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TANGENT GROUPS OF 2-STEP NILPOTENT PRE-LIE GROUPS

MIHAI NICOLAE

Abstract. We investigate several types of topological groups: 2-step nilpotent topological groups, groups with Lie algebra and pre-Lie groups. We find several connections between these groups. We study tangent groups of topological groups with Lie algebra and characterize 2-step nilpotent topological groups for which tangent groups are pre-Lie groups.

1. Introduction

We present some elements of Lie theory for 2-step nilpotent topological groups. That allows us to construct the topological Lie algebra of such a group, and in particular to obtain a detailed proof of Theorem 2.23 which says that every 2-step nilpotent topological group is a group with Lie algebra. We present some results on general topological groups, useful in order to characterize the Lie algebras of 2-step nilpotent topological groups. For clarity we define groups with Lie algebra and pre-Lie groups. We introduce the tangent group of any topological group G with Lie algebra, denoted by $T(G)$. We show that the tangent group of a 2-step nilpotent topological group is in turn a 2-step nilpotent group, and finally as the main result of the present paper we characterize 2-step nilpotent topological groups for which the tangent group is a pre-Lie group(Theorem 2.27).

The main tool used in the present investigation is the differential calculus on topological groups which are not necessarily Lie groups; see for instance [BR80], [BCR81], [HM07], [Ne06], [BN14], [BN15], [Ni15].

2. Lie theory for 2-step nilpotent topological groups

Definition 2.1. Let G be any group. We denote

$$[G,G]=\{xyx^{-1}y^{-1}; x,y\in G\}$$

and

$$Z(G)=\{g\in G; xg=gx, (\forall)x\in G\}$$

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which is called the *center* of G and is a commutative subgroup of G.

We say that G is a *2-step nilpotent* group if $[G,G] \subseteq Z(G)$.

We define the commutator map

$$c : G \times G \to Z(G), \quad c(x,y) := xyx^{-1}y^{-1}.$$

Everywhere in what follows we assume that G is 2-step nilpotent group, unless explicitly stated that G is an arbitrary group.

2.1. Some basic properties of the commutator map.

Lemma 2.2. *The commutator map c is a bi-morphism, namely for all $x, y, a, b \in G$ we have:*
(a) $c(y,x) = (c(x,y))^{-1}$
(b) $c(x,ab) = c(x,a)c(x,b)$
(c) $c(ab,x) = c(a,x)c(b,x)$.

Proof. The proof is based on direct calculations, thus:

a) $(c(x,y))^{-1} = (xyx^{-1}y^{-1})^{-1} = yxy^{-1}x^{-1} = c(y,x)$.

b) We have:

$$\begin{aligned} c(x,ab) = c(x,b)c(x,a) &\iff xabx^{-1}b^{-1}a^{-1} = xbx^{-1}b^{-1}xax^{-1}a^{-1} \\ &\iff abx^{-1}b^{-1} = (bx^{-1}b^{-1}x)ax^{-1} \\ &\iff abx^{-1}b^{-1} = abx^{-1}b^{-1}xx^{-1} \end{aligned}$$

which is true.

c) We have

$$\begin{aligned} c(ab,x) &= (c(x,ab))^{-1} \\ &= (c(x,a)c(x,b))^{-1} \\ &= (c(x,b))^{-1}(c(x,a))^{-1} \\ &= c(b,x)c(a,x) \\ &= c(a,x)c(b,x) \end{aligned}$$

and the proof ends. □

Lemma 2.3. *Let G be any 2-step nilpotent group and $a, b \in G$. Then we have*
(a) $c(a^{-1}, b^{-1}) = c(a,b)$
(b) $c(a, b^{-1}) = c(b,a)$
(c) $c(a^{-1}, b) = c(b,a)$
(d) $c(a^m, b^n) = (c(a,b))^{mn}$ *for any m, n natural numbers.*
(e) $abba = baab = ab^2a = ba^2b$.

Proof. a) $c(a^{-1}, b^{-1}) = c(a,b) \iff a^{-1}b^{-1}ab = aba^{-1}b^{-1} \iff$ $(ba)^{-1}ab = ab(ba)^{-1} \iff abba = baab \iff ab(bab^{-1}a^{-1})a^{-1}b^{-1} = \mathbf{1}$ $\iff aba^{-1}b^{-1}bab^{-1}a^{-1} = \mathbf{1} \iff c(a,b)c(b,a) = \mathbf{1}$ which is true and solve point e).

b) $c(a,b^{-1}) = c(b,a) \iff c(a,b^{-1})c(a,b) = \mathbf{1} \iff c(a,\mathbf{1}) = \mathbf{1}$ which is true.

c) $c(a^{-1},b) = c((a^{-1})^{-1},b^{-1}) = c(a,b^{-1}) = c(b,a)$.

d) $c(a^m,b^n) = c(a^m,b)c(a^m,b)\dots c(a^m,b) = (c(a^m,b))^n =$
$(c(a,b)c(a,b)\dots c(a,b))^n = (c(a,b)^m)^n = (c(a,b))^{mn}$. □

2.2. **Lie brackets for continuous subgroups with one parameter.** We define $\Lambda(G)$ as the set of all continuous homomorphisms $\alpha : \mathbb{R} \to G$ from the additive group $(\mathbb{R},+)$ to G. The elements of $\Lambda(G)$ are called continuous subgroups of G with one parameter. The set $\Lambda(G)$ is endowed with a natural topology as follows.

Let G be any topological group with the set of neighborhoods of $\mathbf{1} \in G$ denoted by $\mathcal{V}_G(\mathbf{1})$.

For arbitrary $n \in \mathbb{N}$ and $U \in \mathcal{V}_G(\mathbf{1})$ denote

$$W_{n,U} = \{(\gamma_1,\gamma_2) \in \Lambda(G) \times \Lambda(G) \mid (\forall t \in [-n,n]) \quad \gamma_2(t)\gamma_1(t)^{-1} \in U\}.$$

For every $\gamma_1 \in \Lambda(G)$ define $W_{n,U}(\gamma_1) = \{\gamma_2 \in \Lambda(G) \mid (\gamma_1,\gamma_2) \in W_{n,U}\}$. Then there exists a unique topology on $\Lambda(G)$ with the property that for each $\gamma \in \Lambda(G)$ the family $\{W_{n,U}(\gamma) \mid n \in \mathbb{N},\ U \in \mathcal{V}_G(\mathbf{1})\}$ is a fundamental system of neighborhoods of γ.

