OPERATOR ALGEBRAS:
THE BANACH ALGEBRA APPROACH

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Dedicated to the memory of Professor Béla Szőkefalvi-Nagy

CONTENTS

PREFACE

Operator algebras are, roughly speaking, noncommutative extensions of the uniformly closed algebras of bounded continuous complex functions on topological spaces, similarly as matrix algebras enlarge the diagonal matrix algebras, that is the function algebras on finite sets. Their theory enables to extend many topics from Topology, Measure Theory, Geometry and Analysis to the noncommutative setting, creating powerful tools for the investigation of several problems in Mathematics and Theoretical Physics.

These notes were written, mainly in the late seventies, with the purpose to develop a detailed self-contained introduction to the theory of operator algebras. They are conceived also as reference text, containing all details we could track down in the literature on each treated subject item. In order to make the overview easier, the material included is described by small titles accompanying the number of the section in listing the Contents. The bibliography and the sections of notes contain only references to works directly used in the exposition.

In the first three chapters we present the basic algebraic, order theoretical and topological properties of the C^*-algebras. In the fourth chapter positive linear forms on C^*-algebras and their relationship with the construction of representations of C^*-algebras on Hilbert spaces is treated. We notice that the treatment is given, whenever possible, in a more general setting than just C^*-algebras, often in the framework of Banach algebras or Banach $*$-algebras.

The subject of the fifth and sixth chapters are two kinds of bounded linear maps between C^*-algebras: completely positive linear maps respectively surjective linear isometries. They are complementary in the sense that surjective linear isometries, which are also completely positive, turn out to be $*$-isomorphisms.

In the seventh chapter weak and strong operator topology on the C^*-algebra of all bounded linear operators on a Hilbert space is discussed. In particular, the basic properties of the weak operator closed, non-degenerate $*$-algebras of bounded linear operators on a Hilbert space, called von Neumann algebras, are presented. Subsequently, the eighth chapter is devoted to the space free theory of the von Neumann algebras, that is to the C^*-algebras which are $*$-isomorphic with von Neumann algebras, called W^*-algebras.

Finally, in the ninth chapter several classes of C^*-algebras defined by order completeness properties, like AW^*-algebras, monotone complete C^*-algebras and their countable variants, are discussed. These classes were introduced by trying to describe W^*-algebras algebraically, but they occur also as universal objects.

The original version of these notes was used for about 20 years by the Romanian school of operator algebras in Bucharest. Until 1995 it had a limited distribution, in Romania and abroad, by means of just copying the typed pages of the manuscript. In 1995 it was multiplied in the frame of the series "Monografii Matematice" of the University of Timişoara, but these copies have been shortly out of stock. Several appeals we got from colleagues to publish these notes convinced us to reasonably update and publish them as a book.

The authors would like to express their sincere gratitude to Prof. Béla Szőkefalvi-Nagy for having encouraged them to write this book and for his constant support.

Warm thanks are due to Prof. Dumitru Gaşpar for the distribution of the original version in the frame of the series "Monografii Matematice". Particular thanks are due also to Sanda Strătilă for having carefully and patiently typed the original manuscript.

Chapter 1

SPECTRAL THEORY

1.1. A $*$-*vector space* is a complex vector space V together with a $*$-*operation* (or *involution*) $* : V \to V$ such that

$$(x^*)^* = x, \quad (\lambda x)^* = \overline{\lambda} x^*, \quad (x+y)^* = x^* + y^*; \quad x, y \in V,\ \lambda \in \mathbb{C}.$$

A $*$-*algebra* is a complex algebra A with a $*$-operation such that A is a $*$-vector space and

$$(xy)^* = y^* x^*; \quad x, y \in A.$$

If A is a unital $*$-algebra with unit 1_A, then $(1_A)^* = 1_A$.

A *normed algebra* is a complex algebra A endowed with a norm such that:

$$\|xy\| \leqslant \|x\|\,\|y\|; \quad x, y \in A.$$

If $A \neq \{0\}$ is a unital normed algebra with unit 1_A, then $\|1_A\| = \|1_A 1_A\| \leqslant \|1_A\|^2$, so $\|1_A\| \geqslant 1$. Remark that there exists an equivalent norm $|\|\cdot\||$ on A, for instance

$$\|1_A\|^{-1}\|x\| \leqslant |\|x\|| = \sup\{\|xy\| : y \in A,\ \|y\| \leqslant 1\} \leqslant \|x\|; \quad x \in A,$$

such that $(A, |\|\cdot\||)$ is a normed algebra and $|\|1_A\|| = 1$. For this reason, by *unital normed algebras* we shall always understand normed algebras with a norm one unit element.

A *Banach algebra* is a complete normed algebra.

A normed algebra which is also a $*$-algebra is called *normed $*$-algebra.* If in addition

$$\|x\| = \|x^*\|; \quad x \in A,$$

then A is called an *involutive normed algebra.* The terms *Banach $*$-algebra* and *involutive Banach algebra* are self-explanatory.

A *pre-C^*-algebra* is a normed $*$-algebra A such that

$$\|x^* x\| \geqslant \|x^2\|; \quad x \in A.$$

Since $\|x\|^2 \leqslant \|x^* x\| \leqslant \|x^*\|\,\|x\|$, it follows that $\|x^*\| = \|x\|$, $(x \in A)$, thus every pre-C^*-algebra is an involutive normed algebra and

$$\|x^* x\| = \|x^2\|; \quad x \in A.$$

If $A \neq \{0\}$ is a unital pre-C^*-algebra, then $(1_A)^* = 1_A$ and $\|1_A\| = \|(1_A)^* 1_A\| = \|1_A\|^2$, so $\|1_A\| = 1$.

A C^*-*algebra* is a complete pre-C^*-algebra.

Remark that the completion of a normed algebra (respectively involutive normed algebra, respectively pre-C^*-algebra) is a Banach algebra (respectively involutive Banach algebra, respectively C^*-algebra).

A subset of a $*$-algebra is called a $*$-*subalgebra* if it is closed under both the algebraic operations and the $*$-operation. A norm-closed $*$-subalgebra of a C^*-algebra is called a C^*-*subalgebra*. If A is a C^*-algebra, and $S \subset A$, then $C^*(S)$ denotes the smallest C^*-subalgebra of A containing S.

Let A, B be $*$-algebras. A $*$-*homomorphism* (respectively a $*$-*antihomomorphism*), $\pi : A \to B$ is a linear mapping such that:

$$\pi(x^*) = \pi(x)^*; \quad x \in A,$$
$$\pi(xy) = \pi(x)\pi(y), \quad (\text{respectively } \pi(xy) = \pi(y)\pi(x)); \quad x, y \in A.$$

A bijective $*$-homomorphism (respectively $*$-antihomomorphism) is called a $*$-*isomorphism* (respectively a $*$-*antiisomorphism*).

1.2. C^*-seminorms on $*$-algebras. Let A be a $*$-algebra and p be a seminorm on A. Then the following conditions are equivalent:

(i) $p(xy) \leqslant p(x)p(y)$ and $p(x^*x) \geqslant p(x)^2$ for all $x, y \in A$;

(ii) there is a $*$-homomorphism π of A into some C^*-algebra such that $p(x) = \|\pi(x)\|$.

Indeed, (ii) $\Rightarrow$ (i) is obvious. If p satisfies (i), then the set $N_p = \{x \in A : p(x) = 0\}$ is a two sided ideal of A and is stable under the $*$-operation. The quotient $*$-algebra A/N_p endowed with the norm induced by p becomes a pre-C^*-algebra, the completion $C_p^*(A)$ of A/N_p is a C^*-algebra and the quotient map $\pi_p : A \to C_p^*(A)$ is the desired $*$-homomorphism.

A seminorm (respectively a norm) on A satisfying the above conditions is called a C^*-*seminorm* (respectively C^*-*norm*). If p is a C^*-seminorm on A, then the above constructed C^*-algebra $C_p^*(A)$ and $*$-homomorphism $\pi_p : A \to C_p^*(A)$ will be said *canonically associated* to p.

1.3. Examples. There are two fundamental examples of C^*-algebras.

First, consider a locally compact Hausdorff space Ω and let $C_0(\Omega)$ be the set of all complex continuous functions on Ω vanishing at infinity. Then $C_0(\Omega)$ becomes a commutative C^*-algebra with the usual algebraic operations, the complex conjugation as $*$-operation and the uniform norm

$$\|f\| = \sup\{|f(t)| : t \in \Omega\}; \quad f \in C_0(\Omega).$$

We will see in 1.14 that every commutative C^*-algebra is isometrically $*$-isomorphic with some $C_0(\Omega)$.

Next, consider a complex Hilbert space H and let $B(H)$ denote the set of all bounded linear operators on H. Then $B(H)$ becomes a C^*-algebra with the usual algebraic operations, the adjunction as $*$-operation and the operator norm

$$\|x\| = \sup\{\|x\xi\| : \xi \in H,\ \|\xi\| = 1\} = \sup\{|(x\xi \mid \eta)| : \xi, \eta \in H,\ \|\xi\| = \|\eta\| = 1\}.$$

Since any norm closed $*$-subalgebra of a C^*-algebra is again a C^*-algebra, the norm closed $*$-subalgebras of $B(H)$ are C^*-algebras. They will be called *Gelfand-Naĭmark algebras*.

We will see in 4.11 that every C^*-algebra is isometrically $*$-isomorphic to a Gelfand-Naĭmark algebra.

A $*$-homomorphism of a $*$-algebra A into $B(H)$ will be called a *$*$-representation* of A on H.

Among Gelfand-Naĭmark algebras we note the two sided ideal $K(H)$ of all compact linear operators on H.

If $n \geqslant 1$ is an integer, then the $*$-algebra $B(\mathbb{C}^n)$ is usually identified with the $*$-algebra M_n of all $n \times n$ matrices with complex entries. Thus, we can consider on M_n the norm transported from $B(\mathbb{C}^n)$, with which M_n becomes a C^*-algebra.

1.4. Let A be a C^*-algebra and let Ω be a topological space. Denote by $C(\Omega, A)$ the vector space of all continuous functions $f : \Omega \to A$ with $\|f\| = \sup\{\|f(t)\| : t \in \Omega\} < +\infty$. Then $C(\Omega, A)$ is a C^*-algebra with pointwise algebraic operations and involution, and the norm $\|\cdot\|$.

The set of all $f \in C(\Omega, A)$ which vanish at infinity, i.e. for every $\varepsilon < 0$ there exists a compact subset $K \subset \Omega$ with $\sup\{\|f(t)\| : t \in \Omega \setminus K\} \leqslant \varepsilon$, is a C^*-subalgebra of $C(\Omega, A)$, denoted by $C_0(\Omega, A)$.

In particular, we can consider the C^*-algebra $C(\Omega) = C(\Omega, \mathbb{C})$ and its C^*-subalgebra $C_0(\Omega) = C_0(\Omega, \mathbb{C})$.

Let $\{A_i\}_{i \in I}$ be a family of C^*-algebras. Then the subset of the direct product $*$-algebra $\prod_{i \in I} A_i$ consisting of all $x = \{x_i\}_{i \in I}$ with $\|x\| = \sup_{i \in I} \|x_i\| < +\infty$ is a $*$-subalgebra which, endowed with the norm $\|\cdot\|$, becomes a C^*-algebra called the *direct product C^*-algebra* of the familly $\{A_i\}_{i \in I}$.

The set of those $x = \{x_i\}_{i \in I} \in \prod_{i \in I} A_i$ which vanish at infinity, i.e. for every $\varepsilon > 0$ there exists a finite subset $F \subset I$ such that $\sup_{i \in I \setminus F} \|x_i\| \leqslant \varepsilon$, is a C^*-subalgebra of the direct product C^*-algebra, called the *restricted direct product C^*-algebra* of $\{A_i\}_{i \in I}$.

Remark that, if $A_i = A$ for all $i \in I$ and I is considered with its discrete topology, then $C(I, A)$ (respectively $C_0(I, A)$) is nothing but the direct product (respectively the restricted direct product) C^*-algebra of $\{A_i\}_{i \in I}$.

1.5. Associate unital algebra. Let A be an algebra. Then $A \oplus \mathbb{C}$ is a unital algebra with the product

$$(x \oplus \lambda)(y \oplus \mu) = (xy + \mu x + \lambda y) \oplus \lambda\mu; \quad x, y \in A,\ \lambda, \mu \in \mathbb{C},$$

and with the unit element $0 \oplus 1$. We call it the *algebra with adjoined unit* corresponding to A. Remark that $A = A \oplus 0$ is a two sided ideal of codimension one in $A \oplus \mathbb{C}$. On the other hand we denote

$$\widetilde{A} = \begin{cases} A & \text{if } A \text{ is unital;} \\ A \oplus \mathbb{C} & \text{if } A \text{ is not unital;} \end{cases}$$

and call $\widetilde{A}$ the *associate unital algebra* of A.

The algebra $\widetilde{A}$ is uniquely determined up to isomorphisms by the following universality property: $\widetilde{A}$ is generated by A and by its unit element, and every injective homomorphism of A into a unital algebra B has an extension to an injective homomorphism of $\widetilde{A}$ into B. Note that every homomorphism π of A into a unital algebra B can be extended to a homomorphism $\widetilde{\pi} : \widetilde{A} \to B$. If A is not unital, then $\widetilde{\pi}$ can be uniquely choosen such that $\widetilde{\pi}(1_{\widetilde{A}}) = 1_B$.

If A is a $*$-algebra then $A \oplus \mathbb{C}$ is also a $*$-algebra with the $*$-operation

$$(x \oplus \lambda)^* = x^* \oplus \overline{\lambda}; \quad x \in A,\, \lambda \in \mathbb{C}.$$

Hence $\widetilde{A}$ is a $*$-algebra and each injective $*$-homomorphism of A in some unital $*$-algebra B can be extended to an injective $*$-homomorphism of $\widetilde{A}$ in B.

If A is a normed algebra (respectively normed $*$-algebra, respectively involutive normed algebra), then the same is true for $A \oplus \mathbb{C}$ with the norm

$$\|x \oplus \lambda\| = \max\{\|x\|, |\lambda|\}; \quad x \in A,\, \lambda \in \mathbb{C}.$$

Hence $\widetilde{A}$ is also a normed algebra (respectively normed $*$-algebra, respectively involutive normed algebra) and it is easy to formulate the corresponding universality property. Remark that if A is complete, then also $A \oplus \mathbb{C}$ and $\widetilde{A}$ are complete.

Now let A be a non-unital C^*-algebra. Consider the Banach algebra $B(A)$ of all bounded linear operators on the underlying Banach space of A and denote by $I \in B(A)$ the identity operator. For any $x \in A$ define $\pi_A(x) \in B(A)$ by

$$\pi_A(x)y = xy; \quad y \in A.$$

Then π_A is an algebra homomorphism and using $\|xx^*\| = \|x\|^2$ it is easy to see that $\|\pi_A(x)\| = \|x\|$ for all $x \in A$. Let $\widetilde{\pi}_A : \widetilde{A} \to B(A)$ be the unique homomorphism extending π_A and carying the unit of $\widetilde{A}$ in I. Then $\widetilde{\pi}_A$ is injective (since A is non-unital) and we can define a norm on $\widetilde{A}$ by

$$\|\widetilde{x}\| = \|\widetilde{\pi}_A(x)\|; \quad \widetilde{x} \in \widetilde{A}.$$

Since $\pi_A(A)$ is a two sided ideal of codimension $\leqslant 1$ in $\widetilde{\pi}_A(\widetilde{A})$ and it is complete, it follows that $\widetilde{\pi}_A(\widetilde{A})$ is complete, so $\widetilde{A}$ is complete with the above norm. Moreover,

$\widetilde{A}$ is a C^*-algebra because for each $x \in A$, $\lambda \in \mathbb{C}$ and $\varepsilon > 0$ there exists $y \in A$, $\|y\| \leqslant 1$, with

$$\begin{aligned}\|\pi_A(x) + \lambda I\|^2 &\leqslant \varepsilon + \|xy + \lambda y\|^2 = \varepsilon + \|(xy + \lambda y)^*(xy + \lambda y)\| \\ &= \varepsilon + \|y^*(x^*xy + \overline{\lambda}xy + \lambda x^*y + \overline{\lambda}\lambda y)\| \\ &\leqslant \varepsilon + \|(\pi_A(x^*) + \overline{\lambda}I)(\pi_A(x) + \lambda I)y\| \\ &\leqslant \varepsilon + \|(\pi_A(x^*) + \overline{\lambda}I)(\pi_A(x) + \lambda I)\|.\end{aligned}$$

If A is a C^*-algebra, then we shall always consider the above norm on $\widetilde{A}$, so that $\widetilde{A}$ becomes a unital C^*-algebra called the *associate unital C^*-algebra* of A.

Remark that if A has a unit e, then we still can define a complete C^*-norm on $A \oplus \mathbb{C}$, namely,

$$\|x \oplus \lambda\| = \max\{\|x + \lambda e\|, |\lambda|\}; \quad x \in A,\ \lambda \in \mathbb{C}.$$

Thus, for every C^*-algebra A there exists a canonical complete C^*-norm on $A \oplus \mathbb{C}$ and, endowed with this norm, $A \oplus \mathbb{C}$ will be called the *C^*-algebra with adjoined unit* corresponding to A.

We stress that for a non unital C^*-algebra the associate unital C^*-algebra coincides with the C^*-algebra with adjoined unit, while for a unital C^*-algebra they differ.

1.6. Special elements. An element x of a $*$-vector space A is called *self-adjoint* (or *hermitian*) if $x^* = x$. The set A_h of all selfadjoint elements of A is a real vector subspace of A. For each $x \in A$ we define the *real part* $\operatorname{Re} x$ and the *imaginary part* $\operatorname{Im} x$ by

$$\operatorname{Re} x = 2^{-1}(x + x^*), \quad \operatorname{Im} x = (2\mathrm{i})^{-1}(x - x^*).$$

Then $\operatorname{Re} x$, $\operatorname{Im} x$ are selfadjoint and

$$x = \operatorname{Re} x + \mathrm{i} \operatorname{Im} x.$$

Thus,

$$A = A_h + \mathrm{i}\, A_h, \quad A_h \cap \mathrm{i}\, A_h = \{0\}.$$

Suppose now that A is a $*$-algebra. Then the product xy of two selfadjoint elements $x, y \in A$ is again selfadjoint if and only if $xy = yx$. An element $x \in A$ is called *normal* if $x^*x = xx^*$. Note that every selfadjoint element is normal. Also, $x \in A$ is normal if and only if $\operatorname{Re} x$ commutes with $\operatorname{Im} x$. A *projection* is a selfadjoint idempotent $p \in A$, i.e. $p^* = p = p^2$ and a *partial isometry* is an element $v \in A$ such that v^*v and vv^* are projections. If $v \in A$ is a partial isometry, then v^*v (respectively vv^*) is called the *initial* (respectively the *final*) projection of v. Two projections $p, q \in A$ are called *orthogonal* if $pq = 0$.

If A is a unital $*$-algebra, then 1_A is necessarily selfadjoint and hence a projection. If $x \in A$ is invertible, then x^* is also invertible and $(x^*)^{-1} = (x^{-1})^*$. An element $u \in A$ with $u^*u = 1_A$ (respectively $uu^* = 1_A$) is called an *isometry* (respectively a *co-isometry*) and if $u^*u = 1_A = uu^*$, then u is called a *unital element*. Every unitary element $u \in A$ is normal and invertible: $u^{-1} = u^*$. The set $U(A)$ of all unitary elements is a multiplicative group, called the *unitary group* of A.

If A is an arbitrary $*$-algebra, then $v \in A$ is called *quasiunitary* if $v^*v = vv^* = v + v^*$. Remark that $v \in A$ is quasi-unitary if and only if $1 - v$ is unitary in $\widetilde{A}$.

A $*$-algebra A is called *U^*-algebra* if A is the linear span of its quasi-unitary elements. If A is a U^*-algebra, then so is $\widetilde{A}$, because the element $2 \in \widetilde{A}$ is quasi-unitary. If A is a unital $*$-algebra, then A is a U^*-algebra if and only if A is the lineary span of $U(A)$.

Finally, assume that A is a C^*-algebra. If $p \in A$ is a nonzero projection then $\|p\|^2 = \|p^*p\| = \|p\|$, thus $\|p\| = 1$. If $v \in A$ and $v^*v = p$ is a projection, then $vv^* = q$ is also a projection, i.e. v is a partial isometry. Indeed,

$$\|q^2 - q\|^2 = \|(q^2 - q)^*(q^2 - q)\| = \|q^4 - 2q^3 - q^2\| = \|vp^3v^* - 2vp^2v^* + vpv^*\| = 0,$$

so $q^* = q = q^2$. Similarly, we get $vv^*v = v$, that is, $vp = qv = v$. For every non-zero partial isometry $v \in A$ we have $\|v\|^2 = \|v^*v\| = 1$, so $\|v\| = 1$. In particular, if A is unital, then $\|1_A\| = 1$ and each unitary element has norm one.

If $x \in A$ is normal, then $\|x^2\| = \|x\|^2$:

$$\|x^2\|^2 = \|(x^2)^*x^2\| = \|(x^*x)^*(x^*x)\| = \|x^*x\|^2 = \|x\|^4.$$

Since the multiplication and the $*$-operation in a C^*-algebra A are continuous, A_h and $\{x \in A : x \text{ is normal}\}$ are closed.

1.7. Spectrum and resolvent. If A is an algebra, then the *spectrum* $\sigma(x) = \sigma_A(x)$ of an element $x \in A$ is the set of all $\lambda \in \mathbb{C}$ such that $\lambda - x$ is not invertible in $\widetilde{A}$ and its *resolvent set* is $\rho(x) = \rho_A(x) = \mathbb{C} \setminus \sigma_A(x)$. If $A \neq \widetilde{A}$, then $0 \in \sigma(x)$ for all $x \in A$.

For every $x, y \in A$ we have

$$\sigma(xy) \cup \{0\} = \sigma(yx) \cup \{0\}. \tag{1}$$

Indeed, suppose that $\lambda \in \mathbb{C}$ is not in $\sigma(xy) \cup \{0\}$. Then there is $u \in \widetilde{A}$ such that $(\lambda - xy)u = u(\lambda - xy) = 1$. Putting $v = \lambda^{-1} + (1 + yux)$ we obtain $(\lambda - yx)v = v(\lambda - yx) = 1$, thus λ is not in $\sigma(yx) \cup \{0\}$.

For each $x \in A$ the number $r(x) = r_A(x) = \sup\{|\lambda| : \lambda \in \sigma_A(x)\}$ is called the *spectral radius* of x. By (1),

$$r(xy) = r(yx); \quad x, y \in A. \tag{2}$$

For $x, y \in A$ put

$$x \circ y = x + y - xy.$$

Then $1 - x$ is right (respectively left) invertible in $\widetilde{A}$ if and only if $x \circ y = 0$ (respectively $y \circ x = 0$) for some $y \in A$, called a *right* (respectively *left*) *quasi-inverse* of x, and, in this case, $1 - y$ is a right (respectively left) inverse of $1 - x$. Thus $1 - x$ is invertible in $\widetilde{A}$ if and only if $x \circ y = y \circ x = 0$ for some $y \in A$, which is then unique and called the *quasi-inverse* of x.

Note that $\lambda \neq 0$ is in $\rho(x)$ if and only if $\lambda^{-1}x$ has quasi-inverse. In particular, if $x \in A$ and $r(x) < 1$, then $1 \in \rho(x)$, so there exists $y \in A$ such that $x \circ y = y \circ x = 0$.

If A is a $*$-algebra, then

$$\lambda \in \sigma(x) \Leftrightarrow \overline{\lambda} \in \sigma(x^*); \quad x \in A,\, \lambda \in \mathbb{C}, \tag{3}$$

$$r(x^*) = r(x); \quad x \in A. \tag{4}$$

Remark that $v \in A$ is quasi-unitary if and only if $v^* \circ v = v \circ v^* = 0$.

Consider now a Banach algebra $A \neq \{0\}$ and an arbitrary Banach algebra norm on $\widetilde{A}$ extending the norm of A.

It is easy to see that if $x \in A$ and $\lambda_0 \in \rho(x)$, then

$$\{\lambda \in \mathbb{C} : |\lambda - \lambda_0| < \|(\lambda_0 - x)^{-1}\|^{-1}\} \subset \rho(x), \tag{5}$$

and that for any $\lambda \in \mathbb{C}$, $|\lambda - \lambda_0| < \|(\lambda_0 - x)^{-1}\|^{-1}$ we have

$$(\lambda - x)^{-1} = \sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n (\lambda_0 - x)^{-n-1}. \tag{6}$$

Thus the resolvent set $\rho(x)$ is open and the *resolvent function*

$$\rho(x) \ni \lambda \mapsto (\lambda - x)^{-1} \in \widetilde{A}$$

is norm analytic, in particular norm continuous.

On the other hand,

$$\{\lambda \in \mathbb{C} : |\lambda| > \|x\|\} \subset \rho(x) \tag{7}$$

and, for any $\lambda \in \mathbb{C}$, $|\lambda| > \|x\|$, we have

$$(\lambda - x)^{-1} = \sum_{n=0}^{\infty} \lambda^{-n-1} x^n. \tag{8}$$

Thus the spectrum $\sigma(x)$ is a compact set and

$$r(x) \leqslant \|x\|. \tag{9}$$

Moreover, using (8) we get $\lim\limits_{|\lambda| \to \infty} \|(\lambda - x)^{-1}\| = 0$. Thus, if it were $\sigma(x) = \emptyset$, then by the Liouville theorem the entire function $\lambda \mapsto (\lambda - x)^{-1}$ would be identically zero, which is impossible. Hence the spectrum $\sigma(x)$ is non void.

PROPOSITION. *If $A \neq \{0\}$ is a Banach algebra and $x \in A$ then*

$$r_A(x) = \lim_n \|x^n\|^{1/n}. \tag{10}$$

Proof. Put

$$\alpha(x) = \inf\{\|x^n\|^{1/n} : n \in \mathbb{N}\}.$$

Fix $\varepsilon > 0$ and choose $n_\varepsilon \in \mathbb{N}$ such that $\|x^{n_\varepsilon}\|^{1/n_\varepsilon} \leqslant \alpha(x) + \varepsilon$. For each $n \in \mathbb{N}$ we can write $n = n_\varepsilon q + r$, $(q, r \in \mathbb{N},\ 0 \leqslant r \leqslant n_\varepsilon - 1)$, and we have

$$\|x^n\| = \|x^{n_\varepsilon q} x^r\| \leqslant \|x^{n_\varepsilon}\|^q \|x\|^r \leqslant (\alpha(x) + \varepsilon)^{n_\varepsilon q} \|x\|^r = (\alpha(x) + \varepsilon)^{n-r} \|x\|^r.$$

Therefore $\|x^n\|^{1/n} \leqslant (\alpha(x) + \varepsilon)^{1-r/n} \|x\|^{r/n}$ and so

$$\alpha(x) \leqslant \liminf_n \|x^n\|^{1/n} \leqslant \limsup_n \|x^n\|^{1/n} \leqslant \alpha(x) + \varepsilon.$$

It folows that

$$\lim_n \|x^n\|^{1/n} = \alpha(x).$$

If $|\lambda| > \alpha(x)$ then the series $\sum\limits_{n=0}^{\infty} \|x^n\|/|\lambda|^n$ converges, and hence, the series $\sum\limits_{n=0}^{\infty} x^n/\lambda^n$ is norm convergent in $\widetilde{A}$. Hence $1 - \lambda^{-1}x$ is invertible and so

$$\alpha(x) \geqslant r(x).$$

Suppose that $\alpha(x) > r = r(x)$. Then $\{\lambda \in \mathbb{C} : |\lambda| > r\} \subset \rho(x)$. Let φ be a bounded linear functional on $\widetilde{A}$. Then the function $\lambda \mapsto \varphi((\lambda - x)^{-1})$ is analytic on $\{\lambda \in \mathbb{C} : |\lambda| > r\}$ and for $|\lambda| > \alpha(x)$ we have

$$\varphi((\lambda - x)^{-1}) = \sum_{n=0}^{\infty} \lambda^{-n-1} \varphi(x^n).$$

Consequently, the equality

$$f(\mu) = \begin{cases} 0 & \text{if } \mu = 0; \\ \varphi((\mu^{-1} - x)^{-1}) & \text{if } 0 < |\mu| < r^{-1}; \end{cases}$$

defines a function f analytic on $\mathbb{D} = \{\mu \in \mathbb{C} : |\mu| < r^{-1}\}$. Since the Taylor expansion of f near 0 is

$$f(\mu) = \sum_{n=0}^{\infty} \mu^{n+1} \varphi(x^n)$$

the same formula holds for any $\mu \in D$.

Consider $\lambda_0 \in \mathbb{C}$, $r < |\lambda_0| < \alpha(x)$. Then $\lambda_0^{-1} \in D$ and

$$\lim_n \lambda_0^{-n-1} \varphi(x^n) = 0,$$

for any bounded liniar functional φ on $\widetilde{A}$. Using the Banach-Steinhauss theorem we infer $\sup\limits_n |\lambda_0|^{-n-1} \|x^n\| = c < +\infty$, therefore $\|x^n\| \leqslant c|\lambda_0|^{n+1}$, $(n \geqslant 0)$, and this yields

$$\alpha(x) = \lim_n \|x^n\|^{\frac{1}{n}} \leqslant \lim_n c^{\frac{1}{n}} |\lambda_0|^{1+n^{-1}} = |\lambda_0| < \alpha(x),$$

a contradiction.

Hence $r(x) = \alpha(x)$. ∎

By the above proposition, if $x, y \in A$, $xy = yx$, then

$$r(xy) \leqslant r(x)r(y), \quad r(x+y) \leqslant r(x) + r(y). \tag{11}$$

Now let $A \neq \{0\}$ be a normed algebra, consider on $\widetilde{A}$ a normed algebra norm extending the norm of A and let B be the Banach algebra completion of $\widetilde{A}$. Then obviously $\sigma_A(x) \supset \sigma_B(x) \neq \emptyset$ for any $x \in A$ and it folows that:

Corollary. *If $A \neq \{0\}$ is a normed algebra and $x \in A$ then*

$$r_A(x) \geqslant \lim_n \|x^n\|^{1/n}. \tag{12}$$

Finally, we note the following property of *lower semicontinuity of the spectrum.* Let A be a Banach algebra, $x \in A$ and let N be a neighborhood of $\sigma(x)$ in $\mathbb{C}$. Then there exists $\varepsilon > 0$ such that $\sigma(y) \subset N$ whenever $y \in A$, $\|x - y\| \leqslant \varepsilon$.

Indeed, if $0 < \varepsilon < \sup\limits_{\lambda \notin N} \|(\lambda - x)^{-1}\|)^{-1}$, then for any $\lambda \notin N$ we successively have: $\|(\lambda - x)^{-1}(x - y)\| < 1$, $-1 \notin \sigma((\lambda - x)^{-1}(x - y))$, $\lambda \notin \sigma(y)$, and

$$(\lambda - y)^{-1} = (\lambda - x)^{-1}(1 + (\lambda - x)^{-1}(x - y))^{-1}.$$

1.8. Whenever A is a C^*-algebra, we shall consider on $\widetilde{A}$ only the C^*-norm defined in 1.5.

Proposition. *Let A be a C^*-algebra and $x \in A$. Then:*

$$x \textit{ is normal} \Rightarrow r(x) = \|x\|; \tag{1}$$

$$x \textit{ is selfadjoint} \Rightarrow \sigma(x) \subset \mathbb{R}; \tag{2}$$

$$x \textit{ is unitary} \Rightarrow \sigma(x) \subset \{\lambda \in \mathbb{C} : |\lambda| = 1\}. \tag{3}$$

Proof. (1) If x is normal, then by 1.6, $\|x^2\| = \|x\|^2$ and using 1.7.(10) we get $r(x) = \lim\limits_n \|x^{2^n}\|^{2^{-n}} = \|x\|$.

(2) Let x be selfadjoint and consider $\lambda = \alpha + \mathrm{i}\,\beta \in \sigma(x)$ with $\alpha, \beta \in \mathbb{R}$. Then $x_n = x - \alpha + \mathrm{i}\,n\beta \in \widetilde{A}$, $\mathrm{i}\,(n+1)\beta \in \sigma(x_n)$ and

$$(n+1)^2\beta^2 \leqslant r(x_n)^2 \leqslant \|x_n\|^2 = \|x_n^* x_n\| = \|(x-\alpha)^2 + n^2\beta^2\| \leqslant \|x - \alpha\|^2 + n^2\beta^2$$

for all $n \in \mathbb{N}$. Clearly, this entails $\beta = 0$, hence $\lambda \in \mathbb{R}$.

(3) Since x and x^{-1} are unitary, we have $\|x\| = \|x^{-1}\| = 1$ so that $\sigma(x)$ and $\sigma(x^{-1}) = \{\lambda^{-1} : \lambda \in \sigma(x)\}$ are both contained in $\{\lambda \in \mathbb{C} : |\lambda| \leqslant 1\}$, which shows that $\sigma(x) \subset \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. ∎

1.9. An important consequence of Proposition 1.8 is that every $*$-homomorphism between C^*-algebras is contractive. More precisely:

THEOREM. *Let π be a $*$-homomorphism of an involutive Banach algebra A onto a C^*-algebra B. Then*

$$\|\pi(x)\| \leqslant \|x\|; \quad x \in A.$$

Proof. Using the caracterization of spectra with quasi-inverses (see 1.7), it is easy to see that $\sigma_B(\pi(x)) \subset \sigma_A(x) \cup \{0\}$ for any $x \in A$. Thus, if x is normal, by 1.8 we get:

$$\|\pi(x)\| = r_B(\pi(x)) \leqslant r_A(x) \leqslant \|x\|.$$

For an arbitrary $x \in A$ we then obtain:

$$\|\pi(x)\|^2 = \|\pi(x)^*\pi(x)\| = \|\pi(x^*x)\| \leqslant \|x^*x\| \leqslant \|x\|^2. \quad \blacksquare$$

COROLLARY 1. *Every $*$-isomorphism of a C^*-algebra onto another C^*-algebra is isometric.* $\blacksquare$

COROLLARY 2. *Any two complete C^*-norms on a $*$-algebra are equal.* $\blacksquare$

In particular, for every C^*-algebra A there is only one complete, C^*-norm on $A \oplus \mathbb{C}$ and $\widetilde{A}$.

1.10. Spectral permanence. The following result is usually called the spectral permanence property for C^*-algebras.

THEOREM. *Let B be a C^*-subalgebra of the C^*-algebra A. Then for every $x \in B$ we have*

$$\sigma_A(x) \cup \{0\} = \sigma_B(x) \cup \{0\}.$$

Proof. Passing to C^*-algebras with adjoined units (1.5) we may assume that A is unital and $1_A \in B$. In this case we show that

$$\sigma_A(x) = \sigma_B(x); \quad x \in B.$$

It is clear that $\sigma_B(x) \supset \sigma_A(x)$. For the converse inclusion suppose first x is selfaldjoint and consider $\lambda \notin \sigma_A(x)$. If $\lambda \notin \mathbb{R}$, then $\lambda \notin \sigma_B(x)$ by 1.8. If $\lambda \in \mathbb{R}$ then $\lambda_n = \lambda + \mathrm{i}/n \notin \sigma_B(x)$, thus $(\lambda_n - x)^{-1} \in B$. Using the equality

$$(\lambda - x)^{-1} - (\lambda_n - x)^{-1} = (\lambda_n - \lambda)(\lambda - x)^{-1}(\lambda_n - x)^{-1},$$

we see that $\{(\lambda_n - x)^{-1}\}$ converges to $(\lambda - x)^{-1}$ as $n \to +\infty$ and hence $(\lambda - x)^{-1} \in B$, which means that $\lambda \notin \sigma_B(x)$. Passing to the general case, consider $\lambda \notin \sigma_A(x)$. Then $y = \lambda - x$ is invertible in A so that the selfadjoint elements y^*y and $yy^* \in B$ are invertible in A and by the preceding discussion they are invertible in B. It follows that y is left and right invertible in B and so $\lambda \notin \sigma_B(x)$. $\blacksquare$

1.11. Regular ideals. Let A be an algebra. A left (respectively right) ideal M of A is called *regular* if there exists a *right* (respectively *left) unit* for M, that is an element $u \in A$ such that $xu - x \in M$ (respectively $ux - x \in M$) for all $x \in A$. Remark that if A is unital then every ideal of A is regular.

A regular left (respectively right) ideal M of A is called *maximal* if $M \neq A$ and for each left (respectively right) ideal N of A with $M \subset N$ we have either $N = M$ or $N = A$.

Lemma 1. *Every regular left (respectively right) ideal $J \neq A$ is contained in some maximal regular left (respectively right) ideal of A.*

Proof. Let J be a regular left ideal and u be a right unit for J. By the Zorn lemma there exists a maximal element M in the set of all left ideals of A containing J but not containing u. Then M is regular with right unit u. If $N \supset M$ is left ideal and $N \neq A$, then $u \notin N$, so $N = M$ by the choice of M. ∎

Lemma 2. *Let $a \in A$ and $0 \notin \lambda \in \mathbb{C}$. Then the following statements are equivalent:*

(i) *$\lambda - a$ has no left (respectively right) inverses in $\widetilde{A}$;*

(ii) *$\lambda^{-1}a$ has no left (respectively right) quasi-inverses in A;*

(iii) *$\lambda^{-1}a$ is a right (respectively left) unit for some maximal regular left (respectively right) ideal in A.*

Proof. (i) $\Leftrightarrow$ (ii) follows immediately using the remarks from 1.7

Suppose that (ii) holds. Then $J = \{x(\lambda^{-1}a) - x : x \in A\}$ is a regular left ideal A with right unit $\lambda^{-1}a$ and $\lambda^{-1}a \notin J$, because otherwise $\lambda^{-1}a$ would have a left quasi-inverse. By Lemma 1, J is contained in some maximal regular left ideal M and $\lambda^{-1}a$ is a right unit for M. Consequently, (ii) $\Rightarrow$ (iii).

We now assume that $\lambda^{-1}a$ is a right unit for the maximal regular left ideal M of A but $\lambda^{-1}a$ has a left quasi-inverse $x \in A$. Then $x + \lambda^{-1}a - x(\lambda^{-1}a) = 0$, so $\lambda^{-1}a = x(\lambda^{-1}a) - x \in M$. Consequently, $y \in A \Rightarrow y(\lambda^{-1}a) - y \in M \Rightarrow y \in M$, that is $M = A$, a contradiction. Thus (iii) $\Rightarrow$ (ii). ∎

Theorem. *The following equalities hold:*

$$
\begin{aligned}
&\bigcap\{M : M \text{ maximal regular left ideal of } A\}\\
&= \bigcap\{N : N \text{ maximal regular right ideal of } A\}\\
&= \bigcup\{J : J \text{ left ideal of } A \text{ and all } a \in J \text{ have left quasi-inverses}\}\\
&= \bigcup\{K : K \text{ right ideal of } A \text{ and all } b \in K \text{ have right quasi-inverses}\}.
\end{aligned}
$$

Proof. If $a \in A$ belongs to all maximal regular left ideals of A, then a is not a right unit for any maximal regular left ideal and hence, by Lemma 2, a has left quasi-inverses in A.

Let J be a left ideal of A such that each element of J has left quasi-inverses in A and let $a \in J$. Then a has a left quasi-inverse $b \in A$ and $b = ba - a \in J$ has also left quasi-inverses in A. It follows that b is quasi-invertible with quasi-inverse a, so a is quasi-invertible. Thus all elements of J are quasi-invertible.

Let J be as above and $a \in J$. For any $y \in \widetilde{A}$, $ya \in J$ has a quasi-inverse $z \in A$ and, as easily verified, $azy - ay$ is a quasi-inverse of ay. Hence all elements of the right ideal generated by a are quasi invertible.

Finally, let K be a right ideal of A such that each element of K has right quasi-inverses in A and let $b \in K$. Assume that $b \notin N$ for some maximal regular right ideal N of A with a left unit $v \in A$. Then $\{y + bx : y \in N,\ x \in \widetilde{A}\}$ is a right ideal of A containing $N \cup \{b\}$, so it coincides with A. In particular, there exist $y \in N$ and $x \in \widetilde{A}$ such that $v = y + bx$. Since $bx \in K$ has a right quasi-inverse $z \in A$, we have

$$v = y + bx = y - z + bxz = y - z + (v - y)z = y - yz + (vz - z) \in N,$$

which is not possible. Therefore K is contained in every maximal regular right ideal of A.

In conclusion:

$$\begin{aligned} &\bigcap\{M : M \text{ maximal regular left ideal of } A\} \\ &\subset \bigcup\{J : J \text{ left ideal of } A \text{ and all } a \in J \text{ have left quasi-inverses}\} \\ &\subset \bigcup\{K : K \text{ right ideal of } A \text{ and all } b \in K \text{ have right quasi-inverses}\} \\ &\subset \bigcap\{N : N \text{ maximal regular right ideal of } A\}. \end{aligned}$$

The proofs of the converse inclusions are completely similar. ∎

The set defined in Lemma 3 is called the *radical* of A and is denoted by $\mathrm{Rad}(A)$. By Lemmas 3 and 2, $\mathrm{Rad}(A)$ is a two sided ideal of A and

$$a \in \mathrm{Rad}(A) \Rightarrow \sigma_A(a) \subset \{0\}. \tag{1}$$

Moreover, $\mathrm{Rad}(A)$ is the greatest left (or right, or two sided) ideal contained in $\{x \in A : \sigma_A(x) \subset \{0\}\}$.

It is easy to see that the radical of a $*$-algebra is a $*$-subalgebra.

LEMMA 4. *Every maximal regular left (respectively right) ideal of a Banach algebra is closed.*

Proof. Let M be a maximal regular left ideal of a Banach algebra A and let u be a right unit for M. Assume that M is not closed. Then its closure is A, so exists $x \in M$ with $r(u - x) \leqslant \|u - x\| < 1$. By 1.7, $u - x$ has a quasi-inverse $y \in A$ and we have:

$$u = (u - x) + x = y(u - x) - y + x = (yu - y) - yx + x) \in M,$$

which is not possible. ∎

By Lemma 4, the radical of a Banach algebra is closed.

An algebra A is called *semisimple* if $\mathrm{Rad}(A) = \{0\}$.

PROPOSITION. *Every pre-C^*-algebra is semisimple.*

Proof. Let A be a pre-C^*-algebra and $x \in \mathrm{Rad}(A)$. Then $a = x^*x \in \mathrm{Rad}(A)$ so, by Lemma 3, $\sigma_A(a) = \{0\}$. If B is the C^*-algebra completion of A, then $\sigma_B(a) = \{0\}$ so, by Proposition 1.8, $a = 0$ and consenquently $x = 0$. ∎

1.12. We have the following continuity result similar to Theorem 1.9:

THEOREM. *Let A be a Banach algebra, B be a pre-C^*-algebra and π be a homomorphism of A onto B. Then π is bounded.*

Proof. By Proposition 1.11

$$\operatorname{Ker}\pi = \bigcap\{\pi^{-1}(M) : M \text{ maximal regular left ideal in } B\}.$$

For any maximal regular left ideal M of B, $\pi^{-1}(M)$ is a maximal regular left ideal of A and hence is closed (Lemma 4 in 1.11). Indeed, $\pi^{-1}(M)$ is obviously a left ideal of A; if $u \in B$ is a right unit for M, then any $v \in \pi^{-1}(\{u\})$ is a right unit for $\pi^{-1}(M)$, so $\pi^{-1}(M)$ is regular; if $N \supset \pi^{-1}(M)$ is a left ideal of A, then either $\pi(N) = M$, in which case $N = \pi^{-1}(M)$, or $\pi(N) = B$, in which case N contains a right unit for N, so $N = A$. We conclude that $\operatorname{Ker}\pi$ is a closed two sided ideal of A.

Denote by $\widehat{\ }$ the canonical homomorphism $A \to A/\operatorname{Ker}\pi$ and consider on $A/\operatorname{Ker}\pi$ the quotient norm

$$\|\widehat{a}\| = \inf\{\|a + x\| : x \in \operatorname{Ker}\pi\}; \quad a \in A.$$

Then $A/\operatorname{Ker}\pi$ becomes a Banach algebra. Since $\widehat{a} \to \pi(a)$ is an isomorphism of $A/\operatorname{Ker}\pi$ onto B, we can consider on B the Banach algebra norm $|||\cdot|||$ defined by

$$|||\pi(a)||| = \|\widehat{a}\|; \quad a \in A.$$

Let $\{b_n\}_{n\in\mathbb{N}} \subset B$ and $c \in B$ be such that

$$\lim_n |||b_n||| = 0, \quad \lim_n |||b_n^* - c||| = 0.$$

Using Corollary 1.7 we get

$$\begin{aligned}\|b_n^* - c\|^2 &= \|(b_n^* - c)^*(b_n^* - c)\| = \lim_k \|[(b_n^* - c)^*(b_n^* - c)]^k\|^{\frac{1}{k}} \\ &\leqslant r_B((b_n^* - c)^*(b_n^* - c)) \leqslant |||b_n - c^*|||\,|||b_n^* - c||| \\ &\leqslant \Big(\sup_n |||b_n||| + |||c^*|||\Big)|||b_n^* - c|||,\end{aligned}$$

so $\lim_n \|b_n^* - c\| = 0$. Similarly, $\lim_n \|b_n^*\| = \lim_n \|b_n\| = 0$. Consequently, $c = 0$. By the closed graph theorem it follows that there exists $\alpha > 0$ such that

$$|||b^*||| \leqslant \alpha|||b|||; \quad b \in B.$$

Now, using again Corollary 1.7, we obtain

$$\begin{aligned}\|\pi(a)\|^2 &= \|\pi(a)^*\pi(a)\| = \lim_k \|[\pi(a)^*\pi(a)]^k\|^{\frac{1}{k}} \\ &\leqslant r_B(\pi(a)^*\pi(a)) \leqslant \alpha|||\pi(a)|||^2 = \alpha\|\widehat{a}\|^2 \leqslant \alpha\|a\|^2.\end{aligned}$$

Thus π is bounded. ∎

COROLLARY. *Let A be a Banach $*$-algebra, B be a C^*-algebra and $\pi : A \to B$ be a $*$-homomorphism. Then π is bounded.*

1.13. The Envelopping C^*-algebra of a Banach $*$-algebra. Let A be a Banach $*$-algebra. For any C^*-seminorm p on A, consider the $*$-homomorphism $\pi_n : A \to C_n^*(A)$ canonically associated to p (1.2).

Using Corollary 1.12, for all $x \in A$ we have

$$p(x)^2 = \|\pi_p(x)\|^2 = \|\pi_p(x^*x)\| = \lim_n \|\pi_p((x^*x)^n)\|^{1/n}$$
$$\leqslant \limsup_n \|\pi_p\|^{1/n} \|(x^*x)^n\|^{1/n} = r_A(x^*x).$$

Hence, for every $x \in A$

$$\|x\|_* = \sup\{p(x) : p \text{ is a } C^*\text{-seminorm on } A\} \leqslant r_A(x^*x)^{\frac{1}{2}} < +\infty.$$

Clearly, $\|\cdot\|_*$ is the *greatest C^*-seminorm* on A.

We denote $C^*_{\mathrm{env}}(A) = C^*_{\|\cdot\|_*}(A)$, $\pi^A_{\mathrm{env}} = \pi_{\|\cdot\|_*}$ and call $C^*_{\mathrm{env}}(A)$ the *enveloping C^*-algebra* of A and π^A_{env} the *canonical $*$-homomorphism* of A into $C^*_{\mathrm{env}}(A)$. By Corollary 1.12, π^A_{env} is bounded.

Let π be an arbitrary $*$-homomorphism of A into some C^*-algebra B. Then $x \mapsto \|\pi(x)\|$ is a C^*-seminorm on A, hence $\|\pi(x)\| \leqslant \|x\|_*$, $(x \in A)$. Consequently, there exists a unique $*$-homomorphism $\rho : C^*_{\mathrm{env}}(A) \to B$ such that

$$\pi = \rho \circ \pi^A_{\mathrm{env}}.$$

Remark that the pair $(C^*_{\mathrm{env}}(A), \pi^A_{\mathrm{env}})$ is uniquely determined up to $*$-isomorphism by the above universality property and by the fact that $\pi^A_{\mathrm{env}}(A)$ is dense in $C^*_{\mathrm{env}}(A)$.

Using the universality property we can improve Corollary 1.12 as follows: for every $*$-homomorphism π of A into some C^*-algebra,

$$\|\pi(x)\| \leqslant \|\pi^A_{\mathrm{env}}(x)\| \leqslant r_A(x^*x)^{1/2}; \quad x \in A, \tag{1}$$

in particular,

$$\|\pi\| \leqslant \|\pi^A_{\mathrm{env}}\|. \tag{2}$$

If $x \in \mathrm{Rad}(A)$, then $\|\pi^A_{\mathrm{env}}(x)\| \leqslant r_A(x^*x)^{1/2} = 0$ (1.11.(1)), so $\mathrm{Rad}(A)$ is contained in $\operatorname{Ker} \pi^A_{\mathrm{env}}$. In particular, if π^A_{env} is injective, then A is semisimple. By the universality property of $(C^*_{\mathrm{env}}(A), \pi^A_{\mathrm{env}})$, π^A_{env} is injective if and only if there exists an injective $*$-homomorphism of A into some C^*-algebra.

Note that if A is an involutive Banach algebra, then $\|\pi^A_{\mathrm{env}}\| \leqslant 1$ by Theorem 1.9. There exist non zero involutive Banach algebras A with $C^*_{\mathrm{env}}(A) = \{0\}$, for instance $A = \mathbb{C}$ with zero multiplication and complex conjugation.

1.14. Gelfand representation. In this section we shall briefly review the Gelfand theory of commutative Banach algebras, which provides a structure theorem for commutative C^*-algebras.

Let A be a commutative Banach algebra.

By Lemma 4 in 1.11, if M is a maximal regular ideal of A, then M is closed, so we can consider the quotient Banach algebra A/M. Then $A/M \neq \{0\}$ is unital, $1_{A/M}$ being the canonical image of any unit for M, and the only ideals of A/M are $\{0\}$ and A/M, so A/M is a field. If $x \in A$, then the canonical image x/M of x in A/M has non void spectrum (1.7) that is, there exists $\lambda_x \in \mathbb{C}$ such that $\lambda_x 1_{A/M} - x/M$ is not invertible, so $x/M = \lambda_x 1_{A/M}$. The map $\omega_M : A \to \mathbb{C}$ defined by $\omega_M(x) = \lambda_x$, $(x \in A)$, is a non zero algebra homomorphism of A into $\mathbb{C}$, i.e. a non zero *character* of A, and $\operatorname{Ker}\omega_M = M$.

Conversely, if ω is a non zero character of A, then $M = \operatorname{Ker}\omega$ is a maximal regular ideal of A and $\omega = \omega_M$. Hence $M \mapsto \omega_M$ and $\omega \mapsto \operatorname{Ker}\omega$ are mutually inverse bijections between all maximal regular ideals M of A and all non zero characters ω of A.

Denote by Ω_A the set of all non-zero characters of A. By Lemma 2 in 1.11 we have

$$\sigma_A(x) \cup \{0\} = \{\omega(x) : \omega \in \Omega_A\} \cup \{0\}; \quad x \in A. \tag{1}$$

In particular, for every $\omega \in \Omega_A$,

$$|\omega(x)| \leqslant r_A(x) \leqslant \|x\|; \quad x \in A,$$

so $\|\omega\| \leqslant 1$. Therefore, the union of Ω_A with the zero character is an A-closed subset of the closed unit ball of A^*, and hence, by the Alaoglu theorem, it is A-compact. It folows that Ω_A endowed with the A-topology is locally compact. The topological space Ω_A is called the *Gelfand spectrum* of A. Every $x \in A$ defines a continuous function $G_A(x)$ on Ω_A by $G_A(x)(\omega) = \omega(x)$, $(\omega \in \Omega_A)$. It is easy to see that $G_A(x)$ vanishes at infinity. The map

$$G_A : A \to C_0(\Omega_A)$$

is a homomorphism, called the *Gelfand representation* of A. Note that $\|G_A\| \leqslant 1$ and, by (1) and the remarks after Lemma 3 in 1.11, $\operatorname{Ker} G_A = \operatorname{Rad}(A)$.

If, in addition, A is unital, then a character ω on A is non zero if and only if $\omega(1_A) = 1$. In this case the Gelfand spectrum Ω_A is compact. Moreover, we have

$$\sigma_A(x) = \{\omega(x) : \omega \in \Omega_A\}; \quad x \in A. \tag{2}$$

Now suppose that A is non unital and consider on $\widetilde{A}$ a Banach algebra norm extending the norm of A. Then each $\omega \in \Omega_A$ can be extended to a unique character $\widetilde{\omega}$ of $\widetilde{A}$ and the map $\omega \mapsto \widetilde{\omega}$ is a homeomorphism of Ω_A onto a subset of $\Omega_{\widetilde{A}}$, so we may consider $\Omega_A \subset \Omega_{\widetilde{A}}$. Then $\Omega_{\widetilde{A}} \setminus \Omega_A$ contains a single element ω_∞ defined by

$$\omega_\infty(x + \lambda) = \lambda; \quad x \in A,\ \lambda \in \mathbb{C}.$$

By (2) we have:

$$\sigma_A(x) = \{\omega(x) : \omega \in \Omega_{\widetilde{A}}\} \ni \omega_\infty(x) = 0; \quad x \in A. \tag{3}$$

Let A be a C^*-algebra. For every selfadjoint $a \in A$ and every $\omega \in \Omega_A$ we have (Proposition 1.8) $\omega(a) \in \sigma_A(a) \cup \{0\} \subset \mathbb{R}$, and hence

$$\omega(x^*) = \overline{\omega(x)}; \quad x \in A, \, \omega \in \Omega_A.$$

Thus G_A is a $*$-homomorphism. Using again Proposition 1.8 we obtain:

$$\|x\| = r_A(x) = \sup\{|\omega(x)| : \omega \in \Omega_A\}; \quad x \in A,$$

so G_A is isometric. This shows that $G_A(A)$ is a closed $*$-subalgebra of $C_0(\Omega_A)$ $(= \{f \in C(\Omega_{\widetilde{A}}) : f(\omega_\infty) = 0\}$, if A is not unital) which separates the points of $\Omega_{\widetilde{A}}$. By the Stone-Weierstrass theorem we obtain that $G_A(A) = C_0(\Omega_A)$. We record these conclusions in the following

THEOREM. *For every commutative C^*-algebra A, the Gelfand representation is a $*$-isomorphism of A onto $C_0(\Omega_A)$.*

In particular, a C^*-algebra A is unital if and only if Ω_A is compact. Remark that this statement is true for arbitrary Banach algebras.

We note the following useful consequence:

COROLLARY. *Let A be C^*-algebra and $x, y \in A$. If $x^*y = xy^* = 0$ then*

$$\|x + y\| = \max\{\|x\|, \|y\|\}.$$

Proof. Since $x^*y = 0$, we have also $y^*x = (x^*y)^* = 0$ so

$$\|x + y\|^2 = \|(x + y)^*(x + y)\| = \|x^*x + y^*y\|.$$

Now $(x^*x)(y^*y) = x^*(xy^*)y = 0$ thus $B = C^*(\{x^*x, y^*y\})$ is comutative and by the above theorem

$$\begin{aligned}\|x^*x + y^*y\| &= \|G_B(x^*x) + G_B(y^*y)\| = \max\{\|G_B(x^*x)\|, \|G_B(y^*y)\|\} \\ &= \max\{\|x^*x\|, \|y^*y\|\} = (\max\{\|x\|, \|y\|\})^2. \quad \blacksquare\end{aligned}$$

1.15. The following result is complementary to Theorem 1.9.

THEOREM. *Let π be an injective $*$-homomorphism of a C^*-algebra A into an involutive normed algebra B. Then*

$$\|\pi(x) \geqslant \|x\|; \quad x \in A. \tag{1}$$

Proof. It is sufficient to prove (1) only for selfadjoint elements since then, for an arbitrary $x \in A$, we deduce

$$\|x\|^2 = \|x^*x\| \leqslant \|\pi(x^*x)\| = \|\pi(x)^*\pi(x)\| \leqslant \|\pi(x)\|^2.$$

In proving (1) for a selfadjoint element $x \in A$ we can assume that A and B are both commutative and unital, B is complete, $\pi(A)$ is dense in B and $\pi(1_A) = 1_B$. Indeed, we can replace A by $C^*(\{x\})$, B by the completion of $\pi(C^*(\{x\}))$ and then, adjoining unit elements to both A and B, we can extend π appropriately.

In this case the Gelfand spectra Ω_A, Ω_B are compact and the transpose map of π

$$ {}^t\pi : \Omega_B \ni \omega \mapsto \omega \circ \pi \in \Omega_A $$

is a well defined continuous mapping. Hence ${}^t\pi(\Omega_B)$ is a closed subset of Ω_A.

We show that ${}^t\pi(\Omega_B) = \Omega_A$. Indeed, assuming the contrary, by Theorem 1.14 there exist $0 \neq a, b \in A$ such that

$$ ab = 0 \quad \text{and} \quad \theta(a) = 1 \quad \text{for all } \theta \in {}^t\pi(\Omega_B). $$

Then by 1.4.(2)

$$ \sigma_B(\pi(a)) = \{\omega(\pi(a)) : \omega \in \Omega_B\} = \{\theta(a) : \theta \in {}^t\pi(\Omega_B)\} = \{1\} \not\ni 0 $$

hence $\pi(a)$ is invertible. On the other hand, by the injectivity of π we have $\pi(b) \neq 0$ and $\pi(a)\pi(b) = \pi(ab) = 0$, a contradiction.

Using again 1.14.(2) we conclude

$$ \|x\| = \sup\{|\theta(x)| : \theta \in \Omega_A\} = \sup\{|\omega(\pi(x))| : \omega \in \Omega_B\} \leqslant \|\pi(x)\|. \quad \blacksquare $$

By the above theorem and by Theorem 1.9 we have

COROLLARY. *If A,B are C^*-algebras and $\pi : A \to B$ is an injective $*$-homomorphism, then π is isometric. In particular, $\pi(A)$ is a C^*-subalgebra of B.* $\blacksquare$

1.16. Continuous functional calculus for normal elements. We begin with brief review of the analytic functional calculus for arbitrary elements of Banach algebras.

Let A be an arbitrary Banach algebra and $x \in A$. If f is an analytic function on some open supset $D_f \supset \sigma_A(x)$ of $\mathbb{C}$ and Γ is a finite union of closed rectifiable Jordan curves in D_f with mutually disjoint interiors encircling counterclockwise $\sigma_A(x)$, then we put

$$ (1) \qquad f(x) = (2\pi \mathrm{i})^{-1} \int_\Gamma (\lambda - x)^{-1} f(\lambda)\, \mathrm{d}\lambda \in \widetilde{A}. $$

By the Cauchy integral theorem, $f(x)$ does not depend on Γ. If f, g are analytic functions defined on open neighborhoods of $\sigma_A(x)$ and $\lambda \in \mathbb{C}$, then

$$ (2) \qquad (f+g)(x) = f(x) + g(x), \quad (\lambda f)(x) = \lambda f(x), \quad (fg)(x) = f(x)g(x). $$

Here the only non trivial verification is the following: if Γ encircles $\sigma_A(x)$ and Δ encircles Γ, then

$$\begin{aligned}(fg)(x) &= (2\pi \mathrm{i})^{-1} \int_\Gamma (\lambda - x)^{-1} f(\lambda) g(\lambda)\, \mathrm{d}\lambda \\ &= (2\pi \mathrm{i})^{-1} \int_\Gamma (\lambda - x)^{-1} f(\lambda) \Big[(2\pi \mathrm{i})^{-1} \int_\Delta (\mu - \lambda)^{-1} g(\mu)\, \mathrm{d}\mu\Big]\, \mathrm{d}\lambda \\ &\quad - (2\pi \mathrm{i})^{-1} \int_\Delta (\mu - x)^{-1} g(\mu) \Big[(2\pi \mathrm{i})^{-1} \int_\Gamma (\mu - \lambda)^{-1} f(\lambda)\, \mathrm{d}\lambda\Big]\, \mathrm{d}\mu \\ &= (2\pi \mathrm{i})^{-2} \int_\Gamma \int_\Delta (\lambda - x)^{-1} (\mu - x)^{-1} f(\lambda) g(\mu)\, \mathrm{d}\lambda \mathrm{d}\mu = f(x) g(x).\end{aligned}$$

If f is analytic on $\{\lambda \in \mathbb{C} : |\lambda| < r_A(x) + \varepsilon\}$ for some $\varepsilon > 0$ and $f(\lambda) = \sum_{n=0}^{\infty} c_n \lambda^n$ is its power series expansion, then

$$f(x) = \sum_{n=0}^{\infty} c_n x^n. \tag{3}$$

If f is an analytic function on an open neighborhood of $\sigma_A(x)$ then for every $\omega \in \Omega_{\widetilde{A}}$

$$\omega(f(x)) = (2\pi \mathrm{i}\,)^{-1} \int_\Gamma (\lambda - \omega(x))^{-1} f(\lambda)\, \mathrm{d}\lambda = f(\omega(x))$$

so, by 1.14.(3),

$$\sigma_A(f(x)) = f(\sigma_A(x)). \tag{4}$$

Note that if $f(0) = 0$ then $f(x) \in A$. Indeed, if A is not unital then

$$\omega_\infty(f(x)) = (2\pi \mathrm{i})^{-1} \int_\Gamma \lambda^{-1} f(\lambda)\, \mathrm{d}\lambda = f(0) = 0,$$

and hence $f(x) \in \operatorname{Ker} \omega_\infty = A$.

Let f be an analytic function on some open neighborhood of $\sigma_A(x)$, g be an analytic function on some open neighborhood of $\sigma_A(f(x)) = f(\sigma_A(x))$. Then $g \circ f$ is defined and analytic on some open neighborhood of $\sigma_A(x)$ and

$$(g \circ f)(x) = g(f(x)). \tag{5}$$

For each $x \in A$ we denote by $\exp x$ the element $f(x) \in \widetilde{A}$ where f is the entire function defined by $f(\lambda) = \mathrm{e}^\lambda$, $(\lambda \in \mathbb{C})$. By (3)

$$\exp x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Suppose that A is a unital Banach algebra and let $x \in A$ be such that

$$\sigma_A(x) \subset \{r\mathrm{e}^{\mathrm{i}\theta} : r > 0,\, \theta_0 - \pi < \theta < \theta_0 + \pi\}$$

for some $\theta_0 \in \mathbb{R}$. Denote by f the branch of the multivalued analytic function ln defined by

$$f(r\mathrm{e}^{\mathrm{i}\theta}) = \ln r + \mathrm{i}\,\theta; \quad r > 0,\, \theta_0 - \pi < \theta < \theta_0 + \pi,$$

and put $y = f(x)$. Then by (5) and (4) we get $x = \exp y$ and

$$\sigma_A(y) \subset \{\ln r + \mathrm{i}\,\theta : r > 0,\, \theta_0 - \pi < \theta < \theta_0 + \pi,\, r\mathrm{e}^{\mathrm{i}\theta} \in \sigma_A(x)\}.$$

Now let A be a C^*-algebra and $x \in A$ be a normal element. Consider the C^*-subalgebra $C^*(\{x, 1_{\widetilde{A}}\})$ of $\widetilde{A}$ and let Ω be its Gelfand spectrum.

First we show that the map $\chi : \Omega \ni \omega \mapsto \omega(x)$ is a homeomorphism of Ω onto $\sigma_A(x)$. Indeed, by Theorem 1.10 and 1.14.(3) we have $\chi(\Omega) = \sigma_A(x)$. If $\omega_1, \omega_2 \subset \Omega$ and $\omega_1(x) = \omega_2(x)$, then $\omega_1 = \omega_2$ since $\{y \in C^*(\{x, 1_{\widetilde{A}}\}) : \omega_1(y) = \omega_2(y)\}$ is a C^*-subalgebra of $C^*(\{x, 1_{\widetilde{A}}\})$ containing x and $1_{\widetilde{A}}$. Thus, χ is bijective and, since it is also continuous and Ω is compact, χ is a homeomorphism.

Let G be a Gelfand representation of $C^*(\{x, 1_{\widetilde{A}}\})$. Using Theorem 1.14 we can define a $*$-isomorphism

$$C(\sigma(x)) \ni f \mapsto f(x) \in C^*(\{x, 1_{\widetilde{A}}\}),$$

by the formula

$$f(x) = G^{-1}(f \circ \chi); \quad f \in C(\sigma(x)).$$

If $f \in C(\sigma(x))$ can be extended to an analytic function, still denoted by f, on an open neighborhood D_f of $\sigma(x)$, and Γ is a finite union of closed rectifiable Jordan curves in D_f encircling counterclockwise $\sigma(x)$, then for every $\omega \in \Omega$ we have

$$\begin{aligned} G(f(x))(\omega) = f(\omega(x)) &= (2\pi\mathrm{i})^{-1} \int_\Gamma (\lambda - \omega(x))^{-1} f(\lambda)\,\mathrm{d}\lambda \\ &= G[(2\pi\mathrm{i})^{-1} \int_\Gamma (\lambda - x)^{-1} f(\lambda\,\mathrm{d}\lambda)](\omega) \end{aligned}$$

and hence

$$f(x) = (2\pi\mathrm{i})^{-1} \int_\Gamma (\lambda - x)^{-1} f(\lambda)\,\mathrm{d}\lambda.$$

Using Theorem 1.10 and 1.14.(3), for each $f \in C(\sigma(x))$ we get

$$\sigma_A(f(x)) = \{\omega(f(x)) : \omega \in \Omega\} = \{G(f(x))(\omega) : \omega \in \Omega\} = f(\sigma_A(x)). \tag{6}$$

Suppose that $0 \notin \sigma_A(x)$. If $1_{\widetilde{A}} \notin C^*(\{x\})$, then by the Hahn-Banach theorem we obtain a linear functional ω on $C^*(\{x, 1_{\widetilde{A}}\})$ which vanishes on $C^*(\{x\})$ and

$\omega(1_{\widetilde{A}}) = 1$. Therefore $\omega \in \Omega$ and hence $0 = \omega(x) \in \sigma_A(x)$, a contradiction. Thus, in this case, $C^*(\{x, 1_{\widetilde{A}}\}) = C^*(\{x\})$ and therefore, $f(x) \in C^*(\{x\})$ for all $f \in C(\sigma(x))$.

If $0 \in \sigma_A(x)$, then $f(x) \in C^*(\{x\})$ only for those $f \in C(\sigma(x))$ with $f(0) = 0$. Indeed there exists a unique $\omega_\infty \in \Omega$ such that $\omega_\infty(x) = 0$ and we have

$$\omega_\infty(f(x)) = G(f(x))(\omega_\infty) = f(\chi(\omega_\infty)) = f(0).$$

For a continuous complex function f defined on some subset of $\mathbb{C}$ containing $\sigma_A(x)$ we denote $f(x) = (f|\sigma_A(x))(x)$. By the above remarks we have

$$f \in C(\sigma_A(x)) \cup \{0\}),\ f(0) = 0 \Rightarrow f(x) \in C^*(\{x\}). \tag{7}$$

The map $f \mapsto f(x)$ is characterized as the unique $*$-homomorphism $\pi : C(\sigma_A(x)) \to \widetilde{A}$ such that:

$$\begin{aligned} &\text{if } f(\lambda) = 1 \text{ for all } \lambda \in \sigma_A(x), \text{ then } \pi(f) = 1_{\widetilde{A}}; \\ &\text{if } f(\lambda) = \lambda \text{ for all } \lambda \in \sigma_A(x), \text{ then } \pi(f) = x. \end{aligned}$$

Indeed, such a π must coincide with $f \mapsto f(x)$ on the algebra of polynomials in λ and $\overline{\lambda}$, which are dense in $C(\sigma_A(x))$ by the Stone-Weierstrass theorem.

