School of Advanced Studies of the Romanian Academy
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SUMMARY OF Ph.D. THESIS

Spectral and algebraic methods in the study of differentiable manifolds

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Abstract

This thesis develops a Selberg trace formula for the Dirac operator on complete hyperbolic surfaces of finite area, from which several results are obtained. First, we investigate the spectrum of the Dirac operator on families of hyperbolic surfaces where a set of disjoint simple geodesics shrink to 0, under the hypothesis that the spin structure is non-trivial along each pinched geodesic. We derive a version of Huber's theorem, a non-standard small time heat asymptotic expansion and a Weyl law for the eigenvalues of the Dirac operator, which is *uniform* in the degenerating parameter. The first main result is the convergence of the Selberg zeta function associated to a non-trivial spin structure.

Secondly, we focus on the behaviour of the spectrum of the Dirac operator on a *typical* hyperbolic surface of finite area. We work on the moduli space of surfaces of genus g with k cusps endowed with the Weil-Petersson measure. In this moduli space there exists a subset $\mathcal{A}_{g,k}$ for which $\mathbb{P}(\mathcal{A}_{g,k}) \to 1$ as $g \to \infty$ such that for every surface in $\mathcal{A}_{g,k}$ endowed with a non-trivial spin structure, the rescaled number of eigenvalues of the Dirac operator between a and b is of order $b - a$. This result refines the Weyl law as the upper bound does not depend on the surface.

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Chapter 2

Introduction

This thesis aims at studying the spectrum of the Dirac operator on hyperbolic surfaces of finite volume. To achieve this purpose, we develop and make use of a Selberg trace formula for the Dirac operator, following the original idea of A. Selberg [44]. There are three main ingredients necessary for this formula, which will be presented in the rest of the chapter. Apart from the introduction, this work consists of two parts. Each one represents a paper ([46] and [36]) the author elaborated specifically for his doctoral studies. We would like to mention that the version of the second paper present here is the one sent to the publishing journal. The rest of the introduction is a crash course in hyperbolic geometry. We present the definitions and notions one needs to know in order to understand the two parts of the thesis.

2.1 Riemann surfaces

A *Riemann surface* is a connected, Hausdorf topological space endowed with a holomorphic atlas. They arise naturally as domains of holomorphic functions. Throughout the second part of the nineteenth century, many famous mathematicians were focused on proving the uniformization of Riemann surfaces. This theorem states that every simply-connected Riemann surface is biholomorphic to either the complex plane $\mathbb{C}$, the unit disk $\mathbb{D}$ or the Riemann sphere $\hat{\mathbb{C}}$. It is arguably the most important result in the field of complex analysis of one variable. In 1907, two rigours proofs appeared independently, thanks to P. Koebe [24] and H. Poincaré [39]. Modern arguments can be found in various books [15, 16, 22] as well as short papers [3].

A remarkable consequence of this theorem is that it build a bridge between the fields of complex analysis and hyperbolic geometry. Let us consider a Riemann surface M. Then $\tilde{M}$, its universal cover, is biholomorphic to either $\mathbb{C}$, $\mathbb{D}$ or $\hat{\mathbb{C}}$. Moreover, we know that $M = \Gamma \setminus \tilde{M}$, where Γ is the fundamental group of our surface, and the induced action is properly discontinuous, without fixed points.

i) Suppose $\tilde{M} \simeq \hat{\mathbb{C}}$. Then:

$$\Gamma \subset \mathrm{Aut}(\hat{\mathbb{C}}) = \left\{ z \mapsto \frac{az+b}{cz+d} : a,b,c,d \in \mathbb{C},\ ad-bc \neq 0 \right\}.$$

The group $\mathrm{Aut}(\hat{\mathbb{C}})$ is also known as the group of *Möbius transformations*. Since each map $z \mapsto \frac{az+b}{cz+d}$ has at least one fixed point, we deduce that Γ can only be the trivial group, hence $M \simeq \hat{\mathbb{C}}$.

ii) Suppose $\tilde{M} \simeq \mathbb{C}$. Then:

$$\Gamma \subset \mathrm{Aut}(\mathbb{C}) = \{ z \mapsto az+b : a,b \in \mathbb{C},\ a \neq 0 \}.$$

