SCIENTIFIC REPORT FOR THE YEAR 2022 FOR THE PROJECT PN-III-P4-ID-PCE-2020-0446 "AXIOMATIC METHODS IN NON-CLASSICAL MODEL THEORY"

The contents of this report is as follows:

(1) Summary of the developments at stage 2 of the project (year 2022) – Section 1. This includes the general scientific and technical description of the results.
(2) The results obtained at stage 2 of the project. These are presented in appropriate detail separately for each of the activities specified in the funding contract – Section 2 for activity 2.1, Section 3 for activity 2.2, Section 4 for activity 2.3, Section 5 for activity 2.4.

All objectives of stage 2 of the project have been 100% accomplished.

1. SUMMARY OF DEVELOPMENTS AT STAGE 2 OF THE PROJECT

1.1. **Bisimulation in stratified institutions.** The goal of this activity is to develop a concept of bisimulation at the level of stratified institutions that is viable in the sense of supporting an institution-independent model-theoretic treatment of bisimilarity. This includes a series of new concepts and results, which we classify in three categories.

1.1.1. *New concepts.*

- From a technical perspective, the most important concept is that of bounded homomorphism, which captures a simplified kind of bisimulation that arises from certain maps from one frame to another. In concrete examples of stratified institutions, such homomorphisms are also called p-morphisms, or pseudo-epimorphisms.
- Bounded homomorphisms allow us to capture another useful concept: generated sub-models and, in particular, rooted (sub-)models. Those are models whose states are all reachable from a given root.
- The main concept of this activity is an institution-independent formalization of the set-theoretic notion of bisimulation. To that end, we define bisimulation representations by means of categorical spans of bounded homomorphisms.
- The notions of image-finite model and maximal sub-model with respect to some forgetful functor, together with preservation and lifting properties, also play an important role in establishing a Hennessy-Milner theorem for stratified institutions.

1.1.2. *Tree unravelling.* Boundedness allows us to capture at the level of stratified institutions an important technique used both in theoretical computer science and in modal logic. It is called tree unravelling, and deals with clearing up the Kripke structure of a model, resulting in a model whose underlying frame is a tree, without altering the information it encompasses. We have developed an institution-independent unravelling theorem, which, together with results on bisimilarity, gives rise to a tree-model property for stratified institutions.

1.1.3. *Bisimilarity versus elementary equivalence.* The fundamental result of this activity is a Hennessy-Milner theorem for stratified institutions, which likens the notions of bisimilarity (i.e., the existence of a bisimulation between two models) and elementary equivalence (referring to the fact that two models satisfy the same sentences). We have studied the two mutual implications of the theorem separately, and we have identified sufficient conditions (which hold in conventional examples of stratified institutions) for each property.

1.2. **Interpolation and definability for the graded consequence relation.** This activity consisted in developing new concepts on the one hand, and on proving results about them on the other hand. The obtained results are classified in three main categories.

1.2.1. *New concepts.*

- The main concept introduced is a many-valued extension / generalisation of the general institution theoretic (crisp) concept of interpolation. Some examples of this new concept have been developed. While some of them may be expected, others go opposite to the to the common understanding from the crisp case. The latter category illustrate the higher level of subtlety of graded interpolation versus crisp interpolation.
- The definition of a graded truth concept of Robinson consistency is another concept that has been introduced. This comes with some specific auxiliary order-theoretic concepts as well as of some specific semantic consistency and compactness concepts.
- The (Beth) definability property (i.e. implicit definability causes explicit definability) is an important model theoretic property, that in the binary situation may have interpolation as its main cause [2, 3, 17, 5, 1]. We have extended the binary institution-theoretic concepts of definability to 'graded definability'. These include explicit and implicit definability, and it has been done at both the semantic and the consequence-theoretic levels.

1.2.2. *Compositionality properties.* In category theory it is well known that pushout / pullback squares enjoy both 'horizontal' and 'vertical' compositionality, in the sense that by 'glueing together' pushout / pullback squares in any of the two ways the result is still a pushout / pullback square. These properties are technically useful in many situations. We have established very similar compositionality properties for interpolation squares. (However these do not bear any technical relationship to the corresponding categorical properties.) They can be used to establish bigger graded interpolation squares from smaller ones.

1.2.3. *Craig-Robinson versus Craig interpolation.* Craig-Robinson interpolation is an extended version of the common interpolation (known as Craig interpolation), this extension being especially relevant in computing science applications, but not only. In the binary situation, in the presence of implications the two versions of interpolation can be established as equivalent (at the general level of abstract institutions a proof can be found in [5]). We have extended this equivalence to graded interpolation.

1.2.4. *Interpolation versus Robinson consistency.* In binary truth model theory, the existence of negations and conjunctions guarantees the equivalence between Craig interpolation and Robinson consistency, which gives the latter the role of a route to establishing interpolation properties [18, 16, 3, 19, 15, 5]. We have extended this result to graded interpolation as follows.

- We have developed proofs for the mutual implications between graded interpolation and graded Robinson consistency.
- A consequence of the equivalence property mentioned above, namely a symmetry property for graded interpolation is also developed.

1.2.5. *Definability by interpolation.* The causality relationship between interpolation and definability is a cornerstone in classical model theory. In the general binary institution-theoretic setting this has been established in [17]. For the graded case we have done the following.

- We have extended the binary institution theoretic concepts of definability to graded / many-valued truth.
- We have established the natural link between consequence-theoretic and model-theoretic levels of graded definability theory.
- We have proved that one form of graded interpolation causes the so called 'definability property' in the graded context, i.e. that the implicit determines the explicit definability. This has been established first at the consequence theoretic level and than on that basis at the semantic level.

1.3. **Representations of 3/2-institutions as stratified institutions.** The fundamental result of this activity is a definition of a representation of 3/2-institutions (introduced and developed in [8]) as stratified institutions.

