INTERPOLATION FOR COMPLETELY POSITIVE MAPS: NUMERICAL SOLUTIONS

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Abstract. We present certain techniques to find completely positive maps between matrix algebras that take prescribed values on given data. To this aim we describe a semidefinite programming approach and another convex minimization method supported by a numerical example.

1. Introduction

The present paper refers to a certain interpolation problem for completely positive maps that take prescribed values on given matrices, closely related to problems recently considered by C.-K. Li and Y.-T. Poon in [24], Z. Huang, C.-K. Li, E. Poon, and N.-S. Sze in [17], T. Heinosaari, M.A. Jivulescu, D. Reeb, and M.M. Wolf in [15] as well as G.M. D'Ariano and P. Lo Presti [12], D.S. Gonçalves et al. [14].

Let M_n denote the C^*-algebra of all $n \times n$ complex matrices. In particular, positive elements (positive semidefinite matrices) in M_n are defined. Recall that a matrix $A \in M_n$ is *positive semidefinite* if all its principal determinants are nonnegative. Let $M_n^+ \subset M_n$ denote the convex cone of all such matrices. A linear map $\varphi\colon M_n \to M_k$ is *positive* if $\varphi(M_n^+) \subset M_n^+$, namely it maps positive semidefinite matrices into positive semidefinite ones. We call φ *completely positive* if the map $I_m \otimes \varphi\colon M_m \otimes M_n \to M_m \otimes M_k$ is positive for all $m \in \mathbb{N}$.

An equivalent notion is that of *positive semidefinite* map, that is, for all $m \in \mathbb{N}$, all $h_1, \ldots, h_m \in \mathbb{C}^k$ and all $A_1, \ldots, A_m \in M_n$ we have $\sum_{i,j=1}^m \langle \varphi(A_j^* A_i) h_j, h_i \rangle \geq 0$. Let $\mathrm{CP}(M_n, M_k)$ denote the convex cone of all completely positive maps $\varphi\colon M_n \to M_k$. If $\varphi : M_n \to M_k$ is completely positive then, cf. K. Kraus [21] and M.D. Choi [11], there are $n \times k$ matrices $V_1, V_2, \ldots, V_m$ with $m \leq nk$ such that

$$\varphi(A) = V_1^* A V_1 + V_2^* A V_2 + \cdots + V_m^* A V_m \text{ for all } A \in M_n \tag{1.1}$$

(and, of course, any map of the form (1.1) is completely positive). The representation (1.1) is called the *Kraus representation* of φ and $V_1, \ldots, V_m$ are called the *operation elements*. The representation (1.1) of a given completely positive map φ is non-unique, with respect to both its operation elements and the number m of these elements. However the minimal

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number of the operation elements in the Kraus form of such a map φ turns to be the rank of its Choi matrix (see subsection 2.1) - the statement is implicit in the original article of M.D. Choi [11]. The following problem has been suggested by C.-K. Li and Y.-T. Poon in [24], where a solution was given in case when the families of matrices $(A_\nu)_\nu$, $(B_\nu)_\nu$ from below are commutative.

Problem A. *Given matrices $A_\nu \in M_n$ and $B_\nu \in M_k$ for $\nu = 1, \ldots, N$, find $\varphi \in \mathrm{CP}(M_n, M_k)$ subject to the conditions*

$$\varphi(A_\nu) = B_\nu, \text{ for all } \nu = 1, \ldots, N. \tag{1.2}$$

Other linear affine restrictions on φ may be added as well, like trace preserving etc. In [5] we dealt with various necessary and/or sufficient conditions for the existence of such solutions φ. Most of the important theoretic results in this sense are related to Arveson's Hahn-Banach type theorem [2] and various techniques of operator spaces [25], some of which being simplified in the present particular context by R.R. Smith and J.D. Ward [28]. In this paper we present some concrete techniques to compute solutions numerically.

Briefly speaking, the existence of solutions to Problem A (or related ones) always turns to be equivalent to the fact that certain affine subspaces of matrices contain at least one positive semidefinite matrix; also, this can be characterized by the positivity of certain related linear functionals. In particular, our Theorem 2.4 and Theorem 2.5 in [5] show such characterizations in terms of a certain density matrix D_φ of φ, see Section 2.4 in [5]. The density matrix D_φ and the Choi matrix Φ_φ are related by the equality $D_\varphi = U^* \overline{\Phi}_\varphi U$ where the symbol $^-$ denotes the complex conjugation and U is a unitary operator coming from the two canonical identifications of $\mathbb{C}^n \otimes \mathbb{C}^k$ with $\mathbb{C}^{nk}$, see Proposition 2.8 in [5]; in this article we chose to use the Choi matrix Φ_φ instead. The first step in our approach is to firstly derive an equivalent formulation in terms of existence of certain positive semidefinite matrices subject to linear affine restrictions, like the matrix X $(= \Phi_\varphi)$ in Problem B from Subsection 2.2.