The map $z \mapsto az+b$ has no fixed point if and only if $a = 1$. Moreover Γ must be discrete, hence the only possible options are:

$$\Gamma = \{\mathrm{Id}\};$$
$$\Gamma = \{ z \mapsto z+nb : n \in \mathbb{Z}, \text{for a fixed } b \in \mathbb{C} \};$$
$$\Gamma = \left\{ z \mapsto z+nb+mb' : n,m \in \mathbb{Z},, \text{ for some fixed } b,b' \in \mathbb{C} \text{ with } \frac{b}{b'} \notin \mathbb{R} \right\}.$$

Factoring the plane $\mathbb{C}$ through these groups we obtain $\mathbb{C}$, a cylinder (which is biholomorphic to $\mathbb{C}^*$ as well), and an elliptic curve.

Clearly, all other Riemann surfaces are covered by the unit disk $\mathbb{D}$. This disk carries one additional structure. It can be equipped with a complete hyperbolic metric $g = \frac{4(dx^2+dy^2)}{(1-x^2-y^2)^2}$, thus becoming the well known *Poincaré disk*. Moreover, we shall see that the group $\mathrm{Aut}(\mathbb{D})$ acts through isometries as well. Therefore, if our initial surface M is covered by $\mathbb{D}$, it automatically inherits a complete hyperbolic metric.

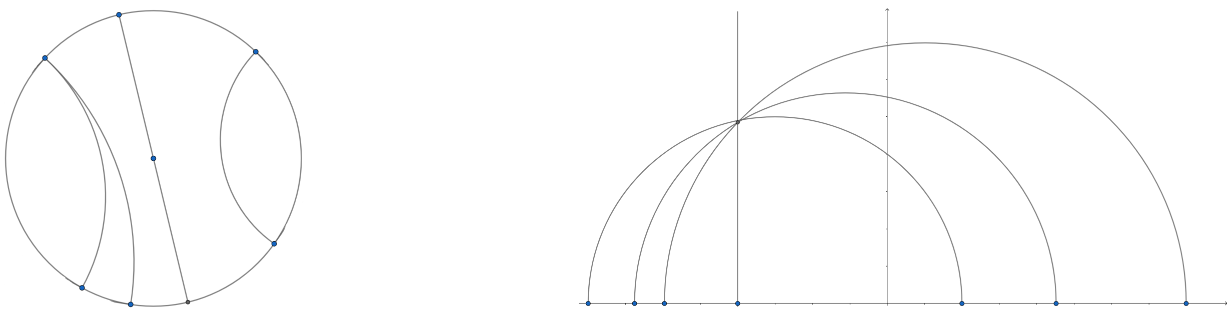
2.2 Hyperbolic surfaces and spin structures

In what follows we restrict ourselves to the study of complete hyperbolic surfaces of finite area. Two models of hyperbolic geometry are usually used: the Poincaré disk mentioned earlier and the *Poincaré half-plane*

$$\mathbb{H} := \left(\{(x,y) \in \mathbb{R}^2 : y > 0\}, g = \frac{dx^2+dy^2}{y^2} \right).$$

One can easily see that these two models are biholomorphic and isometric, using the function:

$$f : \mathbb{H} \longrightarrow \mathbb{D}; \qquad f(z) = \frac{z-i}{z+i},$$



(a) Poincaré disk $\mathbb{D}$ (b) Poincaré half-plane $\mathbb{H}$

Figure 2.1: Geodesics in two models of hyperbolic geometry

where a point $z = x + iy$ is identified with the pair (x, y).

On one hand, for $\mathbb{D}$, the geodesics are straight lines passing through 0 and circle arcs perpendicular on $\partial\mathbb{D}$ on both ends. On the other hand, the geodesics of $\mathbb{H}$ are vertical lines or half-circles centred in points with $y = 0$ (see Figure 2.1).