1.3.1. *New concepts.* The above mentioned representation is conditioned by certain properties of the signature morphisms in the 3/2-institution. We introduce them as special classes of signature morphisms.

1.3.2. *The representation.* This comprises of a stepwise construction of a stratified institution out of any 3/2-institution that admits classes of signature morphisms as mentioned above. We have also proved the correctness of this representation, i.e. that it satisfies the axioms of a stratified institution.

1.3.3. *Consequences of the representation.* We have explored several consequences of representing 3/2-institutions as stratified institutions as follows:

- A further representation to ordinary institution theory via the adjunction from stratified institutions to ordinary institutions defined in [10] (formerly presented as a mere representation in [7]).
- The import of concepts of semantic connectives.
- The relationship between model amalgamation properties in the representation and in the original 3/2-institution.

2. Bisimulation in stratified institutions

Here we document the concepts and results developed under this topic without providing proofs and applications / examples. The full developments can be found in [21]. To simplify the presentation, for the purpose of this report, we consider only binary frames and we denote by λ the corresponding modality; however, all concepts and results outlined below can be adapted with ease to the more general setting of polyadic modalities.

2.1. New concepts.

Definition 2.1 (Bounded homomorphism)**.** *Let $\mathcal{S} = (Sign, Sen, Mod, [\![\_]\!], \models)$ be a stratified institution equipped with a binary frame extraction (L, Fr) and a sentence sub-functor $Sen_0 \subseteq Sen$. We say that a Σ-homomorphism $h\colon M \to N$ is* bounded *when:*

- *for all $w \in [\![M]\!]$ and $\rho \in Sen_0(\Sigma)$, $M \models^w \rho$ if and only if $N \models^{[\![h]\!]w} \rho$;*
- *for all $w \in [\![M]\!]$, $Fr(N)_\lambda([\![h]\!]w) \subseteq [\![h]\!](Fr(M)_\lambda w)$.*

In practice, for every signature Σ, the set $Sen_0(\Sigma)$ typically consists of atomic sentences. So, a homomorphism $h\colon M \to N$ is bounded when it preserves and reflects the satisfaction of atoms, preserves the accessibility of M-states (implicitly via the frame extraction) and, in addition, ensures that the N-states originating from M can only be accessed through transitions that originate from M as well. In the conventional modal-logic literature, this kind of homomorphisms are also known as *p-morphisms* (see, e.g., [14]).

When the category $Mod(\Sigma)$ is endowed with an inclusion system, the concept of bounded homomorphism gives rise to a refined notion of sub-model.

Definition 2.2 (Generated sub-model)**.** *A sub-model M of N is* generated *when the inclusion homomorphism $M \hookrightarrow N$ is bounded.*

Given a set $X \subseteq [\![N]\!]$ of states, a sub-model M of N is generated by *X if it is the least generated sub-model of N such that $X \subseteq [\![M]\!]$. When X is a singleton set $\{x\}$, we also say that the model M is* rooted *and that the state x is its* root*.*

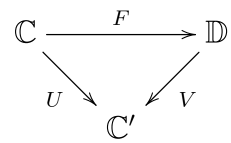
Definition 2.3 (Bisimulation)**.** *Given a stratified institution $\mathcal{S}$ as in Definition 2.1, a* bisimulation (representation) *between two Σ-models M and N is a span $M \xleftarrow{\pi_M} Z \xrightarrow{\pi_N} N$ in $Mod(\Sigma)$ such that both homomorphisms π_M and π_N are bounded.*

We say that the stratified models M and N are bisimilar *if there exists a bisimulation between them. Moreover, for every pair of states $u \in [\![M]\!]$ and $v \in [\![N]\!]$, we say that the pointed models (M, u) and (N, v) are* bisimilar *if there exists a bisimulation $M \leftarrow Z \rightarrow N$ as above and a state $w \in [\![Z]\!]$ such that $[\![\pi_M]\!]w = u$ and $[\![\pi_N]\!]w = v$. In this case, if M and N can be easily inferred, we may also call the states u and v bisimilar, omitting the models.*

Definition 2.4 (Image-finite model)**.** *Given a stratified institution equipped with a binary frame extraction, we say that a Σ-model M is* image finite *when, for all states $w \in [\![M]\!]$, the set of 'successors' $Fr(M)_\lambda w = \{w' \in [\![M]\!] \mid (w, w') \in Fr(M)_\lambda\}$ is finite.*

Definition 2.5 (Maximal sub-objects)**.** *Let $U\colon \mathbb{C} \to \mathbb{C}'$ be an inclusive functor. A sub-object $A \hookrightarrow B$ in $\mathbb{C}$ is* maximal w.r.t. *U when there exists no other sub-object $X \hookrightarrow B$ such that $U(A) = U(X)$ and $A \hookrightarrow X$.*

We say that U maximally lifts sub-objects *when for every inclusion $A' \hookrightarrow U(B)$ in $\mathbb{C}'$ there exists a maximal sub-object A of B in $\mathbb{C}$ such that $U(A) = A'$. And we say that a functor $F\colon \mathbb{C} \to \mathbb{D}$ as in the commuting diagram below, where $V : \mathbb{D} \to \mathbb{C}'$ is also inclusive,*



preserves maximal sub-objects *when $F(A)$ is a maximal sub-object of $F(B)$ w.r.t. V for all maximal sub-objects A of B w.r.t. U.*

2.2. **Tree unravelling.** The first important result on bisimulations in stratified institutions deals with a well-known method in conventional modal logic: *tree unravelling*. This allows the paths in the Kripke frame of a given rooted model to be disentangled into a tree-like structure without altering the model's behaviour.

To that end, we say that a model M is *tree-like* if its underlying frame $Fr(M)$ is a directed tree. In preparation for the tree-unravelling theorem below, we recall that Fr *lifts backwards* a frame homomorphism $f\colon W \to V$ when for every model N such that $Fr(N) = V$ there exists a model homomorphism $h\colon M \to N$ such that $Fr(h) = f$. We say that such a lifting is *final* when for every other homomorphism $h'\colon M' \to N$ such that $Fr(h') = f$ there exists a unique morphism $g\colon M' \to M$ such that $Fr(g) = id_W$ and $g; h = h'$.