Lemma 2.4. *Let G be any 2-step nilpotent topological group and $\alpha,\beta : \mathbb{R} \to G$ two continuos morphisms of groups (from $\Lambda(G)$). Then we have*

$$c(\alpha(1),\beta(t^2)) = c(\alpha(t),\beta(t)), (\forall)t \in \mathbb{R}$$

Proof. The above relation is obvious for $t = 0$.

Let $t = \frac{m}{n}$ with m,n nonzero natural numbers. If we denote

$\alpha(\frac{1}{n}) = a, \beta(\frac{m}{n^2}) = b$ then the formula $c(\alpha(1),\beta(t^2)) = c(\alpha(t),\beta(t))$ is equivalent to $c(a^n,b^m) = c(a^m,b^n) \iff (c(a,b))^{nm} = (c(a,b))^{mn}$ which holds true. Therefore we have shown that $c(\alpha(1),\beta(t^2)) = c(\alpha(t),\beta(t))$ for any t positive rational number.

Let $t \in \mathbb{R}, t > 0$. There existsa a sequence t_n of positive rational numbers such that $\lim_{n\to\infty} t_n = t$. Since α,β are continuous we obtain

$$c(\alpha(1),\beta(t^2)) = \lim_{n\to\infty} c(\alpha(1),\beta(t_n^2)) = \lim_{n\to\infty} c(\alpha(t_n),\beta(t_n)) = c(\alpha(t),\beta(t))$$

If $t \in \mathbb{R}, t < 0$ then we have

$$c(\alpha(1),\beta(t^2)) = c(\alpha(-t),\beta(-t)) = c((\alpha(-t))^{-1},(\beta(-t))^{-1}) = c(\alpha(t),\beta(t))$$

and proof ends. □

Lemma 2.5. *Let G be any 2-step nilpotent group , $a \in G$ and $\beta : \mathbb{R} \to G$ a group morphism. We define $\lambda : \mathbb{R} \to G, \lambda(t) := c(a,\beta(t))$. Then λ is again a group morphism.*

Proof. We have

$$\lambda(t)\lambda(s) = c(a,\beta(t))c(a,\beta(s)) = c(a,\beta(t)\beta(s)) = c(a,\beta(t+s)) = \lambda(t+s)$$

and this concludes the proof. $\square$

Lemma 2.6. *Let G be any 2-step nilpotent topological group and $\alpha,\beta:\mathbb{R}\to G$ be two elements from $\Lambda(G)$. Then we have*

$$[\alpha,\beta](t) = c(\alpha(1),\beta(t)), (\forall)t\in\mathbb{R}$$

and $[\alpha,\beta]\in\Lambda(G)$

Proof. From $(\alpha(\frac{t}{n})\beta(\frac{t}{n})\alpha(-\frac{t}{n})\beta(-\frac{t}{n}))^{n^2} = (c(\alpha(\frac{t}{n}),\beta(\frac{t}{n}))^{n^2} =$ $c((\alpha(\frac{t}{n}))^n,(\beta(\frac{t}{n}))^n) = c(\alpha(\frac{t}{n}n),\beta(\frac{t}{n}n)) = c(\alpha(t),\beta(t)) = c(\alpha(1),\beta(t^2))$ obtain

$[\alpha,\beta](t^2) = c(\alpha(1),\beta(t^2))$ si $[\alpha,\beta](t) = c(\alpha(1),\beta(t)), (\forall)t\in\mathbb{R}, t\geq 0$.

From $[\alpha,\beta](-t^2) = ([\alpha,\beta](t^2))^{-1} = (c(\alpha(1),\beta(t^2)))^{-1} = c(\beta(t^2),\alpha(1)) =$ $c(\alpha(1),(\beta(t^2))^{-1}) = c(\alpha(1),\beta(-t^2))$ we get

$[\alpha,\beta](t) = c(\alpha(1),\beta(t)), (\forall)t\in\mathbb{R}, t<0$.

From the previous lemma it follows that $[\alpha,\beta]:\mathbb{R}\to G$ is morphism.

Since G is a topological group we obtain that $[\alpha,\beta]$ is continuous so $[\alpha,\beta]\in\Lambda(G)$. $\square$

2.3. **Sum of continuous subgroups with a parameter.**

Lemma 2.7. *Let G be any 2-step nilpotent group and $\alpha,\beta:\mathbb{R}\to G$ be two group morphisms. Then $(\alpha(s)\beta(s))^n = \alpha(ns)\beta(ns)c(\alpha(-s),\beta(\frac{n(n-1)}{2}s))$ for any $s\in\mathbb{R}$ and any natural number $n\geq 2$.*

Proof. We will prove the assertion proof by induction on $n\geq 2$.

In the case $n=2$ we have

$$(\alpha(s)\beta(s))^2 = \alpha(s)\beta(s)\alpha(s)\beta(s) = \alpha(s)\alpha(s)\alpha(-s)\beta(s)\alpha(s)\beta(-s)\beta(2s) =$$

$$\alpha(2s)c(\alpha(-s),\beta(s))\beta(2s) = \alpha(2s)\beta(2s)c(\alpha(-s),\beta(s))$$

Induction step n to $n+1$. We have

$$(\alpha(s)\beta(s))^{n+1} = (\alpha(s)\beta(s))^n\alpha(s)\beta(s) =$$

$$\alpha(ns)\beta(ns)\alpha(s)\beta(s)c(\alpha(-s),\beta(\frac{n(n-1)}{2}s)) =$$

$$\alpha(ns)\alpha(s)\alpha(-s)\beta(ns)\alpha(s)\beta(-ns)\beta((n+1)s)c(\alpha(-s),\beta(\frac{n(n-1)}{2}s)) =$$

$$\alpha((n+1)s)c(\alpha(-s),\beta(ns))c(\alpha(-s),\beta(\frac{n(n-1)}{2}s))\beta((n+1)s) =$$

$$\alpha((n+1)s)\beta((n+1)s)c(\alpha(-s),\beta(ns+\frac{n(n-1)}{2}s)) =$$

$$\alpha((n+1)s)\beta((n+1)s)c(\alpha(-s),\beta(\frac{n(n+1)}{2}s))$$

and this concludes the proof. □

Lemma 2.8. *Let G be any 2-step nilpotent topological group and $\alpha, \beta : \mathbb{R} \to G$ be two continuous morfisms of groups. Then we have*

$$c(\beta(s),\alpha(t)) = c(\beta(t),\alpha(s)), (\forall) s,t \in \mathbb{R}$$

Proof. The assertion is obvious for $s = 0$ or $t = 0$. If s, t are positive rational numbers, then $s = \frac{m}{n}, t = \frac{p}{q}$ where m, n, p, q are nonzero natural numbers.