In Subsection 2.3 we briefly describe a method for solving Problem B by known techniques of semidefinite programming. Further, by using results from [3], we describe methods for solving Problem B by convex minimization techniques, see Theorem 2.1. A numerical example that illustrates Theorem 2.1 is performed in Subsection 2.5. The approach we used in [5], through the Smith-Ward linear functional, allows us to point out another numerical method of solving Problem B by means of minimization of linear functionals subject to semidefinite constraints, see Proposition 2.5. Finally, in Subsection 2.7 we show that, under the commutation assumptions, the semidefinite problem that we obtain here turns into a linear programming problem, hence explaining the results in [24] from this perspective as well.

If a more restrictive case of Problem A is considered, for example, when, in addition to the requirement that the solutions φ should be completely positive maps, one imposes the condition of trace preserving, that is, φ must be a quantum channel, we note that this version of Problem A leads to the same type of Problem B since, the additional trace preserving constrained is just another linear constrained. This shows that all the numerical techniques that we describe in this article can be successfully applied to interpolation of quantum channels that take prescribed values on given data, without any essential modification.

Let us mention that the positive semidefinite approach to Problem A has been previously observed also in [12], [14], and [15], in different formulation. With respect to these articles, our present topics, like subsections 2.4, 2.5, and Proposition 2.5 (b), are new.

2. Main results

Consider then the interpolation problem (1.2) for the given matrices $A_\nu \in M_n$ and $B_\nu \in M_k$ where $\nu = 1, \ldots, N$. Firstly, we will translate it below in terms of Choi matrices.

2.1. **The Choi matrix.** Let $\{e_i^{(n)}\}_{i=1}^n$ be the canonical basis of $\mathbb{C}^n$ ($n \in \mathbb{N}$). As usual, the linear space $M_{n,k}$ of all $n \times k$ matrices is identified with the vector space $\mathcal{B}(\mathbb{C}^k, \mathbb{C}^n)$ of all linear transformations $\mathbb{C}^k \to \mathbb{C}^n$ ($n, k \in \mathbb{N}$). Let $\{E_{i,j}^{(n,k)} \mid i = 1, \ldots, n,\ j = 1, \ldots, k\} \subset M_{n,k}$ be the matrix units of size $n \times k$, namely $E_{i,j}^{(n,k)}$ is the $n \times k$ matrix with all entries 0 except for the (i, j)-th entry which is 1. If $n = k$, we note $E_{i,j}^{(k)} = E_{i,j}^{(k,k)}$.

Given any linear map $\varphi\colon M_n \to M_k$, define a $kn \times kn$ matrix Φ_φ by

$$\Phi_\varphi = [\varphi(E_{l,m}^{(n)})]_{l,m=1}^n. \tag{2.1}$$

In what follows we describe the mapping $\varphi \mapsto \Phi_\varphi$, that appears in J. de Pillis [26], A. Jamiołkowski [18], R.D. Hill [16], and M.D. Choi [11]. Use the lexicographic reindexing of $\{E_{i,j}^{(n,k)} \mid i = 1, \ldots, n,\ j = 1, \ldots, k\}$, more precisely

$$\big(E_{1,1}^{(n,k)}, \ldots, E_{1,k}^{(n,k)}, E_{2,1}^{(n,k)}, \ldots, E_{2,k}^{(n,k)}, \ldots, E_{n,1}^{(n,k)}, \ldots, E_{n,k}^{(n,k)}\big) = \big(\mathcal{E}_1, \mathcal{E}_2, ..., \mathcal{E}_{nk}\big) \tag{2.2}$$

Another form of this reindexing is

$$\mathcal{E}_r = E_{i,j}^{(n,k)} \text{ where } r = (j-1)k + i, \text{ for all } i = 1, \ldots, n,\ j = 1, \ldots, k. \tag{2.3}$$

The formula

$$\varphi_{(i-1)k+m,(j-1)k+l} = \langle \varphi(E_{l,m}^{(n)}) e_i^{(j)}, e_m^{(k)} \rangle, \quad i, j = 1, \ldots, k,\ l, m = 1, \ldots, n, \tag{2.4}$$

and its inverse

$$\varphi(C) = \sum_{r,s}^{nk} \varphi_{r,s} \mathcal{E}_r^* C \mathcal{E}_s, \quad C \in M_n, \tag{2.5}$$

establish a linear bijection

$$\mathcal{B}(M_n, M_k) \ni \varphi \mapsto \Phi_\varphi = [\varphi_{r,s}]_{r,s=1}^{nk} \in M_{nk} \tag{2.6}$$

that induces an affine, order preserving bijection

$$\mathrm{CP}(M_n, M_k) \ni \varphi \mapsto \Phi_\varphi \in M_{nk}^+. \tag{2.7}$$

Given $\varphi \in \mathcal{B}(M_n, M_k)$ we call the matrix $\Phi = \Phi_\varphi$ as in (2.1) *the Choi matrix* of φ.

