Continuous parameter Γ-stationary processes

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Abstract

The paper continues the series of presentations started with [17], showing a way to extend the prediction problems to the infinite dimensional case. Stationary Γ-correlated processes with continuous time parameter are analyzed. Various types of continuities for Γ-correlated processes are considered, and via the attached shift group to a continuous time parameter process a time-domain analysis and a spectral analysis are done. Using a discretization procedure, some discrete time techniques can be applied in the study of continuous parameter Γ-correlated processes. Also some nonstationary Γ-correlated processes are considered and relations between the Γ-periodicity and Γ-harmonizability of a continuous time parameter process are analyzed.

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1 Preliminaries

Let $\mathcal{L}(\mathcal{E})$ be the C^*-algebra of all linear bounded operators on a separable Hilbert space $\mathcal{E}$, and $\mathcal{H}$ a right $\mathcal{L}(\mathcal{E})$-module. An *action* of $\mathcal{L}(\mathcal{E})$ on $\mathcal{H}$ is given considering $Ah := hA$ in the sense of the right $\mathcal{L}(\mathcal{E})$-module. The action of $\mathcal{L}(\mathcal{E})$ on $\mathcal{H}$ is *correlated* if there exists a function (the correlation of the action) $\Gamma : \mathcal{H} \times \mathcal{H}$ into $\mathcal{L}(\mathcal{E})$ given by $(h, g) \to \Gamma[h, g]$, such that

(i) $\Gamma[h, h] \geq 0$, $\Gamma[h, h] = 0$ implies $h = 0$;

(ii) $\Gamma[g, h] = \Gamma[h, g]^*$;

(iii) $\Gamma[h, Ag] = \Gamma[h, g]A$

A triplet $\{\mathcal{E}, \mathcal{H}, \Gamma\}$ defined as above was called [8] a *correlated action* of $\mathcal{L}(\mathcal{E})$ on $\mathcal{H}$.

An example of correlated action can be constructed as follows. Take $\mathcal{H} = \mathcal{L}(\mathcal{E}, \mathcal{K})$ – the space of the linear bounded operators from $\mathcal{E}$ into $\mathcal{K}$, where $\mathcal{E}$ and $\mathcal{K}$ are Hilbert spaces. An action of $\mathcal{L}(\mathcal{E})$ on $\mathcal{L}(\mathcal{E}, \mathcal{K})$ is given if we consider $AV := VA$ for each $A \in \mathcal{L}(\mathcal{E})$ and $V \in \mathcal{L}(\mathcal{E}, \mathcal{K})$. It is easy to see that $\Gamma[V_1, V_2] = V_1^* V_2$ is a correlation of the action of $\mathcal{L}(\mathcal{E})$ on $\mathcal{L}(\mathcal{E}, \mathcal{K})$, and the triplet $\{\mathcal{E}, \mathcal{L}(\mathcal{E}, \mathcal{K}), \Gamma\}$ is a correlated action. In [8] was proved that for any correlated action $\{\mathcal{E}, \mathcal{H}, \Gamma\}$ there exists a Hilbert space $\mathcal{K}$ and an algebraic imbedding $h \to V_h$ of $\mathcal{H}$ into $\mathcal{L}(\mathcal{E}, \mathcal{K})$.

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Such a way, if we consider as the *state space* to be a right $\mathcal{L}(\mathcal{E})$-module $\mathcal{H}$, the behaviour of a process $\{f_t\}_{t\in G}$ from $\mathcal{H}$ can be studied knowing the behaviour of the operatorial process $\{V_{f_t}\}$ from $\mathcal{L}(\mathcal{E},\mathcal{K})$. The corresponding space $\mathcal{K}$ is called the *measuring space* and $\mathcal{E}$ is the *parameter space.* If the algebraic imbedding of $\mathcal{H}$ into $\mathcal{L}(\mathcal{E},\mathcal{K})$ is onto, then the correlated action $\{\mathcal{E},\mathcal{H},\Gamma\}$ is a *complete correlated action.*

Since in $\mathcal{H}$ we have not a proper orthogonal projection on a right $\mathcal{L}(\mathcal{E})$-submodule, a Γ-orthogonal projection was constructed, using the following Proposotion, helping to solve specific prediction problems.

PROPOSITION 1.1. *Let $\mathcal{H}_1$ be a submodule in the right $\mathcal{L}(\mathcal{E})$-module $\mathcal{H}$ and*

$$\mathcal{K}_1 = \bigvee_{x\in\mathcal{H}_1} V_x\mathcal{E} \subset \mathcal{K}. \tag{1.1}$$

For each $h \in \mathcal{H}$ there exists a unique element $h_1 \in \mathcal{H}$ such that for each $a \in \mathcal{E}$ we have

$$V_{h_1}a \in \mathcal{K}_1 \qquad \textit{and} \qquad V_{h-h_1}a \in \mathcal{K}_1^{\perp}. \tag{1.2}$$

Moreover, we have

$$\Gamma[h-h_1, h-h_1] = \inf_{x\in\mathcal{H}_1} \Gamma[h-x, h-x], \tag{1.3}$$

where the infimum is taken in the set of all positive operators from $\mathcal{L}(\mathcal{E})$.

For a complete proof see e.g. [9] or [10].

Due the properties (1.2) and (1.3), if we denote the unique element $h_1 \in \mathcal{H}$ by $h_1 = \mathcal{P}h$, we have $\mathcal{P}^2 = \mathcal{P}$ and $\Gamma[\mathcal{P}h, g] = \Gamma[h, \mathcal{P}g]$. Therefore $\mathcal{P}$ is a Γ-orthogonal projection "on" the right $\mathcal{L}(\mathcal{E})$-submodule $\mathcal{H}_1$ of $\mathcal{H}$.

To extend the finite multivariate prediction to the infinite variate case, the main investigation tools was the using of L^2-bounded analytic functions instead of bounded analytic functions, and a study of $\mathcal{L}(\mathcal{E})$-valued semispectral measures. An $\mathcal{L}(\mathcal{E})$-valued semispectral measure is a map $\sigma \to F(\sigma)$ from the family $\mathcal{B}(\mathbb{T})$ of Borel subsets of the unit torus $\mathbb{T}$ from the complex plane $\mathbb{C}$ into $\mathcal{L}(\mathcal{E})$ such that for any $a \in \mathcal{E}$ the map $\sigma \to \langle F(\sigma)a, a\rangle$ is a positive Radon measure on $\mathbb{T}$. An L^2-bounded operator valued analytic function is an analytic function $\Theta(\lambda) = \sum_{n=0}^{\infty} \Theta_n\lambda^n$ on the open unit disc $\mathbb{D}$, where the operator coefficients $\Theta_n \in \mathcal{L}(\mathcal{E},\mathcal{F})$, such that there exists $M > 0$ verifying

$$\sum_{n=0}^{\infty} \|\Theta_n h\|^2 \,\mathrm{d}t \le M^2 \|h\|^2 \qquad (h \in \mathcal{H}) \tag{1.4}$$

or equivalently,

$$\sup_{0\le r<1} \frac{1}{2\pi} \int_0^{2\pi} \left\|\Theta(re^{it})h\right\|^2 \mathrm{d}t \le M^2 \|h\|^2 \qquad (h \in \mathcal{H}). \tag{1.5}$$

An operator valued analytic function is denoted usually by a triplet $\{\mathcal{E},\mathcal{F},\Theta(\lambda)\}$. Also a central role in the study of infinite variate Γ-stationary processes played the generalized Lowdenslager–Sz.-Nagy–Foiaş factorization theorem, which attach to each semispectral measure F a maximal outer L^2-bounded function, so called the maximal function of F. For detailles see [9].

2 Γ-correlated processes

A Γ-correlated process $\{f_t\}_{t\in G}$ is a sequence in $\mathcal{H}$, where G is $\mathbb{Z}$, $\mathbb{R}$, a locally compact group or a hypergroup. To a process $\{f_t\}_{t\in G}$ from $\mathcal{H}$ a *correlation function* is attached by

$$\Gamma_f(s,t) = \Gamma[f_s, f_t]. \tag{2.1}$$

Also, for two processes $\{f_t\}_{t\in G}$ and $\{g_t\}_{t\in G}$ a cross-correlation function is attached by

$$\Gamma_{fg}(s,t) = \Gamma[f_s, g_t]. \tag{2.2}$$

If the correlation function depends only on the difference $t-s$, not on s and t separately, then the process $\{f_t\}_{t\in G}$ is called Γ-*stationary*, otherwise the process is a *nonstationary* one. Similarly, if the cross-correlation function $\Gamma_{fg}(s,t)$ depends only on the difference $t-s$, then $\{f_t\}_{t\in G}$ and $\{g_t\}_{t\in G}$ are stationary cross-correlated processes. Of course, in the Γ-stationary case (2.1) becomes

$$\Gamma_f(t) = \Gamma[f_0, f_t]. \tag{2.3}$$

If $G=\mathbb{Z}$, then $\{f_n\}_{n\in\mathbb{Z}} \in \mathcal{H}$ is a discrete Γ-stationary process and an exhaustive prediction study can be found e.g. in [9], [10]. In the continous case ($G=\mathbb{R}$) some investigations for Γ-correlated processes was done in [12].