The equality $c(\beta(s),\alpha(t)) = c(\beta(t),\alpha(s))$ is equivalent to

$$c(\beta(\frac{m}{n}),\alpha(\frac{p}{q})) = c(\beta(\frac{p}{q}),\alpha(\frac{m}{n})) \iff$$

$$(c(\beta(\frac{1}{n}),\alpha(\frac{1}{q})))^{mp} = c(\beta(\frac{1}{q}),\alpha(\frac{1}{n}))^{mp} \iff$$

$$(c(\beta(\frac{1}{nq}),\alpha(\frac{1}{q})))^{mpq} = c(\beta(\frac{1}{nq}),\alpha(\frac{1}{n}))^{mnp} \iff$$

$$(c(\beta(\frac{1}{nq}),\alpha(\frac{1}{nq})))^{mnpq} = c(\beta(\frac{1}{nq}),\alpha(\frac{1}{nq}))^{mnpq}$$

which holds true.

We thus showed that $c(\beta(s),\alpha(t)) = c(\beta(t),\alpha(s))$ is true for s, t positive rational numbers.

Let $s, t \in \mathbb{R}, s, t > 0$. There exists sequences s_n, t_n of positive rational numbers such that $\lim_{n\to\infty} t_n = t$, $\lim_{n\to\infty} s_n = s$. Since α, β are continuous we obtain

$$c(\beta(s),\alpha(t)) = \lim_{n\to\infty} c(\beta(s_n),\alpha(t_n)) = \lim_{n\to\infty} c(\beta(t_n),\alpha(s_n)) = c(\beta(t),\alpha(s))$$

There remain the cases remain $(s < 0, t > 0), (s < 0, t < 0), (s > 0, t < 0)$.

In the case $(s < 0, t > 0)$ we have

$$c(\beta(s),\alpha(t)) = c(\alpha(t),\beta(-s)) = c(\alpha(-s),\beta(t)) = c(\beta(t),\alpha(s))$$

In the case $(s < 0, t < 0)$ we have

$$c(\beta(s),\alpha(t)) = c(\beta(-s),\alpha(-t)) = c(\beta(-t),\alpha(-s)) = c(\beta(t),\alpha(s))$$

In the case $(s > 0, t < 0)$ we have

$$c(\beta(s),\alpha(t)) = c(\alpha(-t),\beta(s)) = c(\alpha(s),\beta(-t)) = c(\beta(t),\alpha(s))$$

Therefore in all cases we obtain

$$c(\beta(s),\alpha(t)) = c(\beta(t),\alpha(s)), (\forall)s,t \in \mathbb{R}$$

and the proof ends. □

Lemma 2.9. *Let G be any 2-step nilpotent topological group and $\alpha,\beta:\mathbb{R}\to G$ be two continuous morphisms of groups. Then we have*

$$c(\beta(s),\alpha(t)) = c(\beta(1),\alpha(st)), (\forall)s,t \in \mathbb{R}$$

Proof. As in the proof of the previous lemma it is sufficient to prove the required relation for s,t positive rational numbers, $s=\frac{m}{n}, t=\frac{p}{q}$ where m,n,p,q are nonzero natural numbers.

If we denote $\alpha(\frac{1}{nq}) = a, \beta(\frac{1}{nq}) = b$ it remains to prove that

$c(a^{mq}, b^{np}) = c(a^{nq}, b^{mp}) \iff (c(a,b))^{mqnp} = (c(a,b))^{nqmp}$ which is true and the proof ends. □

Lemma 2.10. *Let G be any 2-step nilpotent group and $\alpha,\beta:\mathbb{R}\to G$ be two group morphisms. We define $\lambda:\mathbb{R}\to G, \lambda(t):=\alpha(t)\beta(t)c(\alpha(t),\beta(-\frac{t}{2}))$. Then λ is a group morphism.*

Proof. We must prove that $(\forall s,t\in\mathbb{R})\quad \lambda(s+t)=\lambda(s)\lambda(t)$

The equality $\lambda(s+t)=\lambda(t)\lambda(s)$ is equivalent to

$$\alpha(t)\alpha(s)\beta(t)\beta(s)c(\alpha(t)\alpha(s),\beta(-\frac{t}{2})\beta(-\frac{s}{2}))$$
$$=\alpha(t)\beta(t)c(\alpha(t),\beta(-\frac{t}{2}))\alpha(s)\beta(s)c(\alpha(s),\beta(-\frac{s}{2}))$$

This equality is equivalent to:

$$\alpha(s)\beta(t)\beta(s)c(\alpha(t),\beta(-\frac{t}{2}))c(\alpha(t),\beta(-\frac{s}{2}))c(\alpha(s),\beta(-\frac{t}{2}))c(\alpha(s),\beta(-\frac{s}{2}))$$
$$=\beta(t)c(\alpha(t),\beta(-\frac{t}{2}))\alpha(s)\beta(s)c(\alpha(s),\beta(-\frac{s}{2}))$$

