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Integral Equations
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Nonextreme de Branges–Rovnyak Spaces as Models for Contractions

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Abstract. The de Branges–Rovnyak spaces are known to provide an alternate functional model for contractions on a Hilbert space, equivalent to the Sz.-Nagy–Foias model. The scalar de Branges–Rovnyak spaces $\mathcal{H}(b)$ have essentially different properties, according to whether the defining function b is or not extreme in the unit ball of H^∞. For b extreme the model space is just $\mathcal{H}(b)$, while for b nonextreme an additional construction is required. In the present paper we identify the precise class of contractions which have as a model $\mathcal{H}(b)$ with b nonextreme.

1. Introduction

In order to understand better operators on a Hilbert space, one often tries to find models for certain classes; that is, a subclass of concrete operators with the property that any given operator from the class is unitarily equivalent to an element of the subclass. The typical example is given by normal operators, which by the spectral theorem have multiplication operators on Lebesgue spaces as models.

Going beyond normal operators, there is an extensive theory dealing with models for contractions. The most elaborate form is the Sz.-Nagy–Foias theory [21], that we will shortly describe in the next section. About the same time another model had been devised by de Branges and then developed in detail in [6,7]; its main feature was the extensive use of contractively included subspaces. It turned out in the end that the models are equivalent; an explanation of the relation can be found in [4,18]. One should also note that these so called de Branges–Rovnyak spaces have received new attention in the last years, representing an active area of research (see, for instance, [1,3,5,8,9,14]). There is also an upcoming book on the subject [13].

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In the theory developed by de Branges and Rovnyak, an important starting point is provided by the so called scalar case, when the spaces involved are nonclosed subspaces of the Hardy space H^2. These spaces are determined by a function b in the unit ball of H^∞, and the usual notation is $\mathcal{H}(b)$; later their theory has been extensively developed [2,10–12,15–17,20], an important role being played by the basic monograph of Sarason [19]. It turns out that the study splits quite soon in two disjoint cases, according to whether b is or not an extreme point of the unit ball of H^∞.

From the point of view of model operators, the scalar case corresponds to the situation when the defect spaces of the contraction (see next section for precise definitions) have dimension 1. An important difference appears between the two situations: when b is extreme, the model space is $\mathcal{H}(b)$ itself, and the model operator the backward shift; but when b is not extreme, the model space contains pairs of functions, only the first one being in $\mathcal{H}(b)$, and the model operator acts in a more complicated way.

A natural question then appears: in the nonextreme space, can one also view $\mathcal{H}(b)$ itself as a model space (and the backward shift as a model operator) for a certain class of contractions? The present paper answers this question in the affirmative: we give in Theorem 7.2 precise necessary and sufficient conditions for a contraction on a Hilbert space to be unitarily equivalent to the backward shift acting on some space $\mathcal{H}(b)$ with b nonextreme. However, we should add that the description is rather involved; moreover, different rather distinct functions b may lead to unitarily equivalent models.

The current paper deals with scalar de Branges–Rovnyak spaces. As noted by the referee, the question may be posed also for matrix or operator valued functions b. As shown, for instance, in [18, Section 9], a different condition that replaces extremality can then be formulated, under which the de Branges–Rovnyak space yields a model equivalent to the general Sz.-Nagy–Foias model. If the condition is not satisfied, we are left with the problem of characterizing the completely non unitary contraction for which the backward shift on these spaces is a model; this will be the object of future study.

The plan of the paper is the following. After giving the necessary preliminaries in Sect. 2, we proceed to find necessary conditions for a contraction T to be unitarily equivalent to the backward shift acting on some $\mathcal{H}(b)$ with b nonextreme. Two of these are rather immediate (see Sect. 3), and a third one is not hard to find (this is done in Sect. 4). The last decisive fourth condition requires more work, its discussion being the content of Sects. 5 and 6. The main result is stated in Sect. 7, while Sect. 8 discusses to what extent is the function b determined by the contraction.

2. Preliminaries

2.1. General Notations

We will use the standard notations L^2 for the Lebesgue space of square integrable functions on the unit circle $\mathbb{T}$ and H^2 for the Hardy space, which may be alternately considered either as a closed subspace of L^2 or a space of

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analytic functions in the unit disc $\mathbb{D}$. We will meet also their vector valued variants $L^2(\mathcal{E})$ and $H^2(\mathcal{E})$, with $\mathcal{E}$ a Hilbert space. Multiplication with e^{it} on L^2 will be denoted by Z and its restriction to H^2 by S; for their analogues in the vector valued spaces we will use bold letters $\mathbf{Z}$ and $\mathbf{S}$ respectively (the space $\mathcal{E}$ can be deduced from the context). The action of these operators on the Fourier coefficients of a function explains why $\mathbf{Z}$ is also called the *bilateral shift* and $\mathbf{S}$ the *unilateral shift*.

The Hardy algebra H^∞ of all bounded analytic functions in $\mathbb{D}$ acts by multiplication on H^2; the corresponding operator valued objects are analytic functions in $\mathbb{D}$ with values in $\mathcal{L}(\mathcal{E}_1, \mathcal{E}_2)$ (the linear bounded operators); they map $H^2(\mathcal{E}_1)$ into $H^2(\mathcal{E}_2)$. In fact, we will only meet contractive analytic functions $\Theta : \mathbb{D} \to \mathcal{L}(\mathcal{E}_1, \mathcal{E}_2)$, whose values are contractions from $\mathcal{E}_1$ to $\mathcal{E}_2$. Such a function can be decomposed as

$$\Theta(\lambda) = \begin{pmatrix} \Theta_0(\lambda) & 0 \\ 0 & W \end{pmatrix} : \mathcal{E}_1' \oplus \mathcal{E}_1'' \to \mathcal{E}_2' \oplus \mathcal{E}_2'',$$

where $\mathcal{E}_i = \mathcal{E}_i' \oplus \mathcal{E}_i'' (i = 1, 2), W$ is a unitary constant, and Θ_0 is *pure*, that is, it has no constant unitary part; Θ_0 is called the *pure part* of Θ.