2.2. **Equivalent setting of the problem.** Following the notation in (2.1) – (2.7), the Choi matrix $\Phi = \Phi_\varphi$ of any solution $\varphi : M_n \to M_k$ to (1.2) is given by $\Phi = [\varphi_{rs}]_{r,s}$ where the indices r, s are couples $r \equiv (i, m)$, $s \equiv (j, l)$ for $i, j = 1, \ldots, n$, $l, m = 1, \ldots, k$ and

$$\varphi_{rs} = \langle \varphi(E_{ij}^{(n)}) e_l^{(k)}, e_m^{(k)} \rangle.$$

Since r, s run the cartesian product $\{1, \ldots, n\} \times \{1, \ldots, k\}$ consisting of nk elements that we order lexicographically, we can simply write $\Phi \in M_{nk}$ and $r, s = 1, \ldots, nk$. Set

$$A_\nu = [a_{\nu,i,j}]_{i,j=1}^n = \sum_{i,j=1}^n a_{\nu,i,j} E_{ij}^{(n)}$$

and

$$B_\nu = [b_{\nu,m,l}]_{m,l=1}^k = \sum_{m,l=1}^k b_{\nu,m,l} E_{ml}^{(k)}.$$

Equate the (m, l) entries in the matrix equality $\varphi(A_\nu) = B_\nu$ to get

$$\langle \varphi(A_\nu) e_l^{(k)}, e_m^{(k)} \rangle = b_{\nu,m,l},$$

that is,

$$\langle \varphi(\sum_{i,j=1}^n a_{\nu,i,j} E_{ij}^{(n)}) e_l^{(k)}, e_m^{(k)} \rangle = b_{\nu,m,l}$$

and so

$$\sum_{i,j=1}^n a_{\nu,i,j} \varphi_{(i,m)(j,l)} = b_{\nu,m,l}. \tag{2.8}$$

Write the equality from above using Kronecker's symbol δ_{pq} ($= 1$ if $p = q$ and 0 if $p \neq q$) in the form

$$\sum_{(j,l'),(i,m')} a_{\nu,i,j} \delta_{l'l} \delta_{m'm} \varphi_{(i,m')(j,l')} = b_{\nu,m,l}$$

where (j, l') and (i, m') run $\{1, \ldots, n\} \times \{1, \ldots, k\}$, then set

$$c(\nu, m, l)_{(j,l')(i,m')} := a_{\nu,i,j} \delta_{l'l} \delta_{m'm} = (A_\nu^\tau)_{j,i} (E_{lm}^{(k)})_{l',m'} = (A_\nu^\tau \otimes E_{lm}^{(k)})_{(j,l'),(i,m')} \tag{2.9}$$

where $A \mapsto A^\tau$ denotes the transposition and define

$$C(\nu, m, l) = \left[c(\nu, m, l)_{(j,l')(i,m')} \right]_{(j,l')(i,m')} = A_\nu^\tau \otimes E_{lm}^{(k)} \tag{2.10}$$

that can be represented as an $nk \times nk$ matrix $C(\nu, m, l) \in M_{nk}$

$$C(\nu, m, l) \equiv A_\nu^\tau \otimes E_{lm}^{(k)} \equiv \left[a_{\nu,j,i} E_{lm}^{(k)} \right]_{i,j=1}^n \tag{2.11}$$

via the linear, isometric, order-preserving isomorphisms

$$M_{nk} \equiv M_n \otimes M_k \equiv M_n(M_k).$$

We obtain, using (2.8) – (2.10), the equality

$$\sum_{(j,l'),(i,m')} c(\nu, m, l)_{(j,l')(i,m')} \varphi_{(i,m')(j,l')} = b_{\nu,m,l},$$

namley

$$\operatorname{tr}(C(\nu, m, l)\Phi) = b_{\nu,m,l},$$

that by (2.11) we can write as well

$$\mathrm{tr}\,[(A_\nu^\tau \otimes E_{lm}^{(k)})\Phi] = b_{\nu,m,l}, \tag{2.12}$$

which actually is a particular application of the next formula, easily checked following the lines from above, letting $A = [a_{ij}]_{i,j=1}^n = \sum_{i,j=1}^n a_{ij} E_{ij}^{(n)}$ etc:

$$\varphi(A) = \left[\,\mathrm{tr}\,[(A^\tau \otimes E_{lm}^{(k)})\Phi]\right]_{m,l=1}^k = \left[\,\mathrm{tr}\,[(A \otimes E_{lm}^{(k)})\Phi^\tau]\right]_{m,l=1}^k \quad (A \in M_n). \tag{2.13}$$