The equivalence of equalities can be extended thus:

$$\alpha(s)\beta(t)\beta(s)c(\alpha(t),\beta(-\frac{s}{2}))c(\alpha(s),\beta(-\frac{t}{2}))=\beta(t)\alpha(s)\beta(s) \iff$$

$$\alpha(s)\beta(t)c(\alpha(t),\beta(-\frac{s}{2}))c(\alpha(s),\beta(-\frac{t}{2}))=\beta(t)\alpha(s) \iff$$

$$c(\alpha(s),\beta(t))c(\alpha(t),\beta(-\frac{s}{2}))c(\alpha(s),\beta(-\frac{t}{2}))=\mathbf{1} \iff$$

$$c(\alpha(s),\beta(\frac{t}{2}))c(\alpha(t),\beta(-\frac{s}{2}))=\mathbf{1} \iff$$

$$c(\beta(\frac{s}{2}),\alpha(t))=c(\beta(\frac{t}{2}),\alpha(s)) \iff$$

$$(c(\beta(\frac{s}{2}),\alpha(\frac{t}{2})))^2=(c(\beta(\frac{t}{2}),\alpha(\frac{s}{2})))^2$$

which holds true and the proof ends. □

Lemma 2.11. *Let G be any 2-step nilpotent topological group and $\alpha,\beta:\mathbb{R}\to G$ be two elements of $\Lambda(G)$. Then we have*

$$(\alpha+\beta)(t)=\alpha(t)\beta(t)c(\alpha(t),\beta(-\frac{t}{2})),(\forall)t\in\mathbb{R}$$

and $\alpha+\beta\in\Lambda(G)$.

Proof. We have

$$\begin{aligned}
c(\alpha(-\frac{t}{n}),\beta(\frac{n-1}{2}t)) &= (c(\alpha(-\frac{t}{2n}),\beta(\frac{(n-1)}{2}t))^2\\
&= (c(\alpha(-\frac{t}{2n}),\beta(\frac{n(n-1)}{2}\frac{t}{n}))^2\\
&= (c(\alpha(-\frac{t}{2n}),\beta(\frac{t}{n})))^{n(n-1)}\\
&= c(\alpha(-\frac{t}{2n}n),\beta(\frac{t}{n}(n-1)))\\
&= c(\alpha(-\frac{t}{2}),\beta(\frac{t(n-1)}{n}))
\end{aligned}$$

Further

$$\begin{aligned}
(\alpha+\beta)(t) &= \lim_{n\to\infty}(\alpha(\frac{t}{n})\beta(\frac{t}{n}))^n\\
&= \lim_{n\to\infty}\alpha(n\frac{t}{n})\beta(n\frac{t}{n})c(\alpha(-\frac{t}{n}),\beta(\frac{n-1}{2}t))\\
&= \lim_{n\to\infty}\alpha(t)\beta(t)c(\alpha(-\frac{t}{2}),\beta(\frac{t(n-1)}{n}))\\
&= \alpha(t)\beta(t)c(\alpha(-\frac{t}{2}),\beta(t))\\
&= \alpha(t)\beta(t)c(\beta(t),\alpha(\frac{t}{2}))\\
&= \alpha(t)\beta(t)(c(\beta(\frac{t}{2}),\alpha(\frac{t}{2})))^2\\
&= \alpha(t)\beta(t)c(\beta(\frac{t}{2}),\alpha(t))\\
&= \alpha(t)\beta(t)c(\alpha(-t),\beta(\frac{t}{2}))\\
&= \alpha(t)\beta(t)c(\alpha(t),\beta(-\frac{t}{2})).
\end{aligned}$$

Let us mention that we used above the continuity of the morphisms α,β when we then passed to the limit. From Lemma 2.10 it follows that $\alpha+\beta:\mathbb{R}\to G$ is morphism. Since G is a topological group we obtain that $\alpha+\beta$ is continuous so $\alpha+\beta\in\Lambda(G)$ □

2.4. **Subgroups with one parameter in general topological groups.** Lemmas(2.12–2.15) below are available for general topological groups, so we will not use the hypothesis that the group G is a 2-step nilpotent group.

Lemma 2.12. *Let G be any topological group, $\alpha, \beta : \mathbb{R} \to G$ be two continuous morphisms of groups. We define $\lambda : \mathbb{R} \to G, \lambda(t) := \alpha(t)\beta(t)$. Then λ is a morphism if and only if α, β comute and in this case $\lambda = \alpha + \beta$.*

Proof. The map $\lambda : \mathbb{R} \to G$ is morphism if and only if

$$(\forall s, t \in \mathbb{R}) \quad \lambda(t+s) = \lambda(t)\lambda(s) \iff$$

$$\alpha(t)\alpha(s)\beta(t)\beta(s) = \alpha(t)\beta(t)\alpha(s)\beta(s), (\forall s, t \in \mathbb{R}) \iff$$

$$\alpha(s)\beta(t) = \beta(t)\alpha(s), (\forall s, t \in \mathbb{R}) \iff \alpha, \beta \text{ comute.}$$

We have $(\alpha+\beta)(t) = \lim_{n\to\infty} (\alpha(\frac{t}{n})\beta(\frac{t}{n}))^n = \lim_{n\to\infty} (\alpha(\frac{t}{n}))^n (\beta(\frac{t}{n}))^n = \lim_{n\to\infty} \alpha(n\frac{t}{n})\beta(n\frac{t}{n}) = \alpha(t)\beta(t)$ so $\lambda = \alpha + \beta$. □

In the following lemmas, for all $\alpha \in \Lambda(G)$ and $t \in \mathbb{R}$ we use the notation $t\alpha$ for the element of $\Lambda(G)$ defined by $(t\alpha)(s) = \alpha(ts)$ for all $s \in \mathbb{R}$.

Lemma 2.13. *Let G be any topological group, $\alpha : \mathbb{R} \to G$ be a continuous morphism of groups. Then*

$$\alpha + \alpha + \ldots + \alpha = n\alpha, (\forall) n \geq 2$$

Proof. We will do the proof by induction on $n \geq 2$.

In case $n = 2$ since α and α comute obtain

$$(\alpha+\alpha)(t) = \alpha(t)\alpha(t) = \alpha(t+t) = \alpha(2t) = 2\alpha(t), (\forall) t \in \mathbb{R}$$

whence result $\alpha + \alpha = 2\alpha$.