For the rest of the introduction we will work with the half-plane. Its group of automorphisms consists of those Möbius transformations which fix the upper half-plane $y > 0$. Thus:

$$\text{Aut}(\mathbb{H}) = \text{PSL}_2(\mathbb{R}) := \left\{ z \mapsto \frac{az + b}{cz + d} : a, b, c, d \in \mathbb{R},\ ad - bc > 0 \right\}.$$

One can easily check that these applications are also isometries for the metric on $\mathbb{H}$. There are three standard examples of elements from this group:

i) Dilation: $z \mapsto \lambda z$, for some $\lambda > 0$;

ii) Translation: $z \mapsto z + 1$;

iii) Rotation: $z \mapsto \frac{z\cos\theta + \sin\theta}{-z\sin\theta + \cos\theta}$, for some $\theta \in (0, 2\pi)$.

In fact, a classical result tells us that up to a conjugation, every element in $\text{PSL}_2(\mathbb{R})$ is either a dilation (and is called *hyperbolic*), a translation (and is called *parabolic*) or a rotation. The trace is invariant under conjugation, hence we can find out whether an element is conjugated to either a dilation, a translation or a rotation by looking if its the absolute value of its trace is larger than 2, exactly 2 or smaller than 2 respectively. Since rotations have a fixed point, our fundamental group Γ can only contain hyperbolic and parabolic elements. Moreover, if M is compact, then Γ can only contain hyperbolic elements.

2.2.1 Length spectrum

As we said earlier, if M is covered by $\mathbb{D}$, it has a complete hyperbolic metric. With respect to this metric, there are infinitely many closed geodesics on our surface. Indeed, every hyperbolic element preserves a geodesic δ in $\mathbb{H}$, since it is conjugated to a dilation. The projection of δ on M is a closed geodesic. There is a bijective correspondence between classes of conjugation in Γ and oriented closed geodesics on M.

We say that an element $\gamma \in \Gamma$ is *primitive* if it cannot be written as $\gamma = \mu^n$, with $n \geq 2$. Additionally, we say that a closed geodesic η is *primitive* if γ is primitive, where $[\gamma]$ is the conjugacy class associated to η.

By *length spectrum* we understand the sequence of lengths of closed, oriented geodesics on M. Since we consider oriented geodesics, each length appears an even number of times. This sequence is the first important ingredient in the Sleberg trace formula. A very important mathematical object related to the length spectrum is the Selberg zeta function:

$$Z_\varepsilon(s,(M,g)) = \prod_{[\gamma]} \prod_{m=0}^{\infty} \left(1 - \varepsilon(\gamma) e^{-l(\gamma)(s+m)}\right),$$

where ε is a $\{\pm 1\}$ valued function (which depends on the spin structure) defined in the following section, the lengths of the geodesics are taken with respect to the hyperbolic metric g and the product is taken along all conjugacy classes of hyperbolic, primitive elements $\gamma \in \Gamma$.

2.2.2 Spin structures

Let us came back to the group $\mathrm{PSL}_2(\mathbb{R})$ for a brief moment. Every invertible matrix induces an isometry on $\mathbb{H}$ in the following:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longrightarrow \left(z \mapsto \frac{az+b}{cz+d} \right).$$

Note that a matrix A and λA for $\lambda \in \mathbb{R}^*$ will induce the same isometry, hence:

$$\mathrm{PSL}_2(\mathbb{R}) \simeq \mathrm{SL}_2(\mathbb{R})/\{\pm 1\},$$

where $\mathrm{SL}_2(\mathbb{R})$ is the group of invertible matrices of determinant 1. Thus, we have a natural projection $\pi : \mathrm{SL}_2(\mathbb{R}) \longrightarrow \mathrm{PSL}_2(\mathbb{R})$. If we denote $\tilde{\Gamma}$ the preimage of Γ through π we obtain the short exact sequence:

$$1 \longrightarrow \{\pm 1\} \longrightarrow \tilde{\Gamma} \longrightarrow \Gamma \longrightarrow 1.$$

More details are given in section 3.3, but for the moment, let us define a spin structure as a group morphism $\chi : \tilde{\Gamma} \longrightarrow \{\pm 1\}$, for which $\iota \circ \chi = \mathrm{id}_{\{\pm 1\}}$, where $\iota : \{\pm 1\} \longrightarrow \tilde{\Gamma}$ is the natural inclusion.

