H^2- SPACES OF NON-COMMUTATIVE FUNCTIONS

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Abstract. The p[aper presents some basic results in the study of Hardy H^2 spaces of locally bounded non-commutative functions on certain non-commutative unit balls. We consider the cases of uniform, and row/column operator spaces norm on a finite dimesional vector space.

1. Introduction

1.1. **Haar Unitaries and Free Independence.** Let N be a positive integer and $\mathcal{U}(N)$ be the compact group of the $N \times N$ unitary matrices with complex entries. The Haar measure on $\mathcal{U}(N)$ will be denoted with $d\mathcal{U}_N$.

For each $i, j \in \{1, 2, \ldots, N\}$ we define the maps $u_{i,j} : \mathcal{U}(N) \longrightarrow \mathbb{C}$ giving the i, j-th entry of each element from $\mathcal{U}(N)$. As shown in [1], the maps $u_{i,j}$ are in $L^\infty(\mathcal{U}(N), d\mathcal{U}_N)$. Let S_n be the symmetric group of order n; for $\sigma \in S_n$ denote by $\#(\sigma)$ the number of cycles in a minimal decomposition of the permutation σ. The following result is shown in [2], Corollary 2.4:

Theorem 1.1. *There exists a map $Wg : \mathbb{Z}_+ \times S_n \longrightarrow \mathbb{R}$ such that:*

(1) *For all $\sigma \in S_n$, the limit $\displaystyle\lim_{N\to\infty} \frac{Wg(N,\sigma)}{N^{2n-\#(\sigma)}}$ exists and is finite.*

(2) *For any multiindices $\boldsymbol{i} = (i_1, i_2, \ldots, i_n)$, $\boldsymbol{i'} = (i'_1, i'_2, \ldots, i'_n)$, respectively $\boldsymbol{j} = (j_1, j_2, \ldots, j_n)$, $\boldsymbol{j'} = (j'_1, j'_2, \ldots, j'_n)$ with elements from the set $\{1, 2, \ldots, N\}$ we have that*

$$\int_{\mathcal{U}(N)} u_{i_1,j_1} \cdots u_{i_n,j_n} \overline{u_{i'_1,j'_1}} \cdots \overline{u_{i'_n,j'_n}} d\mathcal{U}_N = \sum_{\sigma,\tau \in S_n} \delta_{\boldsymbol{i},\boldsymbol{i'}} \cdot \delta_{\boldsymbol{j},\boldsymbol{j'}} \cdot Wg(N, \tau\sigma^{-1}).$$

If $m \neq n$, then

$$\int_{\mathcal{U}(N)} u_{i_1,j_1} \cdots u_{i_n,j_n} \overline{u_{i'_1,j'_1}} \cdots \overline{u_{i'_m,j'_m}} d\mathcal{U}_N = 0.$$

An immediate consequence of the result above is the following:

Remark 1.2. *Let $U : \mathcal{U}(N) \longrightarrow \mathbb{C}^{N\times N}$, $U = [u_{i,j}]_{i,j=1}^N$. Then, for all non-zero integers α,*

$$\int_{\mathcal{U}(N)} Tr(U^\alpha) d\mathcal{U}_N = 0.$$

Proof. Suppose that $\alpha > 0$. Then

$$\int_{\mathcal{U}(N)} \mathrm{Tr}(U^\alpha) d\mathcal{U}_N = \sum_{1 \le i_1, \ldots, i_\alpha \le N} \int_{\mathcal{U}(N)} u_{i_1,i_2} \cdots u_{i_{\alpha-1},i_\alpha} u_{i_\alpha,i_1} d\mathcal{U}_N.$$

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From the last part of Theorem 1.1, all the terms in the above summation are zero, hence the conclusion. Since $U^{-1} = U^*$, the case $\alpha < 0$ is similar. $\square$

Suppose that $\mathcal{A}$ is a unital algebra and $\phi : \mathcal{A} \longrightarrow \mathbb{C}$ is a conditional expectation. A family $\{\mathcal{A}_j\}_{j\in J}$ of unital subalgebras of $\mathcal{A}$ are said to be **free independent** if any alternating product of centered (with respect to ϕ) of elements from $\{\mathcal{A}_j\}_{j\in J}$ is centered, i.e. for any $n > 0$, any $\epsilon(k) \in J$ $(1 \leq k \leq n)$ such that $\epsilon(k) \neq \epsilon(k+1)$ and any $a_k \in \mathcal{A}_{\epsilon(k)}$ we have that $\phi(a_1 a_2 \cdots a_n) = 0$.

As shown in the extensive literature on the subject (see [13], [10]), free independence is the natural relation of independence in a non-commutative framework and it is the asymptotical relation satisfied by various classes of random matrices.

Let $\mathfrak{A} = \{A_{j,N}\}_{j\in J, N\geq 1}$ be an ensemble of matrices such that $A_{j,N} \in \mathbb{C}^{N\times N}$ for all $j \in J$. The ensemble $\mathfrak{A}$ is said to have limit distribution if for any $m \in \mathbb{Z}_+$ and $j_1, \ldots, j_m \in J$ the limit

$$\lim_{N\to\infty} \frac{1}{N} \mathrm{Tr}(A_{j_1,N} \cdots A_{j_m,N})$$

exists and it is finite, where Tr denotes the non-normalized trace.

The following result is proved in [14] and, in a more general framework, in [2], [11]:

Theorem 1.3. *Let m be a positive integer; for $1 \leq k \leq m$ and $1 \leq i,j \leq N$ consider the random variables $u_{i,j}^{(k)} : \mathcal{U}(N) \longrightarrow \mathbb{C}$ such that $u_{i,j}^{(k)} \equiv u_{i,j}$ and the families $\{u_{i,j}^{(k)}\}_{i,j=1}^N$ are independent. Finally, for each k and N, consider the matrix $U_{k,N} \in L^\infty(\mathcal{U}(N), d\mathcal{U}_N)^{N\times N}$, having the entries $u_{i,j}^{(k)}$.*

Suppose that $\mathfrak{A} = \{A_{j,N}\}_{j\in J, N\geq 1}$ is an ensemble of complex matrices that has limit distribution. Then the ensembles of random matrices $\{U_{1,N}, U_{1,N}^*\}$, $\{U_{2,N}, U_{2,N}^*\}$, $\ldots$, $\{U_{m,N}, U_{m,N}^*\}$ and $\mathfrak{A}$ are asymptotically free with respect to the functional $\int_{\mathcal{U}(N)} \frac{1}{N} Tr(\cdot) d\mathcal{U}_N$.

Throughout the paper, $\mathcal{F}_m$ will denote the free semigroup with m generators $g_1, \ldots, g_m$. The elements of $\mathcal{F}_m$ are arbitrary reduced words $w = g_{i_l} g_{i_{l-1}} \cdots g_{i_1}$, the semigroup operation is concatenation, the neutral element is the empty word $\emptyset$ and $|w| = l$ will denote the length of the reduced word w. We will also use the notation $\mathcal{F}_m^{[l]}$ for the set of all reduced words from $\mathcal{F}_m$ of length l.

In the next section we will utilize the following consequences of the Theorems 1.1 and 1.3 from above:

Corollary 1.4. *With the notations of Theorem 1.3, let $v, w \in \mathcal{F}_m$, and, if $w = g_{w_t} g_{w_{t-1}} \ldots g_{w_1}$ denote $U_N^w = U_{w_t,N} U_{w_{t-1},N} \cdots U_{w_1,N}$. Then:*

(i) *if $|v| \neq |w|$, we have that, for any positive integer N,*

$$\int_{\mathcal{U}(N)} Tr\left((U_N^w)^* U_N^v\right) d\mathcal{U}_N = 0$$

(ii) $\displaystyle \lim_{N\to\infty} \int_{\mathcal{U}(N)} \frac{1}{N} Tr\left((U_N^w)^* U_N^v\right) d\mathcal{U}_N = \delta_{w,v}.$

Proof. For (i), let

$$L = \{l = (l_{-|v|}, l_{-|v|+1}, \ldots, l_0, l_1, \ldots, l_{|w|}) \in \{1, \ldots, N\}^{|v|+|w|+1} : l_{-|v|} = l_{|w|}\}$$

and $V_k = \{j \in \{1, \dots, |v|\} : v_j = k\}$, respectively $W_k = \{j \in \{1, \dots, |w|\} : w_j = k\}$. Then, from the independence of the families $\{u_{i,j}^{(k)}\}_{i,j=1}^N$,

$$\int_{\mathcal{U}(N)} \mathrm{Tr}\left((U_N^w)^* U_N^v\right) d\mathcal{U}_N = \sum_{l \in L} \int_{\mathcal{U}(N)} \prod_{k=1}^{|w|} \overline{u_{l_{k-1},l_k}^{(w_k)}} \cdot \prod_{k=1}^{|v|} u_{l_{-k+1},l_{-k}}^{(v_k)} d\mathcal{U}_N$$

$$= \sum_{l \in L} \prod_{r=1}^{p} (\int_{\mathcal{U}(N)} \prod_{k \in W_r} \overline{u_{l_{k-1},l_k}^{(w_k)}} \cdot \prod_{k \in V_r} u_{l_{-k+1},l_{-k}}^{(v_k)} d\mathcal{U}_N.$$

From Theorem 1.1, if $\mathrm{card}(W_r) \neq \mathrm{card}(V_r)$, then the corespondent factor in the above product cancels, hence the conclusion.