For a Γ-correlated process (not necessary stationary) the *past-present* at the moment t is the right $\mathcal{L}(\mathcal{E})$-submodule

$$\mathcal{H}_t^f = \Big\{\sum_k A_k f_k;\ A_k \in \mathcal{L}(\mathcal{E}),\ k\le t\Big\}, \tag{2.4}$$

the *future* is

$$\widetilde{\mathcal{H}}_t^f = \Big\{\sum_k A_k f_k;\ A_k \in \mathcal{L}(\mathcal{E}),\ k> t\Big\}, \tag{2.5}$$

the *remote past*

$$\mathcal{H}_{-\infty}^f = \bigcap_t \mathcal{H}_t^f; \tag{2.6}$$

and the *time domain* is

$$\mathcal{H}_{\infty}^f = \Big\{\sum_k A_k f_k;\ A_k \in \mathcal{L}(\mathcal{E}),\ k\in\mathbb{R}\Big\}.$$

By the embedding $h \to V_h$ of $\mathcal{H}$ into $\mathcal{L}(\mathcal{E},\mathcal{K})$, for the corresponding *past*, *remote past*, and the *future* from $\mathcal{H}$ will correspond the closed subspaces of $\mathcal{K}$ given by

$$\mathcal{K}_t^f = \bigvee_{j\le t} V_{f_j}\mathcal{E}, \tag{2.7}$$

$$\mathcal{K}_{-\infty}^f = \bigcap_t \mathcal{K}_t^f, \tag{2.8}$$

$$(2.9)\qquad \widetilde{\mathcal{K}}_t^f = \bigvee_{j>t} V_{f_j}\mathcal{E},$$

respectively, and the *time domain* becomes

$$\mathcal{K}_\infty^f = \bigvee_{j\in\mathbb{R}} V_{f_j}\mathcal{E}.$$

In this paper we are mainly interested in the study of processes $\{f_t\}_{t\in\mathbb{R}}$ in the context of a complete correlated action $\{\mathcal{E}, \mathcal{H}, \Gamma\}$.

A process $\{f_t\}_{t\in\mathbb{R}}$ is said to be norm, strongly, or weakly continuous, if $t \to V_{f_t}$ is norm, strongly, or weakly continuous in $\mathcal{L}(\mathcal{E}, \mathcal{K})$. The process $\{f_t\}_{t\in\mathbb{R}}$ is bounded if there is a positive constant $M > 0$ such that $\|V_{f_t}\| \le M$ for $t \in \mathbb{R}$.

To a process $\{f_t\}_{t\in\mathbb{R}}$ an operatorial valued correlation function $\Gamma(s,t)$ is attached by (2.1), and for any $a, b \in \mathcal{E}$ a scalar correlation function $\tilde{\Gamma}_{ab}(s,t) : \mathbb{R}\times\mathbb{R} \to \mathbb{C}$ is defined by

$$(2.10)\qquad \tilde{\Gamma}_{ab}(s,t) = \langle \Gamma(s,t)a, b\rangle_{\mathcal{E}}\,.$$

PROPOSITION 2.1. *The process $\{f_t\}_{t\in\mathbb{R}} \in \mathcal{H}$ is strongly continuous if and only if far any $a, b \in \mathcal{E}$ the scalar correlation function $\tilde{\Gamma}_{ab}(s,t)$ is continuous.*

Proof. We have only to see that

$$\left|\tilde{\Gamma}_{ab}(s+u, t+v) - \tilde{\Gamma}_{ab}(s,t)\right| = |\langle \Gamma[f_{s+u}, f_{t+v}]a, b\rangle_{\mathcal{E}} - \langle \Gamma[f_s, f_t]a, b\rangle_{\mathcal{E}}| \le$$

$$\begin{aligned}
&\le |\langle \Gamma[f_{s+u} - f_s, f_{t+v}]a, b\rangle_{\mathcal{E}}| + |\langle \Gamma[f_s, f_{t+v} - f_t]a, b\rangle_{\mathcal{E}}| \le \\
&\le [\|\Gamma[f_{s+t} - f_s, f_{t+v}]a\| + \|\Gamma[f_s, f_{t+v} - f_t]a\|]\,\|b\| = \\
&= [\left\|V^*_{f_{s+t}-f_s} V_{f_{t+v}} a\right\| + \left\|V^*_{f_s} V_{f_{t+v}-f_t} a\right\|]\,\|b\| \le \\
&\le [\left\|(V_{f_{s+u}} - V_{f_s})^* V_{f_{t+v}} a\right\| + \|V_{f_s}\|\,\left\|V_{f_{t+v}}a - V_{f_t}a\right\|]\,\|b\| \to 0
\end{aligned}$$

as $u, v \to 0$.

Conversely, if $\tilde{\Gamma}_{ab}(s,t)$ is a continuous function on $\mathbb{R}\times\mathbb{R}$ for any $a, b \in \mathcal{E}$, then for any $u \in \mathbb{R}$ we have

$$\begin{aligned}
\left\|V_{f_{t+u}}a - V_{f_t}a\right\|^2_{\mathcal{K}} &= \left\langle V_{f_{t+u}}a - V_{f_t}a, V_{f_{t+u}}a - V_{f_t}a\right\rangle_{\mathcal{K}} = \\
&= \left\langle V_{f_{t+u}}a, V_{f_{t+u}}a\right\rangle - \left\langle V_{f_{t+u}}a, V_{f_t}a\right\rangle - \left\langle V_{V_{f_t}}a, V_{f_{t+u}}a\right\rangle + \left\langle V_{f_t}a, V_{f_t}a\right\rangle = \\
&= \left\langle V^*_{f_{t+u}} V_{f_{t+u}}a, a\right\rangle - \left\langle V^*_{f_t} V_{f_{t+u}}a, a\right\rangle - \left\langle V^*_{f_{t+u}} V_{f_t}a, a\right\rangle + \left\langle V^*_{f_t} V_{f_t}a, a\right\rangle = \\
&= \langle \Gamma(t+u, t+u)a, a\rangle - \langle \Gamma(t, t+u)a, a\rangle - \langle \Gamma(t+u, t)a, a\rangle + \langle \Gamma(t,t)a, a\rangle = \\
&= \tilde{\Gamma}_{aa}(t+u, t+u) - \tilde{\Gamma}_{aa}(t, t+u) - \tilde{\Gamma}_{aa}(t+u, t) + \tilde{\Gamma}_{aa}(t,t) \to 0
\end{aligned}$$

as u converges to zero. □

PROPOSITION 2.2. *If $\{f_t\}_{t\in\mathbb{R}}$ is a weakly continuous process, then for any $a, b \in \mathcal{E}$ the scalar correlation function $\tilde{\Gamma}_{ab}(s,t)$ is separately continuous on $\mathbb{R}\times\mathbb{R}$.*

Conversely, if $\{f_t\}_{t\in\mathbb{R}}$ is bounded and $\tilde{\Gamma}_{ab}(s,t)$ is separately continuous on $\mathbb{R}\times\mathbb{R}$, then $\{f_t\}_{t\in\mathbb{R}}$ is weakly continuous.

Proof. Obviously we have

$$\left|\tilde{\Gamma}_{ab}(s+u,t)-\tilde{\Gamma}_{ab}(s,t)\right| = \left|\langle\Gamma(s+u,t)a,b\rangle_{\mathcal{E}}-\langle\Gamma(s,t)a,b\rangle_{\mathcal{E}}\right| =$$

$$= \left\langle V^*_{f_{s+u}}V_{f_t}a,b\right\rangle_{\mathcal{E}} - \left\langle V^*_{f_s}V_{f_t}a,b\right\rangle_{\mathcal{E}} = \left\langle (V_{f_{s+u}}-V_{f_s})^*V_{f_t}a,b\right\rangle_{\mathcal{E}} =$$

$$= \left\langle V_{f_t}a,(V_{f_{s+u}}-V_{f_s})b\right\rangle_{\mathcal{K}} \to 0$$

as $u \to 0$.

Analogously we have that for any $a,b\in\mathcal{E}$ and $v\in\mathbb{R}$

$$\left|\tilde{\Gamma}_{ab}(s,t+v)-\tilde{\Gamma}_{ab}(s,t)\right| \to 0$$

as $v \to 0$.