Note that we have as well the formula $\varphi(A) = [\mathrm{tr}\,[(E_{lm}^{(k)} \otimes A)D_\varphi^*]_{m,l=1}^k$ where D_φ denotes the density matrix [5], for which we also omit the details. Conditions (2.4) on φ are thus equivalent to the equations (2.12) from above concerning Φ, via the formulas (2.9), (2.10) and (2.4), (2.6). Denote by $\iota = (\nu, m, l)$ the generic triple consisting of arbitrary $\nu = 1, \ldots, N$ and $m, l = 1, \ldots, k$. Thus ι runs a set of $q := Nk^2$ elements. We may write $\iota = 1, \ldots, q$. Set also $p = nk$. Write $C(\iota) = C(\nu, m, l)$ $(\in M_p)$ and $b_\iota = b_{\nu,m,l}$. Via (2.7), Problem A takes then the form from below.

Problem B *Given $C(\iota) \in M_p$ and numbers b_ι $(1 \le \iota \le q)$, find $X \in M_p$, $X \geq 0$, such that*

$$\mathrm{tr}\,(C(\iota)X) = b_\iota \quad \textit{for all } \iota = 1, \ldots, q. \tag{2.14}$$

Thus, the solvability of Problem A leads to the rather known topic of finding positive semidefinite matrices subject to linear affine conditions and, in particular, establish whether such matrices do exist. These questions often occur and are dealt with by reliable numerical methods in the semidefinite programming, a few elements of which we sketch in what follows.

In addition, a more restrictive case of Problem A is when, in addition to the requirement that the solutions φ should be completely positive maps, one imposes the condition of trace preserving, that is, φ must be a quantum channel. However, this version of Problem A leads to the same type of Problem B since, the additional trace preserving constrained is just another linear constrained. This shows that Problem B can be successfully applied to interpolation of quantum channels that take prescribed values on given data, as well.

2.3. **Solutions by means of semidefinite programming.** Firstly, using $\mathrm{tr}\,(c^*) = \overline{\mathrm{tr}\,(c)}$, $\mathrm{tr}\,(cd) = \mathrm{tr}\,(dc)$ and writing equation (2.14) in terms of $C(\iota) + C(\iota)^*$ and $i(C(\iota) - C(\iota)^*)$ we can asume all matrices $C(\iota)$ to be selfadjoint. We can suppose, without loss of generality, that they are linearly independent over $\mathbb{R}$. *Semidefinite programming* is concerned with minimization of linear functionals subject to the constraint that an affine combination of selfadjoint matrices is positive semidefinite: see in this sense [6], [8], [22], [23], [29], also [9], [13]. Roughly speaking, one sets

$$a(x) = \sum_\iota x_\iota C(\iota) + a_0$$

for the given $C(\iota)$ and a selfadjoint matrix a_0 (that can be suitably chosen, here). Define then

$$p^* = \inf_x \{\sum_\iota b_\iota x_\iota \ : \ a(x) \geq 0\}$$

and

$$q^* = \inf_X \{-\operatorname{tr}(a_0 X) : X \geq 0, \operatorname{tr}(C(\iota)X) = b_\iota \,\forall \iota\}.$$

A problem dual to (2.14) occurs now with respect to p^*, namely to establish if there exist positive definite matrices of the form $a(x)$. Standard algorithms exist to this aim, based on maximizing the minimal eigenvalue of $a(x)$ in the variables $x = (x_\iota)_\iota$, or on interior point methods using barrier functions [23]. In the case when either (2.14) has solutions $X > 0$, or the dual problem has solutions x with $a(x) > 0$, we have

$$p^* = q^*,$$

see for instance [23], [29]. If both conditions hold, the optimal sets for p^* and q^* are nonempty. In this case for every $\lambda \in (p^*, \overline{p})$ where $\overline{p} = \sup_x \{\sum_\iota b_\iota x_\iota : a(x) > 0\}$ there is a unique vector $x^* = (x^*_\iota)_\iota$, the *analytic center* of this linear matrix inequality, such that

$$a(x^*) > 0, \quad \sum_\iota b_\iota x^*_\iota = \lambda$$

and x^* minimizes the *logarithmic barrier function*

$$\ln \det a(x)^{-1}$$

over all x with $\sum_\iota b_\iota x_\iota$ and $a(x) > 0$. It follows by the Lagrange multipliers method that

$$\operatorname{tr}(C(\iota) a(x^*)^{-1}) = \lambda b_\iota \ \ \forall \, \iota,$$

which gives a solution $X = X_*$ of (2.14), namely

$$X_* = \lambda^{-1} a(x^*)^{-1}. \tag{2.15}$$

These techniques provide then a method to find solutions of the form (2.15) to Problem B.