For (ii), note first if $w = v$, then $(U_N^w)^* U_N^v = \mathrm{Id}_N$, and the assertion is trivial. If $w \neq v$, it suffices to prove the the equality for $|w| = |v|$ (according to part (i)) and $v_s \neq w_s$ (since $U_{k,N}^* U_{k,N} = \mathrm{Id}_N$).

From Remark 1.2, $\displaystyle\int_{\mathcal{U}(N)} \frac{1}{N} \mathrm{Tr}\ (U_{k,N}) d\mathcal{U}_N = 0$, for all $N > 1$ and all $1 \leq k \leq m$. The conclusion follows now from Theorem 1.3 and the definition of free independence. $\square$

Corollary 1.5. *Fix m a positive integer and suppose that $U = [u_{i,j}]_{i,j=1}^{mN}$, with the functions $u_{i,j} : \mathcal{U}(mN) \longrightarrow \mathbb{C}$ as defined above. For $1 \leq k \leq m$, consider $U_k \in L^\infty(\mathcal{U}(mN), d\mathcal{U}_{mN})^{N \times N}$ given by $U_k = [u_{(k-1)N+i,j}]_{i,j=1}^N$. (I. e., $U_1, \dots, U_m$ are the $N \times N$ matricial block entries of the first $N \times mN$ matricial row of U).*

Let $v, w \in \mathcal{F}_m$, and, if $w = g_{w_t} g_{w_{t-1}} \dots, g_{w_1}$ denote $U^w = U_{w_t} U_{w_{t-1}} \cdots U_{w_1}$. Then:

(i) *If $|v| \neq |w|$, $\displaystyle\int_{\mathcal{U}(mN)} Tr\left((U^w)^* U^v\right) d\mathcal{U}_{mN} = 0.$*

(ii) *If $|v| = |w|$, $\displaystyle\lim_{N \to \infty} \int_{\mathcal{U}(mN)} \frac{1}{N} Tr\left((U^w)^* U^v\right) d\mathcal{U}_{mN} = \delta_{w,v} \frac{1}{m^{|v|}}.$*

Proof. Part (i) is an immediate consequence of the last equality of Theorem 1.1, since the entries of $U_1, \dots, U_m$ are also entries of U.

For part (ii), let $e_{i,j}$ be the $m \times m$ matrix with the i, j-entry 1 and all others 0, let Id_N be the identity $N \times N$ matrix and $E_{i,j} = \mathrm{Id}_N \otimes e_{i,j} \in \mathbb{C}^{mN \times mN}$. Then for all $1 \leq k \leq m$, we have that $\widetilde{U_k} = U_k \otimes e_{1,1} = E_{1,1} U E_{k,1}$, hence

$$\text{(1)} \quad \mathrm{Tr}\left((U^w)^* U^v\right) = \mathrm{Tr}\left(\widetilde{U_{w_1}^*} \dots \widetilde{U_{w_s}^*} \widetilde{U_{v_s}} \dots \widetilde{U_{v_1}}\right)$$
$$= \mathrm{Tr}\left(E_{1,w_1} U^* E_{1,w_2} U^* \cdots E_{1,w_t} U^* E_{1,1} U E_{v_s,1} U \cdots E_{v_2,1} U E_{v_1,1}\right)$$

To simplify the notations, we shall use the writting

$$E_{i,j}^0 = E_{i,j} - \delta_{i,j} \frac{1}{m} \mathrm{Id}_{mN}. \tag{2}$$

Note that $\mathrm{Tr}(E_{i,j}^0 = 0)$; moreover, for all non-zero integers $\alpha_0, \dots, \alpha_n$ and all indices $i, j, k, l, k_r, l_r \in \{1, \dots, m\}$ we have that

$$\lim_{N \to \infty} \frac{1}{mN} \int_{\mathcal{U}(mN)} \mathrm{Tr}\left(E_{i,j} U^{\alpha_0} E_{k_1,l_1}^0 U_1^\alpha E_{k_2,l_2}^0 \cdots E_{k_n,l_n}^0 U^{\alpha_n} E_{k,l}\right) d\mathcal{U}_{mN} = 0. \tag{3}$$

To see that, we remark that using (2) for $E_{i,j}$ and $E_{k,l}$, the integrand can be written as a linear combination of alternating products of centered (according to Remark

1.2) elements from the algebra generated by U and $U^{-1} = U^*$, respectively from the algebra generated by $\{E_{1,k}\}_{k=1}^m$. According to Theorem 1.3, the two algebras are asymptotically free, hence (2) is proved.

An immediate consequence of (3) is that

$$\lim_{N\to\infty} \frac{1}{mN} \int_{\mathcal{U}(mN)} \mathrm{Tr}\left(E_{1,w_1}U^* \cdots E_{1,w_t}U^* E_{1,1}^0 U E_{v_s,1} U \cdots E_{v_1,1}\right) d\mathcal{U}_{mN} = 0, \tag{4}$$

because, using (2) for $E_{1,w_2}, \ldots, E_{1,w_t}$ and $E_{v_2,1}, \ldots, E_{v_s,1}$, the integrand from (4) is a finite linear combination of integrands from (3).

From part (i), it suffices to prove part (ii) of the Corollary only for $t = s$. For this we will use induction on s. If $s = 1$,

$$\begin{aligned} E_{1,w_1}U^* E_{1,1} U E_{v,1} = & E_{1,w_1} U^* E_{1,1}^0 U E_{v,1} + \frac{1}{m} E_{1,w_1} \cdot E_{v,1} \\ = & E_{1,w_1} U^* E_{1,1}^0 U E_{v,1} + \frac{1}{m} \delta_{v_1,w_1} E_{1,1} \end{aligned}$$

and the conclusion follows from (4) and $\mathrm{Tr}(E_{1,1}) = N$.

For the inductive step, using (2) for $E_{1,1}$ in (1), we obtain

$$\begin{aligned} (U^w)^* U^v = & E_{1,w_1} U^* E_{1,w_2} U^* \cdots E_{1,w_s} U^* E_{1,1}^0 U E_{v_s,1} U \cdots E_{v_2,1} U E_{v_1,1} \\ & + \frac{1}{m} E_{1,w_1} U^* E_{1,w_2} U^* \cdots U^*(E_{1,w_s} \cdot E_{v_s,1}) U \cdots E_{v_2,1} U E_{v_1,1} \end{aligned}$$

The first term of the above is similar to the integrand in (4). For the second term, note that

$$E_{1,w_s} \cdot E_{v_s,1} = \delta_{w_s,v_s} E_{1,1}$$

and the conclusion follows from the induction hypothesis. □

1.2. **Non-Commutative Functions and Taylor-Taylor Expansion.** The following definition for non-commutative functions is similar to [5], [6] and [9].

For $\mathcal{V}$ a (complex) linear space, we will denote by $\mathcal{V}_{\mathrm{nc}}$ the linear space $\coprod_{n=1}^\infty \mathcal{V}^{n\times n}$. A subset Ω of $\mathcal{V}_{\mathrm{nc}}$ is said to be a *non-commutative set* if for all m, n and all $X \in \Omega \cap \mathcal{V}^{m\times m}$ and $Y \in \Omega^{n\times n}$ we have that $X \oplus Y \in \Omega$, where $X \oplus Y$ is the block diagonal matrix from $\mathcal{V}^{(m+n)\times(m+n)}$ with X and Y the block entries of the main diagonal and all other entries zero.

If $\mathcal{V}$ and $\mathcal{W}$ are two linear spaces and Ω a non-commutative set of $\mathcal{V}_{\mathrm{nc}}$, a mapping $f : \Omega \longrightarrow \mathcal{W}_{\mathrm{nc}}$ is said to a *non-commutative function* if it satisfy the following conditions:

- $f(\Omega \cup \mathcal{V}^{n\times n}) \subset \mathcal{W}^{n\times n}$ for all positive integers n;
- $f(X \oplus Y) = f(X) \oplus f(Y)$ for all $X, Y \in \Omega$;
- if $X \in \Omega \cup \mathcal{V}^{n\times n}$ and $T \in \mathbb{C}^{n\times n}$ such that $TXT^{-1} \in \Omega$, then
$$f(TXT^{-1}) = Tf(X)T^{-1}.$$

Non-commutative functions have strong regularity properties - for an introduction to the basic theory see see [5] and [6]. Below we will mention only a particular form of the Taylor-Taylor Expansion property, as shown in Section 7 of [5], that will be extensively utilized in Section 3 of the present work.

