FINAL SCIENTIFIC REPORT FOR THE PROJECT PN-III-P4-ID-PCE-2020-0446
"AXIOMATIC METHODS IN NON-CLASSICAL MODEL THEORY"

1. THE OBJECTIVES OF THE PROJECT

In the funding application the objectives of this project were stated as follows:

O1: To advance significantly the model theory for non-classical generalisations of institution theory;
O2: To establish structural connections between several generalisations;
O3: To establish adequate logic-by-translation structures;
O4: To find new interpretations with applications.

All these objectives have been 100% accomplished. More information on this project is available at its web page.

2. THE RESULTS

The presentation of the results obtained during the implementation of the project follows the structure of the objectives, which in a few cases clashes with the order in which the concepts were introduced. Due to space constraints, we present only the main results and we skip the auxiliary ones. The results are of two kinds: 'new concepts' (consisting of new definitions representing the conceptualisation part of the axiomatisation process) and 'proved results' (consisting of new theorems, propositions, corollaries, etc.). The topics presented correspond to those specified in the work packages in the funding application. For each topic we give the reference of the publications where the full developments and more extensive explanations can be consulted.

2.1. Advance of the model theory for non-classical generalisations of institution theory (O1).

2.1.1. *Model amalgamation in stratified institutions* [12]. Model amalgamation is a fundamental property of institutions that pervades the development of many model-theoretic results. Our study of model amalgamation in the context of stratified institutions had two aims: to define a concept of model amalgamation specific to stratified institutions, and to develop a general result on the existence of this model amalgamation. The latter result is obtained through the decomposition technique introduced in [12], by aggregating model amalgamation properties from the components of the decomposition.

New concepts 2.1.1.1.

- *Stratified model amalgamation.* Consider a stratified institution $\mathcal{S}$ and a commutative square of signature morphisms like below:

(1)
$$\begin{array}{ccc} \Sigma & \xrightarrow{\varphi_1} & \Sigma_1 \\ {\scriptstyle \varphi_2}\downarrow & & \downarrow{\scriptstyle \theta_1} \\ \Sigma_2 & \xrightarrow[\theta_2]{} & \Sigma' \end{array}$$

  Then this square is a *stratified model amalgamation square* when for each Σ_k-model M_k and each $w_k \in \llbracket M_k \rrbracket_{\Sigma_k}$, $k = \overline{1,2}$ such that $\varphi_1(M_1) = \varphi_2(M_2)$ and $\llbracket M_1 \rrbracket_{\varphi_1} w_1 = \llbracket M_2 \rrbracket_{\varphi_2} w_2$ there exists an unique Σ'-model M' and an unique $w' \in \llbracket N' \rrbracket_{\Sigma'}$ such that $\theta_k(M') = M_k$ and $\llbracket M' \rrbracket_{\theta_k} w' = w_k$, $k = \overline{1,2}$. The model M' is called the *(stratified) amalgamation of* M_1 *and* M_2. When all pushout squares of signature morphisms are (stratified) model amalgamation squares we say that $\mathcal{S}$ is *(stratified) semi-exact*.
- *Model amalgamation for constraint models.* Let $\mathcal{B}$ be any institution. A constraint model sub-functor (see also Sec. 2.3.1) $Mod^C \subseteq Mod^{\widetilde{\mathcal{B}}}$ *preserves amalgamation* when for any signature morphisms $\theta_k : \Sigma' \to \Sigma_k$, $k = \overline{1,2}$ and any $\widetilde{\mathcal{B}}$ Σ'-model (W, B'), $\theta_k(W, B') \in |Mod^C(\Sigma_k)|$, $k = \overline{1,2}$, implies $(W, B') \in |Mod^C(\Sigma')|$.

Proved results 2.1.1.1.

- *Stratified model amalgamation as pullback.* A commutative square of signature morphisms like (1) is a stratified model amalgamation square if
  –
  $$\begin{array}{ccc} Mod(\Sigma) & \xleftarrow{Mod(\varphi_1)} & Mod(\Sigma_1) \\ {\scriptstyle Mod(\varphi_2)}\uparrow & & \uparrow{\scriptstyle Mod(\theta_1)} \\ Mod(\Sigma_2) & \xleftarrow[Mod(\theta_2)]{} & Mod(\Sigma') \end{array}$$
  is a pullback in $|\mathbf{CAT}|$, and
  – for each Σ'-model M'
  $$\begin{array}{ccc} [\![\varphi(\theta(M'))]\!]_{\Sigma} & \xleftarrow{[\![\theta_1(M')]\!]_{\varphi_1}} & [\![\theta_1(M')]\!]_{\Sigma_1} \\ {\scriptstyle [\![\theta_2(M')]\!]_{\varphi_2}}\uparrow & & \uparrow{\scriptstyle [\![M']\!]_{\theta_1}} \\ [\![\theta_2(M')]\!]_{\Sigma_2} & \xleftarrow[{[\![M']\!]_{\theta_2}}]{} & [\![M']\!]_{\Sigma'} \end{array}$$
  is a pullback in $\mathbf{Set}$.
- *Transfer of model amalgamation from $\mathcal{B}$ to $\widetilde{\mathcal{B}}$.* For $\widetilde{\mathcal{B}}$ refer to Sec. 2.3.1. Let $\mathcal{B}$ be any institution. Any model amalgamation square in $\mathcal{B}$ is a model amalgamation square in $\widetilde{\mathcal{B}}$ too.
- *Model amalgamation by decomposition.* Consider a decomposition of a stratified institution $\mathcal{S}$
  $$\mathcal{S}^0 \xleftarrow{(\Phi^0,\alpha^0,\beta^0)} \mathcal{S} \xrightarrow{(\Phi,\alpha,\widetilde{\beta})} \widetilde{\mathcal{B}}^C$$
  such that (1) $\mathcal{S}^0$ is strict, (2) Φ and Φ^0 preserve pushouts, (3) $\mathcal{B}$ and $\mathcal{S}^0$ are semi-exact, and (4) Mod^C preserves amalgamation. Then $\mathcal{S}$ is semi-exact too.

2.1.2. *Diagrams in stratified institutions* [12]. In conventional model theory, the method of diagrams is one of the most important methods. The institution-independent method of diagrams pervades the development of a lot of model-theoretic results at the level of abstract institutions, many of its applications being presented in [3]. Under this topic, (1) we have defined the concept of diagram in stratified institutions, (2) we have introduced some preliminary technical concepts supporting the development of the main result of this section, (3) we have formulated and proved a general result on the existence of diagrams in stratified institutions. This comes in two versions: for $\mathcal{S}^*$ (the institution of 'global' satisfaction associated to the stratified institution $\mathcal{S}$) and for $\mathcal{S}^\sharp$ (the institution of the 'local' satisfaction).

New concepts 2.1.2.1.

- *Diagrams in stratified institutions.* A stratified institution $\mathcal{S}$ admits diagrams if and only if its flattening $\mathcal{S}^\sharp$ admits diagrams in the sense of ordinary institution theory.
- *Coherence.* Let $\mathcal{B}$ be an institution and Mod^C be a constraint model sub-functor for $\widetilde{\mathcal{B}}$. A system of diagrams for $\mathcal{B}$ is *coherent with respect to* Mod^C when for each $\mathcal{B}$-signature Σ and each $(W,B) \in |Mod^C(\Sigma)|$ we have that
  (1) For all $i,j \in W$, $\iota_{\Sigma,B^i} = \iota_{\Sigma,B^j}$; in this case all ι_{Σ,B^i}s will be denoted by $\iota_{\Sigma,B} : \ \Sigma \to \Sigma_B$.
  (2) for each $(W,B) \in |Mod^C(\Sigma)$ there exists a canonical isomorphism $i_{\Sigma,(W,B)}$ such that the following diagram commutes:
  $$\begin{array}{ccc} Mod(\Sigma_B, E_{(W,B)}) & \xrightarrow[\cong]{i_{\Sigma,(W,B)}} & (W,B)/Mod^C(\Sigma) \\ & {\scriptstyle Mod^C(\iota_{\Sigma,B})}\searrow \quad \swarrow{\scriptstyle \text{forgetful}} & \\ & Mod^C(\Sigma) & \end{array}$$

where $Mod(\Sigma_B, E_{(W,B)})$ denotes the subcategory of the comma category $[\![M]\!]/[\![\_]\!]_{\Sigma_B}$ (where $[\![\_]\!]_{\Sigma_B}$ is viewed as a restricted functor $Mod^C(\Sigma_B) \to \mathbf{Set}$) induced by those objects $(f: W \to V, (V, N'))$ such that $N'^{f(i)} \models E_{B^i}$ for each $i \in W$.

- *Decompositions with diagrams.* A decomposition of a stratified institution $\mathcal{S}$ (like in Sec. 2.3.1) *admits diagrams* when $\mathcal{S}^{0*}$ and $\mathcal{B}$ have diagrams such that the diagrams of $\mathcal{B}$ are coherent with respect to Mod^C.
- *Notation.* We have introduced the following notational convention. For any decomposition of a stratified institution that admits diagrams for any $\Sigma \in |Sign^{\mathcal{S}}|$ and $M \in |Mod^{\mathcal{S}}(\Sigma)|$, we introduce the following abbreviations: $\Sigma_0 = \Phi^0\Sigma$, $\Sigma_1 = \Phi\Sigma$, $M_0 = \beta^0_\Sigma M$, $M_1 = \widetilde{\beta}_\Sigma M$. We let $\iota_{\Sigma_0,M_0}: \Sigma_0 \to (\Sigma_{0M_0}, E_{M_0})$ and (for each $i \in [\![M]\!]$) $\iota_{\Sigma_1,M_1^i}: \Sigma_1 \to (\Sigma_{1M_1^i}, E_{M_1^i})$ be the diagrams of M_0 and M_1^i, respectively. By the coherence hypothesis we have $\iota_{\Sigma_1,M_1^i} = \iota_{\Sigma_1,M_1^j}$ for all $i, j \in [\![M]\!]$. This allows us to denote all ι_{Σ_1,M_1^i} by ι_{Σ_1,M_1}. We define the $Sign^S$ morphism $\iota_{\Sigma,M}: \Sigma \to \Sigma_M$ by using the product property of (Φ^0, Φ): $\iota_{\Sigma,M} = (\iota_{\Sigma_0,M_0}, \iota_{\Sigma_1,M_1})$.
- *Diagrams denoting stratifications.* Consider a decomposition of a stratified institution that admits diagrams. We say that the diagrams (of the decomposition) *denote the stratification* when (1) $\mathcal{S}$ has a nominal extraction (N, Nm), (2) for each $\mathcal{S}$-signature Σ and each Σ-model M, there exists a function $n_{\Sigma,M}: [\![M]\!]_\Sigma \to N(\Sigma_M)$ such that n is natural in Σ and M, and (3) for each Σ_M-model N such that $N \models \alpha_0 E_{M_0}$, $n_{\Sigma,M}; Nm_{\Sigma_M}(N) = [\![i_{\Sigma_0,M_0} N_0]\!]_{\Sigma_0}$.

Proved results 2.1.2.1.

- *Diagrams of abstract stratified institutions.* For any decomposition of a stratified institution $\mathcal{S}$ that admits diagrams that denote the stratification:
  - $\mathcal{S}^*$ has diagrams when $\mathcal{S}$ has explicit local satisfaction, and
  - $\mathcal{S}^\sharp$ has diagrams when $\mathcal{S}$ has explicit local satisfaction and has i-sentences too.
- *Diagrams in some concrete stratified institutions.* This is a consequence of the corresponding general result. Let *HPL* / *HFOL* denote the stratified institution of hybrid propositional logic / hybrid first-order logic (with sharing of domains and interpretations of constants). Then *HPL**, *HPL*$^\sharp$, *HFOL** and *HFOL*$^\sharp$ have diagrams.

2.1.3. *Quasi-varieties in stratified institutions* [7]. The goal of our study of quasi-varieties of models in stratified institutions was twofold: (1) to establish products of models and sub-models at a general level in stratified institutions; for this we relied on the decomposition technique of [12], and (2) to establish initial semantics for classes of theories by reliance on a general abstract category-theoretic result of existence of initial objects in quasi-varieties; for this, we have developed a modular body of preservation results under products and sub-models.

Proved results 2.1.3.1.

- *Sub-models in $\widetilde{\mathcal{B}}$.* For each signature Σ we assume that I^Σ has small coproducts – denoted $\oplus$ – that are preserved by the inclusion functor $I_\Sigma \subseteq Mod^{\mathcal{B}}(\Sigma)$. Then $Mod^{\widetilde{\mathcal{B}}}(\Sigma)$ admits a canonical inclusion system $(\widetilde{I}_\Sigma, \widetilde{\mathcal{E}}_\Sigma)$ defined as follows:
  - $h: (W, B) \to (W', B')$ belongs to $\widetilde{I}_\Sigma$ when $h_0 = (W \subseteq W')$ is a set inclusion and for each $w \in W$, $(h^w: B^w \to B'^w) \in I_\Sigma$.
  - $h: (W, B) \to (W', B')$ belongs to $\widetilde{\mathcal{E}}_\Sigma$ when $h_0: W \to W'$ is surjective and for each $w' \in W'$
  $$B'^{w'} = \bigoplus_{h_0 w = w'} h^w(B^w).$$
- *Sub-models in $\widetilde{\mathcal{B}}^C$.* Given a constraint model functor Mod^C for $\widetilde{\mathcal{B}}$, for each $\mathcal{B}$-signature Σ we assume that
  (1) the inclusion system $(I_\Sigma, \mathcal{E}_\Sigma)$ *preserve the constraints*, i.e. if $(1_W, f) \in Mod^C(\Sigma)$ and each f^w, $w \in W$, gets factored as $f^w = e^w_f; i^w_f$ with $e^w_f \in \mathcal{E}_\Sigma$ and $i^w_f \in I_\Sigma$ then $(1_W, e_f), (1_W, i_f) \in Mod^C(\Sigma)$,
  (2) for each $L \in |Mod^C(\Sigma)|$, the *diagonal $\widetilde{\mathcal{B}}$ model* (W, L^W) where $(L^W)^w = L$ for each $w \in W$, belongs to $Mod^C(\Sigma)$,

(3) any constraint model $(W, B) \in |Mod^C(\Sigma)|$ admits a coproduct in I_Σ of its components, denoted $\bigoplus_{w \in W} B^w$; moreover this is also initial in the class of the co-cones given by the constraint model homomorphisms $(1_W, f) : (W, B) \to (W, L^W)$. As notation, the mediating homomorphism $\bigoplus_{w \in W} B^w \to L$ is denoted $\sum_W f$.

(4) For each constraint model $(W, B) \in |Mod^C(\Sigma)|$, any $L \in |Mod^{\mathcal{B}}(\Sigma)|$, any constraint model homomorphism $(1_W, f) : (W, B) \to (W, L^W)$, and any family of subsets $(K_j \subseteq W)_{j \in J}$, $(\sum_{K_j} f)_{j \in J}$ is a constraint model homomorphism.

Then $(I^C_\Sigma, \mathcal{E}^C_\Sigma)$ is an inclusion system of $Mod^C(\Sigma)$, where I^C_Σ and $\mathcal{E}^C_\Sigma$ are the restrictions of $\widetilde{I}_\Sigma$ and $\widetilde{\mathcal{E}}_\Sigma$, respectively, to $Mod^C(\Sigma)$.

