Some remarks on the infinite-variate prediction II

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1 Introduction

In a previous paper [23], a way to extend the study of finite multivariate prediction problem to infinite-variate case was presented, also the way how the difficulties to formulate the prediction problems in infinite-variate case was circumvented (see [10]). For stationary processes a complete analysis was done and a Wiener filter for prediction was given, based on a factorization theorem which extends the classical Lowdenslager–Sz.-Nagy–Foias factorization theorem. Also a generalized Wold decomposition was used. The main tool which permited an operatorial setting of the problem was an algebraic embedding of an arbitrary right $\mathcal{L}(\mathcal{E})$-module $\mathcal{H}$, which is the state space, into $\mathcal{L}(\mathcal{E},\mathcal{K})$, where, as usually by $\mathcal{L}(\mathcal{E})$ was denoted the C^*-algebra of linear bounded operators on the complex separable Hilbert space $\mathcal{E}$, and by $\mathcal{L}(\mathcal{E},\mathcal{K})$ the set of all linear bounded operators between the Hilbert spaces $\mathcal{E}$ and $\mathcal{K}$. This paper continues the presentation, extended for some nonstationary cases, especially for the periodically corelated case.

For the beginning, let us remember some necessary preliminaries. By an *action* of $\mathcal{L}(\mathcal{E})$ on $\mathcal{H}$ we mean the map $\mathcal{L}(\mathcal{E})\times\mathcal{H}$ into $\mathcal{H}$ given by $Ah := hA$ in the sense of the right $\mathcal{L}(\mathcal{E})$-module $\mathcal{H}$. We are writting Ah instead of hA to respect the classical notations from the scalar case. A *correlation* of the action of $\mathcal{L}(\mathcal{E})$ on $\mathcal{H}$ is a map Γ from $\mathcal{H}\times\mathcal{H}$ into $\mathcal{L}(\mathcal{E})$ having the properties:

(i) $\Gamma[h,h]\geq 0,$ and $\Gamma[h,h]=0$ implies $h=0$;

(ii) $\Gamma[h,g]^*=\Gamma[g,h]$;

(iii) $\Gamma[h,Ag]=\Gamma[h,g]A$.

In many proofs it is very useful the formula

$$\Gamma\Big[\sum_i A_i h_i, \sum_j B_j g_j\Big] = \sum_{i,j} A_i^*\Gamma[h_i,g_j]B_j$$

obtained by (ii) and (iii) for finite sums of actions of $\mathcal{L}(\mathcal{E})$ on $\mathcal{H}$.

A triplet $\{\mathcal{E},\mathcal{H},\Gamma\}$ defined as above was called a *correlated action* of $\mathcal{L}(\mathcal{E})$ on $\mathcal{H}$.

By the fact that generally in $\mathcal{H}$ we have no topology, the prediction subsets, such as past and present, future, etc., can not be seen as closed subspaces, therefore the powerful tool of the usual orthogonal projection can not be directly used.

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An example of correlated action can be constructed as follows. Take as the right $\mathcal{L}(\mathcal{E})$-module $\mathcal{H} = \mathcal{L}(\mathcal{E}, \mathcal{K})$ – the space of the linear bounded operators from $\mathcal{E}$ into $\mathcal{K}$, where $\mathcal{E}$ and $\mathcal{K}$ are Hilbert spaces. An action of $\mathcal{L}(\mathcal{E})$ on $\mathcal{L}(\mathcal{E}, \mathcal{K})$ is given if we consider $AV := VA$ for each $A \in \mathcal{L}(\mathcal{E})$ and $V \in \mathcal{L}(\mathcal{E}, \mathcal{K})$. It is easy to see that $\Gamma[V_1, V_2] = V_1^* V_2$ is a correlation of the action of $\mathcal{L}(\mathcal{E})$ on $\mathcal{L}(\mathcal{E}, \mathcal{K})$, and the triplet $\{\mathcal{E}, \mathcal{L}(\mathcal{E}, \mathcal{K}), \Gamma\}$ is a correlated action (the *operator model*). It was proved [14] that any abstract correlated action $\{\mathcal{E}, \mathcal{H}, \Gamma\}$ can be embedded into the operator model. Namely, there exists an algebraic embedding $h \to V_h$ of $\mathcal{H}$ into $\mathcal{L}(\mathcal{E}, \mathcal{K})$, where $\mathcal{K}$ is obtained as the Aronsjain reproducing kernel Hilbert space given by a positive definite kernel obtained from the correlation Γ. The generators of $\mathcal{K}$ are elements of the form $\gamma_{(a,h)} : \mathcal{E} \times \mathcal{H} \to \mathbb{C}$, where $\gamma_{(a,h)}(b, g) = \langle \Gamma[g, h]a, b \rangle_{\mathcal{E}}$ and the embedding $h \to V_h$ is given by $V_h a = \gamma_{(a,b)}$.

Due to such an embedding of any correlated action $\{\mathcal{E}, \mathcal{H}, \Gamma\}$ into the operator model, prediction problems can be formulated and solved using operator techniques. In the particular case when the embedding $h \to V_h$ is onto, the correlated action $\{\mathcal{E}, \mathcal{H}, \Gamma\}$ is caled a *complete correlated action*.

In the following the Hilbert space $\mathcal{K}$ uniquely attached to the correlated action $\{\mathcal{E}, \mathcal{H}, \Gamma\}$ will be called the *measuring space* of the correlated action. The name is justified by the fact that having a state h in the state space $\mathcal{H}$, what we can measure is the element $V_h a$ from the Hilbert space $\mathcal{K}$. In prediction problems we are inerested in measuring the closeness between two states, and this fact is not possible to be directely made in the state space $\mathcal{H}$ which is only a right $\mathcal{L}(\mathcal{E})$-module, but it is possible to be done in the measuring space $\mathcal{K}$, and must be interpreted in $\mathcal{H}$. So, we need to have the possibility to "interpret" each element from $\mathcal{K}$ in terms of the state space $\mathcal{H}$. This fact implies a completeness condition imposed to the algebraic imbedding of $\mathcal{H}$ into $\mathcal{L}(\mathcal{E}, \mathcal{K})$. In this paper most of properties are analysed in the complete correlated case.

2 Periodically Γ-correlated processes

In the previous part of this study stationary processes was considered. Here the non-stationary case of a periodically Γ-correlated process is presented and a linear filter for prediction is obtained. To do this, some usefull tools from the stationary prediction, as the imbedding in an operatorial model, or an appropriate "orthogonal" projection, must be extended to the T-variate case.

A process (f_t) from the right $\mathcal{L}(\mathcal{E})$-module $\mathcal{H}$ is periodically Γ-correlated if there exists a positive T such that $\Gamma[f_{s+T}, f_{t+T}] = \Gamma[f_s, f_t]$. In order to make a study of such a process, firstly the cartesian product of T copies of the right $\mathcal{L}(\mathcal{E})$-module $\mathcal{H}$ is considered

$$\mathcal{H}^T = \mathcal{H} \times \mathcal{H} \times \cdots \times \mathcal{H}. \tag{2.1}$$

An element X of $\mathcal{H}^T$ will be seen as a line vector $(h_1, \ldots, h_T)$. On $\mathcal{H}^T$ it is possible to have the action of $\mathcal{L}(\mathcal{E})$ on the components, with the same operator $A \in \mathcal{L}(\mathcal{E})$, or on each component with a different $A_i \in \mathcal{L}(\mathcal{E})$. Also we can consider the action of $\mathcal{L}(\mathcal{E})^{T\times T}$ on $\mathcal{H}^T$, taking for each matrix $A = (A_{ij})_{i,j=1}^T$from $\mathcal{L}(\mathcal{E})^{T\times T}$

$$A(h_1, \ldots, h_T) := (h_1, \ldots, h_T)A \tag{2.2}$$

in the sense of the right module. It is easy to see that $\mathcal{H}^T$ is an $\mathcal{L}(\mathcal{E})^{T\times T}$-right module and the action of $\mathcal{L}(\mathcal{E})$ on $\mathcal{H}^T$ is a particular case of the action of $\mathcal{L}(\mathcal{E})^{T\times T}$ on $\mathcal{H}^T$, taking the particular case of diagonal matrices with the same operator, or different operators on the diagonal.