Let $\mathcal{V}$ be a finite dimensional vector space with basis $e_1, \ldots, e_d$. For $X \in \mathcal{V}^{N\times N}$, there exist some unique $X_1, \ldots, X_d \in \mathbb{C}^{N\times N}$ such that $X = X_1 e_1 + \ldots + X_d e_d$. If $w = g_{i_1} \cdots g_{i_t} \in \mathcal{F}_d$, we write $X^w = X^{g_{i_1}} \cdots X^{g_{i_t}}$.

Suppose that $\Omega \subseteq \mathcal{V}_{\rm nc}$ is a non-commutative set such that for all N, the set $\Omega_N = \Omega \cap \mathcal{V}^{N\times N}$ is open. Let $b \in \Omega_1$, let $\mathcal{W}$ be a Banach space and suppose that $f : \Omega \longrightarrow \mathcal{W}_{\rm nc}$ is a non-commutative function locally bounded on slices separately in every matrix dimension around b, that is for all positive integers N, and all $Y \in \mathcal{V}^{N\times N}$ there exists $\varepsilon > 0$ such that the function $t \mapsto f(X+tY)$ is bounded for $|t| < \varepsilon$.

For n a positive integer also define the set

$$\Upsilon(b,n) = \{X \in \Omega_N :\ bI_n + t(X - bI_n) \in \Omega_n \text{ for all } t \in \mathbb{C} \text{ such that } |t| \leq 1\}.$$

With the notations from above, Theorem 7.2 from [5] gives that for all positive integers n and all $X \in \Upsilon(b,n)$

$$f(X) = \sum_{l=0}^{\infty}[\sum_{|w|=l}(X - bI_n)^w \otimes f_w] \tag{5}$$

series converges absolutely and uniformely (in fact, normally) on compacta of $\Upsilon(b,n)$.

1.3. **Operator space structures on $\mathbb{C}^m$.** An operator space structure on a linear space $\mathcal{V}$ is given (see [3], Proposition 2.3.6) by a family of norms $\{\|\cdot\|_n\}_{n>0}$, such that each $\|\cdot\|_n$ is a norm on $\mathcal{V}^{n\times n}$ and, for all $X \in \mathcal{V}^{n\times n}$, $Y \in \mathcal{V}^{m\times m}$, $T, S \in \mathbb{C}^{n\times n}$, we have that

- $\|X \oplus Y\|_{n+m} = \max\{\|X\|_n, \|Y\|_m\}$
- $\|TXS\| \leq \|T\|\|X\|_n\|S\|$, where $\|\cdot\|$ denotes the usual operator norm of complex matrices.

We will consider the operator spaces structures on $\mathbb{C}^m$ given by the $\|\cdot\|_\infty$, $\|\cdot\|_{\rm col}$, and $\|\cdot\|_{\rm row}$, where, for $X = (X_1, \ldots, X_m) \in (\mathbb{C}^{n\times n})^m \simeq (\mathbb{C}^m)^{n\times n}$ and $\|\cdot\|$ the usual operator norm in $\mathbb{C}^{n\times n}$

$$\|X\|_\infty = \max\{\|X_1\|, \ldots, \|X_m\|\}$$

$$\|X\|_{\rm col} = \|\sum_{i=1}^m X_i^* X_i\|^{\frac{1}{2}}$$

$$\|X\|_{\rm row} = \|\sum_{i=1}^m X_i X_i^*\|^{\frac{1}{2}}.$$

For the norm $\|\cdot\|_\infty$, the non-commutative unit ball is the non-commutative polydisc

$$(\mathbb{D}^m)_{\rm nc} = \coprod_{n=1}^{\infty}\{(X_1, \ldots, X_m) \in (\mathbb{C}^{n\times n})^m : \|X_j\| < 1,\ j = 1, \ldots, m\}.$$

For the norms $\|\cdot\|_{\rm col}$, respectively $\|\cdot\|_{\rm row}$, the non-commutative unit balls are given by

$$(\mathbb{B}^m)_{\rm nc} = \coprod_{n=1}^{\infty}\{(X_1, \ldots, X_m) \in (\mathbb{C}^{n\times n})^m : \sum_{i=1}^m X_i^* X_i < I_n\},$$

respectively by

$$(\mathbb{B}^m_{\rm row})_{\rm nc} = \coprod_{n=1}^{\infty}\{(X_1, \ldots, X_m) \in (\mathbb{C}^{n\times n})^m : \sum_{i=1}^m X_i X_i^* < I_n\}.$$

Identifying the components from $(\mathbb{C}^{n\times n})^m$ of $(\mathbb{D}^m)_{\rm nc}$, $(\mathbb{B}^m)_{\rm nc}$, respectively $(\mathbb{B}^m_{\rm row})_{\rm nc}$ with the correspondent sets from $\mathbb{C}^{mn^2}$, the Shilov boundaries for the commutative algebras of complex analytic functions in mn^2 variables, as shown in [12], Example 1.5.51, are $\mathcal{U}(n)^m$, respectively the set of all isometries and coisometries of $\mathbb{C}^{n\times mn}$, respectively $\mathbb{C}^{mn\times n}$. Since $\mathbb{C}^{n\times mn}$ does not have any coisometries, it follows that for the case of $(\mathbb{B}^m)_{\rm nc}$ the Shilov boundary from above is

$$\partial(\mathbb{B}^m, n) = \{(X_1,\dots,X_m) \in (\mathbb{C}^{n\times n})^m : \sum_{i=1}^m X_i^* X_i = I_n\}.$$

Also, since $\mathbb{C}^{mn\times n}$ does not have any isometries, the correspodent Shilov boundary for $(\mathbb{B}^m_{\rm row})_{\rm nc} \cap (\mathbb{C}^{n\times n})$ is

$$\partial(\mathbb{B}^m_{\rm row}, n) = \{(X_1,\dots,X_m) \in (\mathbb{C}^{n\times n})^m : \sum_{i=1}^m X_i X_i^* = I_n\}.$$

To simplify the writing in the next section, we will denote denote $\mathcal{U}(n)^m$ by $\partial(\mathbb{D}^m, n)$. The natural measure on $\partial(\mathbb{D}^m, n)$ is the m-fold product measure μ_n of the Haar measure on $\mathcal{U}(n)$. Corollary 1.4 gives then, that for any $v, w \in \mathcal{F}_m$, we have:

$$\int_{\partial(\mathbb{D}^m,n)} \mathrm{Tr}\left((X^w)^* X^v\right) d\mu_n = 0 \text{ if } |v| \neq |w| \tag{6}$$

$$\lim_{n\to\infty} \int_{\partial(\mathbb{D}^m,n)} \frac{1}{n}\mathrm{Tr}\left((X^w)^* X^v\right) d\mu_n = \delta_{v,w}. \tag{7}$$

For the case of $(\mathbb{B}^m)_{\rm nc}$, note that

$$\begin{aligned}\partial(\mathbb{B}^m, n) = \{(X_1,\dots,X_m) \in (\mathbb{C}^{n\times n})^m &: \text{there exists some } U \in \mathcal{U}(mn) \\ &\text{such that } (I_n, 0, \dots, 0)\cdot U = (X_1,\dots,X_m)\}.\end{aligned}$$

Note that the group $\mathcal{U}(mn)$ acts transitively on $\partial(\mathbb{B}^m, n)$ by right multiplication. Also, denoting

$$H(m,n) = \{I_n \oplus U : U \in \mathcal{U}((m-1)n)\}$$

we have that $H(m,n)$ is a compact subgroup of $\mathcal{U}(mn)$ which is also the stabilizer of $(I_n, 0, \dots, 0) \in \partial(\mathbb{B}^m, n)$. Hence $\partial(\mathbb{B}^m, n)$ is isomorphic to $\mathcal{U}(mn)/H(m,n)$ and (see [4], Theorem 2.49) there exists a unique Radon measure ν_n of mass 1 on $\partial(\mathbb{B}^m, n)$ invariant at the action of $\mathcal{U}(mn)$ and, for any continuous function $f : \mathcal{U}(mn) \longrightarrow \mathbb{C}$ we have that

$$\int_{\mathcal{U}(mn)} f(U) d\mathcal{U}_{mn}(U) = \int_{\partial(\mathbb{B}^m,n)} \int_{H(m,n)} f(UV) d\mathcal{U}_{(m-1)n}(V) d\nu_n(UH(m,n)). \tag{8}$$

For $1 \leq i \leq n$ and $1 \leq j \leq mn$ and $u_{i,j} : \mathcal{U}(mn) \to \mathbb{C}$ as defined in Section 1.1, a simple verification gives that, for all $U \in \mathcal{U}(mn)$ and $V \in H(m,n)$

$$u_{i,j}(U) = u_{i,j}(U\cdot V). \tag{9}$$

Fix now $f \in \mathrm{Alg}\{u_{i,j}, \overline{u_{i,j}} : 1 \leq i \leq n, 1 \leq j \leq mn\}$. For all $U \in \mathcal{U}(mn)$, equation (9) implies

$$\int_{H(m,n)} f(UV) d\mathcal{U}_{(m-1)n}(V) = \int_{H(m,n)} f(U) d\mathcal{U}_{(m-1)n}(V) = f(U). \tag{10}$$