- *Sub-models in stratified institutions with decomposition.* Consider a decomposition of a stratified institution $\mathcal{S}$

$$\mathcal{S}^0 \xleftarrow{(\Phi^0, \alpha^0, \beta^0)} \mathcal{S} \xrightarrow{(\Phi, \alpha, \widetilde{\beta})} \widetilde{\mathcal{B}}^C$$

such that for each $\mathcal{S}$-signature Σ, $Mod^{\mathcal{S}^0}(\Phi^0\Sigma)$ and $Mod^C(\Phi\Sigma)$ have inclusion systems $(I^0_{\Phi^0\Sigma}, \mathcal{E}^0_{\Phi^0\Sigma})$ and $(I^C_{\Phi\Sigma}, \mathcal{E}^C_{\Phi\Sigma})$, respectively, such that both $[\![\_]\!]^0$ and $[\![\_]\!]^{\widetilde{\mathcal{B}}^C}$ preserve the abstract inclusions and the abstract surjections. Then $(\overline{I}_\Sigma, \overline{\mathcal{E}}_\Sigma)$ is an inclusion system for $Mod^{\mathcal{S}}(\Sigma)$ where $h \in \overline{I}_\Sigma / \overline{\mathcal{E}}_\Sigma$ if and only if $\beta^0_\Sigma h \in I^0_{\Phi^0\Sigma} / \mathcal{E}^0_{\Phi^0\Sigma}$ and $\widetilde{\beta}_\Sigma h \in I^C_{\Phi\Sigma} / \mathcal{E}^C_{\Phi\Sigma}$.
- *Model products in $\widetilde{\mathcal{B}}^C$.* Let $\mathcal{B}$ be an institution such that for each signature $\Sigma \in |Sign^{\mathcal{B}}|$ its category of models $Mod^{\mathcal{B}}(\Sigma)$ has small products. Then $Mod^{\widetilde{\mathcal{B}}}(\Sigma)$ has small products too.
- *Model products by decomposition.* Consider a decomposition of a stratified institution $\mathcal{S}$ like above such that for some $\mathcal{S}$-signature Σ the following conditions hold: (1) $Mod^{\mathcal{B}}$ has small products; (2) $Mod^C(\Phi\Sigma)$ has small products that are preserved by the sub-category inclusion $Mod^C(\Phi\Sigma) \to Mod^{\widetilde{\mathcal{B}}}(\Phi\Sigma)$; (3) $[\![\_]\!]_{\Phi^0\Sigma}$ creates small products. Then $\mathcal{S}$ has products of models that are preserved by the stratification.

2.1.4. *Preservation results in stratified institutions* [7]. The goal of this topic was to extend the preservation theory for stratified institutions developed in [5] from filtered products to quasi-varieties.

New concepts 2.1.4.1.

- *Preservation concepts.* In any stratified institution $\mathcal{S}$, a Σ-sentence ρ is
  - *preserved by model products* when for each family $(M_j)_{j \in J}$ of Σ-models, for any product $(p_j : M \to M_j)_{j \in J}$ and each $w \in [\![M]\!]$ if for each $j \in J$, $M_j \models^{[\![p_j]\!]w} \rho$ then $M \models^w \rho$.
  - if $Mod(\Sigma)$ has an inclusion system $(I_\Sigma, \mathcal{E}_\Sigma)$, ρ is *preserved by sub-models* when for each $(h : M \to N) \in I_\Sigma$, $N \models^{[\![h]\!]w} \rho$ implies $M \models^w \rho$.
  - ρ is *preserved along a Σ-homomorphism* $h : M \to N$ when for each $s \in [\![M]\!]$, $M \models^s \rho$ implies $N \models^{[\![h]\!]s} \rho$.
- *Lifting products.* We say that a signature morphism $\varphi : \Sigma \to \Sigma'$ in a stratified institution $\mathcal{S}$ *lifts forwards / backwards products* when for each product of Σ-models $(p_i : N \to M_i)_{i \in I}$ and each φ-expansion N' / $(M'_i)_{i \in I}$ of N / $(M_i)_{i \in I}$ there exists a φ-expansion $(p'_i : N' \to M'_I)_{i \in I}$ of $(p_I)_{i \in I}$ which is a product.
- *Lifting homomorphisms.* For any signature morphism $\varphi : \Sigma \to \Sigma'$ in a stratified institution $\mathcal{S}$ and for any Σ-homomorphism $h : M \to N$ we say φ *lifts forwards / backwards* h when for each φ-expansion M' / N' of M / N there exists a φ-expansion $h' : M' \to N'$ of h. Moreover, φ *lifts forwards / backwards inclusions* when h' is inclusion if h is inclusion.

Proved results 2.1.4.1.

- *Preservation by decomposition.* Consider a decomposition of a stratified institution $\mathcal{S}$

$$\mathcal{S}^0 \xleftarrow{(\Phi^0, \alpha^0, \beta^0)} \mathcal{S} \xrightarrow{(\Phi, \alpha, \widetilde{\beta})} \widetilde{\mathcal{B}}^C$$

under the conditions of the existence of quasi-varieties. Let Σ be an $\mathcal{S}$-signature. Then

(1) Let $\rho \in Sen^{\mathcal{B}}(\Phi\Sigma)$. If ρ is preserved by products / sub-models / homomorphisms in $\mathcal{B}$ then $\alpha_\Sigma \rho$ is preserved by products / sub-models / homomorphisms in $\mathcal{S}$ too.

(2) Let $\rho^0 \in Sen^0(\Phi^0\Sigma)$. If ρ^0 is preserved by products / sub-models / homomorphisms in $\mathcal{S}^0$ then $\alpha^0\rho^0$ is preserved by products / sub-models / homomorphisms in $\mathcal{S}$ too.

- *Invariance with respect to propositional connectives.* In any stratified institution $\mathcal{S}$
  - the preservation by products, sub-models and homomorphisms is invariant with respect to conjunction;
  - the preservation by sub-models and homomorphisms is invariant with respect to disjunctions.
  - for any Σ-sentences, ρ_1, ρ_2 if ρ_1 is preserved along inclusions / products projections and ρ_2 is preserved by sub-models / products then $\rho_1 \Rightarrow \rho_2$ is preserved by sub-models / products.
- *Invariance of preservation by quasi-varieties under quantifications.* Let $\mathcal{S}$ be a stratified institution, $\chi : \Sigma \to \Sigma'$ be a signature morphism and ρ be a Σ'-sentence.
  (1) If χ lifts forwards inclusions / products / homomorphisms and ρ is preserved by sub-models / products / homomorphisms then $(\forall\chi)\rho$ / $(\forall\chi)\rho$ / $(\exists\chi)\rho$ is preserved by sub-models / products / homomorphisms.
  (2) If χ lifts backwards inclusions / products / homomorphisms and ρ is preserved by sub-models / products / homomorphisms then $(\exists\chi)\rho$ / $(\exists\chi)\rho$ / $(\forall\chi)\rho$ is preserved by sub-models / products / homomorphisms.
- *Invariance of preservation by quasi-varieties under implicit nominals.* Let $\mathcal{S}$ be a stratified institution endowed with a nominal extraction (N, Nm). Let ρ be any Σ-sentence.
  (1) [5] If each Nm_Σ preserves products then each i-sentence is preserved by products.
  (2) [5] If each Nm_Σ preserves products and ρ is preserved by products then for each $i \in N(\Sigma)$, $@_i$ is preserved by products too.
  (3) If each $[\![\_]\!]_\Sigma$ preserves inclusions then each i-sentence is preserved by sub-models.
  (4) If each $[\![\_]\!]_\Sigma$ preserves inclusions and ρ is preserved by sub-models then for each $i \in N(\Sigma)$, $@_i\rho$ is preserved by sub-models too.
  (5) Each i-sentence is preserved along homomorphisms.
  (6) If ρ is preserved along homomorphisms then for each $i \in N(\Sigma)$, $@_i$ is preserved by homomorphisms too.
- *Invariance of preservation by quasi-varieties under implicit modalities.* Let $\mathcal{S}$ be a stratified institution endowed with frame extraction (L, Fr). Then for any $\lambda \in L(\Sigma)_{n+1}$ and $\rho_1, \dots, \rho_n \in Sen(\Sigma)$ we have that:
  (1) [5] If Fr_Σ preserve products and ρ_k, $k = \overline{1,n}$, are preserved by products then $\langle\lambda\rangle(\rho_1, \dots, \rho_n)$ is preserved by products too.
  (2) If ρ_k, $k = \overline{1,n}$, are preserved along homomorphisms then $\langle\lambda\rangle(\rho_1, \dots, \rho_n)$ is preserved along homomorphisms too.
  (3) If ρ_k, $k = \overline{1,n}$ is preserved by sub-models then $[\lambda](\rho_1, \dots, \rho_n)$ is preserved by sub-models too.
  (4) If Fr_Σ preserve products, $\lambda \in L(\Sigma)_2$ and ρ is preserved by products then $[\lambda]\rho$ is preserved by products too.

2.1.5. *Initial semantics in stratified institutions* [11]. The results under this topic are aimed to support the applicability of the general categorical causality between quasi-varieties and initial semantics to stratified institutions.

Proved results 2.1.5.1.

- *Initial semantics by decomposition.* Consider any $\mathcal{S}$ signature Σ and assume that both categories $Mod^0(\Phi^0\Sigma)$ and $Mod^C(\Phi\Sigma)$ have initial objects that share their stratification. Then $Mod^{\mathcal{S}}(\Sigma)$ has initial objects.
- *Properties of inclusion systems by decomposition.* Consider any $\mathcal{S}$ signature Σ and assume that $Mod^0(\Phi^0\Sigma)$ and $Mod^{\mathcal{B}}(\Phi\Sigma)$ admit inclusion systems $(I^0_{\Phi^0\Sigma}, \mathcal{E}^0_{\Phi^0\Sigma})$ and $(I_{\Phi\Sigma}, \mathcal{E}_{\Phi\Sigma})$, respectively, satisfying the conditions of existence of inclusion systems in $\widetilde{\mathcal{B}}^C$ Then the inclusion system $(\overline{I}_\Sigma, \overline{\mathcal{E}}_\Sigma)$ of $Mod^{\mathcal{S}}(\Sigma)$ is epic / co-well-powered when both inclusion systems $(I^0_{\Phi^0\Sigma}, \mathcal{E}^0_{\Phi^0\Sigma})$ and $(I_{\Phi\Sigma}, \mathcal{E}_{\Phi\Sigma})$ are epic / co-well-powered.

2.1.6. *Interpolation in stratified institutions* [11]. Stratified institutions admit two different semantic consequence relations that correspond to the two possible flattenings of the stratified institutions to ordinary institutions. The aim of our study on interpolation was to clarify the two concepts of interpolation emerging from the two concepts of semantic consequence and then to establish a causality relationship between them.

New concepts 2.1.6.1.

- *Craig interpolation in stratified institutions.* Consider a stratified institution $\mathcal{S}$ and a commutative square of signature morphisms like below:

(2)
$$\begin{array}{ccc} \Sigma & \xrightarrow{\varphi_1} & \Sigma_1 \\ {\scriptstyle \varphi_2}\downarrow & & \downarrow{\scriptstyle \theta_1} \\ \Sigma_2 & \xrightarrow[\theta_2]{} & \Sigma' \end{array}$$

  Let $\models$ be any of the two semantic consequence relations, $\models^\sharp$ or $\models^*$. Let E_k be sets of Σ_k-sentences, $k = \overline{1,2}$. If there exists a set E of Σ-sentences such that
$$E_1 \models \varphi_1 E \ \text{ and } \ \varphi_2 E \models E_2$$
  then $\theta_1 E_1 \models \theta_2 E_2$. The commutative square of signature morphisms (2) is a *local / global Craig interpolation square* when the reversal of the above implication holds.
- $(\mathcal{L}, \mathcal{R})$*-interpolation.* Given classes of signature morphisms $\mathcal{L}$ and $\mathcal{R}$, we say that the stratified institution $\mathcal{S}$ has *local / global Craig* $(\mathcal{L}, \mathcal{R})$*-interpolation* when each pushout square (2) with $\varphi_1 \in \mathcal{L}$ and $\varphi_2 \in \mathcal{R}$ is a local / global Craig interpolation square.
- *Signature extensions with nominals.* Any stratified institution has *signature extensions with nominals* when it has nominals extraction (N, Nm) and for each signature Σ there exists a signature morphism $\iota : \ \Sigma \to \Sigma'$, called the signature extension of Σ with the nominal i, such that:
  (1) $N(\iota) : \ N(\Sigma) \to N(\Sigma') = N(\Sigma) \cup \{i\}$ is the extension of $N(\Sigma)$ with one element i,
  (2) for each Σ-model M and each $w \in [\![M]\!]_\Sigma$ there exists a ι-expansion M' of M such that $[\![M']\!]_\iota (Nm_{\Sigma'} M')_i = w$, and
  (3) for each signature morphism $\theta_1 : \ \Sigma_1 \to \Sigma'$ and each signature extension $\iota' : \ \Sigma' \to \Sigma''$ with one nominal i' there exists a signature extension $\iota_1 : \ \Sigma_1 \to \Sigma'_1$ with one nominal i such that
$$\begin{array}{ccc} \Sigma_1 & \xrightarrow{\iota_1} & \Sigma'_1 \\ {\scriptstyle \theta_1}\downarrow & & \downarrow{\scriptstyle \theta'_1} \\ \Sigma' & \xrightarrow[\iota']{} & \Sigma'' \end{array}$$
  is a stratified model amalgamation square.

Proved results 2.1.6.1.

- *Relationship between local and global interpolation in stratified institutions.* Let $\mathcal{S}$ be any stratified institution with signature extensions with nominals, with universal quantifications over those extensions, and with local explicit satisfaction. Then any local Craig interpolation square is a global Craig interpolation square too.

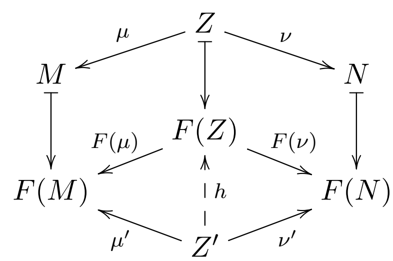
2.1.7. *Bisimulation in stratified institutions* [16]. The aim of this part of the study was, on the one hand, to set the foundations of a theory of abstract bisimulations at the general level of arbitrary stratified institutions, and on the other hand, to examine the relationship between abstract bisimilarity and elementary equivalence. As with other parts of our study, the latter effort has implied identifying sufficient conditions that stratified institutions should meet (and which hold in concrete cases such as *MPL*, *HPL*, *HHPL*, etc.) in order to ensure that the bisimilarity and the elementary-equivalence relations coincide.

New concepts 2.1.7.1.