Theorem 2.1. *Under the notations and hypotheses of Definition 2.1, if $\mathcal{S}$ admits final (backwards) liftings of homomorphisms along Fr and the satisfaction of Sen_0-sentences is invariant along such final liftings, then every rooted model is bisimilar to a tree-like model.*

2.3. **Bisimilarity versus elementary equivalence.** This section comprises two main results relating the concepts of bisimilarity and elementary equivalence. The fist of them states that the satisfaction of sentences is invariant under bisimulations. In other words, bisimilar models satisfy – in bisimilar states – the same sentences.

Theorem 2.2. *If bounded homomorphisms are elementary, then for any two bisimilar pointed Σ-models (M, u) and (N, v), and any Σ-sentence ρ, we have $M \models^u \rho$ if and only if $N \models^v \rho$.*

The hypothesis that bounded homomorphisms are elementary is essential. In concrete examples of stratified institutions, it is usually checked by induction on the structure of sentences. Even so, we can still simplify the proofs by making use of general institution-theoretic results concerning the internal logic of stratified institutions.

For example, if ρ is a semantic conjunction of ρ_1 and ρ_2, and the satisfaction of ρ_1 and ρ_2 is invariant along a homomorphism h (not necessarily bounded), then the satisfaction of ρ is also invariant along h. The same can be said about semantic negations or disjunctions.

The more interesting case of modal operators is discussed in the following result.

Proposition 2.1. *Suppose ρ is a semantic possibility of ρ' and that the satisfaction of ρ' is invariant along a homomorphism h. Then the satisfaction of ρ is preserved along h. Moreover, if h is a bounded homomorphism, the satisfaction of ρ is also reflected along it.*

Combining the invariance of the satisfaction of sentences under bisimulations with the tree-unravelling theorem outlined in Section 2.2, we obtain the following tree-model property.

Corollary 2.1. *Under the hypotheses of Theorems 2.1 and 2.2, every sentence that is locally satisfiable in a rooted model (at its root) is also locally satisfiable in a tree-like model.*

The converse of Theorem 2.2 does not typically hold – not even in conventional examples of stratified institutions. It suffices to consider a stratified institution with only two sentence-building operators: $true$, which holds at all states, and the conventional possibility operator. Then we can define tree-like models that are elementary equivalent but not bisimilar: take, for instance, a model consisting of all finite paths or arbitrary length (starting from the same state – the root of the model) and the model that has, in addition, another infinite path.

However, we can prove a restricted form of the converse of Theorem 2.2, which deals with image-finite models, thus obtaining a Hennessy-Milner theorem for stratified institutions.

For that purpose, we recall that a direct product of models $M \times N$ given by projections $\pi_M \colon M \times N \to M$ and $\pi_N \colon M \times N \to N$ *preserves* the satisfaction of a sentence ρ when, for all states $w \in [\![M \times N]\!]$, $M \models^{[\![\pi_M]\!]w} \rho$ and $N \models^{[\![\pi_N]\!]w} \rho$ imply $M \times N \models^{w} \rho$. We also say that the product *reflects* the satisfaction of ρ when $M \times N \models^{w} \rho$ implies both $M \models^{[\![\pi_M]\!]w} \rho$ and $N \models^{[\![\pi_N]\!]w} \rho$ (this is actually a preservation property along each of the two projections).

Similarly, with respect to (maximal) sub-models, $M \subseteq N$ *preserves* the satisfaction of a sentence ρ at a state $w \in [\![M]\!]$ when $N \models^{w} \rho$ implies $M \models^{w} \rho$, and it *reflects* the satisfaction of ρ when the opposite implication holds (once more, the latter is in fact a preservation property along the inclusion homomorphism $M \hookrightarrow N$).

Theorem 2.3. *Let $\mathcal{S}$ be a stratified institution as in Definition 2.1 such that:*

- *$[\![\_]\!]$ lifts direct products and maximally lifts sub-objects;*
- *Fr preserves direct products and maximal sub-models w.r.t. $[\![\_]\!]$;*
- *direct products and maximal sub-models w.r.t. $[\![\_]\!]$ preserve and reflect the satisfaction of sentences in $Sen_0(\Sigma)$, for every signature Σ;*
- *$\mathcal{S}$ has semantic true ($true$), conjunctions ($\wedge$), and possibilities ($\Diamond$).*

Then any two image-finite and elementary-equivalent pointed Σ-models are bisimilar.

3. Interpolation and definability for the graded consequence relation

The basic concepts of the theory of $\mathcal{L}$-institutions are available in [6]. Examples of the newly developed concepts and proofs of the results in this section can be found in [12]. The assumption on $\mathcal{L}$ is that it is a residuated lattice (where $*$ denotes the residual conjunction. This assumption is common in many-valued truth model theoretic studies and originates from Goguen's seminal work [13].

3.1. **New concepts.**

3.1.1. *Graded interpolation.*

Definition 3.1 (Graded interpolation)**.** *In any $\mathcal{L}$-entailment system, given a commutative square of signature morphisms*

$$\begin{array}{ccc} \Sigma & \xrightarrow{\varphi_1} & \Sigma_1 \\ {\scriptstyle\varphi_2}\downarrow & & \downarrow{\scriptstyle\theta_1} \\ \Sigma_2 & \xrightarrow[\theta_2]{} & \Sigma' \end{array}$$

and finite sets $E_1 \subseteq Sen(\Sigma_1)$ and $E_2, \Gamma_2 \subseteq Sen(\Sigma_2)$ we say that a finite set $E \subseteq Sen(\Sigma)$ is a Craig-Robinson interpolant of E_1, E_2 and Γ_2 *when*

(1) $$\theta_1 E_1 \cup \theta_2 \Gamma_2 \vdash \theta_2 E_2 \ \leq \ (E_1 \vdash \varphi_1 E) * (\varphi_2 E \cup \Gamma_2 \vdash E_2).$$

When Γ_2 is empty then E is called a Craig interpolant *(of E_1 and E_2).*

When interpolants exist for all E_1, E_2 (, Γ_2) the respective commutative square of signature morphisms is called a Craig(-Robinson) interpolation square *(abbr. C(R)i square).*

When $\mathcal{L}$ is a residuated lattice, the concepts introduced in this definition extend also to $\mathcal{L}$-institutions by considering its semantic entailment system.