Having the action of $\mathcal{L}(\mathcal{E})^{T\times T}$ on $\mathcal{H}^T$, various correlations of this action can be constructed. For our goal we are interested in the following two operatorial correlations on $\mathcal{H}^T$, namely:

$$\Gamma_1[X,Y] = \sum_{k=0}^{T-1} \Gamma[x_k, y_k], \tag{2.3}$$

where $X = (x_0, x_1, \ldots x_{T-1})$, $\quad Y = (y_0, y_1, \ldots, y_{T-1})$, and

$$\Gamma_T[X,Y] = \Big(\Gamma[x_i, y_j]\Big)_{i,j\in\{0,1,\ldots,T-1\}}. \tag{2.4}$$

Taking account of (2.2) it is easy to verify the properties (i)-(iii) of a correlation of the action of $\mathcal{L}(\mathcal{E})$, respectively of $\mathcal{L}(\mathcal{E})^{T\times T}$, for Γ_1 and Γ_T.

So, starting with the correlated action $\{\mathcal{E}, \mathcal{H}, \Gamma\}$ of $\mathcal{L}(\mathcal{E})$ on $\mathcal{H}$, we obtain the correlated actions $\{\mathcal{E}, \mathcal{H}^T, \Gamma_1\}$ and $\{\mathcal{E}, \mathcal{H}^T, \Gamma_T\}$ of $\mathcal{L}(\mathcal{E})$, respectively $\mathcal{L}(\mathcal{E})^{T\times T}$, on $\mathcal{H}^T$. As a remark, the correlation Γ_1 is the trace of the matrix given by the correlation Γ_T.

Another $\mathcal{L}(\mathcal{E})^{T\times T}$-right module which will be considered in the study of periodically correlated processes will be $(\mathcal{H}^T)^T$ with an appropriate correlation of the action of $\mathcal{L}(\mathcal{E})^{T\times T}$.

If we consider an arbitrary process $\{X_n\}$ in $\mathcal{H}^T$, the attached prediction submodules have the form:

$$H_n^X = \Big\{ \sum_k A_k X_k; \quad A_k \in \mathcal{L}(\mathcal{E})^{T\times T}, \;\; k \le n \Big\} \qquad \text{(the past)}, \tag{2.5}$$

$$\tilde{H}_n^X = \Big\{ \sum_k A_k X_k; \quad A_k \in \mathcal{L}(\mathcal{E})^{T\times T}, \;\; k > n \Big\} \qquad \text{(the future)}, \tag{2.6}$$

$$H_{-\infty}^X = \bigcap_n H_n^X \qquad \text{(the remote past)}, \tag{2.7}$$

$$H_\infty^X = \Big\{ \sum_k A_k X_k; \quad A_k \in \mathcal{L}(\mathcal{E})^{T\times T} \Big\} \quad \text{(space of the process)}. \tag{2.8}$$

To a periodically Γ-correlated process $\{f_n\}_{n\in\mathbb{Z}}$ from $\mathcal{H}$ we can attach at least two types of stationary processes in $\mathcal{H}^T$ as follows:

1) taking sequences of consecutive T terms starting with f_n, namely the line vector

$$X_n = \Big(f_n, f_{n+1}, \ldots, f_{n+T-1}\Big), \tag{2.9}$$

or

2) taking consecutive blocks of length T

$$X_n^T = \Big(f_{nT}, f_{nT+1}, \ldots, f_{nT+T-1}\Big). \tag{2.10}$$

It is easy to see that $\{X_n\}$ and $\{X_n^T\}$ are respectively Γ_1 and Γ_T stationary processes in $\mathcal{H}^T$. From prediction point of view and the study of the periodically Γ-correlated process $\{f_n\}_{n\in\mathbb{Z}}$ from $\mathcal{H}$, the Γ_1-correlation of $\{X_n\}$ and Γ_T-correlation of $\left\{X_n^T\right\}$ are equivalent, as can be seen from the following

Proposition 2.1. *Let $\{f_n\}_{n\in\mathbb{Z}}$ from $\mathcal{H}$, an integer $T \geq 2$ and $\{X_n\}$, $\{X_n^T\}$ defined by* (2.9) *and* (2.10)*. The following are equivalent:*

(i) $\{f_n\}$ is periodically Γ-correlated in $\mathcal{H}$, with the period T.

(ii) $\{X_n\}$ is stationary Γ_1-correlated in $\mathcal{H}^T$.

(iii) $\{X_n^T\}$ is stationary Γ_T-correlated in $\mathcal{H}^T$.

Proof. $(i) \Rightarrow (ii)$. Having $\{f_n\}$ periodically Γ-correlated , i.e., $\Gamma[f_n, f_m] = \Gamma[f_{n+T}, f_{m+T}]$, it follows that

$$\Gamma_1[X_n, X_m] = \sum_{k=0}^{T-1} \Gamma[f_{n+k}, f_{m+k}] = \Gamma[f_n, f_m] + \sum_{k=1}^{T-1} \Gamma[f_{n+k}, f_{m+k}] =$$

$$= \Gamma[f_{n+T}, f_{m+T}] + \sum_{k=1}^{T-1} \Gamma[f_{n+k}, f_{m+k}] = \sum_{k=1}^{T} \Gamma[f_{n+k}, f_{m+k}] =$$

$$= \sum_{j=0}^{T-1} \Gamma[f_{(n+1)+j}, f_{(m+1)+j}] = \Gamma_1[X_{n+1}, X_{m+1}].$$

Therefore $\{X_n\}_{n\in\mathbb{Z}}$ is stationary Γ_1 -correlated in $\mathcal{H}^T$.

Conversely, $(ii) \Rightarrow (i)$. The process $\{X_n\}$ being stationary Γ_1 -correlated in $\mathcal{H}^T$ we have succesively:

$$\Gamma_1[X_{n+1}, X_{m+1}] = \Gamma_1[f_n, f_m]$$

$$\sum_{k=0}^{T-1} \Gamma[f_{n+1+k}, f_{m+1+k}] = \sum_{k=0}^{T-1} \Gamma[f_{n+k}, f_{m+k}]$$

$$\sum_{j=1}^{T} \Gamma[f_{n+j}, f_{m+j}] = \sum_{k=0}^{T-1} \Gamma[f_{n+k}, f_{m+k}]$$

$$\sum_{j=1}^{T-1} \Gamma[f_{n+j}, f_{m+j}] + \Gamma[f_{n+T}, f_{m+T}] = \Gamma[f_n, f_m] + \sum_{k=1}^{T-1} \Gamma[f_{n+k}, f_{m+k}].$$

It follows that

$$\Gamma[f_{n+T}, f_{m+T}] = \Gamma[f_n, f_m],$$

i.e., $\{f_n\}_{n\in\mathbb{Z}}$ is periodically Γ-correlated in $\mathcal{H}$.

$(i) \Rightarrow (iii)$. Taking account that $\{f_n\}$ from $\mathcal{H}$ is periodically Γ-correlated with the period T, we have

$$\Gamma_T[X_n^T, X_m^T] = \Big(\Gamma[f_{nT+i}, f_{mT+j}]\Big)_{i,j\in\{0,1,\dots,T-1\}} =$$

$$= \Big(\Gamma[f_{nT+i+T}, f_{mT+j+T}]\Big)_{i,j} = \Big(\Gamma[f_{(n+1)T+i}, f_{(m+1)T+j}]\Big)_{i,j} =$$

$$= \Gamma_T[X_{n+1}^T, X_{m+1}^T]$$

and $\{X_n^T\}$ is stationary Γ_T-correlated in $\mathcal{H}^T$.

$(iii) \Rightarrow (i)$. If $\{X_n^T\}$ is stationary Γ_T-correlated in $\mathcal{H}^T$, then for each n, m in $\mathbb{Z}$ we have

$$\Gamma_T[X_n^T, X_m^T] = \Gamma_T[X_{n+1}^T, X_{m+1}^T],$$

i.e., the matrix equality

$$\Big(\Gamma[f_{nT+i}, f_{mT+j}]\Big)_{0\le i,j\le T-1} = \Big(\Gamma[f_{(n+1)T+i}, f_{(m+1)T+j}]\Big)_{0\le i,j\le T-1}.$$

It follows that for each $n, m \in \mathbb{Z}$ and $0 \le i, j \le T-1$ we have

$$\Gamma[f_{nT+i}, f_{mT+j}] = \Gamma[f_{nT+i+T}, f_{mT+j+T}]. \tag{2.11}$$

Taking first $n = m = 0$ in (2.11) obtain that for $0 \le i, j \le T-1$

$$\Gamma[f_i, f_j] = \Gamma[f_{i+T}, f_{j+T}].$$

Then for various other combinations of n and m, denotting $nT+i = p \in \mathbb{Z}$ and $mT+j = q \in \mathbb{Z}$, it follows that for each $p, q \in \mathbb{Z}$ we have

$$\Gamma[f_i, f_j] = \Gamma[f_{i+T}, f_{j+T}],$$

i.e., the process $\{f_n\} \in \mathcal{H}$ is periodically Γ-correlated. □

For the study of the attached line vectors stationary processes from $\mathcal{H}^T$, the corresponding operator model is necessary.