- *Abstract bounded homomorphisms.* A sub-functor $BH \subseteq Mod^{\mathcal{S}}$ is a selection of abstract bounded homomorphisms for a stratified institution $\mathcal{S}$ when $BH(\Sigma)$ is a broad subcategory of $Mod^{\mathcal{S}}(\Sigma)$ for any $\mathcal{S}$ signature Σ.
- *Bisimulation representations* Given a stratified institution $\mathcal{S}$ with bounded homomorphisms, a *bisimulation* (*representation*) between two Σ-models M and N consists in a span $\langle \mu, Z, \nu \rangle$ of bounded homomorphisms.
$$M \xleftarrow{\mu} Z \xrightarrow{\nu} N$$

Two stratified models M and N are *bisimilar* if there exists a bisimulation between them. We say that a bisimulation $\langle \mu, Z, \nu \rangle$ witnesses the bisimilarity of two states $u \in [\![M]\!]$ and $v \in [\![N]\!]$, or that the pointed models $\langle M, u \rangle$ and $\langle N, v \rangle$ are *bisimilar*, if there exists a state $z \in [\![Z]\!]$ such that $[\![\mu]\!]z = u$ and $[\![\nu]\!]z = v$.

- *Hennessy-Milner property.* A stratified institution $\mathcal{S}$ has the *Hennessy-Milner* (*HM*) *property for a signature* Σ when the bisimilarity and the elementary-equivalence relation for Σ are indistinguishable. That is, for any two Σ-models M and N, and for any states $u \in [\![M]\!]$ and $v \in [\![N]\!]$, we have:
  [*bisimilarity*] the pointed models $\langle M, u \rangle$ and $\langle N, v \rangle$ are bisimilar
  if and only if
  [*elementary equivalence*] $M \models^u \rho$ if and only if $N \models^v \rho$ for all Σ-sentences ρ.
  An institution $\mathcal{S}$ has the *HM property*, or is an *HM institution*, if the above condition holds for all signatures.
- *Cover.* Let $F \colon \mathbb{C} \to \mathbb{C}'$ be a functor. We say that a span $\langle \mu, Z, \nu \rangle$ in $\mathbb{C}$ between objects M and N is an F*-cover* of a span $\langle \mu', Z', \nu' \rangle$ in $\mathbb{C}'$ between $F(M)$ and $F(N)$, or that F *covers* $\langle \mu', Z', \nu' \rangle$, when $F(\mu, Z, \nu) \geq \langle \mu', Z', \nu' \rangle$, meaning that there exists $h \colon Z' \to F(Z)$ such that the following diagram commutes.



  An F-cover $\langle \mu, Z, \nu \rangle$ is *strong* when the inequality $F(\mu, Z, \nu) \geq \langle \mu', Z', \nu' \rangle$ is evidenced by an isomorphism $h \colon Z' \to F(Z)$ as in the commutative diagram above.
- *Frame-bounded homomorphism.* Let $\mathcal{S}$ be a stratified institution equipped with a subfunctor $BE \subseteq Sen$ and a binary frame extraction (L, Fr). A Σ-homomorphism $h \colon M \to N$ is *frame-bounded* when, for all $w \in [\![M]\!]$:
  - $M \models^w \rho$ if and only if $N \models^{[\![h]\!]w} \rho$ for every sentence $\rho \in BE(\Sigma)$; and
  - $Fr(N)_\lambda([\![h]\!]w) \subseteq [\![h]\!](Fr(M)_\lambda w)$ for every modality $\lambda \in L(\Sigma)$,

  where $Fr(M)_\lambda w = \{s \in [\![M]\!] \mid (w, s) \in Fr(M)_\lambda\}$ is the set of λ-successors of w in M and likewise $Fr(N)_\lambda([\![h]\!]w) = \{t \in [\![N]\!] \mid ([\![h]\!]w, t) \in Fr(N)_\lambda\}$.
- *Zig-zag relation.* Let M and N be two Σ-models in a stratified institution equipped with a binary frame extraction (L, Fr). A relation $R \subseteq [\![M]\!] \times [\![N]\!]$ is *zig-zag* when, for every $(u, v) \in R$ and $\lambda \in L(\Sigma)$, it satisfies:
  [*zig*] for every $s \in Fr(M)_\lambda u$ there exists $t \in Fr(N)_\lambda v$ such that $(s, t) \in R$;
  [*zag*] for every $t \in Fr(N)_\lambda v$ there exists $s \in Fr(M)_\lambda u$ such that $(s, t) \in R$.
- *Modally saturated model.* Let Σ be signature in a stratified institution equipped with a binary frame extraction (L, Fr). A set Γ of Σ-sentences is (infinitely) *satisfiable* in a set $S \subseteq [\![M]\!]$ of states of a Σ-model M when there exists $w \in S$ such that $M \models^w \Gamma$; and it is κ*-satisfiable* in S, for some cardinal number κ, when every subset of Γ of cardinality less than κ is satisfiable in S.
  A Σ-model M is *modally* κ*-saturated* when, for every $\lambda \in L(\Sigma)$ and $w \in [\![M]\!]$, every set $\Gamma \subseteq Sen(\Sigma)$ that is κ-satisfiable in $Fr(M)_\lambda w$ is satisfiable in $Fr(M)_\lambda w$. M is *finitely-saturated* when it is κ-saturated for every infinite cardinal κ.

Proved results 2.1.7.1.

- *Characterisation of the HM property for arbitrary stratified institutions.* A stratified institution has the HM property for a signature Σ when the following conditions hold:
  - all bounded Σ-homomorphism are elementary; and
  - $(BH(\Sigma) \subseteq Mod(\Sigma))$; $[\![\_]\!]$ covers the elementary-equivalence relation $\equiv_{M,N}$ for all Σ-models M and N.

  Moreover, when the class of bisimulations between any two Σ-models is κ-directed for some infinite cardinal number κ that is larger than the cardinal of the state space of any Σ-model, the two conditions listed above are not only sufficient, but also necessary.

- *Preservation of the HM property along signature morphisms.* In strong stratified institutions, the HM property is preserved along all signature morphisms $\varphi\colon \Sigma \to \Sigma'$ such that:
  - bounded φ-expansions of elementary Σ-homomorphism are elementary; and
  - the model-reduct functor $BH(\varphi)$ covers spans.
- *An HM theorem for stratified institutions with frame-bounded homomorphisms.* Any stratified institution with frame-bounded homomorphisms has the HM property for any signature Σ such that:
  - all frame-bounded Σ-homomorphisms are elementary; and
  - for all Σ-models M and N, the elementary-equivalence relation $\equiv_{M,N}$ is zig-zag and the composite functor $(EH(\Sigma) \subseteq Mod(\Sigma))\, ; Fr$ strongly covers $\langle \pi_M, \lceil \equiv_{M,N} \rceil, \pi_N \rangle$.
- *HM in the context of modally saturated models.* Let $\mathcal{S}$ be a stratified institution with frame-bounded homomorphisms that has maximally consistent state theories, semantic κ-conjunctions, and possibilities. Then $\kappa\text{-}sat(\mathcal{S})$, i.e., the institution of κ-saturated models of $\mathcal{S}$, has the HM property for every signature Σ such that:
  - all frame-bounded Σ-homomorphisms are elementary; and
  - $(EH(\Sigma) \subseteq Mod(\Sigma))\, ; Fr$ strongly covers $\langle \pi_M, \lceil \equiv_{M,N} \rceil, \pi_N \rangle$ for all κ-saturated Σ-models M and N.

2.1.8. *Preservation by filtered products and ultraproducts in $\mathcal{L}$-institutions* [15]. The aim of the research under this topic was to extend the institution-independent method of ultraproducts of [2] to many-valued truth.

New concepts 2.1.8.1.

- *Basic sentences.* Let $\kappa \in L$. A set E of Σ-sentences is *κ-basic* if and only if there exists a Σ-model $M_{E,\kappa}$ such that for each Σ-model M, $(M \models E) \geqslant \kappa$ if and only if there exists a homomorphism $h: M_{E,\kappa} \to M$. When $M_{E,\kappa}$ is ($\mathcal{F}$-)finitely presented in the category of the Σ-models then we say that E is *($\mathcal{F}$-)finitary κ-basic*. Moreover, E is *(($\mathcal{F}$-)finitary) basic* when it is (($\mathcal{F}$-)finitary) κ-basic for each $\kappa \in L$. When $E = \{e\}$ is a singleton set, we say that e is a *($\mathcal{F}$-)finitary (κ-)basic sentence*. Basic sentences have been established in *MVL*$_0$, *MVL*$_1$, *HMVL*, TL_0.
- *Preservation by F-products / factors.* In any $\mathcal{L}$-institution let Σ be any signature and e be any Σ-sentence. Also let $\mathcal{F}$ be any class of filters and κ be any value in $\mathcal{L}$. Then
  - e is *κ-preserved by $\mathcal{F}$-products* when for each F-product $(\mu_J : M_J \to M_F)_{J\in F}$ (where F is a filter over I)
  $$\{i \in I \mid (M_i \models e) \geqslant \kappa\} \in F \ \text{ implies } \ (M_F \models e) \geqslant \kappa$$
  - e is *κ-preserved by $\mathcal{F}$-factors* when for each F-product as above we have the opposite implication than above.

  When $\mathcal{F}$ is the class of all ultrafilters we rather say directly "κ-preserved by ultraproducts / ultrafactors". When $\mathcal{F}$ is the class of all singleton filters we rather say "κ-preserved by direct products / factors".
- *Preservation / invention of F-products by signature morphisms.* Let $\mathcal{F}$ be a class of filters closed under reductions. A signature morphism $\chi : \Sigma \to \Sigma'$ *preserves / invents $\mathcal{F}$-products* when $Mod(\chi)$ preserves / invents $\mathcal{F}$-products.

Proved results 2.1.8.1.

- *Existentially quantified basic sentences are still basic.* In *MVL*$_1$, for any completely join-prime κ, if e is a (finitary) κ-basic $(F + X, P)$-sentence then $\exists X \cdot e$ is a (finitary) κ-basic (F, P)-sentence.
- *Preservation of basic sentences by filtered products / factors.* In any $\mathcal{L}$-institution, each κ-basic sentence
  (1) is κ-preserved by all filtered products;
  (2) is κ-preserved by F-factors when it is F-finitary.
- *Invariance of preservation under propositional connectives.*
  (1) The set of the sentences that are κ-preserved by F-products / F-factors is closed under conjunctions.
  (2) The set of the sentences that are preserved by F-factors are closed under $\wedge$, $\vee$, $*$.
  (3) If ρ is preserved by F-factors and ρ' is preserved by F-products then $\rho \Rightarrow \rho'$ is preserved by F-products.
- *Invariance of preservation under quantifications.* Let $\mathcal{F}$ be a class of filters that is closed under reductions and let $\chi : \Sigma \to \Sigma'$ be a signature morphism Let ρ be a Σ'-sentence.
  (1) If χ invents $\mathcal{F}$-products and ρ is κ-preserved by $\mathcal{F}$-products then $\forall\chi \cdot \rho$ is κ-preserved by $\mathcal{F}$-products too.
  (2) If χ invents $\mathcal{F}$-products and ρ is κ-preserved by $\mathcal{F}$-factors and κ is completely prime-join then $\exists\chi \cdot \rho$ is κ-preserved by $\mathcal{F}$-factors too.

(3) If χ preserves $\mathcal{F}$-products and ρ is κ-preserved by $\mathcal{F}$-products and κ is completely join-prime then $\exists\chi \cdot \rho$ is κ-preserved by $\mathcal{F}$-products too.

Let F be a filter over a set I such that F is closed under arbitrary intersections.

(4) If $\chi : \Sigma \to \Sigma'$ lifts F-products and ρ is κ-preserved by F-factors then $\forall\chi \cdot \rho$ is κ-preserved by F-factors too.
(5) If χ lifts direct products and ρ is κ-preserved by direct products then $\forall\chi \cdot \rho$ is κ-preserved by direct products too.

- *Concrete preservation results.* The general results above were applied to obtain concrete preservation by $\mathcal{F}$-products results in *MVL*$_0$, *MVL*$_1$, for general Horn sentences, and in *FMA*.

2.1.9. *Semantic compactness consequences in $\mathcal{L}$-institutions* [15]. Preservation by ultraproducts has important consequences for compactness, which is one of the most important model theoretic properties. In this part of the study, we have developed those preservation results and consequences for $\mathcal{L}$-institutions.

New concepts 2.1.9.1.

- *Consistency, compactness.* In any $\mathcal{L}$-institution:
  - A Σ-theory T is consistent when there exists a Σ-model M such that $T \leqslant M^*$.
  - For any truth value κ, a set E of Σ-sentences is *κ-consistent* when $T_\kappa|E$ is consistent.
  - The $\mathcal{L}$-institution is *m-compact* when for each Σ-theory T if $T|\Gamma$ is consistent for each finite $\Gamma \subseteq Sen(\Sigma)$ then T is consistent too.
  - The $\mathcal{L}$-institution is *κ-m-compact* when each set E of Σ-sentences is κ-consistent if E_0 is κ-consistent for each finite $E_0 \subseteq E$.

Proved results 2.1.9.1.

- *A fundamental ultraproducts theorem.* Consider any $\mathcal{L}$-institution that has ultraproducts of models and such that its sentences are preserved by ultraproducts. Let Σ be any of its signatures and let $I = \{i \subseteq Sen(\Sigma) \mid i \text{ finite}\}$. For any theory T and any family of Σ-models $(M_i)_{i\in I}$ such that for each $i \in I$, $T|i \leqslant M_i^*$ there exists an ultrafilter U on I such that $T \leqslant M_U^*$.
- *m-compactness by ultraproducts.* In any $\mathcal{L}$-institution with ultraproducts of models
  (1) If each sentence is preserved by ultraproducts then it is m-compact.
  (2) If each sentence is κ-preserved by ultraproducts then the $\mathcal{L}$-institution is κ-m-compact.
- *Concrete consequence of the general compactness results.* The general preservation results together with the general compactness results were applied to obtain concrete compactness results for general Horn sentences, *FMA*, *MVL*$_1$, etc.

2.1.10. *Quasi-varieties and initial semantics in $\mathcal{L}$- institutions* [15]. Initial semantics in $\mathcal{L}$-institutions can be obtained by preservation by direct products and by sub-models. While the former is a consequence of preservation by filtered products (as direct products are a special case of filtered products), the latter needed a specific treatment.

New concepts 2.1.10.1.

- *Epic basic sentences.* Let Σ be a signature in an $\mathcal{L}$-institution such that $Mod(\Sigma)$ admits an initial model and has a designated inclusion system. Then a (κ-)basic set of sentences E is *epic (κ-)basic* when $M_{E,\kappa}$ is reachable.
- *Preservation by sub-models.* A Σ-sentence ρ is *κ-preserved by sub-models* when for each $(N \subseteq M) \in I_\Sigma$ if $(M \models \rho) \geqslant \kappa$ then $(N \models \rho) \geqslant \kappa$. Moreover, ρ is *preserved by sub-models* when it is κ-preserved by sub-models for each $\kappa \in L$.
- *Inventing sub-models.* A signature morphism $\chi : \Sigma \to \Sigma'$ *invents sub-models* when for each $(M \subseteq N) \in I_\Sigma$ and for each χ-expansion M' of M there exists a χ-expansion $(M' \subseteq N') \in I_{\Sigma'}$ of $M \subseteq N$.
- *Quasi-varieties in $\mathcal{L}$-institutions.* In any $\mathcal{L}$-institution, for any signature Σ, a class $\mathcal{C}$ of Σ-models is called a *quasi-variety* if and only if it is closed under small products and sub-models.
- *Strong Horn sentences.* In any $\mathcal{L}$-institution with a designated inclusion system for each of its categories of models, a Horn sentence $\forall\chi \cdot H \Rightarrow e$ is *strong* when
  - χ invents direct products and sub-models,

– H is formed from epic basic sentences by iterations of connectives from the set $\{\wedge, \vee, *\}$, and
– e is an epic basic sentence.