Define $\widehat{f}$ on $\partial(\mathbb{B}^m, n)$ via $\widehat{f}(UH(m,n)) = f(U)$. From (9), $\widehat{f}$ is well-defined. Moreover, equation (10) gives

$$\int_{\partial(\mathbb{B}^m,n)} \widehat{f}(UH(m,n))d\nu_n(UH(m,n)) = \int_{\mathcal{U}(mn)} f(U)d\mathcal{U}_{mn}(U). \tag{11}$$

Hence, Corollary 1.5 implies that, for all $v, w \in \mathcal{F}_m$, we have:

$$\int_{\partial(\mathbb{B}^m,n)} \mathrm{Tr}\left((X^w)^* X^v\right) d\nu_n(X) = 0 \text{ if } |v| \neq |w| \tag{12}$$

$$\lim_{n\to\infty} \int_{\partial(\mathbb{B}^m,n)} \frac{1}{n}\mathrm{Tr}\left((X^w)^* X^v\right) d\nu_n(X) = \delta_{v,w}\frac{1}{m^{|v|}}. \tag{13}$$

Denoting by A^T the transpose of the matrix A, we have that

$$\begin{aligned}\partial(\mathbb{B}^m_{\text{row}}, n) = \{(X_1, \dots, X_m) \in (\mathbb{C}^{n\times n})^m &: \text{there exists some } U \in \mathcal{U}(mn) \\ &\text{such that } U \cdot (I_n, 0, \dots, 0)^T = (X_1, \dots, X_m)^T\}.\end{aligned}$$

In this case the transitive action of $\mathcal{U}(mn)$ on $\partial(\mathbb{B}^m_{\text{row}}, n)$ is by left multiplication and $H(m,n)$ is the stabilizer of $(I_n, 0, \dots, 0)^T$. A similar argument as above gives that there exists ν'_n, a unique Radon measure of mass 1 on $\partial(\mathbb{B}^m_{\text{row}}, n)$, invariant at the action of $\mathcal{U}(mn)$, and the pair $(\partial(\mathbb{B}^m_{\text{row}}, n), \nu'_n)$ also satisfies equalities (12) and (13).

2. Main results

The present section will address properties of certain H^2 Hardy spaces associated to the non-commutative unit balls for the operator norms $\|\cdot\|_\infty$ and $\|\cdot\|_{\text{col}}$ on $\mathbb{C}^m$. Since both $(\partial(\mathbb{B}^m, n), \mu_n)$ and $(\partial(\mathbb{B}^m_{\text{row}}, n), \nu'_n)$ satisfy (12) and (13), similar results to the case of $\|\cdot\|_{\text{col}}$ can be stated for the setting of $\|\cdot\|_{\text{row}}$.

For Ω either $\mathbb{B}^m$ or $\mathbb{D}^m$, consider the algebras

$$\begin{aligned}\mathcal{A}_\Omega = \{f : \Omega_{\text{nc}} \longrightarrow \mathbb{C}_{\text{nc}} \ : \ f = \ &\text{non-commutative function, locally bounded} \\ &\text{on slices separately in every matrix dimension}\}.\end{aligned}$$

Equation (5) gives that for all $f \in \mathcal{A}_\Omega$ there exists a family of complex numbers $\{f_w\}_{w\in\mathcal{F}_m}$ with $f_\emptyset = f(0)$, such that, for all $X \in \Omega$

$$f(X) = \sum_{l=0}^{\infty}\Big(\sum_{w\in\mathcal{F}_m^{[l]}} X^w f_w\Big). \tag{14}$$

Theorem 2.1. (i) *If $f \in \mathcal{A}_{\mathbb{D}^m}$ and $r \in (0,1)$, then:*

$$\begin{aligned}f_w &= \lim_{N\longrightarrow\infty} \frac{1}{r^{|w|}} \int_{\partial(\mathbb{D}^m,N)} \frac{1}{N} Tr\left((X^w)^* f(rX)\right) d\mu_N \\ &= \lim_{r\longrightarrow 1^-} \lim_{N\longrightarrow\infty} \int_{\partial(\mathbb{D}^m,N)} \frac{1}{N} Tr\left((X^w)^* f(rX)\right) d\mu_N.\end{aligned}$$

(ii) *If $f \in \mathcal{A}_{\mathbb{B}^m}$ and $r \in (0,1)$, then*

$$\begin{aligned}f_w &= \lim_{N\longrightarrow\infty} \frac{1}{r^{|w|}} \cdot m^{|w|} \int_{\partial(\mathbb{B}^m,N)} \frac{1}{N} Tr\left((X^w)^* f(rX)\right) d\mu_N \\ &= \lim_{r\longrightarrow 1^-} \lim_{N\longrightarrow\infty} m^{|w|} \int_{\partial(\mathbb{B}^m,N)} \frac{1}{N} Tr\left((X^w)^* f(rX)\right) d\mu_N.\end{aligned}$$

Proof. For any positive integer N, relations (6) and (14) give

$$\begin{aligned}\int_{\partial(\mathbb{D}^m,N)} \mathrm{Tr}\left((X^w)^* f(rX)\right) d\mu_N &= \sum_{l=0}^{\infty}\Big[\sum_{v\in\mathcal{F}_m^{[l]}} \int_{\partial(\mathbb{D}^m,N)} \mathrm{Tr}\left((X^w)^* X^v\right)\cdot r^{|v|} f_v d\mu_N\Big]\\ &= \sum_{\substack{v,w\in\mathcal{F}_m\\ |v|=|w|}} r^{|v|} f_v \cdot \int_{\partial(\mathbb{D}^m,N)} \mathrm{Tr}\left((X^w)^* X^v\right) d\mu_N,\end{aligned}$$

and the equalities from (i) follow from equation (7).

The argument for (ii) is similar, using equations (13), (14) and (12). □

Definition 2.2. *For $(\Omega, d\omega_N)$ either $(\mathbb{B}^m, d\nu_N)$ or $(\mathbb{D}^m, d\mu_N)$, we define*

$$H^2(\Omega_{nc}) = \{f \in \mathcal{A}_\Omega : S(f) = \sup_N \sup_{r<1} \int_{\partial(\Omega,N)} \frac{1}{N} Tr(f(rX)^* f(rX))\, d\omega_N < \infty\}.$$

□

For $f \in \mathcal{A}_\Omega$ as above and $X \in (\mathbb{C}^m)_{\mathrm{nc}}$, denote $f^{[l]}(X) = \sum_{w\in\mathcal{F}_m^{[l]}} f^w X^w$. Remark that:

(i) $f^{[l]}(rX) = r^l \cdot f^{[l]}(X)$

(ii) for $X \in \Omega$, as disscussed is Section 1.2, we have that $f(X) = \sum_{l=0}^{\infty} f^{[l]}(X)$ and the sum is absolutely convergent

(iii) if $l \neq p$, then $\int_{\partial(\Omega,N)} \mathrm{Tr}\left((f^{[l]}(X))^* f^{[p]}(X)\right) d\omega_N = 0$.

Therefore, if $f \in \mathcal{H}^2(\Omega_{\mathrm{nc}})$, with notations

$$\begin{aligned} S_{N,r}(f) &= \int_{\partial(\Omega,N)} \frac{1}{N}\mathrm{Tr}\left(f(rX)^* f(rX)\right) d\omega_N\\ S_N(f) &= \lim_{r\to 1^-} S_{N,r}(f)\end{aligned}$$

we have that $S_{N,r}(f) = \sum_{l=0}^{\infty} S_{N,r}(f^{[l]}) = \sum_{l=0}^{\infty} r^{2l} S_N(f^{[l]})$, and both sums are absolutely convergent; in fact $S_N(f^{[l]}) \geq 0$ for all l.

Theorem 2.3. (i) *If $f \in H^2((\mathbb{D}^m)_{nc})$, then $\sum_{l=0}^{\infty}(\sum_{w\in\mathcal{F}_m^{[l]}} |f_w|^2) < \infty$.*

(ii) *If $f \in H^2((\mathbb{B}^m)_{nc})$, then $\sum_{l=0}^{\infty}(\frac{1}{m^l}\sum_{w\in\mathcal{F}_m^{[l]}} |f_w|^2) < \infty$.*

Proof. For (i), note first that if $r \in (0,1)$, $f \in H^2((\mathbb{D}^m)_{\mathrm{nc}})$ and $X \in \partial(\mathbb{D}^m, N)$, then $f(rX) = \sum_{l=0}^{\infty} r^l f^{[l]}(X)$ and the series is absolutely convergent.