2.2. The Sz.-Nagy–Foias Model and Related Questions

If H is a Hilbert space, we denote by $\mathcal{L}(H)$ the algebra of all bounded operators acting on H. Let then $T \in \mathcal{L}(H)$ be a contraction, that is, $\|T\| \le 1$. We define $D_T = (I - T^*T)^{1/2}$ and $\mathcal{D}_T = \overline{D_T H}$. Obviously T is unitary if and only if $\mathcal{D}_T = \mathcal{D}_{T^*} = \{0\}$. For a general contraction, there exists a unique decomposition $H = H_u \oplus H_c$, where H_u and H_c are invariant with respect to T (and hence reducing), $T|H_u$ is unitary, while $T|H_c$ is *completely nonunitary (c.n.u.)*; that is, it has no reducing space on which it is unitary.

A *dilation* $\widehat{T}$ of T is an operator acting on a space $\widehat{H} \supset H$, such that $P_H \widehat{T}^n | H = T^n$ for all $n \ge 0$. Such a dilation is *minimal* if $\bigvee_{n \ge 0} \widehat{T}^n H = \widehat{H}$. Any dilation $\widehat{T}$ of T "contains" a minimal one: it suffices to restrict $\widehat{T}$ to its invariant subspace spanned by H.

The Sz.-Nagy dilation theorem states that any contraction has a minimal isometric dilation, which is unique up to a unitary equivalence that is the identity on H; a similar result is true for minimal unitary dilations.

The structure of unitary operators can be rather well described by means of the spectral theorem. On the other hand, for a c.n.u. contraction a structure description is given by the "model" theory of Sz.-Nagy and Foias [21] that we describe below. A central role is played by the notion of characteristic function. The characteristic function of a completely nonunitary contraction $T \in \mathcal{L}(H)$ is the contractive valued analytic function $\Theta(\lambda) : \mathcal{D}_T \to \mathcal{D}_{T^*}$, defined by

$$\Theta(\lambda) = -T + \lambda D_{T^*}(I - \lambda T^*)^{-1} D_T | \mathcal{D}_T, \quad \lambda \in \mathbb{D}.$$

The main result states that T is unitarily equivalent with its model $\mathbf{S}_\Theta \in \mathcal{L}(\mathbf{K}_\Theta)$, defined as follows:

$$\begin{aligned}\mathbf{K}_\Theta &= (H^2(\mathcal{D}_{T^*}) \oplus \overline{(I-\Theta^*\Theta)^{1/2}L^2(\mathcal{D}_T)})\\ &\quad \ominus\{\Theta h \oplus (I-\Theta^*\Theta)^{1/2}h : h \in H^2(\mathcal{D}_T)\},\\ \mathbf{S}_\Theta &= P_{\mathbf{K}_\Theta}(\mathbf{S}\oplus\mathbf{Z})|\mathbf{K}_\Theta.\end{aligned}$$

Also, $\mathbf{K}_\Theta$ is invariant with respect to $\mathbf{S}^*\oplus\mathbf{Z}^*$, and so $\mathbf{S}_\Theta^* = \mathbf{S}^*\oplus\mathbf{Z}^*|\mathbf{K}_\Theta$.

An important particular case is obtained when $\dim\mathcal{D}_T = \dim\mathcal{D}_{T^*} = 1$, and the characteristic function is a scalar *inner* function θ. The model space is then $K_\theta = H^2 \ominus \theta H^2$, and we will call $S_\theta \in \mathcal{L}(K_\theta)$ a *scalar model operator.*

Two operator valued analytic functions Θ, Θ' defined in $\mathbb{D}$ are said to *coincide* if there are unitaries τ, τ' such that $\Theta' = \tau\Theta\tau'$. Then two completely nonunitary contractions are unitarily equivalent if and only if their characteristic functions coincide.

We will use the relation between invariant subspaces and characteristic functions developed in the general case in [21, Chapter VII]; since we do not need the general theory, we single out in Lemma 2.1 below the precise consequences that we will need. In short, if $H' \subset H$ is an invariant subspace with respect to T, the decomposition of T with respect to $H' \oplus H'^{\perp}$ being

$$T = \begin{pmatrix} T_1 & X \\ 0 & T_2 \end{pmatrix},$$

there is an associated factorization of the characteristic function Θ such that

$$\Theta = \Theta_2\Theta_1, \tag{2.1}$$

where the characteristic function of T_i is the pure part of Θ_i. Such factorizations satisfy a supplementary condition of regularity (see [21, Theorem VII.1.1]); conversely, any factorization that satisfies this condition is obtained in this way from an invariant subspace.

Lemma 2.1. *Suppose $T \in \mathcal{L}(H)$, $H' \subset H$ is invariant to T, and denote $T' = T|H'$.*

1. *If T has inner characteristic function $\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}$ and T' has scalar characteristic function θ, then θ is a common inner divisor of ϕ_1 and ϕ_2. Conversely, if θ is a common inner divisor of ϕ_1 and ϕ_2, then there exists $H' \subset H$, invariant to T, such that the characteristic function of $T' := T|H'$ is θ.*
2. *If T has scalar characteristic function Θ and T' is an isometry, then T' is a shift of multiplicity 1.*

Proof. We give just a sketch of the proof, based on the results in [21].

(1) If $\Theta = \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}$, it follows from (2.1) that Θ_1 is a column of scalars, and thus it has to be actually the scalar function θ (there is no room for a constant unitary). Therefore

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}\theta, \tag{2.2}$$

whence θ is a common inner divisor for ϕ_1 and ϕ_2.

Conversely, if θ is a common inner divisor for ϕ_1 and ϕ_2, then (2.1) is true for some ψ_i; also, the factorization (2.1) is regular, since all functions are inner [21, Proposition VII.3.3]. Therefore θ is the characteristic function of a restriction of T to an invariant subspace.

(2) This part follows immediately from the description of all factorizations of scalar characteristic functions given in [21, Proposition VII.3.5]. □

2.3. de Branges–Rovnyak Spaces

Suppose $b \in H^\infty$, $\|b\|_\infty \leq 1$, and b is nonextreme; $\Delta = (1 - |b|^2)^{1/2}$, a is the outer function that satisfies $|a| = \Delta$. S is the unilateral shift on H^2, Z the bilateral shift on L^2. We use the notation $\tilde{f}(z) = \overline{f(\bar{z})}$.

