SCIENTIFIC REPORT FOR THE YEAR 2021 FOR THE PROJECT PN-III-P4-ID-PCE-2020-0446 "AXIOMATIC METHODS IN NON-CLASSICAL MODEL THEORY"

The contents of this report is as follows:

(1) Summary of the developments at stage 1 of the project (year 2021) – Section 1. This includes the scientific and technical description of the results.
(2) The results obtained at stage 1 of the project. These are presented separately for each of the activities specified in the funding contract – Section 2 for activity 1.1, Section 3 for activity 1.2, Section 4 for activity 1.3.

All objectives of stage 1 of the project have been 100% accomplished.

1. Summary of developments at stage 1 of the project

1.1. **Axiomatic model theory for stratified institutions.** The goal of our work is to further develop the theory of stratified institutions in several directions that are important either from a model theory as such or a computing science perspective. Our developments consist of

- a new technique for representing stratified institutions,
- on the basis of the above mentioned technique, results establishing a model amalgamation properties and diagrams for stratified institutions,
- a study of quasi-varieties and of their associated initial semantics in stratified institutions, and
- a general study of interpolation in stratified institutions.

In what follows we present these in more detail.

1.1.1. *Decompositions of stratified institutions.* Historically there have been two major approaches to Kripke semantics within institution theory:

(1) The approach introduced in [12] and then used in [14, 7, 11, 9, 13], etc., that considers Kripke semantics as a two-layered concept. In that approach the base layer can be considered as a parameter and treated fully implicitly as an abstract institution while the upper layer considers explicit Kripke structures and modalities that are parameterised by the base layer.
(2) The approach of the stratified institutions that is fully abstract without any explicit Kripke structures and modalities.

The drawback of the former approach is precisely its rigid commitment to a specific common concept of Kripke semantics and modal syntax. On the other hand, due to its high abstraction level, the latter approach is free of such issues and supports a full top down development process where concepts are introduced axiomatically on a by-need basis.

In order to retain some of the benefits of the two-layered approach, such as the hierarchical shape of the respective model theories that at the bottom are based on a concept of possible worlds, here we take a step further in this methodology by introducing a concept of decomposition of a stratified institution. In brief, we associate to a stratified institution a couple of abstract projections to other stratified institutions that in examples correspond to the two layers discussed above. But now the projection corresponding to the upper layer is fully abstract. Most examples / applications of stratified institutions can be presented as decomposed stratified institutions in a meaningful way.

1.1.2. *Model amalgamations and diagrams.* The institution theoretic analysis of model theory has established model amalgamation and the method of diagrams as the most pervasive properties in model theory. At the general level of bare abstract stratified institutions both properties have to be assumed and then established only at the level of the concrete examples. By the decomposition technique discussed above we are able to actually establish these properties at the level of abstract stratified institutions from the corresponding properties of the two "components". The value of these results reside in the fact that, on the one hand they define classes of abstract stratified institutions that admit these properties, and on the other hand they provide an easy way to establish them in concrete situations because the problem is reduced to the two components where things are much simplified.

1.1.3. *Quasi-varieties and initial semantics.* In [7] quasi-varieties and initial semantics have been studied within the framework of the two-layered institution theoretic approach to Kripke semantics with emphasis on the hybrid versions of modal logic. Here we take a step further and study this topic at the higher level of abstract stratified institutions. First we develop a general construction of (direct) products of models and define a concept of sub-models in stratified institutions that is based on the above mentioned decomposition technique. Both products of models and sub-models at the level of stratified institutions are thus obtained as compositions of products of models and of sub-models, respectively, in the components, where they are much easier to establish.

Then we develop a modular body of preservation results for the satisfaction relation of the stratified institutions which enables one to isolate sentences such that their classes of models form a quasi-variety. By applying a general categorical result that establishes initial semantics for quasi-varieties we obtain that certain classes of sentences admit initial semantics. These sentences really depend on the respective particular stratified institution, but our general results provide an uniform way to determine them immediately in the concrete situations.

1.1.4. *Interpolation.* We refine the common institution theoretic approach to interpolation [15, 1, 4, 6, 5] to stratified institutions. Since stratified institutions admit two relevant semantic consequence relations, they admit two main concepts of interpolation. Our study of interpolation in stratified institutions establishes a causal connection between the two concepts of interpolation.

1.2. **Ultraproducts and quasi-varieties in $\mathcal{L}$-institutions.** We study the model theoretic properties of Horn sentences in the highly general axiomatic framework of $\mathcal{L}$-institutions. The goals of our enterprise can be understood from the following different perspectives:

- To provide a many-valued truth generalisation of the binary institution theoretic approach to the model theory of Horn sentences of [6].
- On the other hand, our work can be regarded as an abstract axiomatic approach to the model theory of first order many-valued logic in the sense of [2].
- Finally, the high level of abstraction of the concept of $\mathcal{L}$-institution allows for instances of our results to various concrete many-valued logical systems, that in spite of looking rather unconventional in fact they have a computing science meaning.

The developments under this topic are as follows.

(1) We extend a crucial concept from institutional model theory to many-valued truth and establish some of its most important properties.

(2) Results about preservation of the satisfaction relation by filtered products of models, which are of course considered in their category theoretic form. These results are developed in a modular fashion. By putting together some of them we obtain a preservation result for Horn sentences. However we show how they can also be used to obtain separately preservation results in concrete situations that fall beyond our general concept of Horn sentence. General compactness consequences of our preservation results are developed at the.

(3) Preservation by sub-models is the main topic of another section. In the well established institution theoretic manner [6] our concept of sub-model relies on the *inclusion systems* of [10]. By joining together preservation results by filtered products and by sub-models, through a categorical concept of quasi-variety we obtain a general initial semantics result for an important general class of Horn sentences.

2. Axiomatic model theory for stratified institutions

Here we report the concepts and results developed under this topic without providing proofs and applications / examples. The full developments can be found here.

2.1. **Decompositions of stratified institutions.** Many of the developments under this topic rely on the newly introduced technique reported in this section. This addresses the general structure of Kripke semantics from an abstract axiomatic perspective. The result is a general abstract class of stratified institutions that does not necessarily consider explicitly Kripke frames nor modalised sentences, but which retains in an abstract form the essential idea of a stratified institution in which a "header" institution structures a certain multiplication of a "base" institution.

Definition 2.1. *A* base for a stratified institution $\mathcal{S}$ *is an institution morphism* $(\Phi, \alpha, \beta) : \mathcal{S}^\sharp \to B$. *A base is* shared *when for each signature* Σ, *each* Σ*-model* M *of* $\mathcal{S}$, *and any* $w, w' \in [\![M]\!]_\Sigma$ *we have that* $\beta_\Sigma(M, w) = \beta_\Sigma(M, w')$.

Let $\mathbb{INS}$ be the category of institution morphisms and $\mathbb{SINS}$ be the category of strict stratified institution morphisms.

Proposition 2.1. *Let* $(\_)^\sharp : \mathbb{SINS} \to \mathbb{INS}$ *be the canonical extension of the mapping* $\mathcal{S} \mapsto \mathcal{S}^\sharp$. *Then* $(\_)^\sharp$ *has a right adjoint which we denoted as* $\widetilde{(\_)} : \mathbb{INS} \to \mathbb{SINS}$.