If $\{f_t\}_{t\in\mathbb{R}}$ is bounded and for any $a,b\in\mathcal{E}$ the scalar correlation function $\tilde{\Gamma}_{ab}(s,t)$ is separately continuous, then for $s\in\mathbb{R}$ the function $\tilde{\Gamma}_{ab}(\cdot,s)$ is continuous, and for any $y=\sum\limits_{j=1}^{n}\overline{\alpha_j}V_{f_{s_j}}b$ from $\mathcal{K}^f_\infty$ the function

$$\langle V_{f_\cdot}a,y\rangle_{\mathcal{K}} = \sum_{j=1}^{n}\alpha_j\left\langle V_{f_\cdot}a,V_{f_{s_j}}b\right\rangle = \sum_{j=1}^{n}\alpha_j\tilde{\Gamma}_{ab}(\cdot,s_j)$$

is continuous. Choosing in $\mathcal{K}^f_\infty$ a sequence $\{k_n\}_{n=1}^\infty$, where $k_n=\sum\limits_{j=1}^{n}\alpha_{n,j}V_{f_{s_{n,j}}}b$, such that $\|k_n-y\|_{\mathcal{K}}\to 0$, and taking account that $\tilde{\Gamma}_{ab}(\cdot,t)$ is continuous, then $\langle V_{f_\cdot}a,y\rangle_{\mathcal{K}}=\lim\limits_{n\to\infty}\langle V_{f_\cdot}a,k_n\rangle_{\mathcal{K}}$, which is a uniform limit since $\{f_t\}$ is bounded. Therefore $\{f_t\}$ is weakly continuous. □

3 The shift group

Two continuous parameter Γ-stationary processes $\{f_t\}$ and $\{g_t\}$ are *cross-correlated* if $\Gamma[f_t,g_s]$ depends only on the difference $s-t$. The *cross-correlation function* is given by

$$\Gamma_{fg}(t)=\Gamma[f_s,g_{s+t}]. \tag{3.1}$$

Proposition 3.1. *For any Γ-stationary process $\{f_t\}$ there exists a unique group of unitary operators $(U^f_t)_{t\in\mathbb{R}}$ on $\mathcal{K}^f_\infty$ such that*

$$V_{f_t}=U^f_tV_{f_0}. \tag{3.2}$$

Two continuous parameter Γ-stationary processes $\{f_t\}$ and $\{g_t\}$ are stationatily cross-correlated if and only if there exists a group of unitary operators $(W_t)_{t\in\mathbb{R}}$ on

$$\mathcal{K}^{fg}_\infty=\mathcal{K}^f_\infty\bigvee\mathcal{K}^g_\infty$$

such that

$$W_t|\mathcal{K}^f_\infty=U^f_t \quad \text{and} \quad W_t|\mathcal{K}^g_\infty=U^g_t.$$

The proof runs parallel to that in the discrete case [14] and is omitted.

The group of unitary operators $(U_t^f)_{t\in\mathbb{R}}$ is called the *shift group* of the process $\{f_t\}$, and $(W_t)_{t\in\mathbb{R}}$ is the *extended shift group* of the cross-correlated processes $\{f_t\}$ and $\{g_t\}$. As in the discrete case, let us denote

$$V_f = V_{f_0} \in \mathcal{L}(\mathcal{E}, \mathcal{K}).$$

Then for any $a \in \mathcal{E}$ we have

$$V_f a = P_{\mathcal{K}_\infty^f} V_f a + (I - P_{\mathcal{K}_\infty^f}) V_f a = V_1 a + V_2 a.$$

By (3.2) it follows that $U_t V_2 a$ is the part of $V_{f_t} a$ orthogonal to $\mathcal{K}_\infty^f$ for $t \in \mathbb{R}$, and

$$\mathcal{M} = \bigvee_t U_t V_2 \mathcal{E}$$

is a subspace in $\mathcal{K}_\infty^f$ orthogonal to $\mathcal{K}_t^f$. Hence $\mathcal{M}$ is orthogonal to $\mathcal{K}_{-\infty}^f$. If

$$\mathcal{N} = \mathcal{K}_\infty^f \ominus (\mathcal{K}_{-\infty}^f \oplus \mathcal{M}),$$

then

$$\mathcal{K}_\infty^f = \mathcal{K}_{-\infty}^f \oplus \mathcal{M} \oplus \mathcal{N}. \tag{3.3}$$

When only one term of (3.3) is different from $\{0\}$, then the process is *pure*, Namely, if

$$\mathcal{K}_\infty^f = \mathcal{K}_{-\infty}^f, \tag{3.4}$$

then $\{f_t\}$ is a purely *deterministic* process. If we have

$$\mathcal{K}_\infty^f = \mathcal{M}, \tag{3.5}$$

then $\{f_t\}$ is a purely *discrete innovation* process , and if

$$\mathcal{K}_\infty^f = \mathcal{N}, \tag{3.6}$$

then the process $\{f_t\}$ is a purely *continuous innovation*, or *evanescent*.

The Γ-stationary process $\{f_t\}_{t\in\mathbb{R}}$ is called *continuous*, if the corresponding shift group $(U_t^f)_{t\in\mathbb{R}}$ is a continuous one parameter group of unitary operators on $\mathcal{K}_\infty^f$ for t converging to zero. When no confusion is made, for simplicity, the shift group is denoted with (U_t).

It is obvious that in the continuous case for any $a \in \mathcal{E}$ we have

$$V_{f_t} a \in \mathcal{K}_t^f,$$

and the form of the past-present subspace becomes

$$\mathcal{K}_t^f = \bigvee_{s\le 0} U_s V_f \mathcal{E},$$

also $\mathcal{M} = \{0\}$, which implies that

$$\mathcal{K}_\infty^f = \mathcal{K}_{-\infty}^f \oplus \mathcal{N}. \tag{3.7}$$

In this paper we consider a Γ-stationary process $\{f_t\}_{t\in\mathbb{R}}$ to be continuous, and using the results from the discrete case, a study is made.

4 Time-domain analysis

If U is the cogenerator of the shift group, then U is a unitary operator on $\mathcal{K}^f_\infty$. Putting

$$V_{f'_n} = U^n V_f, \tag{4.1}$$

we obtain a discrete parameter Γ-stationary process $\{f'_n\}_{n\in\mathbb{Z}}$ which has U as the shift operator. The process $\{f'_n\}_{n\in\mathbb{Z}}$ obtained as above is called the *discrete process associated* with $\{f_t\}_{t\in\mathbb{R}}$.

THEOREM 4.1. *Let $\{f_t\}_{t\in\mathbb{R}}$ be a continuous Γ-stationary process and $\{f'_n\}_{n\in\mathbb{Z}}$ be the associated discrete parameter process. Then*

$$\mathcal{K}^{f'}_0 = \mathcal{K}^f_0, \quad \mathcal{K}^{f'}_\infty = \mathcal{K}^f_\infty \quad \textit{and} \quad \mathcal{K}^{f'}_{-\infty} = \mathcal{K}^f_{-\infty}. \tag{4.2}$$

Proof. Generally the proof follows [5]. There exists the following expressions of U_t in term of its cogenerator U.

$$U_{\pm t} = \mathrm{e}^{-it} I + \lim_{n\to\infty} \sum_{k=1}^{n} (1/k!)(-nt/n+1)^k [(I + A_{\pm n})^k - I], \tag{4.3}$$

where

$$A_{\pm n} = \frac{2n}{n+1} \sum_{j=1}^{\infty} (n - 1/n + 1)^{j-1} U^{\pm j}, \qquad n \geq 0.$$

Conversely, U in terms of U_t can be expressed as

$$U^{\pm n} = I + 2 \int_0^\infty L'_n(2t) \mathrm{e}^{-t} U_{\pm t} \mathrm{d}t, \qquad n \geq 0, \tag{4.4}$$

where $L_n(t)$ are the Laguerre polynomials

$$L_n(t) = \sum_{k=1}^{n} (-1)^k / k! \binom{n}{k} t^k, \qquad n \geq 0.$$

From (4.3) and (4.4) it follows that U_t can be seen as a strong limit of U and conversely. It follows that for any subset $\mathcal{A} \subset \mathcal{K}^f_\infty$ one has

$$\bigvee_{n=0}^{\infty} U^{\pm n} \mathcal{A} = \bigvee_{t\geq 0} U_{\pm t} \mathcal{A}. \tag{4.5}$$

If $\mathcal{A} = V\mathcal{E}$, then $\mathcal{K}^{f'}_0 = \mathcal{K}^f_0$ and

$$\mathcal{K}^{f'}_\infty = \mathcal{K}^{f'}_0 \bigvee \Big(\bigvee_{n=0}^{\infty} U^n V_f \mathcal{E} \Big) = \mathcal{K}^f_0 \bigvee \Big(\bigvee_{t\geq 0} U_t V_f \mathcal{E} \Big) = \mathcal{K}^f_\infty.$$

Also, taking $\mathcal{A} = \mathcal{K}^f_{-\infty}$ it results that for any $k \geq 0$

$$U^k \mathcal{K}^f_{-\infty} \subset \bigvee_{0}^{\infty} U^n \mathcal{K}^f_{-\infty} = \bigvee_{t\geq 0} U_t \mathcal{K}^f_{-\infty} = \mathcal{K}^f_{-\infty}.$$

It follows that for any $k \geq 0$ we have

$$\mathcal{K}^f_{-\infty} \subseteq U^{\pm k}\mathcal{K}^f_{-\infty} \subseteq U^{\pm k}\mathcal{K}^f_0 = U^{\pm k}\mathcal{K}^{f'}_0 = \mathcal{K}^{f'}_{-k}.$$

Hence

$$\mathcal{K}^f_{-\infty} \subseteq \bigcap_{k\geq 0} \mathcal{K}^{f'}_{-k} = \bigcap_{k\leq 0} \mathcal{K}^{f'}_{k} = \mathcal{K}^{f'}_{-\infty}.$$

To obtain the converse inclusion, for any $t \geq 0$ we have

$$U_t\mathcal{K}^{f'}_{-\infty} \subseteq \bigvee_{s\geq 0} U_s\mathcal{K}^{f'}_{-\infty} = \bigvee_{n=0}^{\infty} U^n\mathcal{K}^{f'}_{-\infty} = \mathcal{K}^f_{-\infty}.$$

It follows that

$$\mathcal{K}^{f'}_{-\infty} \subseteq U_{-t}\mathcal{K}^{f'}_{-\infty} \subseteq U_{-t}\mathcal{K}^{f'}_0 \subseteq U_{-t}\mathcal{K}^f_0 = \mathcal{K}^f_{-t}.$$

Hence

$$\mathcal{K}^{f'}_{-\infty} \subseteq \bigcap_{t\leq 0} \mathcal{K}^f_t = \mathcal{K}^f_{-\infty}.$$

Therefore $\mathcal{K}^{f'}_{-\infty} = \mathcal{K}^f_{-\infty}$ and the proof is finished. □

As an obvious consequence we obtain that the continuous Γ-stationary process $\{f_t\}$ is deterministic if and only if the associated discrete parameter process is a deterministic one.