Denote by $\mathbf{T}_b$ the Toeplitz operator with symbol b. The de Branges–Rovnyak space $\mathcal{H}(b)$ is defined to be the range of $(I - \mathbf{T}_b\mathbf{T}_b^*)^{1/2}$, with the norm given by

$$\|(g\|_{\mathcal{H}(b)} = \inf\{\|f\|_2 : (I - \mathbf{T}_b\mathbf{T}_b^*)^{1/2} f = g\}.$$

In particular, if $\ker(I - \mathbf{T}_b\mathbf{T}_b^*)^{1/2} = \{0\}$, then

$$\|(I - \mathbf{T}_b\mathbf{T}_b^*)^{1/2} f\|_{\mathcal{H}(b)} = \|f\|_2. \tag{2.3}$$

In the sequel we will suppose that b is nonextreme. Since $\mathbf{T}_b\mathbf{T}_b^* \leq \mathbf{T}_b^*\mathbf{T}_b$, we have, for each $f \in H^2$,

$$\|(I - \mathbf{T}_b\mathbf{T}_b^*)^{1/2} f\|_2^2 = \|f\|_2^2 - \|\mathbf{T}_b^* f\|_2^2 \geq \|f\|_2^2 - \|\mathbf{T}_b f\|_2^2.$$

But, if b is nonextreme, then $|b| < 1$ a.e., whence, for $f \not\equiv 0$, $\|\mathbf{T}_b f\|_2 < \|f\|_2$. Therefore $\ker(I - \mathbf{T}_b\mathbf{T}_b^*)^{1/2} = \{0\}$ and (2.3) is satisfied.

It is proved in [19, II-7] that $\mathcal{H}(b)$ is invariant with respect to S^*, which acts as a contraction on $\mathcal{H}(b)$. This contraction is denoted by X_b; it will be the main character in the sequel, but only in disguise.

Some spaces that will appear in the sequel are:

$$\begin{aligned}
\mathcal{K}_b &= (H^2 \oplus \overline{\Delta H^2}) \ominus \{bh \oplus \Delta h : h \in H^2\},\\
\tilde{\mathcal{K}}_b &= (H^2 \oplus L^2) \ominus \{bh \oplus \Delta h : h \in H^2\},\\
\mathcal{J}_b &= \tilde{\mathcal{K}}_b \ominus \mathcal{K}_b = \{0\} \oplus (L^2 \ominus \overline{\Delta H^2}),\\
Y_b &= P_{\mathcal{K}_b}(S^* \oplus Z^*)|\mathcal{K}_b,\\
\mathbf{Y}_b &= S^* \oplus Z^*|\tilde{\mathcal{K}}_b.
\end{aligned}$$

A basic reason why we introduce them is the next lemma.

Lemma 2.2. *The orthogonal projection onto the first coordinate is a unitary operator from $\mathcal{K}_b$ onto $\mathcal{H}(b)$, that intertwines Y_b with X_b.*

Proof. The lemma is almost completely proved in [19, IV-7]. It is shown therein that the operator

$$B = \begin{pmatrix} \mathbf{T}_b \\ -\mathbf{T}_a \end{pmatrix}$$

is an isometry from H^2 to $H^2 \oplus H^2$, and that the projection Q onto the first coordinate is a unitary from $\mathbf{K}_B = (H^2 \oplus H^2) \ominus BH^2$ onto $\mathcal{H}(b)$, which intertwines the restriction of $S^* \oplus S^*$ to this subspace with X. On the other hand, the map $W : \overline{\Delta H^2} \to H^2$ defined by $W(\Delta h) = -ah$ is easily seen to be an isometry, and it is actually unitary since a is outer. It also commutes with S and therefore, being unitary, with S^*. Then $Q \circ (I_{H^2} \oplus W)$ yields the desired unitary operator. $\square$

As a consequence, we will concentrate on Y_b rather than on X_b in the rest of this paper.

The following result gathers some of the properties of the above spaces and operators. They constitute the basis for the "model theory" that will be investigated in the rest of the paper.

Lemma 2.3. *Suppose $b \not\equiv 0$. With the above notations, the following are true.*

1. *We have $\dim \mathcal{D}_{Y_b} = 2$, $\dim \mathcal{D}_{Y_b^*} = 1$, and $\dim \ker Y_b = 1$. Y_b is unitarily equivalent to $X_b = S^*|\mathcal{H}(b)$. Its characteristic function is*
$$\Theta_{Y_b} = (\tilde{a}\ \tilde{b}) .$$
Consequently, $Y_b \to 0$ strongly.
2. *$\mathbf{Y}_b^*$ is precisely the Nagy–Foias model corresponding to the characteristic function b.*
3. *$\mathbf{Y}_b$ is a nonisometric dilation of Y_b; that is, it satisfies for all $n \in \mathbb{N}$ the relation $Y_b^n = P_{\mathcal{K}_b} \mathbf{Y}_b^n | \mathcal{K}_b$.*
4. *$\mathbf{Y}_b|\mathcal{J}_b$ is an isometry. If $\mathcal{X} \subset \tilde{\mathcal{K}}_b$ is an invariant subspace for $\mathbf{Y}_b$, such that $\mathbf{Y}_b|\mathcal{X}$ is an isometry, then $\mathcal{X} \subset \mathcal{J}_b$. In particular, $\mathbf{Y}_b|\mathcal{J}_b$ is a maximal isometry contained in $\mathbf{Y}_b$, and Y_b has no isometric restriction.*

Proof. (1) The claimed properties of Y_b are proved explicitely in [19, IV-7] for X_b. The only exception is the dimension of $\ker Y_b$. Since $Y_b(\mathcal{D}_{Y_b}) \subset \mathcal{D}_{Y_b^*}$, it has a nonzero kernel. If $f \oplus g \in \ker Y_b$, then $S^* f = 0$, whence $f = c$ (constant). If we had two linearly independent vectors in $\ker Y_b$, some linear combination would have 0 as first coordinate, and thus we would have $0 \oplus g_0 \in \mathcal{K}_b$ for some $g_0 \in \overline{\Delta H^2}$, $g_0 \neq 0$. But the definition of $\mathcal{K}_b$ implies $g_0 \perp \Delta h$ for any $h \in H^2$, whence $g_0 = 0$, which is a contradiction.

(2) is an immediate consequence of the general form of the Sz.-Nagy–Foias model, while (3) follows easily from the fact that $\mathcal{J}_b$ is invariant with respect to $\mathbf{Y}_b$.