A Σ-*theory* T is a *strong Horn theory* when $T\rho = \bot$ for each sentence that is *not* a strong Horn sentence.

Proved results 2.1.10.1.

- *Preservation by sub-models.*
  (1) The set of the sentences that are preserved by sub-models contain the epic basic sentences and is closed under $\wedge, \vee, \Rightarrow, *, \neg$.
  (2) Let $\chi: \Sigma \to \Sigma'$ be a signature morphism that invents sub-models. Then for any Σ'-sentence ρ', $\forall\chi \cdot \rho'$ is κ-preserved by sub-models if ρ' is κ-preserved by sub-models.
- *Models of strong Horn theories.* For any strong Horn sentence ρ and any truth value κ, the class of the models M such that $(M \models \rho) \geqslant \kappa$ is a quasi-variety. Consequently, the models of a strong Horn theory form a quasi-variety.
- *Initial semantics for Horn theories.* If the inclusion system is epic and co-well-powered then any strong Horn theory admits a reachable initial model.

2.1.11. *Interpolation for the graded consequence relation* [13]. The goal of the research under this topic was to extend to many-valued truth the well-known institution-independent concept of interpolation and its fundamental connection to Robinson consistency.

New concepts 2.1.11.1.

- *Graded interpolation.*In any $\mathcal{L}$-entailment system, given a commutative square of signature morphisms

(3)
$$\begin{array}{ccc} \Sigma & \xrightarrow{\varphi_1} & \Sigma_1 \\ {\scriptstyle \varphi_2}\downarrow & & \downarrow{\scriptstyle \theta_1} \\ \Sigma_2 & \xrightarrow[\theta_2]{} & \Sigma' \end{array}$$

and finite sets $E_1 \subseteq Sen(\Sigma_1)$ and $E_2, \Gamma_2 \subseteq Sen(\Sigma_2)$ we say that a finite set $E \subseteq Sen(\Sigma)$ is a *Craig-Robinson interpolant of* E_1*,* E_2 *and* Γ_2 when

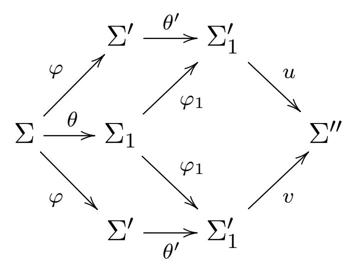
(4)
$$\theta_1 E_1 \cup \theta_2 \Gamma_2 \vdash \theta_2 E_2 \ \leq\ (E_1 \vdash \varphi_1 E) * (\varphi_2 E \cup \Gamma_2 \vdash E_2).$$

When Γ_2 is empty then E is called a *Craig interpolant* (of E_1 and E_2). When interpolants exist for all E_1, E_2 (, Γ_2) the respective commutative square of signature morphisms is called a *Craig(-Robinson) interpolation square* (abbr. C(R)i square). When $\mathcal{L}$ is a residuated lattice, the concepts introduced in this definition extend also to $\mathcal{L}$-institutions by considering its semantic entailment system.
- $\langle \mathfrak{L}, \mathfrak{R}\rangle$*-interpolation.* An $\mathcal{L}$-entailment system (or $\mathcal{L}$-institution) has $\langle \mathfrak{L}, \mathfrak{R}\rangle$-CRi / Ci for $\mathfrak{L}, \mathfrak{R} \subseteq Sign$ classes of signature morphisms, when each pushout square of signature morphisms like (3) with $\varphi_1 \in \mathfrak{L}$ and $\varphi_2 \in \mathfrak{R}$ is a CRi / Ci square.
- *Strong graded interpolation.* In any $\mathcal{L}$-entailment system, for any $\kappa, \kappa_1, \kappa_2 \in L \setminus \{0\}$ such that $\kappa \leq \kappa_1 * \kappa_2$, a commutative square of signature morphisms like (3) is a $(\kappa, \kappa_1, \kappa_2)$*-Ci square* when for any finite sets $E_1 \subseteq Sen(\Sigma_1)$ and $E_2 \subseteq Sen(\Sigma_2)$ such that $\theta_1 E_1 \vdash \theta_2 E_2 \ \geq\ \kappa$ there exists a finite set $E \subseteq Sen(\Sigma)$, called the $(\kappa, \kappa_1, \kappa_2)$*-interpolant of* E_1*,* E_2, such that
$$(E_1 \vdash \varphi_1 E) \ \geq\ \kappa_1 \ \text{ and } \ \varphi_2 E \cup \Gamma_2 \vdash E_2 \ \geq\ \kappa_2.$$
- κ*-reduct of sets of sentences.* In any $\mathcal{L}$-institution, for each signature morphism $\varphi: \Sigma \to \Sigma'$ and each set E' of Σ-sentences and each $\kappa \in L$, we let $[\varphi^{-1}E']_\kappa = \{\rho \in Sen(\Sigma) \mid (E' \models \varphi\rho) \geq \kappa\}$.
- *Inter-consistent sets of sentences.* Consider an $\mathcal{L}$-institution. For any $\kappa \in L$ we say that two sets of Σ-sentences Γ_1, Γ_2 are κ*-inter-consistent* when there exists a Σ-model M such that $(M \models \Gamma_1) * (M \models \Gamma_2) \geq \kappa$. We also just say that they are *inter-consistent* when there exists $\kappa > 0$ such that they are κ-inter-consistent. Otherwise, we say that they are *inter-inconsistent*.

- *Companions.* Let $\mathcal{L}$ be a partially ordered set with bottom (0) and top (1) elements and with a binary commutative monotone operation $*$ which admits 1 as identity. For any $\kappa, \ell \in L$ we say that ℓ is
  - a *lower-companion* to κ when $\{x \mid x * \kappa \neq 0\} \subseteq \{x \mid \ell \leq x\}$;
  - an *upper-companion* to κ when $\{x \mid \ell \leq x\} \subseteq \{x \mid x * \kappa \neq 0\}$; or
  - a *single-companion* to κ when it is both a lower- and and upper-companion to κ.

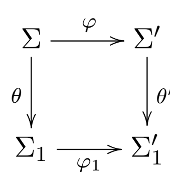
  We say that $\mathcal{L}$ admits lower / upper / single companions when each $\kappa \neq 0$ has a lower / upper / single companion.
- *Graded Robinson consistency.* Consider an $\mathcal{L}$-institution such that $\mathcal{L}$ is a residuated lattice. For any $\ell, \kappa_1, \kappa_2 \in L \setminus \{0\}$, a commutative square of signature morphisms like (3) is a $(\ell, \kappa_1, \kappa_2)$-*Rc* square when for any finite sets E_i of Σ_i-sentences, $i = 1, 2$, if $[\varphi_1^{-1} E_1]_{\kappa_1}$ and $[\varphi_2^{-1} E_2]_{\kappa_2}$ are inter-consistent then $\theta_1 E_1$ and $\theta_2 E_2$ are ℓ-inter-consistent.
- *Compactness.* An $\mathcal{L}$-institution
  - is *compact* when each set E of inconsistent Σ-sentences admits a finite subset $\Gamma \subseteq E$ that is inconsistent too;
  - is *inter-compact* when for any two inter-inconsistent sets E_1, E_2 of Σ-sentences there are finite subsets $\Gamma_i \subseteq E_i$, $i = 1, 2$, such that Γ_1 and Γ_2 are inter-inconsistent.
- *Semantic graded implicit definability.* In any $\mathcal{L}$-institution, for any $\kappa \in L$, a signature morphism $\varphi : \ \Sigma \to \Sigma'$ is *defined κ-implicitly* by a set $E' \subseteq \mathit{Sen}(\Sigma')$ when for any Σ'-models M'_1 and M'_2 if $\mathit{Mod}(\varphi) M'_1 = \mathit{Mod}(\varphi) M'_2$ then $M'_1 = M'_2$.
- *Consequence theoretic graded implicit definability.* In any $\mathcal{L}$-entailment system, for any $\kappa \in L$, a signature morphism $\varphi : \ \Sigma \to \Sigma'$ is *defined κ-implicitly* by a set $E' \subseteq \mathit{Sen}(\Sigma')$ when for any diagram of pushout squares like below

(5)



  and for any Σ'_1-sentence ρ we have that $u(\theta' E') \cup v(\theta' E') \cup u\rho \vdash v\rho \geq \kappa$.
- *Graded explicit definability.*In any $\mathcal{L}$-entailment system, for each $\kappa \in L$, a signature morphism $\varphi : \ \Sigma \to \Sigma'$ is *κ-explicitly defined* by a set of sentences $E' \subseteq \mathit{Sen}(\Sigma')$ when for each pushout square of signature morphisms like below

(6)

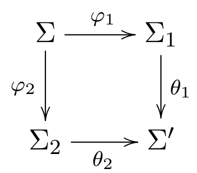


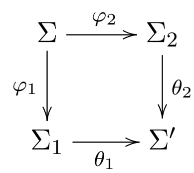
  and each $\rho \in \mathit{Sen}(\Sigma'_1)$ there exists a finite set of sentences $E_\rho \subseteq \mathit{Sen}(\Sigma_1)$ such that
  $(\theta' E' \cup \rho \vdash \varphi_1 E_\rho) * (\theta' E' \cup \varphi_1 E_\rho \vdash \rho) \ \geq \ \kappa.$

Proved results 2.1.11.1.

- *Craig-Robinson versus Craig interpolation.* Let $\mathcal{L}$ be a Heyting algebra. In any $\mathcal{L}$-institution $\mathcal{I}$ with model-theoretic implications any Ci square is a CRi square.
- *Robinson consistency by Craig interpolation.* In any $\mathcal{L}$-institution with conjunctions and negations, if ℓ is a lower-companion to κ then any $(\kappa, \kappa_1, \kappa_2)$-Ci square is a $(\ell, \kappa_1, \kappa_2)$-Rc square.
- *Compactness versus inter-compactness.* On the one hand, any inter-compact $\mathcal{L}$-institution is compact. On the other hand, if $\mathcal{L}$ is a Heyting algebra or $\mathcal{L}$ is a finite total order and $\mathcal{I}$ has semantic residual conjunctions, the compactness property implies inter-compactness.

- *Craig interpolation by Robinson consistency.* In any inter-compact $\mathcal{L}$-institution with conjunctions and negations, if ℓ is an upper-companion to κ and $0 < \kappa \leq \kappa_1 * \kappa_2$, then any $(\ell, \kappa_1, \kappa_2)$-Rc square is a $(\kappa, \kappa_1, \kappa_2)$-Ci square.
- In any inter-compact $\mathcal{L}$-institution with conjunctions and negations, for any κ that admits a single-companion, any commutative square of signature morphism is a $(\kappa^*, \kappa_1, \kappa_2)$-Rc square if and only if it is a $(\kappa, \kappa_1, \kappa_2)$-Ci square.
- *Symmetry of graded interpolation.* In an arbitrary inter-compact $\mathcal{L}$-institution with conjunctions and negations we consider the following commutative squares of signature morphisms:





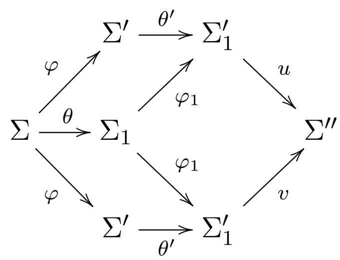
  Then for any $\kappa, \kappa' \in L \setminus \{0\}$ such that $\kappa \leq \kappa' * \kappa'$ we have that the left-hand side square above is a $(\kappa, \kappa', \kappa')$-Ci square if and only if the right-hand side one is a $(\kappa, \kappa', \kappa')$-Ci square.

2.1.12. *Definability by graded interpolation* [13].

Proved results 2.1.12.1.

- *Semantic and consequence-theoretic implicit graded definabilities.* In any semi-exact $\mathcal{L}$-institution let $\kappa, \ell \in L$ such that ℓ is a lower-companion to κ. Then a signature morphism $\varphi\colon\ \Sigma \to \Sigma'$ is defined κ-implicitly by $E' \subseteq Sen(\Sigma')$ in the consequence theoretic sense if it is defined ℓ- implicitly by E' in the semantic sense.
- In any $\mathcal{L}$-entailment system with $*$ monotone, for any signature morphism $\varphi\colon\ \Sigma \to \Sigma'$ and any set of sentences $E' \subseteq Sen(\Sigma')$, for any commuting diagram of signature morphisms like (5) and for all sets of sentences $\Gamma \subseteq Sen(\Sigma_1)$ and all Σ_1'-sentences ρ we have that
$$(\theta' E' \cup \rho \vdash \varphi_1 \Gamma) * (\theta' E' \cup \varphi_1 \Gamma \vdash \rho) \ \leq\ (\theta'; u)E' \cup (\theta'; v)E' \cup u\rho \vdash v\rho.$$
- *Implicit definability causes explicit definability.* In any $\mathcal{L}$-entailment system that has $\langle \mathfrak{L}, \mathfrak{R}\rangle$-CRi for classes $\mathfrak{L}, \mathfrak{R}$ of signature morphisms that are stable under pushouts, for any signature morphism $\varphi\colon\ \Sigma \to \Sigma' \in \mathfrak{L} \cap \mathfrak{R}$ and any set of sentences $E' \subseteq Sen(\Sigma')$, for any diagram of pushout squares:

(7)



  for any Σ_1'-sentence ρ there exists a finite set of sentences $E_\rho \subseteq Sen(\Sigma_1)$ such that
$$(\theta' E' \cup \rho \vdash \varphi_1 E_\rho) * (\theta' E' \cup \varphi_1 E_\rho \vdash \rho) \ \geq\ (\theta'; u)E' \cup (\theta'; v)E' \cup u\rho \vdash v\rho.$$
- *Graded definability by graded interpolation.* In any semi-exact $\mathcal{L}$-institution with $\langle \mathfrak{L}, \mathfrak{R}\rangle$-CRi we let $\kappa, \ell \in L$ such that ℓ is a lower-companion to κ. Then a signature morphism in $\mathfrak{L} \cap \mathfrak{R}$ is defined κ-explicitly when it is defined ℓ-implicitly.

2.2. **Establishing structural connections between several generalisations (O2).** For achieving this objective we have investigated thoroughly all possible structural connections between the three generalisations of ordinary institutions. The only really meaningful such connection that was possible to establish is between 3/2-institutions and stratified institutions. This comes in a form of a representation of the former concept into the latter, which comes with a series of model-theoretic consequences.

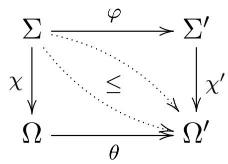
2.2.1. *Representations of 3/2-institutions as stratified institutions* [9].

New concepts 2.2.1.1.

- In any 3/2-institution, a signature morphism χ
  - is *fiber-small* when for each $\chi\Box$-model M we have that $Mod(\chi)M$ is a set; and
  - is *quasi-representable* when for each $\chi\Box$-model homomorphism $h : \; M \to M_0$ and each $N \in Mod(\chi)M$ there exists and unique model homomorphism $h^N \in Mod(\chi)h$ such that $\Box h^N = N$.