Let $\{g'_n\}$ be the maximal white noise contained in the discrete process $\{f'_n\}$ associated to $\{f_t\}$, and $f'_n = u'_n + v'_n$ be the Wold decomposition of $\{f'_n\}$. Then we have

$$\mathcal{K}^{f'}_n = \mathcal{K}^{g'}_n \oplus \mathcal{K}^{f'}_{-\infty} \tag{4.6}$$

and

$$\mathcal{K}^{f'}_\infty = \mathcal{K}^{g'}_\infty \oplus \mathcal{K}^{f'}_{-\infty},$$

where $\mathcal{K}^{g'}_\infty = \mathcal{M}$ coincides with the innovation space of $\{f'_n\}$.

Suppose that $\{f_t\}$ is not deterministic and put

$$V_{g_t} = U_t V_{g'} \qquad (t \in \mathbb{R}), \tag{4.7}$$

where $V_{g'} = V_{g'_0}$. The continuous Γ-stationary process $\{g_t\}$ defined by (4.7) has as a shift group (U_t) and associated discrete parameter process $\{g'_n\}$ - the maximal white noise contained in $\{f'_n\}$. Let us consider the following subspaces of $\mathcal{K}^f_t$

$$\mathcal{N}_t = \mathcal{K}^f_t \ominus \mathcal{K}^f_{-\infty}. \tag{4.8}$$

From (4.2) and (4.6), taking $t = 0$ in (4.8) we have

$$\mathcal{N}_0 = \mathcal{K}^f_0 \ominus \mathcal{K}^f_{-\infty} = \mathcal{K}^{f'}_0 \ominus \mathcal{K}^{f'}_{-\infty} = \mathcal{K}^{g'}_0 = \mathcal{K}^g_0. \tag{4.9}$$

Using again (4.5) it follows that

$$\mathcal{N}_t = U_t\mathcal{N}_0 = U_t\mathcal{K}^g_0 = \bigvee_{s\leq 0} U_{t+s}V_g\mathcal{E} = \bigvee_{s\leq t} U_sV_g\mathcal{E} = \mathcal{K}^g_t.$$

Summing up we have

Proposition 4.2. *Let $\{f_t\}$ be a nondeterministic Γ-stationary process with the shift group (U_t) and $\{f'_n\}$ the associated discrete parameter process. Then*

$$V_{g_t} = U_t V_g$$

gives rise to a continuous Γ-stationary process $\{g_t\}$ which is stationarly cross-correlated with $\{f_t\}$ and has the past and present given by

$$\mathcal{K}_t^g = \mathcal{K}_t^f \ominus \mathcal{K}_{-\infty}^f. \tag{4.10}$$

Proposition 4.3. *For the continuous Γ-stationary process $\{g_t\}$ one has*

$$P_{\mathcal{K}_s^g} V_{g_t} = \mathrm{e}^{s-t} V_{g_s} \qquad (s < t \in \mathbb{R}) \tag{4.11}$$

and the corresponding correlation function is given by

$$\Gamma_g(t) = \mathrm{e}^{-|t|}\Gamma_g(0) \qquad (t \in \mathbb{R}). \tag{4.12}$$

Proof. Let us consider $t \geq 0$. Then by (4.3) we have

$$V_{g_t} = \mathrm{e}^{-t} V_{g'} + \lim_{n\to\infty} \sum_{k=1}^{\infty} (1/k!)(-nt/n+1)^k [(I + A_n)^k - I] V_{g'}.$$

Since $V_{g'_j} = U^j V_{g'}$ is a white noise process it follows that

$$V_{g_t} = \mathrm{e}^{-t} V_{g'} + \eta_t,$$

where $\eta_t a$ is orthogonal to $V_{g'}a, V_{g'_{-1}}a, \ldots$. It follows that $\eta_t a$ is orthogonal on $\mathcal{K}_0^{g'}$. Hence

$$\mathrm{e}^{-t} V_{g'} = P_{\mathcal{K}_0^g} V_{g_t} \qquad (t \geq 0),$$

and so it follows that (using the fact that $V_{g'} = V_g$)

$$\mathrm{e}^{-t} V_{g_s} = P_{\mathcal{K}_s^g} V_{g_{s+t}} \qquad (s \in \mathbb{R}, t \geq 0) \tag{4.13}$$

Taking $s + t = \tau$ it follows that

$$P_{\mathcal{K}_s^g} V_{g_\tau} = \mathrm{e}^{s-\tau} V_{g_s}.$$

Now, for any $a \in \mathcal{E}$, from (4.13) we have

$$\big(\Gamma_g(t)a, a\big) = \big(\Gamma[g_s, g_{s+t}]a, a\big) = \big(V_{g_s}^* V_{g_{s+t}} a, a\big) = \big(V_{g_{s+t}} a, V_{g_s} a\big) =$$

$$= \big(P_{\mathcal{K}_s^g} V_{g_{s+t}} a, V_{g_s} a\big) = \big(\mathrm{e}^{-t} V_g^* a, a\big) = \big(\mathrm{e}^{-t}\Gamma_g(0)a, a\big).$$

Therefore we have for any $t \geq 0$

$$\Gamma_g(t) = \mathrm{e}^{-t}\Gamma_g(0)$$

For $t \leq 0$ one proceeds analogously and it follows that

$$\Gamma_g(t) = \mathrm{e}^{-|t|}\Gamma_g(0) \qquad (t \in \mathbb{R})$$

and the proof is finished. □

By the continuous Γ-stationary process $\{g_t\}$ obtained from the maximal white noise contained in the associated discrete process of $\{f_t\}$, one constructs a process $\{\xi_t\}$ as follows

$$V_{\xi_t} = \tfrac{1}{\sqrt{2}}[V_{g_t} - V_g + \int_0^{2\pi} V_{g_s}\mathrm{d}s] \qquad (t \in \mathbb{R}). \tag{4.14}$$

This process has Γ-orthogonal increments which will play the role of the *differential innovation* of the continuous Γ-stationary process $\{f_t\}$. Firstly let us remark that $\xi_0 = 0$ and for any real a and b we have

$$V_{\xi_b - \xi_a} = \tfrac{1}{\sqrt{2}}[V_{g_b} - V_{g_a} + \int_a^{b} V_{g_s}\mathrm{d}s]. \tag{4.15}$$

PROPOSITION 4.4. *Let $\{\xi_t\}$ be the process defined by* (4.14). *Then*

(i) $\{\xi_t\}$ has increments which are stationary under the group (U_t), i.e.

$$U_t V_{\xi_b-\xi_a} = V_{\xi_{b+t}} - V_{\xi_{a+t}} \qquad (a, b, t \in \mathbb{R}).$$

(ii) $\{\xi_t\}$ has Γ-orthogonal increments, i.e., if $-\infty < a < b \le c < d < +\infty$, then $\Gamma[\xi_b - \xi_a, \xi_d - \xi_c] = 0$.

(iii) For any real a, b one has

$$\Gamma[\xi_b - \xi_a, \xi_b - \xi_a] = |b-a|\,\Gamma_g(0)$$

.

(iv) If $-\infty < a < c < b < d < +\infty$, then

$$\Gamma[\xi_b - \xi_a, \xi_d - \xi_c] = \Gamma[\xi_b - \xi_c, \xi_b - \xi_c] = (b-c)\Gamma_g(0).$$

Proof. The assertion (i) follows from (4.15). To prove (ii) the form of the correlation function and (4.15) are used. If $< a < b \le c < d$ then

$$2\Gamma[\xi_b - \xi_a, \xi_d - \xi_c] = \Gamma\Big[V_{g_b-g_a} + \int_a^b V_{g_s}\mathrm{d}s, V_{g_d-g_c} + \int_c^d V_{g_t}\mathrm{d}t\Big] =$$

$$= \Big[\big(\mathrm{e}^{b-d} - \mathrm{e}^{b-c} + \int_c^d \mathrm{e}^{b-t}\mathrm{d}t\big) - \big(\mathrm{e}^{a-d} - \mathrm{e}^{a-c} + \int_c^d \mathrm{e}^{a-t}\mathrm{d}t\big) +$$

$$+ \int_a^b \big(\mathrm{e}^{s-d} - \mathrm{e}^{s-c} + \int_a^d \mathrm{e}^{s-t}\mathrm{d}t\big)\mathrm{d}s\Big]\Gamma_g(0) = 0.$$

This followed by the fact that each expression in $(\cdots)$ is zero. Hence $\{\xi_t\}$ is a process with Γ-orthogonal increments.