(4) It is immediate that $\mathbf{Y}_b|\mathcal{J}_b$ is an isometry, since it is unitarily equivalent to a restriction of Z^*. Then, if $\mathcal{X}$ has the stated properties, take $f \oplus g \in \mathcal{X}$. We have
$$\|S^{*n} f\|^2 + \|Z^{*n} g\|^2 = \|\mathbf{Y}_b^n (f \oplus g)\|^2 = \|f \oplus g\|^2 = \|f\|^2 + \|g\|^2$$
for any n. Since Z^* is unitary and $S^* f \to 0$, this implies $f = 0$, whence $\mathcal{X} \subset \mathcal{J}_b$. $\square$

It should be kept in mind that, according to Lemma 2.3(2), the model operators in the de Branges–Rovnyak and Sz.Nagy–Foias approaches are mutual adjoints.

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3. A Functional Reformulation

As noted in the introduction, we intend to find necessary and sufficient conditions for a c.n.u. contraction $T \in \mathcal{L}(H)$ to be unitarily equivalent to Y_b for some nonextreme function $b \in H^\infty$, $\|b\| \le 1$. Some necessary conditions follow already from Lemma 2.3(1): we must have

(C1) $\dim \mathcal{D}_T = 2$, $\dim \mathcal{D}_{T^*} = \dim \ker T = 1$;
(C2) $T^n \to 0$ strongly.

We will see later that (C1) and (C2) are not sufficient, but first we will use them in order to give an alternate formulation of the problem.

A general c.n.u. contraction T with $\dim \mathcal{D}_T = 2, \dim \mathcal{D}_{T^*} = 1$ has as characteristic function an arbitrary pure contractive analytic function $\Theta_T : \mathbb{D} \to \mathcal{L}(\mathbb{C}^2, \mathbb{C})$

$$\Theta_T = \begin{pmatrix} \phi_1 & \phi_2 \end{pmatrix}. \tag{3.1}$$

In this case the purity condition (which means that Θ_T has no constant unitary part) is equivalent to the fact that Θ_T does not coincide with the constant function

$$\begin{pmatrix} 0 & \kappa \end{pmatrix},$$

where $\kappa \in \mathbb{C}$, $|\kappa| = 1$. Moreover, the condition $T^n \to 0$ strongly is known to be equivalent to the identity $|\phi_1|^2 + |\phi_2|^2 = 1$ (one says that Θ_T is $*$-*inner*).

Theorem 3.1. *If T is a c.n.u. contraction with characteristic function given by (3.1), then the following are equivalent:*

1. *T is unitarily equivalent with Y_b for some nonextreme b.*
2. *$|\phi_1|^2 + |\phi_2|^2 = 1$ a.e., and there exist $\alpha_1, \alpha_2 \in \mathbb{C}$ with $|\alpha_1|^2 + |\alpha_2|^2 = 1$, such that $\tilde{a} := \alpha_1\phi_1 + \alpha_2\phi_2$ is an outer function.*

Proof. If (1) is true, then Θ_T coincides with Θ_{Y_b}. Using Lemma 2.3(1) it follows that there exists a constant unitary 2×2 matrix $\left(\begin{smallmatrix} \alpha_1 & \alpha_3 \\ \alpha_2 & \alpha_4 \end{smallmatrix}\right)$ such that

$$\begin{pmatrix} \tilde{a} & \tilde{b} \end{pmatrix} = \begin{pmatrix} \phi_1 & \phi_2 \end{pmatrix} \begin{pmatrix} \alpha_1 & \alpha_3 \\ \alpha_2 & \alpha_4 \end{pmatrix}. \tag{3.2}$$

Since $\tilde{a}$ is an outer function, (2) is proved.

Conversely, if (2) is true, then we may choose α_3, α_4 such that $\left(\begin{smallmatrix} \alpha_1 & \alpha_3 \\ \alpha_2 & \alpha_4 \end{smallmatrix}\right)$ is unitary, and (3.2) is satisfied with $\tilde{b} = \alpha_3\phi_1 + \alpha_4\phi_2$. Then $\tilde{b}$ is a function in the unit ball of H^∞ that is nonextreme since $\int \log(1 - |\tilde{b}|^2) = \int \log|\tilde{a}|^2 > -\infty$. Since (3.2) and Lemma 2.3(1) say that Θ_T coincides with Θ_{Y_b}, it follows that T is unitarily equivalent to $Y_{\tilde{b}}$. □

However, characterizing the pairs (ϕ_1, ϕ_2) that satisfy (2) seems an even more difficult problem. Moreover, such a characterization would not use directly properties of the operator T, but rather of its characteristic function. That is why we seek other alternatives.

4. A Third Necessary Condition

As noted above, the conditions $\dim \mathcal{D}_T = 2$, $\dim \mathcal{D}_{T^*} = \dim \ker T = 1$, and $T^n \to 0$ strongly are necessary for the unitary equivalence of T with some Y_b. They are not sufficient; a less obvious condition is given by the next lemma.

Lemma 4.1. *If T is unitarily equivalent to Y_b for some nonextreme b, then:*

(C3) *There is no subspace Y of H invariant with respect to T^*, such that $T^*|Y$ is unitarily equivalent to a scalar model operator.*

Proof. Suppose T has characteristic function given by (3.1); then the characteristic function of T^* is the inner function $\tilde{\Theta}_T = \begin{pmatrix} \tilde{\phi}_1 \\ \tilde{\phi}_2 \end{pmatrix}$. If T is unitarily equivalent to Y_b for some nonextreme b, it follows from Theorem 3.1 that ϕ_1 and ϕ_2 must not have an inner common factor; the same is true also for $\tilde{\phi}_1, \tilde{\phi}_2$. The statement is then a consequence of Lemma 2.1(1). □

It is easy now to give an example of an operator that satisfies (C1) and (C2) but not (C3): take $T = S^* \oplus S_\theta$ for some inner function θ.

However, even all three conditions (C1–C3) are still not sufficient. To show this, it is enough, in view of Theorem 3.1 and Lemma 2.1(1), to find two functions $\phi_1, \phi_2 \in H^\infty$ with $|\phi_1|^2 + |\phi_2|^2 = 1$, such that ϕ_1 and ϕ_2 have no common inner factor and there is no linear combination of ϕ_1 and ϕ_2 which is outer. This is given in the next example.