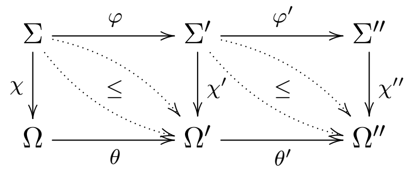
  A 3/2-institution $\mathcal{I} = (Sign, Sen, Mod, \models)$ is fixed and then gradually the entities that will define its associated stratified institution $\mathcal{I}^{\mathrm{s}} = (Sign^s, Sen^s, Mod^s, [\![\_]\!], \models^s)$ are built.
- *The category of the signatures.*The category $Sign^s$ has the objects the fiber-small quasi-representable signature morphisms χ of $Sign$. The arrows $\chi \to \chi'$ in $Sign^s$ are pairs of signature morphisms (φ, θ) such that
  - both φ and θ are total and Mod-strict; and
  - $\chi; \theta \leq \varphi; \chi'$.

(8)



The composition in $Sign^s$ is defined as pairwise composition in $Sign$, i.e. $(\varphi, \theta); (\varphi', \theta') = (\varphi; \varphi', \theta; \theta')$, as shown in the following figure:

(9)



An arrow $(\varphi, \theta) : \; \chi \to \chi'$ is *strict* when $\chi; \theta = \varphi; \chi'$.
- *The sentence translation functor.* For any $Sign^s$ signature χ we define $Sen^s(\chi) = Sen(\Box\chi)$ and for any $Sign^s$-morphism (φ, θ) we define $Sen^s(\varphi, \theta) = Sen(\varphi)$.
- *The model reduct functor.* For any $Sign^s$ signature χ we define $Mod^s(\chi) = Mod(\chi\Box)$ and for any $Sign^s$-morphism (φ, θ) we define $Mod^s(\varphi, \theta) = Mod(\theta)$.
- *The stratification.* For any $Sign^s$ signature χ we define
  - $[\![M]\!]_\chi = Mod(\chi)M$ for any $M \in |Mod^s(\chi)| (= |Mod(\chi\Box)|)$, and
  - $[\![h]\!]_\chi N = h^N\Box$ for any $\chi\Box$-model homomorphism $h \in Mod^s(\chi) (= |Mod(\chi\Box)|)$ and $\Box\chi$-model $N \in Mod(\chi)(\Box h)$.

  For each signature morphism $(\varphi, \theta) : \; \chi \to \chi'$ in $Sign^s$ we define:
  - $[\![M']\!]_{(\varphi,\theta)} N' = Mod(\varphi)N'$ for any $M' \in |Mod^s(\chi')|$ and any $N' \in [\![M']\!]_{\chi'}$.
- *The satisfaction relation.* For each signature χ in $Sign^s$, each $\chi\Box$-model M, each $\Box\chi$-model $N \in [\![M]\!]_\chi$ and each $\Box\chi$-sentence ρ, $\; M (\models^s)^N_\chi \rho \;$ if and only if $\; N \models_{\Box\chi} \rho$.

Proved results 2.2.1.1.

- *The category of the signatures.* $Sign^s$ is a category.
- *The sentence functor.* Sen^s is a functor $Sign^s \to \mathbf{Set}$.
- *The model functor.* Mod^s is a functor $(Sign^s)^{\mathrm{op}} \to \mathbf{CAT}$.
- *The stratification.* $[\![\_]\!]$ is a lax natural transformation $Mod^s \Rightarrow SET$.
- *The satisfaction condition.* For any signature morphism $(\varphi, \theta) : \; \chi \to \chi'$ in $Sign^s$, any χ'-model M', any $N' \in [\![M']\!]_{\chi'}$, and any χ-sentence ρ: $\; M' \models^{N'}_{\chi'} Sen^s(\varphi, \theta)\rho \;$ if and only if $\; Mod^s(\varphi, \theta)M' \models_\chi \rho$.
- *Representing 3/2-institutions as stratified institutions.* $\mathcal{I}^{\mathrm{s}} = (Sign^s, Sen^s, Mod^s, [\![\_]\!], \models^s)$ is a stratified institution.
- *Representing 3/2-institution as ordinary institutions.* Let $\mathcal{I} = (Sign, Sen, Mod, \models)$ be a 3/2-institution. Then

$$(\mathcal{I}^{\mathrm{s}})^\sharp = (Sign^s, Sen^s, (Mod^s)^\sharp, \models^\sharp)$$

defines an ordinary institution where in addition to $Sign^s$ and Sen^s

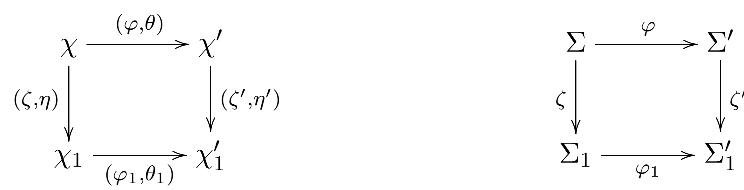
- For each $\chi \in |Sign^s|$:
  * a $(Mod^s)^\sharp$ χ-model is pair (M, N) such that $M \in |Mod(\chi\Box)|$, $N \in Mod(\chi)M$;
  * a χ-model homomorphism $(M, N) \to (M_0, N_0)$ is a model homomorphism $h : M \to M_0$ such that $N_0 = h^N\Box$.
- For each $(\varphi, \theta) : \chi \to \chi'$ and any $(Mod^s)^\sharp$ χ'-model (M', N'):
$$(Mod^s)^\sharp(\varphi, \theta)(M', N') = (Mod(\theta)M', Mod(\varphi)N').$$
- For each $(Mod^s)^\sharp$ χ-model (M, N) and each Sen^s χ-sentence ρ: $(M, N) \models^\sharp_\chi \rho$ if and only if $N \models^{\mathcal{I}}_{\Box\chi} \rho$.

2.2.2. *Model amalgamation by representation* [9].

Proved results 2.2.2.1.

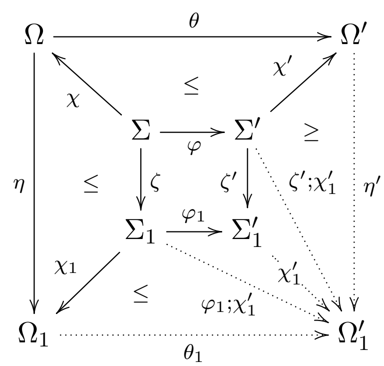
- *Model amalgamation in 3/2-institutions by stratified model amalgamation.* Let $\mathcal{I} = (Sign, Sen, Mod, \models)$ be a 3/2-institution and let $\mathcal{I}^s = (Sign^s, Sen^s, Mod^s, [\![\_]\!], \models^s)$ be its representation as a stratified institution. Let the left hand square below represent a commutative diagram in $Sign^s$ such that its projection on the first component (the right hand side square below) is a model amalgamation square in $\mathcal{I}$.

(10)



Then the square of $\mathcal{I}^s$-signature morphisms is a stratified model amalgamation square if and only if the following lax co-cone of $\mathcal{I}$-signature morphisms has the model amalgamation property.

(11)



- Let us assume a 3/2-institution $\mathcal{I}$ such that when we remove its model homomorphisms it is generated by a 3/2-institutional seed [6]. Let us consider a lax co-cone of $\mathcal{I}$ signature morphisms like in Figure (11) with the following properties:
  - the inner square (aka the right hand side square of Figure (10)) is a 3/2-pushout square;
  - the outer square $(\eta, \theta, \eta', \theta_1)$ is a model amalgamation square; and
  - $(\varphi_1, \theta_1) : \chi_1 \to \chi'_1$ and $(\zeta', \eta') : \chi' \to \chi'_1$ are strict.

  Then the lax co-cone of Figure (11) has the model amalgamation property.
- In the context of a 3/2-institution generated by a 3/2-institutional seed let us assume that Sen preserves and reflects maximality (i.e. φ is maximal if and only if $Sen(\varphi)$ is total). Then any pushout co-cone of signature morphisms determines a model amalgamation square.
- Any pushout square in **Set** is a 3/2-pushout square in **Pfn** (i.e., the 3/2-category of sets and partial functions).

2.3. **Establishing adequate logic-by-translation structures (O3).**

2.3.1. *Decompositions of stratified institutions* [12]. Under this topic, we have introduced a class of stratified institutions that, on the one hand, covers many important concrete cases, and on the other hand, is technically very convenient. It is based on the extension of morphisms of (ordinary) institutions to morphisms of stratified institutions.

New concepts 2.3.1.1.

- *Morphism of stratified institutions.* Given two stratified institutions $\mathcal{S}$ and $\mathcal{S}'$ a *stratified institution morphism* $(\Phi, \alpha, \beta) : \mathcal{S}' \to \mathcal{S}$ consists of
  - a functor $\Phi : Sign' \to Sign$,
  - a natural transformation $\alpha : \Phi; Sen \Rightarrow Sen'$, and
  - a lax natural transformation $\beta : Mod' \Rightarrow \Phi^{\mathrm{op}}; Mod$ such that $\beta; \Phi^{\mathrm{op}}[\![\_]\!] = [\![\_]\!]'$,

  and such that the following Satisfaction Condition holds for any $\mathcal{S}'$-signature Σ', any Σ'-model M', any $w \in [\![M']\!]_{\Sigma'}$ and any $\Phi(\Sigma')$-sentence ρ: $M' \models^{w'} \alpha_{\Sigma'}\rho$ if and only if $\beta_{\Sigma'}M' \models^{w'} \rho$. When β is strict, (Φ, α, β) is called *strict* too.
- *Bases of stratified institutions.* A *base for a stratified institution* $\mathcal{S}$ is an institution morphism $(\Phi, \alpha, \beta) : \mathcal{S}^\sharp \to B$. A base is *shared* when for each signature Σ, each Σ-model M of $\mathcal{S}$, and any $w, w' \in [\![M]\!]_\Sigma$ we have that $\beta_\Sigma(M, w) = \beta_\Sigma(M, w')$.

Proved results 2.3.1.1.

- Let $\mathbf{INS}$ be the category of institution morphisms and $\mathbf{SINS}$ be the category of strict stratified institution morphisms. Let $(\_)^\sharp : \mathbf{SINS} \to \mathbf{INS}$ be the canonical extension of the mapping $\mathcal{S} \mapsto \mathcal{S}^\sharp$. Then $(\_)^\sharp$ has a right adjoint which we denoted as $\widetilde{(\_)} : \mathbf{INS} \to \mathbf{SINS}$.

New concepts 2.3.1.2.

- *Decompositions of stratified institutions.* Let $\mathcal{S}$ be a stratified institution and $(\Phi, \alpha, \beta) : S^\sharp \to \mathcal{B}$ be a base for $\mathcal{S}$. Let $Mod^C \subseteq Mod^{\widetilde{\mathcal{B}}}$ be a sub-functor such that for each signature Σ,

  $$\widetilde{\beta}_\Sigma(Mod^{\mathcal{S}}(\Sigma)) \subseteq Mod^C(\Phi\Sigma),$$

  refereed to as the *constraint model sub-functor*. Let $\widetilde{\mathcal{B}}^C$ denote the stratified sub-institution of $\widetilde{\mathcal{B}}$ induced by Mod^C. A *decomposition of* $\mathcal{S}$ consists of two stratified institution morphisms like below

  $$\mathcal{S}^0 \xleftarrow{(\Phi^0,\alpha^0,\beta^0)} \mathcal{S} \xrightarrow{(\Phi,\alpha,\widetilde{\beta})} \widetilde{\mathcal{B}}^C$$

  such that for each $\mathcal{S}$-signature Σ

  $$\begin{array}{ccccc} Mod^0(\Phi^0\Sigma) & \xleftarrow{\beta^0_\Sigma} & Mod^{\mathcal{S}}(\Sigma) & \xrightarrow{\widetilde{\beta}_\Sigma} & Mod^C(\Phi\Sigma) \\ & {\scriptstyle [\![\_]\!]^0_{\Phi^0\Sigma}} \searrow & \downarrow {\scriptstyle [\![\_]\!]^{\mathcal{S}}_\Sigma} & \swarrow {\scriptstyle [\![\_]\!]^{\widetilde{\mathcal{B}}}_{\Phi\Sigma}} & \\ & & \mathbf{Set} & & \end{array}$$

  is a pullback in $\mathbf{CAT}$.

Proved results 2.3.1.2.

- In the applications the eventual modal structures of $\mathcal{S}$ come from $\mathcal{S}^0$. The following fact clarifies mathematically this situation in a full generality. Consider a decomposition of a stratified institution $\mathcal{S}$. Then any frames / nominals extraction of $\mathcal{S}^0$ induces canonically a frames / nominals extraction of $\mathcal{S}$ by composition with $(\Phi^0, \alpha^0, \beta^0)$.
- Any stratified institution that admits a decomposition is strict.

2.3.2. *Theory morphisms in fuzzy model theory* [1]. The aim of this topic was to establish theory morphisms and their co-limits and model amalgamation properties as foundations for internal logic translations in fuzzy model theory. With respect to institution theory applications to computing science, the concept of theory morphism arguably plays a most important role.

New concepts 2.3.2.1.

- *Morphisms of fuzzy theories.* Consider an $\mathcal{L}$-institution equipped with an $\mathcal{L}$-closure system denoted by $(\_)^\bullet$. A *morphism of fuzzy theories* $\varphi : (\Sigma, T) \to (\Sigma', T')$ is a signature morphism $\varphi : \Sigma \to \Sigma'$ such that $T \leq Sen(\varphi); T'^\bullet$.
- *General specific closure systems.* Two prominent closure systems are Goguen's many-valued interpretation of Modus Ponens and the closure determined by the Galois connection between the syntax and the semantics in $\mathcal{L}$-institutions.

Proved results 2.3.2.1.

- *Categories of fuzzy theory morphisms.* For any choice of a closure system $(\_)^\bullet$ the respective fuzzy theory morphisms form a category under the composition inherited from the category of the signatures.
- *Embedding into the category of flattened theories.* For any $\mathcal{L}$-institution $\mathcal{I}$ we let $\mathrm{Th}^\bullet$ denote the category of theory morphisms with respect to a closure operator $(\_)^\bullet$ and $\mathrm{Th}^\sharp$ denote the category of theory morphisms in $\mathcal{I}^\sharp$. If $(\_)^\bullet$ is a closure system that is lower than the Galois connection closure system $(\_)^{**}$ then there exists an embedding (i.e. injective on objects and faithful) functor $\Phi^\bullet : \mathrm{Th}^\bullet \to \mathrm{Th}^\sharp$ defined for each fuzzy theory (Σ, T) by $\Phi^\bullet(\Sigma, T) = (\Sigma, T^\sharp)$ where $T^\sharp = \{(\rho, T\rho) \mid \rho \in Sen(\Sigma), T\rho \neq 0\}$.
- *Adjoint to embedding into the category of flattened theories.* We assume an $\mathcal{L}$-institution $\mathcal{I}$ such that $\mathcal{L}$ is a complete lattice. For each $E \in Sen(\Sigma) \times L$ let $\overline{E} : Sen(\Sigma) \to L$ be defined by $\overline{E}\rho = \bigvee\{\kappa \mid (\rho, \kappa) \in E\}$. If $(\_)^\bullet$ is a closure system that is higher than the Galois connection closure system $(\_)^{**}$ then there exists a functor $\Phi^\sharp : \mathrm{Th}^\sharp \to \mathrm{Th}^\bullet$ such that $\Phi^\sharp(\Sigma, E) = (\Sigma, \overline{E})$. Moreover, $\Phi^\sharp : \mathrm{Th}^\sharp \to \mathrm{Th}^{**}$ is a right-adjoint right-inverse to $\Phi^{**} : \mathrm{Th}^{**} \to \mathrm{Th}^\sharp$.