Transition from n to $n+1$. Since $n\alpha$ and α comute obtain

$$(\alpha + \alpha + \ldots + \alpha)(t) = (n\alpha + \alpha)(t) = n\alpha(t)\alpha(t) =$$

$$\alpha(nt)\alpha(t) = \alpha(nt+t) = \alpha((n+1)t) = (n+1)\alpha(t), (\forall) t \in \mathbb{R}$$

whence result that $\alpha + \alpha + \ldots + \alpha = (n+1)\alpha$ and proof ends. □

Lemma 2.14. *Let G be any topological group, $a, b \in \mathbb{R}$ and $\alpha : \mathbb{R} \to G$ a morphism of groups. Then $(a+b)\alpha = a\alpha + b\alpha$ and $a(b\alpha) = (ab)\alpha$.*

Proof. Since $a\alpha, b\alpha$ comute we obtain

$$(a\alpha + b\alpha)(t) = a\alpha(t)b\alpha(t) = \alpha(at)\alpha(bt) = \alpha(at+bt) = \alpha((a+b)t) =$$

$(a+b)\alpha(t), (\forall) t \in \mathbb{R}$ so $(a+b)\alpha = a\alpha + b\alpha$.

We have $a(b\alpha)(t) = b\alpha(at) = \alpha(bat) = (ab)\alpha(t)$ so $a(b\alpha) = (ab)\alpha$. □

Lemma 2.15. *Let G be any topological group, $a \in \mathbb{R}$ and $\alpha, \beta : \mathbb{R} \to G$ be two continuous morphisms of groups.*

Then $a(\alpha+\beta) = a\alpha + a\beta$ if $\alpha+\beta : \mathbb{R} \to G$ exist.

Proof. We have $(a(\alpha+\beta))(t) = (\alpha+\beta)(at) = \lim_{n\to\infty}(\alpha(\frac{at}{n})\beta(\frac{at}{n}))^n = \lim_{n\to\infty}(a\alpha(\frac{t}{n})a\beta(\frac{t}{n}))^n = (a\alpha + a\beta)(t), (\forall)t \in \mathbb{R}$ so $a(\alpha+\beta) = a\alpha + a\beta$.

Let us point out that we used only existence in G of the limit which defines $\alpha+\beta$ and not the fact that $\alpha+\beta$ is a morphism or a continuous map. □

2.5. **Topological Lie algebra of a 2-step nilpotent topological group.** In the following we will prove in detail Theorem 2.23, whose prove was only sketched in [Ne06, Th.IV.1.24]. On the way we also get other useful results (for example Proposition 2.20).

Lemma 2.16. *Let G be any 2-step nilpotent topological group and $\alpha, \beta, \gamma \in \Lambda(G)$. Then $\alpha+\beta = \beta+\alpha$ and $(\alpha+\beta)+\gamma = \alpha+(\beta+\gamma)$.*

Proof. We have $\alpha+\beta = \beta+\alpha \iff (\alpha+\beta)(t) = (\beta+\alpha)(t) \iff$

$$\alpha(t)\beta(t)c(\beta(\frac{t}{2}), \alpha(t)) = \beta(t)\alpha(t)c(\alpha(\frac{t}{2}), \beta(t))$$

We denote $\alpha(\frac{t}{2}) = a$, $\beta(\frac{t}{2}) = b$ and the previous relation is equivalent to $a^2b^2c(b, a^2) = b^2a^2c(a^2, b) \iff a^{-2}b^{-2}a^2b^2(c(b,a))^2 = (c(a,b))^2 \iff$

$$c(a^{-2}, b^{-2}) = (c(a,b))^2(c(a,b))^2 \iff c(a^2, b^2) = (c(a,b))^4$$

which holds true and we obtain $\alpha+\beta = \beta+\alpha$.

We have $((\alpha+\beta)+\gamma)(t) = (\alpha+\beta)(t)\gamma(t)c(\gamma(\frac{t}{2}), (\alpha+\beta)(t)) = \alpha(t)\beta(t)\gamma(t)c(\beta(\frac{t}{2}), \alpha(t))c(\gamma(\frac{t}{2}), \alpha(t))c(\gamma(\frac{t}{2}), \beta(t))$

We have $(\alpha+(\beta+\gamma))(t) = \alpha(t)(\beta+\gamma)(t)c((\beta+\gamma)(\frac{t}{2}), \alpha(t)) =$

$$\alpha(t)\beta(t)\gamma(t)c(\gamma(\frac{t}{2}), \beta(t))c(\beta(\frac{t}{2}), \alpha(t))c(\gamma(\frac{t}{2}), \alpha(t)) = ((\alpha+\beta)+\gamma)(t)$$

so $(\alpha+\beta)+\gamma = \alpha+(\beta+\gamma)$. □

Lemma 2.17. *Let G be any 2-step nilpotent topological group and $\alpha, \beta, \gamma \in \Lambda(G)$. Then we have*

(a) $[\beta, \alpha] = -[\alpha, \beta]$

(b) $[[\alpha, \beta], \gamma] + [[\beta, \gamma], \alpha] + [[\gamma, \alpha], \beta] = 0 \in \Lambda(G)$

(c) $[\alpha+\beta, \gamma] = [\alpha, \gamma] + [\beta, \gamma]$

(d) $[a\alpha, \gamma] = a[\alpha, \gamma]$ *for any* $a \in \mathbb{R}$

Proof. For point a) we have

$$\begin{aligned} [\beta, \alpha] = -[\alpha, \beta] &\iff [\beta, \alpha](t) = (-[\alpha, \beta])(t) \\ &\iff c(\beta(1), \alpha(t)) = [\alpha, \beta](-t) = c(\alpha(1), \beta(-t)) \\ &\iff c(\beta(1), \alpha(t)) = c(\beta(t), \alpha(1)) \end{aligned}$$

which holds true.

For b) it is sufficient to show that $[[\alpha,\beta],\gamma]=0\in\Lambda(G)$. We have

$$[[\alpha,\beta],\gamma](t)=c([\alpha,\beta](1),\gamma(t))=c(c(\alpha(1),\beta(1)),\gamma(t))=\mathbf{1}$$

for every $t\in\mathbb{R}$ hence we obtain $[[\alpha,\beta],\gamma]=0\in\Lambda(G)$.