Definition 2.2. *Let $\mathcal{S}$ be a stratified institution and $(\Phi, \alpha, \beta): \ S^\sharp \to \mathcal{B}$ be a base for $\mathcal{S}$. Let $Mod^C \subseteq Mod^{\widetilde{\mathcal{B}}}$ be a sub-functor such that for each signature Σ,*

$$\widetilde{\beta}_\Sigma(Mod^{\mathcal{S}}(\Sigma)) \subseteq Mod^C(\Phi\Sigma),$$

refereed to as the constraint model sub-functor. *Let $\widetilde{\mathcal{B}}^C$ denote the stratified sub-institution of $\widetilde{\mathcal{B}}$ induced by Mod^C. A* decomposition of $\mathcal{S}$ *consists of two stratified institution morphisms like below*

$$\mathcal{S}^0 \xleftarrow{(\Phi^0,\alpha^0,\beta^0)} \mathcal{S} \xrightarrow{(\Phi,\alpha,\widetilde{\beta})} \widetilde{\mathcal{B}}^C$$

such that for each $\mathcal{S}$-signature Σ

$$\begin{array}{ccccc} Mod^0(\Phi^0\Sigma) & \xleftarrow{\beta^0_\Sigma} & Mod^{\mathcal{S}}(\Sigma) & \xrightarrow{\widetilde{\beta}_\Sigma} & Mod^C(\Phi\Sigma) \\ & \searrow^{[\![-]\!]^0_{\Phi^0\Sigma}} & \downarrow{\scriptstyle [\![-]\!]^{\mathcal{S}}_\Sigma} & \swarrow^{[\![-]\!]^{\widetilde{\mathcal{B}}}_{\Phi\Sigma}} & \\ & & \mathbf{Set} & & \end{array}$$

is a pullback in **CAT**.

In the applications the eventual modal structures of $\mathcal{S}$ come from $\mathcal{S}^0$. The following fact clarifies mathematically this situation in a full generality.

Fact 2.1. *Consider a decomposition of a stratified institution $\mathcal{S}$ like in Definition 2.2. Then any frames / nominals extraction of $\mathcal{S}^0$ induces canonically a frames / nominals extraction of $\mathcal{S}$ by composition with $(\Phi^0, \alpha^0, \beta^0)$.*

Fact 2.2. *Any stratified institution that admits a decomposition is strict.*

2.2. **Model amalgamation.** Our study model amalgamation in the context of stratified institutions has two parts as follows:

(1) We define a concept of model amalgamation specific to stratified institutions.
(2) We develop a general result that builds the model amalgamation property in a stratified institution that admits a decomposition, from the model amalgamation properties of the two components.

Definition 2.3. *Consider a stratified institution $\mathcal{S}$ and a commutative square of signature morphisms like below:*

(1)
$$\begin{array}{ccc} \Sigma & \xrightarrow{\varphi_1} & \Sigma_1 \\ {\scriptstyle \varphi_2}\downarrow & & \downarrow{\scriptstyle \theta_1} \\ \Sigma_2 & \xrightarrow[\theta_2]{} & \Sigma' \end{array}$$

Then this square

- *is a* model amalgamation square *when for each Σ_k-model M_k, $k = \overline{1,2}$ such that $\varphi_1(M_1) = \varphi_2(M_2)$ there exists an unique Σ'-model M' such that $\theta_k(M') = M_k$, $k = \overline{1,2}$, and*
- *is a* stratified model amalgamation square *when for each Σ_k-model M_k and each $w_k \in [\![M_k]\!]_{\Sigma_k}$, $k = \overline{1,2}$ such that $\varphi_1(M_1) = \varphi_2(M_2)$ and $[\![M_1]\!]_{\varphi_1} w_1 = [\![M_2]\!]_{\varphi_2} w_2$ there exists an*

unique Σ'-model M' and an unique $w' \in [\![N']\!]_{\Sigma'}$ such that $\theta_k(M') = M_k$ and $[\![M']\!]_{\theta_k} w' = w_k$, $k = \overline{1,2}$.

The model M' is called the (stratified) amalgamation of M_1 and M_2.

When all pushout squares of signature morphisms are (stratified) model amalgamation squares we say that $\mathcal{S}$ is (stratified) semi-exact.

Fact 2.3. *A commutative square of signature morphisms like (1) is a stratified model amalgamation square in $\mathcal{S}$ if and only if it is a model amalgamation square in $\mathcal{S}^\sharp$.*

Proposition 2.2. *A commutative square of signature morphisms like (1) is a stratified model amalgamation square if*

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$$\begin{array}{ccc} Mod(\Sigma) & \xleftarrow{Mod(\varphi_1)} & Mod(\Sigma_1) \\ {\scriptstyle Mod(\varphi_2)}\uparrow & & \uparrow{\scriptstyle Mod(\theta_1)} \\ Mod(\Sigma_2) & \xleftarrow[Mod(\theta_2)]{} & Mod(\Sigma') \end{array}$$

  is a pullback in $|\mathbf{CAT}|$, and
- *for each Σ'-model M'*

$$\begin{array}{ccc} [\![\varphi(\theta(M'))]\!]_{\Sigma} & \xleftarrow{[\![\theta_1(M')]\!]_{\varphi_1}} & [\![\theta_1(M')]\!]_{\Sigma_1} \\ {\scriptstyle [\![\theta_2(M')]\!]_{\varphi_2}}\uparrow & & \uparrow{\scriptstyle [\![M']\!]_{\theta_1}} \\ [\![\theta_2(M')]\!]_{\Sigma_2} & \xleftarrow[{[\![M']\!]_{\theta_2}}]{} & [\![M']\!]_{\Sigma'} \end{array}$$

  is a pullback in $\mathbf{Set}$.

Proposition 2.3. *Let $\mathcal{B}$ be any institution. Any model amalgamation square in $\mathcal{B}$ is a model amalgamation square in $\widetilde{\mathcal{B}}$ too.*

Definition 2.4. *Let $\mathcal{B}$ be any institution. A constraint model sub-functor $Mod^C \subseteq Mod^{\widetilde{\mathcal{B}}}$* preserves amalgamation *when for any signature morphisms $\theta_k : \Sigma' \to \Sigma_k$, $k = \overline{1,2}$ and any $\widetilde{\mathcal{B}}$ Σ'-model (W, B'), $\theta_k(W, B') \in |Mod^C(\Sigma_k)|$, $k = \overline{1,2}$, implies $(W, B') \in |Mod^C(\Sigma')|$.*

The following is the main result of our study of model amalgamation in stratified institutions. Its applications, in conjunction with Proposition 2.3, include model amalgamation properties in a wide range of concrete situations.

Proposition 2.4. *Consider a decomposition of a stratified institution $\mathcal{S}$ like in Definition 2.2 such that*

(1) $\mathcal{S}^0$ is strict,
(2) Φ and Φ^0 preserve pushouts,
(3) $\mathcal{B}$ and $\mathcal{S}^0$ are semi-exact, and
(4) Mod^C preserves amalgamation.

Then $\mathcal{S}$ is semi-exact too.

The following lemma solves the second condition of Proposition 2.4 at the general level, and is applicable to a wide range of concrete situations.

Lemma 2.1. *If the decomposition of the stratified institution has the property that*

$$Sign^0 \xleftarrow{\Phi^0} Sign^{\mathcal{S}} \xrightarrow{\Phi} Sign^{\mathcal{B}}$$

is a product in **CAT**, *then both Φ^0 and Φ preserve pushouts. Moreover any tuple of pushout squares of signatures, one from $\mathcal{S}^0$ and the other one from $\mathcal{B}$, determine canonically a pushout square of $\mathcal{S}$ signatures.*

2.3. **Diagrams.** In conventional model theory the method of diagrams is one of the most important methods. The institution-independent method of diagrams pervades the development of a lot of model theoretic results at the level of abstract institutions, many of its applications being presented in [6]. The structure of our study of diagrams is as follows:

(1) We define the concept of diagram in stratified institutions.
(2) We introduce some preliminary technical concepts that will support the development of the main result of this section.
(3) We formulate and prove a general result on the existence of diagrams in stratified institutions. This comes in two versions: for $\mathcal{S}^*$ and for $\mathcal{S}^\sharp$ (where $\mathcal{S}$ is a stratified institution).

Definition 2.5 (Diagrams in stratified institutions)**.** *A stratified institution $\mathcal{S}$ admits diagrams if and only if its flattening $\mathcal{S}^\sharp$ admits diagrams in the sense of ordinary institution theory.*

The following three definitions define technical conditions necessary for the existence of diagrams in stratified institutions.