Therefore equation (6) implies that

$$\int_{\partial(\mathbb{D}^m,N)} \frac{1}{N}\mathrm{Tr}(f(rX)^*f(rX))d\mu_N = \sum_{l=0}^{\infty} r^{2l}\int_{\partial(\mathbb{D}^m,N)} \frac{1}{N}\mathrm{Tr}(f^{[l]}(X)^*f^{[l]}(X))d\mu_N$$
$$= \sum_{l=0}^{\infty} r^{2l}[a_l + b_{l,N}]$$

where $a_l = \sum_{|w|=l} |f_w|^2$ and

$$b_{l,N} = \sum_{\substack{|v|=|w|=l\\ v\neq w}} \overline{f_w}f_v \cdot \int_{\partial(\mathbb{D}^m,N)} \frac{1}{N}\mathrm{Tr}((X^w)^*X^v)d\mu_N.$$

$f^{[l]}(X)^*f^{[l]}(X)$ is positive in C*-algebra in $\mathbb{C}^{N\times N}$, henceforth $a_l + b_{l,N} \geq 0$. Relation (7) implies that $\lim_{N\to\infty} b_{l,N} = 0$ for all l, hence, since $f \in H^2(\mathbb{D}_{\mathrm{nc}})$, we have that

$$\sup_N \sup_{r<1} \sum_{l=0}^{\infty} r^{2l}\cdot(a_l + b_{l,N}) < \infty.$$

It follows that $\sup_{r<1} \sum_{l=0}^{\infty} r^{2l}a_l < \infty$, and, since $a_l \geq 0$, we have that $\sum_{l=0}^{\infty} a_l < \infty$.

The argument for part (ii) is analogous, utilizing equations (12) and (13). □

Theorem 2.4.
(i) *With the notations from Definition 2.2, $H^2(\Omega_{nc})$ are inner-product spaces, with the inner product given by*

$$\langle f, g\rangle = \lim_{N\longrightarrow\infty} \lim_{r\longrightarrow 1^-} \int_{\partial(\Omega,N)} \frac{1}{N} Tr\left(g(rX)^*f(rX)\right) d\omega_N.$$

(ii) $\{X^w\}_{w\in\mathcal{F}_m}$ *is a complete orthonormal system in $H^2((\mathbb{D}^m)_{nc})$ and, for all $f \in \mathcal{A}_{\mathbb{D}^m}$, we have that $f_w = \langle f, X^w\rangle$ and $f = \sum_{w\in\mathcal{F}_m} f_wX^w$ in $H^2((\mathbb{D}^m)_{nc})$.*

(iii) $\{m^{\frac{|w|}{2}}X^w\}_{w\in\mathcal{F}_m}$ *is a complete orthonormal system in $H^2((\mathbb{B}^m)_{nc})$ and, for all $f \in \mathcal{A}_{\mathbb{B}^m}$, we have that $f_w = \langle f, m^{|w|}X^w\rangle$ and $f = \sum_{w\in\mathcal{F}_m} f_wX^w$ in $H^2((\mathbb{B}^m)_{nc})$.*

Proof. From equations (6) and (12),

$$\int_{\partial(\Omega,N)} \frac{1}{N}\mathrm{Tr}\left(g(rX)^*f(rX)\right)d\omega_N = \sum_{l=0}^{\infty} r^{2l}\int_{\partial(\Omega,N)} \frac{1}{N}\mathrm{Tr}\left(g^{[l]}(X)^*f^{[l]}(X)\right)d\omega_N.$$

For $A, B \in \mathbb{C}^{N\times N}$, $|\mathrm{Tr}(A^*B)| \leq \frac{1}{2}[\mathrm{Tr}(A^*A) + \mathrm{Tr}(B^*B)]$, hence

$$\left|\int_{\partial(\Omega,N)} \frac{1}{N}\mathrm{Tr}\left(g^{[l]}(X)^*f^{[l]}(X)\right)d\omega_N\right| \leq \frac{1}{2}[S_N(f^{[l]} + g^{[l]})].$$

From $f, g \in H^2(\Omega_{\mathrm{nc}})$, the series $\sum_{l=0}^{\infty} S_N(f^{[l]}+g^{[l]})$ is convergent (in fact absolutely convergent, since all terms are positive), hence the limit after $r \to 1^-$ from part (i) does exist and equals

$$\sum_{l=0}^{\infty}\int_{\partial(\Omega,N)}\frac{1}{N}\mathrm{Tr}\left(g^{[l]}(X)^*f^{[l]}(X)\right)d\omega_N.$$

From equations (7) and (13),

$$\lim_{N\to\infty}\int_{\partial(\Omega,N)}\frac{1}{N}\mathrm{Tr}\left(g^{[l]}(X)^*f^{[l]}(X)\right)d\omega_N=\begin{cases}\displaystyle\sum_{w\in\mathcal{F}_m^{[l]}}\overline{g^w}f^w & \text{if } f,g\in H^2((\mathbb{D}^m)_{\mathrm{nc}})\\ \displaystyle\sum_{w\in\mathcal{F}_m^{[l]}}\frac{1}{m^l}\overline{g^w}f^w & \text{if } f,g\in H^2((\mathbb{B}^m)_{\mathrm{nc}})\end{cases}$$

and the last sums are absolutely convergent from Theorem 2.3. Therefore

$$\langle f,g\rangle=\begin{cases}\displaystyle\sum_{w\in\mathcal{F}_m}\overline{g^w}f^w & \text{if } f,g\in H^2((\mathbb{D}^m)_{\mathrm{nc}})\\ \displaystyle\sum_{w\in\mathcal{F}_m}\frac{1}{m^l}\overline{g^w}f^w & \text{if } f,g\in H^2((\mathbb{B}^m)_{\mathrm{nc}})\end{cases}\tag{15}$$

particularly if $\langle f,f\rangle=0$, then $f_w=0$ for all $w\in\mathcal{F}_m$, hence $f=0$.

The parts (ii) and (iii) are immediate consequences of (15) and equations (7) and (13). □

Remark 2.5. *The limit over N in Theorem 2.4(i) is not the supremum. For example, if $m=2$ and $f(X)=X_1X_2+X_2X_1$, then*

$$\int_{\partial(\mathbb{D}^m,N)}\frac{1}{N}Tr(f(rX)^*f(rX))\,d\mu_N=2r^2(1+\frac{1}{N^2})$$

Definition 2.6. *As before, Ω will denote either $\mathbb{D}^m$ or $\mathbb{B}^m$.*

For $X\in\Omega_{nc}$, define the map $E_\Omega^X: H^2(\Omega_{nc})\longrightarrow\mathbb{C}^{N\times N}$ via $E_\Omega^X(f)=f(X)$.

Let $\mathcal{B}_{\Omega,N}=\{X\in\Omega_{nc}\cap\mathbb{C}^{N\times N}:\ E_\Omega^X$ is a bounded map$\}$ and $\mathcal{B}_\Omega=\coprod_{N=1}^{\infty}\mathcal{B}_{\Omega,N}$.

For $p>0$, we define the Hilbert space

$$l_p^2(\mathcal{F}_m)=\{\{f_w\}_{w\in\mathcal{F}_m}:\ f_w\in\mathbb{C},\|\{f_w\}_{w\in\mathcal{F}_m}\|_{2,p}=\sum_{l=0}^{\infty}(\sum_{w\in\mathcal{F}_m^{[l]}}\frac{1}{p^l}\overline{f_w}f_w)<\infty\}.$$

Proposition 2.7. *With the above notations, we have that*

(i) $\mathcal{B}_{\mathbb{D}^m}=\{X\in(\mathbb{D}^m)_{nc}:$ *the series* $\sum_{l=0}^{\infty}(\sum_{w\in\mathcal{F}_m^{[l]}}f_wX^w)$ *converges for any sequence* $\{f_w\}_{w\in\mathcal{F}_m}\in l^2(\mathcal{F}_m)\}$

(ii) $\mathcal{B}_{\mathbb{B}^m}=\{X\in(\mathbb{B}^m)_{nc}:$ *the series* $\sum_{l=0}^{\infty}(\sum_{w\in\mathcal{F}_m^{[l]}}f_wX^w)$ *converges for any sequence* $\{f_w\}_{w\in\mathcal{F}_m}\in l_m^2(\mathcal{F}_m)\}$

Proof. Suppose that $X \in \mathbb{D}^m \cap \mathbb{C}^{N\times N}$ is such that $\sum_{l=0}^{\infty}(\sum_{w\in\mathcal{F}_m^{[l]}} f_w X^w)$ converges for all $\{f_w\} \in l^2(\mathcal{F}_m)$ and consider the linear map $\widetilde{E_{\mathbb{D}^m}^X} : l^2(\mathcal{F}_m) \longrightarrow \mathbb{C}^{N\times N}$, given by

$$\widetilde{E_{\mathbb{D}^m}^X}(\{f_w\}_{w\in\mathcal{F}_m}) = \sum_{l=0}^{\infty}(\sum_{w\in\mathcal{F}_m^{[l]}} f_w X^w).$$

For every l, define also

$$E_{\mathbb{D}^m}^{X,l}(\{f_w\}) = \sum_{s=0}^{l}(\sum_{w\in\mathcal{F}_m^{[l]}} f_w X^w).$$

From the initial assertion, $\widetilde{E_{\mathbb{D}^m}^X}$ is the pointwise limit of $\{E_{\mathbb{D}^m}^{X,l}\}_{l>0}$. Each $E_{\mathbb{D}^m}^{X,l}$ is a bounded linear operator from $l^2(\mathcal{F}_m)$ to $\mathbb{C}^{N\times N}$, so Banach-Steinhaus Theorem gives that $\widetilde{E_{\mathbb{D}^m}^X}$ is bounded.

