Romanian Academy

Institute of Mathematics “Simion Stoilow”

DOCTORAL THESIS

-SUMMARY-

p-adic Distributions, Krasner analytic functions and applications

Advisor: C.S. I Dr. Marian Vâjâitu

PhD Student: Cosmin Constantin Niţu

Bucharest, 2017

Summary

The concept of p-adic number was introduced by Hensel in 1897, but it can be found, without being explicitly named, in previous works of Kummer. The main reason for their introduction was the use of techniques of mathematical analysis, particulary of series theory, in the theory of numbers.

The more we can find a bigger power of p that divides the difference of two p-adic numbers, the closer they are.

Thus, the p-adic numbers can preserve information regarding the congruences modulo p^n and they can have important applications in classical number theory.

Let p be a prime number. The p-adic norm (or p-adic module) is definded for $x \in \mathbb{Q}$ in the following way:

$$|x|_p = \begin{cases} p^{-v_p(x)}, \ x \neq 0 \\ 0 \ , \ x = 0 \end{cases} \quad ,$$

where $v_p(x)$ is the exponent of p in the decomposition of x.

In 1930, Ostrowski proved that every norm defined on the field of rational numbers $\mathbb{Q}$ is equivalent either to the usual module, or to the p-adic module for a certain number p. The topological completion of $\mathbb{Q}$ with respect to the ordinary module is $\mathbb{R}$, and with respect to the p-adic module is $\mathbb{Q}_p$.

There are many similarities between the fields $\mathbb{R}$ and $\mathbb{Q}_p$ (both are completions of $\mathbb{Q}$, $\mathbb{Q}$ is dense in each one of them, they are local compact spaces, they aren't algebrically closed, we can use similar analysis techniques in both of them etc.), however, there are also many differences between them ($\mathbb{R}$ is ordered and the relation of order is compatibile with the algebraic operators "+" and "·", $\mathbb{R}$ is archimedian, but $\mathbb{Q}_p$ is non-archimedian, $\mathbb{R}$ is connex,

while $\mathbb{Q}_p$ is totally disconnected, in $\mathbb{Q}_p$ we cannot define intervals or curves etc.).

Unlike the real case, in the p-adic case the set $\overline{\mathbb{Q}}_p$ (the algebraic closure of $\mathbb{Q}_p$) is not a complete metric space. This is why, one constructs the topological closure of $\overline{\mathbb{Q}}_p$ in relation to the p-adic module, which is denoted by $\mathbb{C}_p$ and it is a complete and algebrically closed field. This is called the Tate field and plays a similar role as $\mathbb{C}$ in the classical analysis.

The analytical techniques mentioned above referred to the local development of an analytical function in a power series. Thus, it began the study of the theory of functions over the p-adic fields, although the great impediment consisted in the fact that these Tate fields are totally disconected, making it difficult and delicate to define the analytical function in a global sense. The first attempt came in 1930 with Schöbe'S PhD thesis, but the successful one was Krasner in the 1950's, inspired by Runge' s theorem of classical analysis regarding the approximation of an analytical function with rational functions, using a simplified Weierstrass method of analytical continuation. Later, in 1961, the study of p-adic analysis triumphed through Tate's works, which used the ideas of Gröthendieck, and gave rise to a rigid topological structure to analytical spaces over p-adic fields.

My PhD thesis is structured in four chapters.

In the first chapter we introduce the p-adic numbers and study some of their analytical, algebraic and topological properties.

In the second chapter we introduce the p-adic distributions and measures. Also, we present the Riemann integral against a distribution. We analyze an important category of distributions: the strongly Lipschitz distributions.

2.1 p-Adic distributions and measures

Let be $(\mathcal{X}_n, \varphi_n)_{n\geq 1}$ a projective system with the property that $\mathcal{X}_n,\ n \geqslant 1$, are finite sets, $\varphi_n : \mathcal{X}_{n+1} \longrightarrow \mathcal{X}_n$ are surjective functions and $\mathcal{X} = \varprojlim \mathcal{X}_n$.

Definition 1. *([21], Definition 1 from [28], pag. 10)*

Let A be an abelian group (additive). A ***distribution*** *on $\mathcal{X}$ with values in A is a sequence of functions $\mu = (\mu_n)_{n\geqslant 1}$, $\mu_n : \mathcal{X}_n \longrightarrow A$ which verifies the compatibility relation:*

$$\mu_n(x) = \sum_{y\in\varphi_n^{-1}(x)} \mu_{n+1}(y),\ \textit{for any } n \geqslant 1 \textit{ and } x \in \mathcal{X}_n. \tag{1}$$

Let $\Omega(\mathcal{X})$ be the set of open and compact subsets of $\mathcal{X}$. Every $D \in \Omega(\mathcal{X})$ can be written as a finite reunion of disjoint balls $D = \bigcup_{i=1}^{m} B_i$. We extend μ pe $\mathcal{X}$ by additivity: $\mu(D) = \sum_{i=1}^{n} \mu(B_i)$. μ is finitely additive, which means that for any $D_i \in \Omega(\mathcal{X}),\ i = \overline{1,n}$, two by two disjoint, if $D = \bigcup_{i=1}^{n} D_i$, then $\mu(D) = \sum_{i=1}^{n} \mu(D_i)$. Conversely, assumming that we have defined $\mu : \Omega(\mathcal{X}) \longrightarrow A$, finitely additive, denoting $\mu_n(x) = \mu(B)$, then $\mu = (\mu_n)_{n\geq 1}$ represents a distribution on $\mathcal{X}$. More generally, we can define the notion of distribution on a compact set $\mathcal{X} \subset \mathbb{C}_p$ without the projective limit, by finite additivity. **The norm** of a distribution μ is defined by: $\|\mu\| = \sup_{D\in\Omega(\mathcal{X})} \|\mu(D)\|$. If $\|\mu\| < \infty$ then we say that μ is a **measure** on $\mathcal{X}$.