Definition 2.6. *Let $\mathcal{B}$ be an institution and Mod^C be a constraint model sub-functor for $\widetilde{\mathcal{B}}$. A system of diagrams for $\mathcal{B}$ is* coherent with respect to Mod^C *when for each $\mathcal{B}$-signature Σ and each $(W,B) \in |Mod^C(\Sigma)|$ we have that*

(1) For all $i,j \in W$, $\iota_{\Sigma,B^i} = \iota_{\Sigma,B^j}$; in this case all ι_{Σ,B^i}s will be denoted by $\iota_{\Sigma,B} : \Sigma \to \Sigma_B$.
(2) for each $(W,B) \in |Mod^C(\Sigma)$ there exists a canonical isomorphism $i_{\Sigma,(W,B)}$ such that the following diagram commutes:

$$\begin{array}{ccc} Mod(\Sigma_B, E_{(W,B)}) & \xrightarrow[\cong]{i_{\Sigma,(W,B)}} & (W,B)/Mod^C(\Sigma) \\ & {\scriptstyle Mod^C(\iota_{\Sigma,B})} \searrow \quad \swarrow {\scriptstyle forgetful} & \\ & Mod^C(\Sigma) & \end{array}$$

*where $Mod(\Sigma_B, E_{(W,B)})$ denotes the subcategory of the comma category $[\![M]\!]/[\![\_]\!]_{\Sigma_B}$ (where $[\![\_]\!]_{\Sigma_B} : Mod^C(\Sigma_B) \to$ **Set**) induced by those objects $(f : W \to V, (V, N'))$ such that $N'^{f(i)} \models E_{B^i}$ for each $i \in W$.*[1]

Definition 2.7. *A decomposition of a stratified institution $\mathcal{S}$ like in Definition 2.2* admits diagrams *when $\mathcal{S}^{0*}$ and $\mathcal{B}$ have diagrams such that the diagrams of $\mathcal{B}$ are coherent with respect to Mod^C.*

[1]Note that unlike E_{B^i}, $E_{(W,B)}$ is *not* a set of sentences.

Notation 2.1. For any decomposition of a stratified institution that admits diagrams (like in Definition 2.7), for any $\Sigma \in |Sign^{\mathcal{S}}|$ and $M \in |Mod^{\mathcal{S}}(\Sigma)|$, we introduce the following abbreviations:

$$\Sigma_0 = \Phi^0\Sigma,\ \Sigma_1 = \Phi\Sigma,\ M_0 = \beta^0_\Sigma M,\ M_1 = \widetilde{\beta}_\Sigma M.$$

We let $\iota_{\Sigma_0,M_0} : \ \Sigma_0 \to (\Sigma_{0M_0}, E_{M_0})$ and (for each $i \in [\![M]\!]$) $\iota_{\Sigma_1,M_1^i} : \ \Sigma_1 \to (\Sigma_{1M_1^i}, E_{M_1^i})$ be the diagrams of M_0 and M_1^i, respectively. By the coherence hypothesis we have $\iota_{\Sigma_1,M_1^i} = \iota_{\Sigma_1,M_1^j}$ for all $i, j \in [\![M]\!]$. This allows us to denote all ι_{Σ_1,M_1^i} by ι_{Σ_1,M_1}.

We define the $Sign^S$ morphism $\iota_{\Sigma,M} : \ \Sigma \to \Sigma_M$ by using the product property of (Φ^0, Φ):

$$\iota_{\Sigma,M} = (\iota_{\Sigma_0,M_0}, \iota_{\Sigma_1,M_1}).$$

Definition 2.8. *Consider a decomposition of a stratified institution that admits diagrams like in Definition 2.7. We say that the diagrams (of the decomposition)* denote the stratification *when*

- *$\mathcal{S}$ has a nominals extraction (N, Nm),*
- *for each $\mathcal{S}$-signature Σ and each Σ-model M, there exists a function*

$$n_{\Sigma,M} : \ [\![M]\!]_\Sigma \to N(\Sigma_M)$$

  such that n is natural in Σ and M, and
- *for each Σ_M-model N such that $N \models \alpha_0 E_{M_0}$,*

$$n_{\Sigma,M}; Nm_{\Sigma_M}(N) = [\![i_{\Sigma_0,M_0} N_0]\!]_{\Sigma_0}.$$

The following is the main result of our study of diagrams for stratified institutions.

Theorem 2.1. *For any decomposition of a stratified institution $\mathcal{S}$ that admits diagrams that denote the stratification:*

- *$\mathcal{S}^*$ has diagrams when $\mathcal{S}$ has explicit local satisfaction, and*
- *$\mathcal{S}^\sharp$ has diagrams when $\mathcal{S}$ has explicit local satisfaction and has i-sentences too.*

The following two results are examples of application of the result of Theorem 2.1.

Corollary 2.1. *Let HPL denote the stratified institution of hybrid propositional logic. Then HPL* and HPL$^\sharp$ have diagrams.*

Corollary 2.2. *Let HFOL denote the stratified institution of hybrid first order logic with sharing of domains and interpretations of constants. Then HFOL* and HFOL$^\sharp$ have diagrams.*

2.4. **Quasi-varieties.** The goal of our study of quasi-varieties of models in stratified institutions is twofold:

- Quasi-varieties of models rely on (direct) products of models and on “sub-models”. Our first goal is to establish products of models and sub-models at a general level in stratified institutions. For this we rely on the decomposition technique of Section 2.1.
- The second goal is to establish initial semantics for classes of theories by reliance on a general abstract category-theoretic result of existence of initial objects in quasi-varieties. For this develop a modular body of preservation results under products and sub-models.

2.4.1. *Sub-models in stratified institutions.* The following developments up to Proposition 2.7 included refer to how to establish inclusion systems in stratified institutions that admit decompositions. For this purpose we assume an institution $\mathcal{B}$ such that for each signature Σ its category of models has an inclusion system $(I_\Sigma, \mathcal{E}_\Sigma)$.

Proposition 2.5. *For each signature Σ we assume that I^Σ has small coproducts – denoted $\oplus$ – that are preserved by the inclusion functor $I_\Sigma \subseteq Mod^{\mathcal{B}}(\Sigma)$. Then $Mod^{\widetilde{\mathcal{B}}}(\Sigma)$ admits a canonical inclusion system $(\widetilde{I}_\Sigma, \widetilde{\mathcal{E}}_\Sigma)$ defined as follows:*

- *$h: (W,B) \to (W',B')$ belongs to $\widetilde{I}_\Sigma$ when $h_0 = (W \subseteq W')$ is a set inclusion and for each $w \in W$, $(h^w: B^w \to B'^w) \in I_\Sigma$.*
- *$h: (W,B) \to (W',B')$ belongs to $\widetilde{\mathcal{E}}_\Sigma$ when $h_0: W \to W'$ is surjective and for each $w' \in W'$*

$$B'^{w'} = \bigoplus_{h_0 w = w'} h^w(B^w).$$

In Proposition 2.6 we extend the result of Proposition 2.5 to the models of $\widetilde{\mathcal{B}}^C$, for some constraint model functor Mod^C. There are two major reasons for this:

(1) On the one hand, model constraints occur in may important examples of decomposed stratified institutions.
(2) On the other hand, for many base institutions $\mathcal{B}$ the lifting of the inclusion systems from $\mathcal{B}$ to $\widetilde{\mathcal{B}}$ can be done only through the concept of Grothendieck inclusion system. This requires conditions that usually fail in the absence of constraints. For instance, if we take $\mathcal{B} = FOL$ the hypothesis of Proposition 2.5 does not hold anymore, and the same happens with many base institutions of interest.