The $*$-isomorphism $C(\sigma_A(x)) \ni f \mapsto f(x) \in C^*(\{x, 1_{\widetilde{A}}\})$ is called the *continuous functional calculus for the normal element $x \in A$*. As we have seen, it extends the "analytic functional calculus".

Using the real version of the Stone-Weierstrass theorem, we obtain the following useful result:

$$\begin{aligned} &\text{if } x = x^* \in A,\ f \in C(\sigma_A(x) \cup \{0\}),\ f \text{ real},\ f(0) = 0 \\ &\text{then } f(x) \in \text{the norm-closed real subalgebra of } A \text{ generated by } x. \end{aligned} \tag{8}$$

Let H be a complex Hilbert space. Since $B(H)$ is a C^*-algebra all the above results apply to normal operators in $B(H)$.

1.17. Fuglede-Putnam theorem. For each element x of a C^*-algebra A we can consider $\exp x = \sum\limits_{n=0}^{\infty} x^n/n! \in \widetilde{A}$ (1.16). Clearly,

$$\begin{aligned} (\exp x)^* &= \exp x^* && \text{for all } x \in A; \\ (\exp x)(\exp y) &= \exp(x + y) && \text{if } x, y \in A \text{ and } xy = yx. \end{aligned}$$

In particular, if $x \in A$ is selfadjoint, then $\exp(\mathrm{i}\, x)$ is unitary.

Theorem. *Let A be a C^*-algebra and $x_1, x_2, y \in A$. If x_1, x_2 are normal and $x_1 y = y x_2$, then $x_1^* y = y x_2^*$.*

Proof. The map $f : \mathbb{C} \ni \lambda \mapsto \exp(-\lambda x_1^*) y \exp(\lambda x_2^*) \in A$ is an entire function with respect to the norm. Using $x_1 y = y x_2$ we infer $y = \exp(\overline{\lambda} x_1) y \exp(-\overline{\lambda} x_2)$, $(\lambda \in \mathbb{C})$, thus

$$\begin{aligned} f(\lambda) &= \exp(-\lambda x_1^*) \exp(\overline{\lambda} x_1) y \exp(-\overline{\lambda} x_2) \exp(\lambda x_2^*) \\ &= \exp(\mathrm{i}\,(\mathrm{i}\,\lambda x_1^* - \mathrm{i}\,\overline{\lambda} x_1)) y \exp(\mathrm{i}\,(\mathrm{i}\,\overline{\lambda} x_2 - \mathrm{i}\,\lambda x_2^*)). \end{aligned}$$

The elements $\mathrm{i}\,\lambda x_1^* - \mathrm{i}\,\overline{\lambda} x_1$ and $\mathrm{i}\,\overline{\lambda} x_2 - \mathrm{i}\,\lambda x_2^*$ being selfadjoint, it follows that the function f is bounded. Therefore f is a constant function, by Liouville's theorem and hence its derivative vanishes:

$$0 = f'(\lambda) = -x_1^* \exp(-\lambda x_1^*) y \exp(\lambda x_2^*) + \exp(-\lambda x_1^*) y \exp(\lambda x_2^*) x_2^*.$$

In particular, $f'(0) = 0$, that is $x_1^* y = y x_2^*$. ∎

1.18. In this section we record some working properties and some consequences of the continuous functionals calculus.

Let A be a C^*-algebra.

(1) *If $x \in A$ is normal, $f \in C(\sigma_A(x))$ and $g \in C(\sigma_A(f(x))) = C(f(\sigma_A(x)))$, then $f(x)$ is normal and*, by the uniqueness of the continuous functional calculus,

$$(g \circ f)(x) = g(f(x)).$$

(2) *If $x, y \in A$ are normal and $xy = yx$ then*, by Theorem 1.17, the C^*-algebra $C^*(\{x, y, 1_A\})$ is commutative and hence, *for every $f \in C(\sigma_A(x))$ and every $g \in C(\sigma_A(y))$ we have $f(x)g(y) = g(y)f(x)$.*

(3) Also, *if $x \in A$ is normal, $y \in A$, $x = xy = y^* x$ and $f \in C(\sigma_A(x))$, then*

$$f(x) = f(x)y + f(0)(1_{\widetilde{A}} - y) = y^* f(x) + f(0)(1_{\widetilde{A}} - y^*).$$

Indeed, this can be easily verified for polynomials in λ and $\overline{\lambda}$ which are uniformly dense in $C(\sigma_A(x))$.

(4) *Let $f \in C([-1, 1])$. For every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $x \in A$, $x = x^*$, $\|x\| \leqslant 1$ and every $y \in A$, $\|y\| \leqslant 1$ we have*

$$\|xy - yx\| \leqslant \delta \Rightarrow \|f(x)y - yf(x)\| \leqslant \varepsilon.$$

Indeed, there exists a polynomial $p(t) = \sum_{k=0}^{n} \lambda_k t^k$, $(t \in \mathbb{R})$, such that

$$\|f - p\| = \sup\{|f(t) - p(t)| t \in [-1, 1]\} \leqslant \frac{\varepsilon}{4}.$$

Define

$$\delta = \frac{\varepsilon}{2} \sum_{k=1}^{n} k|\lambda_k|,$$

and let $x, y \in A$, $\|x\| \leqslant 1$, $\|y\| \leqslant 1$, $x = x^*$ be such that $\|xy - yx\| \leqslant \delta$.

Consider the linear mapping $D : A \to A$ defined by

$$D(a) = ay - ya; \quad a \in A.$$

Clearly, $\|D\| \leqslant 2$. Since

$$\|D(x^{k+1})\| = \|xD(x^k) + D(x)x^k\| \leqslant \|D(x^k)\| + \|D(x)\|,$$

by induction we get

$$\|D(x^k)\| \leqslant k\|D(x)\| \leqslant k\delta; \quad 1 \leqslant k \leqslant n.$$

It follows that

$$\|D\left(p(x)\right)\| = \Big\|\sum_{k=1}^{n} \lambda_k D(x^k)\Big\| \leqslant \Big(\sum_{k=1}^{n} k|\lambda_k|\Big)\delta = \frac{\varepsilon}{2},$$

and hence $\|f(x)y - yf(x)\| = \|D(f(x))\| \leqslant \|D(p(x))\| + 2\|f(x) - p(x)\| \leqslant \varepsilon$.

(5) *Let $\Omega \subset \mathbb{C}$ be such that $\overline{(\overline{\Omega} \setminus \Omega)} \cap \Omega = \emptyset$ and f be a continuous complex function on Ω. Then the map*

$$\{x \in A : x \text{ normal}, \sigma_A(x) \subset \Omega\} \ni x \mapsto f(x) \in \widetilde{A}$$

is norm continuous.

Indeed, let $x \in A$ be normal, $\sigma_A(x) \subset \Omega$ and $\varepsilon > 0$. Since $\overline{(\overline{\Omega} \setminus \Omega)} \cap \sigma_A(x) = \emptyset$, there exists a compact neighbourhood N of $\sigma_A(x)$ such that $\overline{(\overline{\Omega} \setminus \Omega)} \cap N = \emptyset$. Then

$$\Omega \cap N = (\Omega \cup (\overline{\Omega} \setminus \Omega)) \cap N = \overline{\Omega} \cap N$$

is compact and hence by the Stone-Weierstrass theorem there exists a polynomial p in λ and $\overline{\lambda}$ such that $|f(\lambda) - p(\lambda, \overline{\lambda})| \leqslant \varepsilon/3$ for all $\lambda \in \Omega \cap N$. Finally, by the lower semicontinuity of the spectrum (1.7), there exists $\delta > 0$ such that if $y \in A$, $\|x - y\| \leqslant \delta$, then $\sigma_A(y) \subset N$ and $\|p(x, x^*) - p(y, y^*)\| \leqslant \varepsilon/3$. For $y \in A$ normal with $\sigma_A(y) \subset \Omega$ and $\|x - y\| \leqslant \delta$ we have $\sigma_A(y) \subset \Omega \cap N$, hence

$$\|f(x) - f(y)\| \leqslant \|f(x) - p(x, x^*)\| + \|p(x, x^*) - p(y, y^*)\| + \|p(y, y^*) - f(y)\| \leqslant \varepsilon.$$

Note that if $\Omega \subset \mathbb{C}$ is either open or closed or a subinterval of $\mathbb{R}$ or an arc on $\{\lambda \in \mathbb{C} : |\lambda| = 1\}$, then the condition $\overline{(\overline{\Omega} \setminus \Omega)} \cap \Omega = \emptyset$ is satisfied.

(6) *A normal element $x \in A$ is selfadjoint if and only if $\sigma_A(x) \subset \mathbb{R}$.*

Indeed, if x is selfadjoint, then $\sigma_A(x) \in \mathbb{R}$ (Proposition 1.8). Conversely, if $\sigma_A(x) \subset \mathbb{R}$, then $f : \lambda \mapsto \lambda$ and $g : \lambda \mapsto \overline{\lambda}$ concide on $\sigma_A(x)$, hence $x = f(x) = g(x) = x^*$.

Similarly, *a normal element $x \in A$ is unitary if and only if $\sigma_A(x) \subset \{\lambda \in \mathbb{C} : |\lambda| = 1\}$.*

(7) By (6), *if* $x \in A$ *is selfadjoint, then* $\exp(\mathrm{i}\,x)$ *is unitary.*
Conversely, let $u \in \widetilde{A}$ *be unitary and suppose that*

$$\sigma_{\widetilde{A}}(u) \subset \left\{ \mathrm{e}^{\mathrm{i}\theta} : \theta_0 - \pi < \theta < \theta_0 + \pi \right\}$$

for some $\theta_0 \in \mathbb{R}$. *Denote by* f *the continuos function defined on* $\left\{ \mathrm{e}^{\mathrm{i}\theta} : \theta_0 - \pi < \theta < \theta_0 + \pi \right\}$ *by* $f(\mathrm{e}^{\mathrm{i}\theta}) = \theta$ *and put* $x = f(u) \in \widetilde{A}$. *Then* x *is selfadjoint and* $u = \exp(\mathrm{i}\,x)$. *Moreover,*

$$\sigma_{\widetilde{A}}(x) \subset \left\{ \theta : \theta_0 - \pi < \theta < \theta_0 + \pi,\, \mathrm{e}^{\mathrm{i}\theta} \in \sigma_{\widetilde{A}}(u) \right\}.$$

In particular, *every unitary element* $u \in \widetilde{A}$ *with* $\sigma_{\widetilde{A}}(u) \neq \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ *belongs to* $\exp(\mathrm{i}\,\widetilde{A_h}) = \{\exp(\mathrm{i}\,x) : x \in \widetilde{A_h}\}$. This happens, for instance, if $\|\lambda_0 - u\| < 2$ for some $\lambda_0 \in \mathbb{C}$, $|\lambda_0| = 1$ since then $-\lambda_0 \notin \sigma_{\widetilde{A}}(u)$.

(8) *Let* B *be a unital* C^**-algebra and* $\pi : \widetilde{A} \to B$ *be a* $*$*-homomorphism such that* $\pi(1_{\widetilde{A}}) = 1_B$. *Then, for every normal* $x \in \widetilde{A}$, $\pi(x) \in B$ *is normal,* $\sigma_B(\pi(x)) \subset \sigma_{\widetilde{A}}(x)$ *and, by the uniqueness of the continuous functional calculus, we have*

$$\pi(f(x)) = f(\pi(x)); \quad f \in C(\sigma_{\widetilde{A}}(x)).$$

1.19. Notes. For the introduction of C^*-algebras and the main results in this section we refer to the fundamental contributions of I.M. Gelfand and M.A. Naĭmark [105], [106], to the articles [16], [103], [168], [256], [257], [276] and to the monographs [33], [78], [80], [81], [163], [164], [213], [258]. In our exposition we used mainly the books of F.F. Bonsall and J. Duncan [33] and J. Dixmier [78].

I.M. Gelfand and M.A. Naĭmark [106] defined the notion of a C^*-algebra by the following axioms:

(I) A is a $*$-algebra;
(II) A is a Banach space, with the vector structure of (I);
(III) $\|xy\| \leqslant \|x\|\,\|y\|$, for any $x, y \in A$;
(IV) $\|x^*x\| = \|x^*\|\,\|x\|$, for any $x \in A$;
(V) $\|x^*\| = \|x\|$, for any $x \in A$;
(VI) $1 + x^*x$ is invertible in A, for any $x \in A$.

With this definition, they proved that every C^*-algebra is $*$-isomorphic to a norm-closed $*$-subalgebra of $B(H)$ for some Hilbert space H (see 4.11) and conjectured that axiom (VI) and axiom (V) should follow from the other axioms.

The redundancy of (VI) has been proved by I.M. Gelfand and M.A. Naĭmark [106], R. Arens [16] and M. Fukamiya [103] in the commutative case, and by M. Fukamiya [103], I. Kaplansky (see [276]), J.L. Kelley and R.L. Vaught [168] in the general case.

J. Glimm and R.V. Kadison [113] and T. Ono [222] proved that axiom (V) also follows from axioms (I)–(IV) if A has a unit element, and J. Vowden [341] proved the same for the general case, thus solving positively and completely the Gelfand-Naĭmark conjecture.

The term of a "C^*-algebra" has been introduced in fact by I.E. Segal [282], while J. Dixmier proposed the name of "Gelfand-Naĭmark algebras" for this class (cf. [163], p. 5). Here we adopted Dixmier's terminology for concrete C^*-algebras. For a while, the algebras satisfying just the axioms (I)–(IV) were called "B^*-algebras" in contrast to "C^*-algebras", but now there is no more difference between these two terms.

The conjuction of axioms (IV) and (V) is obviously equivalent to

(IV′) $\|x^*x\| = \|x\|^2$, for any $x \in A$,

this being the axiom we used here. H. Araki and G.A. Elliott [15], [90], have shown that axiom (III) is a consequence of (I), (II), and (IV′); moreover, they have also shown that axiom (III) follows from (I), (II), and (IV) provided that the $*$-operation is assumed to be continuous (see also [278], [279], [280], [281]).

The Gelfand-Naĭmark conjecture has led to outstanding achievements in the theory of general Banach $*$-algebras. In what follows we sketch a brief survey of some important results related to the axiomatic theory of C^*-algebras, guided by [28], [33], [253], [258] and using some material from Chapters 2 through 6.

We begin with two general results in Banach algebras (compare with 1.10 and 1.9):

1. Theorem ([257]; see [33], p. 25). *Let B be a closed subalgebra of a Banach algebra A and let $b \in B$. Then $\sigma_A(b) \subset \sigma_B(b) \cup \{0\}$ and the boundary of $\sigma_B(b) \subset$ the boundary of $\sigma_A(b)$.*

2. Theorem ([135]; see [33], p. 130, [137], [256]). *If A is a semisimple Banach algebra, then any other Banach algebra norm on A is equivalent to the original norm on A.*

Now *let A be a Banach $*$-algebra.* From Theorem 2 we get

3. Theorem (see [33], p. 191). *If A is semisimple, then the $*$-operation is continuos.*

Put

$$p(x) = r(x^*x)^{1/2}; \quad x \in A.$$

The *$*$-radical* of A, denoted as $*$-Rad(A), is defined as the intersection of the kernels of all (irreducible) $*$-representation of A on Hilbert spaces. Then

$$\mathrm{Rad}(A) \subset p^{-1}(0) \subset \{x \in A : \|x\|_* = 0\} = *\text{-}\mathrm{Rad}(A).$$

4. Theorem ([103], [168]; see [33], p. 223). *If A is unital, then*

$$*\text{-}\mathrm{Rad}(A) = \{x \in A : -x^*x \in \overline{A^+}\}.$$

A Banach $*$-algebra A is called *hermitian* if $\sigma(a) \subset \mathbb{R}$ for any $a \in A$, $a^* = a$. Then $\widetilde{A}$ is also hermitian. The following theorem contains the solution to a conjecture of I. Kaplansky [159]:

5. Theorem ([252], [288]; see [33], pp. 224–228, [253], 5.10). *If A is unital then the following statements are equivalent:*

(i) *A is hermitian;*
(ii) *$\sigma(x^*x) \subset \mathbb{R}^+$, for all $x \in A$;*
(iii) *$1 + x^*x$ is invertible, for all $x \in A$;*
(iv) *$r(x)^2 \leqslant r(x^*x)$, i.e. $r(x) \leqslant p(x)$, for all $x \in A$;*
(v) *$r(y)^2 = r(y^*y)$, for every normal $y \in A$;*
(vi) *$r(u) \leqslant \beta$, for every unitary $u \in A$ and some $\beta > 0$;*
(vii) *$r(u) = 1$, for every unitary $u \in A$;*
(viii) *p is subadditive;*
(ix) *p is (the greatest) C^*-seminorm on A.*

6. Corollary (see [33], p. 227). *If A is unital and hermitian, then*

$$\mathrm{Rad}(A) = *\text{-}\mathrm{Rad}(A) = p^{-1}(0).$$

7. THEOREM (see [253], 8.4). *If A is unital then the following statements are equivalent:*
(i) *A has an equivalent C^*-norm;*
(ii) *$\|x^*\|\,\|x\| \leqslant \alpha\|x^*x\|$, for all $x \in A$ and some $\alpha > 0$;*
(iii) *$\|u\| \leqslant \beta$, for every unitary $u \in A$ and some $\beta > 0$;*
(iv) *$\|\exp(\mathrm{i}\,a)\| \leqslant \gamma$, for every $a \in A$, $a^* = a$ and some $\gamma > 0$;*
(v) *A is hermitian and $\|a\| \leqslant \delta r(a)$ for every $a \in A$, $a^* = a$ and some $\delta > 0$.*

For related results see [22], [126], [349]. The following result contains the Glimm-Kadison solution of the Gelfand-Naĭmark conjecture:

8. THEOREM (see [253], 10.1). *If A is unital, then the following statements are equivalent:*
(i) *A is a C^*-algebra;*
(ii) *$\|x^*\|\,\|x\| = \|x^*x\|$, for all (normal) $x \in A$;*
(iii) *$\|x\| = p(x)$, for all $x \in A$;*
(iv) *$\|u\| = 1$, for every unitary $u \in A$;*
(v) *$\|\exp(\mathrm{i}\,a)\| = 1$, for every $a \in A$, $a^* = a$.*

An *A^*-algebra* is a Banach $*$-algebra such that the greatest C^*-seminorm is a norm ([33], p. 214). Any semisimple hermitian Banach $*$-algebra is an A^*-algebra ([33], p. 227), but there exists an A^*-algebra which is not hermitian ([107]; cf. [33], p. 227).

The last theorem is also a consequence of the following powerful result, known as the *Vidav-Palmer Theorem*, which characterizes the unital C^*-algebras among all unital Banach algebras by means of a certain "local differential condition".

Let A be a unital Banach algebra. An element $a \in A$ is called a *hermitian element* if

$$\|\exp(\mathrm{i}\,ta)\| = 1 \qquad \text{for every } t > 0.$$

The set of all hermitian elements of A is denoted by $H(A)$.

9. THEOREM ([223], [336]; see also [26], [37], [110], [184], [337]). *A unital Banach algebra A is a C^*-algebra with respect to the involution*

$$(a + \mathrm{i}\,b)^* = a - \mathrm{i}\,b; \quad a, b \in H(A),$$

if and only if $A = H(A) + \mathrm{i}\,H(A)$.

The importance of this theorem justifies a more detailed discussion of the ideas involved in its proof ([33], p. 211). We do not follow the historical line (see for this [32], [33]).

The following outstanding result has considerably simplified the proof:

10. THEOREM ([290]; see [31], [33], pp. 56–57). *The norm of a hermitian element a of a unital Banach algebra A is equal to its spectral radius: $\|a\| = r(a)$.*

The technical instrument used in the proof of Theorem 9 is the *numerical range* $V(x)$ and the *numerical radius* $v(x)$ of an element x of a unital Banach algebra A ([30], [32], [184]) defined by

$$V(x) = \{f(x) : f \in A^*,\ f(1) = 1 = \|f\|\}, \quad v(x) = \sup\{|\lambda| : \lambda \in V(x)\}.$$

Then ([33], pp. 51–55) $V(x)$ is a compact convex subset of $\mathbb{C}$ which does not depend on the ambiant unital Banach algebra and $\sigma(x) \subset V(x)$. Moreover,

$$\begin{aligned} \max \operatorname{Re} V(x) &= \inf_{\alpha>0} \alpha^{-1}(\|1 + \alpha x\| - 1) = \lim_{\alpha \to 0^+} \alpha^{-1}(\|1 + \alpha x\| - 1) \\ &= \sup_{\alpha>0} \alpha^{-1} \log \|\exp(\alpha x)\| = \lim_{\alpha \to 0^+} \alpha^{-1} \log \|\exp(\alpha x)\|. \end{aligned}$$

11. Theorem (see [33], p. 56). *$r(x) \leqslant v(x) \leqslant \|x\| \leqslant \mathrm{ev}(x)$; $x \in A$.*

12. Theorem (see [32], p. 46; [33], p. 55). *The following conditions on $x \in A$ are equivalent:*

(i) *$x \in H(A)$;*

(ii) *$\lim\limits_{\alpha \to 0} \alpha^{-1}(\|1 + \mathrm{i}\,\alpha x\| - 1) = 0$;*

(iii) *$V(x) \subset \mathbb{R}$.*

Using Theorems 10 through 12, it follows that:

13. Corollary (see [33], p. 206). *$V(x) = \mathrm{co}\,\sigma(x)$ for each $x \in A$ of the form $x = a + \mathrm{i}\,b$ with $a, b \in H(A)$, $ab = ba$.*

14. Corollary (see [33], p. 206). *$H(A)$ is a real Banach space and $\mathrm{i}\,(ab - ba) \in H(A)$ whenever $a, b \in H(A)$.*

In general, $H(A) + \mathrm{i}\,H(A)$ need not be a subalgebra of A (see [32], pp. 57–60). However, it is a Banach subspace of A and the $*$-operation $(a+\mathrm{i}\,b) = a-\mathrm{i}\,b$, $(a, b \in H(A))$, is continuous ([32], p. 50).

15. Lemma (see [32], p. 59; [33], p. 207). *If $H(A) + \mathrm{i}\,H(A)$ is an algebra, then it is a Banach $*$-algebra with continuous involution and with $H(A)$ as its set of selfadjoint elements.*

Define

$$P(A) = \left\{x \in A : V(x) \subset \mathbb{R}^+\right\}.$$

By Corollary 13, $x \in P(A)$ if and only if $x \in H(A)$ and $\sigma(x) \subset \mathbb{R}^+$). Also, $P(A)$ is a normal closed cone in $H(A)$ and 1 is an interior point of $P(A)$ ([32], pp. 48–49).

Assume now that $A = H(A) + \mathrm{i}\,H(A)$. The proofs of the next lemmas ([33], pp. 207–208) are similar to those of their analogous results in C^*-algebras:

16. Lemma. *Let $a \in P(A)$ and let C denote the Banach subalgebra of A generated by a and 1. Then there is $b \in C \cap P(A)$ such that $a = b^2$.*

17. Lemma. *For any $x \in H(A)$ there are $a, b \in P(A)$ such that $x = a - b$ and $ab = ba = 0$.*

18. Lemma. *$x^*x \in P(a)$ for any $x \in A$.*

The proof of Theorem 9 is now completed by using the GNS-construction (4.3), Theorem 1.14 and the Russo-Dye Theorem (6.3) (see [33], p. 211). We record two more consequences of Theorem 9:

19. Corollary (see [33], p. 214). *If A is a unital Banach $*$-algebra such that a bounded linear functional f on A is selfadjoint whenever $f(1) = \|f\|$, then A is a C^*-algebra.*

20. Corollary (see [33], p. 213). *A unital Banach $*$-algebra A such that*

(i) *$\sigma(a) \subset \mathbb{R}$ for each $a \in A$, $a^* = a$;*

(ii) *$r(1 + \lambda a) = \|1 + \lambda a\|$ for each $a \in A$, $a^* = a$ and any $\lambda \in \mathbb{C}$,*

is a C^*-algebra.

Also,

21. Corollary ([223], II.3.10). *A unital Banach algebra is isometrically isomorphic to a C^*-algebra if and only if it is linearly isometric to a C^*-algebra.*

For other applications of Theorem 9 see [32], pp. 67–79. A dual characterization of C^*-algebras appears in [199] (see also [20]). There is also a characterization of C^*-algebras just as Banach spaces (see [223], II.4.6). For further results see also [224], [225].

Finally, we record the following interesting result:

22. Theorem ([220]). *Two C^*-algebras which are algebraically isomorphic are also $*$-isomorphic.*

Further refinements of this theorem, suggested by I. Kaplansky ([154], p. 11) appear in [216], [219].

Chapter 2

POSITIVE ELEMENTS

2.1. An element x of the C^*-algebra A is called *positive* if it is normal and $\sigma(x) \subset [0, +\infty)$. By functional calculus (1.18.(5)), x is then selfadjoint.

If $y \in A$ is selfadjoint, then $x = y^2$ is positive since $\sigma(y) \subset \mathbb{R}$, so that $\sigma(x) = \{\lambda^2;\ \lambda \in \sigma(y)\} \subset [0, +\infty)$.

We denote

$$A^+ = \{x \in A;\ x \text{ is positive}\}.$$

Suppose that A is unital. By Gelfand representation (1.14), for $x \in A_h$ and any $\lambda \geqslant \|x\|$ we have

$$x \in A^+ \Leftrightarrow \|\lambda - x\| \leqslant \lambda. \tag{1}$$

PROPOSITION. *For every C^*-algebra A, A^+ is a closed convex cone and $A^+ \cap (-A^+) = \{0\}$.*

Proof. Passing to associate unital C^*-algebra, we may suppose A unital. It is clear that $\lambda x \in A^+$ for any $x \in A^+$, $\lambda \geqslant 0$. If $x, y \in A^+$ then, using (1)

$$\|(\|x\| + \|y\|) - (x + y)\| \leqslant \|\,\|x\| - x\| + \|\,\|y\| - y\| \leqslant \|x\| + \|y\|,$$

so that $x + y \in A^+$. Using again (1) we see also that A^+ is closed. Finally, if $x \in A^+ \cap (-A^+)$, then $x \in A_h$ and $\sigma(x) = \{0\}$, therefore $\|x\| = r(x) = 0$, $x = 0$. ∎

The set A^+ is called the *positive cone* of A.

The above proposition shows that the relation $\leqslant$ defined on A_h by

$$s \leqslant y \Leftrightarrow y - x \in A^+$$

is an *order relation* on A_h.

2.2. PROPOSITION. *Let A be a C^*-algebra. For any $x \in A^+$ there is a unique $a \in A^+$ such that $a^2 = x$.*

Proof. The existence is clear by functional calculus, namely $a = f(x)$ with $f(t) = t^{1/2}$, $t \in [0, +\infty)$. Now let $b \in A^+$, $b^2 = x$. Then b commutes with x, hence it commutes also with $a = f(x)$. Using the Gelfand representation of the C^*-subalgebra generated by a and b, we deduce $b = a$. ∎

The unique $a \in A^+$ with $x = a^2$ will be denoted by $x^{1/2}$. Since $x^{1/2} = f(x)$ as in the above proof, by 1.18.(4) it follows that the mapping

$$A^+ \ni x \mapsto x^{1/2} \in A^+$$

is norm continuous.

2.3. PROPOSITION. *Let A be a C^*-algebra. For any $x \in A_h$ there exist $a, b \in A^+$ uniquely determined such that $x = a - b$, $ab = 0$.*

Proof. Again the existence part is clear by functional calculus, namely $a = f^+(x)$, $b = f^-(x)$, with $f^+(t) = \max(t, 0)$, $f^-(t) = \max(-t, 0)$, $t \in \mathbb{R}$. Now let a, b be as in the statement. Since $ab = 0$ we have $a + b = (a^{1/2} + b^{1/2})^2 \in A^+$ and $(a + b)^2 = x^2$, therefore $a + b = (x^2)^{1/2}$. This, together with $a - b = x$, proves the uniqueness part. ∎

The unique $a, b \in A^+$ with $x = a - b$, $ab = 0$, will be denoted as $x^+ = a$, $x^- = b$. Note that $\|x\| = \max\{\|x^+\|, \|x^-\|\}$.

The proposition shows that $A_h = A^+ - A^+$, therefore every C^*-algebra is the linear span of its positive elements.

2.4. THEOREM. *For an element x of a C^*-algebra A, the following conditions are equivalent:*

(i) $x \in A^+$;

(ii) $x = \sum_{k=1}^{n} y_k^* y_k$ *with* $y_k \in A$, $(1 \leqslant k \leqslant n;\ n \in N)$.

Proof. (i) $\Rightarrow$ (ii) follows with $n = 1$ and $y_1 = x^{1/2}$.

(ii) $\Rightarrow$ (i). Since A^+ is a convex cone, it suffices to show that $x = y^*y \in A^+$ for any $y \in A$. We first remark that:

$$z \in A, \quad z^*z \in -A^+ \Rightarrow z = 0. \tag{1}$$

Indeed, since $\sigma(z^*z) \cup \{0\} = \sigma(zz^*) \cup \{0\}$ by 1.7.(1) we have also $zz^* \in -A^+$. Thus, putting $a = \operatorname{Re} z$, $b = \operatorname{Im} z$, we get

$$z^*z = 2a^2 + 2b^2 + (-zz^*) \in A^+,$$

so that $z^*z \in A^+ \cap (-A^+) = \{0\}$.

Since $x = x^+ - x^-$, $x^+ \in A^+$, $x^- \in A^+$ and $x^+x^- = 0$, we have

$$(yx^-)^*yx^- = x^-(y^*y)x^- = x^-(x^+ - x^-)x^- = -(x^-)^3 \in (-A^+),$$

thus $yx^- = 0$ by (1). Consequently, $(x^-)^2 = -y^*yx^- = 0$, hence $x^- = 0$ and we conclude $x = x^+ \in A^+$. ∎

For an arbitrary $x \in A$ we can therefore define its *modulus*

$$|x| = (x^*x)^{1/2} \in A^+.$$

Note that $|x| = |x^*| \Leftrightarrow x$ is normal. If x is selfadjoint, then $|x| = x^+ + x^-$. Remark that the functions $x \mapsto |x|$, $x \mapsto x^+$, $x \mapsto x^-$ are norm continuous on their domains.

COROLLARY. *If $\pi : A \to B$ is a $*$-homomorphism between C^*-algebras, then $\pi(A^+) = \pi(A) \cap B^+$.*

Proof. Consider $y \in \pi(A) \cap B^+$ and let $x \in A$ be such that $y = \pi(x)$. Then $\pi(|x|) = \pi(|x|^{1/2})^2 \in B^+$ and $\pi(|x|)^2 = \pi(x^*x) = y^2$, hence $y = \pi(|x|) \in \pi(A^+)$ by Proposition 2.2. Conversely, if $x \in A^+$, then $\pi(x) = \pi(x^{1/2})^2 \in \pi(A) \cap B^+$. ∎

2.5. Examples. (i) Let Ω be a topological space, $C(\Omega)$ be the C^*-algebra of all bounded continuous complex functions on Ω and $f \in C(\Omega)$. Then $f \in C(\Omega)^+$ means that $f(t) \geqslant 0$ for all $t \in \Omega$.

(ii) Let $B(H)$ be the C^*-algebra of all bounded linear operators on the Hilbert space H and $x \in B(H)$. Then

$$x \in B(H)^+ \Leftrightarrow (x\xi|\xi) \geqslant 0 \quad \text{for all } \xi \in H.$$

Indeed, if $x = y^*y$, then $(x\xi|\xi) = \|y\xi\|^2 \geqslant 0$. Conversely, if $(x\xi|\xi) \geqslant 0$, $\xi \in H$, then x is selfadjoint and

$$0 \leqslant (x(x^-\xi)|x^-\xi) = -((x^-)^3\xi|\xi) \leqslant 0, \quad \xi \in H,$$

so that $(x^-)^3$ and $x = x^+ \in A^+$.

As an application, we show that for $x \in B(H)$, $x^* = x$,

$$\|x\| = \sup\{|(x\xi|\xi)|;\ \xi \in H,\ \|\xi\| = 1\}. \tag{1}$$

Indeed, $x = x^+ - x^-$, $x^+x^- = 0$ and $\|x\| = \max\{\|x^+\|, \|x^-\|\}$ by 2.3. Let $\varepsilon > 0$. Denote by H^+ the closure of the subspace $\{(x^+)^{1/2}\xi;\ \xi \in H\}$. Then $(x^+)^{1/2}|H \ominus H^+ = 0$ and $x^-|H^+ = 0$. It follows that there is $\eta \in H^+$, $\|\eta\| = 1$ such that

$$(x\eta|\eta) = (x^+\eta|\eta) = \|(x^+)^{1/2}\eta\|^2 \geqslant \|x^+\| - \varepsilon.$$

Similarly, $(x\zeta|\zeta) \geqslant \|x^-\| - \varepsilon$ for some $\zeta \in H$, $\|\zeta\| = 1$. This proves (1).

(iii) Finally, let Ω be a topological space, A be a C^*-algebra and $C(\Omega, A)$ be the C^*-algebra of all bounded continuous A-valued functions on Ω. For $f \in C(\Omega, A)$ we have

$$f \in C(\Omega, A)^+ \Leftrightarrow f(t) \in A^+ \quad \text{for all } t \in \Omega.$$

Indeed, if $f = g^*g$, $(g \in C(\Omega, A))$, then $f(t) = g(t)^*g(t)$ for all $t \in \Omega$. Conversely, if $f(t) \in A^+$ for all $t \in \Omega$, then f is a selfadjoint element of $C(\Omega, A)$ and

$$0 \leqslant f^-(t)f(t)f^-(t) = -(f^-)^3(t) \leqslant 0, \quad t \in \Omega,$$

so that $f^- = 0$, $f = f^+ \in C(\Omega, A)^+$.

2.6. This section assembles some useful properties of the order relation in a C^*-algebra A.

(1) *For* $a, b, x \in A$ *we have* $a \leqslant b \Rightarrow x^*ax \leqslant x^*bx$, since $b - a = y^*y$, $y \in A$, entails $x^*bx - x^*ax = (yx)^*(yx)$.

(2) In particular, *if* $a, b \in A^+$ *and* $ab = ba$ *then* $ab = b^{1/2}ab^{1/2} \in A^+$.

(3) *If* $a \in A^+$ *then*, by Gelfand representation, $\|a\| \leqslant \lambda \Leftrightarrow a \leqslant \lambda 1$ *in* $\widetilde{A}$. Therefore, *if* $0 \leqslant a \leqslant b$, *then* $a \leqslant \|b\|$ *so that* $\|a\| \leqslant \|b\|$.

(4) Again by Gelfand representation, *for* $a \in A^+$, *we have* $\|a\| \leqslant 1 \Leftrightarrow a^2 \leqslant a$.

(5) Once more by Gelfand representation, *if* $a \in A^+$, *then* $\|P(a)\| = P(\|a\|)$ *for every polinomial* P *with positive coefficients.*

(6) *Let* $a, b, c, x, y \in A$, $xx^* \leqslant aa^*$, $y^*y \leqslant b^*b$. *Then* $\|ycx\| \leqslant \|bca\|$. Indeed,

$$\begin{aligned}\|ycx\|^2 &= \|x^*c^*y^*ycx\| \leqslant \|x^*c^*b^*bcx\| = \|(bcx)^*(bcx)\| \\ &= \|(bcx)(bcx)^*\| = \|bcxx^*c^*b^*\| \leqslant \|bcaa^*c^*b^*\| = \|bca\|^2.\end{aligned}$$

(7) *If* $a \in A^+$ *is invertible in* $\widetilde{A}$, *then* $a^{-1} \in \widetilde{A}^+$ *and* $(a^{-1})^{1/2} = a^{-1/2}$. Moreover, *if* $a, b \in A^+$ *are invertible in* $\widetilde{A}$ *and* $a \leqslant b$, *then* $b^{-1} \leqslant a^{-1}$. Indeed,

$$\begin{aligned}a \leqslant b &\Rightarrow b^{-1/2}ab^{-1/2} \leqslant 1 \Rightarrow \|a^{1/2}b^{-1/2}\|^2 = \|b^{-1/2}ab^{-1/2}\| \leqslant 1 \\ &\Rightarrow \|a^{1/2}b^{-1}a^{1/2}\| = \|b^{-1/2}a^{1/2}\|^2 = \|(a^{1/2}b^{-1/2})^*\|^2 \leqslant 1 \\ &\Rightarrow a^{1/2}b^{-1}a^{1/2} \leqslant 1 \Rightarrow b^{-1} \leqslant a^{-1}.\end{aligned}$$

(8) *If* $a, b \in A$, $a = a^*$, $\|a\| \leqslant 1$, $0 \leqslant b \leqslant 1$, *then*

$$a = ab \Rightarrow a \leqslant b.$$

Indeed $ab = a = a^* = ba$ and the assertion follows using the Gelfand representation of $C^*(\{a, b, 1\})$.

Moreover, *if* $a \in A$, $0 \leqslant a \leqslant 1$ *and* $e \in A$ *is a projection, then*

$$a = ae \Leftrightarrow a \leqslant e, quad e = ae \Leftrightarrow e \leqslant a.$$

Indeed, $e \leqslant a \Leftrightarrow 1 - a \leqslant 1 - e$ and

$$\begin{aligned}0 \leqslant a \leqslant e \Rightarrow 0 \leqslant (1-e)a(1-e) \leqslant 0 \Rightarrow a^{1/2}(1-e) = 0 \\ \Rightarrow a - ae = a^{1/2}a^{1/2}(1-e) = 0.\end{aligned}$$

(9) In particular, *for projections* $e_1, e_2 \in A$ *we have*

$$e_1 \leqslant e_2 \Leftrightarrow e_1 = e_1e_2.$$

2.7. Operator monotone functions. Consider an interval (bounded or not) $I \subset \mathbb{R}$ and a real continuous function f defined on I. The function f is called *operator monotone* (increasing) if for every C^*-algebra A and every $x, y \in A_h$ with $\sigma(x), \sigma(y) \subset I$ we have

$$x \leqslant y \Rightarrow f(x) \leqslant f(y).$$

PROPOSITION. *For every $0 \leqslant \alpha \leqslant 1$, the function $t \mapsto t^\alpha$ is operator monotone on $[0, +\infty)$.*

Proof. Let A be a C^*-algebra, $x, y \in A$, $0 \leqslant x \leqslant y$, and let $T = \{\alpha \in [0,1];\ x^\alpha \leqslant y^\alpha\}$. In order to show that $T = [0,1]$ we may assume both x and y invertible in $\widetilde{A}$ since, for each $\varepsilon > 0$, $x + \varepsilon$ and $y + \varepsilon$ are invertible, $x + \varepsilon \leqslant y + \varepsilon$ and from $(x+\varepsilon)^\alpha \leqslant (y+\varepsilon)^\alpha$ we get $x^\alpha \leqslant y^\alpha$ letting $\varepsilon \to 0$.

Thus, let x and y be invertible in $\widetilde{A}$. It is then clear that $0 \in T$, $1 \in T$ and T is closed. We shall show that

$$\alpha, \beta \in T \Rightarrow (\alpha + \beta)/2 \in T,$$

which entails $T = [0,1]$, thus proving the proposition.

If $\alpha, \beta \in T$, then $\|y^{-\alpha/2}x^{\alpha/2}\| \leqslant 1$, $\|x^{\beta/2}y^{-\beta/2}\| \leqslant 1$ (see 2.6). Since the spectral radius of a positive element equals its norm and $r(ab) = r(ba)$ (see 1.7.(2)), we have

$$\begin{aligned} 1 &\geqslant \|y^{-\alpha/2}x^{\alpha/2}x^{\beta/2}x^{-\beta/2}\| \geqslant r(y^{-\alpha/2}x^{(\alpha+\beta)/2}y^{-\beta/2}) \\ &= r(y^{-(\alpha+\beta)/4}x^{(\alpha+\beta)/2}y^{-(\alpha+\beta)/4}) = \|y^{-(\alpha+\beta)/4}x^{(\alpha+\beta)/2}y^{-(\alpha+\beta)/4}\|, \end{aligned}$$

so that $y^{-(\alpha+\beta)/4}x^{(\alpha+\beta)/2}y^{-(\alpha+\beta)/4} \leqslant 1$ and $x^{(\alpha+\beta)/2} \leqslant y^{(\alpha+\beta)/2}$. ∎

If $\gamma > 1$, then the function $t \mapsto t^\gamma$ is not operator monotone (see Corollary 1/4.18). However, if $x, y \in A^+$, $x \leqslant y$ *and* $xy = yx$, then $x^\gamma \leqslant y^\gamma$ for any $\gamma \geqslant 0$, by Gelfand representation.

There are other useful operator monotone functions. For instance, consider $f_\alpha : (-\alpha^{-1}, +\infty) \to \mathbb{R}$, $\alpha > 0$, defined by

$$f_\alpha(t) = t(1+\alpha t)^{-1} = \alpha^{-1}(1 - (1+\alpha t)^{-1}); \quad t \in (-\alpha^{-1}, +\infty).$$

Let A be a C^*-algebra. For any $x \in A_h$, $\sigma(x) \subset (-\alpha^{-1}, +\infty)$, the element $f_\alpha(x) = x(1+\alpha x)^{-1} = \alpha^{-1}(1 - (1+\alpha x)^{-1}) \in \widetilde{A}$ belongs in fact to A since $f_\alpha(0) = 0$. If $y \in A_h$, $\sigma(y) \subset (-\alpha^{-1}, +\infty)$ and $x \leqslant y$, then $0 \leqslant 1 + \alpha x \leqslant 1 + \alpha y$, so that $(1+\alpha x)^{-1} \geqslant (1+\alpha y)^{-1}$ which obviously implies $f_\alpha(x) \leqslant f_\alpha(y)$. Therefore each f_α is operator monotone and the same is true for αf_α.

The functions f_α have the following properties:

(1) $\quad f_\alpha(t) \leqslant \min\{t, \alpha^{-1}\},\ \alpha f_\alpha(t) \leqslant 1 \quad \text{for } t \in (-\alpha^{-1}, +\infty);$

(2) $\quad \alpha \leqslant \beta \Rightarrow f_\alpha(t) \geqslant f_\beta(t) \quad \text{for } t \in (-\beta^{-1}, +\infty);$

(3) $\quad \alpha \leqslant \beta \Rightarrow \alpha f_\alpha(t) \leqslant \beta f_\beta(t) \quad \text{for } t \in [0, +\infty);$

(4) $\quad f_\alpha(f_\beta(t)) = f_{\alpha+\beta}(t) \quad \text{for } t \in (-(\alpha+\beta)^{-1}, +\infty);$

(5) $\quad \lim\limits_{\alpha \to 0} f_\alpha(t) = t$ uniformly on compact subset of $(-\infty, +\infty)$;

(6) $\quad \lim\limits_{\alpha \to +\infty} \alpha f_\alpha(t) = 1$ uniformly on compact subset of $(0, +\infty)$.

Using functional calculus, each of these properties can be translated in a corresponding property of elements of the form $f_\alpha(x)$.

As an application, let $x, y \in A$, $0 \leqslant x \leqslant y$ and $\alpha, \beta \in \mathbb{R}$, $0 < \alpha \leqslant \beta$. Then $\alpha f_\alpha(x) \leqslant \beta f_\beta(x)$ by (3) and $\beta f_\beta(x) \leqslant \beta f_\beta(y)$ since βf_β is operator monotone, so that $\alpha f_\alpha(x) \leqslant \beta f_\beta(y)$. Therefore,

(7) $$0 \leqslant x \leqslant y,\ 0 < \alpha \leqslant \beta \Rightarrow x(\alpha^{-1} + x)^{-1} \leqslant y(\beta^{-1} + y)^{-1}.$$

On the other hand, for $0 < \beta < 1$ and $t > 0$ we have

$$\int_0^\infty f_\alpha(t)\alpha^{-\beta}\,\mathrm{d}\alpha = \int_0^\infty t(1+\alpha t)^{-1}\alpha^{-\beta}\,\mathrm{d}\alpha = t^\beta \int_0^\infty (1+\alpha)^{-1}\alpha^{-\beta}\,\mathrm{d}\alpha = \gamma t^\beta.$$

The second equality is obtained by changing the variable α into αt^{-1} and $\gamma = \pi/\sin \pi\beta > 0$. It follows that for each fixed compact set $K \subset [0, \infty)$ and every $\varepsilon > 0$, there is $n \in \mathbb{N}$ and an equidistant division $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_m = n$ such that

$$\left| t^\beta - \gamma^{-1}\frac{n}{m}\sum_{k=1}^m f_{\alpha_k}(t)\alpha_k^{-\beta} \right| \leqslant \varepsilon, \quad t \in K.$$

This gives *another simple proof of the above proposition.* Indeed, consider $x, y \in A^+$, $x \leqslant y$. Since the functions f_{α_k} are operator monotone, the above inequality with $K = [0, \|y\|]$ yields $y^\beta - x^\beta \geqslant -2\varepsilon$ for all $\varepsilon > 0$, hence $x^\beta \leqslant y^\beta$.

2.8. The preorder structure of a $*$-algebra. Given an arbitrary $*$-algebra A, we still can define the *positive cone* A^+ by

$$A^+ = \Big\{ x \in A;\ x = \sum_{k=1}^n y_k^* y_k,\ y_k \in A,\ n \in \mathbb{N} \Big\}.$$

Then $A^+ \subset A_h$ is a convex cone, but we may have $A^+ \cap (-A)^+ \neq \{0\}$. Therefore the relation

$$x \leqslant y \Leftrightarrow y - x \in A^+$$

is just a preorder relation on A_h.

Another unsatisfactory feature of A^+ is that A^+ does not necessarily span the whole A. However, the following "polarization relation"

(1) $$y^*x = 4^{-1}\sum_{k=0}^3 \mathrm{i}^k (x + \mathrm{i}^k y)^*(x + \mathrm{i}^k y); \quad x, y \in A,$$

shows that $A^2 = \operatorname{lin} A^+$, hence $A = \operatorname{lin} A^+$ whenever $A^2 = A$, which is the case if A is unital.

Note also the following form of the polarization relation:

(2) $$y^*ax = 4^{-1}\sum_{k=0}^3 \mathrm{i}^k (x + \mathrm{i}^k y)^* a (x + \mathrm{i}^k y); \quad x, y, a \in A.$$

Even if A is an involutive Banach algebra, A^+ need not be closed.

Finally, in any non-commutative C^*-algebra, the order is not latticial (see Corollary 1/4.18).

In every $*$-algebra, $(x-y)^*(x-y) \geqslant 0$, which yields $x^*y + y^*x \leqslant x^*x + y^*y$. Using this inequality, we get

$$(3) \qquad \Big(\sum_{k=1}^{n} x_k\Big)^* \Big(\sum_{k=1}^{n} x_k\Big) \leqslant n \sum_{k=1}^{n} x_k^* x_k; \quad x_k \in A,\ 1 \leqslant k \leqslant n.$$

If $\widetilde{A}$ is the associate unital $*$-algebra, then $\widetilde{A}^+ \cap A^+ = A^+$. Indeed, if $\sum\limits_{k=1}^{n} (y_k + \lambda_k)^*(y_k + \lambda_k) \in A$, and A is not unital, then $\sum\limits_{k=1}^{n} |\lambda_k|^2 = 0$, so that $\lambda_k = 0$ for all $k = 1, \ldots, n$.

Note that if B is a $*$-subalgebra of A then we may have $A^+ \cap B \neq B^+$. However, for any C^*-subalgebra B of a C^*-algebra A, the equality $A^+ \cap B = B^+$ holds.

If $\pi : A \to B$ is a $*$-homomorphism between $*$-algebras, then clearly $\pi(A^+) \subset B^+$.

A $*$-algebra A is said to satisfy the *Combes axiom* if for each $x \in A$ there is $\lambda(x) > 0$ such that

$$a^*x^*xa \leqslant \lambda(x)a^*a \quad \text{for all } a \in A.$$

Any C^*-algebra satisfies the Combes axiom with $\lambda(x) = \|x\|^2$, since $x^*x \leqslant \|x\|^2$ in $\widetilde{A}$.

Also, any U^*-algebra A satisfies the Combes axiom. Indeed, each $x \in A$ can be written as $x = \sum\limits_{k=1}^{n} \lambda_k u_k$ with $u_k \in \widetilde{A}$ unitary elements and the Combes axiom is satisfied with $\lambda(x) = n \sum\limits_{k=1}^{n} |\lambda_k|^2$ since $x^*x \leqslant \lambda(x)$ in $\widetilde{A}$ by (3).

In the last part of this section we shall show that any Banach $*$-algebra A is a U^*-algebra and satisfies the Combes axiom with $\lambda(x) = \|x^*x\| + \varepsilon$, where $\varepsilon > 0$ is arbitrary.

For a Banach algebra A, recall the notation

$$x \circ y = x + y - xy; \quad x, y \in A,$$

and the fact for any $y \in A$ with $r(y) < 1$, there exists $x \in A$ such that $x \circ y = 0$ (see 1.7). For $a \in A$ we denote by $A(a)$ the closed subalgebra of A generated by a.

LEMMA. *Let A be a Banach algebra and $a \in A$, $\|a\| < 1$. Then there exists a unique $x \in A$ such that $x \circ x = a$ and $r(x) < 1$. Moreover, $\|x\| < 1$ and $x \in A(a)$.*

Proof. Let $\|a\| = \lambda < 1$, let $E = \{x \in A(a);\ \|x\| \leqslant \lambda\}$ and let $T : E \to E$ be defined by $T(x) = (a + x^2)/2$, $x \in E$. Since $xy = yx$ for any $x, y \in E$, it follows that $\|T(x) - T(y)\| \leqslant \lambda\|x - y\|$ and, by the contraction mapping principle, there exists $x \in E$ with $T(x) = x$, i.e. $x \circ x = a$. Clearly, $r(x) \leqslant \|x\| < 1$.

If $y \in A, r(y) < 1$ and $y \circ y = a$, then $ay = ya$ so that $xy = yx$ since $x \in A(a)$. Thus, letting $u = (x+y)/2$, $v = x - y$, we have $u \circ v = u$ and $r(u) < 1$, so there is $w \in A$ with $w \circ u = 0$. It follows that

$$x - y = v = 0 \circ v = w \circ u \circ v = w \circ u = 0$$

and hence $y = x$. ∎

PROPOSITION (Ford's square root lemma). *Let A be a Banach $*$-algebra and $a \in A$, $a^* = a$, $\|a\| < 1$. Then there exists a unique $x \in A$, $x^* = x$, such that $x \circ x = a$ and $r(x) < 1$. Moreover, $\|x\| < 1$ and $x \in A(a)$.*

Proof. By the above lemma there exists a unique $x \in A$ with $x \circ x = a$ and $r(x) < 1$; moreover, $\|x\| < 1$ and $x \in A(a)$. Since $r(x^*) = r(x) < 1$ and $x^* \circ x^* = a^* = a$, we have also $x^* = x$. ∎

COROLLARY 1. *Every Banach $*$-algebra A is a U^*-algebra.*

Proof. Let $a \in A$, $a^* = a$, $\|a\| < 1$. By the proposition there exists $x \in A$, $x^* = x$, $\|x\| < 1$, $xa = ax$ such that $2x - x^2 = x \circ x = a$. Then $v = x + \mathrm{i}a$ is a quasi-unitary element of A and

$$a = 2^{-1}\mathrm{i}(v^* + v). \quad ∎$$

Let A be a C^*-algebra. In this case Ford's square root lemma is replaced by a stronger result — Proposition 2.2. If $a \in A_h$ and $\|a\| < 1$, then $u = a + \mathrm{i}(1 - a^2)^{1/2} \in \widetilde{A}$ is unitary and

$$a = 2^{-1}u + 2^{-1}u^*.$$

Note that $-\mathrm{i} \ni \sigma(u)$, hence $u \in \exp(\mathrm{i}A_h)$ by 1.18.(6). It follows that every $x \in A$, $\|x\| < 1/2$, is a convex combination of four unitary elements of $\exp(\mathrm{i}A_h) \subset \widetilde{A}$, in particular

$$\{x \in A;\ \|x\| < 1/2\} \subset \operatorname{co}\exp(\mathrm{i}A_h). \tag{4}$$

Similar results hold for Banach $*$-algebra with continuous $*$-operation.

COROLLARY 2. *Let A be a Banach $*$-algebra, $h \in A$, $h^* = h$, and $a \in A$. Then, for any $\varepsilon > 0$,*

$$-(\|h\| + \varepsilon)a^*a \leqslant a^*ha \leqslant (\|h\| + \varepsilon)a^*a.$$

Proof. Let $\varepsilon > 0$. By the proposition there exist $x, y \in A$, $x^* = x$, $y^* = y$ such that $x \circ x = (\|h\| + \varepsilon)^{-1}h$, $y \circ y = -(\|h\| + \varepsilon)^{-1}h$. Denote $u = a - xa$, $v = a - ya$. Then

$$a^*a - (\|h\| + \varepsilon)^{-1}a^*ha = u^*u, \quad a^*a + (\|h\| + \varepsilon)^{-1}a^*ha = v^*v,$$

so that the desired result follows. ∎

Both corollaries show that a Banach $*$-algebra A satisfies the Combes axiom. Corollary 2 shows that for every $x \in A$, the Combes axiom is satisfied with $\lambda(x) = \|x^*x\| + \varepsilon$, where $\varepsilon > 0$ is arbitrary. If additionally A^+ is closed, then we can take $\lambda(x) = \|x^*x\|$.

2.9. Faces. Let A be a $*$-algebra. A subset S of A^+ is called *hereditary* if

$$y \in A^+,\ y \leqslant x \in S \Rightarrow y \in S.$$

A *face* (or an *order ideal*) of A^+ is a hereditary convex subcone F of A^+. A set $F \subset A^+$ is a face of A^+ if and only if for $x, y \in A^+$ we have

$$x + y \in F \Leftrightarrow x \in F \text{ and } y \in F.$$

An *invariant face* of A^+ is a face F of A^+ such that $z^*Fz \subset F$ for all $z \in \widetilde{A}$.

LEMMA 1. *If A is an U^*-algebra, a face F of A^+ is invariant if and only if $u^*Fu \subset F$ for all unitary elements $u \in \widetilde{A}$.*

Proof. By assumption, every $z \in \widetilde{A}$ can be written as a linear combination $z = \sum_{k=1}^{n} \lambda_k u_k$ of unitaries $u_k \in \widetilde{A}$. Then, using 2.8.(3), for any $x \in F$ we obtain

$$z^*xz = \Big(\sum_{k=1}^{n} \lambda_k u_k\Big)^* x \Big(\sum_{k=1}^{n} \lambda_k u_k\Big) \leqslant n \sum_{k=1}^{n} |\lambda_k|^2 u_k^* x u_k,$$

and the desired conclusion follows. ∎

A *strongly invariant face* of A^+ is a face F of A^+ such that

$$x \in A,\ x^*x \in F \Rightarrow xx^* \in F.$$

LEMMA 2. *A convex subcone F of A^+ is a strongly invariant face of A^+ if and only if*

$$x \in A,\ x^*x \leqslant a \in F \Rightarrow xx^* \in F.$$

Proof. Let F be a convex subcone of A^+ which satisfies the condition of the statement. If $b \in A^+$ and $b \leqslant a \in F$ then $b = \sum_{k=1}^{n} x_k^* x_k$, for some $x_1, \ldots, x_n \in A$ and, using repeatedly the assumption we get, for each k,

$$x_k^* x_k \leqslant a \in F \Rightarrow x_k x_k^* \in F \Rightarrow x_k^* x_k \in F.$$

Hence $b \in F$, thus F is a face. By the assumption F is strongly invariant. The converse is clear. ∎

A subalgebra M of A is called a *hereditary subalgebra* or a *facial subalgebra* (respectively a *facial ideal*, respectively a *strongly facial ideal*) if $M \cap A^+$ is a face (respectively an invariant face, respectively a strongly invariant face) of A^+ and $M = \mathrm{lim}(M \cap A^+)$. Due to the last condition, any facial subalgebra is a $*$-subalgebra.

Given a face F of A^+ we shall denote

$$N_F = \{x \in A;\ x^*x \in F\} \quad \text{and} \quad M_F = N_F^* N_F.$$

PROPOSITION. *Let A be a $*$-algebra satisfying the Combes axiom and F be a face of A^+. Then:*

(i) *N_F is a left ideal of A;*
(ii) *$M_F \subset N_F^* \cap N_F$ is a facial subalgebra of A and $M_F \cap A^+ = F$;*
(iii) *N_F is a two-sided ideal $\Leftrightarrow$ M_F is a two-sided ideal $\Leftrightarrow$ F is invariant;*
(iv) *N_F is selfadjoint $\Leftrightarrow$ F is strongly invariant.*

Proof. (i) Consider $x, y \in N_F$ and $z \in A$. We have

$$(x+y)^*(x+y) \leqslant 2(x^*x + y^*y) \in F,$$

hence $x + y \in N_F$. Then using Combes axiom,

$$(zx)^*(zx) = x^*z^*zx \leqslant \lambda(z)x^*x \in F,$$

hence $zx \in N$. Therefore N_F is a left ideal.

(ii) Since N_F is a left ideal, N_F^* is a right ideal, so that M_F is a selfadjoint subalgebra of $N_F^* \cap N_F$. If $a = \sum\limits_k x_k^* x_k \in F$, then $x_k^* x_k \leqslant a \in F$, so that successively: $x_k^* x_k \in F$, $x_k \in N_F$, $x_k^* x_k \in M_F$ and $a \in M_F \cap A^+$. Conversely, let $a = \sum\limits_k y_k^* x_k \in M_F \cap A^+$ where $y_k, x_k \in N_F$. Using the polarization relation (2.8.(1)) we get

$$4a = \sum_k [(x_y + y_k)^*(x_y + y_k) - (x_y - y_k)^*(x_y - y_k)] \leqslant \sum_k (x_y + y_k)^*(x_y + y_k) \in F$$

so that $a \in F$. Thus $M_F \cap A^+ = F$. Using again polarization relation we see that $M_F = \operatorname{lin} F$, therefore M_F is a facial subalgebra.

(iii) If F is invariant, then

$$\begin{aligned} x \in N_F \Rightarrow x^*x \in F \Rightarrow (xz)^*(xz) = z^*x^*xz \in F \quad &\text{for all } z \in A \\ \Rightarrow xz \in N_F \quad \text{for all } z \in A&, \end{aligned}$$

so that N_F is a two-sided ideal. If N_F is a two-sided ideal, then $M_F = N_F^* N_F$ is also a two-sided ideal and this in turn entails that $F = M_F \cap A^+$ is an invariant face.

(iv) This is obvious. ∎

Clearly, a selfadjoint left ideal is two-sided. By the Proposition we infer that *if A satisfies Combes axiom, then every strongly invariant face of A^+ is invariant.*

As the following example shows, the converse is not true even for C^*-algebras. In particular, a two-sided ideal of a C^*-algebra need not be selfadjoint (see however 3.5).

Let A (respectively B) be the C^*-algebra constructed as the restricted direct product (respectively the direct product; 1.4) of a sequence of copies of the C^*-algebra M_2. Consider the elements $v = \{v_n\} \in A$, $p = \{p_n\} \in B$, where

$$v_n = A = \begin{pmatrix} 0 & 0 \\ 1/n & 0 \end{pmatrix}, \quad p_n = A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}; \quad n \in N.$$

Then $A \subset B$. Let F be the smallest invariant face of A^+ containing v^*v. Suppose that $vv^* \in F$. This means that there is a finite set of elements $x_k = \{x_{k,n}\} \in A$ and $\alpha > 0$ such that

$$vv^* \leqslant \sum_k x_k v^* v x_k + \alpha v^* v.$$

We have $vp = 0$, $pv = v$, therefore putting $y_k = x_k p \in A$ we get

$$0 \leqslant vv^* = pvv^*p \leqslant \sum_k y_k^* v^* v y_k.$$

For each $n \in N$ we obtain successively:

$$v_n v_n^* \leqslant \sum_k y_{k,n}^* v_n^* v_n y_{k,n}, \quad \|v_n\|^2 \leqslant \sum_k \|y_{k,n}\|^2 \|v_n\|^2,$$
$$1 \leqslant \sum_k \|y_{k,n}\|^2 \leqslant \sum_k \|x_{k,n}\|^2.$$

Letting $n \to +\infty$, we infer $1 \leqslant 0$, a contradiction. Therefore $vv^* \notin F$ and F is an invariant face which is not strongly invariant.

COROLLARY. *Let A be a $*$-algebra satisfying Combes axiom. Then the mappings $F \mapsto \operatorname{lin} F$, $M \mapsto M \cap A^+$ are mutually inverse correspondences between:*
(i) $\{F$: *faces of* $A^+\}$ *and* $\{M$: *facial subalgebras of* $A\}$*;*
(ii) $\{F$: *invariant faces of* $A^+\}$ *and* $\{M$: *facial ideals of* $A\}$*;*
(iii) $\{F$: *strongly invariant faces of* $A^+\}$ *and* $\{M$: *strongly facial ideals of* $A\}$. ∎

The above results hold in particular if A is a C^*-algebra. In this case we have also

$$M_F = (N_F^* \cap N_F)^2,$$

for every face F of A^+. Indeed, if $a \in M_F \cap A^+$, then $a^{1/2} \in N_F^* \cap N_F$, hence $a = (a^{1/2})^2 \in (N_F^* \cap N_F)^2$.

2.10. PROPOSITION. *Let A be a C^*-algebra and let N be a left or right ideal of A or a facial subalgebra of A. Then the set*

$$\{x \in N \cap A^+;\ \|x\| < 1\}$$

is upward directed.

Proof. Let $x, y \in N \cap A^+$, $\|x\| < 1$, $\|y\| < 1$ and put $u = x(1-x)^{-1}$, $v = y(1-y)^{-1}$, $z = (u+v)(1+u+v)^{-1}$. Then $u, v, z \in A^+$, $\|z\| < 1$ and $x = u(1+u)^{-1}$, $y = v(1+v)^{-1}$. Since the function f_1 in 2.6 is operator monotone, we have $x \leqslant z$, $y \leqslant z$. It remains to prove that $z \in N$. This is clear if N is an ideal. If N is a facial subalgebra, then $N \cap A^+$ is a face of A^+ and

$$u \leqslant \|(1-x)^{-1}\| x \in N \cap A^+$$

so that $u \in N \cap A^+$ and similarly $v \in N \cap A^+$, $z \in N \cap A^+$. ∎

Let N be as in the statement of the proposition. Then for any $x, y \in N \cap A^+$, $\|x\| \leqslant 1$, $\|y\| \leqslant 1$ and any $\varepsilon > 0$ there exist $z \in N \cap A^+$, $\|z\| \leqslant 1$, such that

$$(1-\varepsilon)x \leqslant z, \quad (1-\varepsilon)y \leqslant z.$$

2.11. C^*-seminorms on tensor products. If A, B are $*$-algebras, then their algebraic tensor product $A \otimes B$ becomes a $*$-algebra with $*$-operation

$$\Big(\sum_{k=n}^{n} a_k \otimes b_k\Big)^* = \sum_{k=n}^{n} a_k^* \otimes b_k^*; \quad a_k \in A,\, b_k \in B,$$

and we have

$$(A \otimes B)_h = A_h \otimes B_h.$$

Indeed, if

$$\Big(\sum_{k=1}^{n} a_k \otimes b_k\Big)^* = \sum_{k=1}^{n} a_k \otimes b_k \in (A \otimes B)_h,$$

then

$$\sum_{k=1}^{n} a_k \otimes b_k = \sum_{k=1}^{n} (\operatorname{Re} a_k \otimes \operatorname{Re} b_k - \operatorname{Im} a_k \otimes \operatorname{Im} b_k) \in A_h \otimes B_h.$$

Now, the $*$-algebra $A \otimes B$ has a positive cone $(A \otimes B)^+ \subset (A \otimes B)_h$. It is easy to see that

$$\{a \otimes b;\, a \in A^+,\, b \in B^+\} \subset (A \otimes B)^+.$$

In particular, if $a_1, a_2 \in A^+$, $a_1 \leqslant a_2$ and $b_1, b_2 \in B^+$, $b_1 \leqslant b_2$, then

$$a_1 \otimes b_1 \leqslant a_1 \otimes b_2 \leqslant a_2 \otimes b_2.$$

On the other hand, let A, B be normed spaces. A seminorm p defined on the algebraic tensor product $A \otimes B$ of the underlying vector spaces is called a *subcross seminorm* (respectively *cross seminorm*) if

$$p(a \otimes b) \leqslant \|a\|\,\|b\| \quad (\text{respectively } p(a \otimes b) = \|a\|\,\|b\|); \quad a \in A,\, b \in B.$$

As an application of the study of positive elements in C^*-algebras we prove:

Proposition. *Let A,B be C^*-algebras. Then any C^*-seminorm p on $A \otimes B$ is a subcross seminorm.*

Proof. By 1.2 we may suppose that $p(x) = \|\pi(x)\|$, $x \in A \otimes B$, where π is a $*$-homomorphism of $A \otimes B$ into some C^*-algebra C. Recall that 2.6.(4) for a positive element x of a C^*-algebra we have $\|x\| \leqslant 1 \Leftrightarrow x^2 \leqslant x$.

Consider $a \in A^+$, $\|a\| \leqslant 1$ and $b \in B^+$, $\|b\| \leqslant 1$. Then $a^2 \leqslant a$, $b^2 \leqslant b$, so that $(a \otimes b)^2 = a^2 \otimes b^2 \leqslant a \otimes b$. Then $\pi(a \otimes b) \in C^+$ and $\pi(a \otimes b)^2 \leqslant \pi(a \otimes b)$, hence $\|\pi(a \otimes b)\| \leqslant 1$.

It follows that

$$p(a \otimes b) \leqslant \|a\|\,\|b\|$$

for all $a \in A^+$, $b \in B^+$. Then, for arbitrary $a \in A$, $b \in B$,

$$p(a \otimes b)^2 = p((a \otimes b)^*(a \otimes b)) = p(a^*a \otimes b^*b) \leqslant \|a^*a\|\,\|b^*b\| = \|a\|^2\|b\|^2$$

which completes the proof. ∎

2.12. Consider the $*$-algebra M_n of all $n \times n$ complex matrices. An element of M_n is a colection $[\lambda_{hk}]$ of n^2 complex numbers λ_{hk}, $1 \leqslant h$, $k \leqslant n$. In particular, let

$$e_{ij} = [\delta_{ih}\delta_{jk}] \in M_n; \quad 1 \leqslant i, j \leqslant n,$$

where $\delta_{pq} = 1$ if $p = q$ and $\delta_{pq} = 0$ if $p \neq q$. Then $\{e_{ij}\}_{ij}$ is a *system of matrix units* for M_n, that is

$$e_{ij}e_{rs} = \delta_{jr}e_{is}, \quad e_{ij}^* = e_{ji}; \quad 1 \leqslant i, j, r, s \leqslant n$$

and clearly, $\{e_{ij}\}_{ij}$ is a linear basis of the vector space M_n:

$$[\lambda_{ij}] = \sum_{ij} \lambda_{ij} e_{ij}.$$

Given a $*$-algebra A, we can consider the tensor product $*$-algebra $A \otimes M_n$. An arbitrary element X of $A \otimes M_n$ can be uniquely written as

$$X = \sum_{ij} x_{ij} \otimes e_{ij}, \tag{1}$$

with $x_{ij} \in A$.