For (iii), suppose firstly that $0 = a < b$. Using again (4.12) we obtain

$$2\Gamma[\xi_b - \xi_0, \xi_b - \xi_0] = \Gamma\Big[V_{g_b-g_0} + \int_0^b V_{g_s}\mathrm{d}s, V_{g_0-g_0} + \int_0^b V_{g_t}\mathrm{d}t\Big] =$$

$$= \Gamma_g(0)\Big[1 - \mathrm{e}^{-b} + \int_0^b \mathrm{e}^{t-b}\mathrm{d}t - \mathrm{e}^{-b} + 1 - \int_0^b \mathrm{e}^{-t}\mathrm{d}t +$$

$$+ \int_0^b \big(\mathrm{e}^{s-b} - \mathrm{e}^{-s} + \int_0^b \mathrm{e}^{-|t-s|}\mathrm{d}t\big)\mathrm{d}s\Big] =$$

$$= \Big[2(1-\mathrm{e}^{-b}) + \int_0^b \big(\int_0^s \mathrm{e}^{t-s}\mathrm{d}t + \int_s^b \mathrm{e}^{s-t}\mathrm{d}t\big)\mathrm{d}s\Big]\Gamma_g(0) =$$

$$= \Big[2(1-\mathrm{e}^{-b}) + 2b + 2(\mathrm{e}^{-b}-1)\Big]\Gamma_g(0) = 2b\Gamma_g(0),$$

i.e. it results that

$$\Gamma[\xi_b - \xi_a, \xi_b - \xi_a] = b\Gamma_g(0).$$

From (i), for $a < b$ one has

$$\Gamma[\xi_b - \xi_a, \xi_b - \xi_a] = \Gamma[U_a V_{\xi_{b-a}-\xi_0}, U_a V_{\xi_{b-a}-\xi_0} =$$

$$= \Gamma[\xi_{b-a} - \xi_0, \xi_{b-a} - \xi_0] = (b-a)\Gamma_g(0).$$

Results for $a > b$ follows analogously, and (iii) is verified.

If $-\infty < a < c < b < d < +\infty$ then, taking account of the Γ-orthogonal increments we have

$$\Gamma[\xi_b - \xi_a, \xi_d - \xi_c] = \Gamma[\xi_b - \xi_c + \xi_c - \xi_a, \xi_d - \xi_b + \xi_b - \xi_c] =$$

$$= \Gamma[\xi_b - \xi_c, \xi_d - \xi_b] + \Gamma[\xi_b - \xi_c, \xi_b - \xi_c] + \Gamma[\xi_c - \xi_a, \xi_d - \xi_b] +$$

$$+ \Gamma[\xi_c - \xi_a, \xi_b - \xi_c] = \Gamma[\xi_b - \xi_c, \xi_b - \xi_c] = (b-c)\Gamma_g(0),$$

and the proof is finished. □

By (4.14) we have an expression for ξ_t in terms of the continuous process $\{g_t\}$ corresponding to the maximal white noise contained in the associated discrete process of $\{f_t\}$. In the next Proposition an inverse is obtained.

PROPOSITION 4.5. *For any $a \in \mathcal{E}$ we have*

$$V_{g_t} a = \sqrt{2}\Big(V_{\xi_t} a - \int_{-\infty}^t \mathrm{e}^{s-t} V_{\xi_s} a \mathrm{d}s\Big) = \sqrt{2}\int_{-\infty}^t \mathrm{e}^{s-t}\mathrm{d}V_{\xi_s} a. \tag{4.16}$$

Proof. From the strong continuity of the group (U_t) and the definition of $\{g_t\}$ it follows that $t \to V_{\xi_t}$ is an operator valued continuous function, and, consequently, for any $a \in \mathcal{E}$, $t \to V_{\xi_t} a$ is a continuous function. Hence for $-\infty < \alpha < \beta \le t$, the Rieman integral $\int_\alpha^\beta \mathrm{e}^{s-t} V_{\xi_s} a \mathrm{d}s$ exists. By the fact that

$$\Big\|\int_\alpha^\beta \mathrm{e}^{s-t} V_{\xi_s} a \mathrm{d}s\Big\| \le \int_\alpha^\beta \mathrm{e}^{s-t}\left\|V_{\xi_s} a\right\|\mathrm{d}s = \int_\alpha^\beta \mathrm{e}^{s-t}\sqrt{(\Gamma[\xi_s,\xi_s]a, a)}\mathrm{d}s \le$$

$$\le \Gamma_g(0)^{1/2}\left\|a\right\|\int_\alpha^\beta \mathrm{e}^{s-t}\sqrt{s}\mathrm{d}s,$$

it follows that the integral $\int_{-\infty}^t \mathrm{e}^{s-t} V_{\xi_s} \mathrm{d}t$ is convergent. If we take $t = 0$, then

$$\sqrt{2}\int_{-\infty}^0 \mathrm{e}^s V_{\xi_0 - \xi_s} a \mathrm{d}s = -\int_{-\infty}^0 \mathrm{e}^s \big(V_{g_s - g_0} a + \int_0^s V_{g_\tau} a \mathrm{d}\tau\big)\mathrm{d}s$$

$$= V_{g_0} a - \int_{-\infty}^0 \mathrm{e}^s V_{g_s} a \mathrm{d}s + V_{g_0} a + \int_{-\infty}^0 \int_s^0 \mathrm{e}^s V_{g_\tau} a \mathrm{d}\tau \mathrm{d}s =$$

$$= V_{g_0}a - \int_{-\infty}^{0} \mathrm{e}^s V_{g_s} a\mathrm{d}s + \int_{-\infty}^{0}\int_{-\infty}^{0} \mathrm{e}^s V_{g_\tau} a\mathrm{d}s\mathrm{d}\tau =$$

$$= V_{g_0}a - \int_{-\infty}^{0} \mathrm{e}^s V_{g_s} a\mathrm{d}s + \int_{-\infty}^{0} \Big(\int_{-\infty}^{0} \mathrm{e}^s \mathrm{d}s\Big) V_{g_\tau} a\mathrm{d}\tau =$$

$$= V_{g_0}a - \int_{-\infty}^{0} \mathrm{e}^s V_{g_s} a\mathrm{d}s + \int_{-\infty}^{0} \mathrm{e}^\tau V_{g_\tau} a\mathrm{d}\tau = V_{g_0}a.$$

Therefore

$$V_{g_0}a = \sqrt{2}\int_{-\infty}^{0} \mathrm{e}^s (V_{\xi_0} - V_{\xi_s})a\mathrm{d}s. \tag{4.17}$$

Applying U_t in (4.17) it follows that

$$V_{g_t}a = \sqrt{2}\Big(\int_{-\infty}^{0} \mathrm{e}^s V_{\xi_t} a\mathrm{d}s - \int_{-\infty}^{0} \mathrm{e}^s V_{\xi_{s+t}} a\mathrm{d}s\Big) =$$

$$= \sqrt{2}\Big(V_{\xi_t}a - \int_{-\infty}^{0} \mathrm{e}^s V_{\xi_{s+t}} a\mathrm{d}s\Big) = \sqrt{2}\Big(V_{\xi_t}a - \int_{-\infty}^{t} \mathrm{e}^{s-t} V_{\xi_s} a\mathrm{d}s\Big).$$

This way, the first part of (4.16) is proved. The second part it follows by integrating by parts.

$$\int_{-\infty}^{t} \mathrm{e}^{s-t}\mathrm{d}V_{\xi_s}a = \big[\mathrm{e}^{s-t}V_{\xi_s}\big]_{s\to-\infty}^{s=t} - \int_{-\infty}^{t} V_{\xi_s}a\mathrm{d}_s(\mathrm{e}^{s-t}) =$$

$$= V_{\xi_t}a - \int_{-\infty}^{t} \mathrm{e}^{s-t} V_{\xi_s} a\mathrm{d}s,$$

and the proof is finished. □

As a remark, for any complex valued function $c \in L^2(-\infty,+\infty)$, the integral $\int_{-\infty}^{+\infty} c(s)\mathrm{d}V_{\xi_s}a$ exists. This yelds an expression for the past and present of the continuous process $\{g_t\}$ in terms of the orthogonal increments.

Proposition 4.6. *For any real t, the past and present $\mathcal{K}_t^g$ of the Γ-stationary process $\{g_t\}$ is the set of all integrals of the form $\int_{-\infty}^{+\infty} c(s)\mathrm{d}V_{\xi_s}a$, where $a \in \mathcal{E}$ and $c \in L^2(-\infty,t]$, i.e.,*

$$\mathcal{K}_t^g = \bigvee_{\sigma,\tau\le t} V_{\xi_\sigma - \xi_\tau}\mathcal{E}. \tag{4.18}$$

Proof. Let $\mathcal{K}_t$ be the set of all the integrals of the above form. By Proposition 4.5 it follows that for any $a \in \mathcal{E}$ and $-\infty < \tau \le t < +\infty$ we have

$$V_{g_\tau}a = \sqrt{2}\int_{-\infty}^{\tau} \mathrm{e}^{s-\tau}\mathrm{d}V_{\xi_s}a = \int_{-\infty}^{\tau} c(s)\mathrm{d}V_{\xi_s}a,$$

where $c(s) = \sqrt{2}\mathrm{e}^{s-\tau}$ on $(-\infty,\tau]$ and $c(s) = 0$ on $(\tau,t]$. Because c is from $L^2(-\infty,t]$ it follows that $V_{g_\tau} \in \mathcal{K}_t^\xi$. Therefore

$$\mathcal{K}_t^g = \bigvee_{\tau\le t} V_{g_\tau}\mathcal{E} \subseteq \mathcal{K}_t^\xi.$$