Definition 3.2 ($\langle \mathfrak{L}, \mathfrak{R} \rangle$-interpolation)**.** *An $\mathcal{L}$-entailment system (or $\mathcal{L}$-institution) has $\langle \mathfrak{L}, \mathfrak{R} \rangle$-CRi / Ci for $\mathfrak{L}, \mathfrak{R} \subseteq Sign$ classes of signature morphisms, when each pushout square of signature morphisms*

$$\begin{array}{ccc} \Sigma & \xrightarrow{\varphi_1} & \Sigma_1 \\ {\scriptstyle\varphi_2}\downarrow & & \downarrow{\scriptstyle\theta_1} \\ \Sigma_2 & \xrightarrow[\theta_2]{} & \Sigma' \end{array}$$

with $\varphi_1 \in \mathfrak{L}$ and $\varphi_2 \in \mathfrak{R}$ is a CRi / Ci square.

Definition 3.3 (Strong graded interpolation)**.** *In any $\mathcal{L}$-entailment system, for any $\kappa, \kappa_1, \kappa_2 \in L \setminus \{0\}$ such that $\kappa \leq \kappa_1 * \kappa_2$, a commutative square of signature morphisms*

$$\begin{array}{ccc} \Sigma & \xrightarrow{\varphi_1} & \Sigma_1 \\ {\scriptstyle\varphi_2}\downarrow & & \downarrow{\scriptstyle\theta_1} \\ \Sigma_2 & \xrightarrow[\theta_2]{} & \Sigma' \end{array}$$

is a $(\kappa, \kappa_1, \kappa_2)$-Ci square *when for any finite sets $E_1 \subseteq Sen(\Sigma_1)$ and $E_2 \subseteq Sen(\Sigma_2)$ such that*

$$\theta_1 E_1 \vdash \theta_2 E_2 \ \geq \ \kappa$$

there exists a finite set $E \subseteq Sen(\Sigma)$, called the $(\kappa, \kappa_1, \kappa_2)$-interpolant of E_1, E_2, *such that*

$$(E_1 \vdash \varphi_1 E) \ \geq \ \kappa_1 \ \ \textit{and} \ \ \varphi_2 E \cup \Gamma_2 \vdash E_2 \ \geq \ \kappa_2.$$

3.1.2. *Graded Robinson consistency.*

Definition 3.4 (κ-reduct of sets of sentences)**.** *In any $\mathcal{L}$-institution, for each signature morphism $\varphi : \ \Sigma \to \Sigma'$ and each set E' of Σ-sentences and each $\kappa \in L$, we let $[\varphi^{-1} E']_\kappa = \{\rho \in Sen(\Sigma) \mid (E' \models \varphi\rho) \geq \kappa\}$.*

Definition 3.5 (Inter-consistent sets of sentences)**.** *Consider an $\mathcal{L}$-institution. For any $\kappa \in L$ we say that two sets of Σ-sentences Γ_1, Γ_2 are* κ-inter-consistent *when there exists a Σ-model M such that $(M \models \Gamma_1) * (M \models \Gamma_2) \geq \kappa$. We also just say that they are* inter-consistent *when there exists $\kappa > 0$ such that they are κ-inter-consistent. Otherwise, we say that they are* inter-inconsistent.

Definition 3.6 (Companions)**.** *Let $\mathcal{L}$ be a partially ordered set with bottom (0) and top (1) elements and with a binary commutative monotone operation $*$ which admits 1 as identity. For any $\kappa, \ell \in L$ we say that ℓ is*

- *a* lower-companion *to κ when $\{x \mid x * \kappa \neq 0\} \subseteq \{x \mid \ell \leq x\}$;*
- *an* upper-companion *to κ when $\{x \mid \ell \leq x\} \subseteq \{x \mid x * \kappa \neq 0\}$; or*
- *a* single-companion *to κ when it is both a lower- and and upper-companion to κ.*

We say that $\mathcal{L}$ admits lower / upper / single companions when each $\kappa \neq 0$ has a lower / upper / single companion.

Definition 3.7 (Graded Robinson consistency)**.** *Consider an $\mathcal{L}$-institution such that $\mathcal{L}$ is a residuated lattice. For any $\ell, \kappa_1, \kappa_2 \in L \setminus \{0\}$, a commutative square of signature morphisms*

$$\begin{array}{ccc} \Sigma & \xrightarrow{\varphi_1} & \Sigma_1 \\ {\scriptstyle \varphi_2}\downarrow & & \downarrow{\scriptstyle \theta_1} \\ \Sigma_2 & \xrightarrow[\theta_2]{} & \Sigma' \end{array}$$

*is a $(\ell, \kappa_1, \kappa_2)$-*Rc *square when for any finite sets E_i of Σ_i-sentences, $i = 1, 2$, if $[\varphi_1^{-1} E_1]_{\kappa_1}$ and $[\varphi_2^{-1} E_2]_{\kappa_2}$ are inter-consistent then $\theta_1 E_1$ and $\theta_2 E_2$ are ℓ-inter-consistent.*

Definition 3.8 (Compactness)**.** *An $\mathcal{L}$-institution is* compact *when each set E of inconsistent Σ-sentences admits a finite subset $\Gamma \subseteq E$ that is inconsistent too.*