For c) we have

$$\begin{aligned}[\alpha+\beta,\gamma](t)&=c((\alpha+\beta)(1),\gamma(t))\\&=c(\alpha(1)\beta(1)c(\beta(\frac{1}{2}),\alpha(1)),\gamma(t))\\&=c(\alpha(1)\beta(1),\gamma(t))\\&=c(\alpha(1),\gamma(t))c(\beta(1),\gamma(t))\end{aligned}$$

We have

$$\begin{aligned}([\alpha,\gamma]+[\beta,\gamma])(t)&=[\alpha,\gamma](t)[\beta,\gamma](t)c([\beta,\gamma](\frac{t}{2}),[\alpha,\gamma](t))\\&=c(\alpha(1),\gamma(t))c(\beta(1),\gamma(t))\\&=[\alpha+\beta,\gamma](t)\end{aligned}$$

hence we obtain $[\alpha+\beta,\gamma]=[\alpha,\gamma]+[\beta,\gamma]$.

For d) we have $[a\alpha,\gamma](t)=c(a\alpha(1),\gamma(t))=c(\alpha(a),\gamma(t))$. But $a[\alpha,\gamma](t)=c(\alpha(1),\gamma(at))=c(\alpha(a),\gamma(t))=[a\alpha,\gamma](t)$ hence we obtain $[a\alpha,\gamma]=a[\alpha,\gamma]$. □

Let X be a topological space and G a topological group. On the set $C(X,G)$ of continuous maps from X to$G$ we introduce the topology of uniform convergence on compact subsets of X in which the sets

$$\begin{aligned}E(\beta,K,V)&=\{\gamma\in C(X,G);(\beta(t))^{-1}\gamma(t)\in V,(\forall)t\in K\}\\&=\{\gamma\in C(X,G);\gamma(t)\in\beta(t)V,(\forall)t\in K\}\end{aligned}$$

for every compact subsets $K\subseteq X$ and $V\in V_G(\mathbf{1})$ form a fundamental system of neighborhoods of β. On the set $\Lambda(G)$ we consider the topology induced on $C(\mathbb{R},G)$.

Lemma 2.18. *Let G be any topological group and $a\in\mathbb{R}$. Then the map*

$$f:C(\mathbb{R},G)\to C(\mathbb{R},G),\quad f(\beta):=a\beta$$

is continuous.

Proof. If $a=0$ is evident because f is constant. If $a\neq 0$ continuity result from $E(a\beta,aK,V)=E(\beta,K,V)$. □

The following result is well-known but we prove it here for the sake of completeness

Lemma 2.19. *Let X,Y,T topological space, $y_0\in Y,t_0\in T$ and K compact in X, $f:X\times Y\to T$ a continuous function for which $f(K\times\{y_0\})=\{t_0\}$. Then for every V a neighborhood of t_0 exist U a neighborhood of y_0 such that $f(K\times U)\subseteq V$.*

Proof. Let be $x \in K$. Since f is continuous in (x, y_0) and $f(x, y_0) = t_0$ it follows that there exists D open neighborhood of (x, y_0) such that $f(D) \subseteq V$. We may assume $D = S_x \times U_x$ where S_x is an open neighborhood of x and U_x is an open neighborhood of y_0 and we have $f(S_x \times U_x) \subseteq V$. From $K \subseteq \cup_{t \in K} S_t$ using K compact we obtain a finite sub-covering, so there exists $n \geq 1$ and $x_1, \ldots, x_n \in K$ such that $K \subseteq S_{x_1} \cup S_{x_2} \cup \ldots \cup S_{x_n}$. We denote $U = U_{x_1} \cap \ldots \cap U_{x_n}$ and U is open in Y and $y_0 \in U$, so U is a neighborhood of y_0. We show that $f(K \times U) \subseteq V$. Let $x \in K$ and $y \in U$. There exists $j \in \{1, \ldots, n\}$ such that $x \in S_{x_j}$ and $y \in U_{x_j}$. From $f(x, y) \in f(S_{x_j} \times U_{x_j}) \subseteq V$ we obtain $f(K \times U) \subseteq V$ and the proof ends. □

Proposition 2.20. *Let X be any topological space and G topological group. Then the map $\phi : C(X, G) \times C(X, G) \to C(X, G), \phi(\alpha, \beta) := \alpha\beta$ is continuous, where $\alpha\beta \in C(X, G)$ is defined by $\alpha\beta(t) := \alpha(t)\beta(t)$ for all $t \in \mathbb{R}$.*

Proof. Let $\alpha, \beta \in C(X, G)$, K compact in X and $V \in V_G(\mathbf{1})$. We must find K_1, K_2 compacts in X and $V_1, V_2 \in V_G(\mathbf{1})$ such that

$$\phi(E(\alpha, K_1, V_1) \times E(\beta, K_2, V_2)) \subseteq E(\alpha\beta, K, V)$$

We take $K_1 = K_2 = K$. Let $\alpha_0 \in E(\alpha, K_1, V_1)$ and $\beta_0 \in E(\beta, K_2, V_2)$ and let $t \in K$. Exist $v_1 \in V_1$ and $v_2 \in V_2$ such that $\alpha_0(t) = \alpha(t)v_1$ and $\beta_0(t) = \beta(t)v_2$.

We have $\alpha_0(t)\beta_0(t) = \alpha(t)v_1\beta(t)v_2 = \alpha(t)\beta(t)(\beta(t))^{-1}v_1\beta(t)v_2$.