Take now $f \in H^2((\mathbb{D}^m)_{\rm nc})$. From Theorem 2.3, the sequence $\{f_w\}_{w\in\mathcal{F}_m}$ of its Taylor-Taylor coefficients is in $l^2(\mathcal{F}_m)$ and its norm, according to relation (15), coincides to the norm of f in $H^2((\mathbb{D}^m)_{\rm nc})$, hence the operator $E_{\mathbb{D}^m}^X$ is bounded and $\|E_{\mathbb{D}^m}^X\| \leq \|\widetilde{E_{\mathbb{D}^m}^X}\|$.

For the converse, fix $\{f_w\}_w \in l^2(\mathcal{F}_m)$ and, for all $l > 0$ consider the functions $\alpha_l : ((\mathbb{D}^m)_{\rm nc})_N \longrightarrow \mathbb{C}^{N\times N}$ given by $\alpha_l(X) = \sum_{|w|\leq l} f_w X^w$. The sums are finite, therefore $\alpha_l \in H^2((\mathbb{D}^m)_{\rm nc})$, hence, if $X \in \mathbb{D}^m \cap \mathbb{C}^{N\times N}$, then

$$\begin{aligned}\|\alpha_{l+s}(X) - \alpha_l(X)\| &\leq \|E_{\mathbb{D}^m}^X\| \cdot \|\alpha_{l+s} - \alpha_l\|_{H^2((\mathbb{D}^m)_{\rm nc})}\\ &\leq \|E_{\mathbb{D}^m}^X\| \cdot (\sum_{l<|w|\leq l+s} |f_w|^2).\end{aligned}$$

Since the sequence $\{\sum_{|w|\leq l} |f_w|^2\}_{l\geq 0}$ is Cauchy, it follows that $\{\alpha_l\}_l$ is also a Cauchy sequence, therefore the series $\sum_{l=0}^{\infty}(\sum_{w\in\mathcal{F}_m^{[l]}} f_w X^w)$ converges.

The argument for part (ii) is similar, replacing $l^2(\mathcal{F}_m)$ to $l^2_m(\mathcal{F}_m)$ and using second parts of Theorem 2.3 and of relation (15).

□

Theorem 2.8. *For $p > 0$, define*

$$\Upsilon_p^m = \{X \in (\mathbb{C}^m)_{nc} : \sum_{w\in\mathcal{F}_m} p^{|w|}(X^w)^* X^w \textit{converges}\}.$$

Then $\mathcal{B}_{\mathbb{D}^m} = (\mathbb{D}^m)_{nc} \cap \Upsilon_1^m$ and $\mathcal{B}_{\mathbb{B}^m} = (\mathbb{B}^m)_{nc} \cap \Upsilon_m^m$.

Moreover, if $X \in \Upsilon_p^m \cap \mathbb{C}^{N\times N}$, then $\{X^w\}_{w\in\mathcal{F}_m} \in l^2_{\frac{1}{p}}(\mathcal{F}_m) \otimes \mathbb{C}^{N\times N}$.

Proof. Suppose that $X \in \mathcal{B}_{\mathbb{D}^m,N}$. Since, according to Proposition 2.7(i), the series $\sum_{l=0}^{\infty}(\sum_{w\in\mathcal{F}_m^{[l]}} f_w X^w)$ converges for any $\{f_w\}_{w\in\mathcal{F}_m}$ from $l^2(\mathcal{F}_m)$, it follows that the series $\sum_{l=0}^{\infty}(\sum_{w\in\mathcal{F}_m^{[l]}} f_w e^* X^w \widetilde{e})$ also converges for any $e, \widetilde{e} \in \mathbb{C}^N$. The Riesz Representation Theorem gives that $\{e^* X^w \widetilde{e}\}_{w\in\mathcal{F}_m} \in l^2(\mathcal{F}_m)$, therefore also the series $\sum_{l=0}^{\infty}\sum_{w\in\mathcal{F}_m^{[l]}} e^*(X^w)^*\widetilde{e}X^w$ converges for all $e, \widetilde{e} \in \mathbb{C}^N$.

Taking $e, \widetilde{e}$ from the cannonical basis of $\mathbb{C}^N$, we get that $\sum_{w\in\mathcal{F}_m}(X^w)^*X^w$ converges on each entry, therefore in $\mathbb{C}^{N\times N}$.

The argument for $(\mathbb{B}^m)_{\mathrm{nc}}$ is similar.

Suppose now than $X \in \Upsilon_p^m \cap \mathbb{C}^{N\times N}$. Then $\sum_{w\in\mathcal{F}_m} p^{|w|}(X^w)^*X^w$ also converges entrywise, and, since the (j,j)-entry of the series equals

$$\sum_{w\in\mathcal{F}_m} p^{|w|}(\sum_{l=1}^{N}\overline{x_{l,j}^{(w)}}x_{l,j}^{(w)}) = \sum_{l=1}^{N}(\sum_{w\in\mathcal{F}_m} p^{|w|}\overline{x_{l,j}^{(w)}}x_{l,j}^{(w)})$$

where $x_{l,j}^{(w)}$ is the (l,j)-entry of X^w, it follows that $\{x_{l,j}^{(w)}\}_{w\in\mathcal{F}_m} \in l^2_{\frac{1}{p}}(\mathcal{F}_m)$ for all l,j. In particular $\{X^w\}_{w\in\mathcal{F}_m} \in l^2_{\frac{1}{p}}(\mathcal{F}_m)\otimes\mathbb{C}^{N\times N}$.

□

Remark 2.9. (i) $\Upsilon_1^m \not\subset (\mathbb{D}^m)_{nc}$ *and* $\Upsilon_m^m \not\subset (\mathbb{B}^m)_{nc}$.

(ii) *If* $X = (X_1, X_2, \ldots, X_m) \in (\mathbb{C}^{N\times N})^m$ *is such that*

$$X^*X_1 + X_2^*X_2 + \cdots + X_m^*X_m < \frac{1}{p}$$

then $X \in \Upsilon_p^m$. *In particular,* $\frac{1}{\sqrt{m}}(\mathbb{D}^m)_{nc} \subset \mathcal{B}_{\mathbb{D}^m}$ *and* $\frac{1}{\sqrt{m}}(\mathbb{B}^m)_{nc} \subset \mathcal{B}_{\mathbb{B}^m}$.

Proof. For part (i), it suffices to take $X = (X_1, 0, \ldots, 0)$ with X_1 nilpotent with norm larger than 1. Then $X \in \Upsilon_p^m$ for any $p > 0$, but $X \notin (\mathbb{B}^m)_{\mathrm{nc}}, (\mathbb{D}^m)_{\mathrm{nc}}$.

For part (ii), suppose that $X^*X_1 + X_2^*X_2 + \cdots + X_m^*X_m < \frac{\theta}{p}$ for some $0 < \theta < 1$. Denote by $X^{[l]} = p^l \sum_{w\in\mathcal{F}_m^{[l]}}(X^w)^*X^w$. Then

$$0 \leq X^{[l+1]} = \sum_{w\in\mathcal{F}_m^{[l]}} p^l (X^w)^*\left(p\sum_{k=1}^{m}X_k^*X_k\right)X^w < \theta X^{[l]}$$

hence $\sum_{w\in\mathcal{F}_m}(X^w)^*X^w < \dfrac{1}{1-\theta}$.