Definition 3.9 (Inter-compactness)**.** *An $\mathcal{L}$-institution is* inter-compact *when for any two inter-inconsistent sets E_1, E_2 of Σ-sentences there are finite subsets $\Gamma_i \subseteq E_i$, $i = 1, 2$, such that Γ_1 and Γ_2 are inter-inconsistent.*

3.1.3. *Graded definability.*

Definition 3.10 (Semantic graded implicit definability)**.** *In any $\mathcal{L}$-institution, for any $\kappa \in L$, a signature morphism $\varphi : \Sigma \to \Sigma'$ is* defined κ-implicitly *by a set $E' \subseteq Sen(\Sigma')$ when for any Σ'-models M_1' and M_2' if*

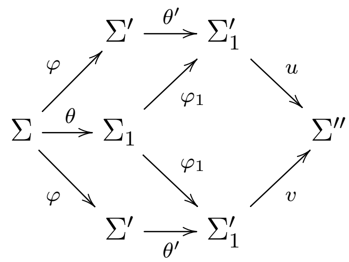
- *$M_1' \models E' \ \wedge \ M_2' \models E' \geq \kappa$ and*
- *$Mod(\varphi) M_1' = Mod(\varphi) M_2'$*

then $M_1' = M_2'$.

Definition 3.11 (Consequence theoretic graded implicit definability)**.** *In any $\mathcal{L}$-entailment system, for any $\kappa \in L$, a signature morphism $\varphi : \Sigma \to \Sigma'$ is* defined κ-implicitly *by a set*

$E' \subseteq Sen(\Sigma')$ *when for any diagram of pushout squares like below*

(2)

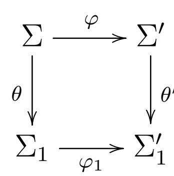


and for any Σ'_1*-sentence* ρ *we have that*

$$u(\theta' E') \cup v(\theta' E') \cup u\rho \vdash v\rho \geq \kappa.$$

Definition 3.12 (Graded explicit definability)**.** *In any* $\mathcal{L}$*-entailment system, for each* $\kappa \in L$*, a signature morphism* $\varphi: \ \Sigma \to \Sigma'$ *is* κ-explicitly defined *by a set of sentences* $E' \subseteq Sen(\Sigma')$ *when for each pushout square of signature morphisms like below*

(3)

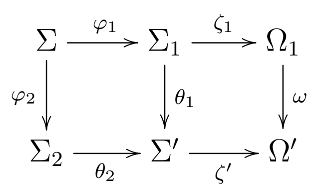


and each $\rho \in Sen(\Sigma'_1)$ *there exists a finite set of sentences* $E_\rho \subseteq Sen(\Sigma_1)$ *such that*

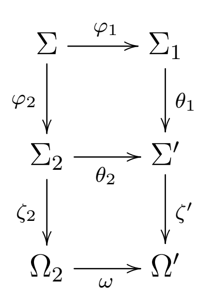
$$(\theta' E' \cup \rho \vdash \varphi_1 E_\rho) * (\theta' E' \cup \varphi_1 E_\rho \vdash \rho) \ \geq \ \kappa.$$

3.2. **Compositionality properties.**

Proposition 3.1 (Horizontal compositionality)**.** *If both the left-hand side and the right-hand side squares below are CRi squares then the outer square is a CRi square too.*



Proposition 3.2 (Vertical compositionality)**.** *If the upper square below is a Ci square and the lower square is a CRi square then the outer square is a CRi square.*



As seen from Propositions 3.1 and 3.2, the vertical compositionality result requires milder conditions than the horizontal one since one of the inner squares is required to be only Ci rather than CRi square.

3.3. **Craig-Robinson versus Craig interpolation.** The following result establishes the sufficient conditions for the equivalence between Craig interpolation and Craig-Robinson interpolation in the graded context.

Theorem 3.1. *Let $\mathcal{L}$ be a Heyting algebra. In any $\mathcal{L}$-institution $\mathcal{I}$ with model-theoretic implications any Ci square is a CRi square.*

3.4. **Interpolation versus Robinson consistency.** The causality relationship between graded interpolation and graded Robinson consistency goes both ways. Theorem 3.2 does one implication. The reverse implication requires an additional condition of compactness which is specific to this result; this is done with Theorem 3.3 below.

Theorem 3.2. *In any $\mathcal{L}$-institution with conjunctions and negations, if ℓ is a lower-companion to κ then any $(\kappa, \kappa_1, \kappa_2)$-Ci square is a $(\ell, \kappa_1, \kappa_2)$-Rc square.*

With respect to the relationship between the compactness of [6] and that given by Definition 3.9, at the general level inter-compactness is stronger than compactness as shown by the following result.

Proposition 3.3. *Any inter-compact $\mathcal{L}$-institution is compact.*

Moreover with Proposition 3.4 we have established general applicable conditions when this follows from the many-valued concept of model-theoretic compactness put forward in [6].

Proposition 3.4. *In any $\mathcal{L}$-institution $\mathcal{I}$, in any of the following two situations*

(1) $\mathcal{L}$ is a Heyting algebra;
(2) $\mathcal{L}$ is a finite total order and $\mathcal{I}$ has semantic residual conjunctions;

the compactness property implies inter-compactness.

The second case of Proposition 3.4 is within the scope of the finite residuated lattices based on the Łukasiewicz arithmetic conjunction.

Theorem 3.3. *In any inter-compact $\mathcal{L}$-institution with conjunctions and negations, if ℓ is an upper-companion to κ and $0 < \kappa \leq \kappa_1 * \kappa_2$, then any $(\ell, \kappa_1, \kappa_2)$-Rc square is a $(\kappa, \kappa_1, \kappa_2)$-Ci square.*

By putting together the results of Theorems 3.2 and 3.3 we obtain an equivalence relationship between Ci and Rc that generalises the corresponding equivalence from the binary institution-independent model theory [19, 5] to $\mathcal{L}$-institutions. This is given by Corollary 3.1 below.