We show that $(\beta(t))^{-1}v_1\beta(t)v_2 \in V$ for every $v_1 \in V_1$ and $v_2 \in V_2$ and any $t \in K$. Let $f : G \times G \times G \to G, f(x, y, z) := x^{-1}yxz$. Applying the previous lemma for the continuous function f and $X = G, Y = G \times G$, $y_0 = (\mathbf{1}, \mathbf{1}), t_0 = \mathbf{1}$ and the compact set $\beta(K) \subseteq G$ we obtain that there exist U an open neighborhood of $(\mathbf{1}, \mathbf{1})$ such that $f(\beta(K) \times U) \subseteq V$. Because U is open neighborhood of $(\mathbf{1}, \mathbf{1})$ in $G \times G$ we may assume $U = V_1 \times V_2$ with $V_1, V_2 \in V_G(\mathbf{1})$. From $f(\beta(K) \times V_1 \times V_2) \subseteq V$ it follows that $(\beta(t))^{-1}v_1\beta(t)v_2 \in V$ for every $v_1 \in V_1, v_2 \in V_2$ and any $t \in K$ and the proof ends. □

Proposition 2.21. *Let G be any 2-step nilpotent topological group. Then*

1. *The map $\phi : \Lambda(G) \times \Lambda(G) \to \Lambda(G), \phi(\alpha, \beta) := \alpha + \beta$ is continuous.*
2. *The map $\psi : \Lambda(G) \times \Lambda(G) \to \Lambda(G), \psi(\alpha, \beta) := [\alpha, \beta]$ is continuous.*

Proof. For (1) since G is a 2-step nilpotent topological group it follows that we have the relationship $(\alpha + \beta)(t) = \alpha(t)\beta(t)c(\alpha(t), \beta(-\frac{t}{2})) =$

$\alpha(t)\beta(t)\alpha(t)\beta(-\frac{t}{2})\alpha(-t)\beta(\frac{t}{2})$ so ϕ is a product of 3 functions from $\Lambda(G) \times \Lambda(G)$ to $C(\mathbb{R}, G)$ which are continuous from previous proposition. Then ϕ is continuous. We used that $C(\mathbb{R}, G)$ is topological group. For (2) we use the relationship $[\alpha, \beta](t) = c(\alpha(\mathbf{1}), \beta(t))$ and continuity of the map ψ to obtain the conclusion as in the previous proposition and with the same reasoning as for the assertion (1). □

Definition 2.22 ([BCR81],[HM07])**.** We say that the topological group G is a *group with Lie algebra* if the topological space $\Lambda(G)$ has the structure of a topological Lie algebra over $\mathbb{R}$ whose algebraic operations satisfy the following conditions for all $t, s \in \mathbb{R}$ and $\lambda, \gamma \in \Lambda(G)$,

$$(t\lambda)(s) = \lambda(ts);$$
$$(\lambda + \gamma)(t) = \lim_{n\to\infty} (\lambda(\frac{t}{n})\gamma(\frac{t}{n}))^n;$$
$$[\lambda, \gamma](t^2) = \lim_{n\to\infty} (\lambda(\frac{t}{n})\gamma(\frac{t}{n})\lambda(-\frac{t}{n})\gamma(-\frac{t}{n}))^{n^2},$$

with uniform convergence on the compact subsets of $\mathbb{R}$.

The topological group G is a *pre-Lie group* if it is a group with Lie algebra and for every nonconstant $\gamma \in \Lambda(G)$ there exists a real-valued function f of class C^∞ on some neighborhood of $\mathbf{1} \in G$ such that $Df(\mathbf{1}; \gamma) \neq 0$.

From previously results for 2-step nilpotent topological groups we obtain the following theorem.

Theorem 2.23 ([Ne06, Th.IV.1.24])**.** *Every 2-step nilpotent topological group is a group with Lie algebra.*

2.6. **Tangent group of topological group with Lie algebra.** In this subsection we introduce the following new notion, which generalizes the tangent bundle of any Lie group has a natural structure of Lie group (see for example [Be06]).

Definition 2.24. Let G be any group with Lie algebra. The *tangent group* of G is $T(G)$,which is the set $G \ltimes \Lambda(G)$ endowed with the group operation $(x, \alpha)(y, \beta) = (xy, \alpha^y + \beta)$.

Here we define $\alpha^y \in \Lambda(G)$ by $\alpha^y(t) = y\alpha(t)y^{-1}$ for all $t \in \mathbb{R}, y \in G, \alpha \in \Lambda(G)$.

Proposition 2.25. *Let G be any group with Lie algebra. Then $T(G)$ is a topological group.*

Proof. Asociativity follows by the relationship

$$(\forall x \in G)(\forall \alpha, \beta \in \Lambda(G)) \quad (\alpha + \beta)^x = \alpha^x + \beta^x$$

which can be verified by direct calculation.

The unit element of $T(G)$ is $(\mathbf{1}, 0) \in T(G)$ and the inverse of (x, α) is $(x^{-1}, -\alpha^{x^{-1}})$. Continuity of operation on $T(G)$ result from the fact that G is a topological group, Lie algebra $\Lambda(G)$ is a topological algebra, and the action of G on $\Lambda(G)$ is continuous. □

Proposition 2.26. *Let G be any 2-step nilpotent topological group. Then $T(G) = G \ltimes \Lambda(G)$ is a 2-step nilpotent group.*

Proof. Let $(g,\alpha),(h,\beta)\in T(G)$. We have

$$\begin{aligned}c((g,\alpha),(h,\beta)) &= (g,\alpha)(h,\beta)(g,\alpha)^{-1}(h,\beta)^{-1}\\ &= (gh,\alpha^h+\beta)(g^{-1}h^{-1},-\beta^{h^{-1}}-\alpha^{g^{-1}h^{-1}})\\ &= (c(g,h),(\alpha^h+\beta)^{g^{-1}h^{-1}}-\beta^{h^{-1}}-\alpha^{g^{-1}h^{-1}})\\ &= (c(g,h),\alpha^{hg^{-1}h^{-1}}+\beta^{g^{-1}h^{-1}}-\beta^{h^{-1}}-\alpha^{g^{-1}h^{-1}})\\ &= (c(g,h),(\alpha^{hg^{-1}h^{-1}}-\alpha^{g^{-1}h^{-1}})+(\beta^{g^{-1}h^{-1}}-\beta^{h^{-1}})).\end{aligned}$$

We now show that

$$Z(T(G))=\{(x,\alpha)\in T(G); x\in Z(G), Im(\alpha)\subseteq Z(G)\}.$$

If $(x,\alpha)\in Z(T(G))$ then $(x,\alpha)(g,\lambda)=(g,\lambda)(x,\alpha)$ for every $(g,\lambda)\in T(G)$.