The Haar distribution on the orbit of an element in $\mathbb{C}_p$ ([28], pag. 13)

Let $\mathbb{Q}_p \subset K \subset \mathbb{C}_p$ be a complete field and $G = Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ **the Galois absolute group** endowed with the Krull topology, which is canonically isomorphic with $Gal_{cont}(\mathbb{C}_p/\mathbb{Q}_p)$, the group of continuos automorphisms of $\mathbb{C}_p$ over $\mathbb{Q}_p$ (see [9]). We denote

$G_K = \{\sigma \in G|\sigma(x) = x,\ \text{for } x \in K\}$. For every closed subgroup H of G, we denote $FixH = \{x \in \mathbb{C}_p|\ \sigma(x) = x,\ \text{for any } x \in H\}$. $FixH$ is a closed subfield of $\mathbb{C}_p$. Also, for $x \in \mathbb{C}_p$ and $\varepsilon > 0$ let $H(x) = \{\sigma \in G|\ \sigma(x) = x\}$ and $H(x,\varepsilon) = \{\sigma \in G|\ |\sigma(x) - x|_p < \varepsilon\}$. Then, $H(x)$ is a subgroup of G and $FixH(x) = \widetilde{\mathbb{Q}_p[x]}$. **The orbit** of an element $T \in \mathbb{C}_p$ in relation with G_K is $O_K(T) = \{\sigma(T)|\ \sigma \in G_K\}$. In the case when $K = \mathbb{C}_p$, we will simply denote $O_{\mathbb{C}_p}(T) = O(T)$. For every $T \in \mathbb{C}_p$, $O_K(T)$ is a compact, equilibrated and ultrametric space on which we can define the Haar distribution:

$$\pi_{T,K}(B(a,\varepsilon)) = \begin{cases} \dfrac{1}{N(T,K,\varepsilon)}, \text{if } B(a,\varepsilon) \cap O_K(T) \neq \phi \\ 0,\ \text{else} \end{cases} . \tag{2}$$

where $N(T,K,\varepsilon)$ is the number of balls which cover the orbit. This distribution becomes a measure when T is p-bounded element (see Definition 5).

Definition 2. *([28]) Let $\mathcal{X} \subset \mathbb{C}_p$, be a compact set and $s > 0$. We say that μ is a distribution of type s on $\mathcal{X}$ or, more simple, a* ***s-distribution*** *if:* $\lim_{n\to\infty} \varepsilon^s \sup_{a\in\mathcal{X}} |\mu(B^*(a,\varepsilon))|_p = 0$, *where the supremum is taken after all the balls $B^*(a,\varepsilon) = B(a,\varepsilon) \cap \mathcal{X}$.*

A 1-distribution is called a ***Lipschitz distribution****.*

Definition 3. *([28]) Let $\mathcal{X} \subset \mathbb{C}_p$, be a compact set and $r > 0$. We say that a function $f : \mathcal{X} \to \mathbb{C}_p$ is Lipschitz of type r or, more simple, **r-Lipschitz**, if there exists $c > 0$ such that:*

$$|f(x) - f(y)|_p \leqslant c|x - y|_p^r. \tag{3}$$

*A 1-Lipschitz function is called, more simple, **Lipschitz**.*

Definition 4. *([2]) An **element** $x \in \mathbb{C}_p$ is called **Lipschitz** if and only if $\lim\limits_{\varepsilon\to 0} \frac{\varepsilon}{|N(x,\varepsilon)|_p} = 0$, where $N(x, \varepsilon)$ is the number of open balls of radius ε which cover $O(x)$.*

Definition 5. *([2]) An element $x \in \mathbb{C}_p$ is called p**-bounded** if there exists $s \in \mathbb{N}$ such that p^s does not divide $N(x, \varepsilon)$, for any $\varepsilon > 0$. In this situation, π_x is a measure.*

A subset $\mathcal{X}$ of $\mathbb{C}_p$ is called equivariant in relation with the p-adic Galois absolute group G or G-**quivariant**, if $\sigma(x) \in \mathcal{X}$ for any $x \in \mathcal{X}$ and $\sigma \in G$. The orbit $O(x)$ is such an example.

Definition 6. *([1, 3, 7]) Let $\mathcal{X}$ be a compact and G-equivariant subset of $\mathbb{C}_p$ and μ a distribution on $\mathcal{X}$ with values in $\mathbb{C}_p$. We say that μ is G**-equivariant** if $\mu(\sigma(B)) = \sigma(\mu(B))$, for any ball B in $\mathcal{X}$ and any $\sigma \in G$.*

Definition 7. *([6]) Let $\mathcal{X}$ be a compact subset of $\mathbb{C}_p$. We say that $\mathcal{X}$ is a **fundamental set** if $\{|x - y|_p \; ; \; x, y \in \mathcal{X}, \; x \neq y\}$ is a sequence $(\varepsilon_n)_{n\geq 1}$ which strictly decreases to 0 . This sequence is called the **fundamental sequence** associated to $\mathcal{X}$.*

Simple examples. $\mathbb{Z}_p$, $\mathbb{Z}_p^\times$ and $O(x)$ are fundamental sets.

Definition 8. *([6]) Let $\mathcal{X}$ be a fundamental set of $\mathbb{C}_p$ and $(\varepsilon_n)_{n\geqslant 1}$ the fundamental sequence associated to $\mathcal{X}$. A Lipschitz **distribution** μ defined on $\mathcal{X}$ to $\mathbb{C}_p$ is called **strongly Lipschitz** if it verifies the following condition: there exists $N(\mu) \in \mathbb{N}$ such that*

$$\left(\varepsilon_n \max_{x\in\mathcal{X}} |\mu(B^*(x, \varepsilon_n))|_p\right)_{n\geqslant N(\mu)} \tag{4}$$

is strictly decreasing to 0.