2.4. **Solutions via a convex minimization technique.** We present another way to obtain particular solutions to Problems A, B, based on the minimization of a certain convex function, see [3]. Suppose that $C(\iota)$ are selfadjoint and linearly independent. Define the function V of q real variables $x = (x_\iota)_{\iota=1}^q$ by

$$V(x) = \operatorname{tr}\left(e^{\sum_{\iota=1}^q x_\iota C(\iota)}\right) - \sum_\iota x_\iota b_\iota. \tag{2.16}$$

Then V is smooth, strictly convex and has strictly positive definite Hessian [3]. Hence whenever it has some critical point this is unique, and necessarily a point of minimum. Generally we may have also an unattained infimum $\inf V > -\infty$, or $\inf V = -\infty$. The following characterization of the existence of the solutions $X > 0$ to Problem B holds.

Theorem 2.1. [3]. *The system of equations (2.14) admits solutions $X > 0$ if and only if the function V defined by (2.16) has a critical point (of minimum), that is, if and only if $\lim_{\|x\| \to \infty} V(x) = +\infty$. In this case, (2.14) has also the (positive) particular solution*

$$X_0 = e^{\sum_\iota x^0_\iota C(\iota)} \tag{2.17}$$

where $x^0 = (x^0_\iota)_\iota$ is the critical point of V.

Remark 2.2. The function V given by (2.16) fulfills the conditions of application of the *method of the conjugate gradients* [10]. This yields, whenever problem (2.14) has solutions $X > 0$, a minimizing sequence of vectors $x = (x_\iota)_\iota$ that is convergent to the critical point x^0 of V and so provides approximations $\widetilde{X}_0 = e^{\sum_\iota x_\iota C(\iota)} \approx X_0$ of the solution (2.17), see [Example 12, Remarks 11, [3]]. Note that the gradient $\nabla V = (\partial V/\partial x_\iota)_{\iota=1}^q$ of V is easily computed to this aim by

$$\frac{\partial V}{\partial x_\kappa}(x) = \mathrm{tr}\,(C(\kappa)e^{\sum_\iota x_\iota C(\iota)}) - b_\kappa,$$

see [3]. We remind also the existence of various versions of Newton's method that can be used as well to approximate the critical point. Certain tests exist [3] also to check if there are no solutions $X \geq 0$ at all.

2.5. **A Numerical Example.** We show how Theorem 2.1 applies to Problems A, B. Suppose one wishes to find $\varphi : M_2 \to M_2$ completely positive such that $\varphi(A_\nu) = B_\nu$ $(\nu = 1, 2)$ for $A_1 = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ and $B_1 = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}$, $B_2 = \begin{bmatrix} 3.5 & 1.5 \\ 1.5 & 2.5 \end{bmatrix}$. Use to this aim the minimization method indicated by Remark 2.2. Formulas (2.9), (2.10) and (2.11) provide the matrices $C(\iota)$ for $\iota = (\nu, m, l)$ where $\nu, m, l = 1, 2$. Due to the symmetry equation (2.8) (or, equivalently, (2.14)) is equivalent to $\sum_{j,i=1}^n \overline{a}_{\nu,ji}\overline{\varphi}_{(j,l)(i,m)} = \overline{b}_{\nu lm}$ (or $\mathrm{tr}\,(C(\iota)^* \Phi) = \overline{b}_\iota$), and so it is enough to consider (2.8) for those couples (m, l) with $m \leq l$. That is, for each $\nu = 1, 2$ we have 3 equations corresponding to $(m, l) = (1, 1), (1, 2), (2, 2)$. The set $\{1, 2\} \times \{1, 2\}$ of indices r, s like $(j, l), (j, l'), (i, m), (i, m')$ with $1 \leq i, j \leq n\,(= 2)$ and $1 \leq m, m', l, l' \leq k\,(= 2)$ from below is ordered lexicographically as $\{(1, 1), (1, 2), (2, 1), (2, 2)\} \equiv \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}\}$. We represent the positive matrix $X = [y_{\alpha\beta}]_{\alpha,\beta\in\{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4}\}} \equiv \Phi = [\varphi_{rs}]_{r,s}$ that we seek for as