PROPOSITION. *Let A be a $*$-algebra. An element of $A \otimes M_n$ belongs to $(A \otimes M_n)^+$ if and only if it is a sum of elements of the form*

$$\sum_{ij} x_i^* x_j \otimes e_{ij}, \tag{2}$$

with $x_1, \ldots, x_n \in A$.

Proof. The elements of the form (2) belong to $(A \otimes M_n)^+$ since

$$\sum_{ij} x_i^* x_j \otimes e_{ij} = \Big(\sum_k x_k \otimes e_{1k}\Big)^* \Big(\sum_k x_k \otimes e_{1k}\Big).$$

Conversely, any element of $(A \otimes M_n)^*$ is a sum of elements of the form X^*X with $X \in A \otimes M_n$ and, if X is as in (1), then

$$\begin{aligned} X^*X &= \Big(\sum_{hk} x_{hk} \otimes e_{hk}\Big)^* \Big(\sum_{rs} x_{rs} \otimes e_{rs}\Big) \\ &= \sum_{hkrs} x_{hk}^* x_{rs} \otimes e_{kh}e_{rs} = \sum_h \sum_{ks} x_{hk}^* x_{hs} \otimes e_{ks}. \quad \blacksquare \end{aligned}$$

Given the $*$-algebra A, we can also consider the set $M_n(A)$ consisting of all collections $x = [x_{ij}]$ of n^2 elements x_{ij} of A, $1 \leqslant i, j \leqslant n$, endowed with the operations

$$[x_{ij}] + [y_{ij}] = [x_{ij} + y_{ij}]; \quad \lambda[x_{ij}] = [\lambda x_{ij}];$$
$$[x_{ij}][y_{ij}] = \Big[\sum_k x_{ik}y_{kj}\Big]; \quad [x_{ij}]^* = [x_{ji}^*].$$

Then $M_n(A)$ becomes a $*$-algebra and the mapping

$$M_n(A) \ni [x_{ij}] \mapsto \sum_{ij} x_{ij} \otimes e_{ij} \in A \otimes M_n$$

is a $*$-isomorphism of $M_n(A)$ onto $A \otimes M_n$.

COROLLARY. *Let A be a $*$-algebra. An element of $M_n(A)$ belongs to $M_n(A)^+$ if and only if it is a sum of elements of the form*

$$[x_i^* x_j]$$

with $x_1, \ldots, x_n \in A$. ∎

In particular, $[e_{ij}] \in M_n(M_n)^+$ since $e_{ij} = (e_{1i})^*(e_{1j})$, $1 \leqslant i, j \leqslant n$. Actually, $[e_{ij}]^2 = n[e_{ij}]$, so that $n^{-1}[e_{ij}]$ is a projection.

Clearly, if $A = M_m$, then $M_n(A)$ can be identified to M_{m+n}. On the other hand, let $A = B(H)$ for some Hilbert space H and let $H^{(n)}$ be the Hilbert space direct sum of n copies of H. Then it is easy to see that $M_n(A)$ can be identified to $B(H^{(n)})$.

Note that *the order relation in $B(H^{(2)})$ determines the norm in $B(H)$*, since for $x \in B(H)$ we have

$$\|x\| \leqslant 1 \Leftrightarrow \begin{pmatrix} 1 & x \\ x^* & 1 \end{pmatrix} \geqslant 0. \tag{3}$$

Indeed, this follows from the following equalities:

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 - x^*x \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ x^* & -1 \end{pmatrix} \begin{pmatrix} 1 & x \\ x^* & 1 \end{pmatrix} = \begin{pmatrix} 0 & x \\ 0 & -1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & x \\ x^* & 1 \end{pmatrix} = \begin{pmatrix} 1 & x \\ x^* & x^*x \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 - x^*x \end{pmatrix}.$$

Finally, if A is the C^*-algebra $C(\Omega)$ of all bounded continuous functions on a topological space Ω, then $M_n(A)$ can be obviously identified to $C(\Omega, M_n)$.

In all above examples A was a C^*-algebra and $M_n(A)$ was identified to a C^*-algebra. Later (4.19) we shall see that $M_n(A)$ is a C^*-algebra whenever A is a C^*-algebra.

2.13. Notes. The main results concerning positive elements in C^*-algebras (2.1–2.4) are due to M. Fukamiya [103], J.L. Kelley and R.L. Vaught [168] and I. Kaplansky (cf. [258], 4.8.8). These results are strongly related to the Gelfand-Naĭmark conjecture (see 1.19). In our exposition of 2.1–2.6 we have used [78].

The study of operator monotone functions appeared with the work of K. Löwner [183], who obtained a complete caracterization of these functions, proving in particular Proposition 2.7. For further developments concerning operator monotone functions and also "operator convex functions" we refer to [14], [23], [38], [68], [69], [74], [79], [129], [165], [169], [180], [181], [182], [287]. The present proofs of Proposition 2.7 are due to G.K. Pedersen ([240]; [242], 1.3).

The Ford square root lemma (2.8), proved in [102], "makes it possible to establish a large part of the theory of Banach $*$-algebras without any assumption of continuity or local continuity of the involution, since that assumption is frequently made for the sole purpose of establishing the square root property" (cf. [102]; compare with [258], 4.1.4). This program is accomplished in [33] which we followed for our exposition in 2.8. The Combes axiom has been introduced in [57].

The notions and the results contained in 2.9 appeared in [57], [76], [84], [233], [251] and our exposition follows that of F. Combes [57]. The Proposition 2.10 is due to J. Dixmier (cf. [61], 3.1).

The fact that any C^*-seminorm on the algebraic tensor product of two C^*-algebras is subcross (2.11) has been asserted by T. Okayasu [215], but there is a gap in his proof. The proof we have presented is due to B.J. Vowden [342] with some simplifications by E.C. Lance [172] (see also [217]).

Chapter 3

APPROXIMATE UNITS AND IDEALS

3.1. Bounded approximate units in Banach algebras. Let A be a Banach algebra. A net $\{u_\iota\}_{\iota\in I}$ in A is called a *bounded left* (respectively *right*) *approximate unit* for A if $\sup\limits_{\iota\in I}\|u_\iota\| < +\infty$ and $\|u_\iota a - a\| \to 0$ (respectively $\|au_\iota - a\| \to 0$) for all $a \in A$. A bounded left and right approximate unit is called simply a *bounded approximate unit.* If $\sup\limits_{\iota\in I}\|u_\iota\| \leqslant 1$, then the word "bounded" will be dropped.

A Banach space X is called a left Banach A-module if there exists a jointly continuous bilinear mapping

$$A \times X \ni (a, x) \to a \cdot x \in X$$

such that $a \cdot (b \cdot x) = (a \cdot b) \cdot x$ for all $a, b \in A$, $x \in X$. Then X is also a left Banach $\widetilde{A}$-module with $1 \cdot x = x$, $(x \in X)$.

THEOREM. *Let A be a Banach algebra having a bounded left approximate unit and let X be left Banach A-module. Then*

$$X_0 = \{a \cdot x;\ a \in A,\ x \in X\}$$

is a closed A-submodule of X.

Proof. Let $\mathbf{u} = \{u_\iota\}$ be a bounded left approximate unit for A and $\lambda = \sup\limits_\iota \|u_\iota\|$. Then Y, the closed linear span of X_0, is a closed submodule of X and

$$Y = \{x \in X;\ \|u_\iota \cdot x - x\| \to 0\}.$$

Put $\gamma = \lambda^{-1}$. Then for any $u \in A$, $\|u\| \leqslant \lambda$, the element $(1+\gamma) - \gamma u$ is invertible in $\widetilde{A}$, $((1+\gamma) - \gamma u)^{-1} = (1+\gamma)^{-1}$ + an element of A, and $\|((1+\gamma) - \gamma u)^{-1}\| \leqslant \lambda$, as easily verified by the usual geometrical series argument.

Let $y \in Y$. We shall define a sequence $\{u_n\}$ of elements of A, $\|u_n\| \leqslant \lambda$, such that

$$a_n = ((1+\gamma) - \gamma u_n)^{-1} \cdots ((1+\gamma) - \gamma u_1)^{-1} = (1+\gamma)^{-n} + b_n$$

$(b_n \in A)$ converges in $\widetilde{A}$ to a limit $a \in A$ and

$$x_n = a_n^{-1} \cdot y$$

converges in X to a limit $x \in X$. The theorem will then follow, since

$$y = a_n a_n^{-1} y = a_n x_n = \lim_n a_n x_n = a \cdot x \in X_0.$$

The sequence $\{u_n\}$ is defined inductively by $\{u_n\} \subset \mathbf{u}$,

$$\|u_n b_{n-1} - b_{n-1}\| \leqslant \lambda(1+\gamma)^{-n} \quad \text{and} \quad \|u_n y - y\| \leqslant \|a_{n-1}^{-1}\|^{-1}\lambda(1+\gamma)^{-n}$$

where $a_0 = 1$, $b_0 = 0$, and all we have to show is that $\{b_n\}$ and $\{x_n\}$ are Cauchy sequences. We have

$$b_{n+1} = a_{n+1} - (1+\gamma)^{-n-1} = ((1+\gamma) - \gamma u_{n+1})^{-1}((1+\gamma)^{-n} + b_n) - (1+\gamma)^{-n-1}$$

thus

$$\begin{aligned} b_{n+1} - b_n &= ((1+\gamma)-\gamma u_{n+1})^{-1}[(1+\gamma)^{-n} + b_n - ((1+\gamma)-\gamma u_{n+1})((1+\gamma)^{-n-1}+b_n)] \\ &= ((1+\gamma) - \gamma u_{n+1})^{-1}[\gamma(u_{n+1}b_n - b_n) + \gamma(1+\gamma)^{-n-1}u_{n+1}] \end{aligned}$$

and therefore

$$\|b_{n+1} - b_n\| \leqslant \lambda(\gamma\lambda(1+\gamma)^{-n-1} + \lambda\gamma(1+\gamma)^{-n-1}) = 2\lambda(1+\gamma)^{-n-1},$$

which entails that $\{b_n\}$ is a Cauchy sequence. On the other hand,

$$x_{n+1} = a_n^{-1}((1+\gamma) - \gamma u_{n+1})y,$$

thus

$$x_{n+1} - x_n = a_n^{-1}((1+\gamma) - \gamma u_{n+1})y - a_n^{-1}y = a_n^{-1}\gamma(y - u_{n+1}y)$$

and therefore

$$\|x_{n+1} - x_n\| \leqslant \|a_n^{-1}\|\gamma\|a_n^{-1}\|^{-1}\lambda(1+\gamma)^{-n-1} = (1+\gamma)^{-n-1}$$

so that $\{x_n\}$ is also a Cauchy sequence. ∎

In the same situation as in the above theorem, we have

COROLLARY 1. *If K is a compact of X_0, then there is $a \in A$ and a compact subset C of X_0 with $K = a \cdot C$.*

Proof. The Banach space $Z = C(K, X_0)$ of all continuous functions $f : K \to X_0$ with the uniform norm and the operation $(a \cdot f)(y) = a \cdot f(y)$, ($a \in A$, $f \in Z$, $y \in K$), is a left Banach A-module. Since $\{u_\iota\}$ is bounded, we have $\|u_\iota \cdot x - x\| \to 0$ uniformly on compact subsets of X_0, so that $\|u_\iota \cdot f - f\| \to 0$ for any $f \in Z$. Therefore $Z_0 = Z$.

Let $g \in Z$ be the function $g(y) = y$, $y \in K$. By the theorem, there are $a \in A$ and $f \in Z$ such that $g = a \cdot f$. Then $K = a \cdot C$ with $C = f(K)$. ∎

Corollary 2. *If $\{x_n\}$ is a sequence in X_0 converging to 0, then there are $a \in A$ and another sequence $\{y_n\}$ in X_0 converging to 0 such that $x_n = a \cdot y_n$ for all n.*

Proof. Take $K = \{0, x_1, \dots, x_n, \dots\}$ and keep the notations from the proof of Corollary 1. As in that proof, there exist $a \in A$ and $f \in Z$ such that $(a \cdot f)(x) = x$ for all $x \in K$. Hence, with $z_n = f(x_n)$, we have $z_n \to f(0)$, $a \cdot z_n = x_n$ and $a \cdot f(0) = 0$. Consequently, $y_n = z_n - f(0) \to 0$ and $a \cdot y_n = x_n$ for all n. ∎

Let A be a Banach algebra with bounded left approximate unit. We can take $X = A$ and then, by the above theorem,

$$A = X_0 = \{ab;\, a, b \in A\}. \tag{1}$$

Thus, for any sequence $\{x_n\}$ in A, $x_n \to 0$, there are $a \in A$ and a sequence $\{y_n\}$ in A, $y_n \to 0$, such that $x_n = ay_n$. If A has also a bounded right approximate unit, then similarly there are $b \in A$ and a sequence $\{z_n\}$ in A, $z_n \to 0$, with $y_n = z_n b$ and hence with $x_n = az_n b$.

3.2. Approximate units in C^*-algebras. Let A be a C^*-algebra and let X be a subset of A. A net $\{u_\iota\}_{\iota \in I}$ in X is called an *increasing left* (respectively *right*) *approximate unit* for X if $u_\iota \in X \cap A^+$, $\|u_\iota\| \leqslant 1$, $\iota \leqslant \kappa \Rightarrow u_\iota \leqslant u_\kappa$ and $\|u_\iota x - x\| \to 0$ (respectively $\|xu_\iota - x\| \to 0$) for every $x \in X$. An increasing left and right approximate unit for X is called simply *increasing approximate unit* for X.

If X is selfadjoint, then any increasing left (right) approximate unit for X is an increasing approximate unit for X since $\|xu_\iota - x\| = \|u_\iota x^* - x^*\|$, $x \in X$.

Each (left, right) approximate unit for X plays the same role for the norm closure $\overline{X}$ of X.

Theorem. *Let A be a C^*-algebra. Then:*

(i) *Every left (respectively right) ideal N of A contains an increasing right (respectively left) approximate unit for N.*

(ii) *Every face F of A^+ contains an increasing right approximate unit for N_F.*

(iii) *For every facial subalgebra M of A, $\{v \in M \cap A^+;\, \|v\| < 1\}$ is an increasing approximate unit for M.*

Proof. (i) Let N be a left ideal. Consider the set Λ of all finite subsets of N, upward directed by inclusion and, for every $\lambda = \{x_1, \dots, x_n\} \in \Lambda$ define

$$v_\lambda = \sum_{k=1}^{n} x_k^* x_k \in N \quad \text{and} \quad u_\lambda = nf_n(v_\lambda) = (n^{-1} + v_\lambda)^{-1} v_\lambda \in N,$$

where f_n are as in 2.7. By the properties of the functions f_n we have $\|u_\lambda\| < 1$, $u_\lambda \geqslant 0$ and $\lambda \subset \mu \Rightarrow u_\lambda \leqslant u_\mu$. For any $\lambda = \{x_1, \dots, x_n\} \in \Lambda$ and any $x \in \lambda$ we have

$$\begin{aligned}(x(1-u_\lambda))^*(x(1-u_\lambda)) &\leqslant \sum_{k=1}^{n} (x_k(1-u_\lambda))^*(x_k(1-u_\lambda)) = (1-u_\lambda)v_\lambda(1-u_\lambda) \\ &= n^{-2}(n^{-1}+v_\lambda)^{-2}v_\lambda \leqslant 4^{-1}n^{-1}.\end{aligned}$$

It follows that $\|xu_\lambda - x\|^2 = \|(x(1-u_\lambda))^*(x(1-u_\lambda))\| \leqslant 4^{-1}n^{-1}$ and hence $\lim\limits_{\lambda\in\Lambda} \|xu_\lambda - x\| = 0$.

(ii) If $N = N_F$ in the proof of (i), then $v_\lambda \in F$ and $u_\lambda \leqslant nv_\lambda \in F$, thus $u_\lambda \in F$.

(iii) Since $\mathbf{v} = \{v \in M \cap A^+; \|v\| < 1\}$ is upward directed (2.10) and, as $M = \mathrm{lin}(M \cap A^+)$, it suffices to show that $\lim\limits_{v\in\mathbf{v}} \|xv - x\| = 0$ for $x \in M \cap A^+$. Then, owing to 2.6, we see that

$$\|vx - x\|^2 = \|x(1-v)^2x\| \leqslant \|x(1-v)x\|$$

and the net $\{\|x(1-v)x\|\}_{v\in\mathbf{v}}$ is monotone decreasing. Therefore it is sufficient to show that its infimum is zero.

With f_n as in 2.7 we have $\|nf_n(x)\| \leqslant 1$ and $nf_n(x) \leqslant nx \in M \cap A^+$. As $M \cap A^+$ is a face we infer that $nf_n(x) \in M \cap A^+$, hence $nf_n(x) \in \mathbf{v}$.

Now,

$$0 \leqslant x(1 - nf_n(x))x = (1+nx)^{-1}x^2 \leqslant n^{-1}x,$$

so that $\|x(1-nf_n(x))x\| \leqslant n^{-1}\|x\| \to 0$ as $n \to +\infty$. ∎

$\{u \in A^+; \|u\| < 1\}$ is called *the canonical approximate unit* of A. If A is separable, then one can take a dense sequence $\{x_n\}$ in N (or in M, or in A) and the proof of (i) shows that $u_n = u_{\{x_1,\ldots,x_n\}}$ is a countable left (right) approximate unit for N (or for M, or for A).

For further information concerning approximate units in C^*-algebras see Corollary 3/4.15.

3.3. Let A be a C^*-algebra and $\mathbf{u} = \{u_\iota\}$ be a bounded increasing net in A. Then clearly

$$N_{\mathbf{u}} = \{x \in A; \lim_\iota \|xu_\iota - x\| = 0\}$$

is a closed left ideal of A.

If $\mathbf{u}$ is an increasing right approximate unit for a left ideal N of A, then

$$N_{\mathbf{u}} = \overline{N}$$

since $N \subset N_{\mathbf{u}}$ so that $\overline{N} \subset N_{\mathbf{u}}$ and $N_{\mathbf{u}} \subset \overline{N}$ as $u_\iota \in N$.

LEMMA. *Let A be a C^*-algebra, $a \in A^+$ and let N_a be the smallest closed left ideal of A containing a. Then*

$$u_n = (n^{-1} + a)^{-1}a = nf_n(a); \quad n \in \mathbb{N},$$

is an increasing right approximate unit for N_a.

Proof. By the properties of the functions f_n (2.7), $\mathbf{u} = \{u_n\}$ is an increasing net in A^+, $u_n \in N_a$ and $\|u_n\| \leqslant 1$. On the other hand, $a - au_n = n^{-1}u_n$ tends to 0 as $n \to +\infty$, so that $a \in N_{\mathbf{u}}$ and $N_a \subset N_{\mathbf{u}}$. ∎

3.4. Proposition. *Let A be a C^*-algebra, $x \in A$, $a \in A^+$ and suppose that $x^*x \leqslant a$. Then $x \in N_a$ and for any $0 < \beta < 1/2$ there is $y \in A$, $\|y\| \leqslant \|a^{1/2-\beta}\|$, such that $x = ya^\beta$.*

Proof. Let $\mathbf{v} = \{v_\iota\}$ be any increasing right approximate unit for N_a. Then

$$(1 - v_\iota)x^*x(1 - v_\iota) \leqslant (1 - v_\iota)a(1 - v_\iota) \to 0,$$

so that $x \in N_{\mathbf{v}} = N_a$ by 3.3.

In particular, taking $x = a^{1/2}$ we have $a^{1/2} \in N_a$. It follows that $a^\gamma \in N_a$ for any $\gamma > 0$ and $N_a = N_{a^\gamma}$.

Now let $0 < \beta < 1/2$. By Lemma 3.3, $u_n = (n^{-1} + a^\beta)^{-1}a^\beta$ is an increasing right approximate unit for $N_a = N_{a^\beta}$, so that

$$x = \lim_n (x(n^{-1} + a^\beta)^{-1})a^\beta.$$

Put $y_n = x(n^{-1} + a^\beta)^{-1}$ and $d_{nm} = (n^{-1} + a^\beta)^{-1} - (m^{-1} + a^\beta)^{-1}$. Then $0 \leqslant (y_n - y_m)^*(y_n - y_m) = d_{nm}x^*xd_{nm} \leqslant d_{nm}ad_{nm} = (a^{(1/2)-\beta}(u_n - u_m))^2 \to 0$. Therefore $\{y_n\}$ is a Cauchy sequence, so it converges to some $y \in A$ and $x = ya^\beta$.

On the other hand,

$$\begin{aligned}\|y_n\|^2 = \|y_n^*y_n\| &= \|(n^{-1} + a^\beta)^{-1}x^*x(n^{-1} + a^\beta)^{-1}\| \\ &\leqslant \|(n^{-1} + a^\beta)^{-1}a(n^{-1} + a^\beta)^{-1}\| \\ &= \|a^{1/2}(n^{-1} + a^\beta)^{-1}\|^2 \leqslant \|a^{(1/2)-\beta}\|^2,\end{aligned}$$

so that $\|y\| \leqslant \|a^{(1/2)-\beta}\|$. ∎

Corollary 1. *If $x \in A$ and $0 < \gamma < 1$, then $x = y|x|^\gamma$ for some $y \in A$, $\|y\| \leqslant \|x\|^{(1-\gamma)}$.* ∎

Corollary 2. *If $a, b \in A^+$ and $b \leqslant a$ then $b = a^{1/4}ca^{1/4}$ for some $c \in A^+$, $\|c\| \leqslant \|a\|^{1/2}$.*

Proof. We have $(b^{1/2})^*(b^{1/2}) = b \leqslant a$, so that $b^{1/2} = ya^{1/4}$ with $y \in A$, $\|y\| \leqslant \|a\|^{1/4}$ and $b = a^{1/4}y^*ya^{1/4}$, $\|y^*y\| \leqslant \|a\|^{1/2}$. ∎

3.5. Algebraic ideals and faces. Let A be a C^*-algebra. A face of A^+ is called an *algebraic face* if $x^{1/2} \in F$ whenever $x \in F$. Then $x^{2^n} \in F$, $(n \in \mathbf{Z})$, so that $x^\alpha \in F$ for all $\alpha > 0$ because F is hereditary.

A left (or right, or two-sided) ideal N of A is called an *algebraic left* (or *right*, or *two-sided*) *ideal* if $x^{1/2} \in N$ whenever $x \in N \cap A^+$. From Proposition 3.4 it follows that *any closed ideal is algebraic.* We shall therefore study algebraic ideals in order to clarify the structure of closed ideals.

For $X, Y \subset A$ we shall denote $X \cdot Y = \{xy;\, x \in X,\, y \in Y\}$ and $XY = \mathrm{lin}(X \cdot Y)$.

3.6. Proposition. *Let A be a C^*-algebra and F be an algebraic face of A^+. Then $N_F \cap A^+ = F$ and*

$$N_F = A \cdot F, \quad M_F = N_F^* N_F = N_F^* \cap N_F = F \cdot A \cdot F.$$

If F is an invariant algebraic face, then F is strongly invariant and $N_F = M_F$.

Proof. By Proposition 2.9 we know that $F \subset M_F = \operatorname{lin} F = N_F^* N_F \subset N_F^* \cap N_F$.

For $x \in A$ we have: $x \in N_F \cap A^+ \Rightarrow x^2 \in F \Rightarrow x \in F \Rightarrow x \in N_F \cap A^+$. Thus, $N_F \cap A^+ = F$.

If $x \in N_F$, then $x^* x \in F$, so that $|x|^{1/2} \in F$ and $x = y|x|^{1/2}$ for some $y \in A$ by Proposition 3.4. Thus, $N_F = A \cdot F$.

Consider $x \in M_F$. Then x is a finite sum, $x = \sum\limits_k \lambda_k x_k$ with $\lambda_k \in \mathbf{C}$, $x_k \in F$. Put $a = \sum\limits_k x_k \in F$. Then $a^{1/4} \in F$ and, by Corollary 2/3.4, $x_k = a^{1/4} c_k a^{1/4}$ with $c_k \in A$, so that

$$x = a^{1/4} \Big(\sum_k \lambda_k c_k \Big) a^{1/4} \in F \cdot A \cdot F.$$

Thus, $M_F \subset F \cdot A \cdot F$ and clearly $F \cdot A \cdot F \subset N_F^* \cap N_F$.

On the other hand, consider $x \in N_F^* \cap N_F$, $x = x^*$ and write $x = x^+ - x^-$. Then

$$(x^+)^2 + (x^-)^2 = x^2 \in F,$$

so that $x^+, x^- \in F$. Thus, $N_F^* \cap N_F \subset \operatorname{lin} F = M_F$.

Let F be an invariant algebraic face. If $x^* x \in F$, then $(x x^*)^2 = x(x^* x) x^* \in F$ since F is invariant and $x^* x \in F$ since F is algebraic. Thus, F is strongly invariant and $M_F = \operatorname{lin} F$ is a two-sided ideal by Proposition 2.9, so that

$$N_F = A \cdot F \subset M_F \subset N_F. \quad \blacksquare$$

Note that an invariant face is algebraic if and only if $N_F = M_F$.

3.7. Proposition. *Let A be a C^*-algebra. If N is an algebraic left ideal of A, then $F = N \cap A^+$ is an algebraic face of A^+, $N = N_F$ and*

$$N = A \cdot (N \cap A^+), \quad N^* N = N^* \cap N = (N \cap A^+) \cdot A \cdot (N \cap A^+). \tag{1}$$

Every algebraic two-sided ideal is a strongly facial ideal.

Proof. If $b \in A^+$ and $b \leqslant a \in N \cap A^+$, then $a^{1/4} \in N \cap A^+$ and $b = a^{1/4} c a^{1/4}$ for some $c \in A^+$ by Corollary 2/3.4, so that $b \in N \cap A^+$. Thus, $F = N \cap A^+$ is a face.

If $x \in N$, then $x^* x \in N \cap A^+ = F$, hence $x \in N_F$. Conversely, by Proposition 3.6, $N_F = A \cdot F = A \cdot (N \cap A^+) \subset N$. Thus, $N = N_F$ and (1) follows using again 3.6.

Finally, if N is an algebraic two-sided ideal, then $F = N \cap A^+$ is an invariant algebraic face, whence a strongly invariant face (3.6) and we have

$$N = N_F = M_F = \operatorname{lin} F = \operatorname{lin}(N \cap A^+)$$

so that N is a strongly facial ideal. $\blacksquare$

3.8. PROPOSITION. *Let A be a C^*-algebra and let N (respectively J) be an algebraic left (respectively two-sided) ideal of A. Then*

$$J \cap N = JN = J \cdot N.$$

Proof. Clearly, $J \cdot N \subset JN \subset J \cap N$. If $x \in J \cap N$ then $|x|^{1/4} \in J \cap N$ and $x = y|x|^{1/2}$ for some $y \in A$, by Corollary 1/3.4. Then

$$x = (y|x|^{1/4})|x|^{1/4} \in J \cdot N. \quad \blacksquare$$

In particular, $J = J \cdot J \cdot \cdots \cdot J$ for any algebraic two-sided ideal J.

Also, if J is an algebraic two-sided ideal of A and I is an algebraic two-sided ideal of J, then I is an algebraic two-sided ideal of A, since

$$A \cdot I \cdot A = A \cdot I \cdot I \cdot I \cdot A \subset J \cdot I \cdot J \subset I.$$

3.9. Closed ideals. Let A be a C^*-algebra. Recall that any closed ideal of A is algebraic (3.5).

(i) If N is a closed left ideal, then by 3.7, $F = N \cap A^+$ is a closed face, $N = N_F$ and

$$N = A \cdot (N \cap A^+), \quad N^*N = N^* \cap N = (N \cap A^+) \cdot A \cdot (N \cap A^+).$$

Conversely, if F is a closed face, then $N_F = \{x \in A;\, x^*x \in F\}$ is a closed left ideal and $N_F \cap A^+ = F$ (2.9, 3.6).

Therefore *the mappings $F \mapsto \{x \in A;\, x^*x \in F\}$ and $N \mapsto N \cap A^+$ are mutually inverse one-to-one correspondences between*

$$\{F : \textit{closed faces of } A^+\} \quad \textit{and} \quad \{N : \textit{closed left ideals of } A\}.$$

(ii) Closed facial subalgebras are called *facial* (or *hereditary*) *C^*-subalgebras.*

If F is a closed face, then $M_F = N_F^* N_F = N_F^* \cap N_F$ is a facial C^*-subalgebra and conversely, if M is a facial C^*-subalgebra, then $M^+ = M \cap A^+$ is closed.

If F is a closed invariant face, then $\operatorname{lin} F = M_F = N_F$ is a closed two-sided ideal (3.6, 3.7) and conversely, the positive part of a closed two-sided ideal is a closed invariant face.

Owing to 2.9, we see that *the mappings $F \mapsto \operatorname{lin} F$, $M \mapsto M \cap A^+$ are mutually inverse one-to-one correspondences between*

$$\{F : \textit{closed faces of } A^+\} \quad \textit{and} \quad \{M : \textit{facial } C^*\textit{-subalgebras of } A\},$$

and also between

$$\{F : \textit{closed invariant faces of } A^+\} \quad \textit{and} \quad \{M : \textit{closed two-sided ideals of } A\}.$$

In particular, *any closed two-sided ideal is selfadjoint.*

(iii) By the way, we have seen that for each closed left ideal N, $M = N^* \cap N$ is a facial C^*-subalgebra and any facial C^*-subalgebra is of this form. Moreover, we have also (see 2.9):

$$M = (N^* \cap N)^2.$$

(iv) For an arbitrary left ideal N we have

$$\overline{N^*N} = \overline{N^* \cap N} = \overline{N}^* \cap \overline{N} = \overline{N}^*\overline{N}.$$

Indeed, $\overline{N}^* \cap \overline{N} = \overline{N}^*\overline{N}$ since $\overline{N}$ is closed and clearly $\overline{N^*N} \subset \overline{N^* \cap N} \subset \overline{N}^* \cap \overline{N}$. Note that $\overline{N}^* \cap \overline{N}$ is a C^*-subalgebra. If $x \in (\overline{N}^* \cap \overline{N})^+$, then $x^{1/2} \in \overline{N}$, so there is a sequence $y_n \in N$ converging to $x^{1/2}$ and it follows that $y_n^* y_n \in N^*N$ tends to x, whence $x \in \overline{N^*N}$.

(v) The closure $\overline{F}$ of a face F is again a face and

$$N_{\overline{F}} = \overline{N_F}, \quad M_{\overline{F}} = \overline{M_F}.$$

Indeed, $F = M_F \cap A^+ = (N_F^* N_F) \cap A^+$, so that $F = (\overline{N_F^* N_F})^+ = (\overline{N_F^*} \cap \overline{N_F})^+ = \overline{N_F} \cap A^+$ is a face and $N_{\overline{F}} = \overline{N_F}$. Also

$$\overline{M_F} = \overline{N_F^* N_F} = \overline{N_F^*}\,\overline{N}_F = N_{\overline{F}}^* N_{\overline{F}} = M_{\overline{F}}.$$

If $\overline{N_F}$ is a two sided ideal, then $\overline{F}$ is a closed invariant face, so that

$$\overline{M_F} = M_{\overline{F}} = N_{\overline{F}} = \overline{N_F}.$$

In particular, N_F is *dense in A if and only if M_F is dense in A.*

3.10. Let X be a Banach space and Y be a closed subspace of X. Denote by $X \ni x \mapsto x/Y \in X/Y$ the quotient mapping of X onto the quotient vector space X/Y. Then X/Y becomes a Banach space with the norm

$$\|x/Y\| = \inf\{\|x + y\|;\, y \in Y\}; \quad x \in X.$$

Proposition. *Let N be a closed left ideal of a C^*-subalgebra A and $\{u_\iota\}_{\iota \in I}$ be an increasing right approximate unit for N. Then*

$$\|x/N\| = \lim_\iota \|x - xu_\iota\|; \quad x \in A.$$

Proof. Let $a \in N$. Since $\|a - au_\iota\| \to 0$, we have

$$\begin{aligned}\|x/N\| = \inf\{\|x + b\|;\, b \in N\} &\leqslant \liminf_\iota \|x - xu_\iota\| \leqslant \limsup_\iota \|x - xu_\iota\| \\ &= \limsup_\iota \|(x + a)(1 - u_\iota)\| \leqslant \|x + a\|.\end{aligned}$$

Taking the infimum over all $a \in N$ we get the desired equality. ∎

COROLLARY. *Let L (respectively R) be a closed left (respectively right) ideal of a C^*-algebra A. Then $L + R$ is closed.*

Proof. Let $\Phi : A \to A/L$ be the quotient map. Since $L + R = \Phi^{-1}(\Phi(R))$, it is enough to show that $\Phi(R)$ is closed in A/L. Consider an increasing right approximate unit $\{u_\iota\}$ for L. Let $y \in \Phi(R)$. Then $y = \Phi(z)$ for some $z \in R$, $\|y\| = \lim_\iota \|z - zu_\iota\|$, $x_\iota = z - zu_\iota \in R$ and $y = \Phi(x_\iota)$ for each ι. Thus, there is $x \in R$ such that

$$\Phi(x) = y \quad \text{and} \quad \|x\| \leqslant 2\|y\|.$$

If a sequence $\{y_n\} \subset \Phi(R)$ converges to $y \in A/L$, then one may assume $\|y_n - y_{n-1}\| \leqslant 2^{-n}$ and, using the above remark, we may define a sequence $\{x_n\} \subset R$ so that

$$\Phi(x_n) = y_n - y_{n-1} \quad \text{and} \quad \|x_n\| \leqslant 2\|y_n - y_{n-1}\| \leqslant 2^{-n+1}.$$

Then $x = \sum_{n=1}^{\infty} x_n \in R$ and $y = \Phi(x) \in \Phi(R)$. ∎

3.11. Quotient C^*-algebras. Another important consequence of Proposition 4.10 is the following theorem.

THEOREM. *Let A be a C^*-algebra and J be a closed two-sided ideal of A. Then the $*$-algebra A/J endowed with the quotient norm is a C^*-algebra.*

Proof. Since J is selfadjoint (3.9.(ii)), A/J is indeed a $*$-algebra. Since J is closed, A/J is Banach algebra. Let $\{u_\iota\}$ be an increasing approximate unit for J. Using Proposition 3.10, for every $x \in A$ we obtain

$$\begin{aligned}\|x/J\|^2 &= \lim_\iota \|x(1 - u_\iota)\|^2 = \lim_\iota \|(1 - u_\iota)x^*x(1 - u_\iota)\| \\ &\leqslant \lim_\iota \|x^*x(1 - u_\iota)\| = \|(x^*x)/J\| = \|(x/J)^*(x/J)\|. \quad ∎\end{aligned}$$

COROLLARY 1. *Let π be a $*$-homomorphism of a C^*-algebra A into a C^*-algebra B. Then $\pi(A)$ is a C^*-algebra of B and*

$$\rho : A/\operatorname{Ker}\pi \ni x/\operatorname{Ker}\pi \mapsto \pi(x) \in \pi(A)$$

is a $*$-isomorphism.

Proof. The map $\rho : A/\operatorname{Ker}\pi \to \pi(A) \subset B$ is an injective $*$-homomorphism. Since $A/\operatorname{Ker}\pi$ and B are C^*-algebras, ρ is isometric by Corollary 1.15, so that $\pi(A) = \rho(A/\operatorname{Ker}\pi)$ is closed. ∎

COROLLARY 2. *Let π be a $*$-homomorphism of a C^*-algebra A into a C^*-algebra B. Then, for every $S \subset A$ we have*

$$\pi(C^*(S)) = C^*(\pi(S)).$$

Proof. Indeed, by Corollary 1, $\pi(C^*(S))$ is a C^*-subalgebra of B containing $\pi(S)$, hence $\pi(C^*(S)) \supset C^*(\pi(S))$.

Conversely, $\pi^{-1}(C^*(\pi(S)))$ is clearly a C^*-subalgebra of A containing S, hence $\pi^{-1}(C^*(\pi(S))) \supset C^*(S)$, so $C^*(\pi(S)) \supset \pi(\pi^{-1}(C^*(\pi(S)))) \supset \pi(C^*(S))$. ∎

COROLLARY 3. *Let A be a C^*-algebra, J be a closed two-sided ideal of A and B be a C^*-subalgebra of A. Then $B+J$ is a C^*-subalgebra of A.*

Proof. Let $\pi : A \to A/J$ be the quotient map. By Corollary 1, $\pi(B)$ is closed, hence $B+J = \pi^{-1}(\pi(B))$ is closed. ∎

Let A be a comutative C^*-algebra with Gelfand spectrum Ω. We shall identify A with $C_0(\Omega)$.

There is a bijective correspondence between the closed ideals J of A and the closed subsets ω of Ω given by

$$\begin{aligned} J &\mapsto \omega_J = \{t \in \Omega;\, x(t) = 0 \text{ for all } x \in J\}, \\ \omega &\mapsto J_\omega = \{x \in A;\, x(t) = 0 \text{ for all } t \in \omega\}. \end{aligned}$$

Indeed, let $\omega \subset \Omega$ be a closed set. Clearly, $\omega \subset \omega_{J_\omega}$. If $t \in \Omega \setminus \omega$, then by Tietze-Urysohn theorem there exists $x \in A$ with $x(\omega) = \{0\}$ and $x(t) \neq 0$. Then $x \in J_\omega$ and $x(t) \neq 0$, hence $t \notin \omega_{J_\omega}$. This shows that $\omega = \omega_{J_\omega}$.

On the other hand, let J be a closed ideal of A. Then J_{ω_J} can be identified with $C_0(\Omega\setminus\omega_J)$, so $J \subset J_{\omega_J}$ can be regarded as a closed $*$-subalgebra of $C_0(\Omega\setminus\omega_J)$. For every $t \in \Omega \setminus \omega_J$ there exists $x \in J$ with $x(t) \neq 0$. If $s \in \Omega \setminus \omega_J$, $s \neq t$, there exists $y \in A$ with $y(t) = 1$ and $y(s) = 0$. Then $xy \in J$, $(xy)(t) \neq 0$ and $(xy)(s) = 0$. By the Stone-Weierstrass theorem we conclude $J = C_0(\Omega \setminus \omega_J)$, that is $J = J_{\omega_J}$.

Let J be a closed ideal of A and $\omega = \omega_J$. By the above, the Gelfand spectrum of the C^*-algebra J is homeomorphic with $\Omega\setminus\omega$. Using again the Stone-Weierstrass theorem and Corollary 1, we see that the map

$$A/J \ni x/J \mapsto x|\omega \in C_0(\omega)$$

is a surjective $*$-isomorphism, so we may identify A/J and $C_0(\omega)$. Thus, the Gelfand spectrum of A/J can be identified with ω. Note that, with the above identifications, the quotient map $A \mapsto A/J$ is simply the restriction map

$$C_0(\Omega) \ni x \mapsto x|\omega \in C_0(\omega).$$

3.12. LEMMA. *Let A be a C^*-algebra. If $x, y \in A$, $a \in A^+$, $x^*x \leqslant a$ and $y \in N_a$, then*

$$z_n = xa^{1/2}(n^{-1}+a)^{-1}y^*$$

is a Cauchy sequence.

Proof. By Lemma 3.3, $u_n = a(n^{-1}+a)^{-1}$ is an increasing right approximate unit for N_a. Put $d_{nm} = ((n^{-1}+a)^{-1} - (m^{-1}+a)^{-1})y^*$. Then

$$\begin{aligned} 0 &\leqslant (z_n - z_m)^*(z_n - z_m) = d_{nm}^* a^{1/2} x^* x a^{1/2} d_{nm} \\ &\leqslant d_{nm} a^2 d_{nm} = (y(u_n - u_m))(y(u_n - u_m))^* \mapsto 0, \end{aligned}$$

which proves the lemma. ∎

3.13. Proposition. *Let A be a C^*-algebra. If*

$$(1)\qquad \sum_{i=1}^{p} x_i^* x_i \leqslant \sum_{j=1}^{q} y_j^* y_j, \quad (x_i, y_j \in A;\ 1 \leqslant i \leqslant p,\ 1 \leqslant j \leqslant q),$$

then there are $z_{ij} \in \overline{x_i A y_j^*}$, $(1 \leqslant i \leqslant p,\ 1 \leqslant j \leqslant q)$, such that

$$(2)\qquad x_i x_i^* = \sum_{j=1}^{q} z_{ij} z_{ij}^*, \quad (1 \leqslant i \leqslant p),$$

$$(3)\qquad y_j y_j^* \geqslant \sum_{i=1}^{p} z_{ij}^* z_{ij}, \quad (1 \leqslant j \leqslant q).$$

Moreover, the equality in (1) *entails the equality in* (3).

Proof. Let $a = \sum_j y_j^* y_j$. By Lemma 3.3 and Proposition 3.4, $u_n = a(n^{-1} + a)^{-1}$ is an increasing right approximate unit for N_a and $x_i, y_j \in N_a$, $(1 \leqslant i \leqslant p$, $1 \leqslant j \leqslant q)$. By Lemma 3.12, the sequences

$$z_{ijn} = x_i a^{1/2} (n^{-1} + a)^{-1} y_j^*$$

are convergent to elements $z_{ij} \in \overline{x_i A y_j^*}$, $(1 \leqslant i \leqslant p,\ 1 \leqslant j \leqslant q)$. We have

$$\begin{aligned} x_i x_i^* &= \lim_n (x_i u_n)(x_i u_n)^* = \lim_n x_i (u_n u_n^*) x_i^* \\ &= \lim_n x_i a^{1/2} (n^{-1} + a)^{-1} \Big(\sum_j y_j^* y_j \Big) (n^{-1} + a)^{-1} a^{1/2} x_i^* \\ &= \lim_n \sum_j z_{ijn} z_{ijn}^* = \sum_j z_{ij} z_{ij}^* \end{aligned}$$

and

$$\begin{aligned} y_j y_j^* &= \lim_n (y_j u_n)(y_j u_n)^* = \lim_n y_j (u_n u_n^*) y_j^* \\ &\geqslant \lim_n y_j (n^{-1} + a)^{-1} a^{1/2} \Big(\sum_j x_i^* x_i \Big) a^{1/2} (n^{-1} + a)^{-1} y_j^* \\ &= \lim_n \sum_i z_{ijn}^* z_{ijn} = \sum_i z_{ij}^* z_{ij}. \quad \blacksquare \end{aligned}$$

If A is *commutative* one obtains in particular *the Riesz decomposition property*:

$a, b, c \in A^+$ and $a \leqslant b + c \Rightarrow$ there are $b', c' \in A^+$ with $b' \leqslant b, c' \leqslant c$ and $a = b' + c'$.

A non-commutative C^*-algebra does not satisfy this property (see Corollary 1/4.18, below), but it satisfies the non-commutative Riesz decomposition property expressed by the above proposition.

3.14. PROPOSITION. *Let A be a C^*-algebra. If $\{J_\iota\}$ is any family of strongly facial ideals of A, then the algebraic sum $\sum_\iota J_\iota$ is a strongly facial ideal of A and*

$$\Big(\sum_\iota J_\iota\Big) \cap A^+ = \sum_\iota (J_\iota \cap A^+).$$

Proof. Denote $F_\iota = J_\iota \cap A^+$ and put $J = \sum_\iota J_\iota$, $F = \sum_\iota F_\iota$. Clearly, F is a convex cone, $F \subset J \cap A^+$ and $J = \operatorname{lin} F$. If F is a face, then the equality $F = J \cap A^+$ is forced. Therefore it is sufficient to prove that F is a strongly invariant face.

Consider $x \in A$, $x^*x \leqslant b \in F$. Then

$$b = y_1^*y_1 + \cdots + y_m^*y_m, \quad (y_j^*y_j \in F_{\iota_j};\ 1 \leqslant j \leqslant m),$$

and, by Proposition 3.13, there are $z_j \in A$, $(1 \leqslant j \leqslant m)$, such that

$$xx^* = z_1z_1^* + \cdots + z_mz_m^* \quad \text{and} \quad z_j^*z_j \leqslant y_jy_j^*, \quad (1 \leqslant j \leqslant m).$$

Since F_{ι_j} is a strongly invariant face and $y_j^*y_j \in F_{\iota_j}$, we have also $y_jy_j^* \in F_{\iota_j}$ and then successively, $z_j^*z_j \in F_{\iota_j}$, $z_j^*z_j \in F_{\iota_j}$, $xx^* \in F$.

By Lemma 2/2.9, it follows that F is indeed a strongly invariant face. ∎

Since any closed two-sided ideal is strongly facial (3.9.(ii)), the above proposition and Corollary 2/3.11 entail:

COROLLARY. *If $J_1, \dots, J_n$ are closed two-sided ideals, then $J_1 + \cdots + J_n$ is a closed two-sided ideal and*

$$(J_1 + \cdots + J_n)^+ = J_1^+ + \cdots + J_n^+. \quad ∎$$

3.15. Let π be a $*$-homomorphism of a C^*-algebra A onto a C^*-algebra B. It is easy to see that π can be uniquely extended to a $*$-homomorphism $\widetilde{\pi}$ of $\widetilde{A}$ onto $\widetilde{B}$. Clearly, $\widetilde{\pi}(1_{\widetilde{A}}) = 1_{\widetilde{B}}$.

PROPOSITION. *Let π be a $*$-homomorphism of a C^*-algebra A onto a C^*-algebra B, $a \in \widetilde{A}^+$ and $b = \widetilde{\pi}(a) \in \widetilde{B}^+$. Then*

$$\pi(\{x \in A;\ x^*x \leqslant a\}) = \{y \in B;\ y^*y \leqslant b\}.$$

Proof. Let $y \in B$, $y^*y \leqslant b$. Since $\pi(A) = B$, there is $z \in A$ with $\pi(z) = y$ and since $\widetilde{\pi}(\widetilde{A^+}) = \widetilde{B}^+$ (2.4), there is $r \in \widetilde{A_h}$, $r \leqslant a$ with $\widetilde{\pi}(r) = y^*y$. Set $h = z^*z - r = h^+ - h^-$. We have $\widetilde{\pi}(h) = 0$, so that $\widetilde{\pi}(h^+) = 0 = \widetilde{\pi}(h^-)$, since $\widetilde{\pi}$ preserves functional calculus (1.18.(7)). As $z^*z \leqslant a + h^+$ and $a \leqslant a + h^+$, the sequence

$$x_n = z(a + h^+)^{1/2}(n^{-1} + a + h^+)^{-1}a^{1/2} \in A$$

is convergent by Lemma 3.12 to an element $x \in A$ and we have:

$$\begin{aligned} x_n^*x_n &= a^{1/2}(n^{-1} + a + h^+)^{-1}(a + h^+)^{1/2}z^*z(a + h^+)^{1/2}(n^{-1} + a + h^+)^{-1}a^{1/2} \\ &\leqslant a^{1/2}(n^{-1} + a + h^+)^{-1}(a + h^+)^2(n^{-1} + a + h^+)^{-1}a^{1/2} \leqslant a, \end{aligned}$$

so that $x^*x \leqslant a$. On the other hand,

$$\pi(x) = \lim_n \pi(x_n) = \lim_n yb(n^{-1} + b)^{-1} = y,$$

since, by 3.3 and 3.4, $\{b(n^{-1}+b)^{-1}\}_n$ is an approximate unit for N_b and $y \in N_b$. ∎

COROLLARY 1. *Let π be a $*$-homomorphism of a C^*-algebra A onto a C^*-algebra B. Then*

$$\pi(\{x \in A;\ \|x\| \leqslant 1\}) = \{y \in B;\ \|y\| \leqslant 1\}. \quad ∎$$

Note that the same assertion with strict inequalities follows immediately from the definition of the quotient norm on $A/\operatorname{Ker}\pi = B$ (Corollary 1/3.11).

COROLLARY 2. *Let π be a $*$-homomorphism of a C^*-algebra A onto a C^*-algebra B and $\{b_n\}_n \subset B^+$ a norm-bounded increasing sequence. Then there exists an increasing sequence $\{a_n\}_n \subset A^+$ such that*

$$\sup_n \|a_n\| = \sup_n \|b_n\|, \quad \pi(a_n) = b_n; \quad n \geqslant 1.$$

Proof. Let $\lambda = \sup\limits_n \|b_n\|$. By the proposition there exists $a_1 \in A^+$ such that

$$a \leqslant \lambda \cdot 1_{\widetilde{A}} \quad \text{and} \quad \pi(a_1) = b_1.$$

Suppose we have already constructed $a_1, \ldots, a_n \in A$ such that $0 \leqslant a_1 \leqslant \cdots \leqslant a_n \leqslant \lambda \cdot 1_{\widetilde{A}}$ and $\pi(a_k) = b_k$, $(1 \leqslant k \leqslant n)$. By the proposition there exists $x \in A^+$ with

$$x \leqslant \lambda \cdot 1_{\widetilde{A}} - a_n \quad \text{and} \quad \pi(x) = b_{n+1} - b_n.$$

Then putting

$$a_{n+1} = a_n + x \in A,$$

we have $0 \leqslant a_n \leqslant a_{n+1} \leqslant \lambda \cdot 1_{\widetilde{A}}$ and $\pi(a_{n+1}) = b_{n+1}$. ∎

COROLLARY 3. *Let π be a $*$-homomorphism of a C^*-algebra A onto a C^*-algebra B.*

(i) *If F is a (closed) face of A^+, then $\pi(F)$ is a (closed) face of B^+ and $M_{\pi(F)} = \pi(M_F)$, $N_{\pi(F)} = \pi(N_F)$.*

(ii) *If M is a (closed) facial subalgebra of A, then $\pi(M)$ is a (closed) facial subalgebra of B.*

(iii) *If N is a closed left ideal of A, then $\pi(N)$ is a closed left ideal of B and $\pi(N^* \cap N) = \pi(N)^* \cap \pi(N)$.*

Proof. The proof is an easy combination of the above proposition with Corollary 1/3.11 and the results in 3.9. ∎

3.16. Consider a $*$-homomorphism π of a C^*-algebra A onto a C^*-algebra B and $a_1, a_2 \in A^+$, $b_1 = \pi(a_1)$, $b_2 = \pi(a_2) \in B^+$. Then

$$\pi(\{x \in A^+;\, x \leqslant a_1,\, x \leqslant a_2\}) \subset \{y \in B^+;\, y \leqslant b_1,\, y \leqslant b_2\},$$

but the equality does not hold in general. Proposition 3.1 shows that the equality is true if $a_1 = a_2$ and the following example shows that it does not hold even if $b_1 = b_2$.

Let H be a separable Hilbert space with a fixed orthonormal basis $\{\xi_n\}_{n\geqslant 1}$ and let $\{\alpha_n\}, \{\beta_n\} \subset \mathbb{R}$ such that

$$0 \neq \alpha_n = \alpha_n^2 + \beta_n^2 \quad \text{and} \quad \lim_n \alpha_n = 0.$$

Let a_1 be the orthogonal projection onto the closed linear span of $\{\xi_{2n}\}_{n\geqslant 1}$, $a_2 = a_1 + c$, where $c \in B(H)$ is defined by

$$c\xi_{2n-1} = \alpha_n \xi_{2n-1} + \beta_n \xi_{2n}, \quad c\xi_{2n} = \beta_n \xi_{2n-1} - \alpha_n \xi_{2n}, \quad (n \geqslant 1),$$

and $A = C^*(\{a_1, a_2\})$, $B = \mathbb{C}$. There is a $*$-homomorphism $\pi : A \mapsto B$ such that $\pi(a_1) = \pi(a_2) = 1$, namely

$$\pi(x) = \lim_n (x\xi_{2n}|\xi_{2n}); \quad x \in A.$$

Then $a_1, a_2 \in A^+$ and $\{y \in B^+;\, y \leqslant \pi(a_1),\, y \leqslant \pi(a_2)\} = [0,1]$, while

$$\{x \in B(H)^+;\, x \leqslant a_1,\, x \leqslant a_2\} = \{0\}.$$

Indeed, let $x \in B(H)^+$, $x \leqslant a_1$, $x \leqslant a_2$. Then, for $n \geqslant 1$,

$$\|x^{1/2}\xi_{2n-1}\|^2 = (x\xi_{2n-1}|\xi_{2n-1}) \leqslant (a_1\xi_{2n-1}|\xi_{2n-1}) = 0,$$

so $x\xi_{2n-1} = 0$. Thus, for $n \geqslant 1$ and $\lambda \in \mathbb{C}$,

$$\begin{aligned}\|x^{1/2}\xi_{2n}\|^2 &= (x(\lambda\xi_{2n-1} + \xi_{2n})|\lambda\xi_{2n-1} + \xi_{2n}) \leqslant (a_2(\lambda\xi_{2n-1} + \xi_{2n})|\lambda\xi_{2n-1} + \xi_{2n}) \\ &= |\lambda|^2\alpha_n + (\lambda + \overline{\lambda})\beta_n + 1 - \alpha_n.\end{aligned}$$

Taking $\lambda = -\beta_n\alpha_n^{-1}$ it follows that $\|x^{1/2}\xi_{2n}\|^2 = (\alpha_n - \alpha_n^2 - \beta_n^2)\alpha_n^{-1} = 0$ so $x\xi_{2n} = 0$. Consequently $x = 0$.

Moreover, with the above notations, the set $\{x \in K(H)^+;\, x \leqslant a_2 + c^-\}$ is not upward directed (compare with Proposition 2.10).

Indeed, $c = c^* \in K(H)$ so $c^+, c^- \in K(H)^+$. Also, $c^- \leqslant a_2 + c^-$, $c^+ = c + c^- \leqslant a_2 + c^-$. Suppose there is $x \in K(H)^+$, $x \leqslant a_2 + c^-$ with $c^+ \leqslant x$, $c^- \leqslant x$. Then $a_2 + c^- - x \in B(H)^+$ and $a_2 + c^- - x \leqslant a_1$, $a_2 + c^- - x \leqslant a_2$ so $a_2 + c^- - x = 0$, that is $x = a_2 + c^- \notin K(H)$, a contradiction.

3.17. PROPOSITION. *Let A be a C^*-algebra and B be a C^*-subalgebra containing an increasing approximate unit $\mathbf{u}$ for A. Then, for any $a \in A^+$ and any $\varepsilon > 0$ there is $b \in B^+$ such that*

$$a \leqslant b \quad \textit{and} \quad \|b\| \leqslant \|a\| + \varepsilon.$$

Proof. We may assume $\|a\| = 1$. We shall construct inductively a sequence $\{u_n;\, n = 1, 2, \ldots\} \subset \mathbf{u}$ and a sequence $\{a_n;\, n = 0, 1, 2, \ldots\} \subset A$ so that

(i) $a_0 = a$; $a_n = a_{n-1} - u_n a_{n-1} u_n$, for $n > 0$;

(ii) $\|a_n\| \leqslant \varepsilon/2^n$, for $n > 0$;

(iii) $u_1 a u_1 \leqslant u_1$; $u_n a_{n-1} u_n \leqslant (\varepsilon/2^{n-1}) u_n$, for $n > 1$.

Indeed if $a_0, \ldots, a_{n-1}$ and $u_1, \ldots, u_{n-1}$ are already constructed, then there is $u_n \in \mathbf{u}$ such that defining a_n by (i) the condition (ii) is satisfied. Since $\|a_{n-1}\| \leqslant \varepsilon/2^{n-1}$, we have $a_{n-1} \leqslant \varepsilon/2^{n-1}$ and condition (iii) follows.

Owing to (iii), set

$$b = u_1 + \sum_{n=2}^{\infty} (\varepsilon/2^{n-1}) u_n.$$

Then $b \in B^+$ and $\|b\| \leqslant 1 + \varepsilon$. Using (i), (ii), (iii) we get

$$a_n = a - \sum_{k=1}^{n} u_k a_{k-1} u_k$$

and then

$$a = \sum_{n=1}^{\infty} u_n a_{n-1} u_n.$$

It follows that $a \leqslant b$. ∎

If in the above proposition A is commutative, then one can choose $b \in B^+$ so that $a \leqslant b$ and $\|b\| = \|a\|$. However, this is not possible in general, as the following example shows.

Let H be a separable Hilbert space with an orthonormal basis $\{\xi_n\}$. Set $A = K(H)$ and $B = \{x \in A;$ each ξ_n is an eigenvector for $x\}$. Denote by q_n the orthogonal projection on $\operatorname{lin}\{\xi_1, \ldots, \xi_n\}$. Then $\{q_n\}$ is an approximate unit for A.

Consider now $\xi = \sum\limits_{n=1}^{\infty} \lambda_n \xi_n \in H$, $\|\xi\| = 1$, $\lambda_n \neq 0$ for all n, and let $p \in A$ be the one-dimensional orthogonal projection corresponding to ξ. Assume that there is $b \in B$, $\|b\| = 1$, $b \geqslant p$.

Then

$$\sum_{n=1}^{\infty} |\lambda_n|^2 (b\xi_n|\xi_n) = (b\xi|\xi) \geqslant (p\xi|\xi) = \sum_{n=1}^{\infty} |\lambda_n|^2,$$

which entails $(b\xi_n|\xi_n) = 1$ for all n, in contradiction with $b \in K(H)$.

Corollary. *Let B be a C^*-subalgebra of the C^*-algebra A and let F (respectively M, respectively N) be the face of A^+ (respectively the facial subalgebra of A, respectively the left ideal of A) generated by B^+ (respectively by B). Then F, M, N are closed and*

$$M_F = M = B^+ \cdot A \cdot B^+, \quad N_F = N = A \cdot B^+.$$

Proof. Let $\mathbf{u}$ be an increasing approximate unit for B. Then $N_{\mathbf{u}}$ is a closed left ideal of A (3.3) and $M_{\mathbf{u}} = N_{\mathbf{u}}^* \cap N_{\mathbf{u}}$ is a facial C^*-subalgebra of A (3.9.(iii)). Since $B \subset M_{\mathbf{u}}$ and $\mathbf{u}$ is an approximate unit for $M_{\mathbf{u}}$, the above proposition tell us that any element of $M_{\mathbf{u}}^+$ is majorized by an element of B^+. Therefore $F = M_{\mathbf{u}}^+$ and F is closed.

By 2.9, it is clear that $M = M_F$, so that $M = M_{\mathbf{u}}$ is closed.

Obviously, $A \cdot B^+ \subset N \subset N_{\mathbf{u}}$. If $a \in N_{\mathbf{u}} \cap A^+ = M_{\mathbf{u}}^+$, then $a \leqslant b$ for some $b \in B^+$ so that, by Corollary 2/3.4, $a = b^{1/4} c b^{1/4}$ for some $c \in A$. It follows that $N_{\mathbf{u}} \cap A^+ \subset A \cdot B^+$ and, by 3.9.(i), $N_{\mathbf{u}} = A \cdot (N_{\mathbf{u}} \cap A^+) \subset A \cdot B^+$. Therefore $N = A \cdot B^+$ is closed.

Also, using again 3.9.(i), we obtain the equality $M = B^+ \cdot A \cdot B^+$. ∎

3.18. Lemma. *Let A be a C^*-algebra and P be a subset of A^+ such that $x \in P \Rightarrow x^{1/2} \in P$. Then the two-sided ideal J of A generated by P is a facial ideal. In particular, J contains the invariant face F of A^+ generated by P.*

Proof. It is sufficient to show that $F \subset J$ since then $\operatorname{lin} F \subset J$ and $\operatorname{lin} F$ is a facial ideal (2.9). If $a \in F$, then there are $\lambda \geqslant 0$, $b_j \in P$ and $u_j \in \widetilde{A}$, u_j unitary elements, $(1 \leqslant j \leqslant m)$, such that

$$a \leqslant \lambda_1 u_1^* b_1 u_1 + \cdots + \lambda_m u_m^* b_m u_m.$$

By Proposition 13 there are $z_j \in A$, $(1 \leqslant j \leqslant m)$, with

$$a = z_1 z_1^* + \cdots + z_m z_m^*$$

and

$$z_j^* z_j \leqslant \lambda_j u_j^* b_j u_j, \quad (1 \leqslant j \leqslant m).$$

By Proposition 3.4 there are $y_j \in A$, $(1 \leqslant j \leqslant m)$, with

$$z_j = y_j (\lambda_j u_j^* b_j u_j)^{1/4} = y_j \lambda_j^{1/4} u_j^* b_j^{1/4} u_j, \quad (1 \leqslant j \leqslant m).$$

Since $b_j^{1/4} \in P \subset J$, it follows that $z_j \in J$ and hence $a \in J$. ∎

3.19. The Pedersen ideal. Let A be a C^*-algebra and consider the set

$$F_0(A) = \{a \in A^+;\ \text{there is } b \in A^+ \text{ such that } a = ab\,(= ba)\}.$$

If A is unital, then clearly $F_0(A) = A^+$.

If A is commutative, that is $A = C_0(\Omega)$ with Ω a locally compact Hausdorff space, then $F_0(A)$ consists of all positive continuous functions on Ω having compact support.

Now, in the general case, an element $a \in A^+$ belongs to $F_0(A)$ if and only if there is a commutative C^*-subalgebra B of A such that $a \in F_0(B)$. Using the Gelfand representation of B and the above remark, it is easy to check that

$$a \in F_0(A) \Leftrightarrow \text{there is } b \in F_0(A) \text{ such that } a = ab. \tag{1}$$

Denote by $C_c((0,+\infty))^+$ the set of all positive continuous functions on $(0,+\infty)$ with compact support. Then

$$x \in A^+,\ f \in C_c((0,+\infty))^+ \Rightarrow f(x) \in F_0(A). \tag{2}$$

Consider also

$$F(A) = \text{the face of } A^+ \text{ generated by } F_0(A),$$
$$K(A) = \operatorname{lin} F(A).$$

Theorem. *For every C^*-algebra A, $K(A)$ is the smallest dense two-sided ideal of a. Moreover,*

(i) *For any $x \in A^+$ there is an increasing sequence $\{x_n\}$ in $K(A)^+$ such that $\|x - x_n\| \to 0$.*

(ii) *The facial C^*-subalgebra (respectively the closed left ideal) of A generated by any finite subset of $K(A)$ is contained in $K(A)$.*

(iii) *The facial C^*-subalgebra of A generated by any finite collection of C^*-subalgebras of A contained in $K(A)$ is again contained in $K(A)$.*

In particular, $K(A)$ is an algebraic strongly facial ideal.

Proof. If $u \in \widetilde{A}$ is unitary and $a \in F_0(A)$, then $a = ab$ for some $b \in A^+$ and $u^*au = (u^*au)(u^*bu) \in F_0(A)$. It follows that $F(A)$ is an invariant face of A^+, so that $K(A)$ is a facial ideal of A (2.9).

Let $x \in A^+$. There is a sequence $\{f_n\} \subset C_c((0,+\infty))^+$ such that $f_n(t) \uparrow t$ uniformly for $t \in \sigma(x)$. Then $f_n(x) \in F_0(A)$, $\{f_n(x)\}$ is increasing by (2), and $\|f_n(x) - x\| \to 0$ by functional calculus. Hence $F(A)$ is dense in A and $K(A)$ is dense in A.

Let J be any dense two-sided ideal of A. Consider

$$P = \{f(x^*x);\ x \in J,\ f \in C_c((0,+\infty))^+\}.$$

As above, P is dense in $\{x^*x;\ x \in J\}$. For any $z \in A$ there is a sequence $\{x_n\}$ in J with $x_n \to z$, so that $x_n^*x_n \to z^*z$. It follows that P is dense in A^+.

If $f \in C_c((0,+\infty))^+$, then there is $g \in C_c((0,+\infty))^+$ with $f(t) = tg(t)$ for all $t \in (0,+\infty)$. Hence $f(x^*x) = x^*xg(x^*x)$ for any $x \in J$. It follows that $P \subset J$.

Clearly, P satisfies the condition of Lemma 3.18, therefore the invariant face F of A^+ generated by P is contained in J and is dense in A^+.

Now, if $a \in F_0(A)$, then $a = ab = ba$ for some $b \in A^+$. Since F is dense in A^+, there is $c \in F$ with $\|b - c\| < \varepsilon < 1$. Then

$$0 \leqslant a^{1/2}(\varepsilon - (b - c))a^{1/2} = a^{1/2}ca^{1/2} - (1 - \varepsilon)a,$$
$$a \leqslant (1 - \varepsilon)^{-1}a^{1/2}ca^{1/2},$$

therefore $a \in F$, because F is an invariant face. Hence $F_0(A) \subset F$, $F(A) \subset F \subset J$ and $K(A) = \operatorname{lin} F(A) \subset J$.

Thus, $K(A)$ is indeed the smallest dense two-sided ideal of A and also (i) is proved.

Now, let $x_1, \dots, x_m \in K(A)$. We shall show that the facial C^*-subalgebra M generated by $\{x_1, \dots, x_m\}$ is contained in $K(A)$. Since every x_i is a linear combination of positive elements, each of them majorized by a sum of elements of $F_0(A)$, we may and we shall assume $x_1, \dots, x_m \in F_0(A)$. Then, by (1), there are $b_i \in F_0(A)$ with $x_i = x_i b_i = b_i x_i$, $(1 \leqslant i \leqslant m)$. Clearly, the facial C^*-subalgebra M generated by $\{x_1, \dots, x_m\}$ is equal to the facial C^*-subalgebra M_a generated by $a = x_1 + \cdots + x_m$. If $y \in M_a^+ = N_a \cap A^+$, then by Lemma 3.12 the sequences

$$y_{in} = x_i^{1/2}a^{1/2}(n^{-1} + a)^{-1}y^{1/2}, \quad (1 \leqslant i \leqslant m;\, n \in \mathbb{N}),$$

are convergent to elements $y_i \in A$, $(1 \leqslant i \leqslant m)$. But $y_{in} = b_i^{1/2}y_{in}$, so that $y_i = b_i^{1/2}y_i$, $(1 \leqslant i \leqslant m)$. As in the proof of 3.13 we obtain

$$y = \sum_{i=1}^{m} y_i^* y_i = \sum_{i=1}^{m} y_i^* b_i y_i \in F(A),$$

since $b_i \in F_0(A) \subset F(A)$ and $F(A)$ is invariant. Thus,

$$M_a = \operatorname{lin} M_a^+ \subset \operatorname{lin} F(A) = K(A).$$

If N is the closed left ideal generated by $\{x_1, \dots, x_m\}$, then $N \cap A^+ = M^+ \subset K(A)$ by the above, so that $N = A(N \cap A^+) \subset K(A)$ (see 3.9).

Let B, C be C^*-subalgebras of A contained in $K(A)$. The closed face Q of A^+ generated by $B^+ \cup C^+$ is the closure of the face of A^+ generated by $B^+ \cup C^+$. Thus, given $x \in Q$, there are:

$$y_n \in B^+, \quad \|y_n\| \leqslant 1, \quad z_n \in C^+, \quad \|z_n\| \leqslant 1; \quad \beta_n, \gamma_n \in \mathbb{R}, \quad \beta_n, \gamma_n \geqslant 0$$

and

$$x_n \in A^+, \quad x_n \leqslant \beta_n y_n + \gamma_n z_n$$

such that $x = \lim\limits_n x_n$. Put

$$y = \sum_n 2^{-n} y_n \in B^+, \quad z = \sum_n 2^{-n} z_n \in C^+.$$

Then $y + z \in K(A)$ and x belongs to the facial C^*-subalgebra of A generated by $y + z$, hence $x \in K(A)$. It follows that the facial C^*-subalgebra of A generated by $B \cup C$ is contained in $K(A)$.