Corollary 3.1. *In any inter-compact $\mathcal{L}$-institution with conjunctions and negations, for any κ that admits a single-companion, any commutative square of signature morphism is a $(\kappa^*, \kappa_1, \kappa_2)$-Rc square if and only if it is a $(\kappa, \kappa_1, \kappa_2)$-Ci square.*

A symmetry property of graded interpolation has been established via graded Robinson consistency:

Corollary 3.2. *In an arbitrary inter-compact $\mathcal{L}$-institution with conjunctions and negations we consider the following commutative squares of signature morphisms:*

$$\begin{array}{ccc} \Sigma & \xrightarrow{\varphi_1} & \Sigma_1 \\ {\scriptstyle\varphi_2}\downarrow & & \downarrow{\scriptstyle\theta_1} \\ \Sigma_2 & \xrightarrow[\theta_2]{} & \Sigma' \end{array} \qquad\qquad \begin{array}{ccc} \Sigma & \xrightarrow{\varphi_2} & \Sigma_2 \\ {\scriptstyle\varphi_1}\downarrow & & \downarrow{\scriptstyle\theta_2} \\ \Sigma_1 & \xrightarrow[\theta_1]{} & \Sigma' \end{array}$$

Then for any $\kappa, \kappa' \in L \setminus \{0\}$ such that $\kappa \leq \kappa' * \kappa'$ we have that the left-hand side square above is a $(\kappa, \kappa', \kappa')$-Ci square if and only if the right-hand side one is a $(\kappa, \kappa', \kappa')$-Ci square.

3.5. **Definability by interpolation.** Proposition 3.5 links semantic and consequence-theoretic implicit graded definabilities. Theorem 3.4 and Corollary 3.3 provide the causality between graded interpolation and graded definability.

Proposition 3.5. *In any semi-exact $\mathcal{L}$-institution let $\kappa, \ell \in L$ such that ℓ is a lower-companion to κ. Then a signature morphism $\varphi : \Sigma \to \Sigma'$ is defined κ-implicitly by $E' \subseteq Sen(\Sigma')$ in the consequence theoretic sense (Definition 3.11) if it is defined ℓ- implicitly by E' in the semantic sense (Definition 3.10).*

The following result shows that at the general graded consequence theoretic level explicit definability implies implicit definability. Note the rather lax conditions of the result: no pushout property required, and any set Γ of sentences instead of E_ρ. This situation is consonant to the rather trivial nature of this part of the definability property, as known from the classical studies of definability where it is often skipped.

Proposition 3.6. *In any $\mathcal{L}$-entailment system with $*$ monotone, for any signature morphism $\varphi : \Sigma \to \Sigma'$ and any set of sentences $E' \subseteq Sen(\Sigma')$, for any commuting diagram of signature morphisms like (2) and for all sets of sentences $\Gamma \subseteq Sen(\Sigma_1)$ and all Σ'_1-sentences ρ we have that*

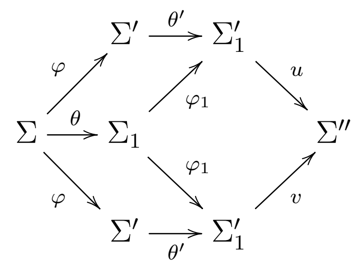
$$(\theta' E' \cup \rho \vdash \varphi_1 \Gamma) * (\theta' E' \cup \varphi_1 \Gamma \vdash \rho) \ \leq \ (\theta'; u) E' \cup (\theta'; v) E' \cup u\rho \vdash v\rho.$$

The other part of the definability property, namely that implicit definability causes explicit definability, and which is the substance of the definability property, can be established on the basis of interpolation. This is what the result of Theorem 3.4 below does at the general graded consequence theoretic level of $\mathcal{L}$-entailment systems.

Theorem 3.4. *In any $\mathcal{L}$-entailment system that has $\langle \mathfrak{L}, \mathfrak{R} \rangle$-CRi for classes $\mathfrak{L}, \mathfrak{R}$ of signature morphisms that are stable under pushouts, for any signature morphism $\varphi : \Sigma \to \Sigma' \in \mathfrak{L} \cap \mathfrak{R}$*

and any set of sentences $E' \subseteq Sen(\Sigma')$, for any diagram of pushout squares:

(4)



for any Σ'_1-sentence ρ there exists a finite set of sentences $E_\rho \subseteq Sen(\Sigma_1)$ such that

$$(\theta' E' \cup \rho \vdash \varphi_1 E_\rho) * (\theta' E' \cup \varphi_1 E_\rho \vdash \rho) \;\geq\; (\theta'; u)E' \cup (\theta'; v)E' \cup u\rho \vdash v\rho.$$

By putting together the results of Proposition 3.5 and of Theorem 3.4 we have obtained the semantic graded definability property:

Corollary 3.3. *In any semi-exact $\mathcal{L}$-institution with $\langle \mathfrak{L}, \mathfrak{R}\rangle$-CRi we let $\kappa, \ell \in L$ such that ℓ is a lower-companion to κ. Then a signature morphism in $\mathfrak{L} \cap \mathfrak{R}$ is defined κ-explicitly when it is defined ℓ-implicitly.*

4. Representations of 3/2-institutions as stratified institutions

The proofs of the results presented in this section can be found in the publication [11], which also provides the involved notations.

4.1. New concepts.

Definition 4.1. *In any 3/2-institution, a signature morphism χ*

- *is* fiber-small *when for each $\chi\Box$-model M we have that $Mod(\chi)M$ is a set; and*
- *is* quasi-representable *when for each $\chi\Box$-model homomorphism $h : \; M \to M_0$ and each $N \in Mod(\chi)M$ there exists and unique model homomorphism $h^N \in Mod(\chi)h$ such that $\Box h^N = N$.*

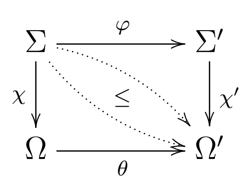
4.2. The general representation.