$$X = \begin{bmatrix} y_{\mathbf{11}} & y_{\mathbf{12}} & y_{\mathbf{13}} & y_{\mathbf{14}} \\ y_{\mathbf{21}} & y_{\mathbf{22}} & y_{\mathbf{23}} & y_{\mathbf{24}} \\ y_{\mathbf{31}} & y_{\mathbf{32}} & y_{\mathbf{33}} & y_{\mathbf{34}} \\ y_{\mathbf{41}} & y_{\mathbf{42}} & y_{\mathbf{43}} & y_{\mathbf{44}} \end{bmatrix} \equiv \Phi = \begin{bmatrix} \varphi_{(1,1)(1,1)} & \varphi_{(1,1)(1,2)} & \varphi_{(1,1)(2,1)} & \varphi_{(1,1)(2,2)} \\ \varphi_{(1,2)(1,1)} & \varphi_{(1,2)(1,2)} & \varphi_{(1,2)(2,1)} & \varphi_{(1,2)(2,2)} \\ \varphi_{(2,1)(1,1)} & \varphi_{(2,1)(1,2)} & \varphi_{(2,1)(2,1)} & \varphi_{(2,1)(2,2)} \\ \varphi_{(2,2)(1,1)} & \varphi_{(2,2)(1,2)} & \varphi_{(2,2)(2,1)} & \varphi_{(2,2)(2,2)} \end{bmatrix}$$

and the given matrices $C(\nu, m, l)$ as follows: $E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ etc and

$$C(1,1,1) = \begin{bmatrix} c(1,1,1)_{(1,1)(1,1)} & c(1,1,1)_{(1,1)(1,2)} & c(1,1,1)_{(1,1)(2,1)} & c(1,1,1)_{(1,1)(2,2)} \\ c(1,1,1)_{(1,2)(1,1)} & c(1,1,1)_{(1,2)(1,2)} & c(1,1,1)_{(1,2)(2,1)} & c(1,1,1)_{(1,2)(2,2)} \\ c(1,1,1)_{(2,1)(1,1)} & c(1,1,1)_{(2,1)(1,2)} & c(1,1,1)_{(2,1)(2,1)} & c(1,1,1)_{(2,1)(2,2)} \\ c(1,1,1)_{(2,2)(1,1)} & c(1,1,1)_{(2,2)(1,2)} & c(1,1,1)_{(2,2)(2,1)} & c(1,1,1)_{(2,2)(2,2)} \end{bmatrix} =$$

$$A_1^\tau \otimes E_{11} = \begin{bmatrix} a_{11}^1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} & a_{21}^1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\ a_{12}^1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} & a_{22}^1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} a_{11}^1 & 0 & a_{21}^1 & 0 \\ 0 & 0 & 0 & 0 \\ a_{12}^1 & 0 & a_{22}^1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$C(1,1,2) = A_1^\tau \otimes E_{21} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ etc. A numerical minimization of V as in Remark 2.2 gave us the matrix $\widetilde{X}_0 \approx X_0$ from below

$$\widetilde{X}_0 = \begin{bmatrix} 1.549937761 & -0.1694804138 & 0.4499571618 & 0.4047411695 \\ -0.1694804138 & 0.1534277390 & -0.06572393508 & -0.1533566973 \\ 0.4499571618 & -0.06572393508 & 0.5249880063 & 0.6652436210 \\ 0.4047411695 & -0.1533566973 & 0.6652436210 & 1.326699194 \end{bmatrix}$$

which approximately satisfies the equations (2.14). Let φ be the map whose Choi matrix $\Phi = \Phi_\varphi = [\varphi_{r,s}]_{r,s}$ is $\widetilde{X}_0 \equiv \Phi$. We got then an approximate solution to our present particular case of problem (1.2), namely $\varphi(A) = \left[\sum_{i,j=1}^{2} \Phi_{(i,m)(j,l)} a_{ij}\right]_{m,l=1}^{2} = \left[\operatorname{tr}[(A^\tau \otimes E_{lm}^{(k)})\widetilde{X}_0\right]_{m,l=1}^{k}$ for every $A = [a_{ij}]_{i,j=1}^{2} \in M_2$, see formula (2.13). For instance, we have

$$\varphi(A_1)_{11} = \operatorname{tr}(C(1,1,1)\widetilde{X}_0) = 2(\widetilde{X}_0)_{11} + (\widetilde{X}_0)_{13} + (\widetilde{X}_0)_{31} = 3.99978984 \approx 4 = b_{(1,1,1)} = (B_1)_{11},$$

$$\varphi(A_1)_{12} = \operatorname{tr}\left[(A_1^\tau \otimes E_{21}^{(2)})\widetilde{X}_0\right] = \operatorname{tr}(C(1,1,2)\widetilde{X}_0) = 0.0000564069 \approx 0 = b_{(1,1,2)} = (B_1)_{12} \text{ etc}$$

Definitely, problems of more sizeable amount can be solved as well by using such semidefinite programming (or related) methods [23], [29], see also [7] , [20], allowing us to consider larger n, k, N.