Conversely, let us consider the elements of the form

$$k = \int_{-\infty}^{t} c(s)\mathrm{d}V_{\xi_s} a, \tag{4.19}$$

where $c \in L^2(-\infty, t]$ and $a \in \mathcal{E}$. Let us take first step functions, i.e.,

$$c(s) = \sum_{k=1}^{n} c_k x_{J_k}(s),$$

where $J_k = [\alpha_k, \beta_k] \subset (-\infty, t]$. Then

$$k = \sum_{k=1}^{n} c_k \big(V_{\xi_{\beta(k)}} a - V_{\xi_{\alpha(k)}} a\big),$$

and from (4.15) it follows that $k \in \mathcal{K}_t^g$. Now, let $c \in L^2(-\infty, t]$ where $c = \lim_{n\to\infty} c^{(n)}$, and $c^{(n)}$ are step functions. it follows that

$$k = \lim_{n\to\infty} \int_{-\infty}^{t} c^{(n)}(s)\mathrm{d}V_{\xi_s} a$$

and, since $\mathcal{K}_t^g$ is closed, it follows that $k \in \mathcal{K}_t^g$. Therefore $\mathcal{K}_t^\xi \subseteq \mathcal{K}_t^g$, and the proof is finished. □

From the above results one may assert the following Wold decomposition type theorem for continuous stationary processes in a complete correlated action $\{\mathcal{E}, \mathcal{H}, \Gamma\}$.

Theorem 4.7. *Let $\{f_t\}$ be a Γ-stationary continuous process and $\{g_n'\}$ be the maximal white noise contained in the discrete time process $\{f_n'\}$ associated with $\{f_t\}$. Let $\{g_t\}$ be the continuous Γ-stationary process corresponding to $\{g_n'\}$ by $V_{g_t} = U_t V_{g'}$ and $\{\xi_t\}$ the process with Γ-orthogonal increments defined by* (4.14). *Then $\{f_t\}$ admits a unique decomposition of the form*

$$f_t = u_t + v_t, \tag{4.20}$$

where $\{u_t\}$ is a moving average, i.e.,

$$V_{u_t} = \int_{0}^{\infty} c(s)\mathrm{d}V_{\xi_{t-s}} \tag{4.21}$$

with $c \in L^2[0,\infty)$ and $\mathcal{K}_t^u = \bigvee_{s\le t} V_{u_s}\mathcal{E} = \mathcal{K}_t^g$, $t \in (-\infty, +\infty)$, and $\{v_t\}$ is a deterministic process with $\mathcal{K}_t^v = \mathcal{K}_{-\infty}^f$.

Proof. We have only to put

$$u_t = \mathcal{P}_{\mathcal{H}_t^g} f_t \tag{4.22}$$

and

$$v_t = \big(I - \mathcal{P}_{\mathcal{H}_t^g}\big) f_t, \tag{4.23}$$

where $\mathcal{P}_{\mathcal{H}_t^g}$ is the Γ-orthogonal projection on the right $\mathcal{L}(\mathcal{E})$-submodule $\mathcal{H}_t^g$ from $\mathcal{H}$.

The requested properties it follows by the above results and the uniqueness can be proved similar as in Theorem 6.8 of [5]. □

Let us remark that $\{u_t\}$ and $\{v_t\}$ have the same shift groups, namely the shift group of $\{f_t\}$.

Finally, between the Wold decomposition for the discrete case and the continuous case there exists the following correspondence.

PROPOSITION 4.8. *The moving average part $\{u'_n\}$ and the deterministic part $\{v'_n\}$ of the Wold decomposition for the discrete process $\{f'_n\}$ associated with the Γ-stationary continuous process $\{f_t\}$ are the discrete processes associated with the moving average part $\{u_t\}$ and the deterministic part $\{v_t\}$, respectively, in the Wold decomposition of the continuous process $\{f_t\}$.*

Proof. From the fact that $\mathcal{K}^f_{-\infty} = \mathcal{K}^{f'}_{-\infty}$ and $V_{f'} = V_f$, it follows that

$$V_{v_0} = P_{\mathcal{K}^f_{-\infty}} V_f = P_{\mathcal{K}^{f'}_{-\infty}} V_{f'} = V_{v'_0},$$

hence $v_0 = v'_0$. Consequently we have also

$$V_{u_0} = V_f - V_{v_0} = V_{f'} - V_{v'_0} = V_{u'_0},$$

and thus we have the desired result. □

5 Spectral analysis

In the remaining of this chapter, some spectral properties of a continuous stationary processes in complete correlated action $\{\mathcal{E}, \mathcal{H}, \Gamma\}$ are analysed.

Let $\{f_t\}$ be a Γ-stationary process with (U_t) the shift group and U the cogenerator on $\mathcal{K}^f_\infty$. From Stone's Theorem, there exists a unique $\mathcal{L}(\mathcal{K}^f_\infty)$-valued spectral measure E on the real line such that

$$(5.1) \qquad U_t = \int_{-\infty}^{+\infty} \mathrm{e}^{-itx} \mathrm{d}E(x).$$

If Γ_f is the correlation function of the process $\{f_t\}$, then for every $a \in \mathcal{E}$ we have

$$\big(\Gamma_f(t)a, a\big) = \big(\Gamma[f_0, f_t]a, a\big) = \big(\Gamma_f[V_f, U_t V_f]a, a\big) = \big(V_f^* U_t V_f a, a\big) =$$

$$= \big(U_t V_f a, V_f a\big) = \int_{-\infty}^{+\infty} \mathrm{e}^{-itx} \mathrm{d}\big(E(x) V_f a, V_f a\big) = \int_{-\infty}^{+\infty} \mathrm{e}^{-itx} \mathrm{d}\big(V_f^* E(x) V_f a, a\big).$$

Hence if we take

$$(5.2) \qquad F = V_f^* E V_f$$

then F is an $\mathcal{L}(\mathcal{E})$-valued semispectral measure on the real line which has as a spectral dilation $[\mathcal{K}^f_\infty, V_f, E]$. Moreover,

$$(5.3) \qquad \Gamma_f(t) = \int_{-\infty}^{+\infty} \mathrm{e}^{itx} \mathrm{d}F(x).$$

The semispectral measure obtained as above is called the *spectral distribution* of the process $\{f_t\}$.

By means of the mapping of the real line onto the unit circle given by

$$\mathrm{e}^{i\theta} = (x-i)(x+i)^{-1} \tag{5.4}$$

(or equivalently, $\theta = -2\arctan x$), to the semispectral measure F on $\mathbb{R}$ corresponds a semispectral measure F_1 on $\mathbb{T}$ which is the spectral distribution of the discrete parameter process $\{f'_n\}$ associated with $\{f_t\}$. Indeed, let $\Gamma_{f'}(n)$ be the correlation function of $\{f'_n\}$ and F' the corresponding spectral distribution. If $[\mathcal{K}^{f'}_\infty, V_{f'}, E']$ is the spectral dilation of E', then the unitary operator U on $\mathcal{K}^{f'}_\infty$ given by

$$U = \int_0^{2\pi} \mathrm{e}^{-i\theta}\mathrm{d}E'(\theta)$$

is the shift operator of the process $\{f'_n\}$. As we know U is the cogenerator of the shift group associated with $\{f_t\}$, which is defined (see e.g. [11]) by

$$U = (H-iI)(H+iI)^{-1},$$

where $H = -iA$ and $A = \lim\limits_{t\to 0}\frac{1}{t}[U_t - I]$ is the infinitesimal generator of one parameter group of unitary operators (U_t). Taking into account that $H = \int_{-\infty}^{+\infty} x\mathrm{d}E(x)$ it follows that

$$\int_0^{2\pi} \mathrm{e}^{-i\theta}\mathrm{d}E_1(\theta) = \int_{-\infty}^{+\infty} (x-i)(x+i)\mathrm{d}E(x) = U.$$

From the uniqueness of the spectral representation of unitary operators, the desired result follows.

Let $L^2_{\mathbb{R}}(\mathcal{E})$ be the space of the square integrable functions from the real axis into $\mathcal{E}$, and $H^2_\Delta(\mathcal{E})$ be the space of all analytic functions on the upper half plane Δ with values in $\mathcal{E}$ such that

$$\sup_{y>0}\int_{-\infty}^{+\infty} \|f(x+iy)\|^2\,\mathrm{d}x < \infty. \tag{5.5}$$

It is known [11] that the space $L^2(\mathcal{E})$ is transformed onto $L^2_{\mathbb{R}}(\mathcal{E})$, if for any g in $L^2(\mathcal{E})$ we associate f in $L^2_{\mathbb{R}}(\mathcal{E})$ by

$$f(x) = (x+i)^{-1}g(x-i/x+i). \tag{5.6}$$

In a similar way a correspondence between $H^2(\mathcal{E})$ and $H^2_\Delta(\mathcal{E})$ is realized.

Let $\mathcal{E}$ and $\mathcal{F}$ be Hilbert spaces. As in the disc case we define an L^2-bounded analytic function $\{\mathcal{E},\mathcal{F},S(z)\}$ on Δ by

$$\sup_{y>0}\frac{1}{\pi}\int_{-\infty}^{+\infty} \|S(x+iy)a\|^2\,\mathrm{d}x \le M^2\,\|a\|^2. \tag{5.7}$$

Via the above identification there exists a one-to-one correspondence between the L^2-bounded analytic function $\Theta(\lambda)$ on the unit disc $\mathbb{D}$ and the L^2-bounded analytic function $S(z)$ on the half plane Δ. The L^2-bounded function $S(z)$ corresponding to the maximal function of the associated discrete process $\{f'_n\}$ is called the maximal function of the continuous process $\{f_t\}$.

