infty \notin D$.

Suppose that also one of the following conditions holds:

3) $K_{0,r}(f_j) \circ g_j \le \eta$ for every $j \in \mathbb{N}$ and $\Delta^r_\eta(y) = 0$ for every $y \in G$.

4) There exists $r < m \le n$ and $T > 0$ such that $(\int\limits_G K_{0,r}(f_j)(g_j(y))^{m/(m-r)}dy)^{\frac{m-r}{m}} < T$ for every $j \in \mathbb{N}$ and if $m < n$, then $\infty \notin G$.

Then either $f(x) = c$ for every $x \in D$ and $c \in \partial G$, or $f : D \to G$ is a generalized ring homeomorphism and its inverse is also a generalized ring homeomorphism.

**References**


1) K. Astala, T. Iwaniecz, P. Koskela and G. Martin, Mappings of BMO-bounded distortion, Math. Annalen, 317, 4(2000), 703-726.

2) P. Caraman, Relations between $p$-capacity and $p$-module (II), Rev. Roumaine Math. Pures Appl., 39, 6(1994), 555-577.

3) M. Cristea, Mappings of finite distortion. Boundary extension. Revue Roumaine Math. Pures Appl., 51, 5-6 (2006), 607-631.

4) M. Cristea, Mappings of finite distortion, Zoric's theorem and equicontinuity results, Revue Roumaine Math. Pures Appl, 52, 5(2007), 539-554.

5) M. Cristea, Local homeomorphisms having local $ACL^n$ inverses, Complex Variables and Elliptic Equations, 53, 1(2008), 77-99.

6) M. Cristea, Open, discrete mappings having local $ACL^n$ inverses, Complex Variables and Elliptic Equations, 55, 1-3(2010), 61-90.

7) M. Cristea, Dilatations of homeomorphisms satisfying some modular inequalities, Rev. Roumaine Math. Pures Appl., 56, 4(2011), 275-282.

8) M. Cristea, Mappings satisfying some modular inequalities, Rev. Roumaine Math. Pures Appl., 57, 2(2012), 117-132.

9) M. Cristea, On generalized quasiregular mappings, Complex Variables and Elliptic Equations, 58, 12(2013), 1745-1764.

10) M. Cristea, Some metric relations of homeomorphisms satisfying generalized modular inequalities (I), Math. Reports, 15(65), 4(2013), 387-396.

11) M. Cristea, On generalized quasiconformal mappings, Complex Variables and Elliptic Equations, 59, 2(2014), 232-246.

12) M. Cristea, Local homeomorphisms satisfying generalized modular inequalities, Complex Variables and Elliptic Equations, 59, 10, (2014), 1363-1387.

13) M. Cristea, Some metric relations of homeomorphisms satisfying generalized modular inequalities (II), Analele Univ. Bucuresti, Matem., 5(LXIII), (2014), 49-60.

14) M. Cristea, Boundary behavior of the mappings satisfying generalized inverse modular inequalities, Complex Variables and Elliptic Equations, DOI:10.1080/17476933.2014.940927.

15) A. Golberg, R. Salimov and E. Sevostyanov, Normal families of discrete, open mappings with controlled p-module, Contemporary Mathematics (to appear).

16) A. Golberg, R. Salimov and E. Sevostyanov, Distortion estimates under mappings with controlled $p$-module, Analele Univ. Bucuresti, Matem., 5 (LXIII), (2014), 95-105.

17) V. I. Kruglikov, Capacity of condensers and spatial mappings quasiconformal in the mean, Mat. Sb.(N.S.) 130(172), 2(6), (1986), 185-206 (in Russian).

18) P. Koskela and J. Onninen, Mappings of finite distortion: the sharp modulus of continuity, Trans. AMS, Vol. 355, 5(2003), 1905-1920.

19) P. Koskela, J. Onninen and K. Rajala, Mappings of finite distortion: Injectivity radius of local homeomorphisms, Future Trends in Geometric Function Theory, Vol. 92, Rep. Univ. Jyväskylä, Dept. Mat. Stat., Univ. of Jyväskylä (2003), 169-174.

20) P. Koskela and J. Onninen, Mappings of finite distortion: Capacity and modulus inequalities, J. Reine Angew. Math., 599(2005), 1-26.

21) D. Kovtoniuk, V. Ryazanov, R. Salimov and E. Sevostyanov, On mappings in the Orlicz-Sobolev speces, Algebra and Analysis, Vol. 25, 6(2013), 49-92 (in Russian).