2.2 On the strongly Lipschitz distributions

In the thesis we present some examples of strongly Lipschitz distributions and an original way to construct new ones. We search for an unbounded distribution $\mu : \mathbb{Z}_p \to \mathbb{C}_p$. Denote $\mu(a + p^n\mathbb{Z}_p) = \alpha_n^{(a)}$, $0 \leqslant a < p^n$. From the compatibility relation we have $\mu(a + p^n\mathbb{Z}_p) = \sum\limits_{b=0}^{p-1} \mu(a + bp^n + p^{n+1}\mathbb{Z}_p)$, $n \geqslant 0$, and we obtain $\alpha_n^{(a)} = \sum\limits_{b=0}^{p-1} \alpha_{n+1}^{(a+bp^n)}$. The following result offers us the possibility to construct new classes of "unbounded distributions" which are strongly Lipschitz.

Lemma 1. *Let $\alpha \in \mathbb{C}_p \setminus \{0\}$, and $t \in p^{\mathbb{Q}}$, $t > 1$. Then there exist $\beta_1, \beta_2, ..., \beta_p \in \mathbb{C}_p$ such that $\alpha = \sum\limits_{i=1}^{p} \beta_i$ and $\max\limits_{i=\overline{1,p}} |\beta_i|_p = t|\alpha|_p$.*

Proposition 1. *On $\mathbb{Z}_p$ there exist strongly Lipschitz distributions which are unbounded.*

Proposition 2. *([6]) Let K be an infinite normal algebraic extention of $\mathbb{Q}_p$. There exists a generic element x of $\tilde{K}$ (where $\tilde{K} = \widetilde{\mathbb{Q}_p[x]}$, [4]) such that the Haar distribution π_x is strongly Lipschitz.*

2.3 The Riemann integral against the p-adic distributions

Definition 9. *([28]) Let $\mathcal{X} = \lim\limits_{\leftarrow} \mathcal{X}_n$. If $B = \theta_n^{-1}(x)$ is a ball in $\mathcal{X}$, we say that x is the **centre** of B and we write B=B(x). A **partition** of $\mathcal{X}$ is a finite set of disjoint balls $\Delta = \{B_1, B_2, ..., B_n\}$ with $\bigcup\limits_{i=1}^{n} B_i = \mathcal{X}$. A **system of intermediate points** is a function $\xi : \bigcup\limits_{n\geqslant 1} \mathcal{X}_n \to \mathcal{X}$ with the property that for any $x \in \mathcal{X}_n$ we have $\xi(x) \in \theta_n^{-1}(x)$. For simplicity, we will denote $\xi(x_i) = \xi_i$ and $B_i = B(x_i)$.*

Definition 10. *Let A be a complete K-vectorial space in relation to a nonarchimedian norm $\|\cdot\|$, $\mu \in \mathcal{D}(\mathcal{X}, K)$, Δ a partition of $\mathcal{X}$, ξ a system of intermediate points and $f : \mathcal{X} \to A$. The **Riemann sum** associated to f, Δ, ξ and μ is: $S(f, \Delta, \xi, \mu) = \sum\limits_{i=1}^{n} \mu(B_i) f(\xi_i)$.*

Definition 11. *Let $(\varepsilon_n)_{n\geqslant 1}$ a strictly descreasing sequence of real numbers tending to 0. We have previously defined the distance which gives the topology on $\mathcal{X}$. For $B = B(x)$ we denote by $\|B\| = \varepsilon_n$ if $x \in \mathcal{X}_n$. The **Norm of a partition** Δ is $\|\Delta\| = \sup\limits_{i=\overline{1,n}} \|B_i\|$.*

Definition 12. *We say that a function $f : \mathcal{X} \to A$ is integrable with respect to a distribution μ, if there exists $I \in A$ with the property that for any $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that for any partition Δ cu $\|\Delta\| < \delta_\varepsilon$ and any system of intermediate points we have*

$$\|S(f, \Delta, \xi, \mu) - I\| < \varepsilon. \tag{2.32}$$

In the case when I exists, it is called the integral of f with respect to μ and is denoted by $\int\limits_{\mathcal{X}} f d\mu$.

Theorem 1. *([28]) Let $\mathcal{X} \subset \mathbb{C}_p$, be compact and $r \geqslant s > 0$. Then, any r-Lipschitz function is Riemann integrabile with respect to any s-distribution.*

The third chapter is about Krasner analytical functions. In the first section we study some properties of the rational functions in the p-adic context.

3.2 Analytical elements

Definition 13. *([24]) Let $D \subset \mathbb{C}_p$ be a closed set. A function $f : D \to \mathbb{C}_p$ is named* ***Krasner analytical*** *(**rigid analytical** or **analytical element**) if there exists a sequence of rational functions $(f_n) \subset R(D)$ which converges uniformly to f on D with respect to the sup norm.*

We denote $H(D) = \{f : D \longrightarrow \mathbb{C}_p \mid f \text{ is Krasner analytical on } D\}$.

As in [2], a Krasner analytical function definited on G-equivariant set $\mathcal{X}$ of $\mathbb{C}_p$ is named **equivariant** if $f(\sigma(x)) = \sigma(f(x))$, for any $x \in \mathcal{X}$ and $\sigma \in G$. For $\mathcal{X} \subset \mathbb{C}_p$, G-equivariant, let $H^G(\mathbb{P} \setminus \mathcal{X})$ be the subset of Krasner analytical functions defined from $\mathbb{P} \setminus \mathcal{X}$ to $\mathbb{C}_p$, and $H_0^G(\mathbb{P} \setminus \mathcal{X})$ its subset that containes the functions whose limit to ∞ is 0. Here $\mathbb{P} = \mathbb{P}^1(\mathbb{C}_p) = \mathbb{C}_p \cup \{\infty\}$.