Due to the one-to-one correspondence between the spectral distribution of $\{f_t\}$ and its associated discrete process $\{f'_n\}$, if we denote by F_S the semispectral measure corresponding to F, then we assert the following

THEOREM 5.1. *Let $\{f_t\}$ be a continuous Γ-stationary process, F its spectral distribution, and $\{\mathcal{E}, \mathcal{F}, S(z)\}$ the attached maximal function. If $f_t = u_t + v_t$ is the corresponding Wold decomposition, then*

(i) F_S is the spectral distribution of the purely nondeterministic process $\{u_t\}$.

(ii) The process $\{f_t\}$ is nondeterministic if there exists a positive function $h \in L^1(dx/1+x^2)$ such that $h \leq h_a$, where h_a is the derivative of $\mathrm{d}(F(x)a, a)$ with respect to $\mathrm{d}x/1 + x^2$ and

$$\int_{-\infty}^{+\infty} \frac{\log h(x)}{1+x^2} \mathrm{d}x > -\infty. \tag{5.8}$$

The condition becomes necessary if the maximal function of the associated process has a scalar multiple.

Proof. The proof is obtained via the above identification between $L^2(\mathcal{E})$ and $L^2_{\mathbb{R}}(\mathcal{E})$. Concernind the last sentence, let $\delta(\lambda)$ be the scalar multiple of the maximal function $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ of the associated discrete process. Then by the definition of scalar multiple, there exists a contractive analitic function $\{\mathcal{F}, \mathcal{E}, \Omega(\lambda)\}$ such that

$$\Omega(\lambda)\Theta(\lambda) = \delta(\lambda) I_{\mathcal{E}} \quad \text{and} \quad \Theta(\lambda)\Omega(\lambda) = \delta(\lambda) I_{\mathcal{F}},$$

and

$$|\delta(\lambda)|^2 \|a\|^2 = \|\Omega(\lambda)\Theta(\lambda)a\|^2 \leq \|\Theta(\lambda)a\|^2 .$$

Taking $h = |\delta(\lambda)|^2$, a simple calculus show that $0 \leq h \leq h_a$ and (5.8) follows. □

6 Periodicity and harmonizability

Let $\{f_t\}_{t\in\mathbb{R}}$ be a *periodically* Γ-*correlated* process, i.e. a process in the complete correlated action $\{\mathcal{E}, \mathcal{H}, \Gamma\}$, whose correlation function $\Gamma_f(s,t)$ given by (2.1) satisfies the periodicity condition

$$\Gamma_f(s+T, t+T) = \Gamma_f(s,t) \qquad (s,t \in \mathbb{R}) \tag{6.1}$$

for a positive real number T. The smallest $T > 0$ satisfying (6.1) is the *period* of the process $\{f_t\}_{t\in\mathbb{R}}$.

Analoguously with the scalar case [1] the notion of an *almost-periodically* Γ-correlated process $\{f_t\}_{t\in\mathbb{R}}$ can be introduced under the condition that its correlation function $\Gamma_f(s,t)$ is uniformly continuous, and is an almost-periodic function with respect to T in the sense of Bohr, but in this note we are concerned mainly on the periodic case.

As in the discrete case, to the continuous parameter process $\{f_t\}_{t\in\mathbb{R}}$, beside the correlation function $\Gamma_f(s,t)$, for any $t \in \mathbb{R}$ the *covariance function* is defined by

$$B(s,t) = \Gamma_f(s+t, s) \qquad (s \in \mathbb{R}), \tag{6.2}$$

and of course we have conversely

$$\Gamma_f(s,t) = B(t, s-t) \qquad (s, \in \mathbb{R}). \tag{6.3}$$

Since $B(s,t)$ is an operator valued periodic function in s with the same period T as $\{f_t\}_{t\in\mathbb{R}}$, there exists the following Fourier representation

$$B(s,t) = \sum_{k\in\mathbb{Z}} B_k(t) \exp\Big(\frac{2\pi i k s}{T}\Big), \tag{6.4}$$

where the $\mathcal{L}(\mathcal{E})$-valued coefficients $B_k(t)$ are given by

$$B_k(t) = \frac{1}{T}\int_0^T B(s,t)\exp\Big(\frac{-2\pi iks}{T}\Big)\mathrm{d}s \qquad (t \in \mathbb{R}, k \in \mathbb{Z}). \tag{6.5}$$

Following the Gladyshev's results [1] the following extension to the complete correlated case $\{\mathcal{E}, \mathcal{H}, \Gamma\}$ can be obtained.

Theorem 6.1. *A norm continuous operator valued function $B(s,t)$ which satisfies condition (6.1) for every $t, s \in \mathbb{R}$ is the covariance function of some continuous periodically Γ-correlated process with the same period $T > 0$ if and only if the $\mathcal{L}(\mathcal{E})$-valued functions B_{jk}, $(j, k \in \mathbb{Z})$, are positive definite, i.e.,*

$$\sum_{p,q=1}^{n} \big\langle A_p^* B_{k_p k_q}(t_p - t_q) A_q a, a\big\rangle_{\mathcal{E}} \geq 0 \qquad (a \in \mathcal{E}), \tag{6.6}$$

for any $n \geq 1$, $k_1, \ldots, k_n \in \mathbb{Z}$, $t_1, \ldots, t_n \in \mathbb{R}$, and $A_1, \ldots, A_n \in \mathcal{L}(\mathcal{E})$, where

$$B_{jk}(t) = B_{k-j}(t)\exp\Big(\frac{2\pi ijt}{T}\Big). \tag{6.7}$$

Proof. Let us remark that the correlation function and the covariance function of an arbitrary Γ-correlated process $\{f_t\}_{t\in\mathbb{R}}$ are positive definite functions. Indeed, for any $n \geq 1$, $a_1, \ldots, a_n \in \mathcal{E}$, $t_1, \ldots, t_n \in \mathbb{R}$, and $A_1, \ldots, A_n \in \mathcal{L}(\mathcal{E})$ we have, taking account that for a finite system of elements $\{a_k\} \subset \mathcal{E}$ there exists a system of operators $S_k \in \mathcal{L}(\mathcal{E})$ such that $a_k = S_k a$,

$$\sum_{p,q=1}^{n} \big\langle A_p^*\Gamma(t_p, t_q)A_q a_q, a_p\big\rangle_{\mathcal{E}} = \sum_{p,q=1}^{n} \big\langle A_p^*\Gamma[f_{t_p}, f_{t_q}]A_q a_q, a_p\big\rangle_{\mathcal{E}} =$$

$$= \sum_{p,q=1}^{n} \big\langle \Gamma[A_p f_{t_p}, A_q f_{t_q}]a_q, a_p\big\rangle_{\mathcal{E}} = \sum_{p,q=1}^{n} \big\langle \Gamma[A_p f_{t_p}, A_q f_{t_q}]S_q a, S_p a\big\rangle_{\mathcal{E}} =$$

$$= \Big\langle \Gamma[\sum_{p=1}^{n} S_p A_p f_{t_p}, \sum_{q=1}^{n} S_q A_q f_{t_q}]a, a\Big\rangle_{\mathcal{E}} = \langle \Gamma[h,h]a, a\rangle_{\mathcal{E}} =$$

$$= \langle V_h^* V_h a, a\rangle_{\mathcal{E}} = \|V_h a\|_{\mathcal{K}}^2 \geq 0.$$

Taking into account (6.2) it follows that (6.6) is verified.

Conversely, if (6.6) is verified, then if we put

$$B_n(s,t) = \frac{1}{n}\sum_{m=0}^{n-1}\sum_{k=-m}^{m} B_k(t)\exp\Big(\frac{2ks}{T}\Big)$$

satisfies that

$$\sum_{p,q=1}^{k} \big\langle A_p^* B_n(t_q - t_p)A_q a, a\big\rangle_{\mathcal{E}} \geq 0$$

for any $k \geq 1$. Since $B_n(s,t)$ converges to $B(s,t)$, it follows that $B(s,t)$ also verifies the above inequality, and, consequently is the covariance function of some periodically Γ-correlated process. □

In the discrete periodically Γ-correlated case [15], similarly with the scalar case, was proved that any periodically Γ-correlated process $\{f_n\}_{n\in\mathbb{Z}}$ is Γ-harmonizable [13] i.e., there exists an $\mathcal{L}(\mathcal{E})$-valued semispectral measure (bimeasure) K on $\mathbb{T}^2$ such that

$$\Gamma[f_m, f_n] = \iint_{\mathbb{T}^2} \chi(t)^m \chi(s)^{-n} K(\mathrm{d}t, \mathrm{d}s), \tag{6.8}$$

where $\chi^n = \mathrm{e}^{-2\pi i n t}$.