A 3/2-institution $\mathcal{I} = (Sign, Sen, Mod, \models)$ is fixed and then gradually the entities that will define its associated stratified institution $\mathcal{I}^s = (Sign^s, Sen^s, Mod^s, [\![\_]\!], \models^s$) are built. The main idea of this representation is that the reducts of a model M are considered to be its states.

Definition 4.2 (The category of the signatures)**.** *The category $Sign^s$ has the objects the fiber-small quasi-representable signature morphisms χ of $Sign$. The arrows $\chi \to \chi'$ in $Sign^s$ are pairs of signature morphisms (φ, θ) such that*

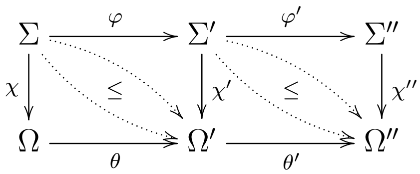
- *both φ and θ are total and Mod-strict; and*
- *$\chi; \theta \leq \varphi; \chi'$.*

(5)



The composition in $Sign^s$ *is defined as pairwise composition in* $Sign$*, i.e.* $(\varphi, \theta); (\varphi', \theta') = (\varphi; \varphi', \theta; \theta')$*, as shown in the following figure:*

(6)



An arrow $(\varphi, \theta) : \ \chi \to \chi'$ *is* strict *when* $\chi; \theta = \varphi; \chi'$.

We have the correctness of Definition 4.1:

Proposition 4.1. $Sign^s$ *is a category.*

Definition 4.3 (The sentence translation functor)**.** *For any* $Sign^s$ *signature* χ *we define* $Sen^s(\chi) = Sen(\Box\chi)$ *and for any* $Sign^s$*-morphism* (φ, θ) *we define* $Sen^s(\varphi, \theta) = Sen(\varphi)$*.*

Proposition 4.2. Sen^s *is a functor* $Sign^s \to \mathbf{Set}$*.*

Definition 4.4 (The model reduct functor)**.** *For any* $Sign^s$ *signature* χ *we define* $Mod^s(\chi) = Mod(\chi\Box)$ *and for any* $Sign^s$*-morphism* (φ, θ) *we define* $Mod^s(\varphi, \theta) = Mod(\theta)$*.*

Proposition 4.3. Mod^s *is a functor* $(Sign^s)^{\mathrm{op}} \to \mathbf{CAT}$*.*

Definition 4.5 (The stratification)**.** *For any* $Sign^s$ *signature* χ *we define*

- $[\![M]\!]_\chi = Mod(\chi)M$ *for any* $M \in |Mod^s(\chi)| (= |Mod(\chi\Box)|)$*, and*
- $[\![h]\!]_\chi N = h^N\Box$ *for any* $\chi\Box$*-model homomorphism* $h \in Mod^s(\chi) (= |Mod(\chi\Box)|)$ *and* $\Box\chi$*-model* $N \in Mod(\chi)(\Box h)$*.*

For each signature morphism $(\varphi, \theta) : \ \chi \to \chi'$ *in* $Sign^s$ *we define:*

- $[\![M']\!]_{(\varphi,\theta)} N' = Mod(\varphi)N'$ *for any* $M' \in |Mod^s(\chi')|$ *and any* $N' \in [\![M']\!]_{\chi'}$*.*

Proposition 4.4. $[\![\_]\!]$ *is a lax natural transformation* $Mod^s \Rightarrow SET$*.*

Definition 4.6 (The satisfaction relation)**.** *For each signature* χ *in* $Sign^s$*, each* $\chi\Box$*-model* M*, each* $\Box\chi$*-model* $N \in [\![M]\!]_\chi$ *and each* $\Box\chi$*-sentence* ρ*,*

$$M (\models^s)^N_\chi \rho \quad \text{if and only if} \quad N \models_{\Box\chi} \rho.$$

Proposition 4.5. *For any signature morphism* $(\varphi, \theta) : \ \chi \to \chi'$ *in* $Sign^s$*, any* χ'*-model* M'*, any* $N' \in [\![M']\!]_{\chi'}$*, and any* χ*-sentence* ρ*:*

$$M' \models^{N'}_{\chi'} Sen^s(\varphi, \theta)\rho \ \text{ if and only if } \ Mod^s(\varphi, \theta)M' \models_\chi \rho.$$

Corollary 4.1. $\mathcal{I}^{\mathrm{s}} = (Sign^s, Sen^s, Mod^s, [\![\_]\!], \models^s)$ *is a stratified institution.*

4.3. **Consequences of the representation.**

4.3.1. *Representing 3/2-institutions as ordinary institutions.* By resorting to the adjunction between institutions and stratified institutions of [10] we can further canonically represent 3/2-institutions as ordinary institutions.

Corollary 4.2. *Let $\mathcal{I} = (Sign, Sen, Mod, \models)$ be a 3/2-institution. Then*

$$(\mathcal{I}^{\mathrm{s}})^\sharp = (Sign^s, Sen^s, (Mod^s)^\sharp, \models^\sharp)$$

defines an ordinary institution where

- *$Sign^s$ and Sen^s are given by Definitions 4.2 and 4.3, respectively.*
- *For each $\chi \in |Sign^s|$:*
  - *a $(Mod^s)^\sharp$ χ-model is pair (M, N) such that $M \in |Mod(\chi\Box)|$, $N \in Mod(\chi)M$;*
  - *a χ-model homomorphism $(M, N) \to (M_0, N_0)$ is a model homomorphism $h : M \to M_0$ such that $N_0 = h^N\Box$.*
- *For each $(\varphi, \theta) : \chi \to \chi'$ and any $(Mod^s)^\sharp$ χ'-model (M', N')*

$$(Mod^s)^\sharp(\varphi, \theta)(M', N') = (Mod(\theta)M', Mod(\varphi)N').$$

- *For each $(Mod^s)^\sharp$ χ-model (M, N) and each Sen^s χ-sentence ρ*

$$(M, N) \models^\sharp_\chi \rho \text{ if and only if } N \models^{\mathcal{I}}_{\Box\chi} \rho.$$

4.3.2. *Semantic connectives.* Institution theory has developed its own general approach to logical connectives [20, 4, 5]. This has been refined in [7] to stratified institution theory. With 3/2-institutions there are two ways to approach this issue.