Example 4.2. Take $\phi_1(z) = \frac{1}{\sqrt{2}} z^2$, $\phi_2(z) = \frac{1}{\sqrt{2}} \frac{z-a}{1-\bar{a}z}$. Then obviously $|\phi_1|^2 + |\phi_2|^2 = 1$. We will show that at least for $0 < a < 1/8$ there is no outer linear combination of ϕ_1 and ϕ_2, and thus, by Theorem 3.1, T is not unitarily equivalent with some Y_b for b nonextreme.

First, since ϕ_1, ϕ_2 themselves are not outer, it is enough to consider linear combinations of the type $\phi_1 + \alpha\phi_2$ for some $\alpha \in \mathbb{C}$. If $|\alpha| < 1$, then $|\alpha\phi_2(z)| < |\phi_1(z)|$ for $z \in \mathbb{T}$, and thus Rouché's Theorem says that $\phi_1 + \alpha\phi_2$ has the same number of zeros in $\mathbb{D}$ as ϕ_1, so it cannot be outer. A similar argument settles the case $|\alpha| > 1$.

Let us now consider $|\alpha| = 1$. The equation $\phi_1(z) + \alpha\phi_2(z) = 0$ can be written

$$z^2 + \alpha z - a\left(\alpha + \frac{\bar{a}}{a} z^3\right) = 0. \tag{4.1}$$

If $|z| = 1/2$, then $|z^2 + \alpha z| = |z(z + \alpha)| > \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$. On the other hand, $|\alpha + \frac{\bar{a}}{a} z^3| \le 1 + \frac{1}{8} < 2$, and thus $|a\left(\alpha + \frac{\bar{a}}{a} z^3\right)| < \frac{1}{4}$ if $0 < a < 1/8$. We may again apply Rouché's Theorem to conclude that (4.1) has a solution in the disc $\{|z| < 1/4\}$, and thus neither is $\phi_1 + \alpha\phi_2$ outer if $|\alpha| = 1$.

We have then to find some other necessary condition, besides (C1)–(C3). This requires a certain construction that will be done in the next section.

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5. Construction of Certain Dilations

We start with an elementary lemma, whose proof we omit.

Lemma 5.1. *Suppose* $0 < \alpha < 1$, $\xi = (\xi_1, \xi_2)$ *with* $|\xi_1|^2 + |\xi_2|^2 = 1$, *and denote*

$$A := \begin{pmatrix} \alpha & 0 \\ a\bar{\xi}_1 & a\bar{\xi}_2 \end{pmatrix}$$

Then

$$a = a_\xi := \left(\frac{1-\alpha^2}{1-\alpha^2|\xi_2|^2} \right)^{1/2}, \tag{5.1}$$

is the only value of a *for which* A *is a contraction with* $\dim \mathcal{D}_A = \dim \mathcal{D}_{A^*} = 1$. *If we denote then by* e_ξ *a unit vector in* $\ker(I - A^*A)$ (*therefore* $\|Ae_\xi\| = \|e_\xi\|$), *then* e_ξ *is determined up to a unimodular constant; moreover, for any* $\eta \in \mathbb{C}^2$ *with* $\|\eta\| = 1$, *there exists* $\xi = (\xi_1, \xi_2)$, $|\xi_1|^2 + |\xi_2|^2 = 1$ *such that* $e_\xi = \eta$.

Note that for $a < a_\xi$ the defects have dimension 2, while for $a > a_\xi$ A is no more a contraction. Also, we may take $e_\xi = \xi$ if and only if ξ is one of the standard basis vectors.

To go beyond the conditions in Sect. 4, we consider a construction that stems from the fact that $\mathbf{Y}_b$ is a nonisometric dilation of Y_b. We have then to discuss a certain general construction of nonisometric dilations. Suppose then that T is a contraction acting on the Hilbert space H with $\dim \mathcal{D}_T = 2$, $\dim \mathcal{D}_{T^*} = \dim \ker T = 1$.

We are interested in dilations $\tilde{T}$ of T with the property that $\dim \mathcal{D}_{\tilde{T}} = \dim \mathcal{D}_{\tilde{T}^*} = 1$. These may be described in the following manner. For clarity of notation, we will denote by T_d and T_u the restrictions $T_d = T : \mathcal{D}_T \to \mathcal{D}_{T^*}$, $T_u = T : \mathcal{D}_T^\perp \to \mathcal{D}_{T^*}^\perp$; note that T_u is unitary and T_d is a strict contraction.

Take a vector $\xi \in \mathcal{D}_T$, with $\|\xi\| = 1$, and consider, for $0 < a \le 1$, the operator

$$A_\xi := \begin{pmatrix} T_d \\ a_\xi \otimes \xi \end{pmatrix} : \mathcal{D}_T \to \mathcal{D}_{T^*} \oplus \mathbb{C}.$$

The operator T_d is a strict contraction with kernel of dimension 1. If we choose in $\mathcal{D}_T$ a basis formed by the eigenvectors of $T_d^* T_d$, then the matrix of T_d is $(\alpha \quad 0)$ for some $0 < \alpha < 1$, and thus A_ξ is precisely the A in Lemma 5.1. Consequently, $\dim \mathcal{D}_{A_\xi} = \dim \mathcal{D}_{A_\xi^*} = 1$. Remember that e_ξ is a normalized vector in $\ker(I - A_\xi^* A_\xi) \cap \mathcal{D}_T$; this notation will be used consistently in the sequel of the paper.