Theorem 2. *(**Mittag-Leffler**, [24]) Let $D \subset \mathbb{C}_p$ a closed bounded and infraconnected subset, $(B_i)_{i \in I}$ its family of holes şi B_i^C the complementary of B_i, $i \in I$. Then, there exists the following descomposition as a direct sum of Banach spaces*

$$H(D) \xrightarrow{\sim} H(B_D) \widehat{\bigoplus_{i \in I}} H_0\left(B_i^C\right), \tag{5}$$

which means that every $f \in H(D)$ can be uniquely written in the form

$$f = f_0 + \sum_{i \in I} f_i, \text{ with } \|f\|_D = \max\left(\|f_0\|, \sup_{i \in I}\|f_i\|\right), \tag{6}$$

$f_0 \in H(B_D)$, $f_i \in H_0\left(B_i^C\right)$ *and* $\|f_i\| = \|f_i\|_{B_i^C} = \|f_i\|_D \xrightarrow[i\to\infty]{} 0.$

3.3 The Shnirelman integral

In this section is introduced the Shnirelman integral, a p-adic analogue of the line integral, a tool we can use to prove the p-adic analogues of some classical theorems of complex analysis: Cauchy's representation formula, residue theorem, maximum modulus principle. At the same time, this integral has applications in the theory of transcendental numbers. Also, Vishik's theorem is presented with its Galois equivariant form.

Definition 14. *([19]) Consider $f : S(a,r) \to \mathbb{C}_p$ and $\Gamma \in \mathbb{C}_p$, $|\Gamma|_p = r$. The Shnirelman integral is defined by the following limit (if it exists!):*

$$\int_{a,\Gamma} f(x)dx \overset{def}{=} \lim_{n\to\infty}{}' \frac{1}{n} \sum_{\xi^n=1} f(a+\xi\Gamma), \tag{7}$$

where the apostroph signifies that the limit is calculated only for the values of n for which $p \not| n$.

In what follows, we will present Vishik's theorem and some extentions of it.

Notations:

$B(\mathcal{X}, r) = \bigcup\limits_{a\in\mathcal{X}} B(a,r)$, $B[\mathcal{X}, r] = \bigcup\limits_{a\in\mathcal{X}} B[a,r]$.

For $\phi \in H_0\left(\mathcal{X}^C\right)$ we define $\|\phi\|_r = \max\limits_{x\in B(\mathcal{X},r)^C} |\phi(z)|_p$. It can be shown that $\|\phi\|_r = \max\limits_{dist(z,\mathcal{X})=r} |\phi(z)|_p$.

A topology on $H_0\left(\mathcal{X}^C\right)$ is given by the neighborhoods of 0: $U(r,\varepsilon) = \left\{\phi \in H_0(\mathcal{X}^C);\ \|\phi\|_r < \varepsilon\right\}$.

$H(\mathcal{X}, r) = \{f : B(\mathcal{X}, r) \to \mathbb{C}_p |\ f$ is Krasner analitical on any B$(a_i, r) \subset B(\mathcal{X}, r)\}$.

$L(\mathcal{X}) = \bigcup\limits_{r>0} H(\mathcal{X}, r)$ (the set of local analytical functions on $\mathcal{X}$).

$L^*(\mathcal{X})$ (the dual space) represents the set of all the continuous linear functionals μ defined on $L(\mathcal{X})$ with the property that for any $r > 0$, $\|\mu\|_r \overset{def}{=} \max\limits_{0\neq f\in H(\mathcal{X},r)} \frac{|\mu(f)|_p}{\|f\|_r}$ is finite.

Definition 15. *For $\mu \in L^*(\mathcal{X})$ we define the* ***Stieltjes transform***

$$S\mu : \mathcal{X}^C \to \mathbb{C}_p, \quad z \to \mu(f_z) \overset{not}{=} (\mu(x), f_z(x)), \tag{8}$$

where $x \in \mathcal{X}$, $f_z : \mathcal{X} \to \mathbb{C}_p$, $f_z(x) = \frac{1}{z-x}$.

Remark 1. *If μ comes from a measure on pe $\mathcal{X}$, then* $S\mu(z) = \int\limits_{\mathcal{X}} \frac{d\mu(x)}{z-x}$.

Definition 16. *For $\phi \in H_0\left(\mathcal{X}^C\right)$ we call the **Vishik transform** the functional $V\phi$ on $L(\mathcal{X})$*

$$f \to \sum_i \int\limits_{a_i,\Gamma} \phi(x)f(x)(x-a_i)dx, \quad f \in H(\mathcal{X}, r), \tag{9}$$

where $\Gamma,\ a_i \in \mathbb{C}_p,\ |\Gamma|_p = r > 0.$

Theorem 3. *(**Vishik**, [30]) V and S are topologically inverse to one another $H_0\left(\mathcal{X}^C\right)$ and $L^*(\mathcal{X})$. In this way, the subspace $M(\mathcal{X}) \subset L^*(\mathcal{X})$ of the measures on $\mathcal{X}$ is in bijective correspundence with the set* $\left\{\phi \in H_0\left(\mathcal{X}^C\right) \mid r\|\phi\|_r \text{ is bounded when } r \to 0\right\}$.