Proposition 2.6. *Given a constraint model functor Mod^C for $\widetilde{\mathcal{B}}$, for each $\mathcal{B}$-signature Σ we assume that*

(1) the inclusion system $(I_\Sigma, \mathcal{E}_\Sigma)$ preserve the constraints, *i.e. if $(1_W, f) \in Mod^C(\Sigma)$ and each f^w, $w \in W$, gets factored as $f^w = e^w_f; i^w_f$ with $e^w_f \in \mathcal{E}_\Sigma$ and $i^w_f \in I_\Sigma$ then $(1_W, e_f), (1_W, i_f) \in Mod^C(\Sigma)$,*
(2) for each $L \in |Mod^C(\Sigma)|$, the diagonal $\widetilde{\mathcal{B}}$ model *(W, L^W) where $(L^W)^w = L$ for each $w \in W$, belongs to $Mod^C(\Sigma)$,*
(3) any $(W,B) \in |Mod^C(\Sigma)|$ admits a coproduct in I_Σ of its components, denoted $\bigoplus_{w \in W} B^w$; moreover this is also initial in the class of the co-cones given by the constraint model homomorphisms $(1_W, f): (W,B) \to (W, L^W)$. As notation, the mediating model homomorphism $\bigoplus_{w\in W} B^w \to L$ is denoted $\sum_W f$.
(4) For each $(W,B) \in |Mod^C(\Sigma)|$, any $L \in |Mod^{\mathcal{B}}(\Sigma)|$, any constraint model homomorphism $(1_W, f): (W,B) \to (W, L^W)$, and any family of subseteq $(K_j \subseteq W)_{j\in J}$, $(\sum_{K_j} f)_{j \in J}$ is a constraint model homomorphism.

Then $(I^C_\Sigma, \mathcal{E}^C_\Sigma)$ is an inclusion system of $Mod^C(\Sigma)$, where I^C_Σ and $\mathcal{E}^C_\Sigma$ are the restrictions of $\widetilde{I}_\Sigma$ and $\widetilde{\mathcal{E}}_\Sigma$ (of Proposition 2.5), respectively, to $Mod^C(\Sigma)$.

Proposition 2.7. *Consider a decomposition of a stratified institution $\mathcal{S}$ like in Definition 2.2 such that for each $\mathcal{S}$-signature Σ, $Mod^{\mathcal{S}^0}(\Phi^0\Sigma)$ and $Mod^C(\Phi\Sigma)$ have inclusion systems $(I^0_{\Phi^0\Sigma}, \mathcal{E}^0_{\Phi^0\Sigma})$ and $(I^C_{\Phi\Sigma}, \mathcal{E}^C_{\Phi\Sigma})$, respectively, such that both $[\![\_]\!]^0$ and $[\![\_]\!]^{\widetilde{\mathcal{B}}^C}$ preserve the abstract inclusions and the abstract surjections. Then $(\overline{I}_\Sigma, \overline{\mathcal{E}}_\Sigma)$ is an inclusion system for $Mod^{\mathcal{S}}(\Sigma)$ where*

- *$h \in \overline{I}_\Sigma$ if and only if $\beta^0_\Sigma h \in I^0_{\Phi^0\Sigma}$ and $\widetilde{\beta}_\Sigma h \in I^C_{\Phi\Sigma}$,*
- *$h \in \overline{\mathcal{E}}_\Sigma$ if and only if $\beta^0_\Sigma h \in \mathcal{E}^0_{\Phi^0\Sigma}$ and $\widetilde{\beta}_\Sigma h \in \mathcal{E}^C_{\Phi\Sigma}$.*

2.4.2. *Model products in stratified institutions.* The following results up to Proposition 2.9 included refer to how to establish model products in stratified institutions that admit decompositions.

Proposition 2.8. *Let $\mathcal{B}$ be an institution such that for each signature $\Sigma \in |Sign^{\mathcal{B}}|$ its category of models $Mod^{\mathcal{B}}(\Sigma)$ has small products. Then $Mod^{\widetilde{\mathcal{B}}}(\Sigma)$ has small products too.*

Proposition 2.9. *Consider a decomposition of a stratified institution $\mathcal{S}$ like in Definition 2.2 such that for some $\mathcal{S}$-signature Σ the following conditions hold:*

(1) $Mod^{\mathcal{B}}$ has small products;
(2) $Mod^C(\Phi\Sigma)$ has small products that are preserved by the sub-category inclusion $Mod^C(\Phi\Sigma) \to Mod^{\widetilde{\mathcal{B}}}(\Phi\Sigma)$;
(3) $[\![\_]\!]_{\Phi^0\Sigma}$ creates small products.

Then $\mathcal{S}$ has products of models that are preserved by the stratification.

2.4.3. *Preservation results.* Two of the following three definitions establish concepts of preservation (of the satisfaction of the sentences) by quasi-varieties. The remaining is an auxiliary preservation concept.

Definition 2.9. *In any stratified institution $\mathcal{S}$, a Σ-sentence ρ is* preserved by model products *when for each family $(M_j)_{j\in J}$ of Σ-models, for any product $(p_j : M \to M_j)_{j\in J}$ and each $w \in [\![M]\!]$ if for each $j \in J$, $M_j \models^{[\![p_j]\!]w} \rho$ then $M \models^w \rho$.*

Fact 2.4. [8] *A sentence is preserved by products in $\mathcal{S}$ if and only if it is preserved by products in $\mathcal{S}^\sharp$.*

Definition 2.10. *In any stratified institution $\mathcal{S}$ such that for some $\mathcal{S}$-signature Σ, $Mod(\Sigma)$ has an inclusion system $(I_\Sigma, \mathcal{E}_\Sigma)$, a Σ-sentence ρ is* preserved by sub-models *when for each $(h : M \to N) \in I_\Sigma$, $N \models^{[\![h]\!]w} \rho$ implies $M \models^w \rho$.*

Fact 2.5. *The inclusion system $(I_\Sigma, \mathcal{E}_\Sigma)$ of $Mod^{\mathcal{S}}(\Sigma)$ provides also a canonical inclusion system of $Mod^\sharp(\Sigma)$, where $h : (M,w) \to (N,v)$ is inclusion / surjection if and only if h belongs to $I_\Sigma/\mathcal{E}_\Sigma$, respectively. Then a Σ-sentence ρ is preserved by sub-models in $\mathcal{S}$ if and only if ρ is preserved by sub-models in $\mathcal{S}^\sharp$.*

Definition 2.11. *In any stratified institution $\mathcal{S}$ a Σ-sentence is* preserved along a Σ-homomorphism *$h : M \to N$ when for each $s \in [\![M]\!]$, $M \models^s \rho$ implies $N \models^{[\![h]\!]s} \rho$.*

Fact 2.6. *In any stratified institution a Σ-sentence ρ is preserved by a Σ-homomorphism $h: M \to N$ if and only if in $\mathcal{S}^\sharp$ it is preserved by each homomorphism $h: (M, w) \to (N, v)$.*

The following result establishes preservation by decomposition. When developing preservation properties in concrete applications by an inductive process on the structure of the sentences the base of the induction is provided by this result.

Proposition 2.10. *Consider a decomposition of a stratified institution $\mathcal{S}$ like in Definition 2.2 under the conditions of Propositions 2.6, 2.7, 2.8, 2.9. Let Σ be an $\mathcal{S}$-signature. Then*

(1) Let $\rho \in Sen^{\mathcal{B}}(\Phi\Sigma)$. If ρ is preserved by products / sub-models / homomorphisms in $\mathcal{B}$ then $\alpha_\Sigma\rho$ is preserved by products / sub-models / homomorphisms in $\mathcal{S}$ too.
(2) Let $\rho^0 \in Sen^0(\Phi^0\Sigma)$. If ρ^0 is preserved by products / sub-models / homomorphisms in $\mathcal{S}^0$ then $\alpha^0\rho^0$ is preserved by products / sub-models / homomorphisms in $\mathcal{S}$ too.