New concepts 2.3.2.2.

- *Models of fuzzy theories.* For any fuzzy theory (Σ, T) in an $\mathcal{L}$-institution $\mathcal{I}$, we say that a Σ-model is a (Σ, T)-*model* when $M^* \geq T$.

Proved results 2.3.2.2.

- *$\mathcal{L}$-institutions of theories.* Let $\mathcal{I}$ be an $\mathcal{L}$-institution endowed with a closure system $(\_)^\bullet$ that is lower than the Galois connection closure system $(\_)^{**}$. Then $\mathcal{I}^\bullet = (\mathrm{Th}^\bullet, Sen^\circ, Mod^\circ, \models^\circ)$ is an $\mathcal{L}$-institution, called the *$\mathcal{L}$-institution of fuzzy theories with respect to* $(\_)^\bullet$, where
  - $Sen^\circ = \Pi^\bullet; Sen$ where $\Pi^\bullet : \mathrm{Th}^\bullet \to Sign$ is the canonical projection functor,
  - $Mod^\circ(\Sigma, T)$ is the full subcategory of $Mod(\Sigma)$ determined by the (Σ, T)-models, and
  - for each (Σ, T)-model M and each $\rho \in Sen^\circ(\Sigma, T) = Sen(\Sigma)$, $(M \models^\circ_{(\Sigma,T)} \rho) = (M \models_\Sigma \rho)$.
- *Preservation of the graded semantic consequence along fuzzy theory morphisms.* If $\mathcal{L}$ is a complete residuated lattice and $\varphi : (\Sigma, T) \to (\Sigma', T')$ is a fuzzy theory morphism with respect to a closure system that is lower than the Galois closure system, then for any set E of Σ-sentences and any Σ-sentence γ, $E \models_{(\Sigma,T)} \gamma \leq \varphi E \models_{(\Sigma',T')} \varphi\gamma$.

2.3.3. *Lifting properties from categories of signatures to categories of theories* [1]. From the point of view of computing science applications, the importance of colimits of theories surpasses that of limits of theories. We have shown that both of them are lifted from the underlying categories of signatures. We fix an $\mathcal{L}$-institution such that $\mathcal{L}$ is a complete lattice and endowed with a closure system $(\_)^\bullet$.

New concepts 2.3.3.1.

- *Theory translations along signature morphisms.* For any signature morphism $\varphi : \Sigma \to \Sigma'$ and any theory (Σ, T) we let the theory $(\Sigma', \varphi T)$ be defined by

$$(\varphi T)\rho' = \bigvee_{\varphi\rho = \rho'} T\rho.$$

Proved results 2.3.3.1.

- *Lifting limits.* $\Pi^\bullet : Sign^\circ \to Sign$ lifts limits.
- *Lifting co-limits.* $\Pi^\bullet : Sign^\circ \to Sign$ lifts colimits.

- *Model amalgamation in $\mathcal{I}^\bullet$ from model amalgamation in $\mathcal{I}$.* Let $\mathcal{I}$ be an $\mathcal{L}$-institution endowed with a closure system $(\_)^\bullet$ that is lower than the Galois connection closure system. If $\mathcal{I}$ has (unique) J-model amalgamation then $\mathcal{I}^\bullet$ has (unique) J-model amalgamation too.

New concepts 2.3.3.2. In logic-based computing languages the most common form of software module import "protects" the imported module. In logical terms this property of the theory morphism is called *conservativity*.

- *Conservative signature morphisms.* A signature morphism $\varphi : \ \Sigma \to \Sigma'$
  - has the *model expansion* property when each Σ-model has a φ-expansion, and
  - is *conservative* when $(\Gamma \models \gamma) \ = \ (\varphi\Gamma \models \varphi\gamma)$ for each $\Gamma \subseteq Sen(\Sigma), \gamma \in Sen(\Sigma)$.
- *Conservative fuzzy theory morphisms.* Let $\mathcal{I}$ be an $\mathcal{L}$-institution endowed with a closure system $(\_)^\bullet$ that is lower than the Galois connection closure system. A fuzzy theory morphism $\varphi : \ (\Sigma, T) \to (\Sigma', T')$ has the model expansion / conservativity property when it has this property as a signature morphism in $\mathcal{I}^\bullet$.

Proved results 2.3.3.2.

- *Strong theory morphisms.* Let $\varphi : \ (\Sigma, T) \to (\Sigma', T')$ be a fuzzy theory morphism. Then the following are equivalent:
  (1) for each theory morphism $\varphi : \ (\Sigma, T) \to (\Sigma', T'_1), T'^\bullet \leq T_1'^\bullet$.
  (2) $T'^\bullet = (\varphi T)^\bullet$.
  When φ satisfies one of the equivalent conditions of above, it is called a *strong* fuzzy theory morphism.
- *Conservative signature morphisms by model expansion.* A signature morphism is conservative when it has the model expansion property.
- *Model expansion for fuzzy theory morphisms.* Any strong fuzzy theory morphism $\varphi : \ (\Sigma, T) \to (\Sigma', T')$ has the model expansion property when its underlying signature morphism $\varphi : \ \Sigma \to \Sigma'$ has the model expansion property.
- *Conservative theory morphisms by model expansion.* Let $\varphi : \ (\Sigma, T) \to (\Sigma', T')$ be a strong fuzzy theory morphism such that $\varphi : \ \Sigma \to \Sigma'$ has the model expansion property. Then $\varphi : \ (\Sigma, T) \to (\Sigma', T')$ is conservative.

2.3.4. *MV-comorphisms* [1]. Comorphisms represent one of the most prominent concepts of 'homomorphisms' between institutions. Due to their role in the institution theoretic method of logic-by-translation they are arguably the most important one. We have extended the concept of comorphism from binary institutions to $\mathcal{L}$-institutions, provided some relevant examples, and established a couple of basic properties of this extension that generalise corresponding properties from binary institution theory. By $\mathcal{I}[\mathcal{L}]$ we mean that $\mathcal{I}$ is an $\mathcal{L}$-institution.

New concepts 2.3.4.1.

- *MV-comorphism.* Let $\mathcal{L}$ and $\mathcal{L}'$ be partially ordered sets and let $\mathcal{I}[\mathcal{L}] = (Sign, Sen, Mod, \models)$ and $\mathcal{I}'[\mathcal{L}'] = (Sign', Sen', Mod', \models')$. Then $(\Phi, \alpha, \beta, \lambda) : \ \mathcal{I}[\mathcal{L}] \to \mathcal{I}'[\mathcal{L}']$ is an *MV-comorphism* when
  - $\Phi : \ Sign \to Sign'$ is functor (called the *signature translation functor*),
  - $\alpha : \ Sen \Rightarrow \Phi; Sen'$ is a natural transformation (called the *sentence translation*),
  - $\beta : \ \Phi; Mod' \Rightarrow Mod$ is a natural transformation (called the *model translation*), and
  - $\lambda : \ \mathcal{L}' \to \mathcal{L}$ is a monotone function

  such that for each $\mathcal{I}$-signature Σ, each Σ-sentence ρ and each $\Phi\Sigma$-model M' the following Satisfaction Condition holds:
  $$(\beta_\Sigma M' \models_\Sigma \rho) = \lambda(M' \models'_{\Phi\Sigma} \alpha_\Sigma \rho).$$
  When λ is an identity it may be omitted and then the MV-comorphism may be called an $\mathcal{L}$-comorphism.

Proved results 2.3.4.1.

- *The category of MV-comorphisms.* Given MV-comorphisms $(\Phi, \alpha, \beta, \lambda) : \ \mathcal{I}[\mathcal{L}] \to \mathcal{I}'[\mathcal{L}']$ and $(\Phi', \alpha', \beta', \lambda') : \ \mathcal{I}'[\mathcal{L}'] \to \mathcal{I}''[\mathcal{L}'']$, the 4-tuple $(\Phi; \Phi', \ \alpha; \Phi\alpha', \ \Phi\beta'; \beta, \ \lambda \circ \lambda')$ is an MV-comorphism $\mathcal{I}[\mathcal{L}] \to \mathcal{I}''[\mathcal{L}'']$ which is called the *composition* of the former MV-comorphisms. Moreover, the composition of MV-comorphisms is associative and has identities.
- *Concrete MV-comorphisms.* We defined the following concrete examples of MV-comorphisms:
  (1) The embedding MVL_0 into MVL_1.

(2) Changing the truth values of an $\mathcal{L}$-institution by a homomorphism of residuated lattices.
(3) The embedding of *MVL* and *FMA* embedded into the generic $\mathcal{L}$-institution $\mathcal{I}(\mathcal{L})$ of [4].
(4) The encoding FEL (fuzzy equational logic of into many-valued first order logic with crisp equality.
(5) The encoding many-valued Horn clause logic into conditional fuzzy equational logic.
(6) The embedding $\mathcal{L}$-institutions into $\mathcal{L}$-institutions of fuzzy theories.

- *Binary flattening of MV-comorphisms.* For any MV-comorphism $(\Phi, \alpha, \beta, \lambda) : \mathcal{I}[\mathcal{L}] \to \mathcal{I}'[\mathcal{L}']$ such that the pair (λ', λ) is a Galois connection between $\mathcal{L}$ and $\mathcal{L}'$ we have that $(\Phi, \alpha^\sharp, \beta)$ is a (binary) comorphism $\mathcal{I}^\sharp \to \mathcal{I}'^\sharp$ where $\alpha^\sharp$ is defined by $\alpha^\sharp_\Sigma(\rho, \kappa) = (\alpha_\Sigma \rho, \lambda' \kappa)$.
- *Preservation of the graded semantic consequence along MV-comorphisms.* Let $(\Phi, \alpha, \beta, \lambda) : \mathcal{I}[\mathcal{L}] \to \mathcal{I}'[\mathcal{L}']$ be an MV-comorphism such that λ is a homomorphism of complete residuated lattices. Then for any $\mathcal{I}$-signature Σ, any $\Gamma \subseteq Sen(\Sigma), \gamma \in Sen(\Sigma)$: $(\Gamma \models_\Sigma \gamma) \leq \lambda(\alpha_\Sigma \Gamma \models'_{\Phi\Sigma} \alpha_\Sigma \gamma)$.

New concepts 2.3.4.2. In the case of MV-comorphisms conservativity guarantees that by translating we cannot deduce more. This is crucial when doing logic-by-translation, once we establish a consequence degree we can return it to the translated (sets of) sentences.

- *Conservative MV-comorphisms.* An MV-comorphism $(\Phi, \alpha, \beta, \lambda) : \mathcal{I}[\mathcal{L}] \to \mathcal{I}'[\mathcal{L}']$
  – has the *model expansion* property when for each Σ-model M in $\mathcal{I}$ there exists a $\Phi\Sigma$-model M' in $\mathcal{I}'$ such that $\beta_\Sigma M' = M$.
  – is *conservative* when $\mathcal{L}$ and $\mathcal{L}'$ are complete residuated lattices and for each $\Sigma \in |Sign^{\mathcal{I}}|$, $\Gamma \subseteq Sen^{\mathcal{I}}(\Sigma)$, $\gamma \in Sen^{\mathcal{I}}(\Sigma)$.
$$\lambda(\Gamma \models_\Sigma \gamma) = (\alpha_\Sigma \Gamma \models'_{\Phi\Sigma} \alpha_\Sigma \gamma).$$

Proved results 2.3.4.2.

- *Conservativity of MV-comorphisms by model expansion.* An MV-comorphism $(\Phi, \alpha, \beta, \lambda) : \mathcal{I}[\mathcal{L}] \to \mathcal{I}'[\mathcal{L}']$ such that λ is a homomorphism of complete residuated lattices is conservative when it has the model expansion property.
- *Concrete conservative MV-comorphisms.* We showed the conservativity of all MV-comorphisms that were introduced above.

2.3.5. *3/2-comorphisms* [10]. 3/2-comorphisms extend the concept of comorphism by considering the ordered categorical aspects of 3/2-institutions.

New concepts 2.3.5.1.

- Given 3/2-institutions $\mathcal{I}$ and $\mathcal{I}'$, a *3/2-comorphism* $(\Phi, \alpha, \beta) : \mathcal{I} \to \mathcal{I}'$ consists of
  – a 3/2-functor $\Phi : Sign \to Sign'$,
  – an oplax natural transformation $\alpha : Sen \Rightarrow \Phi; Sen'$, and
  – a lax natural transformation $\beta : \Phi^{\mathrm{op}}; Mod' \Rightarrow Mod$

  such that the following Satisfaction Condition holds for any $\mathcal{I}$-signature Σ, $\Phi\Sigma$-model M' and Σ-sentence ρ:
$$M' \models' \alpha_\Sigma \rho \text{ if and only if } \beta_\Sigma M' \models \rho.$$

Proved results 2.3.5.1.

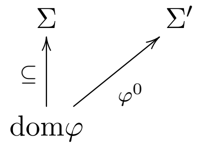
- *The category of 3/2-comorphisms.* Under the obvious component-wise composition, that replicates the composition of ordinary comorphism, 3/2-comorphisms form a category.

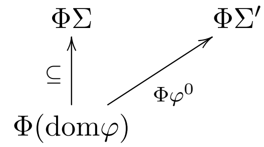
New concepts 2.3.5.2. We have extended the construction from [14] (see also Sec. 2.4.1) to comorphisms. So we assume an (ordinary) institution comorphism $(\Phi, \alpha, \beta) : \mathcal{I} \to \mathcal{I}'$ such that

(1) both $\mathcal{I}$ and $\mathcal{I}'$ satisfy the conditions of the construction of generic partial institutions from [14], i.e. both $Sign$ and $Sign'$ are endowed with inclusion systems and admit pullbacks of semi-inclusive co-spans, and
(2) Φ is inclusive and preserve the pullbacks of semi-inclusive co-spans.

Proved results 2.3.5.2. In this context of partialising institutions we built a 3/2-comorphism $(p\Phi, p\alpha, p\beta) : p\mathcal{I} \to p\mathcal{I}'$ from a comorphism $(\Phi, \alpha, \beta) : \mathcal{I} \to \mathcal{I}'$.

- *Partialising comorphisms.*
  - The mapping $p\Phi : \ pSign \to pSign'$ is defined as follows:
    - on the signatures, $(p\Phi)\Sigma = \Phi\Sigma$, and
    - for each partial $Sign$-morphism $\varphi : \ \Sigma \nrightarrow \Sigma'$ we let $(p\Phi)\varphi = (\Phi\varphi^0 : \ \Phi(\mathrm{dom}\varphi) \to \Phi\Sigma')$ (then $\mathrm{dom}(p\Phi)\varphi = \Phi(\mathrm{dom}\varphi)$).