Moreover, the support of the $\mathcal{L}(\mathcal{E})$-valued semispectral bimeasure attached to a discrete periodically Γ-correlated process with the period $T \geq 1$ is concentrated on $2T-1$ equidistant stright line segments $v = u - 2k\pi k/T$, $k \in \{0, \pm 1, \dots . \pm (T-1)\}$ parallel to the diagonal of the square $[0, 2\pi] \times [0, 2\pi]$. Obvious if $T = 1$ then the Γ-harmonizable process $\{f_n\}_{n\in\mathbb{Z}}$ is stationary Γ-correlated and the support is concentrated only on the diagonal of the square.

In the continuous parameter case, this nice property is no longer valid even in the scalar case (see [1]). Only on supplementary conditions, some particular periodically Γ-correlated processes with continuous time will become Γ-harmonizable, and similarly, the support of the bimeasure will be on parallel equidistant stright lines in the plane.

A process $\{f_t\}_{t\in\mathbb{R}}$ is *strongly* Γ*-harmonizable* if the correlation function $\Gamma_f(s,t)$ can be expressed as

$$\Gamma_f(s,t) = \iint_{\mathbb{R}^2} \mathrm{e}^{i(su-tv)} K(\mathrm{d}u, \mathrm{d}v) \qquad (s, t \in \mathbb{R}), \tag{6.9}$$

for some positive definite $\mathcal{L}(\mathcal{E})$-valued semispectral bimeasure K of bounded variation.

A process $\{f_t\}_{t\in\mathbb{R}}$ is *weakly* Γ*-harmonizable* if its correlation function can be expressed in the form (6.9) for some $\mathcal{L}(\mathcal{E})$-valued semispectral bimeasure K of finite variation.

Similarly as in [3], in the supplementary condition of strongly harmonizability can be proved the following

Proposition 6.2. *Let $\{f_t\}_{t\in\mathbb{R}}$ be a strongly Γ-harmonizable process in the complete correlated action $\{\mathcal{E}, \mathcal{H}, \Gamma\}$. Then $\{f_t\}_{t\in\mathbb{R}}$ is periodically Γ-correlated processes with period T if and only if the support of K is in the set Δ, where*

$$\Delta = \{(u,v) \in \mathbb{R}^2;\ v = u - 2\pi k/T,\ k \in \mathbb{Z}\}. \tag{6.10}$$

Proof. For any $s, t \in \mathbb{R}$, since $\mathrm{e}^{iT(u-v)} = 1$ for $(u,v) \in \Delta$ we have

$$\Gamma_f(s+T, t+T) = \iint_{\Delta} \mathrm{e}^{i[(s+T)u-(t+T)v]} K(\mathrm{d}u, \mathrm{d}v) =$$

$$= \iint_{\Delta} \mathrm{e}^{i(su-tv)} \mathrm{e}^{iT(u-v)} K(\mathrm{d}u, \mathrm{d}v) = \iint_{\Delta} \mathrm{e}^{i(su-tv)} K(\mathrm{d}u, \mathrm{d}v) =$$

$$= \iint_{\mathbb{R}^2} \mathrm{e}^{i(su-tv)} K(\mathrm{d}u, \mathrm{d}v) = \Gamma_f(s,t),$$

therefore $\{f_t\}_{t\in\mathbb{R}}$ is a periodically Γ-correlated process with period T.

Conversely, if $\{f_t\}_{t\in\mathbb{R}}$ is periodically Γ-correlated with period T, then for any $N \geq 1$ and $s, t \in \mathbb{R}$ we have

$$\Gamma_f(s,t) = \frac{1}{2N+1} \sum_{k=-N}^{N} \Gamma_f(s+kT, t+kT) =$$

$$= \frac{1}{2N+1}\sum_{k=-N}^{N}\iint_{\mathbb{R}^2} \mathrm{e}^{i[(s+kT)u-(t+kT)v]}K(\mathrm{d}u,\mathrm{d}v) =$$
$$= \iint_{\mathbb{R}^2} \frac{\sin[(N+\frac{1}{2})T(u-v)]}{2(N+\frac{1}{2})\sin\frac{T(u-v)}{2}}\mathrm{e}^{i(su-tv)}K(\mathrm{d}u,\mathrm{d}v).$$

Since the fraction under the last integral has the property that take the value 1 on Δ, is bounded and continuous on $\mathbb{R}$ and converges pointwise to 1 as $N \to \infty$, by the bounded convergence theorem we have that

$$\Gamma_f(s,t) = \iint_{\Delta} \mathrm{e}^{i(su-tv)}K(\mathrm{d}u,\mathrm{d}v),$$

which implies that the support of K is in the set Δ. □

The previous Proposition is valid in the weakly Γ-harmonizable case, too, and similarly [6] can be proved the following

PROPOSITION 6.3. *Let $\{f_t\}_{t\in\mathbb{R}}$ be a weakly Γ-harmonizable process in the complete correlated action $\{\mathcal{E},\mathcal{H},\Gamma\}$. Then $\{f_t\}_{t\in\mathbb{R}}$ is periodically Γ-correlated processes with period T if and only if the support of K is in the set Δ given by* (6.10).

It is known that in the discrete case, any periodically Γ-correlated process is Γ-harmonizable and, moreover, to each periodically Γ-correlated process we can attach a stationary Γ-correlated process, helping us in solving most of the prediction problems by a "stationarization procedure". In the continuous case, this fact can not be done so simply, but at least in the strongly Γ-harmonizable case a stationarization can be done as follows.

PROPOSITION 6.4. *Let $\{f_t\}_{t\in\mathbb{R}}$ be a strongly Γ-harmonizable process. If $\{f_t\}_{t\in\mathbb{R}}$ is also periodically Γ-correlated processes with period $T>0$, then it can be represented into the form*

$$V_{f_t} = \sum_{k\in\mathbb{Z}} \exp\Big(\frac{2\pi ikt}{T}\Big)V_{g_k(t)} \qquad (t\in\mathbb{R}), \tag{6.11}$$

where $\{g_k(t)\}_{k\in\mathbb{Z}}$ is a family of stationary Γ-cross-correlated processes given by

$$V_{g_k(t)} = \int_0^{\frac{2\pi}{T}} \mathrm{e}^{itu}\xi\big(\mathrm{d}u+\tfrac{2\pi}{T}\big) \qquad (t\in\mathbb{R}), \tag{6.12}$$

ξ being an $\mathcal{L}(\mathcal{E},\mathcal{K})$-valued semispectral measure on $\mathbb{R}$.

Proof. Let $\sigma_k = \big[\frac{2\pi}{T},\frac{2\pi(k+1)}{T}\big)$ for $k\in\mathbb{Z}$. It follows that $\{\sigma_k; k\in\mathbb{Z}\}$ is a countable partition of $\mathbb{R}$. The process $\{f_t\}_{t\in\mathbb{R}}$ being a strongly Γ-harmonizable one, $\Gamma_f(s,t)$ is given by (6.9) with the $\mathcal{L}(\mathcal{E})$-valued semispectral bimeasure K on $\mathbb{R}^2$. If we consider the corresponding Naimark spectral dilation $[\mathcal{K},W,E]$ of K, then, up to a unitary equivalence, the semispectral measure $\xi(\sigma) = E(\sigma,\cdot)W$ is a representing measure of $\{f_t\}_{t\in\mathbb{R}}$, that is

$$V_{f_t} = \int_{\mathbb{R}} \mathrm{e}^{itu}\xi(\mathrm{d}u). \tag{6.13}$$

For $k\in\mathbb{Z}$ let us consider the semispectral measures ξ_k obtained by $\xi_k(\sigma) = \xi(\sigma\cap\sigma_k)$, $\sigma\in\mathcal{B}(\mathbb{R})$, and the process $\{h_k(t)\}$ defined by

$$V_{h_k(t)} = \int_{\sigma_k} \mathrm{e}^{itu}\xi(\mathrm{d}u) = \int_{\mathbb{R}} \mathrm{e}^{itu}\xi_k(\mathrm{d}u). \tag{6.14}$$

It follows that $V_{f_t} = \sum_{k\in\mathbb{Z}} V_{h_k(t)}$ and

$$V_{h_k(t)} = \int_{\frac{2\pi k}{T}}^{\frac{2\pi(k+1)}{T}} \mathrm{e}^{itu}\xi(\mathrm{d}u) = \exp\left(\tfrac{2\pi ikt}{T}\right)\int_0^{\frac{2\pi}{T}} \mathrm{e}^{itu}\xi(\mathrm{d}u + \tfrac{2\pi k}{T}) = \exp\left(\tfrac{2\pi ikt}{T}\right)V_{g_k(t)}.$$

Therefore (6.11) is obtained. Since $\{f_t\}_{t\in\mathbb{R}}$ is a periodically Γ-correlated too, using the discrete case of Gladyshev's theorem in the complete correlated case $\{\mathcal{E}, \mathcal{H}, \Gamma\}$ [15] it follows that

$$\Gamma_{jk}(s,t) = \Gamma[g_j(s), g_k(t)] = \int_0^{\frac{2\pi}{T}} \mathrm{e}^{i(t-s)u} F_{k-j}(\mathrm{d}u + \tfrac{2\pi j}{T}),$$

that is $\{g_k(t)\}_{k\in\mathbb{Z}}$ is a family of stationary Γ-cross-correlated processes attached to $\{f_t\}_{t\in\mathbb{R}}$ as above. □

Taking into account the existence of the stationary dilation in this case, using the Γ-orthogonal projection on a right $\mathcal{L}(\mathcal{E})$-submodule given by the Proposition 1.1, the predictable part can be obtained.

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