The following results up to Proposition 2.13 included establish the invariance of preservation by quasi-varieties under Boolean connectives.

Proposition 2.11. *In any stratified institution $\mathcal{S}$ the preservation by products, sub-models and homomorphisms is invariant with respect to conjunction.*

Proposition 2.12. *In any stratified institution $\mathcal{S}$ the preservation by sub-models and homomorphisms is invariant with respect to disjunctions.*

Proposition 2.13. *Let $\mathcal{S}$ be a stratified institution and ρ_1, ρ_2 be Σ-sentences. If ρ_1 is preserved along inclusions / products projections and ρ_2 is preserved by sub-models / products then $\rho_1 \Rightarrow \rho_2$ is preserved by sub-models / products.*

The following developments up to Proposition 2.14 included establish the invariance of preservation by quasi-varieties under quantifications.

Definition 2.12. *We say that a signature morphism $\varphi: \Sigma \to \Sigma'$ in a stratified institution $\mathcal{S}$* lifts forwards / backwards products *when for each product of Σ-models $(p_i: N \to M_i)_{i\in I}$ and each φ-expansion N' / $(M'_i)_{i\in I}$ of N / $(M_i)_{i\in I}$ there exists a φ-expansion $(p'_i: N' \to M'_I)_{i\in I}$ of $(p_I)_{i\in I}$ which is a product.*

Fact 2.7. *The liftings of products coincide in $\mathcal{S}$ and $\mathcal{S}^\sharp$.*

Definition 2.13. *For any signature morphism $\varphi: \Sigma \to \Sigma'$ in a stratified institution $\mathcal{S}$ and for any Σ-homomorphism $h: M \to N$ we say φ* lifts forwards / backwards h *when for each φ-expansion M' / N' of M / N there exists a φ-expansion $h': M' \to N'$ of h. Moroever, φ* lifts forwards / backwards inclusions *when h' is inclusion if h is inclusion.*

Fact 2.8. *The liftings of homomorphisms and inclusions coincide in $\mathcal{S}$ and $\mathcal{S}^\sharp$.*

Proposition 2.14. *Let $\mathcal{S}$ be a stratified institution, $\chi: \Sigma \to \Sigma'$ be a signature morphism and ρ be a Σ'-sentence.*

(1) *If χ lifts forwards inclusions / products / homomorphisms and ρ is preserved by sub-models / products / homomorphisms then $(\forall\chi)\rho$ / $(\forall\chi)\rho$ / $(\exists\chi)\rho$ is preserved by sub-models / products / homomorphisms.*
(2) *If χ lifts backwards inclusions / products / homomorphisms and ρ is preserved by sub-models / products / homomorphisms then $(\exists\chi)\rho$ / $(\exists\chi)\rho$ / $(\forall\chi)\rho$ is preserved by sub-models / products / homomorphisms.*

The following results up to Proposition 2.16 included establish the invariance of preservation by quasi-varieties under implicit nominals and modalities.

Proposition 2.15. *Let $\mathcal{S}$ be a stratified institution endowed with a nominal extraction (N, Nm). Let ρ be any Σ-sentence.*

(1) [8] *If each Nm_Σ preserves products then each i-sentence is preserved by products.*
(2) [8] *If each Nm_Σ preserves products and ρ is preserved by products then for each $i \in N(\Sigma)$, $@_i$ is preserved by products too.*
(3) *If each $[\![\_]\!]_\Sigma$ preserves inclusions then each i-sentence is preserved by sub-models.*
(4) *If each $[\![\_]\!]_\Sigma$ preserves inclusions and ρ is preserved by sub-models then for each $i \in N(\Sigma)$, $@_i\rho$ is preserved by sub-models too.*
(5) *Each i-sentence is preserved along homomorphisms.*
(6) *If ρ is preserved along homomorphisms then for each $i \in N(\Sigma)$, $@_i$ is preserved by homomorphisms too.*

Proposition 2.16. *Let $\mathcal{S}$ be a stratified institution endowed with frame extraction (L, Fr). Then for any $\lambda \in L(\Sigma)_{n+1}$ and $\rho_1, \ldots, \rho_n \in Sen(\Sigma)$ we have that:*

(1) [8] *If Fr_Σ preserve products and ρ_k, $k = \overline{1,n}$, are preserved by products then $\langle\lambda\rangle(\rho_1, \ldots, \rho_n)$ is preserved by products too.*
(2) *If ρ_k, $k = \overline{1,n}$, are preserved along homomorphisms then $\langle\lambda\rangle(\rho_1, \ldots, \rho_n)$ is preserved along homomorphisms too.*
(3) *If ρ_k, $k = \overline{1,n}$ is preserved by sub-models then $[\lambda](\rho_1, \ldots, \rho_n)$ is preserved by sub-models too.*
(4) *If Fr_Σ preserve products, $\lambda \in L(\Sigma)_2$ and ρ is preserved by products then $[\lambda]\rho$ is preserved by products too.*

2.4.4. *Initial semantics.* The following result establishes a very general causality between quasi-varieties and initial semantics. It plays a crucial role for our developments on initial semantics in stratified institutions.

Proposition 2.17. [6] *Let $\mathcal{C}$ be a category with initial objects and small products and which is endowed with an epic co-well-powered inclusion system. Then any quasi-variety in $\mathcal{C}$ admits initial objects.*

Thus we propose the following stepwise methodology for establishing initial semantics of logical theories in stratified institutions:

(1) Establish initial models, products of models and inclusion systems for the categories of models of the stratified institution eventually by using the result of Proposition 2.18 below and the general results of Sections 2.4.1 and 2.4.2.

(2) Establish that the inclusion systems of the categories of models on the stratified institution has the properties required by Proposition 2.17. This issue can be addressed at the general level by the result of Proposition 2.19 below.
(3) Eventually combine preservation results of Section 2.4.3 in order to obtain preservation by quasi-varieties for certain classes of sentences. Then by applying Proposition 2.17 we obtain initial semantics for them.

For the following couple of results we assume a decomposition of a stratified institution $\mathcal{S}$ like in Definition 2.2.

The following result addresses at a general level the initiality condition in Proposition 2.17.

Proposition 2.18. *Consider any $\mathcal{S}$ signature Σ and assume that both $Mod^0(\Phi^0\Sigma)$ and $Mod^C(\Phi\Sigma)$ have initial objects that share their stratification. Then $Mod^{\mathcal{S}}(\Sigma)$ has initial objects.*

Then following result addresses at a general level the two specific conditions on the inclusion system required by Proposition 2.17.

Proposition 2.19. *Consider any $\mathcal{S}$ signature Σ and assume that $Mod^0(\Phi^0\Sigma)$ and $Mod^{\mathcal{B}}(\Phi\Sigma)$ admit inclusion systems $(I^0_{\Phi^0\Sigma}, \mathcal{E}^0_{\Phi^0\Sigma})$ and $(I_{\Phi\Sigma}, \mathcal{E}_{\Phi\Sigma})$, respectively, satisfying the conditions of Propositions 2.7 and 2.6. Then the inclusion system $(\overline{I}_\Sigma, \overline{\mathcal{E}}_\Sigma)$ of $Mod^{\mathcal{S}}(\Sigma)$ obtained in Proposition 2.7 is epic / co-well-powered when both inclusion systems $(I^0_{\Phi^0\Sigma}, \mathcal{E}^0_{\Phi^0\Sigma})$ and $(I_{\Phi\Sigma}, \mathcal{E}_{\Phi\Sigma})$ are epic / co-well-powered.*

2.5. **Interpolation.** Stratified institutions admit two different semantic consequence relations that correspond to the two possible flattenings of the stratified institutions to ordinary institutions. The aim of our study on interpolation is to clarify the two concepts of interpolation emerging from the two concepts of semantic consequence and then two establish a causality relationship between them. We prove that under some general technical condition, that in the concrete situations means having nominals and quantifications over those, one of the concept of interpolation is stronger than the other one.