Remark 2.3. In order to obtain an exact solution X, we can project $\widetilde{X}_0$ onto the affine subspace defined by (2.14) by a linear affine projection map p, then let

$$X := p\widetilde{X}_0$$

and use $\Phi := X$ instead of $\widetilde{X}_0$. Indeed, since $\widetilde{X}_0 \approx X_0$ then $X = p\widetilde{X}_0 \approx pX_0 = X_0$ and so $X \approx X_0 > 0$ which implies $X > 0$ if a sufficiently good approximation $\widetilde{X}_0 \approx X_0$ was performed.

Remark 2.4. An interesting question is to reduce the number of operation elements in the representation (1.1) of φ, whenever possible. This is equivalent to the minimization of the rank of X. The case of one term for instance would correspond to solutions $X \geq 0$ of rank one, namely to the existence of vectors $v \in \mathbb{C}^{nk}$ such that $\langle C(\iota)v, v\rangle = b_\iota$ for all ι. A first easy step to rank reduction is to find the joint support P of the matrices $C(\iota)$ $(:= C(\iota)^*)$ and consider only solutions X such that $X = PXP$, as follows. Set $K = \{h \in \mathbb{C}^{nk} : C(\iota)h = 0 \,\forall\, \iota\}$. Let P be the orthogonal projection onto $K^\perp$. Then $\operatorname{tr}(C(\iota)PXP) = b_\iota$ for all ι. Indeed, $C(\iota) = C(\iota)P$ and so $C(\iota) = C(\iota)^* = PC(\iota) = PC(\iota)\mathrm{P}$, whence $\operatorname{tr}(C(\iota)X) = \operatorname{tr}(PC(\iota)PX) = \operatorname{tr}(C(\iota)PXP)$. Generally the question to verify if there exist solutions $X \geq 0$ of lower rank and find them is difficult. For certain possibilities of reducing the rank of X see, for instance, Section II.13 in [6], or [27].

2.6. **Characterization in terms of linear functionals.** By Theorem 2.5 from [5] (see also Theorem 6.1 in [25], or [28]), the solvability of (2.14) can be described in terms of the linear functional

$$\sum_\iota x_\iota C(\iota) \mapsto b_\iota x_\iota.$$

We recall this result in the form from below, completed with a version (b) concerning the existence of strictly positive solutions; for the sake of completeness we sketch also the proof.

Proposition 2.5. *Suppose that $C(\iota) \in M_p$ ($\iota = 1, \ldots, q$) are selfadjoint, linearly independent and their linear span contains strictly positive matrices. Then:*

(a) *There exist solutions $X \geq 0$ of the system of equations (2.14) if and only if $\sum_\iota b_\iota x_\iota \geq 0$ for all $(x_\iota)_\iota$ such that $\sum_\iota x_\iota C(\iota) \geq 0$, namely, we have*

$$\inf_{x\,:\,\sum_\iota x_\iota C(\iota)\geq 0} \sum_\iota b_\iota x_\iota \geq 0.$$

(b) *There exist solutions $X > 0$ of the system of equations (2.14) if and only if $\sum_\iota b_\iota x_\iota > 0$ for all $(x_\iota)_\iota \neq 0$ such that $\sum_\iota x_\iota C(\iota) \geq 0$, namely for any norm $\|\cdot\|$*

$$\inf_{x\,:\,\sum_\iota x_\iota C(\iota)\geq 0,\,\|x\|=1} \sum_\iota b_\iota x_\iota > 0.$$