Proposition 2.2. *There exists a unique (up to a unitary equivalence) imbedding $X \to W_X$ of $\mathcal{H}^T$ into $\mathcal{L}(\mathcal{E}, \mathcal{K}^T)$ such that*

$$\Gamma_1[X, Y] = W_X^* W_Y = \sum_{i=1}^{T} V_{x_i}^* V_{y_i} \tag{2.12}$$

where $X = (x_1, \ldots, x_T)$, $Y = (y_1, \ldots, y_T)$.

The subset $\{W_X a; X \in \mathcal{H}^T, a \in \mathcal{E}\}$ is dense in $\mathcal{K}^T$.

Proof. Taking account of the imbedding $h \to V_h$ of $\mathcal{H}$ into $\mathcal{L}(\mathcal{E}, \mathcal{K})$ given by $V_h a = \gamma(a, h)$ and $\Gamma[h_1, h_2] = V_{h_1}^* V h_2$, if we take

$$W_X = \big(V_{x_1}, V_{x_2}, \ldots, V_{x_T}\big), \tag{2.13}$$

then for $a, b \in \mathcal{E}$ we have $W_X a = \big(\gamma_{(a,x_1)}, \ldots, \gamma_{(a,x_T)}\big)$ and

$$\Big(\Gamma_1[X, Y]a, b\Big)_{\mathcal{E}} = \Big(\sum_{i=1}^{T} \Gamma[x_i, y_i]a, b\Big)_{\mathcal{E}} = \Big(\sum_{i=1}^{T} V_{x_i}^* V_{y_i} a, b\Big)_{\mathcal{E}}.$$

By the fact that the usual scalar product on $\mathcal{K}^T$ is the sum of scalar products on components, it follows that

$$\Big(W_X^* W_Y a, b\Big)_{\mathcal{E}} = \Big(W_Y a, W_X b\Big)_{\mathcal{K}^T} = \sum_{i=1}^{T} \Big(\gamma_{(a,y_i)}, \gamma_{(b,x_i)}\Big)_{\mathcal{K}} =$$

$$= \sum_{i=1}^{T} \Big(V_{y_i} a, V_{x_i} b\Big)_{\mathcal{K}} = \sum_{i=1}^{T} \Big(V_{x_i}^* V_{y_i} a, b\Big)_{\mathcal{E}}$$

and (2.12) is proved. Also,

$$\|W_X a\|^2_{\mathcal{K}^T} = \Big(W_X a, W_X a\Big)_{\mathcal{K}^T} = \sum_{i=1}^{T}\Big(V^*_{x_i}V_{x_i}a, a\Big)_{\mathcal{E}} \leq \sum_{i=1}^{T}\|V_{x_i}\|^2 \cdot \|a\|,$$

and W_X is a linear bounded operator from $\mathcal{E}$ into $\mathcal{K}^T$.

If we consider another imbedding W' of $\mathcal{H}^T$ into $\mathcal{L}(\mathcal{E}, \mathcal{K}^T)$ having the property (2.12), then, if we take $\Phi : \mathcal{K}^T \to \mathcal{K}^T$ given by

$$\Phi W'_X a = W_X a,$$

we have

$$\|\Phi W'_X\|^2_{\mathcal{K}^T} = \|W_X a\|^2_{\mathcal{K}^T} = \Big(\sum_{i=1}^{T} V^*_{x_i}V_{x_i}a, a\Big)_{\mathcal{E}} = \|W'_X a\|^2,$$

then for $a, b \in \mathcal{E}$, $W_X a = \Big(\gamma_{(a,x_1)}, \ldots, \gamma_{(a,x_T)}\Big)$ and

$$\Big(\Gamma_1[X,Y]a, b\Big)_{\mathcal{E}} = \Big(\sum_{i=1}^{T}\Gamma[x_i, y_i]a, b\Big)_{\mathcal{E}} = \Big(\sum_{i=1}^{T} V^*_{x_i}V_{y_i}a, b\Big)_{\mathcal{E}}.$$

Also,

$$\|W_X a\|^2_{\mathcal{K}^T} = \big(W_X a, W_X a\big)_{\mathcal{K}^T},$$

i.e., Φ is a unitary operator on $\mathcal{K}^T$. So, the imbedding $X \to W_X$ of $\mathcal{H}^T$ into $\mathcal{L}(\mathcal{E}, \mathcal{K}^T)$ is unique (up to a unitary equivalence). □

Now we are able to introduce an appropriate shift for the periodically Γ_T-correlated process $\{X^T_n\}$. As we have seen, to each periodically correlated process $\{f_n\}$ from $\mathcal{H}$ we can attach its T-shift, a unitary operator U_f on $\mathcal{K}^f_\infty$ such that $U_f V_{f_n} = V_{f_{n+T}}$, where $h \to V_h$ is the usual imbedding of $\mathcal{H}$ into $\mathcal{L}(\mathcal{E}, \mathcal{K})$. Then it is easy to see that the unitary operator on $(\mathcal{K}^f_\infty)^T$ defined by

$$U_T(V_{f_1}, V_{f_2}, \ldots, V_{f_T}) = (U_f V_{f_1}, U_f V_{f_2}, \ldots, U_f V_{f_T}) \tag{2.14}$$

is the shift operator attached to the statoinary Γ_T-correlated process $\{X^T_n\}$ defined by (2.10). Indeed,

$$\begin{aligned} U_T W_{X^T_n} &= U_T(V_{f_{nT}}, V_{f_{nT+1}}, \ldots, V_{f_{nT+T-1}}) = \\ &= (U_f V_{f_{nT}}, U_f V_{f_{nT+1}}, \ldots, U_f V_{f_{nT+T-1}}) = \\ &= (V_{f_{nT+T}}, V_{f_{nT+1+T}}, \ldots, V_{f_{nT+T-1+T}}) = \\ &= (V_{f_{(n+1)T}}, V_{f_{(n+1)T+1}}, \ldots, V_{f_{(n+1)T+T-1}}) = W_{X^T_{n+1}}. \end{aligned}$$

It follows that

$$W_{X^T_n} = U^n W_{X^T_0}.$$

For prediction purposes, we are interested to find the best estimation of an element from $\mathcal{H}^T$ with elements from a subset $\mathcal{M} = \mathcal{H}_1 \times \mathcal{H}_2 \times \cdots \times \mathcal{H}_T \subset \mathcal{H}^T$. To do this, we need the following Proposition.

PROPOSITION 2.3. *Let $\mathcal{M}$ be a subset of $\mathcal{H}^T$. If we take*

$$\mathcal{K}_1^T = \bigvee_{Z \in \mathcal{M}} W_Z \mathcal{E}, \tag{2.15}$$

then for each $X \in \mathcal{H}^T$ there exists a unique element X' in $\mathcal{H}^T$ such that for each $a \in \mathcal{E}$ we have

$$W_{X'} a \in \mathcal{K}_1^T \text{ and } W_{X-X'} a \in (\mathcal{K}_1^T)^\perp. \tag{2.16}$$

Moreover,

$$\Gamma_1[X - X', X - X'] = \inf_{Z \in \mathcal{M}} \Gamma_1[X - Z, X - Z], \tag{2.17}$$

where the infimum is taken in the set of all positive operators from $\mathcal{L}(\mathcal{E})$.

Proof. Let $W_{X'} = P_{\mathcal{K}_1^T} W_X$ where $P_{\mathcal{K}_1^T}$ is the orthogonal projection of $\mathcal{K}^T$ on its closed subset $\mathcal{K}_1^T$. For each $a \in \mathcal{E}$ we have $W_{X'} a \in \mathcal{K}_1^T$ and

$$\begin{aligned} W_{X-X'} a &= \big(\gamma_{(a, x_1 - x_1')}, \ldots, \gamma_{(a, x_T - x_T')}\big) = \big(\gamma_{(a,x_1)} - \gamma_{(a,x_1')}, \ldots, \gamma_{(a,x_T)} - \gamma_{(a,x_T')}\big) \\ &= W_X a - W_{X'} a = W_X a - P_{\mathcal{K}_1^T} W_X a = (I - P_{\mathcal{K}_1^T}) W_X a \in (\mathcal{K}_1^T)^\perp. \end{aligned}$$

If there exists X'' with the property (2.16), then for each $a \in \mathcal{E}$ we have $W_X a = W_{X''} a + W_{X-X''} a$. It follows that $W_{X''} a = P_{\mathcal{K}_1^T} W_X a = W_{h'} a$, i.e., $X'' = X'$.