Definition 2.14. *For any stratified institution $\mathcal{S}$*

- *the* local semantic consequence *relation $\models^\sharp$ is the semantic consequence relation of the institution $\mathcal{S}^\sharp$, and*
- *the* global semantic consequence *relation $\models^*$ is the semantic consequence relation of the institution $\mathcal{S}^*$.*

Fact 2.9. *For any stratified institution $\mathcal{S}$, any $\mathcal{S}$-signature Σ, any set E of Σ-sentences and any Σ-sentence e*

- *$E \models^\sharp e$ if and only if for each Σ-model M and each $w \in [\![M]\!]_\Sigma$,*
$$M \models^w E \text{ implies } M \models^w e,$$
- *$E \models^* e$ if and only if for each Σ-model M,*
$$M \models^* E \text{ implies } M \models^* e.$$

Proposition 2.20. *$E \models^\sharp e$ implies $E \models^* e$.*

Proposition 2.21. *Consider a stratified institution $\mathcal{S}$ and a commutative square of signature morphisms like below:*

(2)
$$\begin{array}{ccc} \Sigma & \xrightarrow{\varphi_1} & \Sigma_1 \\ {\scriptstyle \varphi_2}\downarrow & & \downarrow{\scriptstyle \theta_1} \\ \Sigma_2 & \xrightarrow[\theta_2]{} & \Sigma' \end{array}$$

Let $\models$ be any of the two semantic consequence relations, $\models^\sharp$ or $\models^*$. Let E_k be sets of Σ_k-sentences, $k = \overline{1,2}$. If there exists a set E of Σ-sentences such that

$$E_1 \models \varphi_1 E \ \text{ and } \ \varphi_2 E \models E_2$$

then $\theta_1 E_1 \models \theta_2 E_2$.

Definition 2.15 (Craig interpolation in stratified institutions)**.** *The commutative square of signature morphisms* (2) *is a* local / global Craig interpolation square *when the reversal of the implication of Proposition 2.21 holds.*

Given classes of signature morphisms $\mathcal{L}$ and $\mathcal{R}$ we say that the stratified institution $\mathcal{S}$ has local / global Craig $(\mathcal{L},\mathcal{R})$-interpolation *when each pushout square* (2) *with $\varphi_1 \in \mathcal{L}$ and $\varphi_2 \in \mathcal{R}$ is a local / global Craig interpolation square.*

Definition 2.16 (Signature extensions with nominals)**.** *Any stratified institution has* signature extensions with nominals *when it has nominals extraction (N, Nm) and for each signature Σ there exists a signature morphism $\iota : \ \Sigma \to \Sigma'$ – called the signature extension of Σ with the nominal i – such that*

(1) $N(\iota) : \ N(\Sigma) \to N(\Sigma') = N(\Sigma) \cup \{i\}$ is the extension of $N(\Sigma)$ with one element i, and
(2) for each Σ-model M and each $w \in [\![M]\!]_\Sigma$ there exists a ι-expansion M' of M such that

$$[\![M']\!]_\iota (Nm_{\Sigma'} M')_i = w.$$

(3) for each signature morphism $\theta_1 : \ \Sigma_1 \to \Sigma'$ and each signature extension $\iota' : \ \Sigma' \to \Sigma''$ with one nominal i' there exists a signature extension $\iota_1 : \ \Sigma_1 \to \Sigma'_1$ with one nominal i such that

$$\begin{array}{ccc} \Sigma_1 & \xrightarrow{\iota_1} & \Sigma'_1 \\ {\scriptstyle \theta_1}\downarrow & & \downarrow{\scriptstyle \theta'_1} \\ \Sigma' & \xrightarrow[\iota']{} & \Sigma'' \end{array}$$

is a stratified model amalgamation square.

Proposition 2.22. *Let $\mathcal{S}$ be any stratified institution with signature extensions with nominals, with universal quantifications over those extensions, and with local explicit satisfaction. Let M be a Σ-model, $w \in [\![M]\!]_\Sigma$, E be a set of Σ-sentences and e be a Σ-sentence. Let ι be any extension of Σ with a nominal i. Then*

(1) $M \models^* e$ if and only if $(M, w) \models^\sharp e^\iota$.
(2) $M \models^* e$ if and only if $M \models^* e^\iota$.
(3) $E \models^* e$ implies $E^\iota \models^\sharp e^\iota$.

(4) $E^{\iota} \models^{*} e$ *implies* $E \models^{*} e$.
(5) $E \models^{*} e^{\iota}$ *implies* $E \models^{*} e$.

Proposition 2.23. *Let $\mathcal{S}$ be any stratified institution with signature extensions with nominals, with universal quantifications over those extensions, and with local explicit satisfaction. Let*

$$\begin{array}{ccc} \Sigma_1 & \xrightarrow{\iota_1} & \Sigma'_1 \\ {\scriptstyle \theta_1}\downarrow & & \downarrow{\scriptstyle \theta'_1} \\ \Sigma' & \xrightarrow[\iota']{} & \Sigma'' \end{array} \tag{3}$$

be a stratified model amalgamation square such that both ι_1 and ι' are signature extensions with one nominal. Then for each Σ'-model M' and each $w' \in [\![M']\!]_{\Sigma'}$, for each Σ_1-sentence e_1 we have that

$$(M', w') \models^{\sharp} (\theta_1 e_1)^{\iota'} \text{ if and only if } (M', w') \models^{\sharp} \theta_1(e_1^{\iota_1}).$$

The following is the main result of pour general study of interpolation in stratified institutions.

Proposition 2.24. *Let $\mathcal{S}$ be any stratified institution with signature extensions with nominals, with universal quantifications over those extensions, and with local explicit satisfaction. Then any local Craig interpolation square is a global Craig interpolation square too.*

3. ULTRAPRODUCTS AND QUASI-VARIETIES IN $\mathcal{L}$-INSTITUTIONS

Here we report the concepts and results developed under this topic without providing proofs. The full developments can be found here.

In what follows we use the following abbreviations for some concrete many-valued truth institutions (aka $\mathcal{L}$-institutions): *MVL*$_0$ for many-valued propositional logic, *MVL*$_1$ for many-valued first order logic, *HMVL* for many-valued Horn logic, TL_0 for propositional temporal logic, and *FMA* for fuzzy multi-algebras.

3.1. **Basic sentences.** In the concrete $\mathcal{L}$-institutions the sentences are built from atoms by iterations of connectives. At the abstract level we do not explicitly consider atomic sentences as such but rather consider them implicitly in terms of their model-theoretic properties. The following extends the concept of *basic set of sentences* from ordinary institutions [3, 6] to $\mathcal{L}$-institutions.

Definition 3.1 (Basic sentences)**.** *Let $\kappa \in L$. A set E of Σ-sentences is* κ-basic *if and only if there exists a Σ-model $M_{E,\kappa}$ such that for each Σ-model M*

$$(M \models E) \geqslant \kappa \text{ if and only if there exists a homomorphism } h : M_{E,\kappa} \to M.$$

When $M_{E,\kappa}$ is ($\mathcal{F}$-)finitely presented in the category of the Σ-models then we say that E is ($\mathcal{F}$-)finitary κ-basic. *Moreover, E is* (($\mathcal{F}$-)finitary) basic *when it is (($\mathcal{F}$-)finitary) κ-basic for each $\kappa \in L$. When $E = \{e\}$ is a singleton set, we say that e is a* ($\mathcal{F}$-)finitary (κ-)basic sentence.

Fact 3.1. *If the filter F is a singleton then each κ-basic sentence is trivially F-finitary κ-basic.*

Proposition 3.1 (Basic sentences in *MVL*)**.** *In MVL_0, MVL_1 or in HMVL let E be a set of atomic sentences. Then*

(1) E is basic.
(2) Moreover, if E and κ are finite then E is finitary κ-basic.