□

Definition 2.10. *For* $p > 0$, *we will consider the sets*

$$\mathcal{K}_p = \{(X,Y) \in (\mathbb{C}^m)_{nc} \times (\mathbb{C}^m)_{nc} : \sum_{l=0}^{\infty}[\sum_{w\in\mathcal{F}_m^{[l]}} p^l X^w \otimes (Y^w)^*] \text{ converges}\}$$

and the maps $K_p : \mathcal{K}_p \longrightarrow \mathbb{C}_{nc}$, *given by*

$$K_p(X,Y) = \sum_{l=0}^{\infty}[\sum_{w\in\mathcal{F}_m^{[l]}} p^l X^w \otimes (Y^w)^*].$$

Theorem 2.8 implies the following result:

Remark 2.11. $(\Upsilon_p^m \times \Upsilon_p^m) \subset \mathcal{K}_p$. □

Also, note that from the second part of Theorem 2.8, any sequence $f = \{f_w\}_{w\in\mathcal{F}_m}$ from $l_p^2(\mathcal{F}_m)$ can be identified with a nc-function on Υ_p^m via

$$f(X) = \sum_{l=o}^{\infty}(\sum_{w\in\mathcal{F}_m^{[l]}} f_w X^w).$$

Next, we will consider the following spaces of nc-functions:

Definition 2.12. *For $p > 0$ define $\overline{H^2_{m,p}}$ as follows:*

$$\overline{H^2_{m,p}} = \{f : \Upsilon_p^m \longrightarrow \mathbb{C}_{nc} : \ f \textit{ is nc-function such that there exists some sequence } \{f_w\}_{w\in\mathcal{F}_m} \in l_p^2(\mathcal{F}_m) \textit{ such that } f(X) = \sum_{l=o}^{\infty}\sum_{w\in\mathcal{F}_m^{[l]}} f_w X^w\}.$$

Note that $\overline{H^2_p}$ are Hilbert spaces with the inner-products inherited from $l_p^2(\mathcal{F}_m)$. Proposition 2.13 below shows that in fact they are reproducing kernel Hilbert spaces with respect to K_p.

Proposition 2.13. *Fix $Y \in \Upsilon_p^m \cap \mathbb{C}^{M\times M}$. With the notations above, we have that:*

(i) *The map $K_p(\cdot, Y) : \Upsilon_p^m \longrightarrow (\mathbb{C}^{M\times M})_{nc}$ is a non-commutative function that belongs to $\overline{H^2_{p,m}} \otimes \mathbb{C}^{M\times M}$.*

(ii) *for any $e_1, e_2 \in \mathbb{C}^M$ and any $f \in \overline{H^2_{p,m}}$,*

$$\langle f, e_1^* K_p(\cdot, Y)e_2\rangle_{l_p^2(\mathcal{F}_m)} = e_2^* f(Y)e_1.$$

Proof. For part (i), first note that for any $w \in \mathcal{F}_m$, the map $X \mapsto p^{|w|}X^w \otimes (Y^w)^*$ is a noncommutative function from $\mathbb{C}_{\rm nc}$ to $(\mathbb{C}^{M\times M})_{\rm nc}$, hence it suffices to prove the convergence in $\overline{H^2_{p,m}}$.

Let $y_{i,j}^{(w)}$ be the (i, j)-entry of Y^w.

From Theorem 2.8, the sequences $\{y_{i,j}^{(w)}\}_{w\in\mathcal{F}_m}$ are in $l_p^2(\mathcal{F}_m)$ for all i, j, hence the map $X \mapsto \sum_{w\in\mathcal{F}_m} y_{i,j}^{(w)} X^w$ is a $\mathbb{C}_{\rm nc}$ -valued non-commutative function from $\overline{H^2_{p,m}}$.

For part (ii), let $\{f_w\}_{w\in\mathcal{F}_m} \in l^2(\mathcal{F}_m)$ such that $f(X) = \sum_{w\in\mathcal{F}_m} f_w X^w$. Then

$$\begin{aligned}\langle f, e_1^* K_p(\cdot, Y)e_2\rangle_{l_p^2(\mathcal{F}_m)} &= \langle \{f_w\}_{w\in\mathcal{F}_m}, \{e_1^*(Y^w)^* e_2\}_{w\in\mathcal{F}_m}\rangle_{l_p^2(\mathcal{F}_m)} \\ &= \sum_{w\in\mathcal{F}_m} e_2^* f_w Y^w e_1 = e_2^* f(Y)e_1.\end{aligned}$$

□

Proposition 2.14. *Suppose that f is a non-commutative function locally bounded on slices separately in every matrix dimension around 0 and*

$$\Phi(r) = \lim_{N\longrightarrow\infty} \int_{\partial(\mathbb{D}^m,N)} \frac{1}{N}\, Tr(f(rX)^* f(rX))d\mu_N,$$

$$\Psi(r) = \lim_{N\longrightarrow\infty} \int_{\partial(\mathbb{B}^m,N)} \frac{1}{N}\, Tr(f(rX)^* f(rX))d\nu_N.$$

Then f extends to a function in $\overline{H^2_{1,m}}$, respectively in $\overline{H^2_{m,m}}$, if and only if $\Phi(r)$, respectively $\Psi(r)$, exists for all small r (in which case Φ, respectively Ψ, are also analytic at 0) and it extends analytically to $(0,1)$ and continously to $[0,1]$.

Moreover, $\lim\limits_{r\longrightarrow 1^-}\Phi(r) = \|f\|_{\overline{H^2_{1,m}}}$, *respectively* $\lim\limits_{r\longrightarrow 1^-}\Psi(r) = \|f\|_{\overline{H^2_{m,m}}}$.

Proof. Suppose first that f extends to $\widetilde{f} \in \overline{H^2_{1,m}}$, that is there exists $\{f_w\}_{w\in\mathcal{F}_m}$ such that

$$f(X) = \sum_{l=0}^{\infty}\Big(\sum_{w\in\mathcal{F}_m^{[l]}} f_w X^w\Big) \tag{16}$$

for all $X \in \Upsilon_1^m$; in particular, Remark 2.9(ii) gives that the expansion (16) holds for all $X \in \frac{1}{\sqrt{m}}\mathbb{D}^m$.

As before, consider the non-commutative functions $f^{[l]} : (\mathbb{C}^m)_{\mathrm{nc}} \longrightarrow \mathbb{C}_{\mathrm{nc}}$ given by $f^{[l]}(X) = \sum_{w\in\mathcal{F}_m^{[l]}} f_w X^w$. Then, for $X \in \partial(\mathbb{D}^m, N)$, we have that

$$\|f^{[l]}(\frac{1}{m}X)\| \leq \sum_{w\in\mathcal{F}_m^{[l]}} \frac{1}{m^l} \sup_{w\in\mathcal{F}_m^{[l]}}(|f_w|\cdot\|X^w\|) \leq \sup_{w\in\mathcal{F}_m^{[l]}}|f_w|,$$

therefore, for $r \in (0, \frac{1}{m})$,

$$\int_{\partial(\mathbb{D}^m,N)} \frac{1}{N}\mathrm{Tr}\left(f^{[l]}(rX)^* f^{[l]}(rX)\right) d\mu_N \leq \sup_{w\in\mathcal{F}_m^{[l]}}|f_w|^2.$$

hence, expansion (16) and Corollary 1.4(ii) give that

$$\begin{aligned}&\int_{\partial(\mathbb{D}^m,N)} \frac{1}{N}\mathrm{Tr}\left(f(rX)^*\ f(rX)\right) d\mu_N\\ &\qquad = \sum_{l=0}^{\infty}\int_{\partial(\mathbb{D}^m,N)} \frac{1}{N}\mathrm{Tr}\left(f^{[l]}(rX)^* f^{[l]}(rX)\right) d\mu_N \leq \|f\|^2_{l^2(\mathcal{F}_m)}.\end{aligned}$$

Therefore, using Corollary 1.4(i), we have that for $r \in (0, \frac{1}{m})$,

$$\Phi(r) = \sum_{l=0}^{\infty} r^{2l}\Big(\sum_{w\in\mathcal{F}_m^{[l]}}|f_w|^2\Big)$$

and Φ extends analytically to $(0,1)$ and continously to $[0,1]$.

The proof for Ψ is similar, using Remark 2.9 and Corollary 1.5.

For the converse, suppose that there exists $\delta > 0$ such that $\Phi(r)$ exists for $r < \delta$ and extends analitically to $(0,1)$. In particular there exists some N_0 such that the integral from the definition of $\Phi(\cdot)$ is finite if $N > N_0$. Fix now $N > N_0$; equation (5) gives that there exists some $\alpha > 0$ such that the series $\sum_{w\in\mathcal{F}_m} f_w X^w$ converges absolutely for $X \in \alpha\mathbb{D}^m$, particularly $\{(\frac{\alpha}{m})^{|w|} f_w\}_{w\in\mathcal{F}_m} \in l^1(\mathcal{F}_m) \subset l^2(\mathcal{F}_m)$.

Let $R = \min\{\delta, \frac{\alpha}{m}\}$. Then Corollary 1.4 gives that, for $r \in (0, R)$,

$$\Phi(r) = \sum_{l=0}^{\infty} r^{2l}\Big(\sum_{w\in\mathcal{F}_m^{[l]}}|f_w|^2\Big)$$

and the conclusion follows since $\Phi(\cdot)$ extends analytically to $(0,1)$.

As before, the proof for $\Psi(\cdot)$ is similar, using equation (5) and Corollary 1.5.