Moreover,

$$\Big(\Gamma_1[X - X', X - X'] a, a\Big) = \Big(W^*_{X-X'} W_{X-X'} a, a\Big) = \|W_{X-X'} a\|^2 =$$

$$= \|(I - P_{\mathcal{K}_1^T}) W_X a\|^2 = \inf_{K \in \mathcal{K}_1^T} \|W_X a - K\|^2 = \inf_{\sum_1^n W_{X_j} a_j} \|W_X a - \sum_{i=1}^n W_{X_j} a_j\|^2 =$$

$$= \inf_{\sum_1^n W_{X_j} a_j} \|W_X a - W_{\sum_{j=1}^n X_j} a_j\|^2 = \inf \Big(\Gamma_1[X - \sum_{j=1}^n A_j X_j, X - \sum_{j=1}^n A_j X_j] a, a\Big) =$$

$$= \inf_{Z \in \mathcal{M}} \Big(\Gamma_1[X - Z, X - Z] a, a\Big),$$

where for each finite systems $\{a_1, \ldots, a_n\}$ of elements from $\mathcal{E}$ we choose $A_1, \ldots, A_n \in \mathcal{L}(\mathcal{E})$ such that $A_j a = a_j, \quad j = 1, 2, \ldots, n.$ □

If we denote by $\mathcal{P}_\mathcal{M}$ the endomorphism of $\mathcal{H}^T$ defined by $\mathcal{P}_\mathcal{M} X = X'$,then we have

$$W_{\mathcal{P}^2_\mathcal{M} X} = W_{\mathcal{P}_\mathcal{M} X'} = P_{\mathcal{K}_1^T} W_{X'} = P^2_{\mathcal{K}_1^T} W_X = P_{\mathcal{K}_1^T} W_X = W_{\mathcal{P}_\mathcal{M} X}$$

and also,

$$\begin{aligned} \Gamma_1[\mathcal{P}_\mathcal{M} X, Y] &= W^*_{\mathcal{P}_\mathcal{M} X} W_Y = (P_{\mathcal{K}_1^T} W_X)^* W_Y = \\ &= W_X^* P_{\mathcal{K}_1^T} W_Y = W_X^* W_{\mathcal{P}_\mathcal{M} Y} = \Gamma_1[X, \mathcal{P}_\mathcal{M} Y]. \end{aligned}$$

Hence

$$\mathcal{P}^2_\mathcal{M} X = \mathcal{P}_\mathcal{M} X$$

and

$$\Gamma_1[\mathcal{P}_\mathcal{M} X, Y] = \Gamma_1[X, \mathcal{P}_\mathcal{M} Y].$$

Therefore we can interpret $\mathcal{P}_{\mathcal{M}}$ as an "orthogonal" projection on $\mathcal{M}$, and this will be called the Γ_1-*orthogonal projection* of $\mathcal{H}^T$ on $\mathcal{M} \subset \mathcal{H}^T$.

On the other part, let us remark that we can identify $\mathcal{H}$ as the subset $\mathcal{N} = \mathcal{H} \times \{0\} \times \cdots \times \{0\}$ in $\mathcal{H}^T$. From (2.13) it follows that $W_{(h,0,\ldots,0)} = (V_h, 0, \ldots, 0)$ and the corresponding subspace from $\mathcal{K}^T$ for $\mathcal{N}$ will be

$$\mathcal{K}^T_{\mathcal{N}} = \bigvee_{Z \in \mathcal{M}} W_Z \mathcal{E} = \mathcal{K} \times \{0\} \times \cdots \times \{0\} \subset \mathcal{K}^T.$$

Considering the stationary Γ_1-correlated process $\{X_n\} \subset \mathcal{H}^T$ given by (2.9) we have

$$P_{\mathcal{K}^T_{\mathcal{N}}} W_{X_n} = P_{\mathcal{K}^T_{\mathcal{N}}} (V_{f_n}, \ldots, V_{f_{n+T-1}}) = V_{f_n}$$

and follows that f_n can be identified with $f_n = \mathcal{P}_{\mathcal{N}} X_n$, i.e. the periodically Γ-correlated process from $\mathcal{H}$ admits a stationary Γ_1-correlated *dilation* $\{X_n\}$ in $\mathcal{H}^T$.

Using the imbedding $X \to W_X$ of $\mathcal{H}^T$ into $\mathcal{L}(\mathcal{E}, \mathcal{K})^T$, the corresponding subspaces of the process from $\mathcal{K}^T$ have the form:

$$K^X_n = \bigvee_{k \leq n} W_{X_k} \mathcal{E} \quad \text{(past and present)}, \tag{2.18}$$

$$\tilde{K}^X_n = \bigvee_{k > n} W_{X_k} \mathcal{E} \quad \text{(future)}, \tag{2.19}$$

$$K^X_{-\infty} = \bigcap_{n} K^X_n \quad \text{(remote past)}, \tag{2.20}$$

$$K^X_{\infty} = \bigvee_{-\infty}^{\infty} W_{X_n} \mathcal{E} \quad \text{(the space of the process)}. \tag{2.21}$$

In the following the right $\mathcal{L}(\mathcal{E})^{T \times T}$ -module $\left[\mathcal{L}(\mathcal{E}, \mathcal{K})^T\right]^T$ will be considered, whose elements will be written with capital bold face characters, to avoid the confussion with the elements from $\mathcal{L}(\mathcal{E}, \mathcal{K})^T$ which are only capital letters.

An element

$$\mathbf{Z} = (W_1, \ldots, W_T)$$

is a line vector with $W_k \in \mathcal{L}(\mathcal{E}, \mathcal{K})^T$, and

$$W_k = (V_k^1, V_k^2, \ldots, V_k^T)$$

with $V_k^j \in \mathcal{L}(\mathcal{E}, \mathcal{K})$.

A correlation of the action of $\mathcal{L}(\mathcal{E})^{T \times T}$ on $\left[\mathcal{L}(\mathcal{E}, \mathcal{K})^T\right]^T$ will be done by

$$\Gamma_T[\mathbf{Z}_1, \mathbf{Z}_2] = \Big(\Gamma_1[W_{1j}, W_{2k}]\Big)^T_{j,k=1}, \tag{2.22}$$

where

$$W_{1j} = (V^1_{1j}, \ldots, V^T_{1j})$$

and

$$W_{2k} = (V^1_{2k}, \dots, V^T_{2k})$$

are from $\mathcal{L}(\mathcal{E},\mathcal{K})^T$.

Let $\{f_n\}$ be an arbitrary process in $\mathcal{H}$ and E be the operator of multiplying by $e^{-2\pi i/T}$. Taking account of the construction of the measuring space $\mathcal{K}$ and the action of $\mathcal{L}(\mathcal{E})$ on $\mathcal{H}$ (respectively $\mathcal{L}(\mathcal{E},\mathcal{K})$) in the meaning of the right $\mathcal{L}(\mathcal{E})$-module, to each $\{f_n\}_{n\in\mathbb{Z}}$ from $\mathcal{H}$ we can attach T sequences in $\mathcal{H}^T$ of the form

$$X^k_n = \Big(E^{kn} f_n, E^{k(n+1)} f_{n+1}, \dots, E^{k(n+T-1)} f_{n+T-1}\Big), \tag{2.23}$$

where $k \in \{0, 1, \dots, T-1\}$.

Using the imbeddings $h \longrightarrow V_h$ and $X \longrightarrow W_X$ of $\mathcal{H}$ into $\mathcal{L}(\mathcal{E},\mathcal{K})$, respectively $\mathcal{H}^T$ into $\mathcal{L}(\mathcal{E},\mathcal{K})^T$, we obtain T sequences in $\mathcal{L}(\mathcal{E},\mathcal{K})^T$ taking

$$Z^k_n = W_{X^k_n} \quad ; \quad k \in \{0, 1, \dots, T-1\}, \tag{2.24}$$

and a sequence in $\Big[\mathcal{L}(\mathcal{E},\mathcal{K})^T\Big]^T$, if we take

$$\mathbf{Z}_n = \tfrac{1}{\sqrt{T}}\Big(Z^0_n, Z^1_n, \dots, Z^{T-1}_n\Big). \tag{2.25}$$

Based on Gladyshev's Theorem, in the following a linear predictor for periodically Γ-correlated processes in complete correlated actions will be obtained, generalizing the scalar case [9]. To do this, firstly we will see that the process attached by (2.25) to a periodically Γ-correlated process $\{f_n\}$ from $\mathcal{H}$ is an explicit form of an attached stationary process.