Proof. (a) Assume that $\inf_{x\,:\,\sum_\iota x_\iota C(\iota)\geq 0} \sum_\iota b_\iota x_\iota \geq 0$. The intersection of the closed convex cone of all positive semidefinite $p \times p$ matrices and the linear span S of the $C(\iota)$'s contains a point that is interior to the cone, namely a positive matrix. The linear functional $l : \sum_\iota x_\iota C(\iota) \mapsto \sum_\iota b_\iota x_\iota$ is well defined, and ≥ 0 on this intersection. By Mazur's theorem, see for instance [1], [19], it has a linear extension L to the space M_p^s of all selfadjoint matrices in M_p, such that $LY \geq 0$ for all $Y \geq 0$ in M_p^s. Now L has the form $LY = \operatorname{tr}(XY)$ for some $X \in M_p^s$. Letting $Y = \langle\,\cdot\,, h\rangle h$ for an arbitrary vector $h \in \mathbb{C}^p$ gives $\langle Xh, h\rangle \geq 0$. Hence $X \geq 0$. Since $L|_S = l$, for every ι we have $C(\iota) \in S$ and $\operatorname{tr}(C(\iota)X) = LC(\iota) = lC(\iota) = b_\iota$. Conversely, suppose that there exists an $X \geq 0$ such that $\operatorname{tr}(C(\iota)X) = b_\iota$ for all ι. Then for every $(x_\iota)_\iota$ such that $\sum_\iota x_\iota C(\iota) \geq 0$, we have $\sum_\iota b_\iota x_\iota = \sum_\iota \operatorname{tr}(C(\iota)X) x_\iota = \operatorname{tr}(X \sum_\iota x_\iota C(\iota)) = \operatorname{tr}(X^{1/2} \sum_\iota x_\iota C(\iota) X^{1/2}) \geq 0$.

(b) Assume that $\inf_{x\,:\,\sum_\iota x_\iota C(\iota)\geq 0,\,\|x\|=1} \sum_\iota b_\iota x_\iota > 0$. We proceed as above, except we need the following fact: given a finite dimensional real space M, a linear subspace S and a closed convex cone $C \subset M$ such that $C \cap (-C) = \{0\}$, any linear functional l on S such that $ls > 0$ for all $s \neq 0$ from $S \cap C$ has a linear extension L to M such that $Lm > 0$ for all $m \neq 0$ from C. This is rather a known consequence of the Hahn-Banach, Mazur - type theorems, see for instance [4]. The necessity of the condition follows as in the case (a). □

2.7. **The case of commutative data.** As mentioned before, Problem A was raised in [24] where the commutative case was proven to be equivalent to a linear programming problem, concerned with solving systems in nonnegative variables. Our present approach allows us to put their result into a new perspective.

Firstly, by the commutativity assumption we can suppose, without loss of generality, that all matrices A_ν, B_ν are diagonal. For any matrix $u = [u_{ij}]_{i,j}$, set $\tilde{u} = [u_{ij}\delta_{ij}]_{i,j}$. The Proposition from below shows that in the equations (2.12) we can replace then any positive semidefinite solution X by the (positive semidefinite) diagonal matrix $\tilde{X} = \operatorname{diag}(x_1, \ldots, x_q)$, which leads to the simpler problem of finding some numbers $x_i \geq 0$ satisfying a linear system of nonhomogeneous equations.

Proposition 2.6. *Let the matrices A_ν, B_ν be diagonal. If $X \geq 0$ satisfies (2.12), that is, $\operatorname{tr}[(A_\nu \otimes E^{(k)}_{ml})X] = B_{\nu,lm}$, then $\widetilde{X}$ also is a solution to these equations, namely we have $\operatorname{tr}[(A_\nu \otimes E^{(k)}_{ml})\widetilde{X}] = B_{\nu,lm}$.*

Proof. Represent $X \in M_{nk} \equiv M_n \otimes M_k$ as $X = \sum_\mu Y_\mu \otimes Z_\mu$ with $Y_\mu \in M_n$ and $Z_\mu \in M_k$. Using the easily checked formula $\widetilde{u \otimes v} = \tilde{u} \otimes \tilde{v}$, we obtain $\tilde{X} = \sum_\mu \tilde{Y}_\mu \otimes \tilde{Z}_\mu$. Hence, the equality in the conclusion holds for $l \neq m$ by inserting $\widetilde{X}$ in the left hand side, then using the formula $\operatorname{tr}(u \otimes v) = \operatorname{tr}(u)\operatorname{tr}(v)$ and the equalities $\operatorname{tr}(E^{(k)}_{lm}\tilde{Z}_\mu) = 0$, $B_{\nu,lm} = 0$. To prove it also for $l = m$, use again $\operatorname{tr}(u \otimes v) = \operatorname{tr} u \operatorname{tr} v$ to write the desired conclusion in the form $\sum_\mu \operatorname{tr}(A_\nu \tilde{Y}_\mu) Z_{\mu,ll} = B_{\nu,ll}$. This is equivalent, by means of the equalities $\tilde{A}_\nu = A_\nu$ and the formula $\operatorname{tr}(\tilde{u}\tilde{v}) = \operatorname{tr}(\tilde{u}v)$, to $\operatorname{tr}[(A_\nu \otimes E^{(k)}_{ll}) \sum_\mu Y_\mu \otimes Z_\mu] = B_{\nu,ll}$, that is the case $l = m$ of (2.12) and so holds true by hypotheses. □

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