Finally, if $x \in K(A) \cap A^+$, then $x^{1/2}$ belongs to the (facial) C^*-subalgebra generated by x, so that $x^{1/2} \in K(A)$. Hence $K(A)$ is an algebraic facial ideal and therefore it is an algebraic strongly facial ideal. ∎

$K(A)$ is called *the Pedersen ideal* of A.

Corollary 1. *Let π be a $*$-homomorphism of a C^*-algebra A onto a C^*-algebra B. Then $\pi(K(A)) = K(B)$.*

Proof. $\pi(K(A)) \subset K(B)$ by construction. Since $K(A)$ is a dense facial ideal of A, $\pi(K(A))$ is a dense (facial) ideal of B by Corollary 2/3.15. Using the theorem we infer that $K(B) \subset \pi(K(A))$. ∎

Corollary 2. *Let A,B be C^*-algebras and $\pi_0 : K(A) \to B$ be a $*$-homomorphism. Then π_0 has a unique extension to a $*$-homomorphism $\pi : A \to B$.*

Proof. By the theorem, $K(A)$ is the union of all C^*-subalgebras contained in it. The restriction of π_0 to every C^*-subalgebra of $K(A)$ has norm $\leqslant 1$ (1.9). Therefore $\|\pi_0\| \leqslant 1$ and the result follows. ∎

Corollary 3. *Let A, B be C^*-algebras. If $K(A), K(B)$ are $*$-isomorphic, then A,B are $*$-isomorphic.* ∎

3.20. Notes. The use of approximate units as well as the first factorization theorems appeared in the frame of the classical convolution algebras $L^1(\mathbb{R})$ and $L^1(\mathbb{R}/\mathbb{Z})$ (see [130], Section 32, Notes). The most important contribution to the factorization theory is due to P.J. Cohen [52] who proved a refined version of Theorem 3.1 in the case $X = A$. Actually, if in the statement of Theorem 3.1 the bounded left approximate unit is bounded by some $\lambda > 0$, then, for every y in the closed linear span of X_0 and every $\varepsilon > 0$, there exists $a \in A$, $\|a\| \leqslant \lambda$, and $x \in \overline{A \cdot y} \subset X$, $\|x - y\| \leqslant \varepsilon$, such that $y = ax$ ([130], 32.22). The useful results Corollary 1 and Corollary 2/3.1 appeared in [134], [135]. In 3.1 we followed the lectures of B.E. Johnson [137]. For further results concerning approximate units and factorization in general Banach algebras we refer to [130], Section 32, [177], [230], [291].

The existence of increasing approximate units in arbitrary C^*-algebras has been first proved by I.E. Segal [262] and the refinement of this result for ideals is due to J. Dixmier [76] and F. Combes (cf. [78], 1.8). The canonical approximate unit of a facial subalgebra (3.2) appeared in [242], Section 1.4.

There are two other important results concerning approximate units in C^*-algebras, which we record below.

First, J.F. Aarnes and R.V. Kadison [3] proved that *every separable C^*-algebra A has a countable increasing approximate unit $\{u_n\}_{n\geqslant 1}$ which is "commutative"*, i.e., $u_n u_m = u_m u_n$ for all $m, n \geqslant 1$. Indeed, let $\{x_k\}_{k\geqslant 1}$ be a norm-dense sequence in A and put

$$a = \sum_{k=1}^{+\infty} (2^k \|x_k^* x_k\|)^{-1} x_k^* x_k \in A; \quad u_n = a^{1/n}, \quad (n \geqslant 1).$$

Then $a^{1/n} \uparrow \mathbf{s}_{A^{**}}(a) = 1_{A^{**}}$ (see Corollary 6/8.4 and 7.15.4), so that $\varphi(y_n) = \varphi(a^{1/n}) \to \varphi(1_{A^{**}}) = 1$ for every state φ on A (see 4.7 and Corollary 6/8.4), and hence $\{u_n\}_{n\geqslant 1}$ is a commutative countable increasing approximate unit for A, by Corollary 3/4.15. This argument proves also the "if" part of the following statement: *a C^*-algebra A has a commutative countable increasing approximate unit $\{u_n\}_{n\geqslant 1}$ if and only if there exists a "strictly positive" element $a \in A$*, i.e., such that $\varphi(a) > 0$ for every state φ on A; the "only if" part is immediate with $a = \sum\limits_{n=1}^{+\infty} 2^{-n} u_n$ using again Corollary 3/4.15.

The second result appeared implicitely in the articles of D. Voiculescu [338], D. Olesen and G.K. Petersen [221], [243] and was explicitely stated in [9], [19] as follows: *every two-sided ideal J of a C^*-algebra A has an increasing approximate unit $\{u_\iota\}_{\iota\in I}\subset J$ which is "quasi-central"*, i.e., $\lim\limits_{\iota}\|xu_\iota-u_\iota x\|=0$ for all $x\in A$. For the proof (cf. [19]) the first remark is that given an arbitrary increasing approximate unit $\{v_k\}_{k\in K}$ for J, its convex hull $V=\mathrm{co}\{v_k;\ k\in K\}$ can be still viewed, in a natural way, as an increasing approximate unit for J, and $\inf\limits_{v\in V}\|xv-vx\|=0$ for all $x\in A$; in fact 0 is norm-adherent to the convex set $\{xv-vx;\ v\in V\}$ because in the contrary case there would exist a bounded linear functional φ on A such that $|\varphi(xv-vx|\geqslant\delta>0$ for all $v\in V$ and some $x\in A$, in contradiction with the fact that $\lim\limits_{v\in V}(xv-vx)=0$. Now we assert that for every $x_1,\dots,x_n\in A$, $v\in V$ and $\varepsilon>0$ there is $w\in V$, $w\geqslant v$ such that $\|x_jw-wx_j\|\leqslant\varepsilon$ for each $j=1,\dots,n$; this will clearly enable us to extract a subnet $\{u_\iota\}_{\iota\in I}$ of V such that $\lim\limits_{\iota}\|xu_\iota-u_\iota x\|=0$ for all $x\in A$ and, since V is an approximate unit for J, the same must be true for $\{u_\iota\}_{\iota\in I}$. To prove the assertion, note that the set $W=\{w\in V;\ w\geqslant v\}$ is a convex increasing approximate unit for J and consider the C^*-direct product A_n of n copies of A and $x=(x_1,\dots,x_n)\in A_n$; then the direct product J_n of n copies of J is a two-sided ideal of A_n and $\{\widetilde{w}=w\oplus\dots\oplus w;\ w\in W\}$ is a convex increasing approximate unit for J_n, so that, by the above remark, there exists $w\in W$ such that

$$\max_j\|x_jw-wx_j\|=\|x\widetilde{w}-\widetilde{w}x\|\leqslant\varepsilon.$$

Moreover, *if A is separable, then every two-sided ideal J of A has a countable quasi-central increasing approximate unit*. Indeed, if $\{u_\iota\}_{\iota\in I}$ is any quasi-central increasing approximate unit for J, $\{x_k\}_{k\geqslant1}$ a dense sequence in A and $\{b_k\}_{k\geqslant1}$ a dense sequence in J, then by an obvious induction we can find an increasing sequence $\{\iota_m\}_{m\geqslant1}\subset I$ such that $\|u_{\iota_m}b_k-b_k\|\leqslant1/m$ and $\|x_ku_{\iota_m}-u_{\iota_m}x_k\|\leqslant1/m$ for every $k=1,\dots,m$, and an easy approximation argument shows that the sequence $\{u_{\iota_m}\}_{m\geqslant1}$ has the required properties. These results should be compared with classical result contained in Corollary (ii)/8.7.

The particularities of approximate unites in C^*-algebra (3.2, 3.3, 3.12) give rise to stronger factorization results (3.4, 3.13) near to the classical polar decomposition theorem (see 7.12, 7.13, 9.14). The material in (3.3, 3.4) is due to F. Combes [56], [57] and the non-commutative Riesz decomposition (3.13; see also 9.15, Corollary 1/4.18) is due to G.K. Pedersen ([234]; see also [57]). Some important particular cases of the results presented in 3.5–3.9 have been obtained by I.E. Segal [283], J. Dixmier [71], [73], E.G. Effros [84], R.T. Prosser [251]; in the achieved general form presented here, these results are due F. Combes [57], [58], [59], [60], G.K. Pedersen [233], [234], [237] and N.H. Petersen [244] to which we refer also for further results. We just mention that, generalizing the results obtained by J. Dixmier [73] for two-sided ideals of W^*-algebras (see Proposition 3.(ii)/8.7), F. Combes [58] showed that for every strongly invariant face F of a C^*-algebra A and every $\lambda>0$, the set $F^\lambda=\{x^\lambda;\ x\in F\}$ is still a strongly invariant face, defined for every strongly facial ideal J of A its λ^{th} power as $J^\lambda=\mathrm{lin}(J\cap A^+)^\lambda$ and proved that $J^\lambda J^\mu=J^{\lambda+\mu}$, $(\lambda,\mu>0)$.

The result of Corollary 3.14 has been conjectured by J. Dixmier ([78], first edition, 1.9.12) and proved for the first time by E. Størmer [299]. A very simple proof of this result was given J. Bunce [36]. The proof we have presented and the generalization as Proposition 3.14 are due to G.K. Pedersen [233], III, [234] and F. Combes [57].

The fact that the quotient of a C^*-algebra by a closed two-sided ideal is again a C^*-algebra (Theorem 3.11) has been proved for the first time by I.E. Segal [283] and I. Kaplansky [154]. The results in 3.10 and the present proof of Theorem 3.11 are due to F. Combes [57] (cf. also [78], 1.8). For further results on quotient algebras see [243].

The results in 3.15 are due to G.K. Pedersen [233], III, [234] and F. Combes [57]. The counterexamples in 3.16, given by J.R. Stephansson [292], give in particular a negative answer to a question raised by J. Dixmier ([71], p. 22; [77], Chapter III, Section 1, Exercise 10; see [61] for a detailed discussion of similar properties).

The result of Proposition 3.17 and the example in 3.17 are due to C.A. Akemann [7], while Corollary 3.17 is due to F. Combes [58].

Given a C^*-algebra A, the ideal $K(A)$ (3.19) has been discovered by G.K. Pedersen [233], I, [237] who proved that $K(A)$ is the smallest dense facial ideal of A and, together with N.H. Petersen [244], proved the statements (i)–(iii) from Theorem 3.19 and the Corollaries in 3.19. The minimality and uniqueness of $K(A)$ just as a dense two-sided ideal of A has been pointed out by K.B. Laursen and A.M. Sinclair [175], by proving Lemma 3.18. The terminology of "Pedersen ideal" has been introduced in [56]. For further results concerning the determination of the Pedersen ideal in some concrete C^*-algebras we refer to [56], [176], [237], [248]. Also, G.K. Pedersen [233], [237] introduced and studied a certain locally convex topology τ on $K(A)$ and defined a "C^*-integral"on A as a τ-continuous linear functional on $K(A)$. Several characterizations of C^*-integrals appeared in these works and in the related work of F. Combes [56] on weights on C^*-algebras.

For the elaboration of Section 3 we have used [302] to which we also refer for a comprehensive exposition of the other above mentioned results.

Chapter 4

POSITIVE FORMS AND $*$-REPRESENTATIONS

4.1. $*$-representations. Let A be a $*$-algebra. Recall that a $*$-representation π of A on a Hilbert space H is a $*$-homomorphism $\pi : A \to B(H)$. Then H is called *the space* of π (sometimes denoted as H_π) and the cardinal of any orthonormal basis of H is called the *dimension* of π, denoted as $\dim \pi$. The *commutant* $\pi(A)'$ of $\pi(A)$ in $B(H)$ is the set of those $T' \in B(H)$ which commute with all $\pi(x)$, $x \in A$. Clearly, $\pi(A)'$ is a $*$-subalgebra of $B(H)$ closed in the weak operator topology, in particular $\pi(A)'$ is a C^*-subalgebra of $B(H)$.

Given a $*$-representation $\pi : A \to B(H)$ we obtain a $*$-representation $\widetilde{\pi} : \widetilde{A} \to B(H)$, called the canonical extension of π, by putting $\widetilde{\pi}(x+\lambda) = \pi(x) + \lambda 1_H$ for $x \in A$, $\lambda \in C$ if $A \neq \widetilde{A}$.

Two $*$-representations π_1, π_2 of A on Hilbert spaces H_1, H_2 respectively are called (*unitarily*) *equivalent*, denoted as $\pi_1 \simeq \pi_2$, if there is a unitary operator $U : H_1 \to H_2$ such that $\pi_2(x) = U\pi_1(x)U^*$, for all $x \in A$.

Given a family $\{\pi_\iota : A \to B(H_\iota)\}_{\iota \in I}$ of $*$-representations such that for every $x \in A$ then numerical set $\{\|\pi_\iota(x)\|;\ \iota \in I\}$ is bounded, we can form a new $*$-representation $\pi : A \to B(H)$ where H is the Hilbert space direct sum of the H_ι's and, for each $x \in A$, $\pi(x) \in B(H)$ is the unique bounded linear operator on H such that $\pi(x)|H_\iota = \pi_\iota(x)$, $\iota \in I$. Then π is called the *direct sum* of the $*$-representations π_ι and is denoted by $\bigoplus\limits_{\iota \in I} \pi_\iota$.

In particular, if π_ι are all equal to a fixed $*$-representation π_0, then $\bigoplus\limits_{\iota \in I} \pi_\iota$ is called a *multiple* of π_0 and is denoted by (card I) π_0.

Owing to 1.13.(2), we see that we can form the direct sum of any family of $*$-representations of any Banach $*$-algebra.

Let $\pi : A \to B(H)$ be a $*$-representation and let K be a closed subspace of H, stable under $\pi(A)$. Then $H \ominus K$ is also stable under $\pi(A)$ since for $\xi \in K$, $\eta \in H \ominus K$, $x \in A$, we have $\pi(x^*)\xi \in K$ so that $(\pi(x)\eta|\xi) = (\eta|\pi(x^*)\xi) = 0$. We obtain a $*$-*subrepresentation* $\pi_K : A \to B(H)$ of π by putting $\pi_K(x) = \pi(x)|K$ for $x \in A$. Clearly, $\pi \simeq \pi_K \oplus \pi_{H \ominus K}$.

For a $*$-representation $\pi : A \to B(H)$ consider the closed subspaces H_e, H_0 of H defined by

$$H_e = \text{the closed subspace generated by } \{\pi(x)\xi;\ x \in A,\ \xi \in H\}$$

$$H_0 = \{\eta \in H;\ \pi(x)\eta = 0 \text{ for all } x \in A\}.$$

Then H_e is stable under $\pi(A)$ (also under $\pi(A)'$ and $H_0 = H \ominus H_e$, as easily verified. H_e is called the *essential subspace* of π. If $H = H_e$, then π is called *non-degenerated*. Since π_{H_e} is non-degenerated π_{H_0} is identically zero and π is equivalent to their direct sum, we see that

Lemma 1. *Any $*$-representation of A is equivalent to the direct sum of a non-degenerated $*$-representation and an identically zero representation. Moreover, this decomposition is unique up to a unitary equivalence.* ∎

If $A \subset B(H)$, then A is called *non-degenerated* if the identity representation of A is non-degenerated.

Let $\pi : A \to B(H)$ be a $*$-representation. A vector $\xi \in H$ is called *cyclic* for π if the subspace $\pi(A)\xi$ is dense in H. If there is a cyclic vector for π, then π is called a *cyclic* $*$-representation. A cyclic $*$-representation is non-degenerated. Moreover, an obvious argument based on Zorn's lemma shows that:

Lemma 2. *Any non-degenerated $*$-representation of A is equivalent to a direct sum of cyclic $*$-representations.* ∎

A $*$-representation $\pi : A \to B(H)$ is called *topologically irreducible* if $\pi(A)' = \{\lambda 1_H;\ \lambda \in \mathbb{C}\}$. If K is a closed subspace of H stable under $\pi(A)$, then the orthogonal projection of H onto K belongs to $\pi(A)'$. Therefore, if π is topologically irreducible, then $\{0\}$ and H are the only closed subspace of H stable under $\pi(A)$, that is any non-zero vector in H is cyclic for π. This property characterizes topologically irreducible $*$-representations, but we postpone the proof for Section 7.22, where a much more detailed analysis of irreducibility will be given. Although each finite dimensional $*$-representation is equivalent to a direct sum of (topologically) irreducible $*$-representations, this is not true for all $*$-representations.

The class of all $*$-representations (respectively of all topologically irreducible $*$-representations) of a given $*$-algebra A will be denoted by $\mathrm{Rep}(A)$ (respectively by $\mathrm{Irr}(A)$).

Consider now a Banach $*$-algebra A with a bounded left approximate unit $\{u_\iota\}_{\iota \in I}$ and a $*$-representation $\pi : A \to B(H)$. By Theorem 3.1, $\pi(A)H$ is then a closed subspace of H, i.e.

$$H_e = \pi(A)H. \tag{1}$$

Assume that π is non-degenerated. Then, using Theorem 1.12, for any $x \in A$ and any $\xi \in H$ we obtain

$$\|\pi(u_\iota)\pi(x)\xi - \pi(x)\xi\| \leqslant \|\pi\|\,\|u_\iota x - x\|\,\|\xi\| \to 0. \tag{2}$$

Since by (1) $\pi(A)H = H$, it follows that $\pi(u_\iota)\eta \to \eta$ for all $\eta \in H$ (this can also be obtained with an easy approximation argument). In other words, $\{\pi(u_\iota)\}$ converges to 1_H in the strong operator topology. We state this fact for further reference.

Lemma 3. *If $\pi : A \to B(H)$ is a non-degenerated $*$-representation of a Banach $*$-algebra A, then $\pi(u_\iota) \xrightarrow{\text{so}} 1_H$ for any bounded left approximate unit $\{u_\iota\}$ of A.* ∎

4.2. Forms and weights. Let A be a $*$-algebra. A complex linear functional on A will be called shortly a *form* on A.

For every form φ on A and every $a, b \in A$ we can consider the form $\varphi(a \cdot b) = a \cdot \varphi \cdot b$ on A defined by

$$\varphi(a \cdot b)(x) = (a \cdot \varphi \cdot b)(x) = \varphi(axb); \quad x \in A.$$

The forms $\varphi(a\,\cdot) = a \cdot \varphi$ and $\varphi(\cdot\, b) = \varphi \cdot b$ are defined similarly.

The *adjoint form* φ^* of φ is defined by

$$\varphi^*(x) = \overline{\varphi(x^*)}; \quad x \in A.$$

The form φ is called *selfadjoint* (or *hermitean*) if $\varphi = \varphi^*$. Clearly, φ is selfadjoint if and only if it takes real values on all selfadjoint elements of A. Every form φ has a unique decomposition $\varphi = \operatorname{Re}\varphi + i\operatorname{Im}\varphi$ with $\operatorname{Re}\varphi = (\varphi + \varphi^*)/2$, $\operatorname{Im}\varphi = (\varphi - \varphi^*)/2\mathrm{i}$ selfadjoint forms.

The form φ is called *positive*, denoted as $\varphi \geqslant 0$, if $\varphi(x^*x) \geqslant 0$ for all $x \in A$, i.e. if φ takes positive values on positive elements of A. For two forms φ, ψ we write $\varphi \leqslant \psi$ if $\psi - \varphi \geqslant 0$. If $A_h = A^+ - A^+$, then any positive form is selfadjoint; recall that this happens whenever $A^2 = A$ (2.8), for instance if A is unital or if A is a C^*-algebra.

Given a $*$-representation $\pi : A \to B(H)$ and a vector $\xi \in H$, we obtain a positive form $\varphi_{\pi,\xi}$ on A by the formula

$$\varphi_{\pi,\xi}(x) = (\pi(x)\xi|\xi); \quad x \in A.$$

In particular, if $A \subset B(H)$ and π is the identity representation, then the positive form $\varphi_{\pi,\xi}$ is denoted by ω_ξ, i.e.:

$$\omega_\xi(x) = (x\xi|\xi); \quad x \in A.$$

Thus, in the general case we have $\varphi_{\pi,\xi} = \omega_\xi \circ \pi$. The forms $\omega_\xi \circ \pi$, $\xi \in H$, are called *associated* to π. If $A \subset B(H)$, the forms ω_ξ, $\xi \in H$, are called *vector forms* on A.

A *weight* φ on A^+ is a mapping $\varphi : A^+ \to [0, +\infty]$ such that

$$\varphi(x + y) = \varphi(x) + \varphi(y); \quad x, y \in A^+,$$
$$\varphi(\lambda x) = \lambda\varphi(x); \quad x \in A^+, \quad \lambda \in \mathbb{R}^+, \quad (0 \cdot (+\infty) = 0).$$

In what follows the $*$-algebra A is assumed to satisfy the Combes's axiom (2.8). Then, given a weight φ on A^+,

$$F_\varphi = \{x \in A^+;\ \varphi(x) < +\infty\}$$

is a face of A^+ and, by Proposition 2.9,

$$N_\varphi = N_{F_\varphi} = \{x \in A;\ \varphi(x^*x) < +\infty\}$$

is a left ideal of A and moreover

$$M_\varphi = N_\varphi^* N_\varphi = \operatorname{lin} F_\varphi$$

is a facial subalgebra of A with $M_\varphi \cap A^+ = F_\varphi$. Since $M_\varphi = \operatorname{lin} F_\varphi$, we can extend φ by linearity to a form, still denoted by φ, on the $*$-algebra M_φ.

The weight φ is called *finite* if $F_\varphi = A^+$. In this case $M_\varphi = A^2$. If $A^2 = A$, then $M_\varphi = A$ and φ extends by linearity to a positive form on A, thus the finite weights on A^+ are exactly the restrictions of positive forms on A.

The weight φ is called a *trace* if

$$\varphi(x^*x) = \varphi(xx^*); \quad x \in A.$$

In this case F_φ is a strongly invariant face of A^+ and M_φ, N_φ are selfadjoint (two-sided) ideals of A.

4.3. The GNS construction. In this section we describe a general fundamental construction which gives a canonical way to associate $*$-representations to weights.

Let A be a $*$-algebra. Let N_θ be a left ideal of A and let θ be a *pre-inner product* on N_θ, i.e. a mapping

$$N_\theta \times N_\theta \ni (a,b) \to \theta(a,b) \in \mathbb{C}$$

with the properties

$$\theta\Big(\sum_{i=1}^n \lambda_i a_i,\ \sum_{j=1}^m \mu_j b_j\Big) = \sum_{i=1}^n \sum_{j=1}^m \lambda_i \overline{\mu_j} \theta(a_i, b_j); \quad a_i, b_j \in N_\theta,\ \lambda_i, \mu_j \in \mathbb{C},$$
$$\theta(a,a) \geqslant 0; \quad a \in N_\theta.$$

Then θ defines a pre-Hilbert structure on N_θ, in particular

(1) $$\theta(b,a) = \overline{\theta(a,b)}; \quad a, b \in N_\theta,$$

(2) $$|\theta(a,b)| \leqslant \theta(a,a)^{1/2} \theta(b,b)^{1/2}; \quad a, b \in N_\theta.$$

Assume moreover that θ satisfies the conditions:

(i) for every $x \in A$ there exists $\lambda(x) > 0$ such that

$$\theta(xa, xa) \leqslant \lambda(x)\theta(a,a); \quad a \in N_\theta,$$

(ii) $\theta(xa, b) = \theta(a, x^*b)$; $a, b \in N_\theta$; $x \in A$.

From (i) it follows that $L_\theta = \{a \in N_\theta;\ \theta(a,a) = 0\}$ is a left ideal. For each $a \in N_\theta$ denote by a_θ the canonical image of a in N_θ / L_θ. Then

$$(a_\theta | b_\theta)_\theta = \theta(a,b); \quad a, b \in N_\theta,$$

is a well defined inner product on N_θ/L_θ. Denote by H_θ the associated Hilbert space, i.e. the completion of N_θ/L_θ with respect to the norm

$$\|a_\theta\|_\theta = (a_\theta|a_\theta)_\theta^{1/2}, \quad (a \in N_\theta).$$

Since N_θ and L_θ are left ideals, every $x \in A$ defines a linear operator $\pi_\theta^0(x)$ on N_θ/L_θ by

$$\pi_\theta^0(x)a_\theta = (xa)_\theta; \quad a \in N_\theta,$$

and we have

$$\theta(xa, b) = (\pi_\theta^0(x)a_\theta|b_\theta)_\theta; \quad a, b \in N_\theta. \tag{3}$$

The condition (i) insures that $\pi_\theta^0(x)$ is bounded, so it extends to a unique bounded linear operator $\pi_\theta(x) \in B(H_\theta)$. The condition (ii) shows that $\pi_\theta(x)^* = \pi_\theta(x^*)$.

We thus get a $*$-representation $\pi_\theta : A \to B(H_\theta)$ relied to the original data N_θ and θ by the formula (3).

Now let φ be a weight on the $*$-algebra A, assumed to satisfy Combes' axiom. Then N_φ is a left ideal and

$$(a|b)_\varphi = \varphi(b^*a); \quad a, b \in N_\varphi,$$

defines a pre-inner product $(\cdot|\cdot)_\varphi$ on N_φ. By Combes' axiom, for each $x \in A$ there is $\lambda(x) > 0$ such that $a^*x^*xa \leqslant \lambda(x)a^*a$, for all $a \in A$. It follows that $(\cdot|\cdot)_\varphi$ satisfies condition (i). Also, condition (ii) is satisfied:

$$(xa|b)_\varphi = \varphi(b^*xa) = \varphi((x^*b)^*a) = (a|x^*b)_\varphi; \quad a, b \in N_\varphi,\ x \in A.$$

Consequently, we obtain a $*$-representation $\pi_\varphi : A \to B(H_\varphi)$, called the *GNS* (Gelfand, Naĭmark, Segal) *representation associated to* φ, such that

$$\varphi(b^*xa) = (\pi_\varphi(x)a_\varphi|b_\varphi)_\varphi; \quad x \in A,\ a, b \in N_\varphi. \tag{4}$$

Note also the following particular cases of relations (1), (2):

$$\varphi(a^*b) = \overline{\varphi(b^*a)}; \quad a, b \in N_\varphi; \tag{5}$$

$$|\varphi(b^*a)|^2 \leqslant \varphi(a^*a)\varphi(b^*b); \quad a, b \in N_\varphi. \tag{6}$$

Relation (6) is known as the *Schwarz inequality.* Since $M_\varphi = N_\varphi^* N_\varphi$ relation (5) can be rewritten as

$$\varphi(x^*) = \overline{\varphi(x)}; \quad x \in M_\varphi. \tag{5$'$}$$

A family F of weights on A^+ will be called *sufficient* if

$$x \in A,\ \varphi(a^*x^*xa) = 0 \text{ for all } \varphi \in F,\ a \in N_\varphi \Rightarrow x = 0.$$

If A is a Banach $*$-algebra, then by (4) the family F is sufficient if and only if the direct sum $*$-representation $\bigoplus_{\varphi \in F} \pi_\varphi$ (see 4.1 and 1.13.(2)) is injective.

The family F will be called *separating* if

$$x \in A,\ \varphi(x^*x) = 0 \text{ for all } \varphi \in F \Rightarrow x = 0.$$

Clearly, every separating family of positive forms is sufficient. A weight φ on A^+ is called *faithful* if $\{\varphi\}$ is separating, i.e. if

$$x \in A,\ \varphi(x^*x) = 0 \Rightarrow x = 0.$$

In particular, the above results hold for any positive form φ on A, in which case $N_\varphi = A$. If in addition A is unital, then from (5) and (6) with $b = 1$, we infer

$$\varphi(x^*) = \overline{\varphi(x)}; \quad x \in A, \tag{7}$$

$$|\varphi(x)|^2 \leqslant \varphi(1)\varphi(x^*x); \quad x \in A. \tag{8}$$

Moreover, put $\xi_\varphi = 1_\varphi \in H_\varphi$. Then ξ_φ is a cyclic vector for π_φ and from (4) with $a = b = 1$, we get

$$\varphi(x) = (\pi_\varphi(x)\xi_\varphi|\xi_\varphi)_\varphi; \quad x \in A, \tag{9}$$

that is $\varphi = \omega_{\xi_\varphi} \circ \pi_\varphi$.

Note that every U^*-algebra, in particular every Banach $*$-algebra, in particular ... every C^*-algebra, satisfies Combes's axiom (2.8). Therefore for each weight, in particular for each positive form, on such an algebra there exists the associated GNS representation.

If φ is a positive form on an involutive Banach algebra A with a left approximate unit, but not necessarily unital, then, as we shall see below (4.5), there still exists a cyclic vector $\xi_\varphi \in H_\varphi$ for π_φ such that (9) holds; also (7) holds and a relation similar to (8) is true (4.5).

Whenever it holds, the construction of the $*$-representation $\pi_\varphi : A \to B(H_\varphi)$ and of the cyclic vector $\xi_\varphi \in H_\varphi$ for π_φ is essentially unique, as the following proposition shows.

PROPOSITION. *Let A be a $*$-algebra and let π_1, π_2 be $*$-representations of A on Hilbert spaces H_1, H_2 with cyclic vectors ξ_1, ξ_2 respectively, such that*

$$(\pi_1(x)\xi_1|\xi_1) = (\pi_2(x)\xi_2|\xi_2); \quad x \in A.$$

Then there is a unique unitary operator $U : H_1 \to H_2$ such that

$$U\xi_1 = \xi_2 \quad \textit{and} \quad \pi_2(x) = U\pi_1(x)U^*; \quad x \in A.$$

Proof. For every $y \in A$ we have

$$\|\pi_2(y)\xi_2\|^2 = (\pi_2(y^*y)\xi_2|\xi_2) = (\pi_1(y^*y)\xi_1|\xi_1) = \|\pi_1(y)\xi_1\|^2.$$

Since ξ_1 (respectively ξ_2) is cyclic for π_1 (respectively π_2), it follows that the map $U_0 : \pi_1(y)\xi_1 \mapsto \pi_2(y)\xi_2$, $(y \in A)$, extends to a unitary operator $U : H_1 \to H_2$ and, for $x, y \in A$, we have

$$\pi_2(x)\pi_2(y)\xi_2 = \pi_2(xy)\xi_2 = U\pi_1(xy)\xi_1 = U\pi_1(x)U^*\pi_2(y)\xi_2,$$
$$(\xi_2|\pi_2(y)\xi_2) = (\xi_1|\pi_1(y)\xi_1) = (U\xi_1|U\pi_1(y)\xi_1) = (U\xi_1|\pi_2(y)\xi_2),$$

so that U has the required properties. The uniqueness of U is immediate. ∎

For instance, let $\pi : A \to B(H)$ be a cyclic $*$-representation of the unital $*$-algebra A with cyclic vector $\xi \in H$ and let $\varphi = \omega_\xi \circ \pi$. Then the GNS representation $\pi_\varphi : A \to B(H_\varphi)$ exists (condition (ii) is satisfied with $\lambda(x) = \|\pi(x)\|^2$), we can define $\xi_\varphi = 1_\varphi \in H_\varphi$ and, modulo a unitary equivalence, $\pi_\varphi = \pi$, $\xi_\varphi = \xi$.

4.4. We turn now to the case of Banach $*$-algebras.

Let A be a Banach $*$-algebra and φ be a weight on A^+. Consider the GNS representation $\pi_\varphi : A \to B(H_\varphi)$ associated to φ. Owing to 4.3.(4) and 1.13.(1), for $a, b \in N_\varphi$ and $x \in A$ we get

(1)
$$\begin{aligned}\varphi(b^*xa) &\leqslant \|\pi_\varphi(x)\|\,\|a_\varphi\|_\varphi\|b_\varphi\|_\varphi\|\pi^A_{\mathrm{env}}(x)\|\,\varphi(a^*a)^{1/2}\varphi(b^*b)^{1/2}\\ &\leqslant r_A(x^*x)^{1/2}\varphi(a^*a)^{1/2}\varphi(b^*b)^{1/2}.\end{aligned}$$

Thus, for every $a, b \in N_\varphi$ the linear functional $\varphi(b^* \cdot a)$ on A is bounded and

(2)
$$\|\varphi(b^* \cdot a)\| \leqslant \|\pi^A_{\mathrm{env}}\|\varphi(a^*a)^{1/2}\varphi(b^*b)^{1/2}; \quad a, b \in N_\varphi.$$

Suppose moreover that the Banach $*$-algebra A has bounded left approximate units and also bounded right approximate units. Then every positive form φ on A is bounded.

Indeed, if $\{x_n\}$ is a sequence in A, $x_n \to 0$, then, by the remarks after Corollary 2/3.1, there are $a, b \in A$ and a sequence $\{y_n\}$ in A, $y_n \to 0$, such that $x_n = b^*y_na$ for all n, so $\varphi(x_n) = \varphi(b^*y_na) \to 0$ by the continuity of $\varphi(b^* \cdot a)$.

Now let A be an involutive Banach algebra and φ be a weight on A^+. Then $\|\pi^A_{\mathrm{env}}\| \leqslant 1$, so by (1)

(3)
$$\|\varphi(b^* \cdot a)\| \leqslant \varphi(a^*a)^{1/2}\varphi(b^*b)^{1/2}; \quad a, b \in N_\varphi.$$

In particular, if A is unital and φ is a positive form on A, then $\|\varphi\| = \varphi(1)$.

If φ is a continuous form the involutive Banach algebra A, then φ^* is also continuous and

(4)
$$\|\varphi^*\| = \|\varphi\|.$$

It follows that a form φ on A is continuous if and only if $\mathrm{Re}\,\varphi$ and $\mathrm{Im}\,\varphi$ are both continuous.

If φ is a continuous selfadjoint form on A, then $\varphi|A_h$ is a continuous real functional on A_h and

(5)
$$\|\varphi\| = \|\varphi|A_h\|.$$

Indeed, given $\varepsilon > 0$ there is $x \in A$, $\|x\| \leqslant 1$, with $\varphi(x) \geqslant \|\varphi\| - \varepsilon$. Then $\mathrm{Re}\,x = (x + x^*)/2 \in A_h$, $\|\mathrm{Re}\,x\| \leqslant 1$, so that

$$\|\varphi|A_h\| \geqslant |\varphi(\mathrm{Re}\,x)| = \varphi(x) \geqslant \|\varphi\| - \varepsilon$$

and the assertion follows.

4.5. We now consider the case of an involutive Banach algebra A with a left approximate unit $\{u_\iota\}_{\iota\in I}$. Then $\{u_\iota^*\}_{\iota\in I}$ is a right approximate unit for A.

A weight φ on A^+ is called *lower norm semicontinuous* if for each $\lambda \geqslant 0$ then convex set $\{x \in A^+;\ \varphi(x) \leqslant \lambda\}$ is norm closed. If φ is lower norm semicontinuous and $\{x_\iota\}$ is a net in A^+, $\sigma(A, A^*)$-convergent to $x \in A^+$, then

$$\varphi(x) \leqslant \liminf_\iota \varphi(x_\iota).$$

PROPOSITION. *Let φ be a lower norm semicontinuous weight on A. Then the* GNS *representation associated to φ is non-degenerated.*

Proof. Let $a \in N_\varphi$. Using 4.3.(4) we get

$$\begin{aligned}\|\pi_\varphi(u_\iota^* u_\iota)a_\varphi - a_\varphi\|_\varphi^2 &= \varphi((a^* u_\iota^* u_\iota - a^*)(u_\iota^* u_\iota a - a))\\ &= \varphi(a^*(u_\iota^* u_\iota)^2 a) - 2\varphi(a^* u_\iota^* u_\iota a) + \varphi(a^* a)\\ &\leqslant 2\varphi(a^* a - a^* u_\iota^* u_\iota a).\end{aligned}$$

Since $a^* u_\iota^* u_\iota a \to a^* a$, we have $\varphi(a^* a) \leqslant \liminf_\iota \varphi(a^* u_\iota^* u_\iota a)$, hence

$$0 \leqslant \liminf_\iota \|\pi_\varphi(u_\iota^* u_\iota)a_\varphi - a_\varphi\|_\varphi^2 \leqslant \limsup_\iota \|\pi_\varphi(u_\iota^* u_\iota)a_\varphi - a_\varphi\|_\varphi^2 \leqslant 0,$$

that is $\{\pi_\varphi(u_\iota^* u_\iota)a_\varphi\}$ converge to a_φ in H_φ. ∎

In the case of positive forms, the associated GNS representation has more particularities:

THEOREM. *Let φ be a positive form on A. Then:*

(i) *φ is bounded and $\|\varphi\| = \lim_\iota \varphi(u_\iota) = \lim_\iota \varphi(u_\iota^* u_\iota)$.*

(ii) *For every $x \in A$ we have*

$$\varphi(x^*) = \overline{\varphi(x)}, \tag{1}$$

$$|\varphi(x)|^2 \leqslant \|\varphi\|\varphi(x^* x). \tag{2}$$

(iii) *There is a unique extension of φ to a positive form $\widetilde{\varphi}$ on $\widetilde{A}$ such that $\widetilde{\varphi}(1) = \|\varphi\|$.*

(iv) *Let $\pi_\varphi : A \to B(H_\varphi)$ be the associated* GNS *representation. There exists $\xi_\varphi \in H_\varphi$, $\|\xi_\varphi\|_\varphi = \|\varphi\|^{1/2}$, such that for every net $\{x_\kappa\}_{\kappa\in K}$ in A with $\|x_\kappa\| \leqslant 1$, $(\kappa \in K)$, and $\varphi(x_\kappa) \to \|\varphi\|$, we have*

$$(x_\kappa)_\varphi \to \xi_\varphi \ \textit{in the norm of } H_\varphi. \tag{3}$$

Moreover,

$$\pi_\varphi(x)\xi_\varphi = x_\varphi; \quad x \in A, \tag{4}$$

so ξ_φ is a cyclic vector for π_φ and

$$\varphi(x) = (\pi_\varphi(x)\xi_\varphi|\xi_\varphi); \quad x \in A. \tag{5}$$

Proof. The boundedness of φ was proved in 4.4 in a more general situation. Using the continuity of φ we obtain (1) and (2):

$$\varphi(x^*) = \lim_\iota \varphi(u_\iota x^*) = \lim_\iota \overline{\varphi(xu_\iota^*)} = \overline{\varphi(x)},$$
$$|\varphi(x)|^2 = \lim_\iota \varphi(u_\iota x)|^2 \leqslant \varphi(x^*x) \sup_\iota \varphi(u_\iota u_\iota^*) \leqslant \|\varphi\|\varphi(x^*x).$$

If $A = \widetilde{A}$, then $\|\varphi\| = \varphi(1)$ by (2). Assume $A \neq \widetilde{A}$ and for each $x + \lambda \in \widetilde{A}$ define $\widetilde{\varphi}(x + \lambda) = \varphi(x) + \lambda\|\varphi\|$. Then $\widetilde{\varphi}$ is a form on $\widetilde{A}$ extending φ, $\widetilde{\varphi}(1) = \|\varphi\|$ by definition and $\widetilde{\varphi}$ is positive because, by (1), (2),

$$\begin{aligned}
\widetilde{\varphi}((x+\lambda)^*(x+\lambda)) &= \varphi(x^*x + \overline{\lambda}x + \lambda x^*) + |\lambda|^2\|\varphi\| \\
&= \varphi(x^*x) + 2\mathrm{Re}(\overline{\lambda}\varphi(x)) + |\lambda|^2\|\varphi\| \\
&\geqslant \varphi(x^*x) - 2|\lambda|\, \|\varphi\|^{1/2}\varphi(x^*x)^{1/2} + |\lambda|^2\|\varphi\| \\
&= (\varphi(x^*x)^{1/2} - |\lambda|\, \|\varphi\|^{1/2})^2 \geqslant 0.
\end{aligned}$$

The pre-Hilbert structure of $\widetilde{A}$ associated to $\widetilde{\varphi}$ induces on A its own pre-Hilbert structure, associated to φ. Let $\{x_\kappa\}$ be any net in A with $\|x_\kappa\| \leqslant 1$ for all κ and $\varphi(x_\kappa) \to \|\varphi\|$. Using (2) we get also $\varphi(x_\kappa^* x_\kappa) \to \|\varphi\|$, hence

$$\widetilde{\varphi}((x_\kappa - 1)^*(x_\kappa - 1)) = \varphi(x_\kappa^* x_\kappa) - \varphi(x_\kappa^*) - \varphi(x_\kappa) + \|\varphi\| \to 0.$$

It follows that A is dense in the pre-Hilbert space $\widetilde{A}$. So we may identify H_φ and $H_{\widetilde{\varphi}}$. Then $\xi_\varphi = 1_{\widetilde{\varphi}} \in H_\varphi$, $\|\xi_\varphi\|_\varphi = \|\varphi\|^{1/2}$ and for every $\{x_\kappa\}$ as above, (3) holds. Moreover, for each $x \in A$ we have

$$x_\varphi = x_{\widetilde{\varphi}} = \pi_{\widetilde{\varphi}}(x)1_{\widetilde{\varphi}} = \lim_\kappa (xx_\kappa)_{\widetilde{\varphi}} = \lim_\kappa (xx_\kappa)_\varphi = \pi_\varphi(x)\xi_\varphi,$$

that is (4) holds. Also (5) holds:

$$\varphi(x) = \widetilde{\varphi}(x) = (\pi_{\widetilde{\varphi}}(x)1_{\widetilde{\varphi}}|1_{\widetilde{\varphi}})_{\widetilde{\varphi}} = (\pi_\varphi(x)\xi_\varphi|\xi_\varphi)_\varphi.$$

Since π_φ is non-degenerated, by Lemma 3/4.1 we have

$$\pi_\varphi(u_\iota)\xi_\varphi \to \xi_\varphi,$$

so $\varphi(u_\iota) = (\pi_\varphi(u_\iota)\xi_\varphi|\xi_\varphi)_\varphi \to (\xi_\varphi|\xi_\varphi)_\varphi = \|\varphi\|$ and using (2) we get also $\varphi(u_\iota^* u_\iota) \to \|\varphi\|$. ∎

Note that

$$(u_\iota)_\varphi \to \xi_\varphi, \quad (u_\iota^*)_\varphi \to \xi_\varphi, \quad (u_\iota^* u_\iota)_\varphi \to \xi_\varphi, \quad (u_\iota u_\iota^*)_\varphi \to \xi_\varphi. \tag{6}$$

Also, by (2), for every positive form φ on A we have

$$\|\varphi\| = \sup\{\varphi(x);\, x \in A^+,\, \|x\| \leqslant 1\}. \tag{7}$$

The positive form $\widetilde{\varphi}$ is called the *canonical extension* of φ to $\widetilde{A}$. Note that $\|\widetilde{\varphi}\| = \widetilde{\varphi}(1) = \|\varphi\|$. Let ψ be any positive form on $\widetilde{A}$ with $\psi|A = \varphi$. Then $\psi(1) = \|\psi\| \geqslant \|\varphi\|$ and it follows that $\psi \geqslant \varphi$.

Sometimes it is necessary to consider also the involutive Banach algebra with adjoined unit $A \oplus C$, even if A is unital (1.5). So we remark that given a positive form φ on A, the formula

$$\psi(x + \lambda) = \varphi(x) + \lambda\|\varphi\|; \quad x \in A,\, \lambda \in \mathbb{C},$$

still defines a positive form on $A \oplus C$, the proof being the same as above and clearly, $\|\psi\| = \|\varphi\|$.

Corollary 1. *Let φ, ψ be positive form on A. Then*

$$\|\varphi + \psi\| = \|\varphi\| + \|\psi\| \quad \textit{and} \quad (\widetilde{\varphi + \psi}) = \widetilde{\varphi} + \widetilde{\psi}.$$

Proof. Use (i) and (iii) of the theorem. ∎

As an application, let φ be a positive form on A, let ψ_1 be a positive form on $\widetilde{A}$ with $\psi_1 \leqslant \widetilde{\varphi}$ and $\varphi_1 = \psi_1|A$. Then $\psi_1 = \widetilde{\varphi}_1$. Indeed, $\widetilde{\varphi} = \psi_1 + \psi_2$ for some positive form ψ_2 on $\widetilde{A}$ with restriction $\varphi_2 = \psi_2|A$. We have $\varphi = \varphi_1 + \varphi_2$, $\psi_1(1) \geqslant \|\varphi_1\|$, $\psi_2(1) \geqslant \|\varphi_2\|$ and, by Corollary 1,

$$\widetilde{\varphi}(1) = \psi_1(1) + \psi_2(1) \geqslant \|\varphi_1\| + \|\varphi_2\| = \|\varphi_1 + \varphi_2\| = \|\varphi\| = \widetilde{\varphi}(1),$$

which forces the equality $\psi_1(1) = \|\varphi_1\|$, i.e. $\psi_1 = \widetilde{\varphi}_1$.

Corollary 2. *Let $\pi : A \to B(H)$ be a non-degenerated $*$-representation, $\xi \in H$ and $\varphi = \omega_\xi \circ \pi$. Then $\|\varphi\| = \|\xi\|^2$ and $\widetilde{\varphi} = \omega_\xi \circ \widetilde{\pi}$.*

Proof. By (i) of the theorem and by Lemma 3/4.1 we have

$$\|\varphi\| = \lim_\iota \varphi(u_\iota) = \lim_\iota (\pi(u_\iota)\xi|\xi) = (\xi|\xi) = \|\xi\|^2.$$

Then $(\omega_\xi \circ \widetilde{\pi})|A = \varphi$ and $(\omega_\xi \circ \widetilde{\pi})(1) = \|\xi\|^2 = \|\varphi\|$, hence $\omega_\xi \circ \widetilde{\pi} = \widetilde{\varphi}$. ∎

COROLLARY 3. *Let $\pi : A \to B(H)$ be a $*$-representation. Then*

$$\|\pi(x)\| = \sup \varphi(x^*x)^{1/2}, \quad x \in A,$$

where φ runs over all positive forms associated to π with $\|\varphi\| = 1$.

Proof. By Lemma 1/4.1 we may assume π non-degenerated. If $\varphi = \omega_\xi \circ \pi$ then $\|\varphi\| \leqslant 1 \Leftrightarrow \|\xi\| \leqslant 1$ by Corollary 2 above. The result follows since

$$\|\pi(x)\| = \sup_{\|\xi\| \leqslant 1} \|\pi(x)\xi\| = \sup_{\|\xi\| \leqslant 1} (\omega_\xi \circ \pi)(x^*x)^{1/2}, \quad x \in A. \quad ∎$$

Since any C^*-algebra has an approximate unit (Theorem 3.2) all the above results hold for C^*-algebras. In particular,

COROLLARY 4. *Every positive form on a C^*-algebra is bounded.*

4.6. PROPOSITION. *Let φ be a bounded form on a unital C^*-algebra A. Then*

$$\varphi \text{ is positive} \Leftrightarrow \|\varphi\| = \varphi(1).$$

Proof. If φ is positive, then $\|\varphi\| = \varphi(1)$ by Theorem 4.5.

Conversely, assume that $\varphi(1) = \|\varphi\| = 1$ and $\varphi(x) \notin [0, +\infty)$ for some $x \in A^+$. There is a closed ball $D = \{\lambda \in \mathbb{C};\ |\lambda - \lambda_0| \leqslant r\}$ with $\sigma(x) \subset D$ and $\varphi(x) \notin D$. The element $x - \lambda_0$ is normal and its spectrum is contained in $\{\lambda \in \mathbb{C};\ |\lambda| \leqslant r\}$, so that $\|x - \lambda_0\| = r(x - \lambda_0) \leqslant r$. It follows that

$$|\varphi(a) - \lambda_0| = |\varphi(a - \lambda_0)| \leqslant \|a - \lambda_0\| \leqslant r,$$

a contradiction. ∎

4.7. Let A be an involutive Banach algebra with a left approximate unit $\{u_\iota\}_{\iota \in I}$. The $\sigma(A^*, A)$-topology on A will be abbreviated as A-topology. The convex hull of a subset X of A is denoted by $\operatorname{co} X$ and its A-closure by $\overline{\operatorname{co}}^A X$. If X is convex, $\operatorname{ex} X$ denotes the set of extreme points of X.

Let $Q(A)$ be the set of all positive forms φ on A with $\|\varphi\| \leqslant 1$. Then $Q(A)$ is an A-closed convex subset of the unit ball of A^* so, by Alaoglu's theorem, $Q(A)$ is an A-compact convex subset of A^*.

A positive form φ on A is called a *state* if $\|\varphi\| = 1$. The set of all states is denoted by $S(A)$. Clearly, $S(A) \subset Q(A)$. Using Corollary 1/4.5 we see that $S(A)$ is convex, but $S(A)$ need not be A-closed. However, if A is unital, then for $\varphi \in A^*$, $\|\varphi\| \leqslant 1$, we have $\varphi \in S(A) \Leftrightarrow \varphi(1) = 1$, so that in this case $S(A)$ is also A-compact.

By the Krein-Milman theorem, the set $\operatorname{ex} Q(A)$ of extreme points of $Q(A)$ is non-void and $Q(A)$ is the A-closed convex hull of $\operatorname{ex} Q(A)$.

PROPOSITION. *Let $\varphi \in Q(A)$. Then $\varphi \in \operatorname{ex} Q(A)$ if and only if either $\varphi = 0$, or $\varphi \in S(A)$ and any positive form ψ on A, majorized by φ is of the form $\psi = \lambda\varphi$ for some $0 \leqslant \lambda \leqslant 1$.*

Proof. If $\varphi = 0$, then $\varphi \in \operatorname{ex} Q(A)$ since $\psi \in Q(A)$ and $\psi \leqslant \varphi$ entail $\psi(x^*x) = 0$ for all $x \in A$, so $|\psi(x)| \leqslant \|\psi\|\psi(x^*x) = 0$, hence $\psi = 0$.

If φ satisfies the other condition of the statement and $\varphi = \alpha\varphi_1 + (1-\alpha)\varphi_2$ with $\varphi_1, \varphi_2 \in Q(A)$, $0 < \alpha < 1$, then $\alpha\varphi_1 \leqslant \varphi$ so that $\alpha\varphi_1 = \lambda\varphi$ for some $0 \leqslant \lambda \leqslant 1$. By Corollary 1/4.5 we have $\|\varphi\| = \alpha\|\varphi_1\| + (1-\alpha)\|\varphi_2\|$. Since $\|\varphi\| = 1$ and $\|\varphi_1\| \leqslant 1$, $\|\varphi_2\| \leqslant 1$ we get $\|\varphi_1\| = 1$. It follows that $\alpha = \lambda$, $\varphi_1 = \varphi$, $\varphi_2 = \varphi$. Therefore $\varphi \in \operatorname{ex} Q(A)$.

Conversely, if $\varphi \in \operatorname{ex} Q(A), \varphi \neq 0$, then clearly $\|\varphi\| = 1$, i.e. $\varphi \in S(A)$. Let ψ be a positive form on A, $\psi \leqslant \varphi$. Then $\varphi = \lambda\psi_1 + (1-\lambda)\psi_2$ where $\lambda = \|\psi\|$ and $\psi_1 = \lambda^{-1}\psi \in Q(A)$, $\psi_2 = (1-\lambda)^{-1}(\varphi - \psi) \in Q(A)$. It follows that $\psi_1 = \varphi$, $\psi = \lambda\varphi$. ∎

The non-zero extreme points of $Q(A)$ are called *pure states* of A and the set of all pure states is denoted by $P(A)$. By extension, a non-zero positive form φ on A is called *pure* if $\varphi/\|\varphi\|$ is a pure state. Then φ is pure if and only if any positive form majorized by a multiple of φ is a multiple of φ.

Let φ be a positive form on A and let $\widetilde{\varphi}$ be its canonical extension to a positive form on $\widetilde{A}$. Owing to the remark made after Corollary 1/4.5, it is easy to see that

$$\varphi \text{ is pure} \Leftrightarrow \widetilde{\varphi} \text{ is pure}.$$

Moreover, let $\psi \in P(\widetilde{A})$ with $\varphi = \psi|A \neq 0$. Then $\widetilde{\varphi} \leqslant \psi$ hence $\widetilde{\varphi} = \lambda\psi$ for some $0 \leqslant \lambda \leqslant 1$ and necessarily $\lambda = 1$ since $\widetilde{\varphi}|A = \psi|A \neq 0$. If $A \neq \widetilde{A}$, there is just one pure state ψ_0 on $\widetilde{A}$ with $\psi_0|A = 0$ and this is defined by $\psi_0(x+\lambda) = \lambda$, $(x \in A$, $\lambda \in \mathbb{C})$. Thus

$$P(\widetilde{A}) = \{\widetilde{\varphi};\, \varphi \in P(A)\} \cup \{\psi_0\}. \tag{1}$$

By the Krein-Milman theorem,

$$Q(A) = \overline{\mathrm{co}}^A(P(A) \cup \{0\}). \tag{2}$$

$S(A)$ is a convex set and clearly $\operatorname{ex} S(A) = S(A) \cap \operatorname{ex} Q(A) = P(A)$. A routine approximation argument shows that

$$S(A) \subset \overline{\mathrm{co}}^A(P(A). \tag{3}$$

If A is unital, then $S(A)$ is also A-compact so, again by the Krein-Milman theorem,

$$S(A) = \overline{\mathrm{co}}^A P(A) \text{ if } A \text{ is unital.} \tag{4}$$

Note that the equality (4) is definitely false for non-unital C^*-algebras, namely the zero form is then A-adherent to $P(A)$ (see Proposition 4.17, below); this fact will sharpen (2) and will provide a new proof for (3) in the C^*-algebra case.

COROLLARY. *Let B be a C^*-subalgebra of the C^*-algebra A and*

$$C = \{x \in A;\ xy = yx \text{ for all } y \in B\}.$$

If φ is a state of A and if $\varphi|B$ is a pure positive form on B, then

$$\varphi(yz) = \varphi(y)\varphi(z); \quad y \in B,\ z \in C. \tag{5}$$

Proof. Clearly, $C = \{x \in A;\ xy = yx,\ (\forall)\, y \in B\}$ is a C^*-subalgebra of A, so C is linearly spanned by its positive part. Thus, in proving (5), we may assume $z \in C^+$, $\|z\| \leqslant 1$. Furthermore we may assume $\varphi(z) > 0$ because, by the Schwarz inequality, $\varphi(z) = 0$ entails $\varphi(yz) = 0$ for all $y \in A$. Similarly, we may assume $\varphi(z) \neq 1$. Put $\lambda_1 = \varphi(z)$, $\lambda_2 = 1 - \varphi(z)$.

Now $\varphi|B \in \operatorname{ex} Q(B)$, the mappings

$$\psi_1 : B \ni y \mapsto \lambda_1^{-1}\varphi(yz) = \lambda_1^{-1}\varphi(z^{1/2}yz^{1/2})$$

$$\psi_2 : B \ni y \mapsto \lambda_2^{-1}\varphi(y(1-z)) = \lambda_2^{-1}\varphi((1-z)^{1/2}y(1-z)^{1/2})$$

are states of B and

$$\varphi|B = \lambda_1\psi_1 + \lambda_2\psi_2,$$

hence $\varphi|B = \psi_1$, that is $\varphi(y) = \varphi(z)^{-1}\varphi(yz)$ for all $y \in B$. ∎

4.8. The majorization of positive forms can be expressed in terms of the corresponding GNS representations. This will provide an important caracterization of pure states.

PROPOSITION. *Let A be a $*$-algebra satisfying the Combes axiom and let φ, ψ be two weights on A^+ such that $\psi \leqslant \varphi$, i.e.*

$$\psi(x) \leqslant \varphi(x); \quad x \in A^+.$$

There exists a unique $T' \in \pi_\varphi(A)'$, $0 \leqslant T' \leqslant 1$, such that

$$\psi(b^*a) = (T'a_\varphi | T'b_\varphi)_\varphi; \quad a, b \in N_\varphi. \tag{1}$$

Proof. Since $\psi \leqslant \varphi$, we have $N_\varphi \subset N_\psi$, $L_\varphi \subset L_\psi$, so that we can define a linear mapping

$$S'_0 : N_\varphi/L_\varphi \ni a_\varphi \mapsto a_\psi \in N_\psi/L_\psi; \quad a \in N_\varphi,$$

and since

$$\|S'_0 a_\varphi\|_\psi^2 = \|a_\psi\|_\psi^2 = \psi(a^*a) \leqslant \varphi(a^*a) = \|a_\varphi\|_\varphi^2; \quad a_\varphi \in N_\varphi,$$

S'_0 extends to a bounded linear operator $S' : H_\varphi \to H_\psi$, $\|S'\| \leqslant 1$. We have $S'^*S' \in B(H_\varphi)$, $0 \leqslant S'^*S' \leqslant 1$ and also $T' = (S'^*S')^{1/2} \in B(H_\varphi)$, $0 \leqslant T' \leqslant 1$. For any $a, b \in N_\varphi$ we get

$$\psi(b^*a) = (a_\psi|b_\psi)_\psi = (S'a_\varphi|S'b_\varphi)_\psi = (S'^*S'a_\varphi|b_\varphi)_\varphi = (T'a_\varphi|T'b_\varphi)_\varphi.$$

Then, for any $a, b \in N_\varphi$ and any $x \in A$, we obtain

$$(S'^*S'\pi_\varphi(x)a_\varphi|b_\varphi)_\varphi = \psi(b^*xa) = \psi((x^*b)^*a) = (\pi_\varphi(x)S'^*S'a_\varphi|b_\varphi)_\varphi.$$

This shows that $S'^*S' \in \pi_\varphi(A)'$. Futhermore, $T' \in \pi_\varphi(A)'$ since $\pi_\varphi(A)'$ is a C^*-subalgebra of $B(H_\varphi)$.

From (1) it follows that the numbers $(T'^2a_\varphi|b_\varphi)_\varphi$, $(a, b \in N_\varphi)$, are uniquely determined by ψ, hence T'^2 is determined by ψ and T' is its unique positive square root. ∎

COROLLARY 1. *Let A be an involutive Banach algebra with a left approximate unit $\{u_\iota\}_{\iota\in I}$ and let φ, ψ be positive forms on A with $\psi \leqslant \varphi$. There exists a unique $T' \in \pi_\varphi(A)'$, $0 \leqslant T' \leqslant 1$, such that*

$$\psi(x) = (\pi_\varphi(x)T'\xi_\varphi | T'\xi_\varphi); \quad x \in A. \tag{2}$$

Proof. By the proposition, there is a unique $T' \in \pi_\varphi(A)'$, $0 \leqslant T' \leqslant 1$, such that (1) holds. Then

$$\psi(u_\iota x) = (T'x_\varphi | T'(u_\iota^*)_\varphi)_\varphi; \quad x \in A. \tag{3}$$

By Theorem 4.5 and by 4.5.(6), we have $x_\varphi = \pi_\varphi(x)\xi_\varphi$ and $(u_\iota^*)_\varphi \to \xi_\varphi$ in H_φ, so that (2) follows from (3) taking the limit over $\iota \in I$.

Equation (2) entails equation (1), so that the uniqueness follows by the proposition. ∎

Equation (2) can be rewritten as

$$\psi(x) = (\pi_\varphi(x)X'\xi_\varphi | \xi_\varphi)_\varphi; \quad x \in A, \tag{4}$$

with $X' = T'^2 \in \pi_\varphi(A)'$, $0 \leqslant X' \leqslant 1$, Conversely, given $X' \in \pi_\varphi(A)'$, $0 \leqslant X' \leqslant 1$, (4) defines a positive form ψ on A, $\psi \leqslant \varphi$, which we denote by $\psi = \varphi_{X'}$. Clearly, the assignement $X' \to \varphi_{X'}$ establishes an affine bijection between $\{X' \in \pi_\varphi(A)'; 0 \leqslant X' \leqslant 1\}$ and $\{\psi \in A^*; 0 \leqslant \psi \leqslant \varphi\}$.

We now specialize to the case of C^*-algebras in order to obtain similar results for positive forms majorized by a weight. The restriction is necessary only because we need approximate units for left ideals.

Let φ be a weight on the C^*-algebra A. If f is a positive form on A, $f \leqslant \varphi$, from (1) we then obtain

$$\|T'a_\varphi\|_\varphi \leqslant \|f\|^{1/2}\|a\|; \quad a \in N_\varphi.$$

Let $\mathcal{T}'_\varphi$ be the set all $T' \in \pi_\varphi(A)'$ such that there is a positive real number $\lambda_{T'}$ with

$$\|T'a_\varphi\|_\varphi \leqslant \lambda_{T'}\|a\|; \quad a \in N_\varphi.$$

Clearly, $\mathcal{T}'_\varphi$ is a left ideal of the C^*-algebra $\pi_\varphi(A)'$.

Let $\{u_\iota\}_{\iota\in I}$ be a right approximate unit for the left ideal N_φ of A. For $T' \in \mathcal{T}'_\varphi$ and $a \in N_\varphi$ it follows that

$$\pi_\varphi(a)T'(u_\iota)_\varphi = T'(au_\iota)_\varphi \to T'a_\varphi.$$

Thus, if $a_k \in N_\varphi$, $\xi_k \in H_\varphi$, $(1 \leqslant k \leqslant n)$, and $\zeta = \sum\limits_{k=1}^n \pi_\varphi(a_k^*)\xi_k \in \pi_\varphi(N_\varphi^*)H_\varphi$, then

$$\Big|\sum_{k=1}^n (\xi_k | T'(a_k)_\varphi)_\varphi\Big| = \lim_\iota |(\zeta | T'(u_\iota)_\varphi)_\varphi| \leqslant \lambda_{T'}\|\zeta\|_\varphi.$$

Hence the map $\zeta \mapsto \sum\limits_{k=1}^n (\xi_k | T'(a_k)_\varphi)_\varphi$ defines a bounded linear functional on $\pi_\varphi(N_\varphi^*)H_\varphi$ and therefore there is a unique vector $\eta \in \overline{\pi_\varphi(N_\varphi^*)H_\varphi}$ such that

$$(\xi | T'a_\varphi)_\varphi = (\xi | \pi_\varphi(a)\eta)_\varphi; \quad a \in N_\varphi,\ \xi \in H_\varphi,$$

i.e. $T'a_\varphi = \pi_\varphi(a)\eta$ for all $a \in N_\varphi$. In particular, putting $f = \omega_\eta \circ \pi_\varphi$, we obtain a positive form on A such that $f(b^*a) = (T'a_\varphi | T'b_\varphi)_\varphi$ for all $a, b \in N_\varphi$. We have proved

COROLLARY 2. *Let φ be a weight on the C^*-algebra A.*

(i) *For every positive form f on A, $f \leqslant \varphi$, there is a unique $T' \in \mathcal{T}'_\varphi$, $0 \leqslant T' \leqslant 1$, and a unique vector $\eta \in \overline{\pi_\varphi(N_\varphi^*)H_\varphi}$ such that*

$$f(b^*a) = (T'a_\varphi | T'b_\varphi)_\varphi; \quad a, b \in N_\varphi,$$
$$f(x) = (\omega_\eta \circ \pi_\varphi)(x); \quad x \in M_\varphi.$$

(ii) *For every $T' \in \mathcal{T}'_\varphi$, $0 \leqslant T' \leqslant 1$, there is a unique positive form $f \leqslant \varphi$ on A and a unique vector $\eta \in \overline{\pi_\varphi(N_\varphi^*)H_\varphi}$ such that*

$$f(b^*a) = (T'a_\varphi | T'b_\varphi)_\varphi; \quad a, b \in N_\varphi,$$
$$T'a_\varphi = \pi_\varphi(a)\eta; \quad a \in N_\varphi. \quad ∎$$

4.9. As announced, the next result characterizes the pure positive forms in terms of their associated GNS representations.

PROPOSITION. *Let A be an involutive Banach algebra with a left approximate unit and let φ be a positive form on A. Then φ is pure if and only if π_φ is a non-zero topologically irreducible $*$-representation.*

Proof. Assume that φ is pure and consider $X' \in \pi_\varphi(A)'$, $0 \leqslant X' \leqslant 1$. Then equation 4.8.(4) defines a positive form $\psi = \varphi_{X'} \leqslant \varphi$. The purity of φ entails $\psi = \lambda\varphi$ for some $0 \leqslant \lambda \leqslant 1$, i.e. $\varphi_{X'} = \varphi_{\lambda 1}$, so $X' = \lambda 1$ by (4.8). Since $\pi_\varphi(A)'$ is a C^*-algebra, it follows that $\pi_\varphi(A)'$ consists of scalar operators only and hence π_φ is topologically irreducible. Since $\varphi \neq 0$, we have $(\pi_\varphi(x)\xi_\varphi | \xi_\varphi) = \varphi(x) \neq 0$ for some $x \in A$, hence π_φ is also non-zero.

Conversely, assume that π_φ is non-zero and topologically irreducible. Then clearly $\varphi \neq 0$. By 4.8, every positive form $\psi \leqslant \varphi$ is of the form $\psi = \varphi_{X'}$ with $X' \in \pi_\varphi(A)'$, $0 \leqslant X' \leqslant 1$. Since $\pi_\varphi(A)'$ reduces to scalar operators, it follows that ψ is a multiple of φ. Hence φ is pure. ∎

COROLLARY. *Given a topologically irreducible $*$-representation π of an involutive Banach algebra A with a left approximate unit on a Hilbert space H, every non-zero positive form on A associated to π is pure.*

Proof. Indeed, let $\xi \in H$, $\xi \neq 0$, and $\varphi = \omega_\xi \circ \pi$. Then ξ is cyclic for π and, by Proposition 4.3, we have $\pi_\varphi = \pi$, $\xi_\varphi = \xi$, modulo a unitary equivalence. Thus π_φ is topologically irreducible and φ is pure by the proposition. ∎

In particular, given a pure positive form φ on A and $a \in \widetilde{A}$, the positive form $\varphi(a^* \cdot a)$ is either zero or is also pure since $\varphi(a^* \cdot a) = \omega_{a_\varphi^\sim} \circ \pi_\varphi$.

Let A be a commutative C^*-algebra and Ω be its Gelfand spectrum. Then $P(A) = \Omega$. Indeed, every character $t \in \Omega$ *is* a non-zero one-dimensional, a fortiori topologically irreducible $*$-representation of A hence $t \in P(A)$. Conversely, any non-zero topologically irreducible $*$-representation π of A is one-dimensional since

$\pi(A) \subset \pi(A)'$ and $\pi(A)'$ reduces to the scalars. By the above it follows that every $\varphi \in P(A)$ is multiplicative, hence a character (see also Corollary 4.7).

4.10. Some proofs in the theory of C^*-algebra are base on certain simple general results concerning positive linear functionals on ordered (topological) vector spaces, which we record in this section.

A convex cone X^+ in a vector space X will be always assumed pointed in 0. We write $x \leqslant y$ instead of $y - x \in X^+$ and a linear functional which is $\geqslant 0$ on X^+ will be called positive. Given a subspace Y of X, positivity in Y is understood with respect to the cone $Y^+ = Y \cap X^+$.

PROPOSITION 1. *Let X be a complete metrizable topological real vector space and let X^+ be a closed convex cone in X such that $X = X^+ - X^+$. Then any positive linear functional φ on X is continuous.*

Proof. Let $\mathbf{V}$ be a countable basis of circled neighborhoods of 0 for the topology τ of X. Define

$$V' = (X^+ \cap V) - (X^+ \cap V); \quad V \in \mathbf{V}.$$

Then it is easy to check that $\mathbf{V}' = \{V'; V \in \mathbf{V}\}$ is a countable basis of circled neighborhoods of 0 for a certain vector space topology τ' on X, which is finer than τ. If $\{x_n\}$ is a τ'-Cauchy sequence in X, then we can write $x_n = x_n^+ - x_n^-$ with $\{x_n^+\}, \{x_n^-\} \subset X^+$ both being τ-Cauchy sequence. Since X is τ-complete and X^+ is closed, $\{x_n^+\}$ and $\{x_n^-\}$ are both τ-convergent in X^+, which means that $\{x_n\}$ is τ'-convergent in X. Thus (X, τ), (X, τ') are both complete metrizable topological vector spaces and, since τ' is finer that τ, the open mapping theorem shows that $\tau' = \tau$.