22) T. Iwaniecz, P. Koskela, J. Onninen, Mappings of finite distortion: monotonicity and continuity, Inv. Math., 144 (2001), 507-531.

23) O, Martio, V. Ryazanov, U. Srebro and E. Yakubov, Mappings of finite length distortion, J. D'Analyse Math., 93(2004), 215-236.

24) O. Martio, V. Ryazanov, U. Srebro and E. Yakubov, On Q-homeomorphisms, Ann. Acad. Sci. Fenn., Math., 30(2005), 49-69.

25) O. Martio, V. Ryazanov, U. Srebro and E. Yakubov, Moduli in Modern Mapping Theory, Springer Monographs in Math., Springer, New York, (2009).

26) V. Mazya, Lectures on isoperimetric and isocapacitary inequalities in the theory of Sobolev spaces, Contemporary Mathematics, 338(2003), 307-340.

27) K. Rajala, Mappings of finite distortion: Removable singularities for locally homeomorphic mappings, Proc. Amer. Math. Soc., 123 (2004), 3251-3258.

28) K. Rajala, Mappings of finite distortion: Removability of Cantor sets, Ann. Acad. Sci. Fenn., Math., 26(2004), 269-281.

29) Y. G. Reshetnyak, Space Mappings of Bounded Distortion, Translations of Mathematical Monographs., Vol. 73, AMS, Providence, RI, (1989).

30) S. Rickman, Quasiregular Mappings, Springer Verlag, Berlin, Vol. 26, (1993).

31) V. Ryazanov and R. Salimov, Weakly flat spaces and boundaries in the mapping theory, Ukr. Math. Visn., 4(2), (2007), 199-234, transl. in Ukr. Math. Bull., 4(2), (2007), 199-233.

32) V. Ryazanov, R. Salimov and E. Sevostyanov, On convergence analysis of space homeomorphisms, Siberian Advances in Math., Vol. 23, 4(2013), 263-293.

33) V. Ryazanov and E. Sevostyanov, Towards the theory of ring $Q$-homeomorphisms, Israel J. Math., 168(2008), 101-115.

34) V. Ryazanov and E. Sevostyanov, Equicontinuity of mappings quasiconformal in the mean, Ann. Acad. Sci. Fenn., Math., 36(2011), 1-14.

35) V. Ryazanov and E. Sevostyanov, On convergence and compactness of spatial homeomorphisms, Revue Roumaine Math. Pures Appl., 58, 1(2013), 85-104.

36) R. Salimov, On regular homeomorphisms in the plane, Ann. Acad. Sci. Fenn., Math., 35(2010), 285-289.

37) R. Salimov, On ring $Q$-homeomorphisms with respect to $p$-modulus, Analele Univ. Bucuresti, Matem., 2(LX), (2011), 207-213.

38) R. Salimov, On the Lipschitz property of a class of mappings, Mat. Zametki, 94, 4(2013), 591-599.

39) R. Salimov and E. Sevostyanov, The theory of shell-based $Q$-mappings in geometric function theory, Sbornik: Math., Vol 201(6), (2010), 909-934.

40) R. Salimov and E. Sevostyanov, $ACL$ and differentiability of the open discrete ring mappings, Complex Variables and Elliptic Equations, 55, 1-3, (2010), 49-59.

41) R. Salimov and E. Sevostyanov, The Poletskii and Väisälä inequalities for the mappings with $(p, q)$ distortion, Complex Variables and Elliptic Equations, 59, 2 (2014), 217-231.

42) E. Sevostyanov, Theorems of Liouville, Picard and Sohotskii for ring mappings, Ukrainean Math. Bull., 3(2008), 366-381.

43) E. Sevostyanov, Towards a theory of removable singularities for maps with unbounded characteristics of quasi-conformity, Izvestya, Math., 74(1)(2010), 151-165.

44) E. Sevostyanov, The Väisälä inequality for the mappings with finite distortion, Complex Variables and Elliptic Equations, 55, 1-3, (2010), 91-101.

45) J. *Väisälä*, Lectures on n - dimensional quasiconformal mappings, Lecture Notes in Math., 229, Springer Verlag, Berlin, (1971).

46) M. Vuorinen, Conformal geometry and quasiregular mappings, Lecture Notes in Math., 1319, Springer Verlag, Berlin, (1988).

University of Bucharest
Faculty of Mathematics and Computer Sciences,
Str. Academiei 14, R-010014,
Bucharest, Romania,
Email: mcristea@fmi.unibuc.ro