  - For each $\mathcal{I}$-signature Σ, we let $p\alpha_\Sigma = \alpha_\Sigma$.
  - For each $\mathcal{I}$-signature Σ, we let $p\beta_\Sigma = \beta_\Sigma$.

  Then $(p\Phi, p\alpha, p\beta)$ is a 3/2-comorphism $p\mathcal{I} \to p\mathcal{I}'$.

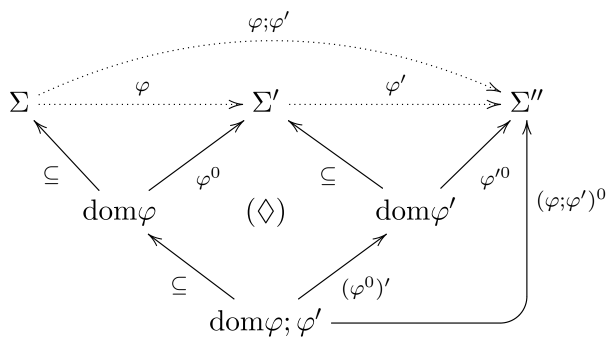
2.4. **New interpretations with applications (O4).**

2.4.1. *Partialising institutions* [14]. We have developed a generic method for constructing 3/2-institutions on top of 1-institutions that is based on extending the category of the signatures by considering partiality for the signature morphisms.

New concepts 2.4.1.1.

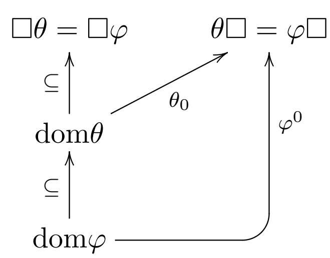
- *The 3/2-category of the abstract partial arrows / morphisms.* Given a category $Sign$ endowed with an inclusion system and which has pullbacks of semi-inclusive cospans, for any $\Sigma, \Sigma' \in |Sign|$, a *partial $Sign$-morphism* $\varphi : \ \Sigma \nrightarrow \Sigma'$ consists of a $Sign$-morphism $\varphi^0 : \ \Sigma_0 \to \Sigma'$ such that $\Sigma_0 \subseteq \Sigma$. We may denote Σ_0 by $\mathrm{dom}\varphi$. Given $\varphi : \ \Sigma \nrightarrow \Sigma'$ and $\varphi' : \ \Sigma' \nrightarrow \Sigma''$ their *composition* $\varphi;\varphi'$ is defined by the following diagram:

(12)



where the square $(\Diamond)$ is the unique pullback of φ^0 and $\mathrm{dom}\varphi' \subseteq \Sigma'$.

Given $\varphi, \theta : \ \Sigma \nrightarrow \Sigma'$, then $\varphi \le \theta$ if and only if $\mathrm{dom}\varphi \subseteq \mathrm{dom}\theta$ and $\varphi^0 = (\mathrm{dom}\varphi \subseteq \mathrm{dom}\theta);\theta^0$.



Proved results 2.4.1.1.

- *The category of the 'partial' signature morphisms.* Let $pSign$ have the same objects as $Sign$ and the partial $Sign$-morphisms as arrows. Under the definitions given above, $pSign$ is a 3/2-category.
- *'Total' signature morphisms as 'partial' signature morphisms.* There is a canonical faithful functor $[\_] : \ Sign \to pSign$ which is the identity on the objects and such that $[\chi]^0 = \chi$ for each arrow $\chi \in Sign$.

New concepts 2.4.1.2.

- *Stability of the abstract surjections.* In any category endowed with an inclusion system and with pullbacks of semi-inclusive cospans, we say that *abstract surjections are stable under semi-inclusive pullbacks* when for each pullback square like in diagram

(13)
$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \subseteq \uparrow & & \uparrow \subseteq \\ A' & \xrightarrow[f']{} & B' \end{array}$$

  if f is an abstract surjection then f' is an abstract surjection too.

Proved results 2.4.1.2.

- *Inclusion systems for abstract partial signature morphisms.* Assuming that in $Sign$ the abstract surjections are stable under inclusive pullbacks, the following gives an inclusion system in $pSign$:
  - abstract inclusions: $[i]$, where i is an abstract inclusion in $Sign$; and
  - abstract surjections: φ, such that φ^0 is an abstract surjection in $Sign$.
- *Concrete examples of inclusion systems for categories of partial signature morphisms.* The categories $pSign^{PL}$ and $pSign^{\mathcal{MSA}}$ have inclusion systems that inherit the respective inclusion systems of $Sign^{PL}$ and of $Sign^{\mathcal{MSA}}$, respectively.
- *Pushouts of abstract partial signature morphisms.* If $Sign$ has (weak) pushouts then $pSign$ has (weak) lax $Sign$-pushouts.

New concepts 2.4.1.3. The following construction represents and extension of the sentence functor Sen of a base institution to a 3/2-institution theoretic sentence functor $pSen$.

- *Partialising the sentence functor.* Given an inclusive functor $Sen : \ Sign \to \mathbf{Set}$, for each partial signature $Sign$-morphism $\varphi \in pSign$ we define a partial function $pSen(\varphi) : \ Sen(\Box\varphi) \nrightarrow Sen(\varphi\Box)$ by letting
  - $\mathrm{dom}\, pSen(\varphi) = Sen(\mathrm{dom}\varphi)$ and
  - for each $\rho \in \mathrm{dom}\, pSen(\varphi)$, $pSen(\varphi)\rho = Sen(\varphi^0)\rho$.

Proved results 2.4.1.3.

- *The sentence functor.* $pSen : \ pSign \to \mathbf{Pfn}$ is an oplax 3/2-functor.
- *Strictness of the sentence functor.* In many concrete situations of interest in fact the sentence 3/2-functor $pSen$ is strict. The following result gives a widely applicable general condition for that. If Sen maps each pullback square of semi-inclusive cospans

$$\begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ \subseteq \downarrow & & \downarrow \subseteq \\ \bullet & \longrightarrow & \bullet \end{array}$$

  to a weak pullback square, then $pSign$ is a strict 3/2-functor.

New concepts 2.4.1.4.

- *Partialising the model functor.* Given any functor $Mod : \ Sign^{\ominus} \to \mathbf{CAT}$ we define
  - for each $\Sigma \in |Sign| \ (= |pSign|)$, $\ pMod(\Sigma) = Mod(\Sigma)$,
  - for each partial $Sign$-morphism $\varphi : \ \Sigma \nrightarrow \Sigma'$,
  $$pMod(\varphi)M' = \{M \in |Mod(\Sigma)| \ \mid \ Mod(\mathrm{dom}\varphi \subseteq \Sigma)M = Mod(\varphi^0)M'\}.$$
  - $pMod(\varphi)$ is defined on the arrows like on the objects.

Proved results 2.4.1.4.

- *The model functor.* $pMod$ is a lax functor $(pSign)^{\oplus} \to$ 3/2$(\mathbf{CAT}_{\mathcal{P}})$.
- *Partialising institutions.* For any institution $\mathcal{I} = (Sign, Sen, Mod, \models)$ such that
  - $Sign$ is endowed with an inclusion system,
  - $Sign$ has pullbacks of semi-inclusive cospans, and

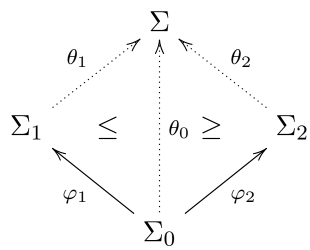
– Sen is an inclusive functor,

3/2$\mathcal{I} = (pSign, pSen, pMod, \models)$ is an oplax 3/2-institution.

2.4.2. *Lax cocones and model amalgamation* [14]. The main proposal of [6] regarding the 3/2-institution theoretic foundations of conceptual blending is based upon two concepts: lax cocones and model amalgamation. Both of them constitute 3/2-institution theoretic extension of corresponding ordinary institution theoretic concepts. In this activity, we have developed a result on the existence of lax cocones and model amalgamation in 3/2$\mathcal{I}$ based upon the existence of cocones and model amalgamation in the base institution $\mathcal{I}$. By considering the mere fact that these properties do hold in concrete institutions that are based on common logical systems, this result is applicable to a wide range of concrete situations.

New concepts 2.4.2.1.

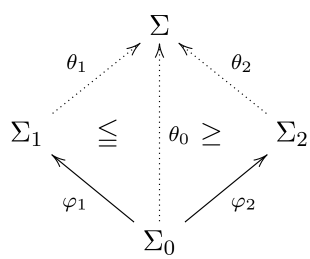
- *Model amalgamation for lax cocones.* In any 3/2-institution, a lax cocone for a span in the 3/2-category of the signature morphisms



  has *model amalgamation* when each model of the span admits an unique completion to a model (called the *amalgamation*) of the lax cocone. When dropping the uniqueness condition, the property is called *weak model amalgamation*.

Proved results 2.4.2.1.

- *Model amalgamation 3/2$\mathcal{I}$ from model amalgamation in $\mathcal{I}$.* Let $\mathcal{I}$ be an inclusive institution with pullbacks of semi-inclusive cospans.
  (1) If each span of signature morphisms in $\mathcal{I}$ admits a cocone then each span of signature morphisms (φ_1, φ_2) in 3/2$\mathcal{I}$ admits a lax cocone.



  (2) If each span of signature morphisms in $\mathcal{I}$ admits a cocone that has (weak) model amalgamation, then each span of signature morphisms in 3/2$\mathcal{I}$ admits a lax cocone that has (weak) model amalgamation.

2.4.3. *A taxonomy of "partial" theory morphisms.* Partial theory morphisms apply both to conceptual blending and software evolution.

New concepts 2.4.3.1.

- In a 3/2-institution given two theories (Σ, E) and (Σ', E'), a signature morphism $\varphi : \Sigma \to \Sigma'$ is
  - a *weak 3/2-theory morphism* $(\Sigma, E) \to (\Sigma', E')$ when $Sen(\varphi)E^\bullet \subseteq E'^\bullet$,
  - a *strong 3/2-theory morphism* $(\Sigma, E) \to (\Sigma', E')$ when for each Σ'-model M' such that $M' \models E'$ there exists $M \in Mod(\varphi)M'$ such that $M \models E$.
- In the light of the above considerations, given an institution $\mathcal{I}$ endowed with an inclusion system and with pullbacks of semi-inclusive cospans, there are four ways to think about partiality for theory morphisms. These four can be grouped as follows:

(1) Consider the closed and the strong inclusion systems for theories in $\mathcal{I}$; each of them determines a concept of partial theory morphisms by considering the category of closed theories in the role of $Sign$. For this it is necessary to recall (from [3]) that in any institution the pullbacks of (closed) theories are inherited from the underlying category of the signatures. Let us call the former concept *closed-partial theory morphism* and the latter one *strong-partial theory morphism*.

(2) Consider 3/2$\mathcal{I}$, the 3/2-institution built on top of $\mathcal{I}$. Then we may consider the two significant concepts of 3/2-theory morphisms discussed above, the weak and the strong one.

Proved results 2.4.3.1.

- Given an inclusive institution $\mathcal{I} = (Sign, Sen, Mod, \models)$ with pullbacks of semi-inclusive cospans, then the category of weak 3/2-theory morphisms in 3/2$\mathcal{I} = (pSign, pSen, pMod, \models)$ is equivalent to the category of closed-partial theory morphisms.
- The following figure shows the relationships between the four concepts of "partial" theory morphisms:

strong-partial theory morphism

weak 3/2-theory morphism = closed-partial theory morphism

strong 3/2-theory morphism

2.4.4. *Stratified parchments* [17]. In this study, we have introduced stratified parchments as a form of model-theoretic parchments that are specifically designed to present stratified institutions. Compared with previous notions of parchment, which rely on many-sorted algebra, stratified parchments integrate initial-algebra semantics with power-set algebras, which can be regarded as algebras with power sets as carriers.

New concepts 2.4.4.1.

- *Stratified parchment.* A *stratified parchment* is a tuple $\langle Sign, L, Mod, K \rangle$ consisting of:
  - a category $Sign$ of *signatures* and *signature morphisms*;
  - a *language* functor $L\colon Sign \to AlgSig$;
  - a *model* functor $Mod\colon Sign^{\mathrm{op}} \to \mathbf{CAT}$;
  - a *stratification* lax natural transformation $K\colon Mod \Rightarrow L^{\mathrm{op}} ; \mathcal{P}Alg$, where $\mathcal{P}Alg\colon AlgSig^{\mathrm{op}} \to \mathbf{CAT}$ is the functor defining, for every algebraic signature Ω, the category $\mathcal{P}Alg(\Omega)$ of power-set Ω-algebras and maps;

  such that, for every signature morphism $\varphi\colon \Sigma \to \Sigma'$ and every Σ'-model M', the map

  $$K_\varphi(M')\colon \mathcal{P}Alg(L(\varphi))(K_{\Sigma'}(M')) \to K_\Sigma(Mod(\varphi)(M'))$$

  preserves and reflects the interpretation of all symbols in $L(\Sigma)$.

  A stratified parchment is *strict* when, for every signature morphism $\varphi\colon \Sigma \to \Sigma'$ and every Σ'-model M', the map $K_\varphi(M')$ is an identity; hence, $\mathcal{P}Alg(L(\varphi))(K_{\Sigma'}(M')) = K_\Sigma(Mod(\varphi)(M'))$.
- *Parchment addendum.* Let $\mathcal{B}$, called *base*, and $\mathcal{F}$, called *feature*, be strict stratified parchments. An $\mathcal{F}$-*addendum* to $\mathcal{B}$ is a functor $\Psi\colon Sign^{\mathcal{B}} \to Sign^{\mathcal{F}}$ such that, for any base signature morphism $\varphi\colon \Sigma \to \Sigma'$, the algebraic signature morphisms $L^{\mathcal{B}}(\varphi)$ and $L^{\mathcal{F}}(\Psi(\varphi))$ are compatible, meaning that the diagram below commutes.

  $$\begin{array}{ccccc} L^{\mathcal{B}}(\Sigma) & \xleftarrow{\supseteq} & L^{\mathcal{B}}(\Sigma) \cap L^{\mathcal{F}}(\Psi(\Sigma)) & \xrightarrow{\subseteq} & L^{\mathcal{F}}(\Psi(\Sigma)) \\ {\scriptstyle L^{\mathcal{B}}(\varphi)}\downarrow & & & & \downarrow{\scriptstyle L^{\mathcal{F}}(\Psi(\varphi))} \\ L^{\mathcal{B}}(\Sigma') & \xrightarrow[\subseteq]{} & L^{\mathcal{B}}(\Sigma') \cup L^{\mathcal{F}}(\Psi(\Sigma')) & \xleftarrow[\supseteq]{} & L^{\mathcal{F}}(\Psi(\Sigma')) \end{array}$$

- *Independent addenda.* Let Ψ_i be an $\mathcal{F}_i$-addendum to a parchment $\mathcal{B}$, for $i \in \{1, 2\}$. We say that Ψ_1 and Ψ_2 are *independent* when $L^{\mathcal{F}_1}(\Psi_1(\varphi))$ and $L^{\mathcal{F}_2}(\Psi_2(\varphi))$ are compatible for all base signature morphisms φ.