Therefore, in order to prove the τ-continuity of φ, it is enough to show that the restriction of φ to X^+ is τ-continuous at 0. Let d be a translation invariant metric for (X, τ) and put $|x| = d(x, 0)$, $(x \in X)$. If $\varphi | X^+$ is not continuous at 0, then there is a sequence $\{x_\kappa\}$ in X^+ with $|x_\kappa| \leqslant 2^{-k}$ and $\varphi(x_\kappa) \geqslant 1$. By the completeness of X, the series $\sum\limits_{k=1}^{\infty} x_\kappa$ converges to an element $x \in X$. Since X^+ is closed,

$$x - \sum_{k=n}^{n} x_\kappa = \sum_{k=n+1}^{\infty} x_\kappa \in X^+$$

so that, by the positivity of φ, $\varphi(x) \geqslant \varphi\Big(\sum\limits_{k=1}^{n} x_\kappa\Big) \geqslant n$, which is a contradiction. ∎

If A is a C^*-algebra, then $X = A_h$ and $X^+ = A^+$ satisfy the assumptions of Proposition 1. We thus obtain another proof of Corollary 4/4.5.

PROPOSITION 2. *Let X be a real locally convex space, X^+ be a closed convex cone in X and $x \in X$, $x \notin X^+$. There is a continuous positive linear functional φ on X with $\varphi(x) < 0$.*

Proof. By the Hahn-Banach theorem we find a continuous linear functional φ on X and a real number α such that $\varphi(x) < \alpha$ and $\varphi(y) \geqslant \alpha$ for all $y \in X^+$. Then $\alpha \leqslant \varphi(0) = 0$, hence $\varphi(x) < 0$. If $\varphi(y) < 0$ for some $y \in X^+$, then $\varphi(\lambda y) < \alpha$ for sufficiently large $\lambda > 0$, which is impossible since $\lambda y \in X^+$. ∎

PROPOSITION 3. *Let X be a real vector space, Y be a subspace of X and X^+ be a convex cone in X such that*

$$Y + X^+ = X.$$

Then every positive linear functional ψ on Y extends to a positive linear functional on X.

Proof. The assumption means that for every $x \in X$ there is $y \in Y$ with $x \leqslant y$. Define

$$p(x) = \inf\{\psi(y);\, y \in Y,\, x \leqslant y\}; \quad x \in X.$$

Then p is a seminorm on X and $p(y) = \psi(y)$ for each $y \in Y$. By the Hahn-Banach theorem, there is a linear functional φ on X such that $\varphi|Y = \psi$ and $\varphi(x) \leqslant p(x)$ for all $x \in X$. If $x \in X^+$, then $p(-x) \leqslant \psi(0) = 0$, so that $\varphi(-x) \leqslant 0$ and $\varphi(x) \geqslant 0$. ∎

4.11. The Gelfand-Naĭmark-Segal theorem. This section contains the fundamental result on C^*-algebras.

THEOREM. *Let A be a C^*-algebra. There is a family $\{\pi_\iota\}_{\iota \in I}$ of topologically irreducible $*$-representation of A such that the direct sum representation $\bigoplus\limits_{\iota \in I} \pi_\iota$ is isometric.*

Proof. Let $x \in A$, $x \neq 0$. By Proposition 2/4.10 applied to $X = A_h$ and $X^+ = A^+$ we get a positive form φ_x on A with $\varphi_x(-x^*x) \neq 0$. Owing to 4.7.(2) we may assume that φ_x is pure, in which case the GNS representation π_{φ_x} is topologically irreducible (Proposition 4.9). Since

$$\|\pi_{\varphi_x}(x)\xi_{\varphi_x}\|^2_{\varphi_x} = \varphi_x(x^*x) \neq 0,$$

we have $\pi_{\varphi_x}(x) \neq 0$. Therefore the direct sum representation of the π_{φ_x}'s, ($x \in A$, $x \neq 0$), is injective and consequently isometric by Corollary 1.15. ∎

The key in the proof was of course the GNS construction together with Theorem 2.4 which clarified the structure of A^+.

As announced (1.3), it follows that *every C^*-algebra is isometrically $*$-isomorphic to a Gelfand-Naĭmark algebra.*

Therefore, in working with a C^*-algebra A we may assume that $A \subset B(H)$ for some Hilbert space H.

4.12. As a first application we obtain formulas giving the norm of an element in a C^*-algebra.

PROPOSITION. *Let A be a C^*-algebra. For every $x \in A$ we have*

$$(1)\qquad \begin{aligned}\|x\| &= \sup\{\|\pi(x)\|;\ \pi \in \mathrm{Rep}(A)\} = \sup\{\|\pi(x)\|;\ \pi \in \mathrm{Irr}(A)\} \\ &= \sup\{\varphi(x^*x)^{1/2};\ \varphi \in Q(A)\} = \sup\{\varphi(x^*x)^{1/2};\ \varphi \in P(A)\}.\end{aligned}$$

For every $x \in A$, $x^* = x$, we have

$$(2)\qquad \|x\| = \sup\{|\varphi(x)|;\ \varphi \in Q(A)\} = \sup\{|\varphi(x)|;\ \varphi \in P(A)\}.$$

Proof. Using Theorem 4.11, Corollary 3/4.5 and 4.7.(2), we obtain (1). To prove (2) we may assume $A \subset B(H)$ for some Hilbert space H. Then, by 2.5.(1),

$$\|x\| = \sup\{|(x\xi|\xi)|;\ \xi \in H,\ \|\xi\| = 1\} \leqslant \sup\{|\varphi(x)|;\ \varphi \in Q(A)\} \leqslant \|x\|.$$

This proves the first equality in (2) and the second follows using again 4.7.(2). ∎

COROLLARY 1. *A C^*-algebra A is commutative if and only if every topologically irreducible $*$-representation of A is one-dimensional.*

Proof. If the condition is satisfied, then $\pi(xy-yx) = 0$ for every topologically irreducible $*$-representation π of A and every $x, y \in A$. Therefore $\|xy - yx\| = 0$, $(x, y \in A)$, by (1). The converse was already proved (4.9). ∎

If A is an involutive Banach algebra with a left approximate unit then the proof of the proposition shows that the right sides of formula (1) are all equal to $\|x\|_*$, where $\|\cdot\|_*$ is the greatest C^*-seminorm on A (1.13). Let $C^*_{\mathrm{env}}(A)$ be the envelopping C^*-algebra of A and $\pi^A_{\mathrm{env}} : A \to C^*_{\mathrm{env}}(A)$ be the canonical $*$-homomorphism (1.13).

COROLLARY 2. *Let A be an involutive Banach algebra with a left approximate unit. The mapping*

$$S(C^*_{\mathrm{env}}(A)) \ni \psi \to \psi \circ \pi^A_{\mathrm{env}} \in S(A)$$

is a bijection, bicontinuous with respect to the corresponding weak topologies.

Proof. Let $\varphi \in S(A)$ and $x \in A$. Using 4.5.(2) and the above remark, we get

$$|\varphi(x)| \leqslant \varphi(x^*x)^{1/2} \leqslant \|x\|_*.$$

Therefore there is a continuous form ψ on $C^*_{\mathrm{env}}(A)$, $\|\psi\| \leqslant 1$, with $\varphi = \psi \circ \pi^A_{\mathrm{env}}$. Also, $1 = \|\varphi\| \leqslant \|\psi\|\,\|\pi^A_{\mathrm{env}}\| \leqslant \|\psi\|$, thus $\|\psi\| = 1$. For each $y \in C^*_{\mathrm{env}}(A)$ there is a sequence $\{x_n\}$ in A such that $\pi^A_{\mathrm{env}}(x_n) \to y$ and we have $\psi(y^*y) = \lim_n \varphi(x_n^*x_n) \geqslant 0$, hence ψ is positive. The bicontinuity assertion is clear. ∎

4.13. Theorem 4.11 allows to characterize the positivity and the selfadjointness in A by means of (pure) states.

Proposition. *Let A be a C^*-algebra and $x \in A$. Then:*

(i) $x \geqslant 0 \Leftrightarrow \varphi(x) \geqslant 0$ *for all* $\varphi \in P(A)$*;*

(ii) $x = 0 \Leftrightarrow \varphi(x) = 0$ *for all* $\varphi \in P(A)$*;*

(iii) $x^* = x \Leftrightarrow \varphi(x) \in \mathbb{R}$ *for all* $\varphi \in P(A)$.

Proof. We may assume $A \subset B(H)$ for some Hilbert space H. For any $\xi \in H$, ω_ξ is a positive form on A. If $\varphi(x) \geqslant 0$ for all $\varphi \in P(A)$, then $\varphi(x) \geqslant 0$ for all positive forms φ on A by 4.7.(2), in particular $(x\xi|\xi) = \omega_\xi(x) \geqslant 0$ for all $\xi \in H$, which means that $x \geqslant 0$ (2.5.(ii)).

This proves (i) and clearly (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). ∎

Corollary 1. *Let A be a C^*-algebra and $x \in A$. Then $x^* = x$ if and only if*

$$\lim_{0 \neq t \in \mathbb{R},\, t \to 0} t^{-1}(\|1 + \mathrm{i}tx\| - 1) = 0 \quad \text{in } \widetilde{A}. \tag{1}$$

Proof. By Gelfand representation, $\|1 + a\| = 1 + \|a\|$ for every $a \in A$, $a \geqslant 0$. If $x \in A$, $x^* = x$, and $t \in \mathbb{R}$ then $t^2x^2 \geqslant 0$ and

$$\|1 + \mathrm{i}tx\|^2 = \|(1 + \mathrm{i}tx)^*(1 + \mathrm{i}tx)\| = \|1 + t^2x^2\| = 1 + t^2\|x^2\|$$

hence

$$\|1 + \mathrm{i}tx\| = (1 + t^2\|x^2\|)^{1/2},$$

and (1) becomes obvious.

Conversely, assume that (1) holds. For every state φ on A and every $t \in \mathbb{R}$, $t > 0$, we have

$$-\varphi(\operatorname{Im} x) = t^{-1}(\operatorname{Re} \widetilde{\varphi}(1 + \mathrm{i}tx) - 1) \leqslant t^{-1}(\|1 + \mathrm{i}tx\| - 1).$$

Using (1) we get $\varphi(\operatorname{Im} x) \geqslant 0$. Arguing similarly with $t \in \mathbb{R}$, $t < 0$, we obtain $\varphi(\operatorname{Im} x) \leqslant 0$. Therefore $\varphi(\operatorname{Im} x) = 0$ for any state φ on A and the proposition shows that $\operatorname{Im} x = 0$, hence $x^* = x$. ∎

Corollary 2. *Let A,B be unital C^*-algebras and $\Phi : A \to B$ be a linear mapping such that $\Phi(1) = 1$ and $\|\Phi(x)\| = \|x\|$ for every normal element $x \in A$. Then Φ is a selfadjoint map, i.e.*

$$\Phi(x^*) = \Phi(x)^* \quad \text{for all } x \in A.$$

Proof. It is sufficient to show that $\Phi(A_h) \subset B_h$. Let $x \in A$, $x^* = x$ and $t \in \mathbb{R}$. Then $1 + \mathrm{i}tx$ is a normal element and, by the assumptions

$$\|1 + \mathrm{i}t\Phi(x)\| = \|\Phi(1 + \mathrm{i}tx)\| = \|1 + \mathrm{i}tx\|.$$

An application of Corollary 1 shows that $\Phi(x)^* = \Phi(x)$. ∎

Corollary 3. *Let $\pi : A \to B$ be an isometric algebra isomorphism of a C^*-algebra A onto a C^*-algebra B. Then π is a $*$-isomorphism.*

Proof. By assumption, A and B are simultaneously unital or not unital. If they are unital, then the result follows at once from Corollary 2. Suppose A and B are not unital and extend π to an algebra isomorphism $\widetilde{\pi} : \widetilde{A} \to \widetilde{B}$ by putting $\widetilde{\pi}(1) = 1$. Owing to the definition of the norm on the associate unital C^*-algebra (1.5), for $x \in A$, $\lambda \in \mathbb{C}$, we get

$$\begin{aligned}\|\widetilde{\pi}(x+\lambda)\| = \|\pi(x)+\lambda\| &= \sup\{\|(\pi(x)+\lambda)\pi(y)\|;\, y \in A,\, \|\pi(y)\| \leqslant 1\} \\ &= \sup\{\|\pi(xy+\lambda y)\|;\, y \in A,\, \|\pi(y)\| \leqslant 1\} \\ &= \sup\{\|xy+\lambda y\|;\, y \in A,\, \|y\| \leqslant 1\} = \|x+\lambda\|.\end{aligned}$$

Hence $\widetilde{\pi}$ is also isometric and the result follows. ∎

Corollary 4. *The $*$-operation in a C^*-algebra A is uniquely determined by the norm and the algebric structure of A.*

Proof. Apply Corollary 3 to the identity mapping on A. ∎

4.14. A result related to Proposition 4.13 is the following

Proposition. *Let A be a unital C^*-algebra and let F be a subset of $S(A)$ such that*

$$x \in A_h \text{ and } \varphi(x) \geqslant 0 \text{ for all } \varphi \in F \Rightarrow x \geqslant 0. \tag{1}$$

Then:

$$\overline{\mathrm{co}}^A F = S(A); \tag{2}$$

$$\overline{F}^A \supset P(A); \tag{3}$$

$$\|x\| = \sup\{\varphi(x^*x)^{1/2};\, \varphi \in F\} \quad \text{for all } x \in A. \tag{4}$$

Proof. For $\varphi \in S(A)$ and $x \in A$ we have $\varphi(x) \leqslant 1$ if and only if $\varphi(1-x) \geqslant 0$. Using the assumption we see that, for $x \in A_{h'}$

$$\varphi(x) \leqslant 1,\, (\forall)\varphi \in F \Leftrightarrow \varphi(x) \leqslant 1,\, (\forall)\varphi \in S(A).$$

Therefore F and $S(A)$ have the same polar set in A_h and (2) follows by the bipolar theorem.

Since $P(A) = \operatorname{ex} S(A)$, (3) follows from (2) by Milman's converse of the Krein-Milman theorem ([81], V.8.5).

Finally, (4) follows from (3) using Proposition 4.12. ∎

Corollary. *Let $\{\pi_\iota : A \to B(H_\iota)\}_{\iota \in I}$ be a family of $*$-representation of C^*-algebra A such that $\pi = \bigoplus_{\iota \in I} \pi_\iota$ is injective and let*

$$E = \{\omega_\xi \circ \pi_\iota;\ \iota \in I,\ \xi \in H_\iota\}.$$

(i) *Every $\varphi \in S(A)$ is an A-limit of states of the form $\varphi_1 + \cdots + \varphi_n$ with $\varphi_k \in E$, $(1 \leqslant k \leqslant n,\ n \in \mathbb{N})$.*

(ii) *Every $\varphi \in P(A)$ is an A-limit of states in E.*

Proof. By Lemma 1/4.1 we may suppose each π_ι non-degenerated. Then π is non-degenerated. Owing to Corollary 1 and Corollary 2/4.5 it is easy to see that, without restricting the generality, we may assume that the C^*-algebra A is unital.

If $x \in A_h$ and $\varphi(x) \geqslant 0$ for all $\varphi \in E$, then $\pi_\iota(x) \geqslant 0$ for $\iota \in I$, so that $\pi(x) \geqslant 0$, and $x \geqslant 0$. Therefore the proposition applies to $F = E \cap S(A)$ and yields the desired results. ∎

Note that the above corollary applies in particular to the identity representation of a Gelfand-Naĭmark algebra.

4.15. Kadison's function representation. Let A be a C^*-algebra. Denote by A_h^* the real Banach space of all self-adjoint forms on A^*. By the last remark in 4.4, A_h^* can be identified to the dual space of the real Banach space A_h. Also, denote by A_+^* the closed convex cone of all positive forms on A. Recall that $Q(A) \subset A_+^* \subset A_h^* \subset A^*$, $Q(A)$ is an A-compact convex set and $\operatorname{ex} Q(A) = P(A) \cup \{0\}$ (4.7).

Denote by $\mathbf{A}(Q(A))$ the set of all A-continuous affine real functions f on $Q(A)$ with $f(0) = 0$. Then $\mathbf{A}(Q(A))$ is an ordered real Banach space with the structure inherited from $C(Q(A))$.

Proposition. *The mapping $\Phi : A_h \to \mathbf{A}(Q(A))$ defined by*

$$[\Phi(x)](\varphi) = \varphi(x); \quad x \in A_h,\ \varphi \in Q(A)$$

is an isometric linear order isomorphism of A_h onto $\mathbf{A}(Q(A))$.

Proof. Clearly, $\Phi(x) \in A(Q(A))$ for all $x \in A_h$ and Φ is linear. Also, Φ is isometric by Proposition 4.12.(2) and $x \geqslant 0 \Leftrightarrow \Phi(x) \geqslant 0$ by Proposition 4.13.

Therefore $\Phi(A_h)$ is a closed real subspace of $\mathbf{A}(Q(A))$. Let $f \in \mathbf{A}(Q(A))$ and $\varepsilon > 0$. Consider the sets

$$X = \{(\varphi, f(\varphi)) \in Q(A) \times \mathbb{R};\ \varphi \in Q(A)\},$$
$$Y = \{(\varphi, f(\varphi) + \varepsilon) \in Q(A) \times \mathbb{R};\ \varphi \in Q(A)\}.$$

Both sets are convex compact subset of $A_h^* \times \mathbb{R}$, where A_h^* is endowed with the A_h-topology, and $X \cap Y = \emptyset$. By the Hahn-Banach theorem, there is a continuous linear functional F on $A_h^* \times \mathbb{R}$ and $\lambda \in \mathbb{R}$ such that

$$\sup F(X) < \lambda < \inf F(Y). \tag{1}$$

Put $\alpha = F(0,1)$. By (1), $0 < \lambda < F(0,\varepsilon) = \varepsilon\alpha$, hence $\alpha > 0$. Consequently, we can define a linear functional g on A_h^* by

$$g(\psi) = -\alpha^{-1}F(\psi, 0); \quad \psi \in A_h^*.$$

Since g is A_h-continuous, there exists $x \in A_h$ such that $g(\psi) = \psi(x)$ for all $\psi \in A_h^*$. For this x and for every $\psi \in A_h^*$ we have

$$\begin{aligned} F(\psi, \psi(x) + \alpha^{-1}\lambda) &= F(\psi, 0) + F(0, g(\psi)) + F(0, \alpha^{-1}\lambda) \\ &= F(\psi, 0) + g(\psi)\alpha + \alpha^{-1}\lambda\alpha = F(\psi, 0) - F(\psi, 0) + \lambda = \lambda. \end{aligned}$$

Hence by (1), for every $\varphi \in Q(A)$ we have

$$\begin{gathered} F(\varphi, f(\varphi)) < F(\varphi, \varphi(x) + \alpha^{-1}\lambda) < F(\varphi, f(\varphi) + \varepsilon), \\ F(\varphi, 0) + f(\varphi)\alpha < F(\varphi, 0) + (\varphi(x) + \alpha^{-1}\lambda)\alpha < F(\varphi, 0) + (f(\varphi) + \varepsilon)\alpha, \\ f(\varphi) < \varphi(x) + \alpha^{-1}\lambda < f(\varphi) + \varepsilon. \end{gathered}$$

Taking $\varphi = 0$, we infer that $0 < \alpha^{-1}\lambda < \varepsilon$, hence

$$-\varepsilon < -\alpha^{-1}\lambda < \varphi(x) - f(\varphi) < \varepsilon - \alpha^{-1}\lambda < \varepsilon; \quad \varphi \in Q(A),$$

that is $\|f - \Phi(x)\| \leqslant \varepsilon$.

Thus $\Phi(A_h)$ is also dense in $\mathbf{A}(Q(A))$ and hence $\Phi(A_h) = \mathbf{A}(Q(A))$. ∎

If A is unital, then $Q(A)$ can be replaced by $S(A)$ in the above proposition. Indeed, every function from $\mathbf{A}(S(A))$ can be extended to an element of $\mathbf{A}(Q(A))$.

It is possible to extend the definition of Φ to the whole A by the same formula: $\Psi(x)(\varphi) = \varphi(x)$, ($x \in A$, $\varphi \in Q(A)$). However, Ψ is no more isometric, as the following example shows. If $A = M_2$ and

$$x = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

then, $\|x\| = 1$ and $\|\Psi(x)\| = 1/2$.

COROLLARY 1. *We have*

$$\{\psi \in A_h^*;\ \|\psi\| \leqslant 1\} = \mathrm{co}(Q(A) \cup (-Q(A))), \tag{2}$$

that is, for every $\psi \in A_h^*$ there exist $\varphi_1, \varphi_2 \in A_h^*$ such that

$$\psi = \varphi_1 - \varphi_2, \quad \|\psi\| = \|\varphi_1\| + \|\varphi_2\|.$$

Proof. Denote the left hand side of (2) by $(A_h^*)_1$. Since $Q(A)$ and $-Q(A)$ are both A_h-compact and convex, it follows that $\mathrm{co}(Q(A)\cup(-Q(A)))$ is an A_h-compact convex subset of $(A_h^*)_1$. If $\psi \in (A_h^*)_1$ does not belong to $\mathrm{co}(Q(A) \cup (-Q(A)))$, then by Hahn-Banach theorem we find an element $x \in A_h$ such that

$$\psi(x) > 1 \quad \text{and} \quad \varphi(x) \leqslant 1 \quad \text{for all } \varphi \in \mathrm{co}(Q(A) \cup (-Q(A))).$$

By the proposition (i.e. by 4.12.(2)) we infer that $\|x\| \leqslant 1$, so that $|\psi(x)| \leqslant \|\psi\| \cdot \|x\| \leqslant 1$, a contradiction. ∎

A more precise result will be presented in 8.11. An obvious consequence of Corollary 1 is: $A_h^* = A_+^* - A_+^*$.

Remark that any lower A-semicontinuous affine real function f on $Q(A)$ attains its lower bound m on $\operatorname{ex} Q(A)$ since the set $K = \{\varphi \in Q(A);\ f(\varphi) = m\}$ is A-compact and convex and it is easy to see that $\emptyset \neq \operatorname{ex} K \subset \operatorname{ex} Q(A)$.

Corollary. *Let $\{x_\iota\}_{\iota \in I}$ be an increasing net in A_h and $x \in A_h$. If $\varphi(x) = \sup_\iota \varphi(x_\iota)$ for all $\varphi \in P(A)$, then $\|x - x_\iota\| \to 0$.*

Proof. By Proposition 4.13.(i), then assumption entails $x_\iota \leqslant x$, $(\iota \in I)$. Thus, for the lower A-semicontinuous affine real function f on $Q(A)$ defined by $f(\varphi) = \sup_\iota \Phi(x_\iota)(\varphi)$, $(\varphi \in Q(A))$, we have $f \leqslant \Phi(x)$. Hence $f - \Phi(x)$ is lower A-semicontinuous, affine, $f - \Phi(x) \leqslant 0$ and $f - \Phi(x) = 0$ on $\operatorname{ex} Q(A)$. By the above remark, $f - \Phi(x)$ attains its lower bound on $\operatorname{ex} Q(A)$, hence $f = \Phi(x)$. Using Dini's theorem, it follows that $\{\Phi(x_\iota)\}$ converges uniformly to $\Phi(x)$. By the proposition, we conclude $\|x - x_\iota\| \to 0$. ∎

The next result is a characterization of increasing approximate units in C^*-algebras.

Corollary. *Let $\{u_\iota\}_{\iota \in I}$ be an increasing net in A^+. Then $\{u_\iota\}$ is an approximate unit for A if and only if $\varphi(u_\iota) \to 1$ for all $\varphi \in P(A)$.*

Proof. By Proposition 4.12.(2) the assumption entails $\|u_\iota\| \leqslant 1$, $\iota \in I$.

Let $a \in A$. If $\varphi \in P(A)$, then either $\varphi(a^* \cdot a) = 0$ or $\varphi(a^* \cdot a)$ is a pure positive form (4.9) and $\|\varphi(a^* \cdot a)\| = \varphi(a^* a)$. By the assumption, $\varphi(a^* u_\iota a) \to \varphi(a^* a)$. Using Corollary 2, we get

$$\|a - u_\iota a\|^2 \leqslant \|(1 - u_\iota)^{1/2} a\|^2 = \|a^* a - a^* u_\iota a\| \to 0. \quad ∎$$

4.16. Proposition. *Let A be a C^*-algebra and B be a C^*-subalgebra of A. Then:*

(i) *every state of B can be extended to a state of A;*
(ii) *every pure state of B can be extended to a pure state of A.*

Proof. (i) Passing to C^*-algebras with *adjoined units* we may assume that A is unital and B contains the unit 1 of A (see the discussion after Theorem 4.5). Then each $x \in A_h$ is majorized by $\|x\| \cdot 1 \in B_h$, so that $B_h + A^+ = A_h$ and Proposition 3/4.10 applies to show that any positive form ψ on B extends to a positive form φ on A and we have

$$\|\varphi\| = \varphi(1) = \psi(1) = \|\psi\|.$$

(ii) If $\psi \in P(B)$, then the set $K = \{\varphi \in Q(A);\ \varphi | B = \psi\}$ is non-void, $\sigma(A_h^*, A_h)$-compact and convex, so that $\operatorname{ex} K \neq \emptyset$. It is easy to see that $\operatorname{ex} K \subset \operatorname{ex} Q(A) = P(A) \cup \{0\}$ and the assertion follows since $0 \notin K$. ∎

Corollary. *Let A be a C^*-algebra, B be a C^*-subalgebra and $\rho : B \to B(K)$ be a (topologically irreducible) $*$-representation. Then there is a (topologically irreducible) $*$-representation $\pi : A \to B(H)$ and an isometric linear operator $V : K \to H$ such that*

$$\rho(y) = V^*\pi(y)V, \quad y \in B.$$

Proof. Owing to Lemma 1 and Lemma 2/4.1, we may assume that ρ has a cyclic vector $\eta \in K$, $\|\eta\| = 1$. Then $\psi = \omega_\eta \circ \rho$ is a (pure) state of B and, by the proposition, there is a (pure) state φ of A with $\psi = \varphi|B$.

Now $\pi = \pi_\varphi$ is a (topologically irreducible) $*$-representation of A on $H = H_\varphi$. Let H_0 be the closure of the subspace $\{\pi(y)\xi_\varphi,\ y \in B\}$. Then $\pi|B$ is a $*$-representation of B on H_0 with cyclic vector $\xi =$ the orthogonal projection of ξ_φ on H_0 and

$$((\pi|B)(y)\xi|\xi) = \varphi(y) = \psi(y) = (\rho(y)\eta|\eta)$$

and the desired result follows by Proposition 4.3. ∎

4.17. Let C be a commutative C^*-subalgebra of a C^*-algebra A and Ω be the Gelfand spectrum of C. Every character $t \in \Omega$ is a pure state of C (4.9) and by Proposition 4.16 it can be extended to a pure state φ_t of A, which is therefore multiplicative on C. A direct computation shows that for every $z \in C$,

$$\|\pi_{\varphi_t}(z)\xi_{\varphi_t} - \varphi_t(z)\xi_{\varphi_t}\|^2_{\varphi_t} = 0,$$

that is,

$$\pi_{\varphi_t}(z)\xi_{\varphi_t} = \varphi_t(z)\xi_{\varphi_t}.$$

Hence the direct sum of all π_φ's with $\varphi \in P(A)$ gives a realization of A on some Hilbert space H such that for every normal $x \in A$ and every $\lambda \in \sigma(x)$ there is $0 \neq \xi \in H$ with

$$x\xi = \lambda\xi.$$

Another consequence of the above remark is:

Proposition. *If A is a non-unital C^*-algebra, then the zero form is $\sigma(A^*, A)$-adherent to $P(A)$.*

Proof. We have to show that for every $x_1, \ldots, x_n \in A$ and every $\varepsilon > 0$ there is $\varphi \in P(A)$ with $|\varphi(x_\kappa)| \leqslant \varepsilon$, $(1 \leqslant k \leqslant n)$. Since A is the linear span of A^+, we may suppose $x_1, \ldots, x_n \in A^+$ and then it suffices to show $\varphi(x_1 + \cdots + x_n) \leqslant \varepsilon$.

Thus, let $x \in A$, $0 \leqslant x \leqslant 1$, let $\varepsilon > 0$ and let $\widetilde{A}$ be the associate unital C^*-algebra of A.

Assume first that $x = p$ is a projection. Since $A \neq \widetilde{A}$, there is $y \in A$ with $(1-p)y(1-p) \neq 0$ so that (4.12) there is a pure state ψ of A with $\psi((1-p)y(1-p)) \neq 0$. Then $\varphi = \psi((1-p)\cdot(1-p))$ is a non-zero pure (4.9) positive form on A and $\varphi(p) = 0$.

In the general case, let $C = C_0(\Omega)$ be the commutative C^*-subalgebra of A generated by x. If C is unital, then its unit p is a projection in A, $0 \leqslant x \leqslant p$ and by the above there is $\varphi \in P(A)$ with $\varphi(p) = 0$, hence $\varphi(x) = 0$. If C is not unital, then Ω is not compact and x is a function vanishing at infinity on Ω, hence $x(t) \leqslant \varepsilon$ for some $t \in \Omega$. Now, there exists $\varphi_t \in P(A)$ with $\varphi_t|C = t$, so $\varphi_t(x) \leqslant \varepsilon$. ∎

Using this proposition we obtain an improvement of 4.7.(2) in the case of non-unital C^*-algebra (compare with 4.7.(4)):

(1) $$Q(A) = \overline{\mathrm{co}}^A P(A) \text{ if } A \text{ is a non unital } C^*\text{-algebra.}$$

Thus a C^*-algebra A is unital if and only if $S(A)$ is A-compact.

4.18. Non-commutativity. The fact that a C^*-algebra is not commutative is best expressed by the following result:

Theorem. *Let A be a C^*-algebra. Then A is not commutative if and only if there is $u \in A$, $u \neq 0$ with $u^2 = 0$.*

Proof. If A is commutative, then every element of A is normal so that, if $u \in A$ and $u^2 = 0$, then $\|u\| = \lim_n \|u^n\|^{1/n} = 0$ and $u = 0$.

Conversely, assume that

(1) $$u \in A,\ u^2 = 0 \Rightarrow u = 0.$$

Moreover, suppose that there is $a \in A$, $a^* = a$, and a topologically irreducible $*$-representation $\pi : A \to B(H)$ such that $\pi(a)$ is not a scalar operator. Then $\sigma(\pi(a))$ contains at least two points, say $s \in \sigma(\pi(a))$, $t \in \sigma(\pi(a))$, $t \neq s$. There are $f, g \in C_0(\mathbb{R})$ such that

$$f(s) \neq 0, \quad g(t) \neq 0 \quad \text{but} \quad fg = 0.$$

We have

(2) $$f(\pi(a)) \neq 0, \quad g(\pi(a)) \neq 0$$

and $(f(a)yg(a))^2 = 0$, $(y \in A)$, which by (1) entails $f(a)yg(a) = 0$, $(y \in A)$. In particular,

(3) $$f(\pi(a))\pi(y)g(\pi(a)) = 0, \quad y \in A.$$

By (2), there is $\xi \in H$ with $g(\pi(a))\xi \neq 0$. Let e be the orthogonal projection of H onto the closed subspace generated by $\pi(A)g(\pi(a))\xi$. Then eH is stable under $\pi(A)$ and, if $\{u_\iota\}$ is an approximate unit for A, then, by Lemma 3/4.1,

$$0 \neq g(\pi(a))\xi = \lim_\iota \pi(u_\iota)g(\pi(a))\xi \in eH.$$

Therefore $e \in \pi(A)'$ and $e \neq 0$. By (3) and (2) we have

$$eH \subset \operatorname{Ker} f(\pi(x)) \neq H,$$

so that $e \neq 1_H$. Thus e is a non-scalar operator in $\pi(A)'$ which contradicts the topological irreducibility of π.

It follows that $\pi(A) = \mathbb{C}1_H$ for every topologically irreducible $*$-representation π of A. Owing to Corollary 2/4.12 we infer that A is commutative. ∎

Recall that an ordered real vector space X is called "lattice ordered" if for each $x \in X$ there is an element $x \vee 0 \in X$ such that $x \vee 0 \geqslant 0$, $x \vee 0 \geqslant x$ and

$$y \in X,\ y \geqslant 0,\ y \geqslant x \Rightarrow y \geqslant x \vee 0.$$

It is easy to see that

$$x = (x \vee 0) - ((-x) \vee 0). \tag{4}$$

Also, X is said to have the "Riesz decomposition property" if given $x, y, z \in X$ with $0 \leqslant x \leqslant y + z$, $y \geqslant 0$, $z \geqslant 0$, there are $y', z' \in X$ such that $x = x' + z'$, $0 \leqslant y' \leqslant y$, $0 \leqslant z' \leqslant z$.

Given a C^*-algebra A, the real vector space A_h and its dual space A_h^* are ordered by the convex cones A^+ and A_+^* respectively.

COROLLARY 1. *Let A be a C^*-algebra. The following conditions are equivalent:*

(i) *A is commutative;*
(ii) *for some $t \in \mathbb{R}$, $t > 1$, we have: $a, b \in A$, $0 \leqslant a \leqslant b \Rightarrow a^t \leqslant b^t$;*
(iii) *A_h is lattice ordered;*
(iv) *A_h has the Riesz decomposition property;*
(v) *A_h^* is lattice ordered;*
(vi) *A_h^* has the Riesz decomposition property.*

Proof. If A is commutative, then $A = C_0(\Omega)$ for some locally compact Hausdorff space Ω and $A^* = M(\Omega)$, the space of all bounded regular Borel measures on Ω, so that assertions (ii) to (vi) are clear in this case.

Conversely, assume that A is not commutative. By the theorem, there is $u \in A$, $\|u\| = 1$, with $u^2 = 0$. Define $p, q, r \in A_h$ by

$$p = u^*u, \quad q = (u^*u)^{1/2}u^* + u(u^*u)^{1/2}, \quad r = uu^*.$$

For any $\alpha, \beta, \gamma \in \mathbb{R}$, $\gamma > 0$, we have

$$\alpha p+\beta q+\gamma r = (\beta\gamma^{-1/2}(u^*u)^{1/2}+\gamma^{1/2}u)(\beta\gamma^{-1/2}(u^*u)^{1/2}+\gamma^{1/2}u)^*+(\alpha\gamma-\beta^2)\gamma^{-1}u^*u.$$

If $\alpha p + \beta q + \gamma r$ is positive, then

$$(\gamma^{1/2}(u^*u)^{1/2} - \beta\gamma^{-1/2}u)^*(\alpha p + \beta q + \gamma r)(\gamma^{1/2}(u^*u)^{1/2} - \beta\gamma^{-1/2}u)$$

is also positive and, using the above equality we get $(\alpha\gamma - \beta^2)(u^*u)^{1/2} \geqslant 0$. Consequently,

$$\alpha p + \beta q + \gamma r \geqslant 0 \Leftrightarrow \alpha\gamma - \beta^2 \geqslant 0. \tag{5}$$

(ii) $\Rightarrow$ (i). If (ii) holds for some $t > 1$ then, by iteration (ii) holds with t replaced by any t^n, ($n \in \mathbb{N}$), hence (ii) holds for some $t > 2$. Owing to Proposition 2.7, it follows that (ii) holds with $t = 2$.

However, assuming A non-commutative, take $a = 4p+2q+r$ and $b = 8p+2r$. Then, by (5), $0 \leqslant a \leqslant b$, but $a^2 \nleqslant b^2$ since $u^*(b^2 - a^2)u = -(uu^*)^3$.

(iii) $\Rightarrow$ (i). First note that if (iii) holds then

$$x \in A_h,\ x = a - b,\ a \geqslant 0,\ b \geqslant 0,\ ab = 0 \Rightarrow x \vee 0 = a.$$

Indeed, let $a' = x \vee 0$ and $b' = (-x) \vee 0$. Since $a \geqslant x$, $a \geqslant 0$, we have $a \geqslant a' \geqslant 0$ and similarly $b \geqslant b' \geqslant 0$. Since $ab = 0$, it follows that $a'b' = 0$. Using (4) we get

$$x = a' - b', \quad a' \geqslant 0, \quad b' \geqslant 0, \quad a'b' = 0.$$

By Proposition 2.3, $a' = a$, $b' = b$.

Now assume that (iii) holds but A is not commutative and take $x = p - r$, $y = 2p + 2^{1/2}q + r$. By the above, $x \vee 0 = p$. However, by (5), $y \geqslant 0$, $y \geqslant x$, $y \ngeqslant p$, a contradiction.

(iv) $\Rightarrow$ (iii). Suppose that (iv) holds and let $x \in A_h$. Write $x = a - b$, $a \geqslant 0$, $b \geqslant 0$, $ab = 0$, by Proposition 2.3. We shall prove that $x \vee 0$ exists, namely $x \vee 0 = a$. Clearly, $a \geqslant 0$, $a \geqslant x$. Let $y \in A_h$ such that $y \geqslant 0$, $y \geqslant x$. Then

$$0 \leqslant a = (a - b) + b \leqslant y + b,$$

so there are $a_1, a_2 \in A_h$ such that $a = a_1 + a_2$, $0 \leqslant a_1 \leqslant y$, $0 \leqslant a_2 \leqslant b$. Since $0 \leqslant a_2 \leqslant a$, $0 \leqslant a_2 \leqslant b$ and $ab = 0$, it follows that $a_2 = 0$. Consequently, $y \geqslant a_1 = a$.

(v) $\Rightarrow$ (i). We may assume that $A \subset B(H)$ for some Hilbert space H. Suppose that (v) holds but A is not commutative. Denote $H_1 = \operatorname{Ker} u$ and $H_2 = H \ominus H_1$. Then $H_2 \subset \operatorname{Ker} u^*$. By (5), $p + 2^{1/2}q + r \ngeqslant 0$, hence there are $\xi_1 \in H_1$, $\xi_2 \in H_2$ such that

$$((p + 2^{1/2}q + r)(\xi_1 + \xi_2)|\xi_1 + \xi_2) < 0.$$

Consider $\varphi, \psi, \theta \in A_h^*$ defined by

$$\begin{aligned}
&\varphi(x) = (x\xi_2|\xi_2) - (x\xi_1|\xi_1), \quad x \in A,\\
&\psi(x) = (x(\xi_1 + 2^{1/2}\xi_2)|\xi_1 + 2^{1/2}\xi_2), \quad x \in A,\\
&\theta(x) = (x\xi_2|\xi_2), \quad x \in A,
\end{aligned}$$

and put $\varphi' = \varphi \vee 0$. Since

$$(\psi - \varphi)(x) = (x(2^{1/2}\xi_1 + \xi_2)|2^{1/2}\xi_1 + \xi_2), \quad x \in A,$$

we have $\psi \geqslant 0$, $\psi \geqslant \varphi$, so $\psi \geqslant \varphi'$. On the other hand, $\theta \geqslant 0$, $\theta \geqslant \varphi$, so $\theta \geqslant \varphi'$.

Now we have successively

$$(p\xi_2|\xi_2) = \varphi(p) \leqslant \varphi'(p) \leqslant \theta(p) = (p\xi_2|\xi_2)$$

hence $\varphi'(p) = (p\xi_2|\xi_2) = \theta(p)$,

$$0 \leqslant \varphi'(r) \leqslant \theta(r) = 0,$$

hence $\varphi'(r) = 0 = \theta(r)$, and since by (5) $p \pm q + r \geqslant 0$,

$$0 \leqslant (\theta - \varphi')(p \pm q + r) = \mp\varphi'(q),$$

hence $\varphi'(q) = 0$. Consequently,

$$0 \leqslant (\psi - \varphi')(p + q + r) = ((p + 2^{1/2}q + r)(\xi_1 + \xi_2)|\xi_1 + \xi_2) < 0,$$

a contradiction.

(vi) $\Rightarrow$ (v). Suppose that (vi) holds and let $\psi \in A_h^*$. By Corollary 1/4.15 there exist $\varphi_1, \varphi_2 \in A_+^*$ such that $\psi = \varphi_1 - \varphi_2$, $\|\psi\| = \|\varphi_1\| + \|\varphi_2\|$. We shall prove that $\psi \vee 0$ exists, namely $\psi \vee 0 = \varphi_1$. Clearly, $\varphi_1 \geqslant 0$, $\varphi_1 \geqslant \psi$. Now let $\varphi \in A_h^*$ be such that $\varphi \geqslant 0$, $\varphi \geqslant \psi$. Then

$$0 \leqslant \varphi_1 = (\varphi_1 - \varphi_2) + \varphi_2 \leqslant \varphi + \varphi_2,$$

hence there are $\theta_1, \theta_2 \in A_h^*$ such that $\varphi_1 = \theta_1 + \theta_2$, $0 \leqslant \theta_1 \leqslant \varphi$, $0 \leqslant \theta_2 \leqslant \varphi_2$. Since $\psi = \varphi_1 - \varphi_2 = \theta_1 - (\varphi_2 - \theta_2)$ and, by Corollary 1/4.5, $\|\varphi_1\| = \|\theta_1\| + \|\theta_2\|$; $\|\varphi_2\| = \|\varphi_2 - \theta_2\| + \|\theta_2\|$, we have

$$\|\psi\| \leqslant \|\theta_1\| + \|\varphi_2 - \theta_2\| = \|\varphi_1\| + \|\varphi_2\| - 2\|\theta_2\| = \|\psi\| - 2\|\theta_2\|.$$

Consequently, $\theta_2 = 0$, thus $\varphi \geqslant \theta_1 = \varphi_1$. ∎

Remark that for an arbitrary ordered real Banach space we have (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) (see [275]), so in the above corollary it would be sufficient to prove only (v) $\Rightarrow$ (i) and (vi) $\Rightarrow$ (v).

Let A be an arbitrary C^*-algebra. For every $a, b \in A_h$, $\|a\| \leqslant 1$, $\|b\| \leqslant 1$, we have

$$\|a + b\| \leqslant 1 + 2\|ab\|. \tag{6}$$

Indeed, for any integer $n \geqslant 0$,

$$\|a^{2^n} + b^{2^n}\| = \|(a^{2^n} + b^{2^n})^2\|^{1/2} \leqslant (2\|ab\| + \|a^{2^n} + b^{2^n}\|)^{1/2}$$

hence, putting $\alpha_0 = \|a + b\|$ and

$$\alpha_n = (2\underbrace{\|ab\| + (2\|ab\| + \cdots + (2\|ab\|}_{n\,\text{times}} + \|a^{2^n} + b^{2^n}\|)^{1/2}\cdots)^{1/2})^{1/2}, \quad n \geqslant 1,$$

we have

$$\|a + b\| = \alpha_0 \leqslant \alpha_1 \leqslant \alpha_2 \leqslant \cdots.$$

Consequently, the sequence $\{\alpha_n\}$ converges and it is easy to see that its limit is $2^{-1}((1 + 8\|ab\|)^{1/2} + 1)$. Thus,

$$\|a + b\| \leqslant 2^{-1}((1 + 8\|ab\|)^{1/2} + 1) \leqslant 1 + 2\|ab\|.$$

Therefore for every C^*-algebra $A \neq \{0\}$ we can define $k_A \geqslant 0$ as the greatest lower bound of all $k \geqslant 0$ such that

$$a, b \in A_h,\ \|a\| \leqslant 1,\ \|b\| \leqslant 1 \Rightarrow \|a + b\| \leqslant 1 + k\|ab\|,$$

and, by the above, $1 \leqslant k_A \leqslant 2$. There is a commutativity criterion for A in terms of k_A:

COROLLARY 2. *Let A be a C^*-algebra. Then A is commutative if and only if, for any selfadjoint $a, b \in A$, $\|a\| = \|b\| = 1$,*

$$\|a + b\| \leqslant 1 + \|ab\|. \tag{7}$$

Proof. If A is commutative, then (7) holds for all $a, b \in A$ with $\|a\| = \|b\| = 1$, as can be seen by Gelfand representation.

If A is not commutative, then by the theorem there is $u \in A$, $\|u\| = 1$, $u^2 = 0$. For $\alpha, \beta \geqslant 0$, $\alpha + \beta = 1$, define

$$a = u^*u, \quad b = \alpha(u^*u + uu^*) + \beta(u + u^*).$$

Clearly, $\|a\| = 1$. Since $(u^*u)(uu^*) = 0 = (uu^*)(u^*u)$, we have

$$\|u + u^*\|^2 = \|(u + u^*)^2\| = \|u^*u + uu^*\| = \max\{\|u^*u\|, \|uu^*\|\} = 1$$

by Gelfand representation, hence $\|b\| \leqslant 1$. Owing to 2.6.(5) we get

$$\|ab\|^2 = \|abba\| = \|\alpha^2(u^*u)^4 + \beta^2(u^*u)^3\| = \alpha^2 + \beta^2,$$

hence $\|ab\| = (\alpha^2 + \beta^2)^{1/2}$.

Using again 2.6.(5) we obtain, for $\gamma \geqslant 0$,

$$\|u^* + \gamma uu^*\|^2 = \|uu^* + \gamma^2(uu^*)^2\| = 1 + \gamma^2 \tag{8}$$

and, for $\lambda, \mu, \nu \geqslant 0$,

$$\begin{aligned} &\|\lambda u^*uu^* + \mu(uu^*)^2 + \nu(uu^*)\|^2 \\ &\quad = \|\lambda^2(uu^*)^3 + \mu^2(uu^*)^4 + 2\mu\nu(uu^*)^3 + \nu^2(uu^*)^2\| = \lambda^2 + (\mu + \nu)^2. \end{aligned} \tag{9}$$

By (8) and (9) we have

$$2\|b\|^2 = \|b\|^2\|u^*+uu^*\|^2 \geqslant \|b(u^*+uu^*)\|^2 = \|(\alpha+\beta)u^*uu^*+\alpha(uu^*)^2+\beta(uu^*)\|^2 = 2$$

so that $\|b\| \geqslant 1$, hence $\|b\| = 1$. Also,

$$\begin{aligned} (1 + \gamma^2)\|a + b\|^2 = \|a + b\|^2\|u^* + \gamma uu^*\|^2 &\geqslant \|(a + b)(u^* + \gamma uu^*)\|^2 \\ &= \|(1 + \alpha + \beta\gamma)u^*uu^* + \alpha\gamma(uu^*)^2 + \beta(uu^*)\|^2 \\ &= (1 + \alpha + \beta\gamma)^2 + (\alpha\gamma + \beta)^2, \end{aligned}$$

so that $\|a + b\|^2 \geqslant ((1 + \alpha + \beta\gamma)^2 + (\alpha\gamma + \beta)^2)(1 + \gamma^2)^{-1}$.

For $\alpha = 2/3$, $\beta = 1/3$ and $\gamma = 1/3$ we get

$$\|a+b\| \geqslant ((1+\alpha+\beta\gamma)^2+(\alpha\gamma+\beta)^2)^{1/2}(1+\gamma^2)^{-1/2} > 1+(\alpha^2+\beta^2)^{1/2} = 1+\|ab\|$$

which contradicts (7). ∎

4.19. Let A be a C^*-algebra and $n \geqslant 1$ be an integer. Then the $*$-algebra $M_n(A)$ (2.12) is a C^*-algebra.

Indeed, by 4.11 we may assume that $A \subset B(H)$ for some Hilbert space H. If $H^{(n)}$ is the Hilbert space direct sum of n copies of H, then every $[x_{ij}] \in M_n(B(H))$ acts on $H^{(n)}$ by

$$[x_{ij}][\xi_k] = \Big[\sum_j x_{kj}\xi_j\Big]; \quad \xi_1, \ldots, \xi_n \in H,$$

and in this way $M_n(B(H))$ can be identified to $B(H^{(n)})$. Hence $M_n(B(H))$ is a C^*-algebra. For $[x_{ij}] \in M_n(B(H))$ we have

$$\|x_{hk}\| \leqslant \|[x_{ij}]\| \leqslant \max_j \Big(\sum_i \|x_{ij}\|^2\Big)^{1/2}; \quad 1 \leqslant h, k \leqslant n. \tag{1}$$

Using (1) it is easy to see that $M_n(A)$ is a closed $*$-subalgebra of $M_n(B(H))$, which proves our assertion.

Since $A \otimes M_n$ is $*$-isomorphic to $M_n(A)$ (2.12), it follows that $A \otimes M_n$ is also a C^*-algebra.

4.20. **C^*-tensor product.** In this section we introduce the most usual notion of tensor product of C^*-algebra. To this end, we first recall the construction and the properties of the Hilbert space tensor product of Hilbert spaces.

Let H, K be complex Hilbert space. On the tensor product $H \otimes K$ of the vector space H, K there exists a unique scalar product such that

$$(\xi_1 \otimes \eta_1 | \xi_2 \otimes \eta_2) = (\xi_1|\xi_2)(\eta_1|\eta_2); \quad \xi_1, \xi_2 \in H,\ \eta_1, \eta_2 \in K,$$

and the Hilbert space completion of $H \otimes K$ with respect to this scalar product is denoted by $H \mathbin{\overline{\otimes}} K$ and called the *Hilbert space tensor product* of H and K.

Let $a \in B(H)$, $b \in B(K)$. Each $\zeta \in H \otimes K$ can be written as $\zeta = \sum_{k=1}^{n} \xi_k \otimes \eta_k$ with $\xi_1, \ldots, \xi_n \in H$ and mutually orthogonal $\eta_1, \ldots, \eta_n \in K$. Then

$$\|(a \otimes 1_K)\zeta\|^2 = \sum_{k=1}^{n} \|a\xi_k\|^2 \|\eta_k\|^2 \leqslant \|a\|^2 \sum_{k=1}^{n} \|\xi_k\|^2 \|\eta_k\|^2 = \|a\|^2 \|\zeta\|^2.$$

Similarly, $\|(1_H \otimes b)\xi\| \leqslant \|b\|\,\|\zeta\|$ for all $\zeta \in H \otimes K$. It follows that $a \otimes b = (a \otimes 1_K)(1_H \otimes b)$ can be uniquely extended to a bounded linear operator $a \mathbin{\overline{\otimes}} b$ on $H \mathbin{\overline{\otimes}} K$. Moreover

$$\|a \mathbin{\overline{\otimes}} b\| = \|a\|\,\|b\|; \quad a \in B(H),\ b \in B(K).$$

It is easy to see that

$$B(H) \times B(K) \ni (a, b) \mapsto a \mathbin{\overline{\otimes}} b \in B(H \mathbin{\overline{\otimes}} K)$$

is a bounded bilinear mapping with norm equal to one and

$$(a_1 \mathbin{\overline{\otimes}} b_1)(a_2 \mathbin{\overline{\otimes}} b_2) = a_1 a_2 \mathbin{\overline{\otimes}} b_1 b_2; \quad a_1, a_2 \in B(H);\ b_1, b_2 \in B(K),$$
$$(a \mathbin{\overline{\otimes}} b)^* = a^* \mathbin{\overline{\otimes}} b^*; \quad a \in B(H),\ b \in B(K).$$

In particular, if $a \in B(H)$, $b \in B(K)$ are normal (respectively selfadjoint, respectively positive, respectively unitary, respectively projection), then the same is true for $a \mathbin{\overline{\otimes}} b \in B(H \mathbin{\overline{\otimes}} K)$.

Lemma 1. *The mapping*

$$B(H) \otimes B(K) \ni \sum_{k=1}^{n} a_k \otimes b_k \mapsto \sum_{k=1}^{n} a_k \mathbin{\overline{\otimes}} b_k \in B(H \mathbin{\overline{\otimes}} K)$$

is an injective $*$*-homomorphism.*

Proof. Let $x \in B(H) \otimes B(K)$, $x \neq 0$. There exist linearly independent $a_1, \ldots, a_n \in B(H)$ and non-zero $b_1, \ldots, b_n \in B(K)$ such that $x = \sum\limits_{k=1}^{n} a_k \otimes b_k$. Then there exists $\eta \in K$ with $b_1\eta \neq 0$. Assume that $\{b_1\eta, \ldots, b_m\eta\}$, $(1 \leqslant m \leqslant n)$, is a maximal linearly independent subset of $\{b_1\eta, \ldots, b_n\eta\}$ and write

$$b_k\eta = \sum_{j=1}^{m} \lambda_{kj} b_j \eta; \quad 1 \leqslant k \leqslant n.$$

Since $\lambda_{11} = 1$ and $\{a_1, \ldots, a_n\}$ are linearly independent, we have

$$a = \sum_{k=1}^{n} \lambda_{k1} a_k \neq 0.$$

Thus there exists $\xi \in H$ with $a\xi \neq 0$. Then $\xi \otimes \eta \in H \mathbin{\overline{\otimes}} K$ and

$$\begin{aligned}\Big(\sum_{k=1}^{n} a_k \mathbin{\overline{\otimes}} b_k\Big)(\xi \otimes \eta) &= \sum_{k=1}^{n} a_k\xi \otimes \Big(\sum_{j=1}^{m} \lambda_{kj} b_j \eta\Big) = \sum_{j=1}^{m} \Big(\sum_{k=1}^{n} \lambda_{kj} a_k\Big)\xi \otimes b_j\eta \\ &= a\xi \otimes b_1\eta + \sum_{j=2}^{m} \Big(\sum_{k=1}^{n} \lambda_{kj} a_k\Big)\xi \otimes b_j\eta \neq 0\end{aligned}$$

because $a\xi \neq 0$ and $\{b_1\eta, \ldots, b_m\eta\}$ are linearly independent.

This proves that the map in question is injective and this is the only non-trivial assertion of the lemma. ∎

We shall identify $B(H) \otimes B(K)$ with a $*$-subalgebra of $B(H \mathbin{\overline{\otimes}} K)$.

Now let A, B be C^*-algebras. By Theorem 4.11 there exist isometric $*$-representations $\rho_0 : A \to B(H_0)$, $\sigma_0 : B \to B(K_0)$. Put

$$\|x\|_{\rho_0,\sigma_0} = \|(\rho_0 \otimes \sigma_0)(x)\|_{B(H_0 \mathbin{\overline{\otimes}} K_0)}; \quad x \in A \otimes B.$$

By Lemma 1, $\|\cdot\|_{\rho_0,\sigma_0}$ is a C^*-norm and a cross-norm on $A \otimes B$. The corresponding C^*-algebra completion of $A \otimes B$ is denoted by

$$A \otimes_{\rho_0,\sigma_0} B.$$

Clearly, this C^*-algebra is $*$-isomorphic to the norm-closure of $\rho_0(A) \otimes \sigma_0(B)$ in $B(H_0 \mathbin{\overline{\otimes}} K_0)$.

LEMMA 2. *For every $\varphi \in S(A)$ and every $\psi \in S(B)$ we have*

$$|(\varphi \otimes \psi)(x)| \leqslant \|x\|_{\rho_0,\sigma_0}; \quad x \in A \otimes B.$$

Proof. Without restricting the generality, we may suppose that ρ_0 and σ_0 are non-degenerated. Then by Corollary 4.14.(i) it is suficient to consider only φ, ψ of the form

$$\varphi = \sum_{i=1}^{n} \omega_{xi_i} \circ \rho_0 \quad \text{with} \quad \xi_1, \dots \xi_n \in H_0, \quad \sum_{i=1}^{n} \|\xi_i\|^2 = 1,$$
$$\psi = \sum_{j=1}^{m} \omega_{\eta j} \circ \sigma_0 \quad \text{with} \quad \eta_1, \dots, \eta_m \in K_0, \quad \sum_{j=1}^{m} \|\eta_j\|^2 = 1.$$

In this case, for every $x \in A \otimes B$ we have

$$\begin{aligned} |(\varphi \otimes \psi)(x)| &= \Big| \sum_{i,j} ((\rho_0 \otimes \sigma_0)(x)(\xi_i \otimes \eta_j) | \xi_i \otimes \eta_j) \Big| \\ &\leqslant \sum_{ij} \|(\rho_0 \otimes \sigma_0)(x)\| \, \|\xi_i\|^2 \|\eta_j\|^2 = \|x\|_{\rho_0,\sigma_0}. \quad \blacksquare \end{aligned}$$

LEMMA 3. *For every $*$-representations $\rho : A \to B(H)$, $\sigma : B \to B(K)$, we have*

$$\|(\rho \otimes \sigma)(x)\|_{B(H \overline{\otimes} K)} \leqslant \|x\|_{\rho_0,\sigma_0}; \quad x \in A \otimes B. \tag{1}$$

If in addition ρ, σ are both isometric, then

$$\|(\rho \otimes \sigma)(x)\|_{B(H \overline{\otimes} K)} = \|x\|_{\rho_0,\sigma_0}; \quad x \in A \otimes B. \tag{2}$$

Proof. By Lemma 1 and Lemma 2/4.1 we may suppose that ρ and σ are cyclic $*$-representations with cyclic vectors $\xi \in H$, $\|\xi\| = 1$ and $\eta \in K$, $\|\eta\| = 1$, respectively. Then $\varphi = \omega_\xi \circ \rho \in S(A)$, $\psi = \omega_\eta \circ \sigma \in S(B)$ and, by Proposition 4.3, $\rho \simeq \pi_\varphi$, $\sigma \simeq \pi_\psi$. It follows that the $*$-representations $\rho \otimes \sigma$ and $\pi_\varphi \otimes \pi_\psi$ of the $*$-algebra $A \otimes B$ are unitarily equivalent, in particular

$$\|(\rho \otimes \sigma)(x)\|_{B(H \overline{\otimes} K)} = \|(\pi_\varphi \otimes \pi_\psi)(x)\|_{B(H_\varphi \overline{\otimes} H_\psi)}; \quad x \in A \otimes B. \tag{3}$$

Now, by Lemma 2, $\varphi \otimes \psi$ is $\| \cdot \|_{\rho_0,\sigma_0}$-continuous on $A \otimes B$ and hence it extends to a positive form θ on the C^*-algebra $A \otimes_{\rho_0,\sigma_0} B$. By the construction of the GNS representations, H_θ can be identified with $H_\varphi \overline{\otimes} H_\psi$ in such a way that

$$\pi_\theta(x) = (\pi_\varphi \otimes \pi_\psi)(x); \quad x \in A \otimes B. \tag{4}$$

Since π_θ is a $*$-representation of the C^*-algebra $A \otimes_{\rho_0,\sigma_0} B$, by Theorem 1.9 we get

$$\|\pi_\theta(x) \leqslant \|x\|_{\rho_0,\sigma_0}; \quad x \in A \otimes B. \tag{5}$$

Thus, (1) follows form (3), (4), (5) and (2) follows form (1) by interchanging the roles of ρ, ρ_0 and σ, σ_0 respectively. $\blacksquare$

By Lemma 3 we conclude that there exists a unique C^*-norm $\|\cdot\|_{C^*}$ on $A\otimes B$ such that

$$\|x\|_{C^*} = \|(\rho\otimes\sigma)(x)\|; \quad x\in A\otimes B,$$

for every injective $*$-representation ρ,σ of A, B respectively. The C^*-algebra completion of $A\otimes B$ with respect to $\|\cdot\|_{C^*}$ is denoted by $A\otimes_{C^*} B$ and is called the C^*-*tensor product* (or the *spatial tensor product*) of the C^*-algebra A and B.

We record the main conclusions in the following

THEOREM. *Let A, B, M, N be C^*-algebra. If $\rho : A\to M$ and $\sigma : B\to N$ are $*$-homomorphisms, then the $*$-homomorphism*

$$\rho\otimes\sigma : A\otimes B \to M\otimes N$$

can be uniquely extended to a $*$-homomorphism

$$A\otimes_{C^*} B \to M\otimes_{C^*} N$$

still denoted by $\rho\otimes\sigma$.

If ρ and σ are injective $*$-homomorphisms, then the extension $\rho\otimes\sigma$ is an injective $*$-homomorphism.

If ρ and σ are $*$-homomorphisms, then the extension $\rho\otimes\sigma$ is a $*$-homomorphism. ∎

In particular, if A is a C^*-subalgebra of M and B is a C^*-subalgebra of N, the $A\otimes_{C^*} B$ can be identified with the C^*-subalgebra $C^*(A\otimes B)$ of $M\otimes_{C^*} N$, that is we may consider

$$A\otimes_{C^*} B\subset M\otimes_{C^*} N.$$

Let A, B be C^*-algebras.

COROLLARY 1. *For every $x\in A\otimes B$ we have*

$$\|x\|_{C^*} = \sup\{\|(\rho\otimes\sigma)(x)\|;\ \rho\in\mathrm{Rep}(A),\ \sigma\in\mathrm{Rep}(B)\}, \tag{6}$$

$$\|x\|_{C^*} = \sup\{\|(\rho\otimes\sigma)(x)\|;\ \rho\in\mathrm{Irr}(A),\ \sigma\in\mathrm{Irr}(B)\}. \tag{7}$$

Proof. Use the definition of $\|\cdot\|_{C^*}$, the above theorem and Theorem 4.11. ∎

COROLLARY 2. *For every bounded linear functionals $\varphi\in A^*$, $\psi\in B^*$ the linear functional $\varphi\otimes\psi$ on $A\otimes B$ can be uniquely extended to a bounded linear functional on $A\otimes_{C^*} B$, still denoted by $\varphi\otimes\psi$ and we have*

$$\|\varphi\otimes\psi\|\leqslant 4\|\varphi\|\,\|\psi\|. \tag{8}$$

If, φ,ψ are positive forms on A, B respectively, then $\varphi\otimes\psi$ is a positive form on $A\otimes_{C^*} B$ and we have

$$\|\varphi\otimes\psi\| = \|\varphi\|\,\|\psi\|. \tag{9}$$

Proof. Let $\varphi\in A^*_+$, $\psi\in B^*_+$. By Theorem 4.5 we have $\varphi=\omega_{\xi_\varphi}\circ\pi_\varphi$ and $\psi=\omega_{\xi_\psi}\circ\pi_\psi$ where $\xi_\varphi\in H_\varphi$, $\|\xi_\varphi\|=\|\varphi\|^{1/2}$ and $\xi_\psi\in H_\psi$, $\|\xi_\psi\|=\|\psi\|^{1/2}$. Then

$$(\varphi\otimes\psi)(x) = (\omega_{\xi_\varphi\otimes\xi_\psi}\circ(\pi_\varphi\otimes\pi_\psi))(x); \quad x\in A\otimes B.$$

By the Theorem it follows that $\varphi\otimes\psi$ is $\|\cdot\|_{C^*}$-continuous, so (10) holds for all $x\in A\otimes_{C^*} B$. Using Corollary 2/4.5 we get

$$\|\varphi\otimes\psi\| = \|\xi_\varphi\otimes\xi_\psi\|^2 = \|\xi_\varphi\|^2\|\xi_\psi\|^2 = \|\varphi\|\,\|\psi\|.$$

Now, for arbitrary $\varphi\in A^*$, $\psi\in B^*$, the desired conclusion follows by using Corollary 1/4.15. ∎

The relation (8) will be improved in Proposition 4/8.11 and the assertion concerning the tensor product of positive forms will be extended in Proposition 5.4.

By Corollary 2, the vector space $A^* \otimes B^*$ can be identified with a vector subspace of $(A \otimes_{C^*} B)^*$.

COROLLARY 3. *For every $x \in A \otimes B$ we have*

$$\|x\|_{C^*} = \sup\{\theta(x^*x)^{1/2};\ \theta \in S(A \otimes_{C^*} B) \cap (A^* \otimes B^*)\}, \tag{11}$$

$$\|x\|_{C^*} = \sup\{\theta(x^*x)^{1/2};\ \theta \in Q(A \otimes_{C^*} B) \cap (A^* \otimes B^*)\}, \tag{12}$$

$$\|x\|_{C^*} = \sup\Big\{\frac{(\varphi \otimes \psi)(y^*x^*xy)^{1/2}}{(\varphi \otimes \psi)(y^*y)^{1/2}};\ \varphi \in P(A),\ \psi \in P(B),\ y \in A \otimes B,\ (\varphi \otimes \psi)(y^*y) \neq 0\Big\}. \tag{13}$$

Proof. Let $\{u_\iota\}_\iota$, $\{v_\kappa\}_\kappa$ be approximate units for A, B respectively. Using the subcrose property of $\|\cdot\|_{C^*}$, it is easy to check that $\{u_\iota \otimes v_\kappa\}_{\iota,\kappa}$ is an approximate unit for $A \otimes_{C^*} B$. Using this remark and Theorem 4.5 it follows that

$$(\varphi \otimes \psi)(y^*y)/(\varphi \otimes \psi)(y^*y) \in S(A \otimes_{C^*} B) \cap (A^* \otimes B^*)$$

for all $\varphi \in A^*_+$, $\psi \in B^*_+$ and all $y \in A \otimes B$ with $(\varphi \otimes \psi)(y^*y) \neq 0$. On the other hand, by Proposition 4.12, $\theta(x^*x)^{1/2} \leqslant \|x\|_{C^*}$ for all $\theta \in Q(A \otimes_{C^*} B)$. These prove the inequalities $\geqslant$ in (11), (12), (13).

Conversely, it is sufficient to prove the inequality $\leqslant$ in (13). If $\rho \in \mathrm{Irr}(A)$, $\sigma \in \mathrm{Irr}(B)$ and $\xi \in H_\rho$, $\eta \in H_\sigma$ with $\|\xi\| = 1$, $\|\eta\| = 1$, then $\varphi = \omega_\xi \circ \rho \in P(A)$, $\psi = \omega_\eta \circ \sigma \in P(B)$ and the vector subspace $\{(\rho \otimes \sigma)(y)(\xi \otimes \eta);\ y \in A \otimes B\}$ is dense in $H_\rho \mathbin{\overline{\otimes}} H_\sigma$. Hence

$$\begin{aligned}\|(\rho \otimes \sigma)(x)\| &= \sup\Big\{\frac{\|(\rho \otimes \sigma)(xy)(\xi \otimes \eta)\|}{\|(\rho \otimes \sigma)(y)(\xi \otimes \eta)\|};\ y \in A \otimes B,\ (\rho \otimes \sigma)(y)(\xi \otimes \eta) \neq 0\Big\} \\ &= \sup\Big\{\frac{(\varphi \otimes \psi)(y^*x^*xy)^{1/2}}{(\varphi \otimes \psi)(y^*y)^{1/2}};\ y \in A \otimes B,\ (\varphi \otimes \psi)(y^*y) \neq 0\Big\}.\end{aligned}$$

Thus, the desired inequality in (13) follows form (7). ∎

COROLLARY 4. *For every $x \in A \otimes_{C^*} B$, $x \neq 0$, there exist $\varphi \in P(A)$, and $\psi \in P(B)$ such that $(\varphi \otimes \psi)(x) \neq 0$.*

Proof. Suppose A, B act on Hilbert spaces H, K respectively. Since $x \neq 0$, there exist $\xi_1, \xi_2 \in H$, $\eta_1, \eta_2 \in K$ such that

$$(\omega_{\xi_1,\xi_2} \otimes \omega_{\eta_1,\eta_2})(x) = (x(\xi_1 \otimes \eta_1)|\xi_2 \otimes \eta_2) \neq 0.$$

Thus, there are $\varphi \in A^*$, $\psi \in B^*$ with $(\varphi \otimes \psi)(x) \neq 0$. Since every bounded linear form on a C^*-algebra is a linear combination of states (Corollary 1/4.15) and every state is a pointwise limit of convex combinations of pure states (4.7.(3)), the desired result follows. ∎

By Corollary 4, $A^* \otimes B^*$ separates the points of $A \otimes_{C^*} B$. It follows that $A^* \otimes B^*$ is $(A \otimes_{C^*} B)$-dense in $(A \otimes_{C^*} B)^*$.