Proposition 3.2 (Basic sentences in TL_0)**.** *In TL_0, for any finite $\kappa \in \mathcal{L}$, each set E of atoms for a signature P is finitary κ-basic.*

Proposition 3.3 (Unions of sets of basic sentences)**.** *If the $\mathcal{L}$-institution coproducts of models then the set of the (κ-)basic sets of sentences is closed under unions. Moreover, when κ is finite, this property extends to finitary κ-basic sets of sentences.*

Proposition 3.4 (Coproducts of *MVL*-models)**.** *Both MVL_0 and MVL_1 have coproducts of models.*

Proposition 3.5 (Existentially quantified basic sentences are still basic)**.** *In MVL_1, for any completely join-prime κ, if e is a (finitary) κ-basic $(F + X, P)$-sentence then $\exists X \cdot e$ is a (finitary) κ-basic (F, P)-sentence.*

3.2. **Filtered (reduced) products.** The following results up to Corollary 3.2 included establish the existence of categorical filtered (reduced) products in several concrete $\mathcal{L}$-institutions of interest.

Proposition 3.6 (Directed co-limits of MVL_0 models)**.** *If $\mathcal{L}$ has directed joins then MVL_0 has directed co-limits of models.*

Corollary 3.1 (F-products in MVL_0)**.** *If $\mathcal{L}$ has all meets and directed joins then for each filter F, MVL_0 has F-products of models.*

Proposition 3.7 (F-products in MVL_1)**.** *If $\mathcal{L}$ has all meets and directed joins then MVL_1 has F-products of models for each filter F.*

Proposition 3.8 (F-products in TL_0)**.** *The category of TL_0 P-models has all filtered products.*

Fact 3.2. *Let (F, C) be any FMA signature. For each $n \in \omega$ we define $\widetilde{F}_{n+1} = F_n$ and $\widetilde{F}_0 = \emptyset$. Then there exists a canonical isomorphism $Mod^{FMA}(F, C) \cong Mod^{MVL_1}(C, \widetilde{F})$.*

Corollary 3.2 (F-products in *FMA*)**.** *The category of FMA (F, C)-models has all filtered products.*

3.3. **Preservation properties.**

3.3.1. *Preservation by filtered products.* The following definition extends the corresponding concept of preservation from binary institution theory to $\mathcal{L}$-institutions.

Definition 3.2. *In any $\mathcal{L}$-institution let Σ be any signature and e be any Σ-sentence. Also let $\mathcal{F}$ be any class of filters and κ be any value in $\mathcal{L}$. Then*

- e *is* κ-preserved by $\mathcal{F}$-products *when for each* F*-product* $(\mu_J : M_J \to M_F)_{J\in F}$ *(where* F *is a filter over* I*)*

$$\{i \in I \mid (M_i \models e) \geqslant \kappa\} \in F \ \textit{ implies } \ (M_F \models e) \geqslant \kappa$$

- e *is* κ-preserved by $\mathcal{F}$-factors *when for each* F*-product as above we have the opposite implication than above.*

When $\mathcal{F}$ *is the class of all ultrafilters we rather say directly "*κ*-preserved by ultraproducts / ultrafactors". When* $\mathcal{F}$ *is the class of all singleton filters we rather say "*κ*-preserved by direct products / factors".*

Proposition 3.9 (Preservation by F-factors)**.** *For any filter* F *and any* Σ*-sentence* ρ *the following are equivalent:*

(1) ρ *is preserved by* F*-factors.*
(2) For each F*-product* $(\mu_J : M_J \to M_F)_{J\in F}$

$$\{i \mid (M_F \models \rho) \leqslant (M_i \models \rho)\} \in F.$$

The following couple of results establish the preservation (of the satisfaction) of basic sentences by F-products / factors.

Proposition 3.10 (Preservation of basic sentences by filtered products)**.** *Each* κ*-basic sentence is* κ*-preserved by all filtered products.*

Proposition 3.11 (Preservation of basic sentences by filtered factors)**.** *Each* F*-finitary* κ*-basic sentence is* κ*-preserved by* F*-factors.*

The following results up to Proposition 3.14 included establish the invariance of preservation under propositional connectives.

Proposition 3.12 (Invariance of preservation under conjunction)**.** *The set of the sentences that are* κ*-preserved by* F*-products /* F*-factors is closed under conjunctions.*

Proposition 3.13 (Invariance of preservation by factors under propositional connectives)**.** *The set of the sentences that are preserved by* F*-factors are closed under* $\wedge$, $\vee$, $*$.

Proposition 3.14 (Invariance of preservation under implication)**.** *If* ρ *is preserved by* F*-factors and* ρ' *is preserved by* F*-products then* $\rho \Rightarrow \rho'$ *is preserved by* F*-products.*

The following series of results up to Proposition 3.20 establishes the invariance of preservations under quantification connectives.

Definition 3.3. *Let* $\mathcal{F}$ *be a class of filters closed under reductions. A signature morphism* $\chi : \Sigma \to \Sigma'$ preserves / invents $\mathcal{F}$-products *when* $Mod(\chi)$ *preserves / invents* $\mathcal{F}$*-products.*

Proposition 3.15 (Invariance of preservation under quantifications (I))**.** *Let* $\mathcal{F}$ *be a class of filters that is closed under reductions and let* $\chi : \Sigma \to \Sigma'$ *be a signature morphism that invents* $\mathcal{F}$*-products. Let* ρ *be a* Σ'*-sentence.*

(1) If ρ *that is* κ*-preserved by* $\mathcal{F}$*-products then* $\forall\chi \cdot \rho$ *is* κ*-preserved by* $\mathcal{F}$*-products too.*

(2) If ρ is κ-preserved by $\mathcal{F}$-factors and κ is completely prime-join then $\exists\chi \cdot \rho$ is κ-preserved by $\mathcal{F}$-factors too.

Proposition 3.16 (Invention of filtered products in *MVL*$_1$)**.** *In MVL$_1$ all signature extensions with new constants invent all filtered products.*

Corollary 3.3 (Invention of filtered products in *FMA*)**.** *In FMA all signature extensions with deterministic constants invent all filtered products.*

Proposition 3.17 (Invariance of preservation under quantifications (II))**.** *Let $\mathcal{F}$ be a family of filters that is closed under reductions and let $\chi : \Sigma \to \Sigma'$ be a signature morphism that preserves $\mathcal{F}$-products. Let ρ be any Σ'-sentence. If ρ is κ-preserved by $\mathcal{F}$-products and κ is completely join-prime then $\exists\chi \cdot \rho$ is κ-preserved by $\mathcal{F}$-products too.*

Proposition 3.18 (Preservation of filtered products in *MVL*$_1$/*FMA*)**.** *In MVL$_1$ / FMA each signature morphism preserve all filtered products.*

Proposition 3.19. *Let F be a filter over a set I such that F is closed under arbitrary intersections. Let $\chi : \Sigma \to \Sigma'$ be a signature morphism that lifts F-products and let ρ be any Σ'-sentence. If ρ is κ-preserved by F-factors then $\forall\chi \cdot \rho$ is κ-preserved by F-factors too.*

Corollary 3.4. *Let χ be a signature morphism that lifts direct products and let ρ be a Σ'-sentence. If ρ is κ-preserved by direct products then $\forall\chi \cdot \rho$ is κ-preserved by direct products too.*

Proposition 3.20 (Lifting filtered products in *MVL*$_1$/*FMA*)**.** *In MVL$_1$ / FMA each signature extension with constants / deterministic constants lifts all filtered products.*

The following defines Horn sentences in $\mathcal{L}$-institution and the next two results show they are preserved by filtered products.