  We also say that two addenda Ψ_1 and Ψ_2 (over the same base parchment) are *strongly independent* when $L^{\mathcal{F}_1}(\Psi_1(\Sigma)) \cap L^{\mathcal{F}_2}(\Psi_2(\Sigma)) \subseteq L^{\mathcal{B}}(\Sigma)$ for all base signatures Σ.
- *Split addendum.* An $\mathcal{F}$-addendum Ψ to a parchment $\mathcal{B}$ *splits* when there exists a subfunctor $S \subseteq L^{\mathcal{F}}$, called a *split* of Ψ, such that $\Psi ; S = L^{\mathcal{B}} \cap \Psi ; L^{\mathcal{F}}$.

- *Stratified rooms and corridors.* A *stratified-parchment room* is a triple $\langle \Omega, \mathbb{M}, K \rangle$ where Ω is an algebraic signature of *sentences*, $\mathbb{M}$ is a category of *models*, and $K\colon \mathbb{M} \to \mathcal{P}Alg(\Omega)$ is a *stratification* functor.

  A *corridor* $\langle \Omega, \mathbb{M}, K \rangle \to \langle \Omega', \mathbb{M}', K' \rangle$ is a triple $\langle \alpha, \beta, \kappa \rangle$ consisting of an algebraic signature morphism $\alpha\colon \Omega \to \Omega'$, a functor $\beta\colon \mathbb{M}' \to \mathbb{M}$, and a natural transformation $\kappa\colon K' \mathbin{;} \mathcal{P}Alg(\alpha) \Rightarrow \beta \mathbin{;} K$ such that, for every model $M' \in |\mathbb{M}'|$, the map $\kappa_{M'}\colon \mathcal{P}Alg(\alpha)(K'(M')) \to K(\beta(M'))$ is an Ω-algebra homomorphism.
- *Stratified-parchment morphism.* A morphism between stratified parchments $\mathcal{S}$ and $\mathcal{T}$ consists of:
  - a *signature* functor $\Phi\colon Sign^{\mathcal{S}} \to Sign^{\mathcal{T}}$;
  - a *language-translation* natural transformation $\alpha\colon \Phi \mathbin{;} L^{\mathcal{T}} \Rightarrow L^{\mathcal{S}}$;
  - a *model-reduction* natural transformation $\beta\colon Mod^{\mathcal{S}} \Rightarrow \Phi^{\mathrm{op}} \mathbin{;} Mod^{\mathcal{T}}$; and
  - a modification $\kappa\colon K^{\mathcal{S}} \mathbin{;} \alpha\mathcal{P}Alg \Rightarrow \beta \mathbin{;} \Phi^{\mathrm{op}} K^{\mathcal{T}}$ with algebra homomorphisms as components; i.e., for every signature Σ in $\mathcal{S}$ and every Σ-model M, the many-sorted map $\kappa_\Sigma(M)\colon \mathcal{P}Alg(\alpha_\Sigma)(K^{\mathcal{S}}_\Sigma(M)) \to K^{\mathcal{T}}_{\Phi(\Sigma)}(\beta_\Sigma(M))$ preserves and reflects the interpretation of all symbols in the grammar $L^{\mathcal{T}}(\Phi(\Sigma))$.

Proved results 2.4.4.1.

- *Every stratified parchment presents a stratified institution.* Every stratified parchment $\langle Sign, L, Mod, K \rangle$ can be 'flattened' to a stratified institution $\langle Sign^\flat, Sen^\flat, Mod^\flat, [\![\_]\!]^\flat, \models \rangle$ where:
  - The signatures in $Sign^\flat$ are pairs $\langle \Sigma, s \rangle$ consisting of a signature $\Sigma \in |Sign|$ and a type $s \in ST(\Sigma)$, while the morphisms $\varphi\colon \langle \Sigma, s \rangle \to \langle \Sigma', s' \rangle$ in $Sign^\flat$ are morphisms $\varphi\colon \Sigma \to \Sigma'$ in $Sign$ such that $\varphi(s) = s'$.
  - The sentence functor is defined on signatures by $Sen^\flat(\Sigma, s) = T_{L(\Sigma),s}$. For every signature morphism $\varphi\colon \langle \Sigma, s \rangle \to \langle \Sigma', s' \rangle$, the function $Sen^\flat(\varphi)$ is given by the s-component of the unique $L(\Sigma)$-homomorphism $T_{L(\Sigma)} \to Alg(L(\varphi))(T_{L(\Sigma')})$ that arises from the universal property of the term algebra.
  - The model functor is defined by $Mod^\flat(\Sigma, s) = Mod(\Sigma)$, discarding the type.
  - For every signature $\langle \Sigma, s \rangle$ in $Sign^\flat$ and every Σ-model M, $[\![M]\!]^\flat_\Sigma = K_\Sigma(M)_s$.
  - The satisfaction relations coincide with those of the parchment.
- *Parchment extensions along addenda.* Any $\mathcal{F}$-addendum Ψ to a parchment $\mathcal{B}$ gives rise to a strict 'extended' stratified parchment $\mathcal{B} + \Psi = \langle Sign^{\mathcal{B}}, L^{\mathcal{B}} \cup \Psi \mathbin{;} L^{\mathcal{F}}, Mod^{\mathcal{B}+\Psi}, K^{\mathcal{B}+\Psi} \rangle$ where:
  - For every base signature Σ, the objects in $Mod^{\mathcal{B}+\Psi}(\Sigma)$ are pairs (M, F) consisting of a base Σ-model M and a feature $\Psi(Sigma)$-model F in such a way that $\mathcal{P}Alg(\eta^{\mathcal{B}}_\Sigma)(K^{\mathcal{B}}_\Sigma(M)) = \mathcal{P}Alg(\eta^{\mathcal{F}}_\Sigma)(K^{\mathcal{F}}_{\Psi(\Sigma)}(F))$. The arrows in $Mod^{\mathcal{B}+\Psi}(\Sigma)$ are defined in a similar manner and they compose componentwise.
  - For every base signature Σ and every extended Σ-model (M, F), $K^{\mathcal{B}+\Psi}_\Sigma(M, F)$ is the unique power-set algebra arising from the amalgamation of $K^{\mathcal{B}}_\Sigma(M)$ and $K^{\mathcal{F}}_{\Psi(\Sigma)}(F)$; the same applies to Σ-homomorphisms.
- *Addenda translations under parchment extensions.* If Ψ_1 and Ψ_2 are independent addenda to a parchment $\mathcal{B}$, then Ψ_2 is also an addendum to the extended parchment $\mathcal{B} + \Psi_1$.
- *Preservation under parchment extensions of the addenda-independence property.* If Ψ_1, Ψ_2, and Ψ_3 are pairwise (strongly) independent addenda to a base parchment $\mathcal{B}$, then Ψ_2 and Ψ_3 are also (strongly) independent addenda to the extended parchment $\mathcal{B} + \Psi_1$.
- *Stratified parchments as functors.* Stratified-parchment rooms and corridors form a category denoted $SPRoom$.

  There is a one-to-one correspondence between stratified parchments $\langle Sign^{\mathcal{S}}, L^{\mathcal{S}}, Mod^{\mathcal{S}}, K^{\mathcal{S}} \rangle$ and (homonymous) functors $\mathcal{S}\colon Sign^{\mathcal{S}} \to SPRoom$ given by:
  - $\mathcal{S}(\Sigma) = \langle L^{\mathcal{S}}(\Sigma), Mod^{\mathcal{S}}(\Sigma), K^{\mathcal{S}}_\Sigma \rangle$ for every signature Σ; and
  - $\mathcal{S}(\varphi) = \langle L^{\mathcal{S}}(\varphi), Mod^{\mathcal{S}}(\varphi), K^{\mathcal{S}}_\varphi \rangle$ for every signature morphism φ.
- *Categories of parchments.* Strict and lax stratified parchments form categories $\int sParch \subseteq \int Parch$.
- *Cospan representations of addenda.* Every split S of an $\mathcal{F}$-addendum Ψ to a parchment $\mathcal{B}$ determines a cospan in the Grothendieck category $\int sParch$ of strict stratified parchments of the form

$$\mathcal{B} \xrightarrow{\langle \Psi, \alpha^{\mathcal{B}}, \beta^{\mathcal{B}} \rangle} Triv(S) \xleftarrow{\langle 1_{Sign^{\mathcal{F}}}, \alpha^{\mathcal{F}}, \beta^{\mathcal{F}} \rangle} \mathcal{F}$$

  where the language translations $\alpha^{\mathcal{B}}$ and $\alpha^{\mathcal{F}}$ correspond to the natural inclusions $\Psi \mathbin{;} S \subseteq L^{\mathcal{B}}$ and $S \subseteq L^{\mathcal{F}}$, and the model reductions $\beta^{\mathcal{B}}$ and $\beta^{\mathcal{F}}$ are given by the composite transformations $K^{\mathcal{B}} \mathbin{;} \alpha^{\mathcal{B}}\mathcal{P}Alg$ and $K^{\mathcal{F}} \mathbin{;} \alpha^{\mathcal{F}}\mathcal{P}Alg$.

- *Parchment extensions as pullbacks.* Suppose Ψ is an $\mathcal{F}$-addendum to a parchment $\mathcal{B}$ and let $S\colon Sign^{\mathcal{F}} \to AlgSig$ be one of its splits. Then the extended stratified parchment $\mathcal{B} + \Psi$ is the vertex of a pullback in $\int sParch$ of cospan representation $\langle \Psi, \alpha^{\mathcal{B}}, \beta^{\mathcal{B}} \rangle$ and $\langle 1_{Sign^{\mathcal{F}}}, \alpha^{\mathcal{F}}, \beta^{\mathcal{F}} \rangle$ of Ψ given by

$$\mathcal{B} \xleftarrow{\langle 1_{Sign^{\mathcal{B}}}, \theta^{\mathcal{B}}, \pi^{\mathcal{B}} \rangle} \mathcal{B} + \Psi \xrightarrow{\langle \Psi, \theta^{\mathcal{F}}, \pi^{\mathcal{F}} \rangle} \mathcal{F}$$

  where $\theta^{\mathcal{B}}$ and $\theta^{\mathcal{F}}$ are the natural inclusions $L^{\mathcal{B}} \subseteq L^{\mathcal{B}} \cup \Psi \,;\, L^{\mathcal{F}} \supseteq \Psi \,;\, L^{\mathcal{F}}$ and $\pi^{\mathcal{B}}$ and $\pi^{\mathcal{F}}$ are the obvious projections $Mod^{\mathcal{B}} \Leftarrow Mod^{\mathcal{B}+\Psi} \Rightarrow \Psi \,;\, Mod^{\mathcal{F}}$.
- *Completeness of parchment categories.* The parchment categories $\int sParch$ and $\int Parch$ have all small limits.
- *Sequential extensions.* Suppose $(\Psi_i \mid 1 \leq i \leq n)$ is a sequence of split and strongly independent addenda to a parchment $\mathcal{B}$. Then the extended parchment $\mathcal{B} + \Psi_1 + \cdots + \Psi_n$ is well formed and, moreover, the vertex of a limit in $\int sParch$ of the cospan representations of $\Psi_1, \ldots, \Psi_n$.

  Consequently, if Ψ_1, Ψ_2, ..., Ψ_n are split and strongly independent addenda to $\mathcal{B}$, then for any permutation γ of their indices, the extended parchments $\mathcal{B} + \Psi_1 + \cdots + \Psi_n$ and $\mathcal{B} + \Psi_{\gamma(1)} + \cdots + \Psi_{\gamma(n)}$ are isomorphic.

3. Results indicators

The results indicators that have been promised in the funding application have been significantly surpassed by the realised results indicators, as follows.

3.1. **Promised indicators.**

- Submission of 10 articles to journals, most of them indexed in Web of Science, with good Impact Factors.
- We estimated that by the end of the project at least 5 of the submissions will have been published while at most 5 articles will still be under evaluation. This had to do with the length of time required for a final decision in the submission process at such kind of journals.

3.2. **Realised indicators.**

- We have 7 articles already published as follows: [14, 11, 13, 15, 12, 8, 9]. Besides [11], all of them are Web of Science indexed. However, 'Logics' is a new journal but with a very prestigious editorial board. For instance, its Editor-in-Chief is the famous logician Val Goranko.
- Another 4 articles have been submitted to Web of Science indexed journals and are currently under evaluation. They are as follows: [1, 7, 16, 17].
- Moreover, a Second Edition of [3] has been agreed with Springer. Two chapters of this book are based to a very large extent on the results of this project. These chapters are "Models with states" (based on the work on stratified institutions) and "Many-valued truth institutions" (based on the work on $\mathcal{L}$-institutions). It is common knowledge that dissemination through monographs is stronger than dissemination through articles.

4. Estimated impact of the results

The estimated impact of the project's result are analysed in a broader context of the scientific domain, by focusing on: (1) the potential to significantly advance the knowledge of the field, by introducing new concepts or approaches and by opening new areas or research directions (if applicable); (2) the potential impact of the project and / or of the applied research directions explored in the project (if applicable) on the scientific, social, economic or cultural environment.

- The results will contribute to *maturing non-classical model theories* by using advanced mathematical techniques. While first-order model theory has reached a high degree of scientific maturity, this is not yet the case with non-classical model theories. The results of this project bring the scientific level of the latter one step closer to that of the former. Moreover, they open up a path characterised by abstraction and axiomatisation, to a further development of non-classical model theory.
- The results of this project will firmly *establish non-classical institutional model theory* as a new research direction, which will be long lived because the completion of the objectives of the project by no means do exhaust

the potential of this research direction; on the contrary, it will probably rise a lot of new interesting and relevant research questions.

- In model theory, the axiomatic method based on categorical abstraction will gain momentum due to the results obtained within this project. These may serve as an example of what can be achieved with this method. On the other hand, they also provide a body of techniques supporting the method.
- It will have a strong *foundational impact on several applied areas*, especially related to computing science which, as past experience tells us, is an avid consumer of such kind of foundations:
  - Models with states, which are the central piece in the theory of stratified institutions, are very relevant for all areas of computing science dealing with *dynamics of systems*.
  - Many-valued truth – central to $\mathcal{L}$-institutions – plays an important role in *fuzzy engineering*. An important chapter of this is soft AI. Our results in $\mathcal{L}$-institution theory, such as initial semantics and translation structures, will provide strong foundational support for *fuzzy specification and programming*.
  - The theory of 3/2-institutions has the potential to become the foundational structure for concept blending in *computational creativity*, a modern branch of AI. This is a badly needed development at the current stage of soft AI as the creation of new meaningful concepts seems to be inherently beyond the machine learning paradigm. And without this capability the AI grand ideal is problematic.

4.1. **The most significant result.** This may be the development of the theory of graded interpolation, reported in [13]. There are at least two reasons supporting this choice.

- The importance of interpolation for reasoning, in general, especially computer-based formal reasoning. In particular this study is done in the context of approximate reasoning, which is an important topic in soft AI. In this area of research, our work has brought-in interpolation in a mathematically solid way.
- Its mathematical finesse, which is very apparent especially in the axiomatic investigations of the dependencies between graded interpolation, on the one side, and Robinson consistency and definability, on the other side. The aspect of mathematical elegance is highly relevant for any mathematical project.

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