COROLLARY 5. *Let A, B be commutative C^*-algebras with Gelfand spectra Ω_A, Ω_B respectively. Then the Gelfand spectrum of $A \otimes_{C^*} B$ is homeomorphic to $\Omega_A \times \Omega_B$.*

In other words

$$C_0(\Omega_A) \otimes_{C^*} C_0(\Omega_B) \textit{ is } *\textit{-isomorphic to } C_0(\Omega_A \times \Omega_B).$$

Proof. We identify A with $C_0(\Omega_A)$ and B with $C_0(\Omega_B)$. Then every element $x = \sum_{k=1}^{n} a_k \otimes b_k \in A \otimes B$ defines a function $x(\cdot\,,\cdot)$ on $\Omega_A \times \Omega_B$ by

$$x(s,t) = \sum_{k=1}^{n} a_k(s) b_k(t); \quad s \in \Omega_A,\ t \in \Omega_B$$

and the map $x \mapsto x(\cdot\,,\cdot)$ is a $*$-homomorphism of $A \otimes B$ into $C_0(\Omega_A \times \Omega_B)$. Using the Stone-Weierstrass theorem, it is easy to check that

$$\{x(\cdot\,,\cdot);\ x \in A \otimes B\} \text{ is a norm-dense } *\text{-subalgebra of } C_0(\Omega_A \times \Omega_B).$$

On the other hand, using formula (13) and the last remark in 4.9, for every $x \in A \otimes B$ we get

$$\|x\|_{C^*} = \sup\{|x(s,t)|;\ s \in \Omega_A,\ t \in \Omega_B\} = \|x(\cdot\,,\cdot)\|_{C_0(\Omega_A \times \Omega_B)}\}.$$

Hence the map $x \to x(\cdot\,,\cdot)$ extends to a $*$-isomorphism of $A \otimes_{C^*} B$ onto $C_0(\Omega_A \times \Omega_B)$. ∎

Note that usual associativity and distributivity properties are valid for the above defined C^*-tensor product and the direct product of C^*-algebras.

In general, given two C^*-algebras A, B, on the $*$-algebra $A \otimes B$ there exist several different C^*-norms. A C^*-algebra A is called *nuclear* if for every C^*-algebra B the only C^*-norm on $A \otimes B$ is $\|\cdot\|_{C^*}$. The variety of C^*-norms on $A \otimes B$, as well as the property of nuclearity will be analysed in another place. For other properties of the C^*-tensor product $A \otimes_{C^*} B$ see Corollary 1/5.3, Sections 6.10 and 8.8, and Proposition 4/8.11.

4.21. Notes. For the classical theory of positive forms and $*$-representations (4.1, 4.2, 4.5, 4.6, 4.7, 4.9, 4.11, 4.12, 4.14, 4.16) we refer to the fundamental contribution of I.M. Gelfand, M.A. Naĭmark [106] and I.E. Segal [282], to the articles [27], [107], [108], [114], [154], [210], [211], [212] and to the monographs [33], [78], [80], [213], [258]. In our exposition of these topics we used mainly the book of J. Dixmier [78].

The GNS construction for weights (4.3) and the study of majorization of weights (4.8) are due to F. Combes [56]. The main result in 4.4 belongs to N.Th. Varopoulos [335] (see also [353]). The characterization of positive functionals given in 4.6 has been noticed by H.F. Bohnenblust and S. Karlin [27]. They also proved Corollary 3 and Corollary 4/4.13, but in our exposition we followed the approach of G. Lumer [184] which contains also the results of Corollary 1 and Corollary 2/4.13. For the Kadison function representation we refer to [141], [58], [246], [247]. The result of Corollary 1/4.15 is due to Z. Takeda [312] and A. Grothendieck [119] (see 8.10). The characterization of increasing approximate units (Corollary 3/4.15 and Corollary 2/4.15) have been obtained by C.A. Akemann [5]. The main result in 4.17 is due to J. Glimm (cf. [78], 2.12.13) and the precise characterization of non-commutativity (Theorem 4.18) is due to I. Kaplansky (cf. [78], second edition, 2.12.21). The results contained in Corollary 1/4.18 appeared in [104], [214], [287] but the proof is that from [65] (see also [39]). The inequality 4.18.(6) is due to D.C. Taylor [322] and C.M. McGregor [65]. For the general results in 4.10 we used [49], [81], [83].

The C^*-tensor product (or the spatial tensor product) of two C^*-algebras A, B (4.20) has been introduced by T. Turumaru [333]. Subsequently, M. Takesaki [318] proved that $\|\cdot\|_{C^*}$ is the smallest C^*-norm on the algebraic tensor product $A \otimes B$ and A. Guichardet [120] considered also the greatest C^*-norm on $A \otimes B$. Moreover, M. Takesaki [318] introduced (under a different terminology) the notion of nuclear C^*-algebra and showed that every "type I" ([78], [112], [157]; in particular, commutative, or finite dimensional) C^*-algebra is nuclear (see [121]). An important contribution to the theory of tensor products is due to E.G. Effros [85] and E.C. Lance [171] and has been further developed in [44], [45], [46], [47], [88], [172], [329]. In our exposition we have used [346].

J.R. Ringrose [206] conjectured that a linear functional on a C^*-algebra A, which is bounded on each commutative $*$-subalgebra of A, is also bounded on the whole of A and proved this conjecture in several particular cases. The general case has been settled affirmatively by J. Cuntz [66] for a class of operator algebras strictly larger than the class of C^*-algebras.

R.V. Kadison [146] proved a remarkable tranzitivity theorem (see [113]; [78], 2.8; [274], 1.21.16) which entails in particular that every topologically irreducible $*$-representation π of a C^*-algebra A on a Hilbert space H is algebraically irreducible, i.e., $\pi(A)\xi = H$ for any non-zero $\xi \in H$. An approach to this theorem will be given in 7.22.

Further references: [3], [7], [53], [54], [55], [124], [148], [153], [282], [314].

Chapter 5

COMPLETELY POSITIVE LINEAR MAPPINGS

5.1. Let Φ be a linear mapping between $*$-algebras A, B. Then Φ is called *selfadjoint* if $\Phi(A_h) \subset B_h$ and is called *positive* if $\Phi(A^+) \subset B^+$. If $A^2 = A$ (e.g., if A is unital or if A is a C^*-algebra), then any positive linear mapping is selfadjoint since $A_h = A^+ - A^+$ in this case. Clearly, Φ is selfadjoint if and only if $\Phi(x^*) = \Phi(x)^*$ for all $x \in A$.

Let $n \in \mathbb{N}$. Recall that $M_n(A)$ is also a $*$-algebra $*$-isomorphic to the $*$-algebra $A \otimes M_n$ (2.12):

$$M_n(A) \ni [x_{ij}] \mapsto \sum_{i,j=1}^{n} x_{ij} \otimes e_{ij} \in A \otimes M_n$$

where $\{e_{ij}\}$ is the system of matrix units in M_n. Consequently we shall identify $M_n(A)$ and $A \otimes M_n$. Given a linear mapping $\Phi : A \to B$, we define $\Phi_n : M_n(A) \to M_n(B)$ by

$$\Phi_n([x_{ij}]) = [\Phi(x_{ij})]; \quad [x_{ij}] \in M_n(A).$$

Clearly, $\Phi_n = \Phi \otimes \mathrm{id}_{M_n}$, that is

$$\Phi_n\Big(\sum_{ij} x_{ij} \otimes e_{ij}\Big) = \sum_{ij} \Phi(x_{ij}) \otimes e_{ij}; \quad \sum_{ij} x_{ij} \otimes e_{ij} \in A \otimes M_n.$$

Then Φ is called *n-positive* if Φ_n is positive. The set of all n-positive linear mappings $\Phi : A \to B$ is denoted by $P_n(A, B)$. It is easy to see that $P_n(A, B) \subset P_{n-1}(A, B)$, but the converse is not generally true.

The linear mapping Φ is called *completely positive* if it is n-positive for all $n \in \mathbb{N}$. The set of all completely positive linear mappings $\Phi : A \to B$ is denoted by $CP(A, B)$.

Since any positive element of $M_n(A)$ is a sum of elements of the form $[x_i^* x_j]$, with $x_i, \ldots, x_n \in A$, (Proposition 2.12), it follows that $\Phi \in P_n(A, B)$ if and only if

$$(1) \qquad [\Phi(x_i^* x_j)] \in M_n(B)^+ \quad \text{for all } x_1, \ldots, x_n \in A.$$

Assume now that $B = B(H)$ for some Hilbert space H. Then $M_n(B) = B(H^{(n)})$ where $H^{(n)}$ is the Hilbert space direct sum of n copies of H (2.12). An element $Y \in B(H^{(n)})$ is positive if and only if $(Y\xi|\xi) \geqslant 0$, $\xi \in H^{(n)}$, and any

$\xi \in H^{(n)}$ has the form $\xi = [\xi_k]$ with $\xi_1, \dots, \xi_n \in H$. It follows that a linear mapping $\Phi : A \to B(H)$ is n-positive if and only if, for every $[x_{ij}] \in M_n(A)^+$ and every $\xi_1, \dots, \xi_n \in H$,

$$\sum_{ij} (\Phi(x_{ij})\xi_j | \xi_i) \geqslant 0 \tag{2}$$

or equivalently

$$\sum_{ij} (\Phi(x_i^* x_j)\xi_j | \xi_i) \geqslant 0 \quad \text{for all } x_1, \dots, x_n \in A,\ \xi_1, \dots, \xi_n \in H. \tag{3}$$

For any $x_1, \dots, x_n \in A$ and any $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ we have

$$\sum_{ij} \lambda_j \overline{\lambda}_i x_i^* x_j = \Big(\sum_k \lambda_k x_k\Big)^* \Big(\sum_k \lambda_k x_k\Big) \in A^+.$$

Using this it readily follows that every positive form on A is completely positive.

In what follows we shall consider only mappings between C^*-algebras. Recall that if A is a C^*-algebra then $M_n(A)$ is also a C^*-algebra (4.19).

5.2. Proposition. *Any positive linear mapping $\Phi : A \to B$ between C^*-algebras is bounded.*

Proof. For every positive form ψ on B, $\psi \circ \Phi$ is a positive, hence continuous (Corollary 4/4.5), form on A. Since any form on B is a linear combination of positive ones (Corollary 1/4.5), it follows that $\{\psi \circ \Phi;\ \psi \in B^*,\ \|\psi\| \leqslant 1\}$ is a family of bounded linear mappings $A \to \mathbb{C}$. Moreover,

$$|(\psi \circ \Phi)(x)| \leqslant \|\Phi(x)\| \quad \text{for all } \psi \in B^*,\ \|\psi\| \leqslant 1.$$

By the Banach-Steinhauss Theorem, there is $\mu > 0$ such that $\|\psi \circ \Phi\| \leqslant \mu$ for all $\psi \in B^*$, $\|\psi\| \leqslant 1$. Hence

$$\|\Phi(x)\| = \sup\{|\psi(\Phi(x))|;\ \psi \in B^*,\ \|\psi\| \leqslant 1\} \leqslant \mu\|x\|, \quad x \in A$$

and Φ is bounded. ∎

A general fact is contained in the last part of the proof: any weakly continuous linear mapping between Banach spaces is norm continuous.

5.3. The Stinespring dilation. For completely positive linear mappings into $B(H)$ there is an important extension of the GNS construction which is described below.

Theorem. *Let A be a C^*-algebra, H be a Hilbert space and $\Phi : A \to B(H)$ be a linear mapping. Then Φ is completely positive if and only if there exist a Hilbert space H, a $*$-representation $\pi : A \to B(K)$ and a bounded linear operator $V : H \to K$ such that:*

(1) $\Phi(x) = V^*\pi(x)V$ *for all* $x \in A$,

(2) $K =$ *the closed linear span of* $\pi(A)VH$,

(3) $\|V\| = \|\Phi\|^{1/2}$.

Moreover, conditions (1) *and* (2) *determine the triple* $\{\pi, V, K\}$ *uniquely up to a unitary equivalence.*

Proof. It is clear that (1) defines a linear mapping Φ of A into $B(H)$ which satisfies 5.1.(3), hence $\Phi \in CP(A, B(H))$.

Conversely, let $\Phi \in CP(A, B(H))$ and $\{u_\iota\}$ be an approximate unit for A. Consider the vector space tensor product $A \otimes H$ and, for $a_1, \dots, a_n, b_1, \dots, b_m \in A$, $\xi_1, \dots \xi_n, \eta_1, \dots \eta_m \in H$, define

$$\Big(\sum_{j=1}^{n} a_j \otimes \xi_j \,\Big|\, \sum_{i=1}^{m} b_i \otimes \eta_i\Big)_\Phi = \sum_{ij} (\Phi(b_i^* a_j)\xi_j | \eta_i)_H.$$

Since $\Phi \in CP(A, B(H)), (\cdot\,|\,\cdot)_\Phi$ is a pre-inner product on $A \otimes H$.

For each $x \in A$ define a linear operator $\pi_0(x)$ on $A \otimes H$ by

$$\pi_0(x)\Big(\sum_{k=1}^{n} a_k \otimes \xi_k\Big) = \sum_{k=1}^{n} x a_k \otimes \xi_k; \quad \sum_{k=1}^{n} a_k \otimes \xi_k \in A \otimes H.$$

On the other hand, for each $p = \sum_{k=1}^{n} a_k \otimes \xi_k \in A \otimes H$, define

$$\varphi_p(x) = \sum_{ij} (\Phi(a_i^* x a_j)\xi_j | \xi_i), \quad x \in A.$$

Since $\Phi \in CP(A, B(H))$, φ_p is a positive form on A. Using Theorem 4.5 and the continuity of Φ (Proposition 5.2), we get

$$\|\varphi_p\| = \lim_\iota \varphi_p(u_\iota) = \sum_{ij} (\Phi(a_i^* a_j)\xi_j | \xi_i)_H = (p|p)_H,$$

so that, for any $x \in A$,

(4) $$(\pi_0(x)p | \pi_0(x)p)_\Phi = \varphi_p(x^*x) \leqslant \|\varphi_p\|\, \|x^*x\| = \|x\|^2 (p|p)_\Phi.$$

This shows that the subspace $L = \{p \in A \otimes H;\, (p|p)_\Phi = 0\}$ is stable under $\pi_0(x)$, $x \in A$, so that every $\pi_0(x)$ can be factored to a linear operator, still denoted by $\pi_0(x)$, on the quotient space $(A \otimes H)/L$. Moreover, $(\cdot\,|\,\cdot)_\Phi$ induces an inner product on $(A\otimes H)/L$ and (4) shows that every $\pi_0(x)$ is bounded with respect to it on $(A\otimes H)/L$. In order to avoid notational complications we will denote by the same symbol an element of $A\otimes H$ and its canonical image in $(A\otimes H)/L$.

Let K be the Hilbert space completion of $(A\otimes H)/L$. Then each $\pi_0(x)$; $x\in A$, extends to a bounded linear operator $\pi(x)$ on K and, as easily verified, $\pi : A\to B(K)$ is a non-degenerated $*$-representation.

For every $\xi\in H$, $\omega_\xi\circ\Phi$ is a positive form on A, hence $\{u_\iota\}$ is convergent with respect to the pre-Hilbert structure defined on A by $\omega_\xi\circ\Phi$ (4.5). Since

$$\|u_\iota\otimes\xi - u_\kappa\otimes\xi\|_K^2 = (\Phi(u_\iota - u_\kappa)^*(u_\iota-u_\kappa)\xi|\xi)_H = \|u_\iota - u_\kappa\|^2_{\omega_\xi\circ\Phi}$$

it folows that $\{u_\iota\otimes\xi\}$ converges in K to some $V\xi\in K$. Since

$$\|u_\iota\otimes\xi\|_K^2 = (\Phi(u_\iota^* u_\iota)\xi|\xi)_H \leqslant \|\Phi\|\,\|\xi\|_H^2,$$

it follows that $\|V\xi\|_K\leqslant \|\Phi\|^{1/2}\|\xi\|_H$.

We thus obtain a bounded linear operator $V : H\to K$,

$$\|V\|\leqslant\|\Phi\|^{1/2}. \tag{5}$$

For $b\in A$, $\eta\in H$, $\xi\in H$ we have

$$(V^*(b\otimes\eta)|\xi)_H = (b\otimes\eta|V\xi)_K = \lim_\iota (b\otimes\eta|u_\iota\otimes\xi)_K = \lim_\iota(\Phi(u_\iota^* b)\eta|\xi)_H = (\Phi(b)\eta|\xi)_H$$

hence $V^*(b\otimes\eta)=\Phi(b)\eta$. For $x\in A$; $\xi\in H$, it follows that

$$V^*\pi(x)V\xi = \lim_\iota V^*\pi(x)(u_\iota\otimes\xi) = \lim_\iota V^*(xu_\iota\otimes\xi) = \lim_\iota \Phi(xu_\iota)\xi = \Phi(x)\xi.$$

This proves (1). Then (2) can be satisfied simply replacing K by the closed linear span of $\pi(A)VH$ and (3) follows from (1) and (5).

For the uniqueness assertion, remark that condition (1) entails

$$\Big\|\sum_k \pi(x_k)V\xi_k\Big\|_K^2 = \sum_{ij}(\Phi(x_i^* x_j)\xi_j|\xi_i)_H$$

for any $x_1,\dots,x_n\in A$, $\xi_1,\dots,\xi_n\in H$. Therefore, if $\{\pi',V',K'\}$ is another triple satisfying (1) and (2), then the mapping

$$\pi(A)VH \ni \sum_k \pi(x_k)V\xi_k \mapsto \sum_k \pi'(x_k)V'\xi_k \in \pi'(A)V'H$$

extends to a unitary operator $U : K\to K'$ such that

$$V' = UV;\quad \pi'(x) = U\pi(x)U^*,\quad x\in A. \quad ∎$$

We shall refer to the triple $\{\pi, V, K\}$ as to *the Stinespring dilation* of Φ. It will be also denoted by $\{\pi_\Phi, V_\Phi, K_\Phi\}$.

Since π is non-degenerated, $\{\pi(u_\iota)\}$ is strongly operator convergent to 1_K by Lemma 3/4.1. It follows that

$$\Phi(u_\iota) \xrightarrow{\text{so}} V^*V, \quad \Phi(u_\iota^* u_\iota) \xrightarrow{\text{so}} V^*V. \tag{6}$$

In particular, if A is unital,

$$\Phi(1) = V^*V, \quad \|\Phi\| = \|\Phi(1)\| \tag{7}$$

and if moreover $\Phi(1) = 1_H$, then V is an isometry, H can be identified with a subspace of K and π appears clearly as a dilation of Φ.

Also, (1) shows that Φ can be extended to a unique element, called the *canonical extension.* $\widetilde{\Phi} \in CP(\widetilde{A}, B(H))$ such that $\widetilde{\Phi}(1) = V^*V$. Moreover, it is easy to see that $A \otimes H$ is dense in $\widetilde{A} \otimes H$ with respect to $(\cdot\,|\,\cdot)_\Phi$, namely, for each $\xi \in H$, $\{u_\iota \otimes \xi\}_\iota$ converges to $1 \otimes \xi$. Thus,

$$V\xi = 1 \otimes \xi, \quad \xi \in H. \tag{8}$$

As in the case of positive forms (Corollary 1/4.5), for $\Phi, \Psi \in CP(A, B(H))$ we have $\Phi + \Psi \in CP(A, B(H))$ and

$$V_{\Phi+\Psi}^* V_{\Phi+\Psi} = V_\Phi^* V_\Phi + V_\Psi^* V_\Psi, \quad (\widetilde{\Phi + \Psi}) = \widetilde{\Phi} + \widetilde{\Psi}. \tag{9}$$

Corollary 1. *Let A, B, M, N be C^*-algebras. If $\Phi : A \to M$, $\Psi : B \to N$ are completely positive linear mappings, then the linear mapping*

$$\Phi \otimes \Psi : A \otimes B \to M \otimes N$$

can be uniquely extended to a completely positive linear mapping

$$A \otimes_{C^*} B \to M \otimes_{C^*} N,$$

still denoted by $\Phi \otimes \Psi$, and we have

$$\|\Phi \otimes \Psi\| = \|\Phi\|\,\|\Psi\|.$$

Proof. By 4.11 and 4.20 we may suppose, without restricting the generality, that $M = B(H)$, $N = B(K)$ for some Hilbert spaces H, K. Let $\{\rho, U, H'\}$, $\{\sigma, V, K'\}$ be the Stinespring dilations of Φ, Ψ respectively. Then

$$(\Phi \otimes \Psi)(x) = (U \mathbin{\overline{\otimes}} V)^*(\rho \otimes \sigma)(x)(U \mathbin{\overline{\otimes}} V); \quad x \in A \otimes B. \tag{10}$$

Using Theorem 4.20, we infer that

$$\|(\Phi \otimes \Psi)(x)\| \leqslant \|U\|^2 \|V\|^2 \|(\rho \otimes \sigma)(x)\| \leqslant \|\Phi\|\,\|\Psi\|\,\|x\|_{C^*}; \quad x \in A \otimes B,$$

hence $\Phi \otimes \Psi$ is $\|\cdot\|_{C^*}$-bounded on $A \otimes B$. Therefore $\Phi \otimes \Psi$ extends to a bounded linear mapping on $A \otimes_{C^*} B$ and equation (10) is valid for all $x \in A \otimes_{C^*} B$. By the theorem it follows that

$$\Phi \otimes \Psi : A \otimes_{C^*} B \to M \otimes_{C^*} N$$

is completely positive. It is easy to check that $\{\rho \otimes \sigma, U \overline{\otimes} V, H' \overline{\otimes} K'\}$ is the Stinespring dilation of $\Phi \otimes \Psi$. Using (3) we get

$$\|\Phi \otimes \Psi\| = \|U \overline{\otimes} V\|^2 = \|U\|^2 \|V\|^2 = \|\Phi\| \, \|\Psi\|. \quad \blacksquare$$

In particular, if $\Phi \in CP(A, B)$, then $\Phi_n \in CP(M_n(A), M_n(B))$ and

$$\|\Phi_n\| = \|\Phi\|; \quad n \in \mathbb{N}. \tag{11}$$

For a positive form φ on A we have $|\varphi(x)|^2 \leqslant \|\varphi\| \varphi(x^* x)$, $(x \in A)$. For $\Phi \in CP(A, B)$ the analogous result is

$$[\Phi(x_i)^* \Phi(x_j)] \leqslant \|\Phi\| [\Phi(x_i^* x_j)]; \quad x_1, \ldots, x_n \in A, \, n \in \mathbb{N}, \tag{12}$$

where the inequality is understood in $M_n(B)$. Indeed, we may suppose $B = B(H)$. For $\xi_1, \ldots, \xi_n \in H$ we have

$$\begin{aligned}
([\Phi(x_i)^* \Phi(x_j)][\xi_k] | [\xi_k])_{H^{(n)}} &= \sum_{ij} (\Phi(x_i)^* \Phi(x_j) \xi_j | \xi_i)_H = \Big\| \sum_k \Phi(x_k) \xi_k \Big\|_H^2 \\
&\leqslant \|V_\Phi\|^2 \Big\| \sum_k \pi_\Phi(x_k) V_\Phi \xi_k \Big\|_K^2 \\
&= \|\Phi\| \sum_{ij} (\Phi(x_i^* x_j) \xi_j | \xi_i)_H \\
&= \|\Phi\| ([\Phi(x_i^* x_j)][\xi_k] | [\xi_k])_{H^{(n)}}.
\end{aligned}$$

In particular, if $\Phi \in CP(A, B)$, then

$$\Phi(x)^* \Phi(x) \leqslant \|\Phi\| \Phi(x^* x), \quad x \in A. \tag{13}$$

Every positive form on a C^*-subalgebra B of A has a norm preserving extension to a positive form on A (Proposition 4.16). The analogous result is

Corollary 2. *If B is a C^*-subalgebra of A and $\Psi \in CP(B, B(H))$, then there exists $\Phi \in CP(A, B(H))$ such that $\Phi | B = \Psi$ and $\|\Phi\| = \|\Psi\|$.*

Proof. Let $\{\pi_\Psi, V_\Psi, K_\Psi\}$ be the Stinespring dilation of Ψ. By Corollary 4.16 there is a Hilbert space K, an isometric linear operator $W : K_\Psi \to K$ and a $*$-representation $\pi : A \to B(K)$ such that

$$\pi_\Psi(y) = W^* \pi(y) W, \quad y \in B.$$

Then the desired extension $\Phi \in CP(A, B(H))$ of Ψ is given by

$$\Phi(x) = V_\Psi^* W^* \pi(x) W V_\Psi; \quad x \in A. \quad \blacksquare$$

5.4. The Arveson theorem. Let A be a unital C^*-algebra and S be a selfadjoint vector subspace of A which contains the unit of A. A linear functional on S which is positive on $S \cap A^+$ can be still extended to a positive form on A, the proof being the same as for Proposition 4.16. There is an important generalization of this result for completely positive linear mappings into $B(H)$. A slight extension of the definitions is necessary.

Let $\Psi : S \to B(H)$ be a linear mapping. Then Ψ is called *positive* if $\Psi(S \cap A^+) \subset B(H)^+$. Note that in this case Ψ is automatically selfadjoint, i.e. $\Psi(a)^* = \Psi(a)$ for $a^* = a \in S$. Indeed, both $\|a\| \cdot 1 - a$ and $\|a\| \cdot 1$ belong to $S \cap A^+$ so $\Psi(a) = \Psi(\|a\| \cdot 1) - \Psi(\|a\| \cdot 1 - a)$ is a selfadjoint operator.

For each $n \in \mathbb{N}$, $M_n(S) = S \otimes M_n$ is a unital selfadjoint vector subspace of $M_n(A) = A \otimes M_n$ and $\Psi_n = \Psi \otimes \mathrm{id}_{M_n}$ is a linear mapping of $M_n(S)$ into $M_n(B(H)) = B(H^{(n)})$. Then Ψ is called *completely positive* if every Ψ_n is positive, that is if

$$\sum_{ij} (\Psi(x_{ij})\xi_j | \xi_i) \geqslant 0, \quad \xi_1, \ldots, \xi_n \in H$$

whenever $[x_{ij}] \in M_n(S)$ is positive. The set of all completely positive linear mappings $S \to B(H)$ is denoted by $CP(S, B(H))$.

Theorem. *Let S be a unital selfadjoint subspace of a unital C^*-algebra A. For every $\Psi_0 \in CP(S, B(H))$ there exists $\Phi_0 \in CP(A, B(H))$ such that $\Phi_0 | S = \Psi_0$ and $\|\Phi_0\| = \|\Psi_0\|$.*

Proof. Let $\Omega = CP(A, B(H))$ and let $B_{\mathbb{C}}(\Omega)$ denote the vector space of all complex functions on Ω. The subset $B_{\mathbb{R}}(\Omega)$ of all real functions is an ordered real vector space under pointwise ordering.

Consider the subset $Y_{\mathbb{C}}$ of $B_{\mathbb{C}}(\Omega)$ consisting of all functions g of the form

$$(1) \qquad g(\Phi) = \sum_{ij} (\Phi(x_{ij})\xi_j | \xi_i); \quad \Phi \in \Omega$$

with $n \in \mathbb{N}$, $[x_{ij}] \in M_n(S)$ and $\xi_1, \ldots, \xi_n \in H$.

Consider also the subset $X_{\mathbb{C}}$ of $B_{\mathbb{C}}(\Omega)$ consisting of all functions f such that there exists $g \in Y_{\mathbb{C}}$ with

$$(2) \qquad |f(\Phi)| \leqslant g(\Phi); \quad \Phi \in \Omega.$$

Then $Y_{\mathbb{C}}$ (and consequently $X_{\mathbb{C}}$) is a vector subspace of $B_{\mathbb{C}}(\Omega)$. Evidently, $\lambda g \in Y_{\mathbb{C}}$ whenever $g \in Y_{\mathbb{C}}$, $\lambda \in \mathbb{C}$. Let $g' \in Y_{\mathbb{C}}$ and $g'' \in Y_{\mathbb{C}}$ be defined as in (1) by

n', $[x'_{ij}]$, $\xi'_1, \dots, \xi'_{n'}$ and n'', $[x''_{ij}]$, $\xi''_1, \dots, \xi''_{n''}$ respectively. Then $g = g' + g'' \in Y_{\mathbb{C}}$, g being defined as in (1) by $n = n' + n''$ and

$$x_{ij} = \begin{cases} x'_{ij} & \text{if } 1 \leqslant i,\, j \leqslant n' \\ x''_{i-n',j-n'} & \text{if } n'+1 \leqslant i,\, j \leqslant n'+n'' \\ 0 & \text{otherwise;} \end{cases}$$

(3)

$$\xi_k = \begin{cases} \xi'_k & \text{if } 1 \leqslant k \leqslant n' \\ \xi''_{k-n} & \text{if } n'+1 \leqslant k \leqslant n'|n''. \end{cases}$$

Let $Y_{\mathbb{R}} = Y_{\mathbb{C}} \cap B_{\mathbb{R}}(\Omega)$, $X_{\mathbb{R}} = X_{\mathbb{C}} \cap B_{\mathbb{R}}(\Omega)$. Then $X_{\mathbb{R}}$ is an ordered real vector space, $Y_{\mathbb{R}}$ is a subspace of $X_{\mathbb{R}}$ and, clearly

$$Y_{\mathbb{C}} = Y_{\mathbb{R}} + \mathrm{i}Y_{\mathbb{R}}, \quad X_{\mathbb{C}} = X_{\mathbb{R}} + \mathrm{i}X_{\mathbb{R}}.$$

Now, for $g \in Y_{\mathbb{C}}$ defined as in (1), put

(4) $$\psi_{\mathbb{C}}(g) = \sum_{ij} (\Psi_0(x_{ij})\xi_j|\xi_i)$$

where $\Psi_0 \in CP(S, B(H))$ is the given map.

We claim that $\psi_{\mathbb{R}} = \psi_{\mathbb{C}}|Y_{\mathbb{R}}$ is a well defined positive $\mathbb{R}$-linear functional on Y. To prove this it is sufficient to show that

(5) $$g \in Y_{\mathbb{R}},\ g \geqslant 0 \Rightarrow \psi_{\mathbb{R}}(g) \geqslant 0,$$

since then $g = 0 \Rightarrow \psi_{\mathbb{R}}(g) = 0$, i.e. $\psi_{\mathbb{R}}$ is well defined and the linearity follows using (3).

Let $g \in Y_{\mathbb{R}}$ be as in (1) and let $\zeta_1, \dots, \zeta_m$, $(m \leqslant n)$ be an orthonormal basis of the vector subspace of H spanned by $\xi_1, \dots, \xi_n$. Write $\xi_i = \sum_{h=1}^{m} \lambda_{ih}\zeta_h$, $(1 \leqslant i \leqslant n)$. Then an elementary computation shows that, for some $[z_{hk}] \in M_m(S)$,

$$g(\Phi) = \sum_{hk} \Phi(z_{hk}\zeta_k|\zeta_h), \qquad \text{and} \qquad \psi_{\mathbb{C}}(g) = \sum_{hk} (\Psi_0(z_{hk})\zeta_k|\zeta_h).$$

Thus, in proving (5) we may and shall assume that the vectors $\xi_1, \dots, \xi_n$ appearing in (1) and (4) form an orthonormal system in H. In this case we show that the matrix $[x_{ij}] \in M_n(S)$ appearing in (1) and (4) is positive whenever $g \geqslant 0$. This obviously entails $\psi_{\mathbb{R}}(g) \geqslant 0$ by the complete positivity of Ψ_0.

So, let $A \subset B(K)$ for some Hilbert space K and let $\eta_1, \dots, \eta_n \in K$. Since $\xi_1, \dots, \xi_n$ are linearly independent, there is a unique bounded linear operator $V : H \to K$ such that $V\xi_k = \eta_k$, $(1 \leqslant k \leqslant n)$, and $V = 0$ on the orthogonal complement of the vector subspace spanned by $\xi_1, \dots, \xi_n$ in H. Then the mapping $\Phi_V : A \to B(H)$ defined by

$$\Phi_V(x) = V^*xV; \quad x \in A \subset B(K),$$

belongs to $CP(A, B(H))$ (Theorem 5.3), i.e. $\Phi_v \in \Omega$. Since $g \geqslant 0$ we get

$$([x_{ij}][\eta_k]|[\eta_k])_{K^{(n)}} = \sum_{ij}(x_{ij}\eta_j|\eta_i)_K = \sum_{ij}(\Phi_V(x_{ij})\xi_j|\xi_i)_H = g(\Phi_V) \geqslant 0.$$

This shows that $[x_{ij}]$ is positive and hence proves (5).

Therefore $\psi_{\mathbb{R}}$ is indeed a positive $\mathbb{R}$-linear functional on $Y_{\mathbb{R}}$. It is clear that $X_{\mathbb{R}}$ and $Y_{\mathbb{R}}$ satisfy the condition of Proposition 3/4.10 so that $\psi_{\mathbb{R}}$ has an extension to a positive $\mathbb{R}$-linear functional denoted by $\varphi_{\mathbb{R}}$ on $X_{\mathbb{R}}$. Furthermore, $\varphi_{\mathbb{R}}$ can be uniquely extended to a $\mathbb{C}$-linear functional denoted by $\varphi_{\mathbb{C}}$ on $X_{\mathbb{C}}$ and $\varphi_{\mathbb{C}}|Y_{\mathbb{C}} = \psi_{\mathbb{C}}$. Note that whenever $f \in X_{\mathbb{C}}$ and $g \in Y_{\mathbb{R}}$ are related by (2) we have

$$|\varphi_{\mathbb{C}}(f)| \leqslant 2\psi_{\mathbb{R}}(g). \tag{6}$$

We now want to define the required extension Φ_0 of Ψ_0. To this end, for $a \in A$ and $\xi, \eta \in H$ we put

$$f_{a;\xi,\eta}(\Phi) = (\Phi(a)\xi|\eta); \quad \Phi \in \Omega.$$

It is easy to see that $|f_{a;\,\xi,\eta}(\Phi)| \leqslant g_{a;\xi,\eta}(\Phi)$, $(\Phi \in \Omega)$, with $g_{a;\,\xi,\eta} \in Y_{\mathbb{R}}$ defined as in (1) by $n = 4$ and $x_{ij} = \delta_{ij}\|a\|/2$, $\xi_k = \xi + \mathrm{i}^k\eta$, $(1 \leqslant i, j, k \leqslant 4$; δ_{ij} = Delta Kronecker; $\mathrm{i}^2 = -1)$. Hence $f_{a;\xi,\eta} \in X_{\mathbb{C}}$ and, owing to (6)

$$|\varphi_C(f_{a;\,\xi,\eta})| \leqslant 2\psi_R(g_{a;\,\xi,\eta}) \leqslant 2\|\Psi_0\|\,\|a\|(\|\xi\|^2 + \|\eta\|^2). \tag{7}$$

For fixed $a \in A$, the assignment $H \times H \ni (\xi, \eta) \mapsto \varphi_{\mathbb{C}}(f_{a;\,\xi,\eta}) \in \mathbb{C}$ is linear in ξ and conjugate linear in η, so that (7) entails

$$|\varphi_{\mathbb{C}}(f_{a;\,\xi,\eta})| = |\varphi_{\mathbb{C}}(f_{a;\,s\xi,t\eta})| \leqslant 4\|\Psi_0\|\,\|a\|\,\|\xi\|\,\|\eta\|; \quad \big(t^{-1} = s = (\|\eta\|/\|\xi\|)^{1/2}\big).$$

It follows that there exists a unique $\Phi_0(a) \in B(H)$ with

$$(\Phi_0(a)\xi|\eta) = \varphi_{\mathbb{C}}(f_{a;\,\xi,\eta}); \quad \xi, \eta \in H. \tag{8}$$

Evidently, the mapping $\Phi_0 : A \to B(H)$ is linear. If $a \in S$, then $f_{a;\,\xi,\eta} \in Y_{\mathbb{C}}$ so that

$$(\Phi_0(a)\xi|\eta) = \varphi_{\mathbb{C}}(f_{a;\,\xi,\eta}) = \psi_{\mathbb{C}}(f_{a;\,\xi,\eta}) = (\Psi_0(a)\xi|\eta); \quad \xi, \eta \in H$$

and hence $\Phi_0(a) = \Psi_0(a)$. To see that Φ_0 is completely positive, let $[a_{ij}] \in M_n(A)$ be positive and let $\xi_1, \ldots, \xi_n \in H$. Then

$$\Big(\sum_{ij} f_{a_{ij};\,\xi_j,\xi_i}\Big)(\Phi) = \sum_{ij}(\Phi(a_{ij})\xi_j|\xi_i) \geqslant 0; \quad \Phi \in \Omega,$$

hence

$$\sum_{ij}(\Phi_0(a_{ij})\xi_j|\xi_i) = \sum_{ij}\varphi_{\mathbb{C}}(f_{a_{ij};\,\xi_j,\xi_i}) = \varphi_{\mathbb{R}}\Big(\sum_{ij} f_{a_{ij};\,\xi_j,\xi_i}\Big) \geqslant 0$$

by the positivity of $\varphi_{\mathbb{R}}$.

Finally, $\|\Phi_0\| = \Phi_0(1) = \Psi_0(1) \leqslant \|\Psi_0\| \leqslant \|\Phi_0\|$ so that Ψ_0 is automatically bounded and $\|\Phi_0\| = \|\Psi_0\|$. ∎

Remark that the above proof yields a one-to-one correspondence between $CP(S, B(H))$ and the positive linear functionals on $Y_{\mathbb{R}}$.

5.5. We have seen that any positive form is completely positive. Moreover,

PROPOSITION. *Let A, B be C^*-algebras. If B is commutative, then any positive linear mapping $\Phi : A \to B$ is completely positive.*

Proof. By Gelfand representation we may assume $B = C_0(\Omega)$ for some locally compact Hausdorff space Ω. Let $x_1, \dots, x_n \in A$ and $\lambda, \dots, \lambda_n \in \mathbb{C}$. Then $\sum_{ij} \overline{\lambda}_i \lambda_j x_i^* x_j \in A^+$ and the positivity of Φ entails $\sum_{ij} \overline{\lambda}_i \lambda_j \Phi(x_i^* x_j) \geqslant 0$, that is

$$\sum_{ij} \overline{\lambda}_i \lambda_j \Phi(x_i^* x_j)(t) \geqslant 0; \quad t \in \Omega.$$

This means that, for every $t \in \Omega$, the matrix $[\Phi(x_i^* x_j)(t)] \in M_n$ is positive. By 2.5.(iii) it follows that the element

$$[\Phi(x_i x_j)] \in C_0(\Omega, M_n) = M_n(C_0(\Omega)) = M_n(B)$$

is positive. This proves the proposition. ∎

5.6. On the other hand,

PROPOSITION. *Let A, B be C^*-algebras. If A is commutative, then any positive linear mapping $\Phi : A \to B$ is completely positive.*

Proof. Let $A = C_0(\Omega)$ for some locally compact Hausdorff space Ω and let $B \subset B(H)$ for some Hilbert space H.

Consider $\xi_1, \dots, \xi_n \in H$. Using the Riesz-Kakutani theorem we get a bounded positive Borel measure μ on Ω such that

$$\int_\Omega x(t) \,\mathrm{d}\mu(t) = \sum_k (\Phi(x)\xi_k | \xi_k); \quad x \in A.$$

Using the classical Randon-Nikodym theorem, we obtain μ-integrable functions d_{ij} on Ω such that

$$\int_\Omega x(t) d_{ij}(t) \,\mathrm{d}\mu(t) = (\Phi(x)\xi_j | \xi_i); \quad x \in A,\ 1 \leqslant i, j \leqslant n. \tag{1}$$

For any $x \in A$, $x \geqslant 0$ and any $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ we have

$$\begin{aligned} \int_\Omega x(t) \Big(\sum_{ij} \overline{\lambda}_i \lambda_j d_{ij}(t) \Big) \mathrm{d}\mu(t) &= \sum_{ij} \overline{\lambda}_i \lambda_j (\Phi(x)\xi_j | \xi_i) \\ &= \Big(\Phi(x) \Big(\sum_k \lambda_k \xi_k \Big) \Big| \Big(\sum_k \lambda_k \xi_k \Big) \Big) \geqslant 0. \end{aligned}$$

By a routine measure-theoretic argument we infer that there is a μ-negligible subset N of Ω such that

$$(2) \qquad \sum_{ij} \overline{\lambda}_i \lambda_j d_{ij}(t) \geqslant 0 \quad \text{for all } \lambda_1, \dots \lambda_n \in \mathbb{C} \text{ and all } t \in \Omega \setminus N.$$

Let $x_1, \dots, x_n \in A$. Owing to (1) and (2) we obtain

$$\sum_{ij} (\Phi(x_i^* x_j)\xi_j | \xi_i) = \int_\Omega \Big(\sum_{ij} \overline{x_i(t)} x_j(t) d_{ij}(t) \Big) \mathrm{d}\mu(t) \geqslant 0.$$

Therefore $[\Phi(x_i^* x_j)] \in M_n(B)^+$ for all $x_1, \dots, x_n \in A$ and $\Phi \in CP(A, B)$. ∎

5.7. By the last two sections we have $P_1(A, B) = CP(A, B)$ whenever either A or B is commutative. Conversely,

PROPOSITION. *Let A, B be C^*-algebras. If $P_1(A, B) = P_2(A, B)$, then either A or B is commutative.*

Proof. Let $A \subset B(H)$, $B \subset B(K)$ for some Hilbert spaces H, K. Assume that neither A nor B is commutative. Since A is not commutative, there is $u \in A$, $u \neq 0$, $u^2 = 0$, by Theorem 4.18. Let $\zeta \in H$, ζ orthogonal to $\operatorname{Ker} u$, $\xi_1 = u\zeta \neq 0$ and $a_1 = u^* u$. Then $a_1 \in A^+$, $a_1 \xi_1 = 0$ and $a_1 u^* \xi_1 \neq 0$. Since u^* is a linear combination of positive elements, there is $a_2 \in A^+$ with $a_1 a_2 \xi_1 = \xi_2 \neq 0$. Thus,

$$(1) \qquad a_1, a_2 \in A^+,\ \xi_1, \xi_2 \in H \quad \text{and} \quad a_1 \xi_1 = 0,\ a_1 a_2 \xi_1 = \xi_2 \neq 0.$$

Similarly we get

$$(2) \qquad b_1, b_2 \in B^+,\ \eta_1, \eta_2 \in K \quad \text{and} \quad b_2 \eta_1 = 0,\ b_2 b_1 \eta_1 = \eta_2 \neq 0.$$

Define the mappings $\Phi : A \to M_2$, $\Psi : M_2 \to B$ by

$$\Phi(x) = [(x\xi_h | \xi_k)] \in M_2; \quad x \in A,$$
$$\Psi([\lambda_{hk}]) = \sum_{hk} \lambda_{hk} b_h b_k \in B; \quad [\lambda_{hk}] \in M_2.$$

It is clear that Φ and Ψ are positive linear mappings, hence

$$\Psi \circ \Phi \in P_1(A, B).$$

The element $[a_i a_j] \in M_2(A)$ is positive. Put $[y_{ij}] = (\Psi \circ \Phi)_2([a_i a_j]) \in M_2(B)$. Then

$$y_{ij} = \sum_{nk} (a_i a_j \xi_h | \xi_k) b_h b_k; \quad i, j \in \{1, 2\},$$

and an easy computation based on (1) shows that

$$(3) \qquad y_{11} = \|a_1 \xi_2\|^2 b_2^2, \quad y_{12} = \|\xi_2\|^2 b_1 b_2 + \alpha\, b_2^2, \quad y_{21} = \|\xi_2\|^2 b_1 b_2 + \overline{\alpha}\, b_2^2$$

where $\alpha = (a_2\xi_2|a_1\xi_2)$. Let $\varepsilon > 0$, $\zeta_1 = \eta_1 \in K$, $\zeta_2 = -\varepsilon\eta_2 \in K$. Then, by (3) and (2),

$$\begin{aligned}([y_{ij}][\zeta_k]|[\zeta_k]) &= (y_{11}\zeta_1|\zeta_1) + (y_{12}\zeta_2|\zeta_1) + (y_{21}\zeta_1|\zeta_2) + (y_{22}\zeta_2|\zeta_2)\\ &= 0 - \varepsilon\|\xi_2\|^2\|\eta_2\|^2 - \varepsilon\|\xi_2\|^2\|\eta_2\|^2 + \varepsilon^2(y_{22}\eta_2|\eta_2)\\ &= -2\varepsilon\|\xi_2\|^2\|\eta_2\|^2 + \varepsilon^2(y_{22}\eta_2|\eta_2).\end{aligned}$$

For sufficiently small $\varepsilon > 0$, the result is not positive, so that

$$\Psi \circ \Phi \notin P_2(A, B). \quad ∎$$

5.8. The Kadison inequality. Combining Proposition 5.6 with the Stinespring dilation theorem we obtain an important consequence:

PROPOSITION. *Let Φ be a positive linear mapping between C^*-algebras A and B. For every normal element $x \in A$ we have*

$$\Phi(x)^*\Phi(x) \leqslant \|\Phi\|\Phi(x^*x). \tag{1}$$

Proof. The restriction of Φ to the commutative C^*-algebra C generated by x is a positive, and hence a completely positive (5.6), linear mapping, so that the result follows by 5.3.(13). ∎

Remark that (1) holds with $\|\Phi\|$ replaced by $\|\Phi|C\|$.

In particular, if $a \in A$ is selfadjoint, then

$$\Phi(a)^2 \leqslant \|\Phi\|\Phi(a^2). \tag{2}$$

5.9. PROPOSITION. *Let A, B be C^*-algebras and $\Phi \in P_n(A, B)$. For every $x_1, \dots, x_{n-1}$, $y \in A$ and every $\varepsilon > 0$ we have, in $M_{n-1}(B)$:*

$$[\Phi(x_i^*y)(\Phi(y^*y) + \varepsilon)^{-1}\Phi(y^*x_j)] \leqslant [\Phi(x_i^*x_j)], \tag{1}$$

$$[\Phi(x_i^*)\Phi(y^*x_j)] \leqslant \|\Phi(y^*y)\|\,[\Phi(x_i^*x_j)], \tag{2}$$

$$[\Phi(x_i)^*\Phi(x_j)] \leqslant \|\Phi\|\,[\Phi(x_i^*x_j)]. \tag{3}$$

Proof. Let $B \subset B(H)$ for some Hilbert space H. Let $\xi_1, \dots, \xi_{n-1} \in H$, $\xi_n = -(\Phi(y^*y) + \varepsilon)^{-1} \sum\limits_{k=1}^{n-1} \Phi(y^*x_k)\xi_k \in H$ and $x_n = y$. Since Φ is n-positive, we have

$$\sum_{i,j=1}^{n} (\Phi(x_i^*x_j)\xi_j|\xi_i) \geqslant 0. \tag{4}$$

Substituing in (4) the definitions of ξ_n, x_n and rearranging the terms, one obtains the first inequality below, the others being clear:

$$\begin{aligned}
&\sum_{i,j=1}^{n-1} \big(\Phi(x_i^* x_j)\xi_j|\xi_i\big) \\
&\quad\geqslant \sum_{i,j=1}^{n-1} \big(\Phi(x_i^* y)\left\{2(\Phi(y^*y)+\varepsilon)^{-1} - (\Phi(y^*y)+\varepsilon)^{-2}\Phi(y^*y)\right\}\Phi(y^*x_j)\xi_j|\xi_i\big) \\
&\quad= \sum_{i,j=1}^{n-1} \big(\Phi(x_i^* y)\left\{2(\Phi(y^*y)+\varepsilon)^{-1} + \varepsilon(\Phi(y^*y)+\varepsilon)^{-2}\right\}\Phi(y^*x_j)\xi_j|\xi_i\big) \\
&\quad\geqslant \sum_{i,j=1}^{n-1} \big(\Phi(x_i^* y)(\Phi(y^*y)+\varepsilon)^{-1}\Phi(y^*x_j)\xi_j|\xi_i\big).
\end{aligned}$$

This proves (1). Since $(\Phi(y^*y)+\varepsilon)^{-1} \geqslant \|\Phi(y^*y)+\varepsilon\|^{-1}$, (2) follows from (1) letting $\varepsilon \to 0$.

Let $\{u_\iota\}_{\iota\in I}$ be an approximate unit for A. Then (3) follows from (2) replacing y by u_ι and taking the limit over $\iota \in I$. ∎

In particular, if $\Phi \in P_2(A,B)$, then for every $x \in A$

$$\Phi(x)^*\Phi(x) \leqslant \|\Phi\| \cdot \Phi(x^*x). \tag{5}$$

Corollary. *Let A, B be C^*-algebras and $\Phi : A \to B$ be an isometric linear isomorphism such that Φ and Φ^{-1} are both 2-positive. Then Φ is a $*$-isomorphism.*

Proof. Indeed, using (5) for Φ and Φ^{-1}, we get

$$\Phi(x^*x) = \Phi(x)^*\Phi(x)\ ; \qquad x \in A,$$

and then by polarization (2.8.(1)) we obtain the multiplicativity of Φ. ∎

Owing to (2) we see that a linear isomorphism Φ between unital C^*-algebras such that $\Phi(1) = 1$ and Φ, Φ^{-1} are both 2-positive is automatically isometric (see also Proposition 6.4), hence a $*$-isomorphism.

A linear isomorphism Φ between C^*-algebras is called a (*complete*) *order isomorphism* if both Φ and Φ^{-1} are (completely) positive. Thus in particular, any isometric (or unit preserving) complete order isomorphism is a $*$-isomorphism.

5.10. The linear mappings $\Phi : A \to B$ which satisfy

$$\Phi(x)^*\Phi(x) \leqslant \Phi(x^*x); \quad x \in A,$$

have received special attention and were called *Schwartz maps*. Clearly, every Schwartz map is positive and hence selfadjoint. We have seen that any $\Phi \in P_2(A,B)$, $\|\Phi\| \leqslant 1$, is a Schwartz map.

PROPOSITION. *Let A, B be C^*-algebras and $\Phi : A \to B$ be a Schwartz map. For $a \in A$ the following conditions are equivalent:*

(i) $\Phi(a)^*\Phi(a) = \Phi(a^*a)$;

(ii) $\Phi(x^*a) = \Phi(x)^*\Phi(a)$ *and* $\Phi(a^*x) = \Phi(a)^*\Phi(x)$ *for all* $x \in A$.

Proof. Clearly (ii) $\Rightarrow$ (i). Conversely assume that (1) holds and let $x \in A$, $t \in \mathbb{R}$. Since Φ is a Schwartz map we have

$$\begin{aligned} t(\Phi(a)^*\Phi(x) + \Phi(x)^*\Phi(a)) &= \Phi(ta+x)^*\Phi(ta+x) - t^*\Phi(a)^*\Phi(a) - \Phi(x)^*\Phi(x) \\ &\leqslant \Phi((ta+x)^*(ta+x)) - t^2\Phi(a^*a) - \Phi(x)^*\Phi(x) \\ &= t\Phi(a^*x + x^*a) + (\Phi(x^*x) - \Phi(x)^*\Phi(x)). \end{aligned}$$

Dividing this inequality by $t \gtrless 0$ and letting $|t| \to \infty$ we get

$$\Phi(a)^*\Phi(x) + \Phi(x)^*\Phi(a) = \Phi(a^*x) + \Phi(x^*a). \tag{1}$$

Replacing here a by $-\mathrm{i}a$ and then multiplying by i we obtain

$$\Phi(a)^*\Phi(x) - \Phi(x)^*\Phi(a) = \Phi(a^*x) - \Phi(x^*a). \tag{2}$$

Then (ii) follows from (1) and (2). ∎

The set $M(\Phi)$ of all $a \in A$ satisfying (i) is a closed subalgebra (not necessarily selfadjoint) of A and the restriction of Φ to $M(\Phi)$ is an algebra homomorphism. $M(\Phi)$ is sometimes called "the multiplicative domain" of Φ.

Given a C^*-subalgebra B of a C^*-algebra A, a linear mapping $\Phi : A \to B$ is called a *linear projection* of A onto B if $\Phi(A) = B$ and $\Phi \circ \Phi = \Phi$. Then clearly $\Phi(b) = b$ for all $b \in B$.

COROLLARY. *Let $\Phi : A \to B$ be a linear projection of C^*-algebra A onto its C^*-subalgebra B. The following conditions are equivalent:*

(i) Φ *is a Schwartz map;*

(ii) Φ *is positive and, for all* $a \in A$, $b \in B$

$$\Phi(ab) = \Phi(a)b, \qquad \Phi(ba) = b\Phi(a). \tag{3}$$

Proof. If $b \in B$, then $b^*b \in B$ and $\Phi(b)^*\Phi(b) = b^*b = \Phi(b^*b)$ so that (i) $\Rightarrow$ (ii) by the proposition. Conversely, if Φ satisfies (ii), then for any $a \in A$,

$$0 \leqslant \Phi((\Phi(a) - a)^*(\Phi(a) - a)) = -\Phi(a)^*\Phi(a) + \Phi(a^*a). \quad ∎$$

5.11. Let A be a C^*-algebra, H be a Hilbert space, and $\Phi : A \to B(H)$ be a linear mapping. Assume that $\bigoplus_{\iota \in I} H_\iota$ with each H_ι stable under $\Phi(A)$. Then Φ is completely positive if and only if, for every $\iota \in I$, the mapping

$$\Phi_\iota : A \ni x \mapsto \Phi(x)|H_\iota \in B(H_\iota)$$

is completely positive.

Indeed, if $[x_{ij}] \in M_n(A)^+$ and $\xi_k = \bigoplus_{\iota\in I} \xi_k^\iota \in H$, $\xi_k^\iota \in H_\iota$, $(1 \leqslant k \leqslant n;\ \iota \in I)$, then

$$\sum_{ij} (\Phi(x_{ij})\xi_j|\xi_i) = \sum_{\iota\in I}\sum_{ij} (\Phi(x_{ij})\xi_j^\iota|\xi_i^\iota) \geqslant 0.$$

Owing to Theorem 4.11 we infer that, in proving the complete positivity of a linear mapping $\Phi : A \to B$, we may assume $B \subset B(H)$ for some Hilbert space H and $\overline{B\xi} = H$ for some $\xi \in H$.

The following proposition gives more examples of completely positive linear mappings, including those considered in Corollary 5.10.

Proposition. *Let A, B be C^*-algebras and $\Phi : A \to B$ be a positive linear mapping such that:*

$$(1) \qquad (\forall)\ a \in A,\ (\exists)\ a' \in A \quad \textit{with} \quad \Phi(xa') = \Phi(x)\Phi(a),\ (\forall)\ x \in A.$$

Then Φ is completely positive.

Proof. Due to (1), $\Phi(A)$ is a $*$-subalgebra of B and we may assume $\Phi(A)$ dense in B. By the above remark, we may consider $B \subset B(H)$ for some Hilbert space H and $\overline{B\xi} = H$ for some $\xi \in H$. Then $\overline{\Phi(A)\xi} = H$.

Let $x_k \in A$, $a_k \in A$ and $\xi_k = \Phi(ak\xi)$, $(1 \leqslant k \leqslant n)$. We have

$$\sum_{ij}(\Phi(x_i^* x_j)\xi_j|\xi_i) = \sum_{ij}(\Phi(a_i^*)\Phi(x_i^* x_j)\Phi(a_j)\xi|\xi) = \sum_{ij}(\Phi({a_i'}^* x_j a_j')\xi|\xi)$$
$$= \Big(\Phi\Big(\sum_{ij} {a_i'}^* x_i^* x_j a_j'\Big)\xi|\xi\Big) \geqslant 0,$$

since $\sum_{ij} {a_i'}^* x_i^* x_j a_j' \leqslant 0$ and Φ is positive.

This proves that Φ is completely positive. ∎

5.12. Matrix ordered spaces. Completely positive linear mappings can be considered in a more general frame-work which appears to be useful in studying C^*-algebra, especially for tensor products.

We begin by some definitions and notation. Let V, W be complex vector spaces. Then $L(V, W)$ denotes the vector space of all linear mappings $V \to W$ and $V_d = L(V, \mathbb{C})$. If V, W are normed vector spaces, then $B(V, W)$ denotes the normed vector space of all bounded linear mappings $V \to W$ and $V^* = B(V, \mathbb{C})$.

The vector spaces V and V' are in duality if there exists a bilinear map $V \times V' \ni (v, v') \mapsto \langle v, v'\rangle \in \mathbb{C}$ such that for every $v \in V$, $v \neq 0$, there exists $v' \in V'$ with $\langle v, v'\rangle \neq 0$ and for every $v' \in V'$, $v' \neq 0$, there exists $v \in V$ with $\langle v, v'\rangle \neq 0$. For instance, $\mathbb{C}$ and $\mathbb{C}$ are put in duality by the multiplication map. If (V, V') and (W, W') are pairs of vector spaces in duality, then $B_{V',W'}(V, W)$ denotes the vector space of all (V', W')-continuous linear mappings $V \to W$ and $V^\delta = B_{V',\mathbb{C}}(V, \mathbb{C})$. Clearly, V^δ can be identified to V' and $V^{\delta\delta}$ can be identified to

V. For every $\Phi \in B_{V^\delta,W^\delta}(V,W)$ we denote by $\Phi^\delta \in B_{W,V}(W^\delta,V^\delta)$ the transposed map. Then

$$\delta : B_{V^\delta,W^\delta}(V,W) \ni \Phi \mapsto \Phi^\delta \in B_{W,V}(W^\delta,V^\delta)$$

is a linear isomorphism.

By $M_m(V)$ we denote the vector space of all $m \times m$ matrices $v = [v_{ij}]$ with $v_{ij} \in V$, $(1 \leqslant i,j \leqslant m)$. For $v^1 \in M_{m_1}(V)$ and $v^2 \in M_{m_2}(V)$ we define their direct sum $v^1 \oplus v^2 \in M_{m_1+m_2}(V)$ by

$$v^1 \otimes v^2 = \begin{bmatrix} v^1 & 0 \\ 0 & v^2 \end{bmatrix}.$$

Every linear mapping $\Phi : V \to W$ defines a linear mapping $\Phi_m : M_m(V) \to M_m(W)$ by $\Phi([v_{ij}]) = [\Phi(v_{ij})]$, $([v_{ij}] \in M_m(V))$.

If (V,V^δ) is a pair of vector spaces in duality, then $(M_m(V),\ M_m(V^\delta))$ is also a pair of vector spaces in duality with pairing

$$\langle [v_{ij}],[f_{ij}]\rangle = \sum_{i,j} \langle v_{ij},f_{ij}\rangle; \quad [v_{ij}] \in M_m(V),\ [f_{ij}] \in M_m(V^\delta),$$

and we can identify $M_m(V)^\delta$ to $M_m(V^\delta)$. In particular, taking $V = V^\delta = \mathbb{C}$, we see that there exists a canonical identification of $(M_m)^\delta = (M_m)^d$ with M_m,

$$\Delta : M_m \to (M_m)^d$$

such that $\langle [\beta_{ij}],\Delta([\alpha_{ij}])\rangle = \sum\limits_{i,j} \alpha_{ij}\beta_{ij}$, $([\alpha_{ij}],[\beta_{ij}] \in M_m)$.

If (V,V^δ), (W,W^δ) are pairs of vector spaces in duality and $\Phi \in B_{V^\delta,W^\delta}(V,W)$, then $(\Phi_m)^\delta = (\Phi^\delta)_m$ for each $m \in \mathbb{N}$.

If V is a $*$-vector space (1.1), then $M_m(V)$ becomes a $*$-vector space with $*$-operation $[v_{ij}]^* = [v_{ji}^*]$. If V,W are $*$-vector spaces, then $L(V,W)$ becomes a $*$-vector space with $*$-operation $\Phi^*(v) = \Phi(v^*)^*$, $(v \in V,\ \Phi \in L(V,W))$. Note that if $\Phi \in L(V,W)$ is selfadjoint, then also $\Phi_m \in L(M_m(V),M_m(W))$ is selfadjoint. Clearly, $\mathbb{C}$ with complex conjugation is a $*$-vector space, hence V^d is also a $*$-vector space.

Let (V,V^δ) be a pair of vector spaces in duality and assume that V is a $*$-vector space. Then V^δ is called a *$*$-vector dual* of V if $f \in V^\delta \subset V^d \Rightarrow f^* \in V^\delta$. In this case V^δ is itself a $*$-vector space and $v \in V$ (respectively $f \in V^\delta$) is selfadjoint if and only if $\langle v,f\rangle \in \mathbb{R}$ for all $f \in (V^\delta)_h$ (respectively $v \in V_h$). Note that $(V_h,(V^\delta)_h)$ is a pair of real vector spaces in duality, V_h is V^δ-closed in V and $(V^\delta)_h$ is V-closed in V^δ. If V,W are $*$-vector spaces with $*$-vector duals V^δ,W^δ respectively and $\Phi \in B_{V^\delta,W^\delta}(V,W)$, then $(\Phi^*)^\delta = (\Phi^\delta)^*$.

An *ordered $*$-vector space* is a $*$-vector space V endowed with a convex cone $V^+ \subset V_h$. Given two ordered $*$-vector spaces V,W, a linear mapping $\Phi : V \to W$ is called *positive* if $\Phi = \Phi^*$ and $\Phi(V^+) \subset W^+$. The set of all positive linear mappings $V \to W$ is a convex cone $L(V,W)_+ \subset L(V,W)_h$, thus $L(V,W)$ is an ordered $*$-vector space. A linear isomorphism $\Phi : V \to W$ such that $\Phi = \Phi^*$ and $\Phi(V^+) = W^+$ is called an *order isomorphism.*

Let V be an ordered $*$-vector space with a $*$-vector dual $V^\delta \subset V^d$. Then V^δ becomes an ordered $*$-vector space with the dual positive cone $V_+^\delta = V^\delta \cap V_+^d$ and with this structure V^δ is called an *ordered $*$-vector dual of* V. Note that $V_+^\delta = -(V^+)^0$, where $(V^+)^0$ is the polar set of V^+ in V^δ. Thus, regarding $V = V^{\delta\delta}$ as a $*$-vector dual of the ordered $*$-vector space V^δ and using the bipolar theorem we get

$$V_+^{\delta\delta} = \text{the } V^\delta\text{-closure of } V^+. \tag{1}$$

Hence V is an ordered $*$-vector dual of V^δ if and only if V^+ is V^δ-closed in V.

Let V be a vector space, $v = [v_{ij}] \in M_m(V)$ and $\gamma = [\gamma_{kl}]$ be a complex $m \times n$ matrix. Then an element $\gamma^* \cdot v \cdot \gamma \in M_n(V)$ is defined by the formal multiplication of matrices:

$$\gamma^* \cdot v \cdot \gamma = \Big[\sum_{i,j} \overline{\gamma}_{ir}\gamma_{js}v_{ij}\Big] \in M_n(V).$$

We now introduce the central notion of this section. A *matrix ordered space* is a $*$-vector space V together with a convex cone $M_m(V)^+ \subset M_m(V)_h$ in each $M_m(V)$ such that, for every $m, n \in \mathbb{N}$ and every complex $m \times n$ matrix γ,

$$v \in M_m(V)^+ \Rightarrow \gamma^* \cdot v \cdot \gamma \in M_n(V)^+.$$

The example motivating this definition is the following. Let H be a Hilbert space and N be a selfadjoint vector subspace of $B(H)$. For each $m, M_m(N)$ is a selfadjoint subspace of $B(H^{(m)})$, hence an ordered $*$-vector space with operator involution and order. Every complex $m \times n$ matrix γ defines a bounded linear operator $\Gamma_\gamma : H^{(n)} \to H^{(m)}$.

$$\Gamma_\gamma([\xi_k]) = \Big[\sum_j \gamma_{ij}\xi_j\Big]; \quad \xi_1, \ldots, \xi_n \in H.$$

If $x \in M_m(N) \subset B(H^{(m)})$, then $\gamma^* \cdot x \cdot \gamma \in M_n(N) \subset B(H^{(n)})$ is nothing but the operator $\Gamma_\gamma^* \cdot x \cdot \Gamma_\gamma$, hence $x \geqslant 0 \Rightarrow \gamma^* \cdot x \cdot \gamma \geqslant 0$. Thus, every selfadjoint vector subspace of $B(H)$ is a matrix ordered space. In particular, $B(H), M_n$, *and every C^*-algebra is a matrix ordered space.*

Let V be a matrix ordered space. Then

$$v^1 \in M_{m_1}(V)^+,\ v^2 \in M_{m_2}(V)^+ \Rightarrow v^1 \otimes v^2 \in M_{m_1+m_2}(V)^+. \tag{2}$$

Indeed, $v^1 \otimes v^2 = (\gamma^1)^* v^1 \gamma^1 + (\gamma^2)^* v^2 \gamma^2$, where

$$\gamma^1 = \left[\left.\begin{matrix} \underbrace{\begin{matrix} 1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 1 \end{matrix}}_{m} & \underbrace{\begin{matrix} 0 & \ldots & 0 \\ \vdots & & \vdots \\ 0 & \ldots & 0 \end{matrix}}_{n} \end{matrix}\right\} m\right], \quad \gamma^2 = \left[\left.\begin{matrix} \underbrace{\begin{matrix} 0 & \ldots & 0 \\ \vdots & & \vdots \\ 0 & \ldots & 0 \end{matrix}}_{m} & \underbrace{\begin{matrix} 1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 1 \end{matrix}}_{n} \end{matrix}\right\} n\right].$$

Let V, W be matrix ordered spaces. A linear mapping $\Phi : V \to W$ is called *completely positive* if Φ_n is positive for all $n \in \mathbb{N}$. The convex cone $CP(V, W)$ of all completely positive linear mappings $V \to W$ is contained in $L(V, W)_+$. Thus we have two different structures of ordered $*$-vector space on $L(V, W)$ defined by the convex cones $L(V, W)_+$ and $CP(V, W)$ respectively. A linear isomorphism $\Phi : V \to W$ is called a *complete order isomorphism* if both Φ and Φ^{-1} are completely positive.

Let V be a matrix ordered space with a $*$-vector dual V^δ. Then V^δ, together with the convex cones $M_m(V)_+^\delta$ in $M_m(V^\delta) = M_m(V)^\delta$ is a matrix ordered space and with this structure V^δ will be called a *matrix ordered dual* of V. Indeed, let $f = [f_{ij}] \in M_m(V^\delta)^+$, let $\gamma = [\gamma_{kl}]$ be a complex $m \times n$ matrix and denote by ${}^t\gamma$ the transposed $n \times m$ matrix. Then $\gamma^* \cdot f \cdot \gamma = \Big[\sum\limits_{i,j} \overline{\gamma}_{ir} \gamma_{js} f_{ij} \Big] \in M_n(V^\delta)$ and, if $v = [v_{rs}] \in M_n(V)^+$, then

$$\langle v, \gamma^* \cdot f \cdot \gamma \rangle = \sum_{r,s} \sum_{i,j} \overline{\gamma}_{ir} \gamma_{js} \langle v_{rs}, f_{ij} \rangle = \sum_{i,j} \Big\langle \sum_{r,s} \overline{\gamma}_{ir} \gamma_{js} v_{rs},\ f_{ij} \Big\rangle = \langle ({}^t\gamma)^* v ({}^t\gamma), f \rangle \geqslant 0,$$

hence $\gamma^* \cdot f \cdot \gamma \in M_n(V^\delta)^+$.