Theorem 2.4. *The process $\{f_n\}$ from $\mathcal{H}$ is periodically Γ-correlated with period T if and only if $\{\boldsymbol{Z}_n\}_{n\in\mathbb{Z}}$ from $\Big[\mathcal{L}(\mathcal{E},\mathcal{K})^T\Big]^T$ attached by (2.25) is a stationary Γ_T-correlated process.*

Proof. If $\{f_n\}$ from $\mathcal{H}$ is periodically Γ-correlated with period T, then in each element of the matrix

$$\Gamma_T[\mathbf{Z}_m, \mathbf{Z}_n] = \Big(\tfrac{1}{T}\Gamma_1[Z^j_m, Z^k_n]\Big)^{T-1}_{j,k=0}$$

we have

$$\Gamma_1[Z^j_m, Z^k_n] = \Gamma_1[W_{X^j_m}, W_{X^k_n}] =$$

$$= \sum_{p=0}^{T-1} \Gamma[E^{j(m+p)} V_{f_{m+p}}, E^{k(n+p)} V_{f_{n+p}}] =$$

$$= \sum_{p=0}^{T-1} V^*_{f_{m+p}} V_{f_{n+p}} E^{-j(m+p)+k(n+p)} =$$

$$= V^*_{f_m} V_{f_n} E^{-jm+kn} + \sum_{p=1}^{T-1} V^*_{f_{m+p}} V_{f_{n+p}} E^{-j(m+p)+k(n+p)} =$$

$$= \Gamma[V_{f_m}, V_{f_n}] E^{-jm+kn} + \sum_{p=1}^{T-1} \Gamma[V_{f_{m+p}}, V_{f_{n+p}}] E^{-j(m+p)+k(n+p)} =$$

$$= \Gamma[V_{f_{m+T}}, V_{f_{n+T}}]E^{-j(m+T)+k(n+T)} + \sum_{p=1}^{T-1} \Gamma[V_{f_{m+p}}, V_{f_{n+p}}]E^{-j(m+p)+k(n+p)} =$$

$$= \sum_{p=1}^{T} \Gamma[V_{f_{m+p}}, V_{f_{n+p}}]E^{-j(m+p)+k(n+p)} =$$

$$= \sum_{s=0}^{T-1} \Gamma[V_{f_{m+s+1}}, V_{f_{n+s+1}}]E^{-j(m+s+1)+k(n+s+1)} =$$

$$= \sum_{s=0}^{T-1} \Gamma[E^{j(m+1+s)}V_{f_{m+1+s}}, E^{k(n+1+s)}V_{f_{n+1+s}}] =$$

$$= \Gamma_1[W_{X^j_{m+1}}, W_{X^k_{n+1}}] = \Gamma_1[Z^j_{m+1}, Z^k_{n+1}].$$

This implies that

$$\Gamma_T[\mathbf{Z}_m, \mathbf{Z}_n] = \Gamma_T[\mathbf{Z}_{m+1}, \mathbf{Z}_{n+1}],$$

i.e., the process $\{\mathbf{Z}_n\}_{n\in\mathbb{Z}}$ is stationary Γ_T-correlated in $\left[\mathcal{L}(\mathcal{E},\mathcal{K})^T\right]^T$.

Conversely, if $\{\mathbf{Z}_n\}_{n\in\mathbb{Z}}$ is stationary Γ_T-correlated, then each element of the Γ_T-correlation matrix verifies the relation

$$\tfrac{1}{T}\Gamma_1[Z^j_m, Z^k_n] = \tfrac{1}{T}\Gamma_1[Z^j_{m+1}, Z^k_{n+1}].$$

Taking the element corresponding to $j = k = 0$ we have

$$\Gamma_1[W_{X^0_m}, W_{X^0_n}] = \Gamma_1[W_{X^0_{m+1}}, W_{X^0_{n+1}}],$$

$$\sum_{p=0}^{T-1} \Gamma[V_{f_{m+p}}, V_{f_{n+p}}] = \sum_{s=0}^{T-1} \Gamma[V_{f_{m+1+s}}, V_{f_{n+1+s}}],$$

and follows (reducing the similar terms) that

$$\Gamma[V_{f_m}, V_{f_n}] = \Gamma[V_{f_{m+T}}, V_{f_{n+T}}],$$

i.e., the process $\{f_n\}_{n\in\mathbb{Z}}$ is periodically Γ-correlated with the period T. □

In the following we will see that for a periodically Γ-correlated process $\{f_n\}_{n\in\mathbb{Z}}$, the stationary Γ_T-correlated process $\{\mathbf{Z}_n\}_{n\in\mathbb{Z}}$ defined by (2.25) verifies the conditions from Gladyshev's Theorem. Indeed, taking account by the coefficients of the Γ_T-correlation matrix function, have the form

$$B_{jk}(t) = \tfrac{1}{T}\Gamma_1[Z^j_t, Z^k_0] = \tfrac{1}{T}\Gamma_1[W_{X^j_t}, W_{X^k_0}] =$$

$$= \tfrac{1}{T}\sum_{p=0}^{T-1} \Gamma[V_{f_{t+p}}, V_{f_p}]E^{-j(t+p)+kp} = \tfrac{1}{T}\sum_{p=0}^{T-1} \Gamma(t+p,p)E^{-j(t+p)+kp} =$$

$$= E^{-jt}\tfrac{1}{T}\sum_{p=0}^{T-1} B(p,t)E^{p(k-j)} = E^{-jt}B_{k-j}(t).$$

So,

$$B_{jk}(t) = B_{k-j}(t)\exp(2\pi ijt/T)$$

and the condition of Gladyshev's Theorem is verified.

According with the definition of the *past and present* given in the previous section, the past and present of the process $\{\mathbf{Z}_n\}_{n\in\mathbb{Z}}$ from $\left[\mathcal{L}(\mathcal{E},\mathcal{K})^T\right]^T$ will be a subspace from $(\mathcal{K}^T)^T$ given by

$$K_n^{\mathbf{Z}} = \bigvee_{k\le n} \mathbf{Z}_k\mathcal{E}, \tag{2.26}$$

where the elements are of the form

$$\mathbf{Z} = \sum_{k\le n} A_k\mathbf{Z}_k a_k, \tag{2.27}$$

while the action of $A_k \in \mathcal{L}(\mathcal{E})^{T\times T}$ being understand in the sense of the right $\mathcal{L}(\mathcal{E})^{T\times T}$ -module $\left[\mathcal{L}(\mathcal{E},\mathcal{K})^T\right]^T$.

Also for the process $\{\mathbf{Z}_n\}$ from $\left[\mathcal{L}(\mathcal{E},\mathcal{K})^T\right]^T$ another *"past and present"* denoted by $\mathcal{K}_n^{\mathbf{Z}}$ can be considered in $\mathcal{K}^T$ as the linear span of the finite sums of the form

$$\sum_{k\le n} A_k Z_k^j a_k \quad ; \ 0\le j\le T-1, \tag{2.28}$$

or:

$$\mathcal{K}_n^{\mathbf{Z}} = \bigvee_{k\le n} Z_k^j\mathcal{E}. \tag{2.29}$$

Due to the particular form of the stationary Γ_T-correlated process attached to a periodically Γ-correlated process, the geometry of the past and present spaces is given as in the following theorem.

THEOREM 2.5. *The past and present of $\{\boldsymbol{Z}_n\}_{n\in\mathbb{Z}}$ has the following structure*

$$K_n^{\boldsymbol{Z}} = \left(\mathcal{K}_n^{\boldsymbol{Z}}\right)^T \tag{2.30}$$

and

$$\mathcal{K}_n^{\mathbf{Z}} = \mathcal{K}_n^f \times \mathcal{K}_{n+1}^f \times \cdots \times \mathcal{K}_{n+T-1}^f. \tag{2.31}$$

Proof. . For each finite linear combination from $K_n^{\mathbf{Z}}$ we have in$(\mathcal{K}^T)^T$

$$\sum_{k\le n} A_k\mathbf{Z}_k a_k = \tfrac{1}{\sqrt{T}}\sum_{k\le n} A_k(Z_k^0, Z_k^1,\dots,Z_k^{T-1})a_k =$$

$$= \tfrac{1}{\sqrt{T}}\sum_{k\le n}\left(A_k^{ij}\right)_{i,j=0}^{T-1}(Z_k^0, Z_k^1,\dots,Z_k^{T-1})a_k =$$

$$= \tfrac{1}{\sqrt{T}}\sum_{k\le n}\Big(\sum_{j=0}^{T-1} Z_k^j A_k^{j0}a_k, \sum_{j=0}^{T-1} Z_k^j A_k^{j1}a_k,\dots,\sum_{j=0}^{T-1} Z_k^j A_k^{j,T-1}a_k\Big).$$

It follows that each component of the considered linear combination is of the form $\sum_{k\le n}\sum_{j=0}^{T-1} A_k^{ji} Z_k^j a_k$, i.e., belongs to $\mathcal{K}_n^{\mathbf{Z}}$. Therefore

$$K_n^{\mathbf{Z}} \subset (\mathcal{K}_n^{\mathbf{Z}})^T.$$

Conversely, each linear combination from $(\mathcal{K}_n^{\mathbf{Z}})^T$ has the form

$$\mathbf{Z} = (Z^0, Z^1, \ldots, Z^{T-1}),$$

where

$$Z^i = \sum_{k\le n}\sum_{j=0}^{T-1} A_k^{ji} Z_k^j a_k,$$

and follows that

$$\mathbf{Z} = \sum_{k\le n} A_k \mathbf{Z}_k a_k \in K_n^{\mathbf{Z}}.$$

Therefore

$$(\mathcal{K}_n^{\mathbf{Z}})^T \subset K_n^{\mathbf{Z}},$$

and the equality (2.30) is proved.