Let us now consider an element $\gamma \in \Gamma$ and denote $\tilde{\gamma}$ the preimage in $\mathrm{SL}_2(\mathbb{R})$ of positive trace. We define the *class function* ε as $\varepsilon(\gamma) = \chi(\tilde{\gamma})$ (more details are given in section 3.3.4). It is a class function since the trace is invariant to conjugation. We say that a spin structure is *non-trivial* if $\varepsilon(\gamma) = -1$ for every parabolic element $\gamma \in \Gamma$. This ε is the second important ingredient in the trace formula.

The third and final ingredient is the spectrum of D, the Dirac operator, a differential operator of order 1 on a vector bundle, constructed using the spin structure (see section 3.3.2). In $\mathbb{R}^n$, the square of the Dirac operator is exactly the the Laplacian acting on spinors. In arbitrary curvature, the Lichnerowicz formula tells us that the difference between the square of the Dirac operator and the Laplacian is exactly one forth of the scalar curvature. From the theory of pseudodifferential operators, we know that on compact surfaces the spectrum of the Dirac operator is discrete. Bär [6] showed that, under some technical conditions on the spin structure, the spectrum is discrete on hyperbolic surfaces of finite area as well.

2.2.3 Selberg trace formula for the Dirac operator

We are now in a position where we can state the trace formula. The complete proof can be found in Chapter 3. Consider $\{r_j\}_{j\in\mathbb{N}}$ the ordered sequence of eigenvalues of $|\,\mathrm{D}\,|$. Let u be an *admissible* (see Definition 3.22) function. If $M = \Gamma \setminus \mathbb{H}$ is a complete hyperbolic surface with k cusps and a non-trivial spin structure, we have:

$$\sum_{j=0}^{\infty} u(\xi_j) = \frac{\mathrm{Area}(M)}{2\pi} \int_{\mathbb{R}} \xi u(\xi) \coth(\pi\xi) d\xi + \sum_{[\mu]} \sum_{n=1}^{\infty} \frac{l(\mu)\varepsilon^n(\mu)\check{u}(nl(\mu))}{\sinh\left(\frac{nl(\mu)}{2}\right)} - 2k\log(2)\check{u}(0),$$

where $[\mu]$ runs along all conjugacy classes of primitive, hyperbolic elements in Γ and $\check{u}$ is the inverse Fourier transform of u. From a philosophical point of view one can say that it is a bridge between classical mechanics and quantum mechanics. It relates closed trajectories of classical particles (i.e., closed geodesics) to periodic states of quantum particles (i.e., eigenvalues and eigenfunctions of D). For this thesis, the above formula represents the foundation on which we will build our applications. Compared to the classical trace formula developed by Selberg [44], here the function coth appears instead of tanh.

2.2.4 Pinching process

A *pinching process* is obtained when the length of a simple closed geodesic on M shrinks to 0. Rigorously we do this by choosing a family of complete hyperbolic metrics on our surface, as explained in Definition 3.1. But intuitively, the process can be seen in Figure 2.2. At the limit, the geodesic η disappears so the surface is no longer compact. We also mention that, because of the Gauss-Bonnet formula, the area of the surface remains constant throughout this process.

A first result in this direction appears in Chapter 3 and says that the right hand-side of the trace formula behaves well when the length of the geodesic shrinks to 0 (Theorem 3.24). From here, we obtain a Weyl law (Theorem 3.4) which is *uniform* in the pinching parameter:

$$\lim_{\lambda\to\infty} \frac{N_{\mathrm{D}_t^2}(0,\lambda)}{\lambda} = \frac{\mathrm{Area}(M)}{2\pi}; \qquad \textit{uniformly} \text{ for } t \in [0,1].$$

This immediately implies that:

$$N_{D_t}(-\xi,\xi) = \xi^2 \frac{\text{Area}(M)}{2\pi} + o(\xi^2),$$

uniformly in $t \in [0,1]$ (the pinching parameter), where $N_{D_t}(-\xi,\xi)$ is the counting function of D_t-eigenvalues between $-\xi$ and ξ. The above result greatly improves the existing estimates known [6, Theorem 2].