Consider then the space

$$K_\xi = H \oplus \mathbb{C} \oplus \mathbb{C} \oplus \cdots$$

on which acts the operator

$$T_\xi := \begin{pmatrix} T & 0 & 0 & 0 & \dots \\ a_\xi \otimes \xi & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \end{pmatrix}. \tag{5.2}$$

Lemma 5.2. 1. *With the above notations, T_ξ is a minimal contractive dilation of T satisfying $\dim \mathcal{D}_{T_\xi} = \dim \mathcal{D}_{T_\xi^*} = 1$.*

2. *Suppose $\widehat{T} \in \mathcal{L}(\widehat{H})$ is a dilation of T, such that $\dim \mathcal{D}_{\widehat{T}} = 1$ and $\widehat{T}|\widehat{H} \ominus H$ is a pure isometry of multiplicity 1. Then $\widehat{T}$ is unitarily equivalent to some T_ξ as above.*

Proof. (1) Let us denote $\mathcal{I}_\xi = K_\xi \ominus H$. With respect to the two decompositions

$$K_\xi = \mathcal{D}_T^\perp \oplus \mathcal{D}_T \oplus \mathbb{C} \oplus \mathbb{C} \oplus \cdots = \mathcal{D}_{T^*}^\perp \oplus \mathcal{D}_{T^*} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \cdots$$

T_ξ has the matrix

$$T_\xi := \begin{pmatrix} T_u & 0 & 0 & 0 & \dots \\ 0 & T_d & 0 & 0 & \dots \\ 0 & a_\xi \otimes \xi & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \ddots & \end{pmatrix}, \tag{5.3}$$

which means that, if in the range space we consider together the second and the third space ($\mathcal{D}_{T^*} \oplus \mathbb{C}$), the matrix of T_ξ is diagonalized, and we have

$$T_\xi = T_u \oplus A_\xi \oplus 1 \oplus 1 \oplus \cdots.$$

Since all operators except the second are unitary, T_ξ is a contraction and the dimensions of its defects are the same as those of A_ξ, that is 1. Moreover, T_ξ is a minimal dilation of T.

(2) Suppose $\widehat{T}$ is a dilation of T with $\dim \mathcal{D}_{\widehat{T}} = 1$, acting on $\widehat{H} \supset H$. Since $\widehat{T}|\widehat{H} \ominus H$ is a shift of multiplicity 1, $\widehat{T}$ must have the form

$$\widehat{T} := \begin{pmatrix} T & 0 & 0 & \dots \\ X & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

with a nonnull $X : H \to \mathbb{C}$, $X = a \otimes \xi$ for some $\xi \in H$ with $\|\xi\| = 1$ and some a. We have $\mathcal{D}_{\widehat{T}} \subset H$, and $D_{\widehat{T}}|H = I - T^*T - X^*X$. Since $I - T^*T$ has rank 2, while $I - T^*T - X^*X$ has rank 1, it follows from Lemma 5.1 that $X = a_\xi \otimes \xi$, with $\xi \in \ker D_{\widehat{T}} \cap \mathcal{D}_T$. ☐

Lemma 5.3. *If ξ is an eigenvector of $\mathcal{D}_T$, and T_ξ is completely nonunitary, then:*

1. *The characteristic function b_ξ of T_ξ is nonextreme.*
2. *If $\mathcal{I}_\xi \subset \mathcal{Y} \subset K_\xi$, $\mathcal{Y}$ is invariant with respect to T_ξ, and $T_\xi|\mathcal{Y}$ is an isometry, then $T_\xi|\mathcal{Y}$ is a shift of multiplicity 1.*

Proof. By Lemma 5.2, we have $\dim \mathcal{D}_{T_\xi} = \dim \mathcal{D}_{T_\xi^*} = 1$, so T_ξ has a scalar characteristic function b_ξ. This has to be nonextreme since T_ξ has an isometric restriction (namely, $T|\mathcal{I}_\xi$).

For the second statement, apply Lemma 2.1(2) to the contraction T_ξ and its invariant subspace $\mathcal{Y}$. □

At this point we may give another reformulation of the main question.

Theorem 5.4. *If T is a c.n.u. with characteristic function given by* (3.1)*, then the following are equivalent:*

1. *T is unitarily equivalent with Y_b for some nonextreme b.*
2. *There exists $\xi \in \mathcal{D}_T$, $\|\xi\| = 1$, such that the contraction T_ξ defined by* (5.2) *is completely nonunitary and $T_\xi|\mathcal{I}_\xi$ is a maximal isometry.*

Proof. If T is unitarily equivalent with Y_b for some nonextreme b, then, by Lemma 2.3, $\mathbf{Y}_b$ is a completely nonunitary dilation of Y_b with the required properties in the assumptions of 5.2(2), whence it has to be unitarily equivalent to some T_ξ. By Lemma 2.3 we know that $T_\xi|\mathcal{I}_\xi$ is a maximal isometry.

Conversely, if (2) is true, the given completely nonunitary contraction T_ξ has a nonextreme characteristic function b_ξ by Lemma 5.3(i). There exists therefore a unitary $W : K_\xi \to \widehat{\mathcal{K}}_{b_\xi}$, such that $\mathbf{Y}_{b_\xi} W = W T_\xi$. By Lemma 2.3(4), $\mathcal{J}_b$ is the space on which acts the unique maximal isometry contained in $\mathbf{Y}_{b_\xi}$, and therefore it has to be equal to $W\mathcal{I}_\xi$. Passing to orthogonals, W maps H onto $\mathcal{K}_{b_\xi}$, and commutes with the respective compressions there. This says precisely that T is unitarily equivalent to Y_{b_ξ}. □

We have then to investigate the two properties in point (2) of the above proposition.

6. T_ξ Completely Nonunitary

We prove in this section that conditions (C1)–(C3) imply that T_ξ is completely nonunitary.

Proposition 6.1. *Suppose T is a c.n.u. contraction on H that satisfies conditions (C1)–(C3). Then T_ξ is completely nonunitary for all $\xi \in \mathcal{D}_T$, $\|\xi\| = 1$.*

Proof. Denote by V the minimal isometric dilation of T_ξ, acting on the space $K \supset H$. Since T_ξ is a minimal dilation of T, it follows easily that V is also a minimal isometric dilation of T.

We will use the Sz.-Nagy–Foias model of the contraction T, which is the space

$$\mathbf{H} = (H^2 \oplus \Delta L^2(\mathbb{C}^2)) \ominus \{\Theta_T h \oplus \Delta h : h \in H^2(\mathbb{C}^2)\}$$

and the operator unitarily equivalent to T is $\mathbf{T} = P_{\mathbf{H}}(S \oplus Z)|\mathbf{H}$. The minimal unitary dilation $\mathbf{V}$ is just $S \oplus Z$ acting on $\mathbf{K} = H^2 \oplus \Delta L^2(\mathbb{C}^2)$, and its unitary part acts on the space $\{0\} \oplus \Delta L^2(\mathbb{C}^2)$. Let us denote by Ω the unitary that implements the equivalence; that is, $\Omega : K \to \mathbf{K}$, $\Omega(H) = \mathbf{H}$, $\Omega V = \mathbf{V}\Omega$.