To prove (2.31), let see that from the definition (2.24) of a component Z_m^k, where $m \le n$, we have

$$Z_m^k a = W_{X_m^k} a = (E^{km} V_{f_m} a, E^{k(m+1)} V_{f_{m+1}} a, \ldots, E^{k(m+T-1)} V_{f_{m+T-1}} a)$$

as elements from $\mathcal{K}_n^f \times \mathcal{K}_{n+1}^f \times \cdots \times \mathcal{K}_{n+T-1}^f$. Therefore

$$\mathcal{K}_n^{\mathbf{Z}} \subset \mathcal{K}_n^f \times \mathcal{K}_{n+1}^f \times \cdots \times \mathcal{K}_{n+T-1}^f \subset \mathcal{K}^T.$$

Conversely, the linear combination of the form

$$Z = \sum_{j=0}^{T-1} E^{-j(m+k)} Z_m^j a,$$

where Z_m^j are the components of $\mathbf{Z}_m$, for each $m \le n$ and $0 \le k \le T-1$ has the form

$$Z = \sum_{j=0}^{T-1} W_{X_m^j} E^{-j(m+k)} a = \sum_{j=0}^{T-1} (V_{f_{m+l}} E^{j(m+l-j(m+k)} a)_{l=0}^{T-1} =$$

$$= \sum_{j=0}^{T-1} (V_{f_{m+l}} E^{j(l-k)} a)_{l=0}^{T-1} = (V_{fm+l} a \sum_{j=0}^{T-1} E^{j(l-k)})_{l=0}^{T-1} = (V_{fm+l} a \cdot T\delta_{lk})_{l=0}^{T-1}.$$

Therefore for each $0 \le k \le T-1$ we have that

$$\{0\} \times \cdots \times \{0\} \times \mathcal{K}_{n+k}^f \times \{0\} \times \cdots \times \{0\} \subset \mathcal{K}_n^{\mathbf{Z}},$$

and consequently

$$\mathcal{K}_n^f \times \mathcal{K}_{n+1}^f \times \cdots \times \mathcal{K}_{n+T-1}^f \subset \mathcal{K}_n^{\mathbf{Z}}.$$

This complete the proof. $\square$

In the following we suppose that the $\mathcal{L}(\mathcal{E})^{T\times T}$ -valued semispectral measure attached to the Γ_T-correlation function of the process $\{\mathbf{Z}_n\}$ satisfies a Harnack type boundedness condition i.e., $c^{-1}\mathrm{d}t \le F \le c\mathrm{d}t$. Then the inversable maximal function matrix

$$\Theta(\lambda) = \Big(\Theta_{ij}(\lambda)\Big)_{i,j=0}^{T-1} \tag{2.32}$$

has a bounded inverse

$$\Omega(\lambda) = \Big(\Omega_{ij}(\lambda)\Big)_{i,j=0}^{T-1} \tag{2.33}$$

and the predictable part of $\mathbf{Z}_{n+1}$ can be obtained as

$$\hat{\mathbf{Z}}_{n+1} = \sum_{k=0}^{\infty} A_k \mathbf{Z}_{n-k}, \tag{2.34}$$

where the Wiener filter for prediction

$$A_k = \Big(A_k^{ij}\Big)_{i,j=0}^{T-1} \tag{2.35}$$

is given in terms of the coefficients of its maximal function in a similar way as in the discrete one-parameter case.

Theorem 2.6. *If $\{f_n\}_{n\in\mathbb{Z}}$ is a periodically Γ-correlated process and the predictible part of the attached stationary Γ_T-correlated process $\{\boldsymbol{Z}_n\}_{n\in\mathbb{Z}}$ is given by* (2.34) *and* (2.35)*, then the predictible part of $\{f_n\}_{n\in\mathbb{Z}}$ can be found as*

$$\hat{f}_{n+1} = \sum_{k=0}^{\infty} C_k f_{n-k}, \quad \text{where } C_k = \sum_{j=0}^{T-1} \tfrac{1}{\sqrt{T}} A_k^{j0} E^{j(n-k)}. \tag{2.36}$$

Proof. Let consider the predictible part

$$\hat{\mathbf{Z}}_{n+1} = (\hat{Z}_{n+1}^0, \hat{Z}_{n+1}^1, \ldots, \hat{Z}_{n+1}^{T-1}) =$$
$$= P_{K_n^{\mathbf{Z}}} \mathbf{Z}_{n+1} = P_{K_n^{\mathbf{Z}}} (Z_{n+1}^0, Z_{n+1}^1, \ldots, Z_{n+1}^{T-1}).$$

Taking the zero component of $\hat{\mathbf{Z}}_{n+1}$ we have

$$\hat{Z}_{n+1}^0 = P_{\mathcal{K}_n^{\mathbf{Z}}} Z_{n+1}^0 = P_{\mathcal{K}_n^{\mathbf{Z}}} (V_{f_{n+1}}, V_{f_{n+2}}, \ldots, V_{f_{n+T}}) =$$
$$= (P_{\mathcal{K}_n^f} V_{f_{n+1}}, \ldots, P_{\mathcal{K}_{n+T-1}^f} V_{f_{n+T}}).$$

On the other way, using (2.34)

$$\hat{\mathbf{Z}}_{n+1} = \sum_{k=0}^{\infty} A_k \mathbf{Z}_{n-k} = \sum_{k=0}^{\infty} A_k^{ij} \tfrac{1}{\sqrt{T}} (Z_{n-k}^0, Z_{n-k}^1, \ldots, Z_{n-k}^{T-1}) =$$

$$= \tfrac{1}{\sqrt{T}} \sum_{k=0}^{\infty} \Big(\sum_{j=0}^{T-1} Z_{n-k}^j A_k^{j0}, \sum_{j=0}^{T-1} Z_{n-k}^j A_k^{j1}, \ldots, \sum_{j=0}^{T-1} Z_{n-k}^j A_k^{j,T-1} \Big) =$$

$$= \Big(\tfrac{1}{\sqrt{T}} \sum_{k=0}^{\infty} \sum_{j=0}^{T-1} Z_{n-k}^{j} A_k^{j0}, \dots, \tfrac{1}{\sqrt{T}} \sum_{k=0}^{\infty} \sum_{j=0}^{T-1} Z_{n-k}^{j} A_k^{j,T-1} \Big) =$$

$$= \Big(\hat{Z}_{n+1}^{0}, \hat{Z}_{n+1}^{1}, \dots, \hat{Z}_{n+1}^{T-1} \Big).$$

It follows that

$$\hat{Z}_{n+1}^{0} = \tfrac{1}{\sqrt{T}} \sum_{k=0}^{\infty} \sum_{j=0}^{T-1} Z_{n-k}^{j} A_k^{j0} = \tfrac{1}{\sqrt{T}} \sum_{k=0}^{\infty} \sum_{j=0}^{T-1} A_k^{j0} Z_{n-k}^{j} =$$

$$= \sum_{k=0}^{\infty} \sum_{j=0}^{T-1} \tfrac{1}{\sqrt{T}} A_k^{j0} W_{X_{n-k}^{j}} = \sum_{k=0}^{\infty} \sum_{j=0}^{T-1} \tfrac{1}{\sqrt{T}} A_k^{j0} \Big(E^{j(n-k+s} V_{f_{n-k+s}} \Big)_{s=0}^{T-1} =$$

$$= \Big(\sum_{k=0}^{\infty} \sum_{j=0}^{T-1} \tfrac{1}{\sqrt{T}} A_k^{j0} E^{j(n-k+s} V_{f_{n-k+s}} \Big)_{s=0}^{T-1}.$$

Therefore

$$V_{\mathcal{P}_{\mathcal{H}_n^f} f_{n+1}} = P_{\mathcal{K}_n^f} V_{f_{n+1}} = \sum_{k=0}^{\infty} \sum_{j=0}^{T-1} \tfrac{1}{\sqrt{T}} A_k^{j0} E^{j(n-k} V_{f_{n-k}},$$

where $\mathcal{P}_{\mathcal{H}_n^f}$ is the "Γ-orthogonal projection" on the submodule $\mathcal{H}_n^f$ of $\mathcal{H}$.