We now present an original extension of Theorem 3 (see [30]). Let σ be a continous automorphism of $\mathbb{C}_p$. There exists a canonical isomorphism, denoted in the same way: $\sigma : L(\mathcal{X}) \to L(\sigma\mathcal{X})$, where $\sigma\mathcal{X} = \sigma(\mathcal{X})$. $G = Gal_{cont}(\mathbb{C}_p/\mathbb{Q}_p)$ acts on $L(\mathcal{X})$ in the following way: $G \times L(\mathcal{X}) \to L(\mathcal{X})$, $(\sigma, f) \to \sigma * f$ where $(\sigma * f)(u) \overset{def}{=} \sigma f\left(\sigma^{-1}u\right)$, and $\sigma^{-1}u = \sigma^{-1}(u)$. Also, we define the dual isomorfismul denoted in the same way: $\sigma : L^*(\mathcal{X}) \to L^*(\mathcal{X}^\sigma)$. Thus, for any $\mu \in L^*(\mathcal{X})$ and $f \in L(\mathcal{X})$ we have the equality $(\sigma\mu, \sigma * f) = \sigma(\mu, f)$. Obviously, σ acts naturally between $H_0(\mathcal{X}^C)$ and $H_0(\sigma\mathcal{X}^C)$, that is for any $\varphi \in H_0(\mathcal{X}^C)$ one has: $\sigma * \varphi(z) = \sigma\varphi\left(\sigma^{-1}z\right)$. The following diagram is commutative.

$$\begin{array}{ccc} L^*(\mathcal{X}) & \overset{\sigma}{\to} & L^*(\sigma\mathcal{X}) \\ S\downarrow & & \downarrow S \\ H_0(\mathcal{X}^C) & \overset{\sigma}{\to} & H_0\left(\sigma X^C\right) \end{array} \tag{10}$$

3.4 The Galois equivariant form of the theorem of Vishik

In this section we present some original results from the article [23].

Let $\mathcal{X} \subset \mathbb{C}_p$ be compact and G-equivariant and $L(\mathcal{X}) = \bigcup\limits_{n\geqslant 0} H(\mathcal{X}, r)$.

Proposition 3. *Consider $\tau \in G = Gal_{cont}(\mathbb{C}_p/\mathbb{Q}_p)$. If f is Schnirelman integrable, then*

$$\tau\left(\int\limits_{a,\Gamma} f(x)dx\right) = \int\limits_{\tau a,\tau\Gamma} (\tau * f)(x)dx. \tag{11}$$

Consider $H_0^G(\mathcal{X}^C) \subseteq H_0(\mathcal{X}^C)$ the subspace of the analitical G-equivariant functions (i.e. which satisfy the condition $\tau\phi(x) = \phi(\tau x),\ \phi \in H_0(\mathcal{X}^C)$). Let $L_G^*(\mathcal{X}) \subseteq L^*(\mathcal{X})$ be the subspace of the functionals $\mu \in L^*(\mathcal{X})$ which verify the equality $\tau(\mu(f)) = \mu(\tau * f),$ for any $\tau \in G$ and $f \in L(\mathcal{X})$ (i.e. the subspace of the G-equivariant functionals).

Theorem 4. *There exists the isomorfism of topological spaces $H_0^G\left(\mathcal{X}^C\right) \simeq L_G^*(\mathcal{X})$.*

Also, we prove that $\tau S\mu(z) = S\mu(\tau z),\ \tau \in G$.

In the forth chapter we study a special class of Krasner analytical functions, in particular the trace functions, which have applications in obtaining original transcendence results for certain functions (for example Diamond's p-adic log Gamma Function).

4.1 The trace of an element

Every element $\alpha \in \overline{\mathbb{Q}}_p$ is p-bounded, so the distribution π_α is a measure. Moreover, for any function $f : O(\alpha) \to \mathbb{C}_p$ we have: $\int\limits_{O(\alpha)} f d\pi_\alpha = \frac{1}{deg(\alpha)} \sum\limits_\sigma f(\sigma(\alpha))$.

Definition 17. *([2]) Consider $\alpha \in \overline{\mathbb{Q}}_p$. The **trace** of α is:*

$$Tr(\alpha) = \frac{1}{deg(\alpha)} tr_{\mathbb{Q}_p(\alpha)/\mathbb{Q}_p} = \int\limits_{O(\alpha)} x d\pi_\alpha(x) \tag{12}$$

More general, the trace of an element $T \in \mathbb{C}_p$ (if it exists!) is given by

$$Tr(T) = \int\limits_{O(T)} x d\pi_T(x) \tag{13}$$

4.2 An estimation of the norm

Let $\mathcal{X}$ be a fundamental subset $\mathbb{C}_p$ and μ a strongly Lipschitz distribution defined on $\mathcal{X}$. Consider the Cauchy transfom:

$$F_\mu(z) = \int\limits_{\mathcal{X}} \frac{1}{z-t} \mathrm{d}\mu(t) \in H(\mathbb{P} \setminus \mathcal{X}). \tag{14}$$

For $\mathcal{X} = O(x)$, with $x \in \mathbb{C}_p$ and $\mu = \pi_x$ we have that F_μ is trace function of x associated to π_x, see [2] and [29]. For any $F \in H(\mathbb{P} \setminus \mathcal{X})$, $\varepsilon > 0$ and $\mathcal{X}(\varepsilon) = \{y \in \mathbb{C}_p \mid \text{there exists } t \in$

$\mathcal{X}$ such that $|y-t|_p < \varepsilon\}$ a ε-**neighborhood** of $\mathcal{X}$ we denote $\|F\|_{\mathbb{P}\setminus\mathcal{X}(\varepsilon)}$ the sup norm of F on $\mathbb{P}\setminus\mathcal{X}(\varepsilon)$. Let $(\varepsilon_n)_{n\geqslant 1}$ be the fundamental sequence assciated to $\mathcal{X}$. For $n \geqslant 1$, let $N(\varepsilon_n)$ be the number of open balls of radius ε_n which cover $\mathcal{X}$. Also, we consider $a_i^{(n)}$, $1 \leqslant i \leqslant N(\varepsilon_n)$, a convenient choice for the centers of theese balls.