(1) The straightforward way that mimics the semantic treatment of connectives from ordinary institution theory.
(2) By using the stratified institution theoretic approach via the representation result given by Corollary 4.1.

In [11] it is argued that the straightforward approach does not work, which means that in order to have sound semantic connectives we have to rely on the representation result.

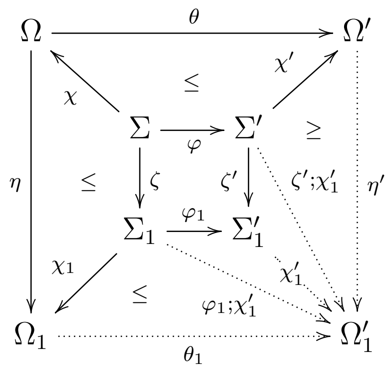
4.3.3. *Model amalgamation.* Model amalgamation is a crucial property in the institution theoretic computer science and model theory alike. It has been instrumental in the proposal for the institution theoretic treatment of conceptual blending put forward in [8]. The following is a general result that explains model amalgamation in 3/2-institutions as a model amalgamation property specific to stratified institutions.

Proposition 4.6. *Let $\mathcal{I} = (Sign, Sen, Mod, \models)$ be a stratified institution and let $\mathcal{I}^{\mathrm{s}} = (Sign^s, Sen^s, Mod^s, [\![\_]\!], \models^s)$ be its representation as a stratified institution. Let the left hand square below represent a commutative diagram in $Sign^s$ such that its projection on the first component (the right hand side square below) is a model amalgamation square in $\mathcal{I}$.*

(7)
$$\begin{array}{ccc} \chi & \xrightarrow{(\varphi,\theta)} & \chi' \\ {\scriptstyle(\zeta,\eta)}\downarrow & & \downarrow{\scriptstyle(\zeta',\eta')} \\ \chi_1 & \xrightarrow[(\varphi_1,\theta_1)]{} & \chi'_1 \end{array} \qquad\qquad \begin{array}{ccc} \Sigma & \xrightarrow{\varphi} & \Sigma' \\ {\scriptstyle\zeta}\downarrow & & \downarrow{\scriptstyle\zeta'} \\ \Sigma_1 & \xrightarrow[\varphi_1]{} & \Sigma'_1 \end{array}$$

Then the square of $\mathcal{I}^{\mathrm{s}}$-signature morphisms is a stratified model amalgamation square if and only if the following lax co-cone of $\mathcal{I}$-signature morphisms has the model amalgamation *property.*

(8)



Propositions 4.7 / 4.8 / 4.9 support the applicability of the general result given by Proposition 4.6.

Proposition 4.7. *Let us assume a 3/2-institution $\mathcal{I}$ such that when we remove its model homomorphisms it is generated by a 3/2-institutional seed* [8]. *Let us consider a lax co-cone of $\mathcal{I}$ signature morphisms like in Figure* (8) *with the following properties:*

- *the inner square (aka the right hand side square of Figure* (7)*) is a 3/2-pushout square;*
- *the outer square $(\eta, \theta, \eta', \theta_1)$ is a model amalgamation square; and*
- $(\varphi_1, \theta_1) : \ \chi_1 \to \chi'_1$ *and* $(\zeta', \eta') : \ \chi' \to \chi'_1$ *are strict.*

Then the lax co-cone of Figure (8) *has the model amalgamation property.*

In concrete situations it is quite common that the 3/2-pushout condition on the inner square in Proposition 4.7 implies the model amalgamation condition on the same square in Proposition 4.6, a fact that enhances the applicability of Proposition 4.7 within the context of the equivalence established by Proposition 4.6.

Proposition 4.8. *In the context of a 3/2-institution generated by a 3/2-institutional seed let us assume that Sen preserves and reflects maximality (i.e. φ is maximal if and only if $Sen(\varphi)$ is total). Then any pushout co-cone of signature morphisms determines a model amalgamation square.*

With respect to the conditions underlying Proposition 4.8 note that:

- There are no restrictions on the signature morphisms that form the pushout square.
- Then the condition of Proposition 4.8 that Sen preserves and reflects maximality applies well in concrete situations. As an example let us consider the case of $3/2PL$. There $Sign = \mathbf{Pfn}$ and therefore it is evident that for any signature morphism φ, $Sen(\varphi)$ is total if and only if φ is total.
- The 3/2-pushout condition of Proposition 4.7 in general is stronger than the pushout condition of Proposition 4.8.

Often, in concrete situations that are related to the basic context of Proposition 4.6, the pushout squares of signature morphisms are already 3/2-pushout squares. The following result illustrates such a case that is emblematic for the concrete applications not only because it is sometimes involved as such (e.g. in $3/2PL$) but also because when it is not the case then the respective category of signature morphisms can be often treated in a similar way.

Proposition 4.9. *Any pushout square in* $\mathbf{Set}$ *is a 3/2-pushout square in* $\mathbf{Pfn}$ *(i.e., the 3/2-category of sets and partial functions).*

5. ARTICLES 2022

The publications [11, 12, 9] have emerged from the research reported here. The article "Interpolation concepts in stratified institutions" that contains results obtained at stage 1 of the project has been submitted for publication in 2022. According to the latest version of the planned activities (following the buget reduction for the stage 2 of the project, in 2022) the results obtained at activity 2.1 (reported in Section 2) will constitute the subject of an article to be prepared and submitted for publication to a journal in 2023 at stage 3 of the project.

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