So, the corresponding operatorial Wiener filter for the prediction of a periodically Γ-correlated process from $\mathcal{H}$ is given by (2.36). □

As a remark, in the periodic case the prediction error

$$\Delta(n) = \Gamma[f_{n+1} - \hat{f}_{n+1}, f_{n+1} - \hat{f}_{n+1}] \tag{2.37}$$

will be a periodic function, not an operator like in the stationary case. Therefore we have

$$\Delta(n) = \sum_{k=0}^{T-1} \Delta_k \exp(2\pi ijk/T) \tag{2.38}$$

and conversely, the coefficients Δ_k can be obtained by

$$\Delta_k = \frac{1}{T} \sum_{j=0}^{T-1} \Delta(j) \exp(-2\pi ijk/T). \tag{2.39}$$

The following theorem gives a characterization of the prediction error for a periodically Γ-correlated process $\{f_n\}$ in terms of the coefficients of the maximal function of the attached Γ_T-correlated process $\{\mathbf{Z}_n\}$.

Theorem 2.7. *The prediction error $\Delta(n)$ of a periodically Γ-correlated process $\{f_n\}$ has the form*

$$\Delta(n) = \sum_{k=0}^{T-1} D_k E^{-k(n+1)}, \tag{2.40}$$

where the operator coefficients $D_k \in \mathcal{L}(\mathcal{E})$ are the elements from the zero line of the prediction error matrix of the attached stationary process $\{\boldsymbol{Z}_n\}$, namely

$$D_k = \sum_{s=0}^{T-1} \Theta_{s0}^* \Theta_{sk}, \tag{2.41}$$

where $\Theta_{ij} = \Theta_{ij}(0)$ from the maximal function (2.32) of the process $\{\boldsymbol{Z}_n\}$.

Proof. Let Δ be the prediction error matrix of the stationary Γ_T-correlated process $\{\mathbf{Z}_n\}$ attached to $\{f_n\}$ by (2.25). Then for each $n \in \mathbb{Z}$

$$\Delta = \Gamma_T[\mathbf{Z}_{n+1} - \hat{\mathbf{Z}}_{n+1}, \mathbf{Z}_{n+1} - \hat{\mathbf{Z}}_{n+1}] = \Big(\Delta_{ij}\Big)_{i,j=0}^{T-1}$$

where the operators Δ_{ij} are given by

$$\Delta_{ij} = \tfrac{1}{T}\Gamma_1[Z_{n+1}^i - \hat{Z}_{n+1}^i, Z_{n+1}^j - \hat{Z}_{n+1}^j].$$

As we know, if $\Theta(\lambda)$ is the maximal function of a stationary process, then the prediction error is given by

$$\Delta = \Theta^*(0)\Theta(0) \tag{2.42}$$

and from (2.32), putting $\Theta_{ij} = \Theta_{ij}(0)$ we have that

$$\Delta ij = \sum_{s=0}^{T-1} \Theta_{si}^* \Theta_{sj}. \tag{2.43}$$

On the other way, from (2.39) we have

$$\begin{aligned}
\Delta_k &= \tfrac{1}{T}\sum_{j=0}^{T-1} \Delta(j)E^{kj} = \tfrac{1}{T}\sum_{j=0}^{T-1} \Gamma[f_{j+1} - \hat{f}_{j+1}, f_{j+1} - \hat{f}_{j+1}]E^{kj} = \\
&= \tfrac{1}{T}\sum_{j=0}^{T-1} V_{f_{j+1}-\hat{f}_{j+1}}^* V_{f_{j+1}-\hat{f}_{j+1}} E^{kj} = \tfrac{1}{T}E^{-k}\sum_{j=0}^{T-1} V_{f_{j+1}-\hat{f}_{j+1}}^* V_{f_{j+1}-\hat{f}_{j+1}} E^{k(j+1)} = \\
&= \tfrac{1}{T}E^{-k}\sum_{j=0}^{T-1} \Gamma[V_{f_{j+1}-\hat{f}_{j+1}}, E^{k(j+1)} V_{f_{j+1}-\hat{f}_{j+1}}] = \\
&= E^{-k}\tfrac{1}{T}\Gamma_1[Z_1^0 - \hat{Z}_1^0, Z_1^k - \hat{Z}_1^k] = E^{-k}\Delta_{0k} = E^{-k}\sum_{s=0}^{T-1} \Theta_{s0}^*\Theta_{sk}.
\end{aligned}$$

It follows from (2.38) that

$$\Delta(n) = \sum_{k=0}^{T-1} E^{-k} \sum_{s=0}^{T-1} \Theta_{s0}^* \Theta_{sk} E^{-kn} = \sum_{k=0}^{T-1} D_k E^{-k(n+1)},$$

where

$$D_k = \sum_{s=0}^{T-1} \Theta_{s0}^* \Theta_{sk},$$

and the proof is finished. □

3 Some more remarks

In the remaining of this paper, some geometrical aspects are analysed, especially about the angle between the past and the future of a Γ-correlated process. Actually, the study of the angle between the past and the future of a process is a major problem of the prediction theory. Starting with the studies of Helson and Szegő [4] and Helson and Sarason [6], the results was generalized in various contexts, helping in the characterization of stationary and some nonstationary processes. Here a generalization in the stationary Γ-correlated case as in [24] is given, and some results for periodically case are analysed.

The notions of the angles between two subspaces of a Hilbert space arise in [2] and [1], starting from the general definition of the scalar product of two vectors into the form $\langle h, g\rangle = \|h\|\,\|g\|\cdot\cos\alpha$. The *angle* (sometimes called the Dixmier angle) between two subspaces $\mathcal{M}$ and $\mathcal{N}$ of a Hilbert space $\mathcal{K}$ is given by its cosine

$$\rho(\mathcal{M},\mathcal{N}) := \sup\big\{\, |\langle h, g\rangle|\,;\ h\in\mathcal{M}\cap B_{\mathcal{K}}, g\in\mathcal{N}\cap B_{\mathcal{K}}\big\}. \tag{3.1}$$

where $B_{\mathcal{K}}$ is the unit ball of $\mathcal{K}$.

In the context of a complete correlated action $\{\mathcal{E},\mathcal{H},\Gamma\}$ the cosine between the submodules $\mathcal{M}$ and $\mathcal{N}$ of the right $\mathcal{L}(\mathcal{E})$-module $\mathcal{H}$ is given by

$$\rho(\mathcal{M},\mathcal{N}) = \sup\big\{\, |\langle\Gamma[g,h]a,b\rangle|\,;\|\Gamma[h,h]a\|\le 1, \|\Gamma[g,g]b\|\le 1\big\},$$

where $h\in\mathcal{M}, g\in\mathcal{N},\ a,b\in\mathcal{E}$.

We say that $\mathcal{M}$ and $\mathcal{N}$ have a *positive angle* if $\rho(\mathcal{M},\mathcal{N})<1$, or equivalently, if there exists $\rho<1$ such that for any $h\in\mathcal{M}, g\in\mathcal{N},\ a,b\in\mathcal{E}$

$$|\langle\Gamma[g,h]a,b\rangle_{\mathcal{E}}| \le \rho\,\|V_h a\|\,\|V_g b\|\,. \tag{3.2}$$

In the study of prediction problems we are interested in the case when the angle between past and future is positive, i.e., when $\rho(n)=\rho(\mathcal{H}^f_n,\widetilde{\mathcal{H}}^f_n)<1$, which will give the possibility of finding the predictor.

A nice geometrical aspect of stationary Γ-correlated process is the fact that the angle between the past and future is constant.

Generalizing to stationary Γ-correlated case a result of [4] we have

Proposition 3.1. *Let (f_n) be a stationary Γ-correlated process in $\mathcal{H}$. The angle between past and future of (f_n) is positive if and only if there exists a finite constant C which depends only by (f_n) such that for each element of the form $\sum V_{f_n}a_n$ from the time domain $\mathcal{K}^f_\infty$ and for each $-\infty\le n_1\le n_2<\infty$ we have*

$$\left\|\sum_{k=n_1}^{n_2} V_{f_k}a_k\right\| \le C\left\|\sum V_{f_k}a_k\right\|, \tag{3.3}$$

where in the second term the sum has finitely many non-zero elements.