Theorem 5. *Let $\mathcal{X}$ be a fundamental set of lui $\mathbb{C}_p$ and μ a strongly Lipschitz distribution defined on $\mathcal{X}$. Consider $(\varepsilon_n)_{n\geqslant 1}$ the fundamental sequence associated to $\mathcal{X}$ and $\mathcal{X}(\varepsilon_n)$ the open ε_n-neighborhood of $\mathcal{X}$ in $\mathbb{C}_p$. Then, there exists $N(\mu) \in \mathbb{N}$, which only depends on μ, such that for any $n \geq N(\mu)$,*

$$\|F_\mu\|_{\mathbb{P}\setminus\mathcal{X}(\varepsilon_n)} = \frac{1}{\varepsilon_n} \cdot \max_{x\in\mathcal{X}} |\mu(B^*(x,\varepsilon_n))|_p \ . \tag{15}$$

Remark 2. *Let k be a fixed natural number. In the same hypotheses as in the previous theorem, by integrating $\frac{1}{(z-t)^k}$ instead of $\frac{1}{z-t}$, the main result from Theorem 5 remains unchanged, beside the fact that $\frac{1}{z-t}$ becomes $\frac{1}{(z-t)^k}$ in the left side, and $\frac{1}{\varepsilon_n}$ which becomes $\frac{1}{\varepsilon_n^k}$ in the right side. A particular case of Theorem 5 este studied in [5].*

Next we will see that a large class of functions that are Cauchy transforms by integration with respect to strongly Lipschitz distributions defined on $\mathcal{X}$ are transcendental over $\mathbb{C}_p(Z)$ and consequently we will obtain transcendence results regarding the the twisted p-adic log gamma (respectively regularized) and regarding the trace functions.

Proposition 4. *Let $\mathcal{X} \subset \mathbb{C}_p$ be a compact subset and $f : \mathbb{P}\setminus\mathcal{X} \to \mathbb{C}_p$ a function with the property that there exists an infinite subset $\mathcal{S}$ of $\mathcal{X}$ such that $\limsup_{z\to x} |f(z)|_p = \infty$, for any $x \in \mathcal{S}$. Then f is transcendental over $\mathbb{C}_p(Z)$.*

Remark 3. *The Proposition 4 is a more refined version the first part of Theorem 6 from [26].*

Lemma 2. *Let $\mathcal{X}$ be a compact subset of $\mathbb{C}_p$ with no isolated points and $f : \mathbb{C}_p\setminus\mathcal{X} \to \mathbb{C}_p$ is a local analytical function and algebraic on $\mathbb{C}_p(Z)$. Then, f', which is defined on $\mathbb{C}_p\setminus\mathcal{X}$, except, possibly, a discrete set, is algebraic over $\mathbb{C}_p(Z)$.*

Corollary 1. *Let $\mathcal{X}$ be a compact compact subset of $\mathbb{C}_p$ with no isolated points, $f : \mathbb{C}_p\setminus\mathcal{X} \to \mathbb{C}_p$ local analytical function such that its derivative is Krasner analytical over $\mathbb{C}_p\setminus\mathcal{X}$ and transcendental over $\mathbb{C}_p(Z)$. Then f is transcendental over $\mathbb{C}_p(Z)$.*

Lemma 3. *Let $\mathcal{X}$ be a compact subset of $\mathbb{C}_p$ with no isolated points and $f : \mathbb{C}_p \setminus \mathcal{X} \to \mathbb{C}_p$ a Krasner analytical function. If there exists a ball $B(\alpha, \varepsilon) \subset \mathbb{C}_p \setminus \mathcal{X}$ such that $f|_{B(\alpha,\varepsilon)} : B(\alpha, \varepsilon) \to \mathbb{C}_p$ is algebraic over $\mathbb{C}_p(Z)$, then f is algebraic over $\mathbb{C}_p(Z)$.*

4.3 Aplications to transcendence results

Let $\mathcal{X}$ be a fundamental subset of $\mathbb{C}_p$ and μ a strongly Lipschitz distributions defined on $\mathcal{X}$ to $\mathbb{C}_p$. Consider $(\varepsilon_n)_{n\geqslant 1}$ the fundamental sequence associated to $\mathcal{X}$. It can easily be observed that the sequence $\left(\max\limits_{x\in\mathcal{X}} |\mu(B^*(x, \varepsilon_n))|_p\right)_{n\geq 1}$ is ascending, not necessarily bounded. For every $k \geq 1$, we denote $F_{k,\mu}(z) = \int\limits_{\mathcal{X}} \frac{1}{(z-t)^k} \mathrm{d}\mu(t)$. From Remark 2 we obtain

$$\|F_{k,\mu}\|_{\mathbb{P}\setminus\mathcal{X}(\varepsilon_n)} = \frac{1}{\varepsilon_n^k} \cdot \max_{x\in\mathcal{X}} |\mu(B^*(x, \varepsilon_n))|_p \geqslant \frac{1}{\varepsilon_n^k} \cdot \max_{x\in\mathcal{X}} |\mu(B^*(x, \varepsilon_{N(\mu)}))|_p \ , \tag{16}$$

for any $n \geq N(\mu)$. For $n \to \infty$ in (16) we get $\lim\limits_{n\to\infty} \|F_{k,\mu}\|_{\mathbb{P}\setminus\mathcal{X}(\varepsilon_n)} = \infty$. Then, there exist two sequences $(z_n)_{n\geqslant 1}$ in $\mathbb{P} \setminus \mathcal{X}$ and $(x_n)_{n\geqslant 1}$ in $\mathcal{X}$ such that $\operatorname{dist}(z_n, \mathcal{X}) = |z_n - x_n|_p \to 0$ and $\lim\limits_{n\to\infty} |f(z_n)|_p = \infty$. Because $\mathcal{X}$ is sequentially compact, there exists a subsequence $(x_{n_m})_{m\geqslant 1}$ of $(x_n)_{n\geqslant 1}$ which converges to $x \in \mathcal{X}$. The subsequence $(z_{n_m})_{m\geqslant 1}$ converges to x and $\lim\limits_{m\to\infty} |f(z_{n_m})|_p = \infty$. Obviously, x is a singular point of $F_{k,\mu}$ like in Proposition 4. If $F_{k,\mu}$ verifies a "functional equation" on $\mathcal{X}$ and certain "properties" (for example, a compact subgrup or a Galois echivariant and compact subset of $\mathbb{C}_p$) it's easy to see that $F_{k,\mu}$ has a finite number of singular points like in $\mathcal{X}$, so from Theorem 5 and Proposition 4 it is transcendental over $\mathbb{C}_p(Z)$. A good exemple for this situation is the trace function of a strongly Lipschitz distribution. This function is transcendental over $\mathbb{Q}_p(Z)$ and, moreover, its derivatives are linearly independent over $\mathbb{Q}_p(Z)$. In particular, the trace function cannot satisfy a differential equation over $\mathbb{Q}_p(Z)$, see [5]. Subsequently, we present an interesting example for this case. In 1977, Diamond introduced the p-adic log gamma function, the analogue of the classical gamma function $\frac{\log\Gamma(x)}{\sqrt{2\pi}}$, which is defined by the equality:

$$G_p(z) = \lim_{n\to\infty} \frac{1}{p^n} \sum_{0\leq i<p^n} (z+i)(\log_p(z+i) - 1), \tag{17}$$

which makes sense for $z \in \mathbb{C}_p \setminus \mathbb{Z}_p$, see[10], where $\log_p$ is the Iwasawa logaritm ([12]). Koblitz [16] introduced a twisted version of the log gamma function to give a simple proof to Leopold's formula for $L_p(1, \chi)$, where $L_p(s, \chi)$ is the p-adic L-function of the character χ

and the formulas expressing $L_p'(0,\chi)$ and $L_p(k,\chi)$, $k \geq 1$, in terms of the p-adic log gamma function. The twisted p-adic log gamma function is defined by the formula:

$$G_{p,\xi}(z) = \lim_{n\to\infty} \frac{1}{rp^n} \sum_{0\leq i<rp^n} \xi^i (z+i)(\log_p(z+i)-1), \tag{18}$$

where $\xi^r = 1$, r is the order of ξ which is prime with p and $z \in \mathbb{C}_p \setminus \mathbb{Z}_p$. Particularly, $G_{p,1} = G_p$. The twisted p-adic log gamma function is the convolution of $-\log_p$ with μ_ξ (Koblitz's distribution with $d = 1$, $z = \xi \neq 1$):

$$G_{p,\xi}(z) = -\int_{\mathbb{Z}_p} \log_p(z+t)\mathrm{d}\mu_\xi(t),\ z \in \mathbb{C}_p \setminus \mathbb{Z}_p, \tag{19}$$

which is locally analytical and verifies the functional equation

$$\xi G_{p,\xi}(z+1) - G_{p,\xi}(z) = \log_p z,\ z \in \mathbb{C}_p \setminus \mathbb{Z}_p. \tag{20}$$

The derivative of order $k \geqslant 1$ of $G_{p,\xi}$ is

$$G_{p,\xi}^{(k)}(z) = (-1)^k (k-1)! \int_{\mathbb{Z}_p} \frac{1}{(z+t)^k} \mathrm{d}\mu_\xi(t) \in H(\mathbb{C}_p \setminus \mathbb{Z}_p). \tag{21}$$

Theorem 6. *The twisted p-adic log gamma function $G_{p,\xi}$ and its derivatives of any order are transcendental over $\mathbb{C}_p(Z)$. Furthermore, all their zeros are algebraic.*

Theorem 7. *The twisted p-adic log gamma function $G_{p,\xi}$ and its derivatives are linearly independent over $\mathbb{C}_p(Z)$. In particular, $G_{p,\xi}$ cannot be the solution of a differential equation of the form $\sum_{k=0}^{m} P_k G_{p,\xi}^{(k)} = 0$, where for any $0 \leqslant k \leqslant m$ one has $P_k \in \mathbb{C}_p(Z)$, not all equal to 0, and $m \in \mathbb{N}$.*

Proposition 5. *Let μ be a strongly Lipschitz distribution defined on the orbit of a transcendent element $x \in \mathbb{C}_p$. Then, for any $s \in \mathbb{N}^*$*

$$F_{s,\mu}(z) = \int_{O(x)} \frac{1}{(z-t)^s} d\mu(t) \in H_0^G(\mathbb{P}^1(\mathbb{C}_p) \setminus O(x)) \tag{22}$$

and is transcendental over $\mathbb{Q}_p(\mathbb{Z})$.

Proposition 6. *Consider* $k \in \mathbb{N}^*$. *For any natural numbers* $s_1 < s_2 < ... < s_k$ *let* $\mu_1, \mu_2, ..., \mu_k$ *be strongly Lipschitz distributions on the orbit of a transcendental element* $x \in \mathbb{C}_p$. *Then, the functions*

$$F_{s_i,\mu_i}(z) = \int\limits_{O(x)} \frac{1}{(z-t)^{s_i}} d\mu_{s_i}(t), \ i = \overline{1,k} \tag{23}$$

are linearly independent over $\mathbb{Q}_p(\mathbb{Z})$. *In particular, it results that no function* $F_{s,\mu}$ *can verify a differential equation of the form* $\sum\limits_{j=0}^{m} P_j F_{s,\mu}(j) = 0$, *where* $m \in \mathbb{N}^*$ *and* $P_j \in \mathbb{Q}_p(\mathbb{Z}), \ j = \overline{0,m}$, *are not equal to 0.*