If $T^n \to 0$ strongly, then the characteristic function of T is given by (3.1), with $|\phi_1|^2 + |\phi_2|^2 = 1$ a.e. Then

$$\Theta_T^* \Theta_T = \begin{pmatrix} |\phi_1|^2 & \bar{\phi}_1 \phi_2 \\ \bar{\phi}_2 \phi_1 & |\phi_2|^2 \end{pmatrix} = \begin{pmatrix} \bar{\phi}_1 \\ \bar{\phi}_2 \end{pmatrix} (\phi_1 \ \phi_2)$$

is almost everywhere on $\mathbb{T}$ a one-dimensional projection in $\mathbb{C}^2$. Therefore $\Delta(e^{it})$ is also a one-dimensional projection a.e. If we write $J(e^{it}) = \begin{pmatrix} \bar{\phi}_1(e^{it}) \\ \bar{\phi}_2(e^{it}) \end{pmatrix} : \mathbb{C} \to \mathbb{C}^2$, then the map $f \mapsto J(f)$ is a unitary operator from L^2 to $JL^2 = \Delta L^2(\mathbb{C}^2)$. Moreover, J intertwines multiplication with e^{it} in the corresponding L^2 spaces.

Consider now the operator $\mathbf{T}_\xi$ corresponding in the Sz.-Nagy–Foias model to T_ξ, that is, $\mathbf{T}_\xi = \Omega T_\xi \Omega^*$. Its unitary part is a reducing subspace of the unitary part of $\mathbf{V}$, and thus has to be a reducing subspace of $\{0\} \oplus \Delta L^2(\mathbb{C}^2)$ with respect to $S \oplus Z$, which means a reducing subspace of JL^2 with respect to multiplication by e^{it}. Therefore it is $J(L^2(E))$ for some measurable subset $E \subset \mathbb{T}$, or, equivalently, $\Delta L^2(E)$.

Consider now the vector e_ξ introduced in the previous section. Since $\|T_\xi e_\xi\| = \|e_\xi\|$, we must also have $T_\xi e_\xi = V e_\xi$, and therefore

$$\mathbf{T}_\xi \Omega e_\xi = \mathbf{V} \Omega e_\xi = (S \oplus Z) e_\xi \in (S \oplus Z)\mathbf{H} \subset \mathbf{H} \oplus \{\Theta_T c_1 \oplus \Delta c_2 : c_1, c_2 \in \mathbb{C}\}.$$

By (5.2), $T_\xi e_\xi$ belongs to $H \oplus \mathbb{C}$ (it has no components on the subsequent copies of $\mathbb{C}$ in the formula of K_ξ), and the second component is $a_\xi \neq 0$. So the projection of $T_\xi e_\xi$ onto $\mathcal{I}_\xi$ is a nonzero vector on the first component of $\mathcal{I}_\xi$, which is a wandering vector for $T_\xi|\mathcal{I}_\xi$. Applying Ω to this projection, we obtain that a wandering vector for $\mathbf{T}_\xi|\Omega(\mathcal{I}_\xi)$ is of the form $\Theta_T c_1 \oplus \Delta c_2$. After a change of basis in $\mathcal{D}_T$, we may assume that $c_2 = 0$.

It follows then that $\mathbf{T}_\xi$ is the compression of $S \oplus Z$ to the space

$$\begin{aligned} \mathbf{K}_\xi &= \mathbf{H} \oplus \left\{ \Theta_T \begin{pmatrix} h \\ 0 \end{pmatrix} \oplus \Delta \begin{pmatrix} h \\ 0 \end{pmatrix} : h \in H^2 \right\} \\ &= \mathbf{K} \ominus \left\{ \Theta_T \begin{pmatrix} 0 \\ h \end{pmatrix} \oplus \Delta \begin{pmatrix} 0 \\ h \end{pmatrix} : h \in H^2 \right\}. \end{aligned} \tag{6.1}$$

Now, if $\{0\} \oplus \Delta L^2(E) \subset \mathbf{K}_\xi$, it has to be orthogonal to $\Delta \left(\begin{smallmatrix} 0 \\ H^2 \end{smallmatrix} \right)$, whence $\Delta(e^{it})$ must be a.e. on E the projection on the first coordinate. That means that $\phi_1 = 0$ a.e. on E, whence $\phi_1 \equiv 0$, ϕ_2 inner. This is excluded by the last part of the hypothesis. □

7. The Final Result

We need only one more ingredient to obtain the final result.

Lemma 7.1. *Suppose T is a c.n.u. contraction on H that satisfies conditions (C1)–(C3). The following are equivalent:*

1. *$T_\xi|\mathcal{I}_\xi$ is a maximal isometry.*
2. *For any $H' \subset H$ such that $TH' \subset H'$ and $T' := T|H'$ is a scalar model operator, we have $e_\xi \notin H'$.*
3. *For any $H' \subset H$ such that $TH' \subset H'$ and $T' := T|H'$ is a scalar model operator, we have $e_\xi \notin \mathcal{D}_{T'}$.*

Proof. (1) $\Longrightarrow$ (2). Suppose (1) is true, and let $H' \subset H$ such that $TH' \subset H'$, $e_\xi \in H'$, and $T' := T|H'$ is a scalar model operator. Then $\mathcal{D}_{T'}$ having dimension 1, is spanned by e_ξ. It may then be checked that the space $\mathcal{Y} = \mathcal{I}_\xi \oplus H'$ is invariant with respect to T_ξ, and $T_\xi|\mathcal{Y}$ is an isometry that strictly extends $T_\xi|\mathcal{I}_\xi$. Therefore $e_\xi \notin H'$.

(2) $\Longrightarrow$ (3) is immediate. Let us assume that (3) is true, and suppose $\mathcal{I}_\xi \subset \mathcal{Y} \subset \mathcal{K}_\xi$, $T_\xi\mathcal{Y} \subset \mathcal{Y}$, and $T_\xi|\mathcal{Y}$ is an isometry. If $\mathcal{Y}' = \mathcal{Y} \cap H \neq \{0\}$ and $T' = P_{\mathcal{Y}'}T'_\xi|\mathcal{Y}'$, then T'_ξ is an isometric dilation of T', which is a shift of multiplicity 1 by Lemma 5.3(2). Thus T' is the compression of a shift of multiplicity one to a coinvariant subspace, which is precisely unitarily equivalent to a scalar model operator.