Proof. It is known [4] that for two subspaces $\mathcal{M}$ and $\mathcal{N}$ from a Hilbert space we have $\rho(\mathcal{M},\mathcal{N})<1$ if and only if there exists a finite constant C such that $\|x\|\le C\,\|x+y\|$ for x and y generators in $\mathcal{M}$ and $\mathcal{N}$, respectively. Therefore for any sum of the form $\sum V_{f_n}a_n$ from the time domain

$\mathcal{K}^f_\infty$, taking into account that $\rho(\mathcal{H}^f_n, \widetilde{\mathcal{H}}^f_n) < 1$, we have

$$\left\| \sum_{k \le n} V_{f_k} a_k \right\| \le C \left\| \sum_{k \le n} V_{f_k} a_k + \sum_{k > n} V_{f_k} a_k \right\| = C \left\| \sum V_{f_k} a_k \right\|,$$

where $\sum V_{f_k} a_k$ has finitely many non-zero elements. Since (f_n) is stationary Γ-correlated, for any $m \in \mathbb{Z}$ we have

$$\begin{aligned}\left\| \sum_{k \le m} V_{f_k} a_k \right\|^2_{\mathcal{K}} &= \left\langle \sum_{k \le m} V_{f_k} a_k, \sum_{p \le m} V_{f_p} a_p \right\rangle = \sum_{k,p \le m} \left\langle V^*_{f_p} V_{f_k} a_k, a_p \right\rangle_{\mathcal{E}} = \\ &= \sum_{k,p \le m} \langle \Gamma[f_p, f_k] a_k, a_p \rangle = \sum_{k,p \le m} \langle \Gamma[f_{p-(m-n)}, f_{k-(m-n)}] a_k, a_p \rangle = \\ &= \sum_{i,j \le n} \langle \Gamma[f_j, f_i] a_i, a_j \rangle = \left\| \sum_{k \le n} V_{f_k} a_k \right\|^2_{\mathcal{K}} \le C^2 \left\| \sum V_{f_k} a_k \right\|^2_{\mathcal{K}}.\end{aligned}$$

Therefore

$$\begin{aligned}\left\| \sum_{k=n_1}^{n_2} V_{f_k} a_k \right\| &= \left\| \sum_{k \le n_2} V_{f_k} a_k - \sum_{k < n_1} V_{f_k} a_k \right\| \le \\ &\le \left\| \sum_{k \le n_2} V_{f_k} a_k \right\| + \left\| \sum_{k \le n_1} V_{f_k} a_k \right\| \le 2C \left\| \sum V_{f_k} a_k \right\|\end{aligned}$$

and (3.3) is proved. □

Based on the previous result, the property of representing the elements from the time domain of a process as a series (Schauder basis) was generalized for Γ-correlated processes [24] as follows

PROPOSITION 3.2. *The angle between past and future of a stationary Γ-correlated process (f_n) is positive if and only if each element k from the time domain $\mathcal{K}^f_\infty$ can be uniquely represented in the form $k = \sum_{n=-\infty}^{\infty} k_n$ where k_n are elements from $\overline{V_{f_n}\mathcal{E}}$.*

We have seen that each periodically Γ-correlated process $(f_n)_{n\in\mathbb{Z}}$ from $\mathcal{H}$ has a stationary Γ_1-correlated dilation (X_n) in $\mathcal{H}^T$ and an explicit stationary dilation can be constructed to help in obtaining the Wiener filter for prediction and the prediction-error operator function for a periodically Γ-correlated process, in terms of the operator coefficients of its attached maximal function. This stationary dilation preserves the positivity of the angle between the past and the future of the considered periodically Γ-correlated process.

PROPOSITION 3.3. *If (f_n) from $\mathcal{H}$ is a periodically Γ-correlated process with a positive angle between its past and future, then the angle between the past and the future of its stationary Γ_1-correlated dilation (X_n) from $\mathcal{H}^T$ it is also positive.*

Proof. If (f_n) from $\mathcal{H}$ is a periodically Γ-correlated process having a positive angle between its past and future, then at each time $t = n$ there exists $\rho(n) < 1$ such that

$$|\langle \Gamma[g,h]a, b\rangle_{\mathcal{E}}| \le \rho(n) \|V_h a\| \|V_g b\|$$

for each $h \in \mathcal{H}_n^f$ and $g \in \tilde{\mathcal{H}}_n^f$. For each element $X = \sum\limits_{k\le n} A_k X_k$ from the past H_n^X and $Y = \sum\limits_{p>n} B_p X_p$ from the future $\tilde{H}_n^X$ of the Γ_1-correlated process (X_n) given by (2.9), and for any $a, b \in \mathcal{E}$ we have

$$
\begin{aligned}
|\langle \Gamma_1[X,Y]a,b\rangle_{\mathcal{E}}| &= \left|\left\langle \Gamma_1\Big[\sum_{p>n} B_pX_p, \sum_{k\le n} A_kX_k\Big]a, b\right\rangle_{\mathcal{E}}\right| = \\
&= \left|\sum_{p>n}\sum_{k\le n} \langle \Gamma_1[B_pX_p, A_kX_k]a,b\rangle_{\mathcal{E}}\right| = \\
&= \left|\sum_{p>n}\sum_{k\le n}\sum_{i=0}^{T-1} \langle \Gamma[B_pf_{p+i}, A_kf_{k+i}]a,b\rangle_{\mathcal{E}}\right| = \\
&= \left|\sum_{p>n}\sum_{k\le n}\sum_{i=0}^{T-1} \big\langle B_p^*\Gamma[f_{p+i}, f_{k+i}]A_ka,b\big\rangle_{\mathcal{E}}\right| = \\
&= \left|\sum_{i=0}^{T-1} \left\langle \Gamma\Big[\sum_{p>n} B_pf_{p+i}, \sum_{k\le n} A_kf_{k+i}\Big]a,b\right\rangle_{\mathcal{E}}\right| \le \\
&\le \sum_{i=0}^{T-1} \rho_i(n) \left\|\sum_{k\le n} A_kf_{k+i}a\right\| \left\|\sum_{p>n} B_pf_{p+i}b\right\| \le \\
&\le \rho(n) \sum_{i=0}^{T-1} \left\|\sum_{k\le n} A_kf_{k+i}a\right\| \left\|\sum_{p>n} B_pf_{p+i}b\right\| \le \\
&\le \rho(n) \Big(\sum_{i=0}^{T-1} \left\|\sum_{k\le n} A_kf_{k+i}a\right\|^2\Big)^{1/2} \Big(\sum_{i=0}^{T-1} \left\|\sum_{p>n} B_pf_{p+i}b\right\|^2\Big)^{1/2} = \\
&= \rho \left\|\sum_{k\le n} A_kW_{X_k}a\right\| \left\|\sum_{p>n} B_pW_{X_p}b\right\| = \rho\, \|W_Xa\|\, \|W_Yb\|,
\end{aligned}
$$

where $\rho(n)$ is the maximum of $\rho_i(n) < 1$; $i = 0, 1, \ldots, T-1$, and we used the embedding $X \to W_X$ of $\mathcal{H}^T$ into $\mathcal{L}(\mathcal{E}, \mathcal{K}^T)$ and the fact that $\rho(n) = \rho$ for stationary Γ_1-correlated proces (X_n). Therefore $|\langle \Gamma_1[X,Y]a,b\rangle_{\mathcal{E}}| \le \rho\, \|W_Xa\|\, \|W_Yb\|$ for each $X \in H_n^X$, $Y \in \tilde{H}_n^X$, and the angle between the past and the future of the stationary Γ_1-correlated dilation (X_n) is positive. □

Another angle between two subspaces M_1 and M_2 of a Hilbert space $\mathcal{K}$ is the *Friedrichs angle* [2] defined to be the angle in $[0, \pi/2]$ whose cosine is given by

$$c(M_1, M_2) := \sup\{|\langle k_1, k_2\rangle|\,; k_i \in M_i \cap M^\perp \cap B_{\mathcal{K}},\ i \in \{1,2\}\}, \tag{3.4}$$

where $M = M_1 \cap M_2$ and $B_{\mathcal{K}}$ is the unit ball of $\mathcal{K}$.

By (3.1) and (3.4) it follows that $c(M_1, M_2) \le \rho(M_1, M_2)$. Obviously we have $c(M_1, M_2) = \rho(M_1 \cap M^\perp, M_2 \cap M^\perp)$, and of course $c(M_1, M_2) = c(M_1^\perp, M_2^\perp)$. More geometrical aspects and generalizations of the angles between the past and the present of periodically Γ-correlated processes can be found in [24].

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