Definition 3.4 (Horn sentences)**.** *Any sentence $\forall\chi \cdot H \Rightarrow e$ where χ is a signature morphism, H is a quantifier-free sentence formed from basic sentences by iterations of $\wedge$, $\vee$, $*$, and e is a basic sentence, is called a* Horn sentence*.*

*Given a class $\mathcal{F}$ of filters, an $\mathcal{F}$-*Horn sentence *is a Horn sentence $\forall\chi \cdot H \Rightarrow e$ such that each basic sentence of H is $\mathcal{F}$-finitary. When $\mathcal{F}$ is the class of all filters we rather say "finitary Horn sentence".*

Corollary 3.5 (Preservation of Horn sentences)**.** *Let $\mathcal{F}$ be a class of filters closed under reductions. Then any $\mathcal{F}$-Horn sentence $\forall\chi \cdot H \Rightarrow e$ such that χ invents $\mathcal{F}$-products is preserved by $\mathcal{F}$-products.*

Corollary 3.6 (Preservation in *FMA*)**.** *In FMA let us assume that each truth value is completely prime-join. Then each atomic sentence $t \prec t'$ is preserved by filtered products.*

3.3.2. *Preservation by sub-models.* The following defines a general sub-class of basic sentences that are preserved by sub-models.

Definition 3.5 (Epic basic sentences)**.** *Let Σ be a signature in an $\mathcal{L}$-institution such that $Mod(\Sigma)$ admits an initial model and has a designated inclusion system. Then a (κ-)basic set of sentences E is* epic (κ-)basic *when $M_{E,\kappa}$ is reachable.*

the following developments up to Proposition 3.22 establish preservation by sub-models properties in $\mathcal{L}$-institutions.

Definition 3.6 (Preservation by sub-models)**.** *A Σ-sentence ρ is* κ-preserved by sub-models *when for each $(N \subseteq M) \in I_\Sigma$ if $(M \models \rho) \geqslant \kappa$ then $(N \models \rho) \geqslant \kappa$. Moreover, ρ is* preserved by sub-models *when it is κ-preserved by sub-models for each $\kappa \in L$.*

Proposition 3.21 (Preservation by sub-models (I))**.** *The set of the sentences that are preserved by sub-models contain the epic basic sentences and is closed under $\wedge$, $\vee$, $\Rightarrow$, $*$, $\neg$.*

Definition 3.7 (Inventing sub-models)**.** *A signature morphism $\chi : \ \Sigma \to \Sigma'$* invents sub-models *when for each $(M \subseteq N) \in I_\Sigma$ and for each χ-expansion M' of M there exists a χ-expansion $(M' \subseteq N') \in I_{\Sigma'}$ of $M \subseteq N$.*

Proposition 3.22 (Preservation by sub-models (II))**.** *Let $\chi : \ \Sigma \to \Sigma'$ be a signature morphism that invents sub-models. Then for any Σ'-sentence ρ', $\forall \chi \cdot \rho'$ is κ-preserved by sub-models if ρ' is κ-preserved by sub-models.*

3.4. **Semantic compactness consequences.** Preservation by filtered products has the important consequences for compactness, which is one of the most important model theoretic properties. The following developments up to Proposition 3.23 define notions of compactness for $\mathcal{L}$-institutions.

Definition 3.8 (Consistent theory)**.** *In any $\mathcal{L}$-institution, a Σ-theory T is consistent there exists a Σ-model M such that $T \leqslant M^*$.*

Definition 3.9 (κ-consistency)**.** *In any $\mathcal{L}$-institution, for any truth value κ, a set E of Σ-sentences is* κ-consistent *when $T_\kappa | E$ is consistent.*

Definition 3.10 (Many-valued semantic compactness)**.** *An $\mathcal{L}$-institution is* m-compact *when for each Σ-theory T if $T|\Gamma$ is consistent for each finite $\Gamma \subseteq Sen(\Sigma)$ then T is consistent too.*

Definition 3.11 (κ-m-compactness)**.** *In an $\mathcal{L}$-institution let $\kappa \in L$ be any truth value. Then the $\mathcal{L}$-institution is* κ-m-compact *when each set E of Σ-sentences is κ-consistent if E_0 is κ-consistent for each finite $E_0 \subseteq E$.*

Proposition 3.23 (κ-m-compactness by m-compactness)**.** *Any m-compact $\mathcal{L}$-institution is κ-m-compact for each truth value κ.*

The following is an important theorem that extends a corresponding result from ordinary to $\mathcal{L}$-institution theory.

Theorem 3.1 (A fundamental ultraproducts theorem)**.** *Consider any $\mathcal{L}$-institution that has ultraproducts of models and such that its sentences are preserved by ultraproducts. Let Σ be any of its signatures and let*

$$I = \{i \subseteq Sen(\Sigma) \mid i \text{ finite}\}.$$

For any theory T and any family of Σ-models $(M_i)_{i \in I}$ such that for each $i \in I$, $T|i \leqslant M_i^*$ there exists an ultrafilter U on I such that $T \leqslant M_U^*$.

The following series of results up to Corollary 3.10 included establishes general and particular compactness properties for $\mathcal{L}$-institutions.

Corollary 3.7 (m-compactness by ultraproducts)**.** *Any $\mathcal{L}$-institution with ultraproducts of models such that each sentence is preserved by ultraproducts is m-compact.*

Corollary 3.8 (κ-m-compactness by ultraproducts)**.** *In any $\mathcal{L}$-institution, if each sentence is κ-preserved by ultraproducts then the $\mathcal{L}$-institution is κ-m-compact.*

Corollary 3.9 (m-compactness for Horn sentences)**.** *Let $\mathcal{I}$ be an $\mathcal{L}$-institution such that each sentence is semantically equivalent to a finitary Horn sentence $\forall\chi \cdot H \Rightarrow e$ such that χ invents ultraproducts. Then $\mathcal{I}$ is m-compact.*

Corollary 3.10 (Compactness in *FMA*)**.** *In FMA let us assume that each truth value is completely prime-join. Let us consider the sub-institution FMA$'$ of FMA obtained by restricting the sentences to those formed from the atoms $t \prec t'$ by iterations of conjunctions and universal and existential quantifications with deterministic variables. Then FMA$'$ is m-compact.*

3.5. **Quasi-varieties and initial semantics.** The final part of our study on the model theory of $\mathcal{L}$-institutions develops a general initial semantics result for a sub-class of Horn sentences. The result is based on preservation by quasi-varieties.

Definition 3.12 (Quasi-varieties in $\mathcal{L}$-institutions)**.** *In any $\mathcal{L}$-institution, for any signature Σ, a class $\mathcal{C}$ of Σ-models is called a* quasi-variety *if and only if it is closed under small products and sub-models.*

Definition 3.13 (Strong Horn sentences)**.** *In any $\mathcal{L}$-institution with a designated inclusion system for each of its categories of models, a Horn sentence $\forall\chi \cdot H \Rightarrow e$ is* strong *when*
- *χ invents direct products and sub-models,*
- *H is formed from epic basic sentences by iterations of connectives from the set $\{\wedge, \vee, *\}$, and*
- *e is an epic basic sentence.*

Corollary 3.11. *For any strong Horn sentence ρ and any truth value κ, the class of the models M such that $(M \models \rho) \geqslant \kappa$ is a quasi-variety.*

Definition 3.14 (Strong Horn theory)**.** *A* Σ-theory *T is a* strong Horn theory *when $T\rho = \bot$ for each sentence that is* not *a strong Horn sentence.*

Corollary 3.12 (Quasi-varieties of Horn theories)**.** *The models of a strong Horn theory form a quasi-variety.*

Corollary 3.13 (Initial semantics for Horn theories)**.** *If the inclusion system is epic and co-well-powered then any strong Horn theory admits a reachable initial model.*

4. Articles 2021

The following articles have emerged from the research reported here:

(1) "Decompositions of stratified institutions",

(2) “Quasi-varieties in stratified institutions”, and
(3) “Horn sentences in many-valued truth institutions”.

With respect to dissemination, the first and the third articles have been submitted to publication to important journals in the area of the research. The second article, albeit being ready, will be submitted only after the first one is accepted to publication because it refers to it.

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