Using (1) we see that, given a matrix ordered space V with a $*$-vector dual V^δ, V is identical as a matrix ordered space with the matrix ordered dual $V^{\delta\delta}$ of V^δ if and only if each convex cone $M_m(V)^+$ is $M_m(V)^\delta$-closed. In this case, we say that (V, V^δ) are *dual matrix ordered spaces*. Note that (V, V^δ) are dual matrix ordered spaces whenever V is finite dimensional.

Also, *for every C^*-algebra A, (A, A^*) are dual matrix ordered spaces.*

Proposition 1. *Let V be a matrix ordered space and $n \in \mathbb{N}$. Then the mapping $\Theta : M_m(V) \to L(M_m, V)$ defined by*

$$\Theta(v)\alpha = \sum_{i,j} \alpha_{ij} v_{ij}; \quad \alpha = [\alpha_{ij}] \in M_m, \quad V = [v_{ij}] \in M_m(V)$$

is an order isomorphism with respect to the convex cones $M_m(V)^+$ and $CP(M_m, V)$.

Proof. Clearly, Θ is a linear isomorphism and

$$\Theta^{-1}(\Phi) = [\Phi(e_{ij}] \in M_m(V); \quad \Phi \in L(M_m, V),$$

where $\{e_{ij}\}$ is the system of matrix units in M_m. If $\Phi \in L(M_m, V)$ is completely positive, then $\Phi_m : M_m(M_m) \to M_m(V)$ is positive, so $\Theta^{-1}(\Phi) = \Phi([e_{ij}]) \in M_m(V)^+$ because $[e_{ij}] \in M_m(M_m)^+$ (see 2.12).

Let $v = [v_{ij}] \in M_m(V)^+$, $n \in \mathbb{N}$, and $\alpha \in M_n(M_m)^+$. We have to show that $\Theta(v)_n(\alpha) \in M_n(V^+)$. By Corollary 2.12 we may suppose $\alpha = [\alpha_r^* \alpha_s]$ with $\alpha_r = [\alpha_{rij}] \in M_m$, $(1 \leqslant r \leqslant n)$. Then

$$\begin{aligned} \Theta(v)_n(\alpha) &= [\Theta(v)(\alpha_r^* \alpha_s)] = \Big[\sum_{i,j} (\alpha_r^* \alpha_s)_{ij} v_{ij} \Big] = \Big[\sum_{i,j} \Big(\sum_h \overline{\alpha}_{rhi} \alpha_{shj} \Big) v_{ij} \Big] \\ &= \Big[\sum_h \sum_{i,j} \overline{\alpha}_{rhi} \alpha_{shj} v_{ij} \Big] = \sum_h \Big[\sum_{i,j} \overline{\alpha}_{rhi} \alpha_{shj} v_{ij} \Big] = \sum_h (\gamma^h)^* v\, \gamma^h, \end{aligned}$$

where γ^h is the complex $m \times n$ matrix with entries $\gamma^h_{kl} = \gamma_{lhk}$, $(1 \leqslant k \leqslant m$, $1 \leqslant l \leqslant n$; $1 \leqslant h \leqslant m)$. Since V is a matrix ordered space, it follows that $\Theta(v)_n(\alpha) \in M_m(V)^+$. ∎

Using Corollary 2.12 as in the above proof it is easy to see that

PROPOSITION 2. *For each $m \in \mathbb{N}$, the mapping $\Delta : M_m \to M_m^d$ defined by*

$$\langle \beta, \Delta(\alpha) \rangle = \sum_{i,j} \alpha_{ij} \beta ij; \quad \alpha = [\alpha_{ij}],\ \beta = [\beta_{ij}] \in M_m$$

is a complete order isomorphism. ∎

PROPOSITION 3. *Let (V, V^δ) and (W, W^δ) be two dual matrix ordered spaces. Then $\Phi \in B_{V^\delta, W^\delta}(V, W)$ is completely positive if and only if $\Phi^\delta \in B_{W,V}(W^\delta, V^\delta)$ is completely positive. In other words, the map*

$$\delta : B_{V^\delta, W^\delta}(V, W) \to B_{W,V}(W^\delta, V^\delta)$$

is an order isomorphism relative to complete positivity.

Proof. Since (V, V^δ) and (W, W^δ) are dual matrix ordered spaces, it is sufficient to show that Φ^δ is completely positive whenever Φ is. So, let $n \in \mathbb{N}$, $F \in M_n(W^\delta)^+$ and $X \in M_n(V)^+$. Then $\Phi_n(X) \in M_n(W)^+$ by the complete positivity of Φ and hence

$$\langle X, (\Phi^\delta)_n(F) \rangle = \langle X, (\Phi_n)^\delta(F) \rangle = \langle \Phi_n(X), F \rangle \geqslant 0. \quad ∎$$

PROPOSITION 4. *Let (V, V^δ) be dual matrix ordered spaces and $m \in \mathbb{N}$. Then the mapping $\Lambda : M_m(V) \to B_{V, M_m^d}(V^\delta, M_m)$ defined by*

$$\Lambda(v)f = [f(v_{ij})]; \quad f \in V^\delta,\ v = [v_{ij}] \in M_m(V)$$

is an order isomorphism with respect to the convex cones $M_m(V)^+$ and $CP(V^\delta, M_m) \cap B_{V, M_m^d}(V^\delta, M_m)$.

Proof. This follows, using Propositions 1, 2 and 3, from the commutative diagram

$$\begin{tikzcd} M_m(V) \arrow[r, "\Theta"] \arrow[dr, "\Lambda"'] & L(M_m, V) = B_{M_m^d, V^\delta}(M_m, V) \arrow[d, "\delta"] \\ & B_{V, W_m^d}(V^\delta, M_m) = B_{V, M_m}(V^\delta, M_m^d) \end{tikzcd}$$

where we have identified M_m and M_m^d *via* Δ. ∎

5.13. As an application, in this section we explicitate some consequences of the results from 5.12 for linear mappings between a C^*-algebra A and a full matrix algebra M_m, $(m \in \mathbb{N})$.

Recall that (A, A^*) is a dual matrix ordered space (5.12) and $M_m(A)$ is a C^*-algebra (4.19).

By $\{e_{ij}\}_{1 \leqslant i,j \leqslant m}$ we denote the system of matrix units in M_m.

Proposition 1. *Let A be a C^*-algebra, $m \in \mathbb{N}$ and $\Phi : M_m \to A$ be a linear mapping. The following statements are equivalent:*

(i) *Φ is completely positive;*

(ii) *Φ is m-positive;*

(iii) *$[\Phi(e_{ij})] \in M_m(A)^+$;*

(iv) *there exist $x_{ij} \in A$, $(1 \leqslant i, j \leqslant m)$, such that*

$$\Phi(e_{ij}) = \sum_{k=1}^{m} x_{ki}^* x_{kj}; \quad 1 \leqslant i, j \leqslant m.$$

Proof. (i) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (iii) because $[\Phi(e_{ij})] = \Phi_m([e_{ij}])$, $[e_{ij}] \in M_m(M_m)^+$ by 2.12 and Φ_m is positive by assumption.

(iii) $\Rightarrow$ (i) by Proposition 1/5.12.

(iii) $\Leftrightarrow$ (iv) because every positive element of the C^*-algebra $M_m(A)$ is of the form $[x_{ij}]^*[x_{ij}]$ with $x_{ij} \in A$, $(1 \leqslant i, j \leqslant m)$. ∎

Since every selfadjoint element of $M_m(A)$ is the difference of two positive elements (2.3) it follows that every selfadjoint linear mapping $M_m \to A$ is the difference of two completely positive linear mappings $M_m \to A$.

Corollary. *Let A, B be C^*-algebras, $\pi : A \to B$ be a surjective $*$-homomorphism and $m \in \mathbb{N}$. For every completely positive linear mapping $\Psi : M_m \to B$ there exists a completely positive linear mapping $\Phi : M_m \to A$ such that*

$$\Psi = \pi \circ \Phi \quad \textit{and} \quad \|\Phi\| \leqslant \|\Psi\|.$$

If in addition A is unital and $\Psi(1_{M_n}) = 1_B$, then Φ can be chosen so that $\Phi(1_{M_n}) = 1_A$.

Proof. Assume that $\|\Psi\| = 1$. Passing, if necessary, to C^*-algebras with adjoint units (1.5), we may also assume that A is unital. Then B is also unital and $\pi(1_A) = 1_B$.

By the proposition, there exist $y_{ij} \in B$, $(1 \leqslant i, j \leqslant m)$, with

$$\Psi(e_{ij}) = \sum_{k=1}^{m} y_{ki}^* y_{kj}; \quad 1 \leqslant i, j \leqslant m.$$

Since π is surjective, there exist $x_{ij} \in A$ such that $\pi(x_{ij}) = y_{ij}$, $(1 \leqslant i, j \leqslant m)$. Again by the proposition, the linear mapping $\Theta : M_m \to A$ defined by

$$\Theta(e_{ij}) = \sum_{k=1}^{m} x_{ki}^* x_{kj}; \quad 1 \leqslant i, j \leqslant m,$$

is completely positive and clearly $\Psi = \pi \circ \Theta$.

Consider the continuous function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(t) = 1 \quad \text{for } t \leqslant 1, \quad f(t) = t^{-1/2} \quad \text{for } t > 1.$$

Let $a = f(\Theta(1_{M_m})) \in A^+$ and define $\Phi : M_n \to A$ by

$$\Phi(\cdot) = a\,\Theta(\cdot)\,a.$$

Then Φ is linear and completely positive. Since $\|\Psi\| = 1$, we have

$$\pi(a) = \pi(f(\Theta(1_{M_m}))) = f(\pi(\Theta(1_{M_m}))) = f(\Psi(1_{M_m})) = 1_B$$

hence $\Psi = \pi \circ \Phi$. On the other hand, using 5.3.(7) we obtain

$$\|\Phi\| = \|\Phi(1_{M_m})\| = \|f(\Theta(1_{M_m}))\Theta(1_{M_m})f(\Theta(1_{M_m}))\| \leqslant \|1_A\| = 1$$

because $0 \leqslant tf(t)^2 \leqslant 1$ for $t \geqslant 0$.

Finally, assume that $\Psi(1_{M_m}) = 1_B$. Then the equality $\|\Psi\| = 1$ is automatic. Let $\Theta : M_m \to A$ be an arbitrary contractive completely positive linear mapping such that $\pi \circ \Theta = \Psi$ and let φ be an arbitrary state of M_m. Then the mapping $\Phi : M_m \to A$ defined by

$$\Phi(\cdot) = \varphi(\cdot)(1_A - \Theta(1_{M_m})) + \Theta(\cdot)$$

is linear, completely positive, $\Phi(1_{M_m}) = 1_A$ and $\pi \circ \Phi = \Psi$. ∎

Consider now a linear mapping $\Phi : A \to M_m$. Then Φ defines a matrix $[\varphi_{ij}]$ of linear forms on A by

$$\Phi(x) = [\varphi_{ij}(x)]; \quad x \in A.$$

It is easy to see that Φ is bounded if and only if all φ_{ij} are bounded. In this case $\Phi^\delta : M_m \to A$ is defined by

$$\Phi^\delta(e_{ij}) = \varphi_{ij}; \quad 1 \leqslant i, j \leqslant m.$$

PROPOSITION 2. *Let A be a C^*-algebra, $m \in \mathbb{N}$ and $\Phi : A \to M_m$ be a bounded linear mapping. The following statements are equivalent:*

(i) Φ *is completely positive;*

(ii) Φ *is m-positive;*

(iii) $[\varphi_{ij}] = \Phi_m^\delta([e_{ij}]) \in M_m(A^*)^+$*;*

(iv) *for every $x_1, \ldots, x_m \in A$ we have*

$$\sum_{i,j=1}^m \varphi_{ij}(x_i^* x_j) \geqslant 0.$$

Proof. (i) ⇒ (ii) is obvious.

(ii) ⇒ (iii). By assumption, $\Phi_m : M_m(A) \to M_m(M_m)$ is positive. Then also $\Phi_m^\delta : M_m(M_m) \to M_m(A)^\delta = M_m(A^*)$ is positive, hence $\Phi_m^\delta([e_{ij}]) \in M_m(A^*)^+$.

(iii) ⇒ (i). By Proposition 1/5.12, from (iii) we infer that $\Phi^\delta : M_m \to A^*$ is completely positive. Since $A = A^{\delta\delta} = (A^*)^\delta$ and $\Phi^{\delta\delta} = \Phi$, by Proposition 3/5.12 it follows that also $\Phi : A \to M_m$ is completely positive.

(iii) ⇔ (iv) because $M_m(A^*) = M_m(A)^*$ and every positive element of $M_m(A)$ is a sum of matrices of the form $[x_i^* x_j]$ with $x_1, \ldots, x_n \in A$ (2.12). ∎

Since every selfadjoint element of $M_m(A)^*$ is the difference of two positive elements (Corollary 1/4.15), it follows that every selfadjoint linear mapping $A \to M_m$ is the difference of two completely positive linear mappings $A \to M_m$.

Remark that, for M_m-valued completely positive linear mappings, the result analogous to the corollary of Proposition 1 is nothing but a particular case of Corollary 2/5.3.

5.14. Tensor Products of Matrix Ordered Spaces. Let (V, V^δ), (W, W^δ) be pairs of vector spaces in duality. Denote by $V \otimes W$ the vector space tensor product. For $\varphi \in (V \otimes W)^d$, $v \in V$ and $w \in W$ we define $\varphi_w \in V^d$ and ${}_v\varphi \in W^d$ by

$$\varphi_w(v') = \varphi(v' \otimes w); \quad v' \in V,$$
$${}_v\varphi(w') = \varphi(v \otimes w'); \quad w' \in W.$$

Put

$$(V \otimes W)^\delta = \{\varphi \in (V \otimes W)^d;\ \varphi_w \in V^\delta,\ {}_v\varphi \in W^\delta \text{ for all } v \in W,\ v \in V\}.$$

We shall identify $V^d \otimes W^d$ with a vector subspace of $(V \otimes W)^d$. Under this identification it is clear that

$$V^\delta \otimes W^\delta \subset (V \otimes W)^\delta. \tag{1}$$

Moreover, if V (or W) is finite dimensional, then

$$V^\delta \otimes W^\delta = (V \otimes W)^\delta. \tag{2}$$

Indeed, let $v_1, \ldots, v_n$ be a linear basis in V and let $f_j \in V^d = V^\delta$ be the j^{th}-coordinate function on V relative to this basis. If $\varphi \in (V \otimes W)^\delta$, then $g_j = {}_{v_j}\varphi \in W^\delta$, $(1 \leqslant j \leqslant n)$, and $\varphi = \sum\limits_{j=1}^n f_j \otimes g_j \in V^\delta \otimes W^\delta$.

Note that $(V \otimes W, (V \otimes W)^\delta)$ and $(V \otimes W, V^\delta \otimes W^\delta)$ are pairs of vector spaces in duality. Indeed, let $u \in V \otimes W$, $u \neq 0$, and write $u = \sum\limits_{k=1}^n v_k \otimes w_k$ with $v_1, \ldots, v_n \in V$ linearly independent and $u_1, \ldots, w_n \in W$, $w_1 \neq 0$. There is $g \in W^\delta$ with $g(w_1) \neq 0$ and there is $f \in V^\delta$ with $f(v_1) \neq 0$ and $f(v_k) = 0$ for $k \neq 1$. Then $\varphi = f \otimes g \in V^\delta \otimes W^\delta \subset (V \otimes W)^\delta$ and $\varphi(u) = f(v_1)g(w_1) \neq 0$.

If V, W are $*$-vector spaces, then $V \otimes W$ endowed with the $*$-operation

$$\Big(\sum_{k=1}^n v_k \otimes w_k\Big)^* = \sum_{k=1}^n v_k^* \otimes w_k^*; \quad v_k \in V,\ w_k \in W,\ n \in \mathbb{N},$$

is a $*$-vector space. Note that if $u = \sum\limits_k v_k \otimes w_k$ is selfadjoint, then $u = \sum\limits_k (\operatorname{Re} v_k \otimes \operatorname{Re} w_k - \operatorname{Im} v_k \otimes \operatorname{Im} w_k)$, so

$$(V \otimes W)_h = V_h \otimes W_h. \tag{3}$$

If V, W are $*$-vector spaces with $*$-vector duals V^δ, W^δ respectively, then $(V \otimes W)^\delta$ and $V^\delta \otimes W^\delta$ are $*$-vector duals of $V \otimes W$. This is clear for $V^\delta \otimes W^\delta$ because $(f \otimes g)^* = f^* \otimes g^*$, for all $f \in V^d$, $g \in W^d$. If $\varphi \in (V \otimes W)^\delta$, then $(\varphi^*)_w = (\varphi_{w^*})^* \in V^\delta$ and ${}_v(\varphi^*) = ({}_{v^*}\varphi)^* \in W^\delta$ for all $v \in V$, $w \in W$, hence $\varphi^* \in (V \otimes W)^\delta$.

Now let V, W be matrix ordered spaces. For $v = [v_{ij}] \in M_m(V)$, $w = [w_{ij}] \in M_m(W)$ denote

$$v \times w = \sum_{i,j} v_{ij} \otimes w_{ij} \in V \otimes W.$$

Note that, for $v \in M_m(V)$, $v^1 \in M_{m_1}(V)$, $v^2 \in M_{m_2}(V)$ and $w \in M_m(W)$, $w_1 \in M_{m_1}(W)$, $w^2 \in M_{m_2}(W)$, we have

$$(v \times w)^* = v^* \times w^*,$$
$$v^1 \times w^1 + v^2 \times w^2 = (v^1 \oplus v^2) \times (w^1 \oplus w^2).$$

Owing to 5.12.(2), it follows that

$$(V \otimes W)^+ = \{v \times w;\, v \in M_m(V)^+,\, w \in M_m(W)^+,\, m \in \mathbb{N}\}$$

is a convex cone contained in $(V \otimes W)_h$. Thus $V \otimes W$ becomes an ordered $*$-vector space.

Let A, B be C^*-algebras. Then A, B are matrix ordered spaces, so $A \otimes B$ becomes an ordered $*$-vector space. On the other hand, by 2.11, $A \otimes B$ is a $*$-algebra and, by 2.8, a natural preorder structure is defined on $A \otimes B$. These two structures coincide because an element $u \in A \times B$ is positive with respect to the $*$-algebra preorder structure if and only if it is of the form

$$u = \sum_{k=1}^{n} \Big(\sum_{i=1}^{m} a_{ik} \otimes b_{ik}\Big)^* \Big(\sum_{i=1}^{m} a_{ik} \otimes b_{ik}\Big) = \sum_{k=1}^{n} \sum_{i,j=1}^{m} a_{ik}^* a_{jk} \otimes b_{ik}^* b_{jk}$$

with $a_{ik} \in A$, $b_{ik} \in B$, $(1 \leqslant i \leqslant m,\, 1 \leqslant k \leqslant n)$, that is

$$u = \sum_{k=1}^{n} [a_{ik}^* a_{jk}] \times [b_{ik}^* b_{jk}]$$

with $[a_{ik}^* a_{jk}] \in M_m(A)^+$, $[b_{ik}^* b_{jk}] \in M_m(B)^+$, which means that w is a positive element of the ordered $*$-vector space $A \otimes B$.

If V, W are matrix ordered spaces with $*$-vector duals V^δ, W^δ respectively, then $(V \otimes W)^\delta$ is an ordered $*$-vector dual of $V \otimes W$. For $\varphi \in (V \otimes W)^\delta$ we define

$$L_\varphi : V \to W^\delta \quad \text{and} \quad R_\varphi : W \to V^\delta$$

by

$$\langle w, L_\varphi(v)\rangle = \langle v \otimes w, \varphi\rangle = \langle v, R_\varphi(w)\rangle; \quad v \in V,\, w \in W.$$

Note that the mappings

$$L:(V\otimes W)^{\delta}\ni\varphi\mapsto L_{\varphi}\in B_{V^{\delta},W}(V,W^{\delta}),$$
$$R:(V\otimes W)^{\delta}\ni\varphi\mapsto R_{\varphi}\in B_{W^{\delta},V}(W,V^{\delta}),$$

are linear bijections. Moreover, it is easy to see that L (respectively R) is a homeomorphism with respect to the $(V\otimes W)$-topology on $(V\otimes W)^{\delta}$ and the point W-topology on $B_{V^{\delta},W}(V,W^{\delta})$ (respectively the point V-topology on $B_{W^{\delta},V}(W,V^{\delta})$).

Consider again two C^*-algebras A,B. The dual spaces $A^{\delta}=A^*$, $B^{\delta}=B^*$ are $*$-vector duals, thus they are also matrix ordered spaces. By definition, $(A\otimes B)^{\delta}$ is then the set of all bilinear functionals $\varphi:A\times B\to\mathbb{C}$ which are separately norm continuous and an easy application of the Banach-Steinhauss theorem shows that $(A\times B)^{\delta}$ consists of those $\varphi\in(A\otimes B)^{\delta}$ such that

$$|||\varphi|||=\sup\{|\varphi(a\otimes b)|;\ a\in A,\ \|a\|\leqslant 1,\ b\in B,\ \|b\|\leqslant 1\}<+\infty.$$

Furthermore, $\varphi\mapsto|||\varphi|||$ is a norm on $(A\otimes B)^{\delta}$ and the mappings

$$L:(A\otimes B)^{\delta}\ni\varphi\mapsto L_{\varphi}\in B(A,B^*),$$
$$R:(A\otimes B)^{\delta}\ni\varphi\mapsto R_{\varphi}\in B(B,A^*),$$

are isometric linear (surjective) isomorphisms.

Let $\varphi\in(A\otimes B)^{d}_{+}$. For each fixed $b\in B^{+}$ (respectively $a\in A^{+}$), the map φ_b (respectively ${}_a\varphi$) is a positive form on the C^*-algebra A (respectively B). Owing to Corollary 4/4.5 we infer that $\varphi\in(A\otimes B)^{\delta}$, hence

$$(A\otimes B)^{d}_{+}=(A\otimes B)^{\delta}_{+}. \tag{4}$$

An element $\varphi\in(A\otimes B)^{d}_{+}$ is called a *state* on $A\otimes B$ if $|||\varphi|||=1$. By (4), every positive linear functional on the $*$-algebra $A\otimes B$ is a multiple of some state.

Owing to Corollary 2.12 it is easy to check that a linear mapping $\Phi:A\to B^*$ is completely positive if and only if

$$\sum_{i,j=1}^{n}\langle b_i^*b_j,\Phi(a_i^*a_j)\rangle\geqslant 0;\quad a_1,\ldots,a_n\in A,\ b_1,\ldots,b_n\in B,\ n\in\mathbb{N}. \tag{5}$$

Proposition 1. *Let V,W be matrix ordered spaces with matrix ordered duals V^{δ},W^{δ} respectively and let $\varphi\in(V\otimes W)^{\delta}$. Then the following conditions are equivalent:*

(i) $\varphi\in(V\otimes W)^{\delta}_{+}$;

(ii) $L_{\varphi}:V\to W^{\delta}$ *is completely positive;*

(iii) $R_{\varphi}:W\to V^{\delta}$ *is completely positive.*

Proof. First note that $L_{\varphi^*}=L_{\varphi}^*$, hence φ is selfadjoint if and only if L_{φ} is selfadjoint. Then $\varphi\in(V\otimes W)^{\delta}_{+}\Leftrightarrow\langle u,\varphi\rangle\geqslant 0$, $(\forall)\,u\in(V\otimes W)^{+}\Leftrightarrow\langle v\times w,\varphi\rangle\geqslant 0$, $(\forall)\,v\in M_m(V)^{+}$, $w\in M_m(W)^{+}$, $m\in\mathbb{N}\Leftrightarrow\langle w,(L_{\varphi})_m(v)\rangle\geqslant 0$, $(\forall)\,v\in M_m(V)^{+}$, $w\in M_m(W)^{+}$, $m\in\mathbb{N}\Leftrightarrow L_{\varphi}$ is completely positive.

This proves (i) $\Leftrightarrow$ (ii) and similarly, (i) $\Leftrightarrow$ (iii). ∎

Let V, W be matrix ordered spaces with matrix ordered duals V^δ, W^δ respectively. Then $V \otimes W \subset (V^\delta \otimes W^\delta)^\delta$, so every $u \in V \otimes W$ defines linear mappings

$$L_u : V^\delta \to W^{\delta\delta} = W \quad \text{and} \quad R_u : W^\delta \to V^{\delta\delta} = V,$$

namely, if $u = \sum_k v_k \otimes w_k$, $f \in V^\delta$ and $g \in W^\delta$, then

$$L_u(f) = \sum_k f(v_k) w_k \quad \text{and} \quad R_u(g) = \sum_k g(w_k) v_k.$$

Remark that the mappings

$$L : V \otimes W \ni u \mapsto L_u \in B_{V, W^\delta}(V^\delta, W)$$
$$R : V \otimes W \ni u \mapsto R_u \in B_{W, V^\delta}(W^\delta, V)$$

are linear injections.

Clearly, $(V \otimes W)^+ \subset (V \otimes W) \cap (V^\delta \otimes W^\delta)^\delta_+$, but this inclusion can be strict.

PROPOSITION 2. *Let V, W be matrix ordered spaces with matrix ordered duals V^δ, W^δ respectively and let $u \in V \otimes W$. Then the following conditions are equivalent:*

(i) $u \in (V \otimes W)^+$;

(ii) *there exist $m \in \mathbb{N}$ and completely positive mappings $\Phi \in L(M_m, W)$, $\Psi \in B_{V, M_m^d}(V^\delta, M_m)$ such that $L_u = \Phi \circ \Psi$:*

$$\begin{tikzcd} & M_m \arrow[dr, "\Phi"] & \\ V^\delta \arrow[ur, "\Psi"] \arrow[rr, "L_u"] & & W; \end{tikzcd}$$

(iii) *there exist $m \in \mathbb{N}$ and completely positive mappings $\Phi \in L(M_m, V)$, $\Psi \in B_{W, M_m^d}(W^\delta, M_m)$ such that $R_u = \Phi \circ \Psi$:*

$$\begin{tikzcd} & M_m \arrow[dr, "\Phi"] & \\ W^\delta \arrow[ur, "\Psi"] \arrow[rr, "R_u"] & & V. \end{tikzcd}$$

Proof. (i) $\Rightarrow$ (ii). If $u \in (V \otimes W)^+$, then $u = v \times w$ for some $v \in M_m(V)^+$, $w \in M_m(W)^+$, $m \in \mathbb{N}$. By Propositions 1 and 4/5.12, $\Phi = \Theta(w) \in L(M_m, W)$ and $\Psi = \Lambda(v) \in B_{V, M_m^d}(V_\delta, M_m)$ are completely positive and it is easy to check that $L_u = \Phi \circ \Psi$.

(ii) $\Rightarrow$ (i). Let $L_u = \Phi \circ \Psi$ for some completely positive $\Phi \in L(M_m, W)$, $\Psi \in B_{V, M_m^d}(V^\delta, M_m)$. Using again Propositions 1 and 4/5.12 we get $v \in M_m(V)^+$, $w \in M_m(W)^+$ such that $\Phi = \Theta(w)$ and $\Phi = \Lambda(v)$. Then $L_u = \Phi \circ \Psi = \Theta(w) \circ \Lambda(v) = L_{(v \times w)}$, hence $u = v \times w \in (V \otimes W)^+$.

The proof of (i) $\Leftrightarrow$ (iii) is similar. ∎

COROLLARY 1. *Let V be a matrix ordered space and $m \in \mathbb{N}$. Then:*

(i) *the mapping $R : V \otimes M_m \ni u \mapsto R_u \in L(M_m^d, V)$ is an order isomorphism with respect to the convex cones $(V \otimes M_m)^+$ and $CP(M_m^d, V)$;*

(ii) *the mapping $\Omega : V \otimes M_m \ni \sum\limits_{i,j} v_{ij} \otimes e_{ij} \mapsto [v_{ij}] \in M_m(V)$ is an order isomorphism with respect to the convex cones $(V \otimes M_m)^+$ and $M_m(V)^+$.*

Proof. (i) If $u \in (V \otimes M_m)^+$, then $R_u \in CP(M_m^d, V)$ by Proposition 2. Conversely, if $u \in V \otimes M_m$ and R_u is completely positive, then $\Psi = \Delta^{-1} : M_m^d \to M_m$ is a complete order isomorphism (Proposition 2/5.12), $\Phi = R_u \circ \Delta : M_m \to V$ is completely positive and $R_u = \Phi \circ \Psi$, hence $u \in (V \otimes M_m)^+$ by Proposition 2.

(ii) It is easy to check that the diagram

$$\begin{array}{ccc} V \otimes M_m & \xrightarrow{\Omega} & M_m(V) \\ R\Big\downarrow & & \Big\downarrow\Theta \\ L(M_m^d, V) & = & L(M_m, V) \end{array}$$

where M_m and M_m^d are identified via Δ, is commutative. Thus the desired result following using (1) and Proposition 1/5.12. ∎

COROLLARY 2. *Let V, W be matrix ordered spaces with matrix ordered duals V^δ, W^δ respectively and let $u \in V \otimes W$. Assume that either V or W is finite dimensional. Then the following conditions are equivalent:*

(i) *u belongs to the $(V \otimes W)^\delta$-closure of $(V \otimes W)^+$;*

(ii) *$L_u : V^\delta \to W$ is a limit of mappings of the form $\Phi \circ \Psi$ with $\Phi \in L(M_m, W)$, $\Psi \in B_{V,M_m^d}(V^\delta, M_m)$ completely positive, $m \in \mathbb{N}$, relative to the point W^δ-topology on $L(V^\delta, W)$;*

(iii) *$R_u : W^\delta \to V$ is a limit of mappings of the form $\Phi \circ \Psi$ with $\Phi \in L(M_m, V)$, $\Psi \in B_{W,M_m^d}(W^\delta, M_m)$ completely positive, $m \in \mathbb{N}$, relative to the point V^δ-topology on $L(W^\delta, V)$.*

Proof. Since either V or W is finite dimensional, we have $(V \otimes W)^\delta = V^\delta \otimes W^\delta$ (5.12.(2)). Thus, a net $\{u_\iota\}$ from $(V \otimes W)^+$ is $(V \otimes W)^\delta$-convergent to u if and only if

$$\langle u_\iota, f \otimes g \rangle \to \langle u, f \otimes g \rangle \quad \text{for all } f \in V^\delta,\ g \in W^\delta,$$

that is, if and only if

$$\langle L_{u_\iota}(f), g \rangle \to \langle L_u(f), g \rangle \quad \text{for all } f \in V^\delta,\ g \in W^\delta.$$

Thus (i) $\Leftrightarrow$ (ii) follows from Proposition 2. Similarly, (i) $\Leftrightarrow$ (iii). ∎

5.15. Matrix quotients. Let $m \in \mathbb{N}$. We shall identify M_m and M_m^d via the map $\Delta : M_m \Rightarrow M_m^d$ (see Proposition 2/5.12). Consider a *matrix system*

$N \subset M_m$, i.e. a selfadjoint subspace N of M_m which contains the unit of M_m. Then N, as well as N^d are matrix ordered spaces. Set

$$K = N^0 = \Big\{\beta \in M_m;\ \sum_{i,j} \alpha_{ij}\beta_{ij} = 0,\ \text{for all } \alpha \in N\Big\}.$$

Then K is a selfadjoint subspace of M_m. Let $\pi : M_m \to M_m/K$ be the canonical quotient map. We can define a $*$-operation on M_m/K by putting $\pi(\alpha)^* = \pi(\alpha^*)$, $(\alpha \in M_m)$, and then M_m/K becomes a $*$-vector space. For every $n \in \mathbb{N}$, the map $\pi_n : M_n(M_m) \to M_n(M_m/K)$ is linear, selfadjoint and $\operatorname{Ker}\pi_n = M_n(K)$. Thus we may identify $M_m(M_m)/M_n(K)$ and $M_n(M_m/L)$ as $*$-vector spaces. With this identification we define the positive cone in $M_n(M_m/K)$ by $M_n(M_m/K)^+ = \pi_n(M_n(M_m)^+)$. The $*$-vector space M_m/K endowed with these convex cones will be called a *matrix-quotient*.

Let $\eta = \operatorname{res} : M_m^d \to N^d$ be the restriction map. Then the map $\xi : M_m/K \to N^d$ defined by

$$\xi(\pi(\alpha)) = \eta(\Delta(\alpha)); \quad \alpha \in M_m,$$

is a well defined linear isomorphism of M_m/K onto N^d. Moreover,

PROPOSITION. *$\xi : M_m/K \to N^d$ is a complete order isomorphism.*

Proof. Fix $n \in \mathbb{N}$ and consider the diagram

$$\begin{tikzcd} M_n(M_m) \arrow[r, "\Delta_n"] \arrow[d, "\pi_n"'] & M_n(M_m^d) \arrow[d, "\eta_n"] \\ M_n(M_m/K) \arrow[r, "\xi_n"] & M_n(N^d) \end{tikzcd}$$

By identifying $M_n(M_m^d)$ with $M_n(M_m)^d$ and $M_n(N^d)$ with $M_n(N)^d$, η_n identifies to the restriction map $\text{rès} : M_n(M_m)^d \to M_n(N)^d$. Thus, we have to show that an element $\psi \in M_n(N)^d$ is positive if and only if $\psi = \text{rès}\,\varphi$ for some positive $\varphi \in M_n(M_m)^d$.

Clearly, $\varphi \in M_n(M_m)^d_+$ entails $\psi = \text{rès}\,\varphi \in M_n(N)^d_+$.

Conversely, let $\psi \in M_n(N)^d$. If $\alpha \in M_n(N)$, $\|\alpha\| \leqslant 1$ and $\lambda \in \mathbb{C}$, $|\lambda| = 1$, then $\operatorname{Re}(\lambda\alpha) \leqslant \|\operatorname{Re}(\lambda\alpha)\| \leqslant 1$, hence $\operatorname{Re}\lambda\psi(\alpha) \leqslant \psi(1)$ and for a suitable choice of λ we get $|\psi(\alpha)| \leqslant \psi(1)$. It follows that $\|\psi\| = \psi(1)$. By the Hahn-Banach theorem there exists $\varphi \in M_n(M_m)^d$ such that $\text{rès}\,\varphi = \psi$ and $\|\varphi\| = \|\psi\|$. Since $\|\varphi\| = \varphi(1)$ and $M_n(M_m)$ is a C^*-algebra, by Proposition 4.6 we infer that φ is positive. ∎

COROLLARY. *Every matrix-quotient is a matrix ordered space.* ∎

5.16. Notes. The concept of a completely positive linear mapping between C^*-algebras has been introduced by W.F. Stinespring [293] who proved Theorem 5.3 for unital C^*-algebras as well as Proposition 5.5 for positive linear functionals and Proposition 5.6, thus showing that Theorem 5.3 contains as particular cases both the GNS construction and the Naĭmark dilation theorem [209]. In the general case, Theorem 5.3 has been considered in [93], 2.1, [172], Section 4 and Proposition 5.5 appeared in [17], I, 1.2.2, [296], 6.1. For some extensions of these results to more general $*$-algebras we refer to [93], [94], [228], [229], [250], II. Recall that another important extension of the Naĭmark dilation theorem is the known unitary dilation of a linear contraction, due to B. Sz.-Nagy [311].

The Cauchy-Schwarz type inequality (5.8.(1)), which is now an easy consequence of 5.6, 5.3, has been previously discovered by R.V. Kadison [143] who used it in studying the linear isometries between operator algebras (see Chapter 6). In Proposition 5.9 we collected several related inequalities ([42], [92], [182]) and in 5.10 we listed the main properties of the Schwartz maps ([42], [226]; this term has been introduced in [226]). There is another extension of the Kadison inequality (5.8.(2)) which says ([14], [42], [68], [69]) that if $\Phi : A \to B$ is a unit-preserving positive linear map between unital C^*-algebras and $f : (-\lambda, \lambda) \to \mathbb{R}$ is an "operator convex function" ([14], [23], [68], [69], [79]; for instance, $f(t) = t$ is operator convex), then $\Phi(f(a)) \geqslant f(\Phi(a))$ for all $a \in A_h$, $\|a\| \leqslant \lambda$; also ([42]), $\Phi(a^{-1}) \geqslant \Phi(a)^{-1}$ for any invertible $a \in A^+$. For another property of Schwartz maps we refer to [362].

In proving the complete positivity of positive linear "pseudo-multiplicative" mappings (5.11) we followed the arguments of M.D. Choi [40] (cf. [301], Theorem 2.2). The complete positivity of "conditional expectations" (or linear projections of norm one; see 8.3 and Corollary 5.10) is also asserted in [204], Theorem 5.

The simplest example of a positive but not 2-positive linear map is the transposition map on M_2 ([17], I, p. 144). M.D. Choi [40], [41] has made a sistematic study of n-positive linear maps. Then, M.D. Choi [41] proved Proposition 5.7 and, using the map $\Phi : M_n \to M_n$ defined by $\Phi(x) = (n-1)(\operatorname{trace} x) - x$, showed that $P_{n-1}(M_n, M_n) \neq P_n(M_n, M_n)$. On the other hand, M.D. Choi [41] extended Proposition 5.5 and Proposition 5.6 by showing that $P_n(A, M_n(C)) = CP(A, M_n(C))$ and $P_n(M_n(C), B) = CP(M_n(C), B)$ whenever C is a commutative C^*-algebra. Moreover, M.D. Choi [41] made the following nice conjecture which seems to be still unsolved: "if $P_n(A, B) = P_{n+1}(A, B)$, then $P_n(A, B) = CP(A, B)$, $(n \geqslant 1)$"; also he conjectured the following extension of Proposition 5.7: "if $P_n(A, B) = P_{n+1}(A, B)$, then either A is a quotient space or B is a subalgebra of $M_n(C)$ for certain commutative C, $(n \geqslant 1)$". Of course, a first example of a positive but not completely positive linear mapping was given by W.F. Stinespring [293].

Together with a detailed discussion of the Stinespring dilation theorem in the unital case, W.B. Arveson ([17], I) extended the notion of complete positivity to linear mappings defined on selfadjoint vector subspaces, proved the remarkable extension theorem (5.4), introduced the concept of "boundary representations" and, as an application, obtained a classification up to unitary equivalence of certain Hilbert space operators which are neither normal, nor compact. The proof of Theorem 5.4 we have presented (cf. [308]) retains the main ideas of the original proof ([17]), but avoids the complications related to weak topologies. As remarked in [17], the requirement $1_A \in S$ in Theorem 5.4 can be weakened. A generalization of the Arveson theorem removing this requirement appears in [250], II. For C^*-subalgebras, the Arveson theorem reduces to Corollary 2/5.3 which is a simple application of the Stinespring dilation theorem combined with the corresponding result for $*$-representations (cf. [172], Section 4).

The Arveson theorem has important implications in the structure theory of operator algebras and, more generally, of "operator systems" (i.e., unital selfadjoint vector subspaces of $B(H)$). The relatively simple structure of completely positive linear mappings (5.3), together with the result of M.D. Choi [42] contained in Corollary 5.9, showed that they should be the appropriate morphisms in the category of operator systems. An operator system S is called injective if, whenever $R \subset T$ are operator systems, any morphism $R \to S$ can be extended to a morphism $T \to S$. A consequence of the Arveson theorem is that an operator system $S \subset B(H)$ is injective if and only if there is a linear projection of norm one of $B(H)$ onto S. The interest of such and related properties appeared previously in several papers dealing with examples of non-injective operator algebras (e.g. [123], [277]) and with tensor products of operator algebras ([85], [88], [171]). On the other hand, W.B. Arveson [18] pointed out a very useful connection of the "completely positive lifting problem" (see Corollary 5.13, for a sample) with the problem whether the Brown-Douglas-Fillmore Ext of a C^*-algebra is a group ([19], [34]).

Based on these studies, M.D. Choi and E.G. Effros [44] developed an extensive theory of injectivity in operator spaces and ([45], [46], [47]) applied it in order to clarify the notion of a nuclear C^*-algebra. Subsequently, A. Connes [63] proved, among other fundamental results, that for W^*-algebras, the injectivity is equivalent to the very strong and concrete property of "hyperfiniteness". Extending from the commutative case the result of [34], D. Voiculescu [338] proved that $\mathrm{Ext}(A)$ is a unital semigrup for any separable C^*-algebra, M.D. Choi and E.G. Effros [48] showed that $\mathrm{Ext}(A)$ is even a group if A is nuclear and simpler proofs of this last result appeared in [19], [339]. Excellent accounts on these topics can be found in [19], [86], [87].

Almost all the material included in 5.12–5.15 is borrowed from [44]. The statements of Proposition 1 and Proposition 2/5.13 are implicit in the article of M.D. Choi [43] who also showed that a linear map $\Phi : M_n \to M_m$ is completely positive if and only if it admits an expression $\Phi(a) = \sum\limits_i v_i^* a v_i$ where v_i are $n \times m$ matrices. Also, Corollary 5.13 is a first step in proving that $\mathrm{Ext}(A)$ is a group for A nuclear (cf. [339]).

There are several applications of completely positive linear mappings in problems related to mathematical physics (see, e.g., [96], [97], [98], [99], [100], [178], [179], [180], [181], [182]).

Further references: [51], [296], [301], [343], [344].

Chapter 6

LINEAR ISOMETRIES

6.1. In studying linear isometries between C^*-algebras one is primarily interested in the structure of the closed unit ball A_1 of a C^*-algebra A. As usually, the set of all extreme points of a convex set $S \subset A$ is denoted by $\mathrm{ex}(S)$.

LEMMA. *Let A be a C^*-algebra* and *X be a closed real vector subspace of A such that for every $x \in X$, X contains the ring generated by x and x^*. If $x \in \mathrm{ex}(X \cap A_1)$, then x is a partial isometry and*

$$X \cap [(1 - xx^*)A(1 - xx^*)] = \{0\}.$$

Proof. Put $e = x^*x$ and suppose $e^2 \neq e$. Then there exists $\lambda_0 \in \sigma(e)$, $0 < \lambda_0 < 1$. Let f be a positive continuous function on $[0, 1]$ such that $f(0) = 0$, $f(\lambda_0) \neq 0$ and $\sup\{\lambda(1 \pm f(\lambda))^2;\ \lambda \in [0, 1]\} \leqslant 1$. With $a = f(e)$ we have $ea \neq 0$. On the other hand, $\|e(1 \pm a)^2\| \leqslant 1$, hence

$$\|x \pm xa\|^2 = \|(x^* \pm ax^*)(x \pm xa)\| = \|e(1 \pm a)^2\| \leqslant 1.$$

By the assumption on X we have $xa \in X$, hence $x \pm xa \in X \cap A_1$. Since $x = 2^{-1}(x + xa) + 2^{-1}(x - xa)$, by the extremality of x, it follows that $x = x + xa = x - xa$. Thus $xa = 0$ hence $ea = 0$, a contradiction. Consequently, $e^2 = e$ so that x is a partial isometry.

Now suppose that there exists $b \in X \cap [(1 - xx^*)A(1 - x^*x)]$, $\|b\| = 1$. Since x^*x and xx^* are projections, we have successively $xx^*b = bx^*x = 0$, $b^*xx^*b = bx^*xb^* = 0$, $x^*b = xb^* = 0$. By Corollary 1.14 we infer that $x \pm b \in X \cap A_1$. Since $x = 2^{-1}(x + b) + 2^{-1}(x - b)$, the extremality of x entails $x = x + b = x - b$, so $b = 0$, a contradiction. ∎

THEOREM. *Let A be a C^* algebra and $x \in A$. Then*

(i) $\mathrm{ex}(A_1) \neq \emptyset \Leftrightarrow A$ *is unital;*

(ii) $x \in \mathrm{ex}(A_1) \Leftrightarrow (1 - xx^*)A(1 - x^*x) = \{0\}$.

Proof. Suppose that there exists $x \in \mathrm{ex}(A_1)$. Denote $e = x^*x$, $f = xx^*$, $u = f + e - fe$ and consider an approximate unit $\{u_\iota\}$ for A. By the above lemma, $(1 - f)A(1 - f) = \{0\}$, so

$$u_\iota = fu_\iota + u_\iota e - fu_\iota e \to f + e - fe = u.$$

It follows that u is a unit element for A.

Conversely assume that A is unital. Let $a, b \in A_1$, $a + b = 2$. Then $\operatorname{Re} a + \operatorname{Re} b = 2$, hence $\operatorname{Re} a, \operatorname{Re} b$ commute. By Gelfand representation we get $\operatorname{Re} a = \operatorname{Re} b = 1$. Since $a, b \in A_1$, this entails $a = b = 1$. Consequently $1 \in \operatorname{ex}(A_1)$.

By the above lemma, $x \in \operatorname{ex}(A_1) \Rightarrow (1 - xx^*)A(1 - x^*x) = \{0\}$.

Let $x \in A$ be such that $(1 - xx^*)A(1 - x^*x) = \{0\}$. Then $x^*x(1 - x^*x)^2 = x^*(1 - xx^*)x(1 - x^*x) = 0$, so $e = x^*x$ is a projection. Hence $f = xx^*$ is also a projection, $x \in A_1$ and $x = fx = xe$ (1.6). Suppose that $2x = p + q$ for some $p, q \in A_1$. Then

$$\begin{gathered}(pe + qe)^*(pe + qe) = (ep^*pe + eq^*qe) + (ep^*qe + eq^*pe),\\ 4e = 4ex^*xe = (pe + qe)^*(pe + qe) \leqslant 2(ep^*pe + eq^*qe) \leqslant 4e,\end{gathered}$$

so that

$$ep^*pe + eq^*qe = 2e = ep^*qe + eq^*pe.$$

Since e is the unit of eAe, hence an extreme point in $(eAe)_1$,

$$ep^*pe = eq^*qe = ep^*qe + eq^*pe = e,$$

thus $((p - q)e)^*((p - q)e) = 0$ and $pe = qe$. Similarly, $fp = fq$. It follows that

$$p - q = (1 - f)(p - q)(1 - e) \in (1 - xx^*)A(1 - x^*x) = \{0\},$$

hence $p = q = x$. Therefore $x \in \operatorname{ex}(A_1)$. ∎

In particular, every isometry or coisometry in a unital C^*-algebra A is an extreme point of A_1. It is easy to check that *unitaries are the only normal and the only invertible extreme points of* A_1.

6.2. Jordan algebras. Let A be a C^*-algebra. We say that J is a Jordan algebra in A if J is a real vector subspace of A_h and for every $x, y \in J$ the "Jordan product" $2^{-1}(xy + yx)$ belongs to J. Since

$$x^2 = 2^{-1}(xx + xx) \quad \text{and} \quad xy + yx = (x + y)^2 - x^2 - y^2,$$

a real vector subspace J of A_h is a Jordan algebra if and only if $x^2 \in J$ whenever $x \in J$.

Let J be a Jordan algebra in A. Consider also the "Lie product"

$$[x, y] = xy - yx; \quad x, y \in A.$$

Then

(1) $x \in J$, $n \geqslant 1$ integer $\Rightarrow x^n \in J$;

(2) $x, y \in J \Rightarrow xyx \in J$;

(3) $x, y, z \in J \Rightarrow xyz + zyx \in J$;

(4) $x, y, z \in J \Rightarrow [[x, y], z] \in J$;

(5) $x, y \in J \Rightarrow [x, y]^2 \in J$.

Indeed, (1) can be proven by induction using

$$x^{n+1} = 2^{-1}(x^n x + x x^n); \tag{6}$$

assertion (2) follows from (1) using

$$2xyx = (y+x)^3 + (y-x)^3 - 2y^3 - 2(yx^2 + x^2 y); \tag{7}$$

assertion (3) follows from (2) using

$$xyz + zyx = (x+z)y(x+z) - xyx - zyz; \tag{8}$$

assertion (4) follows from (3) using

$$[[x,y],z] = (xyz + zyx) - (yxz + zxy); \tag{9}$$

and finally (5) follows from (2) using

$$[x,y]^2 = (x(yxy) + (yxy)x) - xy^2x - yx^2y. \tag{10}$$

We say that J is unital if there exists $u \in J$ such that $ux = xu = x$ for all $x \in J$. If J is unital, then its unit $1_J = u$ is unique and is a projection of A. Moreover, 1_J is the unit element of the C^*-subalgebra $C^*(J)$ of A generated by J.

Let J be a norm-closed Jordan algebra in A. By 1.16.(8) we have

$$x \in J, \quad f \in C\left(\sigma(x) \cup \{0\}\right), \quad f \text{ real}, \quad f(0) = 0 \Rightarrow f(x) \in J \tag{11}$$

that is

$$x \in J \Rightarrow C^*(\{x\})_h \subset J.$$

More generally, if S is a family of mutually commuting elements of J, then

$$C^*(S)_h \subset J.$$

In particular, if J is a unital norm-closed Jordan algebra in A, then

$$x \in J,\ x \text{ invertible in } C^*(J) \Rightarrow x^{-1} \in J. \tag{12}$$

Remark that if J is a norm closed vector subspace of A_h generated by a convex cone C such that $x^2 \in J$ for every $x \in C$, then J is a Jordan algebra.

Indeed, if y is an element of the real vector subspace generated by C, then $y = x_1 - x_2$ for some $x_1, x_2 \in C$, so

$$y^2 = 2x_1^2 + 2x_2^2 - (x_1 + x_2)^2 \in J.$$

Since X is norm dense in J, it follows that $z \in J \Rightarrow z^2 \in J$.

Also, let J be a norm closed real vector subspace of A_h which contains an element u such that $ux = xu = x$ for all $x \in J$ and assume that J is the norm closed linear hull of a convex cone C such that $u \in C$ and $x^{-1} \in J$ whenever $x \in C$ is invertible in $C^*(J)$. Then J is a Jordan algebra, because for every $x \in C$ we have

$$x^2 = \underset{t \to 0}{\text{norm-lim}}\, t^{-2}((u + tx)^{-1} - u + tx) \in J.$$

The most typical example of norm-closed Jordan algebras in A are the real parts of C^*-subalgebras of A. A norm-closed Jordan algebra J in A is the real part of a C^*-subalgebra of A if and only if

$$x, y \in J \Rightarrow \mathrm{i}(xy - yx) \in J. \tag{13}$$

Indeed, $B = J + \mathrm{i}J$ is then a C^*-subalgebra of A and $J = B_h$.

Note that not every norm-closed Jordan algebra in A is the real part of a C^*-subalgebra of A. For instance, let $A = M_2$. Then the set $J \subset A$ of all real symmetric matrices is a Jordan algebra in A, but J does not satisfy (13).

THEOREM. *Let A be a C^*-algebra, J be a norm-closed Jordan algebra in A and $x \in J$. Then*
(i) $\mathrm{ex}(J_1) \neq \emptyset \Leftrightarrow J$ *is unital;*
(ii) $x \in \mathrm{ex}(J_1) \Leftrightarrow x^2$ *is the unit of J.*

Proof. Let $x \in \mathrm{ex}(J_1)$. By (1) we can apply Lemma 6.1 with $X = J$ to get $J \cap [(1-x^2)A(1-x^2)] = \{0\}$. But using (1) and (2), for any $y \in J$ we obtain

$$(1-x^2)y(1-x^2) = y - (x^2y + yx^2) + x^2yx^2 \in J,$$

so $(1-x^2)J(1-x^2) = \{0\}$. Consequently, for any $y \in J$ we get successively $(1-x^2)y^2(1-x^2) = 0$, $y(1-x^2) = 0$, $y = yx^2 = x^2y$. We conclude that J is unital and x^2 is the unit of J.

Conversely, assume that J is unital and let $x \in J$ such that x^2 is the unit of J. Then x^2 is the unit of the C^*-subalgebra B of A generated by J. By Theorem 6.1 it follows that $x \in \mathrm{ex}(B_1) \cap J \subset \mathrm{ex}(J_1)$. ∎

In particular, if A is a C^*-algebra and $x \in \mathrm{ex}(A_h \cap A_1)$, then the Jordan algebra A_h is unital with unit x^2, i.e. A is unital and $x^2 = 1$. Therefore *selfadjoint unitaries are the only extreme points of $A_h \cap A_1$.*

Also, remark that *the extreme points of $A^+ \cap A_1$ are exactly the projections of A.* Indeed, if $e \in A$ is a projection and $2e = a + b$ with $a, b \in A^+ \cap A_1$, then $a, b \in (eAe)_1$ so that $a = b = e$ since by Theorem 6.1 e is an extreme point of $(eAe)_1$. Conversely, if $x \in \mathrm{ex}(A^+ \cap A_1)$, then $x \in \mathrm{ex}(B^+ \cap B_1)$, where B is the C^*-subalgebra generated by x, and using the Gelfand representation of B we see that x is a projection.

6.3. Russo-Dye theorem. Let A be a unital C^*-algebra. Denote by $U(A)$ the unitary group of A, by $\mathrm{co}\, U(A)$ its convex hull and by $\overline{\mathrm{co}}\, U(A)$ its norm closed convex hull. Consider also

$$\exp(\mathrm{i}A_h) = \{\exp(\mathrm{i}a);\, a \in A,\, a = a^*\} \subset U(A)$$

and its convex hull $\mathrm{co}\, \exp(\mathrm{i}A_h)$, respectively its norm-closed convex hull $\overline{\mathrm{co}}\, \exp(\mathrm{i}A_h)$. Recall that (2.8.(4))

$$\{x \in A;\, \|x\| < 2^{-1}\} \subset \mathrm{co}\, \exp(\mathrm{i}A_h) \subset \mathrm{co}\, U(A). \tag{1}$$

The following theorem contains a much stronger result:

THEOREM. *Let A be a unital C^*-algebra. Then*
(i) $\{x \in A;\, \|x\| < 1\} \subset \mathrm{co}\, \exp(\mathrm{i}A_h) \subset \mathrm{co}\, U(A)$;
(ii) $\{x \in A;\, \|x\| \leqslant 1\} = \overline{\mathrm{co}}\, \exp(\mathrm{i}A_h) = \overline{\mathrm{co}}\, U(A)$.

Proof. Let $x \in A, \|x\| < r < 1$. For $\lambda \in \mathbb{C}, \lambda < \frac{1}{r}$, define

$$u_x(\lambda) = (1 - xx^*)^{-\frac{1}{2}}(\lambda + x)(1 + \lambda x^*)^{-1}(1 - xx^*)^{\frac{1}{2}}.$$

Then $\{\lambda \in \mathbb{C};\, |\lambda| < \frac{1}{r}\} \ni \lambda \mapsto u_x(\lambda) \in A$ is an analytic function so that, by the Cauchy integral formula,

$$u_x(0) = \frac{1}{2\pi}\int_0^{2\pi} u_x(\mathrm{e}^{\mathrm{i}t})\mathrm{d}t. \tag{2}$$

Using successively the elementary identities

$$\begin{gathered}(1+\lambda x^*)^{-1}(\lambda + x) = x + \lambda(1+\lambda x^*)^{-1}(1 - xx^*); \quad |\lambda| \leqslant 1,\\ (1 - xx^*)x = x(1 - x^*x)\\ (\lambda + x)(1+\lambda x^*)^{-1} = x + \lambda(1 - xx^*)(1+\lambda x^*)^{-1}; \quad |\lambda| \leqslant 1,\end{gathered}$$

it is easy to verify that, for $\lambda \in \mathbb{C}$, $|\lambda| = 1$,

$$\begin{aligned}(u_x(\lambda))^{-1} &= (1 - xx^*)^{\frac{1}{2}}(1+\lambda x^*)^{-1}(\lambda + x)(1 - x^*x)^{-\frac{1}{2}}\\ &= (1 - xx^*)^{-\frac{1}{2}}[(1 - xx^*)(1+\lambda x^*)^{-1}(\lambda + x)(1 - x^*x)^{-1}](1 - x^*x)^{\frac{1}{2}}\\ &= (1 - xx^*)^{-\frac{1}{2}}(\lambda + x)(1+\lambda x^*)^{-1}(1 - x^*x)^{\frac{1}{2}} = u_x(\lambda),\end{aligned}$$

hence $u_x(\lambda) \in U(A)$. From $(1 - xx^*)x = x(1 - x^*x)$ by functional calculus we infer $(1 - xx^*)^{-\frac{1}{2}}x = x(1 - x^*x)^{-\frac{1}{2}}$, which in turn gives $u_x(0) = x$.

Thus (2) and the classical approximation of an integral by Riemann sums shows that $x \in \overline{\mathrm{co}}\,\{u_x(\lambda);\, \lambda \in \mathbb{C},\, |\lambda| = 1\}$. Hence

$$\{x \in A;\, \|x\| < 1\} \subset \overline{\mathrm{co}}\, U(A). \tag{3}$$

Let $v \in U(A)$, $0 < t < 1$ and put $x = tv$. Since $x^*x = xx^* = t^2$, for each $\lambda \in \mathbb{C}$, $|\lambda| = 1$ we have $u_x(\lambda) = (\lambda + x)(1 + \lambda x^*)^{-1}$, so

$$(\lambda + u_x(\lambda))(1 + \lambda x^*) = 2\lambda(1 + 2^{-1}(\lambda x^* + \overline{\lambda} x)).$$

As $\|x\| < 1$, this shows that $\lambda + u_x(\lambda)$ is invertible, hence $u_x(\lambda)$ is a unitary with $\sigma(u_x(\lambda)) \neq \mathbb{C}$, so that $u_x(\lambda) \in \exp(\mathrm{i}A_h)$ by 1.18.(6) for all $\lambda \in \mathbb{C}$, $|\lambda| = 1$. It follows that

$$tv = x \in \overline{\mathrm{co}}\,\{u_x(\lambda);\, \lambda \in \mathbb{C},\, |\lambda| = 1\} \subset \overline{\mathrm{co}}\, \exp(\mathrm{i}A_h).$$

Since $0 < t < 1$ was arbitrary, we deduce

$$U(A) \subset \overline{\mathrm{co}}\, \exp(\mathrm{i}A_h). \tag{4}$$

By (3) and (4),

$$\{x \in A;\, \|x\| \leqslant 1\} = \overline{\mathrm{co}}\, \exp(\mathrm{i}A_h) = \overline{\mathrm{co}}\, U(A). \tag{5}$$

Consider again $x \in A$, $\|x\| < 1$. Then $\|sx\| < 1$ for some $s > 1$. By (5) there is $x_1 \in \operatorname{co}\exp(\mathrm{i}A_h)$ with $\|sx - x_1\| < 2^{-1}(s-1)$. By (1) there is $x_2 \in \operatorname{co}\exp(\mathrm{i}A_h)$ with $sx - x_1 = (s-1)x_2$, that is,

$$x = s^{-1}x_1 + (1 - s^{-1})x_2 \in \operatorname{co}\exp(\mathrm{i}A_h). \quad \blacksquare$$

In particular $A_1 = \overline{\operatorname{co}}\operatorname{ex}(A_1)$, a rather surprising fact since A_1 is not necessarily compact in some vector space topology on A.

Since A is the linear span of $U(A)$ we can, for each $x \in A$, define

$$\|x\|_U = \inf\Big\{\sum_{k=1}^{n} |\lambda_k|;\, x = \sum_{k=1}^{n} \lambda_k u_k,\ \lambda_k \in \mathbb{C},\ u_k \in U(A),\ n \in \mathbb{N}\Big\}.$$

COROLLARY 1. *For all $x \in A$, $\|x\|_U = \|x\|$.*

Proof. Clearly, $\|x\| \leqslant \|x\|_U$ and $\|\cdot\|_U$ is positive homogeneous. For any $0 < t < 1$, by the above theorem, $t\|x\|^{-1}x \in \operatorname{co} U(A)$, hence $t\|x\|^{-1}\|x\|_U = \big\|t\|x\|^{-1}x\big\|_U \leqslant 1$. Since $0 < t < 1$ was arbitrary, it follows that $\|x\|^{-1}\|x\|_U \leqslant 1$, i.e. $\|x\|_U \leqslant \|x\|$. $\blacksquare$

COROLLARY 2. *Let Φ be a linear mapping of A into a normed space X. Then*

$$\|\Phi\| = \sup\{\|\Phi(u)\|;\, u \in U(A)\}.$$

Proof. Let $\alpha = \sup\{\|\Phi(u)\|;\, u \in U(A)\}$. Clearly, $\alpha \leqslant \|\Phi\|$. On the other hand it is easy to see that $\|\Phi(x)\| \leqslant \alpha\|x\|_U$, $x \in A$, hence $\|\Phi\| \leqslant \alpha$ by Corollary 1. $\blacksquare$

6.4. An important consequence of the Russo-Dye theorem is the following

PROPOSITION. *Let A, B be unital C^*-algebras and $\Phi : A \mapsto B$ be a linear mapping such that $\Phi(1) = 1$. Then*

$$\Phi \textit{ is positive} \Leftrightarrow \|\Phi\| = 1.$$

Proof. Assume that Φ is positive. Using the Kadison inequality (Proposition 5.8), we get for each $u \in U(A)$

$$\|\Phi(u)\|^2 = \|\Phi(u)^*\Phi(u)\| \leqslant \|\Phi(u^*u)\| = \|\Phi(1)\|,$$

hence, by Corollary 2 of 6.3, $\|\Phi\| = \|\Phi(1)\| = 1$.

Conversely, assume $\|\Phi\| = 1$. For any $\psi \in S(B)$ we have $(\psi \circ \Phi)(1) \leqslant \|\psi \circ \Phi\| \leqslant 1 = (\psi \circ \Phi)(1)$, hence $\psi \circ \Phi \in S(A)$ by Proposition 4.6. Owing to Proposition 4.13.(i) we infer that Φ is positive. $\blacksquare$

Remark that in the first part of the proof we have in fact proved that if A is a unital C^*-algebra, B is an arbitrary C^*-algebra and $\Phi : A \mapsto B$ is a positive linear mapping, then $\|\Phi\| = \|\Phi(1)\|$.

6.5. Function representation for Jordan algebras. Let A be a C^*-algebra and J be a norm closed Jordan algebra in A. Denote by J^+ the closed convex cone of all $x \in J$, $x \geqslant 0$. Since $x \in J \Rightarrow x^+,\ x^- \in J$, we have $J = J^+ - J^+$. By Proposition 1/4.10 it follows that every positive linear functional on J is bounded.

As easily verified, a linear functional φ on J is positive if and only if for each $x \in J$ there exists a positive form on $C^*(\{x\})$ which coincides with φ on $C^*(\{x\})_h \subset J$. Using 4.5.(6) we infer that, for every positive linear functional φ on J we have

$$\|\varphi\| = \sup\{\varphi(x);\ x \in J^+,\ \|x\| \leqslant 1\}.$$

If J has a unit element u, then a linear functional φ on J is positive if and only if $\|\varphi\| = \varphi(u)$.

Denote by J^* the dual space of the real Banach space J, by J^*_+ the J-closed convex cone of all positive linear functionals on J, by $Q(J)$ the J-compact convex set of all $\varphi \in J^*_+$ with $\|\varphi\| \leqslant 1$ and by $S(J)$ the set of all $\varphi \in J^*_+$ with $\|\varphi\| = 1$; if J is unital, then $S(J)$ is a J-compact convex set. Finally, let $\mathbf{A}(Q(J))$ be the ordered real Banach space of all J-continuous affine real functions on $Q(J)$ with $f(0) = 0$.

Let B be the C^*-algebra obtained from A by *adjoining the unit.* Then $K = \text{lin}(J \cup \{1\})$ is a norm closed Jordan algebra in B and every $\varphi \in J^*_+$ can be extended to some $\theta \in K^*_+$ with $\theta(1) = \|\varphi\| = \|\theta\|$. Thus, using Proposition 3/4.10 as in the proof of Proposition 4.16, we see that *every $\varphi \in J^*_+$ can be extended to some $\psi \in A^*_+$ with $\|\psi\| = \|\varphi\|$.*

Using this last remark and arguments similar to the proof of Proposition 4.15 we obtain the following extensions of Proposition 4.15:

Proposition. *Let A be a C^*-algebra and J a norm closed Jordan algebra in A. Then the mapping $\Phi : J \mapsto \mathbf{A}(Q(J))$ defined by*

$$[\Phi(x)](\varphi) = \varphi(x); \quad x \in J,\ \varphi \in Q(J)$$

is an isometric linear order isomorphism of J onto $\mathbf{A}(Q(J))$. ∎

6.6. Jordan homomorphisms. Let A, B be C^*-algebras and J, K be Jordan algebras in A, B respectively. A linear mapping $\Phi : J \mapsto K$ is called a *Jordan homomorphism* if it preserves the Jordan product, i.e.,

$$\Phi(xy + yx) = \Phi(x)\Phi(y) + \Phi(y)\Phi(x); \quad x, y \in J.$$

As in 6.2 we see that Φ is a Jordan homomorphism if and only if

$$\Phi(x^2) = \Phi(x)^2; \quad x \in J.$$

Moreover, if Φ is a Jordan homomorphism, then, using 6.2.(6)–(10) we obtain

(1) $x \in J,\ n \geqslant 1$ integer $\Rightarrow \Phi(x^n) = \Phi(x)^n$;

(2) $x, y \in J \Rightarrow \Phi(xyx) = \Phi(x)\Phi(y)\Phi(x)$;

(3) $x, y, z \in J \Rightarrow \Phi(xyz + zyx) = \Phi(x)\Phi(y)\Phi(z) + \Phi(z)\Phi(y)\Phi(x)$;

(4) $x, y, z \in J \Rightarrow \Phi([[x, y], z]) = [[\Phi(x), \Phi(y)], \Phi(z)]$;

(5) $x, y \in J \Rightarrow \Phi([x, y]^2) = [\Phi(x), \Phi(y)]^2$.

We have also

(6) $$x, y \in J,\ [x, y] = 0 \Rightarrow [\Phi(x), \Phi(y)] = 0.$$

Indeed, $[\Phi(x), \Phi(y)]^*[\Phi(x), \Phi(y)] = -[\Phi(x), \Phi(y)]^2 = 0$ by (5), so $[\Phi(x), \Phi(y)] = 0$. Furthermore, if J is norm closed, then

(7) $$x, y \in J,\ [x, y] = 0 \Rightarrow \Phi(xy) = \Phi(x)\Phi(y).$$

Indeed, if x commutes with y, then also $(x^+)^{\frac{1}{2}}, (x^-)^{\frac{1}{2}} \in J$ commute with y, so by (2), (6) and (1) we have

$$\Phi(x^+y) = \Phi((x^+)^{\frac{1}{2}} y (x^+)^{\frac{1}{2}}) = \Phi((x^+)^{\frac{1}{2}})\Phi(y)\Phi((x^+)^{\frac{1}{2}}) = \Phi(x^+)\Phi(y)$$

and similarly $\Phi(x^-y) = \Phi(x^-)\Phi(y)$. A bijective Jordan homomorphism is called a *Jordan isomorphism.*

PROPOSITION. *Let A, B be C^*-algebras, J be a norm closed Jordan algebra in A and $\Phi : J \mapsto B_h$ be a Jordan homomorphism. Then*

(i) $\|\Phi\| \leqslant 1$*;*

(ii) $\Phi(J)$ *is a norm-closed Jordan algebra in B;*

(iii) *if $x \in J$ and $f \in C(\sigma(x) \cup \{0\})$ is real with $f(0) = 0$ then*

$$\Phi(f(x)) = f(\Phi(x));$$

(iv) *if J is unital, then $\Phi(J)$ is unital and $\Phi(1_J) = 1_{\Phi(J)}$. In this case, if $x \in J$ is invertible in $C^*(J)$, then $\Phi(x)$ is invertible in $C^*(\Phi(J))$ and $\Phi(x)^{-1} = \Phi(x^{-1})$;*

(v) *if $a, b \in J$, $a \leqslant b$, and $y \in \Phi(J)$, $\Phi(a) \leqslant y \leqslant \Phi(b)$, then there exists $x \in J$, $a \leqslant x \leqslant b$, such that $\Phi(x) = y$.*

Proof. If S is a family of mutually comuting elements of J, then $C^*(S)_h \subset J$ and, by (7), Φ coincides on $C^*(S)$ with a $*$-homomorphism of $C^*(S)$ into B.