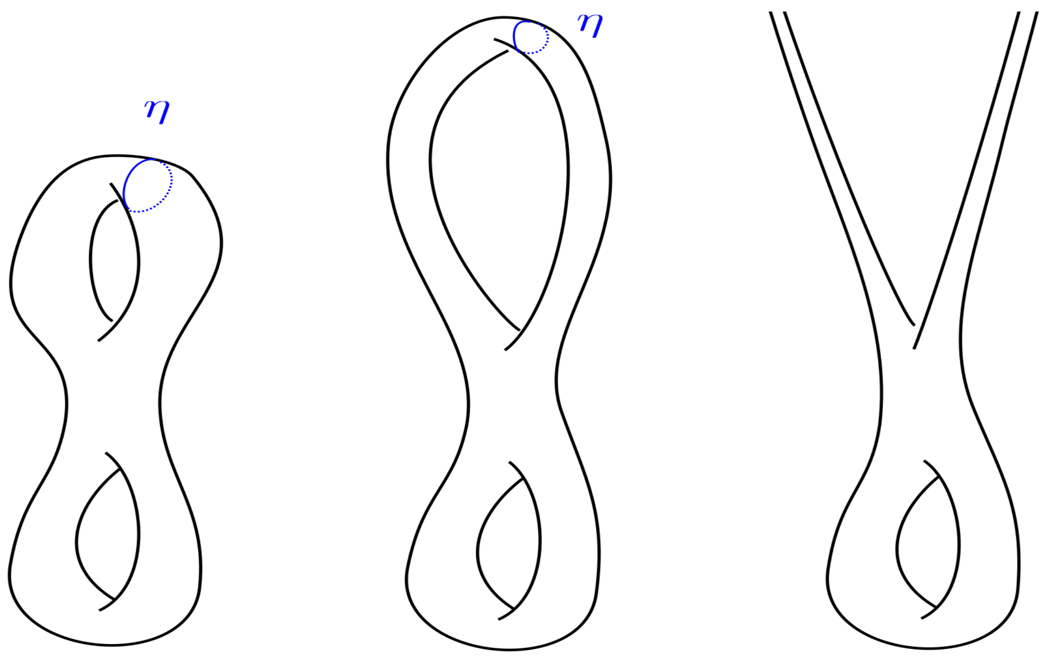


Figure 2.2: Pinching process

Finally, the main theorem of the next chapter is the convergence of the Selberg zeta function defined above during a pinching process (Theorem 3.5):

$$\lim_{t\to 0} Z_\varepsilon(s,(M,g_t)) \exp\left(-\sum_{j=1}^{\kappa} \frac{\pi^2}{6l_t(\eta_j)}\right) = Z_\varepsilon(s,(M,g_0))2^{\kappa(1-2s)},$$

uniformly on compacts in $\mathbb{C}$, where g_t is the hyperbolic metric on M at time $t \in [0,1]$.

2.2.5 Counting eigenvalues on random surfaces

Chapter 4 is concerned with the spectral properties of the Dirac operator on a *typical* hyperbolic surface of finite area. Since similar results are already known in the case of the Laplacian, let us briefly explain them. In [35] Monk studied the distribution of eigenvalues of the Laplacian on a random compact hyperbolic surface. She worked with the Weil-Petersson volume, which induces a probability measure on $\mathcal{M}_g$, the moduli space of surfaces of genus g. The main result is that for a typical hyperbolic surface X and $0 \leq a \leq b$ one has:

$$\frac{N_X^{\Delta}(a,b)}{\text{Area}(X)} = \mathcal{O}\left(b - a + \sqrt{\frac{b+1}{\log g}}\right),$$

where $N_X^{\Delta}(a,b)$ is the *counting function* of Δ-eigenvalues between a and b on the hyperbolic surface X. In [25] Le Masson and Sahlsten extend this result to hyperbolic surfaces of finite area, provided that the number of cusps $k = k(g)$ is of order $\mathcal{O}(g^{\kappa})$, for $0 < \kappa < 1/2$. Under the same hypothesis for the number of cusps, we obtain that (Theorem 4.1):

$$\frac{N_X^{\mathrm{D}^2}(a,b)}{\text{Area}(X)} = \mathcal{O}\left(\frac{1}{4\pi}\int_a^b \coth(\pi\sqrt{r})dr + \frac{\sqrt{b}+1}{\sqrt{\log g}}\right).$$

This result is obtained in collaboration with Laura Monk. As in [35], the proof consists of applying the Selberg trace formula for a specific family of functions.

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