So $x\in Z(G)$ and $\alpha^g+\lambda=\lambda^x+\alpha,(\forall)g\in G,(\forall)\lambda\in\Lambda(G)$. From $x\in Z(G)$ we obtain that $\lambda^x=\lambda$ and $\alpha^g=\alpha,(\forall)g\in G$ hence it follows that $Im(\alpha)\subseteq Z(G)$ and we obtain relationship required.

Since G is 2-step nilpotent group we have that if $\alpha,\beta\in\Lambda(G)$ and $Im(\alpha),Im(\beta)\subseteq Z(G)$ then $Im(\alpha+\beta)\subseteq Z(G)$. To complete the proof we will show that if $x,y\in G$ and $\beta\in\Lambda(G)$ then $Im(\beta^x-\beta^y)\subseteq Z(G)$.

We have

$$\begin{aligned}(\beta^x-\beta^y)(t) &= \beta^x(t)\beta^y(-t)c(\beta^x(t),\beta^y(\frac{t}{2}))\\ &= x^{-1}\beta(t)xy^{-1}\beta(-t)yc(\beta^x(t),\beta^y(\frac{t}{2}))\\ &= c(x^{-1},\beta(t))c(\beta(t),y^{-1})c(\beta^x(t),\beta^y(\frac{t}{2}))\end{aligned}$$

which belongs to $Z(G)$ because G is a 2-step nilpotent group.

It follows that $Im(\beta^x-\beta^y)\subseteq Z(G)$ and the proof ends. □

Now we can obtain the main result of the present paper.

Theorem 2.27. *Let G be any 2-step nilpotent topological group. Then $T(G)=G\ltimes\Lambda(G)$ is a pre-Lie group if and only if G is pre-Lie group and for every $\alpha\in\Lambda(G),\alpha\neq 0$ exists a continuous linear functional $\psi:\Lambda(G)\to\mathbb{R}$ with $\psi(\alpha)\neq 0$.*

Proof. We first assume that $T(G)=G\ltimes\Lambda(G)$ is a pre-Lie group. Let $\gamma=(\gamma_1,\gamma_2)\in\Lambda(T(G))$ with $\gamma_1\neq 0\in\Lambda(G)$. Since G is pre-Lie group it follows that there exists $f:G\to\mathbb{R}$ of class C^∞ on a neighborhood U of $\mathbf{1}\in G$ such that $Df(\mathbf{1};\gamma_1)\neq 0$.

We define $h:T(G)\to\mathbb{R}$ by $h(x,\lambda):=f(x)$ which is of class C^∞ on the neighborhood $U\times\Lambda(G)$ of $(\mathbf{1},0)\in T(G)$. We have

$$Dh((\mathbf{1},0);(\gamma_1,\gamma_2))=Df(\mathbf{1};\gamma_1)\neq 0.$$

Now let $\gamma=(\gamma_1,\gamma_2)\in\Lambda(T(G))$ with $\gamma_1=0\in\Lambda(G)$ and $\gamma_2:\mathbb{R}\to\Lambda(G)$ for which there exists $t_0\in\mathbb{R}$ such that $\gamma_2(t_0)\neq 0\in\Lambda(G)$. In addition γ_2

verifies $\gamma_2(t+s) = \gamma_2(t) + \gamma_2(s), (\forall)t, s \in \mathbb{R}$ so γ_2 is continuous morphism of group and we have $\gamma_2(t) = t\gamma_2(1), (\forall)t \in \mathbb{R}$.

From $\gamma_2(t_0) \neq 0 \in \Lambda(G)$ it follows that there exists $\psi : \Lambda(G) \to \mathbb{R}$ linear and continuous such that $\psi(\gamma_2(t_0)) \neq 0$.

We define $h : T(G) \to \mathbb{R}$ by $h(x, \lambda) := \psi(\lambda)$ which is of class C^∞ on $T(G)$.

We have

$$\begin{aligned} Dh((\mathbf{1}, 0); (\gamma_1, \gamma_2)) &= \lim_{t \to 0} \frac{h(\mathbf{1}, \gamma_2(t)) - h(\mathbf{1}, 0)}{t} \\ &= \lim_{t \to 0} \frac{\psi(\gamma_2(t)) - \psi(0)}{t} \\ &= \lim_{t \to 0} \frac{t\psi(\gamma_2(\mathbf{1}))}{t} \\ &= \psi(\gamma_2(\mathbf{1})) \\ &= \frac{\psi(\gamma_2(t_0))}{t_0} \neq 0 \end{aligned}$$

so $T(G) = G \ltimes \Lambda(G)$ is pre-Lie group.

Conversely, suppose that $T(G) = G \ltimes \Lambda(G)$ is pre-Lie group and proof that G is pre-Lie group. Let $\alpha \in \Lambda(G), \alpha \neq 0$. Since $T(G) = G \ltimes \Lambda(G)$ is pre-Lie group for $(\alpha, 0) \in \Lambda(T(G))$ exist $h : T(G) \to \mathbb{R}$ of class C^∞ on neighborhood $U \times V$ of $(\mathbf{1}, 0) \in T(G)$ and $Dh((\mathbf{1}, 0); (\alpha, 0)) \neq 0$.

We define $f : G \to \mathbb{R}$ by $f(x) := h(x, 0)$.

Since $Df(\mathbf{1}; \alpha) = Dh((\mathbf{1}, 0); (\alpha, 0))$ it follows that $Df(\mathbf{1}; \alpha) \neq 0$ hence we obtain that G is pre-Lie group.

Let $\alpha \in \Lambda(G), \alpha \neq 0$. We must find $\psi : \Lambda(G) \to \mathbb{R}$ linear and continuous such that $\psi(\alpha) \neq 0$. Since G is pre-Lie group it follows that exist $f : G \to \mathbb{R}$ of class C^∞ on a neighborhood U of $\mathbf{1} \in G$ such that $Df(\mathbf{1}, \alpha) \neq 0$. Let $\psi : \Lambda(G) \to \mathbb{R}$ given by $\psi(\lambda) := Df(\mathbf{1}, \lambda)$.

Since $\psi(\alpha) = Df(\mathbf{1}, \alpha) \neq 0$ and derivative of f is linear and continuous it follows that ψ verifies the requirement and the proof ends. □

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