□

Proposition 2.15. *For $f \in \overline{H^2_{1,m}}$ and $g \in \overline{H^2_{m,m}}$, respectively $Y \in \Upsilon_1^m \cap \mathbb{C}^{M\times M}$ and $Y' \in \Upsilon_m^m \cap \mathbb{C}^{M\times M}$, we have that*

$$\varphi_{f,Y}(r) = \lim_{N\longrightarrow\infty} \int_{\partial(\mathbb{D}^m,N)} \frac{1}{N} Tr \otimes Id_{\mathbb{C}^{M\times M}} \left(f(rX)K_1(rX, Y^*)^*\right) d\mu_N(X)$$

$$\psi_{g,Y'}(r) = \lim_{N\longrightarrow\infty} \int_{\partial(\mathbb{B}^m,N)} \frac{1}{N} Tr \otimes Id_{\mathbb{C}^{M\times M}} \left(g(rX)K_m(rX, (Y')^*)^*\right) d\nu_N(X)$$

are analytic functions of r for r small and they extend analytically to $(0, 1)$ and continously to $[0, 1]$. Moreover, $\lim_{r\longrightarrow 1^-} \varphi_{f,Y}(r) = f(Y)$ and $\lim_{r\longrightarrow 1^-} \psi_{g,Y'}(r) = g(Y')$.

Proof. Let $p \geq 1$ and $r \in (0, (2m^2p)^{-1})$, let $\{f_w\}_{w\in\mathcal{F}_m} \in l^2_p(\mathcal{F}_m)$, and consider $X \in \mathbb{C}^{N\times N}$, $Y \in \mathbb{C}^{M\times M}$ such that $\sup_{w\in\mathcal{F}_m} \|Z^w\| = m(Z) < \infty$ and $\sup_{w\in\mathcal{F}_m} \|X^w\| \leq 1$.

First, note that the series $\sum_{l=0}^{\infty}(\sum_{w\in\mathcal{F}_m^{[l]}} r^{2|w|} f_w p^{|w|} Y^w)$ converges asolutely, since $\|\{(rp)^{2|w|}\}_w\|_{l^2(\mathcal{F}_m)} < 2$ and

$$\begin{aligned}\sum_{l=0}^{\infty}\Big(\sum_{w\in\mathcal{F}_m^{[l]}} \|r^{2|w|} f_w p^{|w|} Y^w\|\Big) &\leq m(Y) \cdot \sum_{l=0}^{\infty}\Big(\sum_{w\in\mathcal{F}_m^{[l]}} (rp)^{2|w|} \cdot |p^{-|w|} f_w|\\ &\leq m(Y) \cdot \|\{f_w\|_w\|^{\frac{1}{2}}_{l^2_p(\mathcal{F}_m)} \cdot \|\{(rp)^{2|w|}\}_w\|^{\frac{1}{2}}_{l^2(\mathcal{F}_m)}\end{aligned}$$

Also, we have that $\|\{(rmp)^{|w|}\}_w\|_{l^2(\mathcal{F}_m)} < 2$ and

$$\begin{aligned}&[\sum_{k=0}^{\infty}(\sum_{w\in\mathcal{F}_m^{[k]}} \|f_w \cdot r^{|w|} X^w\|)] \cdot [\sum_{l=0}^{\infty}(\sum_{v\in\mathcal{F}_m^{[l]}} p^l \cdot r^l \|X^v \otimes (Y^v)^*\|)]\\ &\leq [\sum_{k=0}^{\infty} m^k r^k (\sum_{w\in\mathcal{F}_m^{[k]}} |f_w|)] \cdot (\sum_{l=0}^{\infty} p^l r^l m^l \cdot m(Y))\\ &\leq [\sum_{k=0}^{\infty}(r\cdot mp)^k \cdot (\sum_{w\in\mathcal{F}_m^{[k]}} p^{-|w|}|f_w|)] \cdot 2m(Y)\\ &\leq \|\{f_w\}_w\|^{\frac{1}{2}}_{l^2_p(\mathcal{F}_m)} \cdot \|\{(rmp)^{|w|}\}_w\|^{\frac{1}{2}}_{l^2(\mathcal{F}_m)} \cdot 2m(Y).\end{aligned}$$

Therefore, for $p = 1$, if f and Y are as in the statement of 2.15, we have that

$$\begin{aligned}&\int_{\partial(\mathbb{D}^m,N)} \frac{1}{N}\mathrm{Tr} \otimes \mathrm{Id}_{\mathbb{C}^{M\times M}} \left(f(rX)K_1(rX, Y^*)^*\right) d\mu_N(X)\\ &= \int_{\partial(\mathbb{D}^m,N)} \frac{1}{N}\mathrm{Tr} \otimes \mathrm{Id}_{\mathbb{C}^{M\times M}} \Big(\sum_{k,l=0}^{\infty} \sum_{w\in\mathcal{F}_m^{[k]}} \sum_{v\in\mathcal{F}_m^{[l]}} f_w \cdot r^k X^w \cdot (X^v)^* \otimes Y^v\Big) d\mu_N(X).\end{aligned}$$

Since $\frac{1}{N}\mathrm{Tr} \otimes \mathrm{Id}_{\mathbb{C}^{M\times M}}$ is a bounded linear map, using Corollary 1.4, the right hand side of the equation above equals $\sum_{l=0}^{\infty}(\sum_{w\in\mathcal{F}_m^{[l]}} f_w Y^w)$. and the conclusion for $\varphi_{Y,f}$ follows.

The proof for $\psi_{g,Y'}$ is analogous letting $p = m$ and using Corollary 1.5.

∎

Definition 2.16. *For Ω either $\mathbb{B}^m$ or $\mathbb{D}^m$, define*

$$H^\infty(\Omega_{nc}) = \{f : \Omega_{nc} \longrightarrow \mathbb{C}_{nc} : f \textit{ is a non-commutative function and } \sup_{Z\in\Omega} \|f(Z)\| < \infty\}$$

$$H^\infty(\mathcal{B}_\Omega) = \{f : \mathcal{B}_\Omega \longrightarrow \mathbb{C}_{nc} : f \textit{ is a non-commutative function and } \sup_{Z\in\mathcal{B}_\Omega} \|f(Z)\| < \infty\}.$$

Obviously, $H^\infty(\Omega_{\mathrm{nc}}) \subset H^\infty(\mathcal{B}_\Omega)$, since $\mathcal{B}_\Omega \subset \Omega_{\mathrm{nc}}$. We will further detail this inclusion below.

Definition 2.17. *For Ω either $\mathbb{B}^m$ or $\mathbb{D}^m$, define*

$$\begin{aligned}\mathcal{M}(\Omega_{nc}) = \{f : \Omega_{nc} \longrightarrow \mathbb{C}_{nc} :\ & f \textit{ is a non-commutative function which is also a}\\ & \textit{bounded left multiplier for } H^2(\Omega_{nc})\}\end{aligned}$$

$$\begin{aligned}\mathcal{M}(\mathcal{B}_\Omega) = \{f : \mathcal{B}_\Omega \longrightarrow \mathbb{C}_{nc} :\ & f \textit{ is a non-commutative function which is also a}\\ & \textit{bounded left multiplier for } \overline{H^2_{1,m}}, \textit{if } \Omega = \mathbb{B}^m, \textit{ respectively } \overline{H^2_{m,m}} \textit{ if } \Omega = \mathbb{D}^m\}.\end{aligned}$$

where the multiplier norms are the natural ones.

Proposition 2.18. *With the notations above, we have that*

$$H^\infty(\Omega_{nc}) \subseteq \mathcal{M}(\Omega_{nc}) \subseteq \mathcal{M}(\mathcal{B}_\Omega) \subseteq H^\infty(\mathcal{B}_\Omega).$$

Proof. From the consideration above, we only need to prove the last inclusion. Consider $g \in \mathcal{M}(\mathcal{B}_\Omega)$, denote M_g the left multiplier with g and take $X \in \mathcal{B}_\Omega \cap \mathbb{C}^{M\times M}$, $Y \in \Upsilon_p^m \cap \mathbb{C}^{N\times N}$. From Poposition 2.13, for any $e_1, e_2 \in \mathbb{C}^M$ and $f_1, f_2 \in \mathbb{C}^N$, we have that

$$\begin{aligned}\langle (M_g)^* e_1^* K(\cdot, X)e_2, f_1^* K(\cdot, Y)f_2\rangle &= \langle e_1^* K(\cdot, X)e_2, M_g f_1^* K(\cdot, Y)f_2\rangle\\ &= \langle g(\cdot) f_1^* K(\cdot, Y)f_2, e_1^* K(\cdot, X)e_2\rangle^*\\ &= (e_2^* g(X) f_1^* K(X, Y)f_2 e_1)^*,\end{aligned}$$

hence $(M_g)^* K(\cdot, X) = K(\cdot, X)g(X)^*$ and since $\|(M_g)^* K(\cdot, X)\| \leq \|M_g\| \cdot \|K(\cdot, X)\|$ and $K(\cdot, \cdot)$ is a reproducing kernel, it follows that $\|g(X)\| \leq \|M_g\|$. ☐

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