Proposition 7. *Let* $(s_i)_{i\geqslant 1}$ *be a strict increasing sequence of positive integers and* $\mu_{s_i}, \ i \geqslant 1$, *strongly Lipschitz distributions defined on the orbit of a transcendent element* $x \in \mathbb{C}_p$. *If a function* $G : \mathbb{P}^1(\mathbb{C}_p) \setminus O(x) \to \mathbb{C}_p$ *can be written in the form of a series* $G(z) = \sum\limits_{i=1}^{\infty} P_i(z) F_i(z)$, *that converges on* $E(x, \varepsilon_n)$, *where* $(\varepsilon_n)_{n\geqslant 1}$ *is the fundamental sequence associated to the orbit* $O(x)$, $P_i(z) \in \mathbb{Q}_p(z)$ *and* $F_i(z) = F_{s_i,\mu_{s_i}}(z) = \int\limits_{O(x)} \frac{1}{(z-t)^{s_i}} d\mu_{s_i}(t), \ i = \overline{1,k}$, *then for* n *sufficiently large one has* $\|G\|_{E(x,\varepsilon_n)} = \sup\limits_{i\geqslant 1} \|P_i F_i\|_{E(x,\varepsilon_n)}$ *and the representation is unique.*

References

[1] V. Alexandru, N. Popescu, A. Zaharescu, *On the closed subfields of* $\mathbb{C}_p$, J. Number Theory 68, 2 (1998), p. 131-150.

[2] V. Alexandru, N. Popescu, A. Zaharescu, *Trace on* $\mathbb{C}_p$, J. Number Theory 88, 1 (2001), p. 13-48.

[3] V. Alexandru, E.L. Popescu, N. Popescu, *On the continuity of the trace*, Proceedings of the Romanian Academy, Series A, vol. bf 5, nr. 1 (2005), p. 11-16.

[4] V. Alexandru, N. Popescu, M. Vâjâitu and A. Zaharescu,*The p-adic measure on the orbit of an element of* $\mathbb{C}_p$, Rend. Sem. Mat. Univ. Padova, Vol. 118 (2007), p. 197-216.

[5] V. Alexandru, C.C. Niţu and M. Vâjâitu, *On the norm of the trace functions and applications*, Bull. Math. Soc. Sci. Math. Roumanie Tome 56(104) No. 1 (2013), p. 47-54.

[6] V. Alexandru, C.C. Niţu and M. Vâjâitu and A.Zaharescu *On the norm of Krasner analytic functions with applications to transcendence results*, Journal of Pure and Applied Algebra, vol. 219 (2015), p. 4607 - 4618.

[7] V. Alexandru, N. Popescu, M. Vâjâitu and A. Zaharescu,*On the zeros of Krasner analytic functions*, Algebr. Represent. Theor., Vol. 16, 3 (2013), p. 895-904.

[8] Y. Amice, *Les nombres p-adiques*, Presse Univ. de France, Collection Sup., 1975.

[9] J. Ax, *Zeros of polynomials over local fields-The Galois action*, J. Algebra 15 (1970),p. 417-428

[10] J. Diamond, *The p-adic log gamma function and p-adic Euler constants*, Trans. Amer. Math. Soc. 233 (1977), p. 321-337.

[11] J. Fresnel, M. van der Put, *Rigid Analytic Geometry and its Applications*, Birkhauser, 2004.

[12] K. Iwasawa, *Lectures on p-Adic L-Functions*, Princeton University Press, 1972.

[13] F. Gouvea, *p-adic Numbers - An introduction*, Springer - Verlag Berlin Heidelberg, 1997.

[14] G. Groza, A. Popescu,*Extinderi de corpuri valuate*, Editura Academiei Române, Bucureşti, 2011.

[15] N. Koblitz, *Interpretation of the p-adic log gamma function and Euler constants using the Bernoulli measure*, Trans. Amer. Math. Soc. bf 242 (1978), p. 261-269.

[16] N. Koblitz, *A new proof of certain formulas for p-adic L-functions*, Duke Math. J. 46, 2 (1979), p. 455-468.

[17] N. Koblitz, *Interpretation of the p-Adic Log Gamma Function and Euler Constants Using the Bernoulli Measure*, Transactions of the American Mathematical Society, 242 (1978), p. 261-269

[18] N. Koblitz, *p-adic Numbers, p-adic Analysis and Zeta - Functions* (2 ed.), Springer, 1984.

[19] N. Koblitz, *p-adic Analysis: A Short Course on Recent Work*, Cambridge University Press, 1980.

[20] T. Kubota and H. Leopold, *Eine p-adische Theorie der Zetawerte.* I, J. Reine Angew Math. 214/215 (1965), p. 328-339.

[21] B. Mazur, P. Swinnerton-Dyer, *Arithmetic of Weil curves*, Invent. Math. 25 (1974), p. 1-61.

[22] R. Murty, *Introduction to p-adic Analytic Number Theory*, American Mathematical Society/ International Press, 2002.

[23] C.C. Niţu, M. Vâjâitu, *On a theorem of Vishik*, in progress.

[24] A. M. Robert, *A course in p-adic analysis*, Springer-Verlag New-York, Inc., 2000.

[25] W.H. Schikhov *Ultrametric calculus. An Introduction to p-adic analysis*, Cambridge University Press, 1984.

[26] M. Vâjâitu, *Integral Representations and the Behavior of Krasner Analytic Functions Around Singular Points*, Algebr. Repres. Theor., Vol. 16, 6 (2013), p. 1611-1620.

[27] M. Vâjâitu, *On a class of Krasner analytic functions and applications*, Bull. Math. Soc. Sci. Math. Roumanie Tome 58(106) No. 4 (2015), p. 475-482.

[28] M. Vâjâitu, A. Zaharescu, *Non-Archimedean Integration and Applications*, The publishing house of the Romanian Academy, 2007.

[29] M. Vâjâitu, A. Zaharescu, *Trace functions and Galois invariant p-adic measures*, Publ. Mat. 50 (2006), p. 43-55.

[30] M.M. Vishik, *Nonarchimedian spectral theory*, Journal of Soviet Mathematics, September 1985, Volume 30, Issue 6, p. 2513-2555.

[31] A. Wiles, *Modular elliptic curves and Fermat's Last Theorem*, Annals of Mathematics, (142) 1995, p. 443-551.