Since $\mathcal{D}_{T'} = \{x \in \mathcal{Y}' : \|T'x\| < \|x\|\}$, we have $\mathcal{D}_{T'} \subset \mathcal{D}_T$. Suppose then $x \in \mathcal{D}_{T'}$, $x = x_1 + x_2$, with $x_1 \in \ker T$, x_2 multiple of e_ξ. We have then

$$\|x_1\|^2 + \|x_2\|^2 = \|x\|^2 = \|T_\xi x\|^2 = \|Tx_1\|^2 + \|Tx_2\|^2 \le \|x_2\|^2,$$

whence $x_1 = 0$. Therefore x is a multiple of e_ξ, which contradicts assumption (3). It follows that $\mathcal{Y} = \mathcal{I}_\xi$, ending the proof of the lemma. □

In the light of Lemma 7.1, we may now state the last necessary condition:

(**C4**) There exists $\eta \in \mathcal{D}_T$ such that, if $\mathcal{Y}' \subset H$, $T\mathcal{Y}' \subset \mathcal{Y}'$, and $T' := T|\mathcal{Y}'$ is unitarily equivalent to a scalar model operator, then $\eta \notin \mathcal{Y}'$.

The desired characterization is then given by the next theorem.

Theorem 7.2. *Suppose T is a c.n.u. contraction on H. The following are equivalent:*

1. *T is unitarily equivalent to X_b for some nonextreme function $b \not\equiv 0$.*
2. *T satisfies conditions (C1)–(C4).*

Proof. If T is unitarily equivalent to X_b for some nonextreme function $b \not\equiv 0$, then (C1)–(C3) have already been proved. To prove (C4), note that, since $\mathbf{Y}_b$ is a dilation of Y_b with $\dim \mathcal{D}_{Y_b} = 1$ and $\mathbf{Y}_b|\tilde{K}_b \ominus K_b$ is a maximal isometry, it follows from Lemma 5.2(2) that $\mathbf{Y}_b$ is unitarily equivalent to T_ξ, with $\xi \in \mathcal{D}_{Y_b}$. Then (C4) follows from Lemma 7.1.

For the reverse implication, choose a vector ξ such that $\eta = e_\xi$; its existence is ensured by Lemma 5.1. The dilation T_ξ is a completely nonunitary contraction by Proposition 6.1. Lemma 7.1 ensures that $T_\xi|\mathcal{I}_\xi$ is a maximal isometry, and then Theorem 5.4 implies that (1) is true. □

Condition (C3) can be reformulated as

(C3′) There exists no subspace $\mathcal{Y} \subset H$ such that $T^*\mathcal{Y} \subset \mathcal{Y}$ and, if $T_\mathcal{Y} := T^*|\mathcal{Y}$, then $\dim \mathcal{D}_{T_\mathcal{Y}} = \dim \mathcal{D}_{T_\mathcal{Y}^*} = 1$.

Indeed, we have $T_\mathcal{Y}^{*n} = P_\mathcal{Y} T^n \to 0$ strongly by (a). Similarly, condition (C4) can be reformulated as

(C4′) There exists $\eta \in \mathcal{D}_T$ such that, whenever $\mathcal{Y}' \subset H$, $T\mathcal{Y}' \subset \mathcal{Y}'$, and, if $T' := T|\mathcal{Y}'$, $\dim \mathcal{D}_{T'} = \dim \mathcal{D}_{T'^*} = 1$, we have $\eta \notin \mathcal{Y}'$.

8. Freedom in the choice of b

A natural question when considering model theory is whether a given operator determines its model (up to some simple transformation). Let us then suppose that a contraction $T \in \mathcal{L}(H)$ is unitarily equivalent to X_{b_1} as well as to X_{b_2} for some b_1, b_2 in the unit ball of H^∞. Since $X_{b_1}^*$ and $X_{b_2}^*$ have to be unitarily equivalent, their characteristic functions must coincide. Also, by looking at the dimensions of the defect spaces of T, it follows immediately that b_1, b_2 are simultaneously extreme or nonextreme, so we have to discuss two cases.

If b is extreme, then the characteristic function of X_b^* is precisely b. So the answer is simple: if T is unitarily equivalent to X_{b_1} as well as to X_{b_2}, then $b_1 = \kappa b_2$ for some unimodular constant κ.

If b_1, b_2 are nonextreme, the characteristic functions of $X_{b_1}^*$ and $X_{b_2}^*$ are given by Lemma 2.3, and if they coincide we must have

$$\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \tag{8.1}$$

for some unitary constant matrix $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$. This is possible for rather different functions b_1, b_2, as shown by the following example. Take $b_1 = z/\sqrt{2}$ (so $a_1 = 1/\sqrt{2}$), and $\alpha = \beta = \gamma = -\delta = 1/\sqrt{2}$; it follows that $b_2 = \frac{1-z}{2}$. We have then X_{b_1} unitarily equivalent to X_{b_2}, but b_1 is a constant multiple of an inner function, while b_2 is outer. There seems to be no simple criterion that could decide when X_{b_1} unitarily equivalent to X_{b_2} without involving the associated outer functions a_1 and a_2.

A natural question is then whether there exist cases when, as in the extreme case, b is uniquely determined up to a unimodular constant. If X_{b_1} and X_{b_2} are unitarily equivalent, then (8.1) implies, in particular, that $a_2 = \alpha a_1 + \beta b_1$ is outer. Conversely, suppose b_1 is given, a_1 is the associated outer function, and a certain combination $a_2 = \alpha a_1 + \beta b_1$ is outer. We may suppose $|\alpha|^2 + |\beta|^2 = 1$; if we take $\gamma = \bar{\beta}$, $\delta = -\bar{\alpha}$, then $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ is unitary and b_2 defined by (8.1) has the property that X_{b_2} is unitarily equivalent to X_{b_1}.

We may then reformulate the last problem as follows:

Question: Does there exist a nonextreme function b such that, if a is the associated outer function, then $\alpha a + \beta b$ outer implies $\beta = 0$?

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