Hence (i), (iii), (iv) follow from the corresponding statements for C^*-algebras and $*$-homomorphisms.

Clearly, $\Phi(J)$ is a Jordan algebra in B. Let $y \in B_h$ be norm-adherent to $\Phi(J)$. Then there is a sequence $\{x_n\}_n$ in J such that $\sum\limits_{n=1}^{\infty} \|\Phi(x_n)\| < +\infty$ and $y = \sum\limits_{n=1}^{\infty} \Phi(x_n)$. By the remark at the beginning of the proof and by Corollary 1/3.15, for each n there exists an element $z_n \in C^*(x_n)_h \subset J$ such that $\|z_n\| \leqslant \|\Phi(x_n)\|$ and $\Phi(z_n) = \Phi(x_n)$. Since $\sum\limits_{n=0}^{\infty} \|z_n\| < +\infty$, we can consider $z = \sum\limits_{n=0}^{\infty} z_n \in J$ and then we have $\Phi(z) = \sum\limits_{n=1}^{\infty} \Phi(z_n) = \sum\limits_{n=1}^{\infty} \Phi(x_n) = y$. Consequently, $\Phi(J)$ is norm closed.

Finally, let $a, b \in J$, $a \leqslant b$, and $y \in \Phi(J)$, $\Phi(a) \leqslant y \leqslant \Phi(b)$. We must show that there exists $d \in J$, $0 \leqslant d \leqslant b - a$, such that $\Phi(d) = y - \Phi(a)$. Let $z \in J$ be such that $\Phi(z) = y$ and denote $v = b - a - |b - z|$. Then $v = v^+ - v^- \leqslant b - a$, so $v^+ \leqslant b - a + v^-$. Define, for each integer $n \geqslant 1$,

$$\begin{aligned} d_n = {} & (b-a)^{\frac{1}{2}}(b-a+v^-)^{\frac{1}{2}} \left(\tfrac{1}{n} + b - a + v^-\right)^{-1} v^+ \\ & \times \left(\tfrac{1}{n} + b - a + v^-\right)^{-1} (b-a+v^-)^{\frac{1}{2}}(b-a)^{\frac{1}{2}}. \end{aligned}$$

By 6.2.(2), all d_n belong to J and $0 \leqslant d_n \leqslant b - a$. By (iii),

$$\Phi(v) = \Phi(b) - \Phi(a) - |\Phi(b) - y| = y - \Phi(a) \geqslant 0,$$

so $\Phi(v^+) = \Phi(v)^+ = y - \Phi(a)$ and $\Phi(v^-) = \Phi(v)^- = 0$. Using (2) and the remark at the beginning of the proof we infer that

$$\begin{aligned} \Phi(d_n) = {} & (\Phi(b) - \Phi(a)) \left(\tfrac{1}{n} + \Phi(b) - \Phi(a)\right)^{-1} (y - \Phi(a)) \\ & \times \left(\tfrac{1}{n} + \Phi(b) - \Phi(a)\right)^{-1} (\Phi(b) - \Phi(a)). \end{aligned}$$

It is easy to see that $\{d_n\}$ is a Cauchy sequence, so it converges to some $d \in J$ and $0 \leqslant d \leqslant b - a$. Since $\{\Phi(d_n)\}$ converges to $y - \Phi(a)$, we get $\Phi(d) = y - \Phi(a)$. ∎

Let A, B be C^*-algebras. If $\Phi : A_h \mapsto B_h$ is a Jordan homomorphism, then Φ can be extended to a unique linear mapping $\Psi : A \mapsto B$ such that

$$\Psi(x^*) = \Psi(x)^*, \quad x \in A;$$

$$\Psi(xy + yx) = \Psi(x)\Psi(y) + \Psi(y)\Psi(y); \quad x, y \in A.$$

Conversely, if $\Psi : A \mapsto B$ is a linear mapping satisfying the above conditions, then the restriction of Ψ to A_h is a Jordan homomorphism of A_h into B_h. Such a mapping Ψ is called a *Jordan $*$-homomorphism* of A into B.

A Jordan $*$-homomorphism satisfies statements (1) to (6) for all $x, y, z \in A$. Only the proof of (6) is somewhat different. If $x, y \in A$ and $[x, y] = 0$, then $[\Psi(x), \Psi(y)]$ commutes with all $\Psi(z)$, $z \in A$, by (4), hence it is normal. On the other hand, by (5), $[\Psi(x), \Psi(y)]^2 = 0$, so the spectral radius of $[\Psi(x), \Psi(y)]$ is 0 and consequently $[\Psi(x), \Psi(y)] = 0$.

In particular, the Jordan $*$-homomorphism Ψ maps normal elements into normal elements and preserves continuous functional calculus for normal elements.

6.7. Isometries of Jordan algebras. Let A be a C^*-algebra and J be a Jordan algebra in A. We say that z is a central element of J if $z \in J$ and $zx = xz$ for all $x \in J$. Then z is central in the C^*-subalgebra of A generated by J.

THEOREM. *Let A, B be C^*-algebras, J be a unital norm closed Jordan algebra in A, K be a Jordan algebra in B and $T : J \mapsto K$ be a real linear bijection. The following statements are equivalent:*
(i) *T is a unitary isometry.*
(ii) *K is unital and there exist a unitary central element v of K and a Jordan isomorphism $\Phi : J \mapsto K$ such that*

$$T(x) = v\Phi(x); \quad x \in J.$$

Proof. (ii) $\Rightarrow$ (i). Let $x \in J$. Then Φ coincides on $C^*(\{x\})_h$ with an injective $*$-homomorphism of $C^*(\{x\})$ into B, hence $\|T(x)\| = \|\Phi(x)\| = \|x\|$ by Corollary 1.15.

(i) $\Rightarrow$ (ii). K is norm closed and using Theorem 6.2 we successively get: $1_J \in \mathrm{ex}(J_1)$, $T(1_J) \in \mathrm{ex}(K_1)$, K is unital and $T(1_J)$ is unitary. Put $v = T(1_J)$ and define a real linear mapping $\Phi : J \mapsto K$ by

$$\Phi(x) = \mathrm{Re}\,(vT(x)) = 2^{-1}(vT(x) + T(x)v); \quad x \in J.$$

Then Φ is injective. Indeed, if $x \in J$ and $vT(x) = \mathrm{i}b$ for some $b \in B_h$, then for every $t \in \mathbb{R}$ we have

$$\|1_J + tx\| = \|T(1_J + tx)\| = \|v + tT(x)\| = \|v(1_B + \mathrm{i}tb)\| = (1 + t^2\|b\|^2)^{\frac{1}{2}}$$

so that $\lim\limits_{t\to 0} t^{-1}(\|1_J + tx\| - 1) = 0$. But $\|x^+\| \leqslant t^{-1}(\|1_J + tx\| - 1)$ for $t > 0$ and $\|x^-\| \leqslant t^{-1}(\|1_J + tx\| - 1)$ for $t < 0$, hence $x^+ = x^- = 0$, $x = x^+ - x^- = 0$.

Now let $x \in J$ be arbitrary. Then $v\Phi(x) = 2^{-1}(T(x) + vT(x)v) \in K$, hence there exists $y \in J$ with $v\Phi(x) = T(y)$, that is, $\Phi(x) = vT(y)$. Since $vT(y)$ is selfadjoint,

$$\Phi(x) = vT(y) = \mathrm{Re}\,(vT(y)) = \Phi(y).$$

By the injectivity of Φ it follows that $x = y$, so

$$\Phi(x) = vT(x); \quad x \in J.$$

Since $vT(x)$ and $T(x)$ are selfadjoint, v commutes with $T(x)$ for all $x \in J$, so v is a central element of K. Thus $\Phi : J \mapsto K$ is a linear bijection and, by the definition of v, $\Phi(1_J) = \Phi(1_K)$. Since Φ is an isometry, both Φ and Φ^{-1} are positive. By the Kadison inequality (5.8) we have, for any $x \in J$, $\Phi(x)^2 \leqslant \Phi(x^2)$ and $x^2 = \Phi^{-1}(\Phi(x))^2 \leqslant \Phi^{-1}(\Phi(x)^2)$, $\Phi(x^2) \leqslant \Phi(x)^2$, hence $\Phi(x^2) = \Phi(x)^2$. Thus Φ is a Jordan homomorphism. ∎

Combining the above theorem with Proposition 6.5 we get

COROLLARY. *Let A, B be C^*-algebras, J, K be norm closed Jordan algebras in A, B respectively and assume that J is unital. Then the following statements are equivalent:*
(i) *there exists a Jordan isomorphism of J onto K;*
(ii) *there exists an affine homeomorphism of $Q(J)$ (with J-topology) onto $Q(K)$ (with K-topology);*
(iii) *there exists a linear isometry of J onto K.* ∎

Clearly, we can replace in (ii) $Q(J), Q(K)$ by $S(J), S(K)$ respectively. Note that the above results hold in particular for $J = A_h$, $K = B_h$. The results of this section will be extended in 9.32 also for not necessarily unital norm-closed Jordan algebras.

6.8. Isometries of C^*-algebras. We shall need a characterization of Jordan $*$-homomorphisms between unital C^*-algebras which is of independent interest:

LEMMA. *Let A, B be unital C^*-algebras and $\Psi : A \mapsto B$ be a linear mapping such that $\Psi(1) = 1$. Then Ψ is a Jordan $*$-homomorphism if and only if $\Psi(U(A)) \subset U(B)$ and in this case $\|\Psi\| = 1$.*

Proof. If Ψ is a Jordan $*$-homomorphism, then by 6.6 for each $u \in U(A)$ we have $\Psi(u)^*\Psi(u) = \Psi(u^*u) = \Psi(1) = 1$ and similarly $\Psi(u)\Psi(u)^* = 1$, so $\Psi(u) \in U(B)$.

Now suppose that $\Psi(U(A)) \subset U(B)$. Then, by Corollary 2/6.3, $\|\Psi\| = 1$ and, by Proposition 6.4, Ψ is positive. In particular, $\Psi(A_h) \subset B_h$. Let $x \in A_h$, $\|x\| < 1$. By 2.8.(4) there exists $u \in U(A)$ such that $x = u + u^*$. Consider also $y = \mathrm{i}(u - u^*)$. Since $\Psi(U(A)) \subset U(B)$, using the Kadison inequality (5.8) we obtain

$$\Psi(u^2) + \Psi((u^*)^2) = \Psi(x^2) - 2 \geqslant \Psi(x)^2 - 2 = \Psi(u)^2 + \Psi(u^*)^2,$$

$$\Psi(u^2) + \Psi((u^*)^2) = -\Psi(y^2) + 2 \leqslant -\Psi(y)^2 + 2 = \Psi(u)^2 + \Psi(u^*)^2,$$

hence $\Psi(x^2) = \Psi(x)^2$. Thus Ψ is a Jordan $*$-homomorphism. ∎

THEOREM. *Let A be a unital C^*-algebra, B be a C^*-algebra and $T : A \mapsto B$ be a linear bijection. Then the following statements are equivalent:*

(i) *T is an isometry;*

(ii) *B is unital and there exist a unitary $v \in B$ and a Jordan $*$-homomorphism $\Psi : A \mapsto B$ such that*

$$T(x) = v\Phi(x); \quad x \in A.$$

Proof. (ii) $\Rightarrow$ (i). By the above lemma, $\|\Psi\| = \|\Psi^{-1}\| = 1$, so Ψ, and also T is an isometry.

(i) $\Rightarrow$ (ii). Using Theorem 6.1 we get successively $1 \in \mathrm{ex}(A_1)$, $T(1) \in \mathrm{ex}(B_1)$, B is unital and $T(1)$ is a partial isometry. Let $x \in A$ be such that $T(x) = 1 - T(1)^*T(1)$. Then $T(x)T(1)^* = 0$ and $T(x)T(x)^* = 1 - T(1)^*T(1) \leqslant T(1)T(1)^*$, so for each $\lambda \in \mathbb{C}$ we get

$$\begin{aligned}\|1 + \lambda x\| &= \|T(1) + \lambda T(x)\| = \|(T(1) + \lambda T(x))(T(1)^* + \overline{\lambda} T(x))^*\|^{\frac{1}{2}} \\ &= \|T(1)T(1)^* + |\lambda|^2 T(x)T(x)^*\|^{\frac{1}{2}} = (1 + |\lambda|^2 \|T(x)\|^2)^{\frac{1}{2}}.\end{aligned}$$

It follows that

$$\lim_{\mathbb{R} \ni t \to 0} t^{-1}(\|1 + \mathrm{i}tx\| - 1) = 0, \quad \lim_{\mathbb{R} \ni t \to 0} t^{-1}(\|1 + \mathrm{i}t(\mathrm{i}x)\| - 1) = 0,$$

and by Corollary 1/4.13 we infer that both x and ix are selfadjoint, that is $x = 0$. Hence $T(1)^*T(1) = 1$. Similarly, $T(1)T(1)^* = 1$.

Thus $v = T(1) \in B$ is unitary. Define a linear bijection $\Psi : A \mapsto B$ by

$$\Psi(x) = v^*T(x); \quad x \in A.$$

Then Ψ is an isometry and $\Psi(1) = 1$, so Ψ is positive by Proposition 6.4, in particular $\Psi(A_h) = B_h$. Applying Theorem 6.7 to the restriction of Ψ to A_h we infer that Ψ is a Jordan $*$-homomorphism. ∎

COROLLARY. *Let A, B be unital C^*-algebras and $\Psi : A \mapsto B$ be a linear bijection with $\Psi(1) = 1$. Then the following statements are equivalent:*
(i) Ψ *is an isometry;*
(ii) Ψ *is a Jordan $*$-isomorphism;*
(iii) Ψ *is an order isomorphism;*
(iv) $\Psi(U(A)) = U(B)$.

Proof. Clearly, (i) $\Leftrightarrow$ (ii) by the theorem and (ii) $\Leftrightarrow$ (iv) by the lemma. Since a Jordan $*$-homomorphism Ψ satisfies $\Psi(x^2) = \Psi(x)^2$, $x \in A_h$, and each positive element is of the form x^2 for some $x \in A_h$, it is also clear that (ii) $\Rightarrow$ (iii). Finally, if Ψ and Ψ^{-1} are both positive, then $\|\Psi\| = \|\Psi^{-1}\| = 1$ by Proposition 6.4, so Ψ is isometric, i.e. (iii) $\Rightarrow$ (i). ∎

Remark that, by Theorem 6.7 and by the above theorem, if A, B are unital C^*-algebras, then each real linear isometry of A_h onto B_h can be extended (by complexification) to a complex linear isometry of A onto B.

The results of this section will also be extended for general C^*-algebras (see 9.32) and a deep structure theorem for Jordan $*$-homomorphisms will be given (see 9.31).

6.9. PROPOSITION. *Let Ψ be a linear mapping between C^*-algebras A, B. Then Ψ is a $*$-homomorphism if and only if Ψ is a Jordan $*$-homomorphism and a Schwartz map.*

Proof. Let $a \in A_h$. If Ψ is a Jordan $*$-homomorphism, then $\Psi(a^2) = \Psi(a)^2$ and if Ψ is also a Schwartz map, this entails $\Psi(ax) = \Psi(a)\Psi(x)$ for all $x \in A$, by Proposition 5.10. This proves the sufficiency of the conditions and their necessity is obvious. ∎

COROLLARY. *Every 2-positive Jordan $*$-homomorphism between C^*-algebras is a $*$-homomorphism.*

Proof. Use 5.9.(2). ∎

6.10. Let A, B be Banach spaces. A uniformly cross-norm on $A \otimes B$ is a norm p on $A \otimes B$ such that for every bounded linear mappings $\Phi : A \mapsto A$, $\Psi : B \mapsto B$ the linear mapping $\Phi \otimes \Psi : A \otimes B \mapsto A \otimes B$ is p-bounded and

$$\|\Phi \otimes \Psi\| = \|\Phi\| \, \|\Psi\|.$$

If A, B are C^*-algebras, then a C^*-norm on $A \otimes B$ is not necessarily uniformly cross (compare with 2.11).

In fact if A and B are both unital, non-commutative and if there exists a $*$-antiisomorphism $\tau : B \mapsto B$, then no C^*-norm on $A \otimes B$ is uniformly cross.

Indeed, assume to the contrary that there exists a uniformly cross C^*-norm p on $A \otimes B$ and denote by $A \otimes_p B$ the C^*-algebra completion of $A \otimes B$. Let $\iota : A \mapsto A$ be the identity mapping. Since

$$\|\iota\| = \|\iota^{-1}\| = \|\tau\| = \|\tau^{-1}\| = 1$$

and p is uniformly cross, it follows that the linear mappings $\iota \otimes \tau$ and $(\iota \otimes \tau)^{-1} = \iota^{-1} \otimes \tau^{-1}$ are p-bounded and

$$\|\iota \otimes \tau\| = \|(\iota \otimes \tau)^{-1}\| = 1.$$

Thus $\iota \otimes \tau$ extends to a unit preserving linear isometry on the C^*-algebra $A \otimes_p B$. By Corollary 6.8, $\iota \otimes \tau$ is a Jordan $*$-isomorphism, in particular

$$(\iota \otimes \tau)(x^2) = ((\iota \otimes \tau)(x))^2; \quad x \in A \otimes B.$$

However, there exist $a_1, a_2 \in A$, $b_1, b_2 \in B$ with $a_1 a_2 \neq a_2 a_1$, $b_1 b_2 \neq b_2 b_1$ and for

$$x = a_1 \otimes b_1 + a_2 \otimes b_2 \in A \otimes B$$

we have

$$(\iota \otimes \tau)(x^2) \neq ((\iota \otimes \tau)(x))^2,$$

a contradiction.

At the same time, the above discussion shows that the tensor product of two Jordan $*$-homomorphisms is not necessarily a Jordan $*$-homomorphism.

Moreover, we shall provide an example where $\iota \otimes \tau$ is not $\| \cdot \|_{C^*}$-bounded on $A \otimes B$.

Let H be a separable infinite dimensional Hilbert space with an orthonormal basis $\{\xi_n\}_{n=0,1,2,\ldots}$. It is easy to check that for every $x \in B(H)$ there exists a unique $\tau(x) \in B(H)$ such that

$$(\tau(x)\xi_i | \xi_j) = (x\xi_j | \xi_i); \quad i, j \in \{0, 1, 2, \ldots\},$$

and the map $\tau : B(H) \mapsto B(H)$ is a $*$-antiisomorphism. Let ι be the identity mapping on $B(H)$ and $A = B = B(H)$.

For $k \in \{1, 2, \ldots\}$ define $v_k \in B(H)$ by

$$v_k \xi = (\xi | \xi_0) \xi_k; \quad \xi \in H$$

and remark that

$$v_k^* \xi = (\xi | \xi_k) \xi_0, \ (\xi \in H), \quad \text{and} \quad \tau(v_k^*) = v_k.$$

Then $x_n = \sum\limits_{k=1}^{n} v_k \otimes v_k^* \in A \otimes B$, $(\iota \otimes \tau)(x_n) = \sum\limits_{k=1}^{n} v_k \otimes v_k$ and, as easily verified,

$$\|x_n\|_{C^*} = \|x_n\|_{B(H \overline{\otimes} H)} = 1$$
$$\|(\iota \otimes \tau)(x_n)\|_{C^*} = \|(\iota \otimes \tau)(x_n)\|_{B(H \overline{\otimes} H)} = n^{\frac{1}{2}},$$

hence $\iota \otimes \tau$ is not $\| \cdot \|_{C^*}$-bounded.

Thus, the result expressed by Corollary 1/5.3 appears as a remarkable property of completely positive linear mappings, especially because a similar result is not true for very particular positive linear mappings.

6.11. Notes. The characterization of extreme points of the unit ball of an operator algebra (6.1, 6.2) is due to R.V. Kadison [142]. The fact that the existence of extreme points is equivalent to the existence of a unit element has been pointed out by S. Sakai [267]. For the exposition in 6.1, 6.2 we used these sources and [242], 1.4.

The same characterization of extreme points (6.1) holds in every pre-C^*-algebra ([197]). Also, P.E. Miles [197] gives a detailed description of extreme points in concrete operator algebras, as well as some applications to the type analysis of AW^*-algebras.

The set $E_A = \{x \in A;\ (1 - x^*x)A(1 - xx^*) = \{0\}\}$ is considered by D. Yood [348] for objects A which are not necessarily C^*-algebras. For instance, $x \in E_A \Rightarrow x^n \in E_A$, $(n \geqslant 1)$, in any unital $*$-ring, $x \in E_A \Rightarrow r(x) \geqslant 1$ in any unital Banach $*$-algebra, and $x \in E_A \Rightarrow r(x) = 1$ in C^*-algebras.

Let A be a Banach space. An element $u \in A$, $\|u\| = 1$, is called a *vertex* if $\{f \in A^*;\ f(u) = 1 = \|f\|\}$ is a total subset of A^*. Any vertex is an extreme point of the unit ball. An element $u \in A$, $\|u\| = 1$, is called a *point of local uniform convexity* if $\lim\limits_n x_n = \lim\limits_n y_n = u$ whenever $x_n, y_n \in A$, $\|x_n\| = \|y_n\| = 1$ and $\lim\limits_n (x_n + y_n) = 2u$. The use of numerical ranges ([32], p. 38) allows an easy proof of the following result: "*the unit element of any unital Banach algebra is a vertex and a point of local uniform convexity; an element of a unital C^*-algebra is a vertex if and only if it is a unitary element*" ([27], [184]). Further results can be found in [10], [27], [126], [184], [197], [348], [359], [360], [361].

Also the results concerning the relationship between linear isometries, order isomorphisms and Jordan isomorphisms (6.7, 6.8) are due to R.V. Kadison [142], [143]. The proof of Theorem 6.7 completes the arguments from [143], Theorem 2. For the proof of Theorem and Corollary 6.8, which is simpler than the original proof of R.V. Kadison (because of the Russo-Dye theorem), we used [142], [143], [227], [231], [262] (see also [196], [261]). Note that, in the commutative case, the results contained in 6.7, 6.8 give several variants of the Banach-Stone theorem (see [81], V.8.8). In section 6.9 we included some results from [42], [226]. The counterexamples given in 6.10 are from [218]. As we already mentioned in the main text, the results from 6.7, 6.8 will be extended in 9.32 to arbitrary (not necessarily unital) operator algebras and further results and references will be given in 9.31, 9.42.

For the basic results concerning Jordan algebras and Jordan homomorphisms (6.2, 6.5, 6.6) we used [35], [58], [133], [141], [142], [145], [233], III, Section 4.

The remarkable equality $\{x \in A;\ \|x\| \leqslant 1\} = \overline{\mathrm{co}}\, U(A)$ from Theorem 6.3, as well as its consequences (Corollary 1 and Corollary 2/6.3; Proposition 6.4) are due to B. Russo and H.A. Dye [262]. The stronger result expressed by Theorem 6.3 is due to T.W. Palmer [223] and the proof of Theorem 6.3 is that given by L.A. Harris [126]. As mentioned in [33], p. 211, a more elementary proof of Theorem 6.3 on the same lines, can be obtained via the standard approximation to the integral using n^{th} roots of unity. The refined form of the Russo-Dye theorem was an important tool in establishing the Vidav-Palmer theorem (see 1.19). For further details and extensions to more general Banach $*$-algebras we refer to [33], [126], [223], [224], [225], [226], [227], [253], [361].

Chapter 7

$B(H)$

As we have seen in Chapter 4, every C^*-algebra can be realized as a Gelfand-Naĭmark algebra, i.e. as a norm closed $*$-algebra of bounded linear operators on a complex Hilbert space. It is therefore natural to expect that the study of closures of Gelfand-Naĭmark algebras with respect to topologies weaker then the norm topology will be useful for the general theory of C^*-algebras. We are thus led to the study of von Neumann algebras, which are Gelfand-Naĭmark algebras closed in the weak operator topology. For this purpose, we begin with an examination of all reasonable topologies on $B(H)$. This is prepared by some general considerations.

7.1. Lemma. *Let X be a complex vector space, $1 \geqslant \nu < +\infty$, $\{p_\iota\}_{\iota\in I}$ be a family of seminorms on X with the property*

$$\sum_{\iota\in I} p_\iota(x) < +\infty; \quad x \in X \tag{1}$$

and φ be a linear functional on X with the property

$$|\varphi(x)| \leqslant \Big(\sum_{\iota\in I} p_\iota(x)^\nu\Big)^{\frac{1}{\nu}}; \quad x \in X. \tag{2}$$

Then there exists a family $\{\varphi_\iota\}_{\iota\in I}$ of linear functionals on X such that

$$|\varphi_\iota(x)| \leqslant p_\iota(x); \quad \iota \in I,\, x \in X \tag{3}$$

$$\sum_{\iota\in I} |\varphi_\iota(x)| \leqslant \Big(\sum_{\iota\in I} p_\iota(x)^\nu\Big)^{\frac{1}{\nu}}; \quad x \in X \tag{4}$$

$$\varphi(x) = \sum_{\iota\in I} \varphi_\iota(x); \quad x \in X. \tag{5}$$

Proof. Let $\mathcal{X}$ be a vector space of all families $\{x_\iota\}_{\iota\in I}$ with $x_\iota \in X$ such that $\sum\limits_{\iota\in I} p_\iota(x_\iota)^\nu < +\infty$ and define a seminorm p on $\mathcal{X}$ by

$$p(\{x_\iota\}_{\iota\in I}) = \Big(\sum_{\iota\in I} p_\iota(x_\iota)^\nu\Big)^{\frac{1}{\nu}}$$

The set $\mathcal{D}$ of all $\{x_\iota\}_{\iota\in I} \in \mathcal{X}$ with $x_\iota = x_\kappa$ for all $\iota, \kappa \in I$, is a vector subspace of $\mathcal{X}$ and the mapping

$$\psi_0 : \mathcal{D} \ni \{x_\iota\}_{\iota\in I} \mapsto \varphi(x_{\iota_0}); \quad (\iota_0 \in I,\ \text{arbitrary}),$$

is a linear functional on $\mathcal{D}$. Using (2) and the Hahn-Banach theorem, we can extend ψ_0 to a linear functional ψ on $\mathcal{X}$ majorized by the seminorm p.

Define, for each $\iota \in I$, a linear functional φ_ι on X by $\varphi_\iota(x) = \psi(\{x_\iota\}_{\iota\in I})$, where $x_\iota = x$ and $x_\kappa = 0$ if $\kappa \neq \iota$; $x \in X$.

It is clear that (3) holds. Let $x \in X$ and, for each $\iota \in I$, let $\alpha_\iota \in \mathbb{C}$, $|\alpha_\iota| = 1$ such that $|\varphi_\iota(x)| = \alpha_\iota\varphi_\iota(x) = \varphi_\iota(\alpha_\iota x)$. Then

$$\sum_\iota |\varphi_\iota(x)| = \sum_\iota \varphi_\iota(\alpha_\iota x) = \psi(\{\alpha_\iota x\}_\iota) \leqslant p(\{\alpha_\iota x\}_\iota) = \Big(\sum_\iota p_\iota(x)^\nu\Big)^{\frac{1}{\nu}}.$$

This proves (4) and now (5) is imediate. ∎

7.2. Let X be a complex Banach space, X^* be its dual space and F be a vector subspace of X^*. Then the F-topology on X is defined by the seminorms p_φ, $(\varphi \in F)$ where

$$p_\varphi(x) = |\varphi(x)|; \quad x \in X.$$

We denote by X_1 the closed unit ball of X and by $\overline{F}$ the norm closure of F in X^*.

Proposition. *Let X, F be as above and φ be a linear functional on X. Then*
(i) *$\varphi \in F \Leftrightarrow \varphi$ is F-continuous;*
(ii) *$\varphi \in \overline{F} \Leftrightarrow$ the restriction of φ to X_1 is F-continuous.*

Proof. (i) If φ is F-continuous, then there exist non zero $\psi_1, \dots, \psi_n \in F$ such that

$$|\varphi(x)| \leqslant \sum_{j=1}^n p_{\psi_j}(x); \quad x \in X.$$

By Lemma 7.1 there are linear functionals $\varphi_1, \dots, \varphi_n$ on X such that $\varphi = \sum\limits_{j=1}^n \varphi_j$ and $|\varphi_j(x)| \leqslant |\psi_j(x)|$; $x \in X$, $1 \leqslant j \leqslant n$.

If $x_j \in X$ and $\psi_j(x_j) = 1$, then for any $x \in X$,

$$|\varphi_j(x - \psi_j(x)x_j)| \leqslant |\psi_j(x - \psi_j(x)x_j)| = 0$$

so $\varphi_j = \varphi_j(x_j)\psi_j \in F$ and $\varphi = \sum\limits_{j=1}^n \varphi_j \in F$. The converse is trivial.

(ii) It is easy to check that the restriction of any $\varphi \in \overline{F}$ to X_1 is F-continuous. Conversely, assume that the restriction of φ to X_1 is F-continuous. Then φ is norm continuous, i.e. $\varphi \in X^*$. Let $0 < \varepsilon < 1$. Since $\varphi|X_1$ is F-continuous at 0, there exist $\psi_1, \dots, \psi_n \in F$ such that

$$\|x\| \leqslant 1, \quad \sum_{j=1}^n p_{\psi j}(x) \leqslant 1 \Rightarrow |\varphi(x)| \leqslant \varepsilon.$$

Then $\left|\varphi\Big(\Big(\|x\| + \sum\limits_{j=1}^{n} p_{\psi j}(x)\Big)^{-1} x\Big)\right| \leqslant \varepsilon$ for all $x \in X$, thus

$$|\varphi(x)| \leqslant \varepsilon\Big(\|x\| + \sum_{j=1}^{n} p_{\psi j}(x)\Big) \leqslant \varepsilon\|x\| + \sum_{j=1}^{n} p_{\psi_j}(x); \quad x \in X.$$

Using Lemma 7.1 we find linear functionals φ_1, φ_2 on X such that

$$\varphi = \varphi_1 + \varphi_2$$
$$|\varphi_1(x)| \leqslant \varepsilon\|x\|,\ (x \in X), \qquad \text{and} \qquad |\varphi_2(x)| \leqslant \sum_{j=1}^{n} p_{\psi j}(x),\ (x \in X).$$

It follows that $\varphi_2 \in F$ and $\|\varphi - \varphi_2\| = \|\varphi_1\| \leqslant \varepsilon$. Since $0 < \varepsilon < 1$ was arbitrary, we obtain $\varphi \in \overline{F}$. ∎

In particular, if F is norm closed, then the linear functional φ on X is F-continuous if and only if $\varphi|X_1$ is F-continuous.

Corollary. *Let X, F be as above. Then, on X_1, the F-topology coincides with the $\overline{F}$-topology.*

7.3. For symmetry reasons we introduce the following definition: a pair (X, F) is called a *dual pair of normed* (respectively *Banach) spaces* if X, F are complex normed (respectively Banach) spaces and there exists a bilinear mapping

$$X \times F \ni (x, \varphi) \mapsto \langle x, \varphi\rangle \in \mathbb{C}$$

such that

$$\|x\| = \sup\{|\langle x, \varphi\rangle|;\ \varphi \in F,\ \|\varphi\| \leqslant 1\}; \quad x \in X$$
$$\|\varphi\| = \sup\{|\langle x, \varphi\rangle|;\ x \in X,\ \|x\| \leqslant 1\}; \quad \varphi \in F.$$

In this case F can be identified to a norm closed vector subspace of X^* such that the pairing $\langle\cdot,\cdot\rangle$ is induced by the usual one between X and X^*. Moreover, the F-topology on X is the Hausdorff. The following uniform boundedness result is often useful.

Proposition. *Let Y be a complex Banach space, (X, F) be a dual pair of Banach spaces and $\{T_\iota\}_{\iota \in I}$ be a family in $B(Y, X)$. Then the following statements are equivalent:*

(i) $\sup\{\|T_\iota\|;\ \iota \in I\} < +\infty$*;*

(ii) $\sup\{|\langle T_\iota(y), \varphi\rangle|;\ \iota \in I\} < +\infty$ *for all* $y \in Y$, $\varphi \in F$.

Proof. Clearly (i) $\Rightarrow$ (ii). Conversely assume that (ii) holds. Consider the Banach space $\mathcal{X} = Y \times F$ ($\|(y, \varphi)\| = \max\{\|y\|, \|\varphi\|\}$, $(y, \varphi) \in \mathcal{X}$) and, for each $n \in \mathbb{N}$, define

$$F_n = \{(y, \varphi) \in \mathcal{X};\ |\langle T_\iota(y), \varphi\rangle| \leqslant n \text{ for all } \iota \in I\}$$

Since $\mathcal{X} = \bigcup_n F_n$, by the Baire property of $\mathcal{X}$ there is $n_0 \in \mathbb{N}$ such that F_{n_0} has non empty interior. Thus, there is $(y_0, \varphi_0) \in \mathcal{X}$ and $\varepsilon > 0$ such that, for every $y \in Y$, $\|y\| \leqslant 1$, every $\varphi \in F$, $\|\varphi\| \leqslant 1$ and every $\alpha, \beta \in \mathbb{C}$ with $|\alpha|, |\beta| \leqslant \varepsilon$, we have

$$(y_0 + \alpha y, \varphi_0 + \beta\varphi) \in F_{n_0}$$

that is

$$|\langle T_\iota(y_0), \varphi_0\rangle + \alpha\langle T_\iota(y), \varphi_0\rangle + \beta\langle T_\iota(y_0), \varphi\rangle + \alpha\beta\langle T_\iota(y), \varphi\rangle| \leqslant n_0$$

($\iota \in I$). Taking succesively $\alpha = \beta = 0$, $\alpha = \varepsilon$ and $\beta = 0$, $\alpha = 0$ and $\beta = \varepsilon$, $\alpha = \beta = \varepsilon$, we get $|\langle T_\iota(y_0), \varphi_0\rangle| \leqslant n_0$, $\varepsilon|\langle T_\iota(y), \varphi_0\rangle| \leqslant 2n_0$, $\varepsilon|\langle T_\iota(y_0), \varphi\rangle| \leqslant 2n_0$, $\varepsilon^2|\langle T_\iota(y), \varphi\rangle| \leqslant 6n_0$. Thus for all $\iota \in I$ we have

$$\|T_\iota\| = \sup\{|\langle T_\iota(y), \varphi\rangle|;\ \|y\| \leqslant 1,\ \|\varphi\| \leqslant 1\} \leqslant 6n_0/\varepsilon^2. \quad \blacksquare$$

Corollary. *Let Y be a complex Banach space, (X, F) be a dual pair of Banach spaces, Ω be an open subset of $\mathbb{C}$ and $F : \Omega \to B(Y, X)$. Then the following statements are equivalent:*

(i) *F is analitic for the norm topology on $B(Y, X)$;*

(ii) *for all $y \in Y$, $\varphi \in F$ the function $\Omega \ni \alpha \mapsto \langle F(\alpha)y, \varphi\rangle \in \mathbb{C}$ is analitic.*

Proof. Let $\alpha \in \Omega$ and $V \subset \Omega$ be a compact neighborhood of α. For $\beta, \gamma \in V$, $\alpha \neq \beta$, $\beta \neq \gamma$, $\gamma \neq \alpha$, define

$$G(\alpha; \beta, \gamma) = \frac{1}{\beta - \gamma}\left[\frac{1}{\beta - \alpha}(F(\beta) - F(\alpha)) - \frac{1}{\gamma - \alpha}(F(\gamma) - F(\alpha))\right].$$

If (ii) holds, then for each $y \in Y$, $\varphi \in F$, we have

$$\sup\{|\langle G(\alpha; \beta, \gamma)y, \varphi\rangle|;\ \beta, \gamma \in V,\ \alpha \neq \beta,\ \beta \neq \gamma,\ \gamma \neq \alpha\} < +\infty$$

using the above proposition we get

$$c = \sup\{\|G(\alpha; \beta, \gamma)\|;\ \beta, \gamma \in V,\ \alpha \neq \beta,\ \beta \neq \gamma, \gamma \neq \alpha\} < +\infty$$

which means that

$$\left\|\frac{1}{\beta - \alpha}(F(\beta) - F(\alpha)) - \frac{1}{\gamma - \alpha}(F(\gamma) - F(\alpha))\right\| \leqslant c|\beta - \gamma|$$

for all $\beta, \gamma \in V$, $\alpha \neq \beta$, $\beta \neq \gamma$, $\gamma \neq \alpha$. It follows that F is norm derivable at α.

Thus (i) $\Rightarrow$ (ii) and the converse is trivial. $\blacksquare$

In particular, with $Y = \mathbb{C}$, it follows that for every dual pair of Banach spaces (X, F) and every $S \subset X$ we have

$$\sup\{\|x\|;\, x \in S\} < +\infty \Rightarrow \sup\{|\varphi(x)|;\, x \in S\} < +\infty \quad \text{for all } \varphi \in F$$

and, for each open subset Ω of $\mathbb{C}$, the analycity of a function $F : \Omega \to X$ with respect to norm topology is the same as the F-analycity.

7.4. The Krein-Shmulyan Theorem. Let (X, F) be a dual pair of Banach spaces. Assume that X_1 is F-compact. Then each norm bounded F-closed part of X is F-compact.

Define a locally convex topology τ on X by the seminorms

$$p_{\{\varphi_n\}}(x) = \sup_n |\langle x, \varphi_n\rangle|; \quad x \in X,$$

where $\{\varphi_n\}$ runs over all sequences in F with $\|\varphi_n\| \to 0$

Lemma. *Let (X, F) be as above. If S is a subset of X such that $S \cap \lambda X_1$ is F-closed for all $\lambda > 0$ then S is τ-closed.*

Proof. We have to show that for each $x_0 \notin S$ there is a τ-neighborhood V of 0 such that $S \cap (x_0 + V) = \emptyset$. We may assume without loss that $x_0 = 0$. Since $S \cap X_1$ is F-closed and $0 \notin S \cap X_1$, there exists a finite set $F_0 \subset F$ such that

$$(S \cap X_1) \cap \{x \in X;\, |\langle x, \varphi\rangle| \leqslant 1 \text{ for all } \varphi \in F_0\} = \emptyset.$$

Suppose that for some integer $n \geqslant 1$ there are mutually disjoint finite subsets $F_0, \dots, F_{n-1}$ of F such that $\|\varphi\| \leqslant 1/j$ whenever $\varphi \in F_j$, $(1 \leqslant j \leqslant n-1)$ and

$$(S \cap nX_1) \cap \Big\{x \in X;\, |\langle x, \varphi\rangle| \leqslant 1 \text{ for all } \varphi \in \bigcup_{j=1}^{n-1} F_j\Big\} = \emptyset.$$

If for every finite subset F' of $F \setminus \bigcup_{j=0}^{n-1} F_j$ with $\sup\big\{\|\varphi\|;\, \varphi \in F'\big\} \leqslant \frac{1}{n}$, it were

$$(1) \qquad (S \cap (n+1)X_1) \cap \Big\{x \in X;\, |\langle x, \varphi\rangle| \leqslant 1 \text{ for all } \varphi \in \bigcup_{j=0}^{n-1} F_j \cup F'\Big\} \neq \emptyset$$

then by the F-compactness of $(n+1)X_1$ it would exist $x_{n+1} \in S \cap (n+1)X_1$ such that $|\langle x_{n+1}, \varphi\rangle| \leqslant 1$ for all $\varphi \in \bigcup_{j=0}^{n-1} F_j$ and for all $\varphi \in F \setminus \bigcup_{j=0}^{n-1} F_j$ with $\|\varphi\| \leqslant \frac{1}{n}$. This would imply $\|x_{n+1}\| \leqslant n$ so the existence of x_{n+1} would contradict (1).

Therefore, by induction we can find a sequence $F_0, F_1, \dots, F_n, \dots$ of mutually disjoint finite subsets of F such that $\sup\{\|\varphi\|;\, \varphi \in F_n\} \leqslant \frac{1}{n}$ and such that (1) holds for all $n \geqslant 1$. Then

$$V = \Big\{x \in X;\, |\langle x, \varphi\rangle| < 1 \text{ for all } \varphi \in \bigcup_{n=0}^{\infty} F_n\Big\}$$

is a τ-neighborhood of 0 and $S \cap V = \emptyset$. ∎

Theorem. *Let (X, F) be a dual pair of Banach spaces such that X_1 is F-compact. If S is a convex subset of X such that $S \cap \lambda X_1$ is F-closed for all $\lambda > 0$, then S is F-closed.*

Proof. By the above lemma, S is τ-closed. Clearly, the τ-topology is stronger than the F-topology. On the other hand, it is easy to see that the two topologies coincide on X_1. Using Proposition 7.2 it follows that every τ continuous linear functional on X is also F-continuous. By a well known application of the Hahn-Banach theorem we infer that the τ-closed convex set S is also F-closed. ∎

7.5. We record a general compactness criterion which is often useful in proving that the closed unit ball of a Banach space is compact with respect to an appropriate weak topology.

Proposition. *Let Y, X be a complex vector space, F be a vector space of linear functionals on X which separate the elements of X, $\{S_\iota\}_{\iota\in I}$ be a family of subsets of Y such that $\operatorname{lin}\Big(\bigcup\limits_{\iota\in I} S_\iota\Big) = Y$ and $\{R_\iota\}_{\iota\in I}$ be a family of F-compact subsets of X. Then the set*

$$L = \{T \in L(Y, X);\ TS_\iota \subset R_\iota \text{ for all } \iota \in I\}$$

is compact with respect to the topology of pointwise F-convergence on $L(Y, X)$.

Proof. For $\iota \in I$ and $y \in S_\iota$, let $R_{\iota,y} = R_\iota$ endowed with the F-topology. Then let $Q_\iota = \prod\limits_{y\in S_\iota} R_{\iota,y}$ and $Q = \prod\limits_{\iota\in I} Q_\iota$ be endowed with the product topologies. Then Q is compact and, as easily verified, the map

$$\Phi : L \ni T \mapsto \{\{Ty\}_{y\in S_\iota}\}_{\iota\in I} \in Q$$

is a homeomorphism of L, endowed with the topology of pointwise F-convergence, onto a subset $\Phi(L)$ of Q. Thus we have to prove that $\Phi(L)$ is closed.

Let $\{\{x_{\iota,y}\}_{y\in S_\iota}\}_{\iota\in I} \in Q$ be adherent to $\Phi(L)$. We have to show that $x_{\iota,y} = T_0(y)$ for some $T_0 \in L(X, Y)$ and all $y \in S_\iota$, $\iota \in I$. It is sufficient to prove that for every family $\{\lambda_{\iota,y}\}_{\iota\in I,\, y\in S_\iota} \subset \mathbb{C}$ such that the set $\{(\iota, y);\ \iota \in I,\ y \in S,\ \lambda_{\iota,y} \neq 0\}$ is finite and

$$\sum_{\iota\in I}\sum_{y\in S_\iota} \lambda_{\iota,y} y = 0 \tag{1}$$

we have

$$\sum_{\iota\in I}\sum_{y\in S_\iota} \lambda_{\iota,y} x_{\iota,y} = 0. \tag{2}$$

Let $\varphi \in f$, $\varepsilon > 0$ and put $\lambda = \sum\limits_{\iota\in I}\sum\limits_{y\in S_\iota} |\lambda_{\iota,y}|$. By assumption there exists $T \in L(Y, X)$ such that $|\varphi(x_{\iota,y} - T(y))| < \varepsilon/\lambda$ whenever $\lambda_{\iota,y} \neq 0$ ($\iota \in I$, $y \in S_\iota$). Then, by (1),

$$\Big|\varphi\Big(\sum_{\iota\in I}\sum_{y\in S_\iota} \lambda_{\iota,y} x_{\iota,y}\Big)\Big| = \Big|\sum_{\iota\in I}\sum_{y\in S_\iota} \lambda_{\iota,y}\varphi(x_{\iota,y} - T(y))\Big| \leqslant \varepsilon$$

and (2) it follows. ∎

For instance, if $X = \mathbb{C}$, I is a singleton, $S = Y_1$ and $R = \{\alpha \in \mathbb{C};\ |\alpha| \leqslant 1\}$, then it follows that $(Y^*)_1$ is Y-compact, i.e. the Alaoglu theorem.

7.6. Finally, we give a characterisation of those dual pairs (X, F) of a Banach spaces for which X_1 is F-compact.

THEOREM. *Let X be a complex normed space and F be a vector subspace of X^*. The following statements are equivalent:*

(i) *the map $\Phi : X \to F^*$ defined by $\Phi(x)\varphi = \varphi(x)$, $(x \in X,\ \varphi \in F)$ is a linear isometry of X onto F^*;*

(ii) *F separates the elements of X and X_1 is F-compact;*

(iii) *(X, F) is a dual pair of normed spaces with respect to the natural pairing and no norm closed vector subspace G of F, $G \neq F$ separates the elements of X.*

Proof. (i) $\Rightarrow$ (ii) by Alaoglu's theorem.

(ii) $\Rightarrow$ (ii). By (ii), Φ is an injective contraction. Note that Φ is continuous with respect to the F-topologies so, again by (ii), $\Phi(X_1)$ is a convex F-compact subset of $(F^*)_1$. If $f \in (F^*)_1 \setminus \Phi(X_1)$, then by the Hahn-Banach theorem there exist $\varphi \in F$, $c \in \mathbb{R}$ and $\varepsilon > 0$ such that $\operatorname{Re} f(\varphi) \geqslant c + \varepsilon$ and $\operatorname{Re} \varphi(x) \leqslant c$ for all $x \in X_1$. Since X_1 is circled, the second inequality entails $c \geqslant 0$ and $\|\varphi\| \leqslant c$, in contradiction with the first inequality. Consequently, $\Phi(X_1) = (F^*)_1$ and so Φ is an isometry of X onto F^*.

(i) $\Rightarrow$ (iii) by a simple application of the Hahn-Banach theorem.

(iii) $\Rightarrow$ (i). Let $0 \neq f \in F^*$. Since $\operatorname{Ker} F$ is a proper norm closed vector subspace of F, (iii) shows that $\operatorname{Ker} f$ does not separate the elements of X so there is $0 \neq x \in X$ such that $\Phi(x)\varphi = \varphi(x) = 0$ whenever $\varphi \in F$ and $f(\varphi) = 0$. It follows that $f = \alpha\Phi(x)$ for some $\alpha \in \mathbb{C}$, thus $f \in \Phi(X)$. Hence Φ is surjective and, again by (iii), Φ is isometric. ∎

Roughly speaking, for a dual pair (X, F) of Banach spaces the following conditions are equivalent: (i) $X = F^*$, (ii) X_1 is F-compact, (iii) F is "minimal separating".

7.7. Now let H be a complex Hilbert space. Besides the *norm topology*, the following topologies are usually considered on $B(H)$: the *weak-operator topology* or *wo-topology*, defined by the seminorms

$$B(H) \ni x \mapsto |(x\xi|\eta)|; \quad \xi, \eta \in H;$$

the *strong-operator topology* or *so-topology* defined by the seminorms

$$B(H) \ni x \mapsto \|x\xi\|; \quad \xi \in H;$$

the *so*-topology*, defined by the seminorms

$$B(H) \ni x \mapsto \|x\xi\| + \|x^*\xi\|; \quad \xi \in H.$$

For $\xi, \eta \in H$ let $\omega_{\xi,\eta}$ denote the linear functional on $B(H)$ defined by

$$\omega_{\xi,\eta}(x) = (x\xi|\eta); \quad x \in B(H).$$

We have already considered the functionals $\omega_\xi = \omega_{\xi,\xi}$, $(\xi \in H)$.

Clearly, the *wo*-topology is the weak topology on $B(H)$ defined by the functionals $\omega_{\xi,\eta}$, $(\xi, \eta \in H)$. Also, the *wo*-topology is weaker than the *so*-topology and the *so*-topology is weaker than the *so**-topology:

$$wo \prec so \prec so^*.$$

Theorem. *A linear functional φ on $B(H)$ is wo-continuous if and only if it is so*-continuous.*

Proof. If φ is *so**-continuous, then

$$|\varphi(x)| \leqslant \sum_{j=1}^{n}(\|x\xi_j\| + \|x^*\xi_j\|); \quad x \in B(H)$$

for some non zero vectors $\xi_1, \ldots, \xi_n \in H$. By Lemma 7.1 there exist linear functionals $\varphi_1, \ldots, \varphi_n, \psi_1, \ldots, \psi_n$ on $B(H)$ such that

$$\varphi = \sum_{j=1}^{n} \varphi_j + \sum_{j=1}^{n} \psi_j, \tag{1}$$

$$|\varphi_j(x)| \leqslant \|x\xi_j\| \quad \text{and} \quad |\psi_j(x)| \leqslant \|x^*\xi_j\| \quad \text{for all } x \in B(H),\ 1 \leqslant j \leqslant n.$$

For each j, the map $H = B(H)\xi_j \ni x\xi_j \mapsto \varphi_j(x) \in \mathbb{C}$ is a well defined bounded linear functional on H, hence there exists $\eta_j \in H$ such that

$$\varphi_j(x) = (x\xi_j|\eta_j) = \omega_{\xi_j,\eta_j}(x); \quad x \in B(H).$$

Similarly, for each j there exists $\zeta_j \in H$ such that

$$\psi_j(x) = x(x\zeta_j|\xi_j) = \omega_{\zeta_j,\xi_j}(x); \quad x \in B(H).$$

By (1) it follows that φ is *wo*-continuous.

The converse is clear because $wo \prec so^*$. ∎

Let τ_{wo} denote the Mackey topology on $B(H)$ associated to the *wo*-topology. The above theorem shows that:

$$wo \prec so \prec so^* \prec \tau_{wo}.$$

In particular, the closures of a convex subset of $B(H)$ are the same in all these topologies.

7.8. Owing to 7.2 we are led to consider the norm closure $B(H)_*$ of the vector space of all *wo*-continuous linear functionals in $B(H)$ and to define the *w-topology* on $B(H)$ by the seminorms

$$B(H) \ni x \mapsto |\varphi(x)|; \quad \varphi \in B(H)_*.$$

Clearly, $wo \prec w$.

LEMMA 1. *$B(H)_1$ is w-compact.*

Proof. The wo-topology on $B(H)$ is the topology of pointwise weak convergence and, by Alaoglu's theorem, H_1 is weakly compact. Thus applying Proposition 7.5 with $Y = H$, $X = H$, $F = H^*$, I a singleton, $S = H_1$ and $R = H_1$, it follows that $B(H)_1$ is wo-compact. But, by Corollary 7.2, on $B(H)_1$ the w-topology coincides with the wo-topology, hence $B(H)_1$ is w-compact. ∎

LEMMA 2. *For every $\varphi \in B(H)_*$ there exist $\psi \in B(H)_*$, $\psi \geqslant 0$, and $a, b \in B(H)_1$ such that $\varphi = \psi(a \cdot b)$ and $\psi = \varphi(a^* \cdot b^*)$.*

Proof. Since $B(H)_1$ is w-compact (Lemma 1), the set $\{u \in B(H)_1;\, \varphi(u) = \|\varphi\|\}$ is non void, w-compact and convex. By the Krein-Milman theorem it contains an extreme point v which, as easily verified, is also an extreme point of $B(H)_1$. By Theorem 6.1 we have $(1 - vv^*)B(H)(1 - v^*v) = \{0\}$. Since for every $\xi, \eta \in H$ there exists $x \in B(H)$ with $x\xi = \eta$, it follows that either $vv^* = 1$ or $v^*v = 1$.

Suppose that $vv^* = 1$ and denote $\psi = \varphi(v\,\cdot)$, $a = v^*$, $b = 1$. Then $\psi(1) = \varphi(v) = \|\varphi\| \geqslant \|\psi\|$, hence $\|\psi\| = \|\psi(1)\|$ and, by Proposition 4.6, $\psi \geqslant 0$. For every $x \in B(H)$ we have $\psi(x) = \varphi(vx) = \varphi(a^*xb^*)$ and $\varphi(x) = \varphi(vv^*x) = \psi(v^*x) = \psi(axb)$, which proves the lemma in this case. If $v^*v = 1$, then we can take $\psi = \varphi(\cdot\, v)$, $a = 1$, $b = v^*$, and a similar argument works. ∎

As easily verified, the so-topology is defined by the seminorm

$$B(H) \ni x \mapsto \psi(x^*x)^{\frac{1}{2}}; \quad \psi \ wo\text{-continuous}, \psi \geqslant 0$$

and the so^*-topology is defined by the seminorms

$$B(H) \ni x \mapsto \psi(x^*x)^{\frac{1}{2}} + \psi(xx^*)^{\frac{1}{2}}; \quad \psi \ wo\text{-continuous}, \psi \geqslant 0.$$

It is therefore natural to consider, on $B(H)$,
the *s-topology*, defined by the seminorms

$$B(H) \ni x \mapsto \varphi(x^*x)^{\frac{1}{2}}; \quad \varphi \in B(H)_*, \varphi \geqslant 0$$

the *s^*-topology*, defined by the seminorms

$$B(H) \ni x \mapsto \varphi(x^*x)^{\frac{1}{2}} + \varphi(xx^*)^{\frac{1}{2}}; \quad \varphi \in B(H)_*, \varphi \geqslant 0.$$

Clearly,

$$so \prec s, \quad so^* \prec s^*.$$

On the other hand, let $\varphi \in B(H)_*$ and let ψ, a, b be as in Lemma 2. Then by the Schwartz inequality, for each $x \in B(H)$ we get

$$|\varphi(x)| = |\psi(axb)| \leqslant \psi(aa^*)^{\frac{1}{2}}\psi(b^*x^*xb)^{\frac{1}{2}} = \psi(aa^*)^{\frac{1}{2}}[\psi(b^* \cdot b)](x^*x)^{\frac{1}{2}}$$

and consequently

$$w \prec s \prec s^*.$$

Theorem. *A linear functional φ on $B(H)$ is w-continuous if and only if it is s^*-continuous.*

Proof. If φ is s^*-continuous, then

$$|\varphi(x)| \leqslant [\rho(x^*x) + \rho(xx^*)]^{\frac{1}{2}}; \quad x \in B(H)$$

for some $\rho \in B(H)_*$, $\rho \geqslant 0$. By the definition of $B(H)_*$, there is a sequence $\{\rho_n\}$ of wo-continuous linear functionals on $B(H)_1$ such that $\sum\limits_{n=1}^{\infty} \|\rho_n\| < +\infty$ and

$$\rho(x) = \sum_{n=1}^{\infty} \rho_n(x); \quad x \in B(H).$$

By Lemma 2, for each n there exist $\theta_n \in B(H)_*$, $\theta_n \geqslant 0$, and $a_n, b_n \in B(H)_1$ such that $\rho_n = \theta_n(a_n \cdot b_n)$ and $\theta_n = \rho_n(a_n^* \cdot b_n^*)$. Then θ_n are wo-continuous, $\sum\limits_{n=1}^{\infty} \|\theta_n\| \leqslant \sum\limits_{n=1}^{\infty} \|\rho_n\| < +\infty$ and, for every $x \in B(H)$ we have

$$\begin{aligned}
|\varphi(x)| &\leqslant [\rho(x^*x) + \rho(xx^*)]^{\frac{1}{2}} = \Big[\sum_n \theta_n(a_n x^* x b_n) + \sum_n \theta_n(a_n x x^* b_n)\Big]^{\frac{1}{2}} \\
&\leqslant \Big[\sum_n \theta_n(a_n x^* x a_n^*)^{\frac{1}{2}} \theta_n(b_n^* x^* x b_n)^{\frac{1}{2}} + \sum_n \theta_n(a_n x x^* a_n^*)^{\frac{1}{2}} \theta_n(b_n^* x x^* b_n)^{\frac{1}{2}}\Big]^{\frac{1}{2}} \\
&\leqslant \Big[\Big(\sum_n \theta_n(a_n x^* x a_n^*) + \theta_n(a_n x x^* a_n^*)\Big)^{\frac{1}{2}} \cdot \\
&\qquad \cdot \Big(\sum_n \theta_n(b_n^* x^* x b_n) + \theta_n(b_n^* x x^* b_n)\Big)^{\frac{1}{2}}\Big]^{\frac{1}{2}} \\
&\leqslant 2^{-1}\Big[\Big(\sum_n \theta_n(a_n(x^*x + xx^*)a_n^*)\Big)^{\frac{1}{2}} + \Big(\sum_n \theta_n(b_n^*(x^*x + xx^*)b_n)\Big)^{\frac{1}{2}}\Big] \\
&\leqslant \Big[\sum_n \theta_n(a_n(x^*x + xx^*)a_n^*) + \sum_n \theta_n(b_n^*(x^*x + xx^*)b_n)\Big]^{\frac{1}{2}}.
\end{aligned}$$

For each $n \in \mathbb{N}$, $\psi_n = \theta_n(a_n \cdot a_n^*) + \theta_n(b_n^* \cdot b_n)$ is wo-continuous, positive, $\sum\limits_{n=1}^{\infty} \|\psi_n\| < +\infty$ and, by the above computation,

$$|\varphi(x)| \leqslant \Big[\sum_{n=1}^{\infty} \psi_n(x^*x + xx^*)\Big]^{\frac{1}{2}}; \quad x \in B(H).$$

Applying Lemma 7.1 with $\nu = 2$ we get a sequence φ_n of linear functionals on $B(H)$ such that

$$|\varphi_n(x)| \leqslant \psi_n(x^*x + xx^*)^{\frac{1}{2}}; \qquad x \in B(H),\ n \in \mathbb{N},$$

$$\sum_{n=1}^{\infty} |\varphi_n(x)| \leqslant \Big[\sum_{n=1}^{\infty} \psi_n(x^*x + xx^*)\Big]^{\frac{1}{2}}; \quad x \in B(H),$$

$$\varphi(x) = \sum_{n=1}^{\infty} \varphi_n(x); \qquad x \in B(H).$$

Then each φ_n is so^*-continuous, hence wo-continuous (Theorem 7.7), $\|\varphi_n\| \leqslant 2^{\frac{1}{2}}\|\psi_n\|$, thus $\sum\limits_{n=1}^{\infty} \|\varphi_n\| < +\infty$ and $\varphi = \sum\limits_{n=1}^{\infty} \varphi_n$ is w-continuous.

The converse is clear because $w \prec s^*$. ∎

Let τ_w denote the Mackey topology on $B(H)$ associated to the w-topology. The above theorem shows that

$$w \prec s \prec s^* \prec \tau_w.$$

In particular, the closures of a convex subset of $B(H)$ are the same in all these topologies.

By Corollary 7.2, on norm bounded subsets of $B(H)$, the w-topology coincides with the wo-topology. It follows that, on norm bounded sets, the s-topology (respectively the s^*-topology) coincides with the so-topology (respectively the so^*-topology).

On the other hand, by Proposition 7.2 and Theorem 7.4, the w-continuity of linear functionals and the w-closedness of convex sets may be verified just on bounded parts of $B(H)$.

Using Lemma 1, it is easy to check that $B(H)_1$ is complete relative to the uniform structure associated to the $s = so$ (respectively $s^* = so^*$)-topology.

Let H be a separable complex Hilbert space and $\{\xi_n\}$ be a norm-dense sequence of non-zero vectors in H. Then the topologies w and wo on $B(H)_1$ are defined by the metric

$$(x, y) \mapsto \sum_{j,k=1}^{\infty} (2^{j+k}\|\xi_j\|\,\|\xi_k\|)^{-1}|((x-y)\xi_j|\xi_k)|$$

hence $B(H)_1$ is metrizable in this topology. The topologies so and s on $B(H)_1$ are defined by the metric

$$(x, y) \mapsto \sum_{j=1}^{\infty}(2^j\|\xi_j\|)^{-1}\|(x-y)\xi_j\|$$

hence $B(H)_1$ is metrizable also in this topology. Similarly, $B(H)_1$ is metrizable in the topologies so^* and s^*. Note that all these metrics are complete.

7.9. We record the following relation between the above topologies and the algebraic operations in $B(H)$

(1) for each $a \in B(H)$, the mapings

$$B(H) \ni x \mapsto ax \in B(H) \quad \text{and} \quad B(H) \ni x \mapsto xa \in B(H)$$

are continuous with respect to the topologies wo, so, so^*, w, s, s^*;

(2) the mappings

$$B(H)_1 \times B(H) \ni (x, y) \mapsto xy \in B(H)$$

and

$$B(H)_1 \times B(H)_1 \ni (x, y) \mapsto xy \in B(H)$$

are continuous with respect to the topologies so, s, respectively so^*, s^*;

(3) the $*$-operation

$$B(H) \ni x \mapsto x^* \in B(H)$$

is continuous with respect to the topologies wo, so^*, w, s^*.

Let H be an infinite dimensional complex Hilbert space, $\{\xi_n\}$ be an orthonormal sequence in H and $\{v_n\} \subset B(H)$, be defined by

$$v_n(\xi) = (\xi|\xi_n)\xi_1; \quad \xi \in H, n \in \mathbb{N}$$

then $\|v_n\| = 1, v_n \xrightarrow{so} 0$, hence $v_n \xrightarrow{s} 0$, and $v_n^*\xi_1 = \xi_n$, $v_n v_n^* \xi_1 = \xi_1$. By (1), $v_n \xrightarrow{w} 0$, but $(v_n v_n^* \xi_1|\xi_1) = \|v_n^*\xi_1\| = 1$ for all $n \in \mathbb{N}$. It follows that the mapping

$$B(H)_1 \times B(H)_1 \ni (x, y) \mapsto xy \in B(H)_1$$

is not w (or wo)-continuous and also that the $*$-operation is not continuous with respect to the topologies s and so.

A remarkable connection between topologies and algebraic operations in $B(H)$ is the following result:

THEOREM (Kaplansky density theorem). *Let A be a $*$-subalgebra of $B(H)$ and $x \in B(H)$. If x is wo-adherent to A, then there exists a net $\{x_\iota\}_{\iota \in I}$ in A such that $\|x_\iota\| \leqslant \|x\|$ $(\iota \in I)$, and $x_\iota \overset{s^*}{\longmapsto} x$. If moreover x is selfadjoint (respectively positive), then the x_ι can be chosen also selfadjoint (respectively positive).*

Proof. The norm closure B of A is a C^*-subalgebra of the wo-closure M of A and we have to show that B_1 is s^*-dense in M_1. By Theorem 7.8 and Corollary 7.2, it suffices to show that B_1 is wo-dense in M_1. Moreover, by Lemma 1/7.8, M_1 is wo-compact so, by the Krein-Milman theorem, it suffices to prove that every extreme point $v \neq 0$ of M_1 is wo-adherent to B_1.

By 7.7 v is so^*-adherent to B, hence there exists a net $\{b_\iota\}_{\iota \in I}$ in B such that $b_\iota \xrightarrow{so^*} v$. Then

$$2b_\iota(1 + b_\iota^* b_\iota)^{-1} \xrightarrow{wo} 2v(1 + v^*v)^{-1}.$$

Indeed, for any $\xi, \eta \in H$,

$$
\begin{aligned}
&|([b_\iota(1+b_\iota^*b_\iota)^{-1} - v(1+v^*v)^{-1}]\xi|\eta)| \\
&\quad \leqslant |((b_\iota - v)(1+b_\iota^*b_\iota)^{-1}\xi|\eta)| + |(v[(1+b_\iota^*b_\iota)^{-1} - (1+v^*v)^{-1}]\xi|\eta)| \\
&\quad = |((1+b_\iota^*b_\iota)^{-1}\xi|(b_\iota^* - v^*)\eta)| \\
&\qquad + |((1+b_\iota^*b_\iota)^{-1}(v^*v - b_\iota^*b_\iota)(1+v^*v)^{-1}\xi|v^*\eta)| \\
&\quad \leqslant |((1+b_\iota^*b_\iota)^{-1}\xi|(b_\iota^* - v^*)\eta)| \\
&\qquad + |((1+b_\iota^*b_\iota)^{-1}(v^* - b_\iota^*)v(1+v^*v)^{-1}\xi|v^*\eta)| \\
&\qquad + |((1+b_\iota^*b_1)^{-1}b_\iota^*(v - b_\iota)(1+v^*v)^{-1}\xi|v^*\eta)| \\
&\quad \leqslant \|(1+b_\iota^*b_\iota)^{-1}\|\,\|\xi\|\,\|(b_\iota^* - v^*)\eta\| \\
&\qquad + \|(1+b_\iota^*b_\iota)^{-1}\|\,\|(v^* - b_\iota^*)v(1+v^*v)^{-1}\xi\|\,\|v^*\|\,\|\eta\| \\
&\qquad + \|(1+b_\iota^*b_\iota)^{-1}b_\iota^*\|\,\|(v - b_\iota)(1+v^*v)^{-1}\xi\|\,\|v^*\|\,\|\eta\| \\
&\quad \leqslant \|\xi\|\,\|(b_\iota^* - v^*)\eta\| \\
&\qquad + \|\eta\|\,\|(v^* - b_\iota^*)v(1+v^*v)^{-1}\xi\| \\
&\qquad + 2^{-1}\|\eta\|\,\|(v - b_\iota)(1+v^*v)^{-1}\xi\|.
\end{aligned}
$$

As $2b_\iota(1+b_\iota^*b_\iota)^{-1} \in B_1$, we see that $2v(1+v^*v)^{-1}$ is *wo*-adherent to B_1.

Since $v = 2^{-1}[2v(1+v^*v)^{-1}] + 2^{-1}[2v - 2v(1+v^*v)^{-1}]$ and $2v(1+v^*v)^{-1} \in M_1$, $2v - 2v(1+v^*v)^{-1} \in M_1$, we have $v = 2v(1+v^*v)^{-1}$ by the extremality of v in M_1. Thus v is *wo*-adherent to B_1.

If x is selfadjoint, then replacing x_ι by $2^{-1}(x_\iota + x_\iota^*)$ we may assume that the x_ι's are selfadjoint.

If x is positive, then $x = y^*y$ for some $y \in M$ and by the above there exists a net $\{y_\iota\}$ in A with $\|y_\iota\| \leqslant \|y\|$ and $y_\iota \xrightarrow{s^*} y$. Then

$$x_\iota = y_\iota^*y_\iota \in A, \quad x_\iota \geqslant 0, \quad \|x_\iota\| \leqslant \|x\| \quad \text{and} \quad x_\iota \xrightarrow{s^*} x. \quad \blacksquare$$

If H is separable, then nets can be replaced by sequences in the statement of the theorem because of the so^*-metrizability of bounded subsets in $B(H)$ (see 7.8).

7.10. An essential step in proving the Kaplansky density theorem (7.9) was the continuity of the mapping $x \mapsto x(1+x^*x)^{-1}$ from $B(H)$ with so^*-topology to $B(H)$ with *wo*-topology. The restriction of the above map to normal operators has stronger continuity properties and this will be extended in this section.

Let $\Omega \subset \mathbb{C}$ and $f : \Omega \to \mathbb{C}$ be continuous. We say that f is *operator continuous* if for every Hilbert space H the mapping

$$\{x \in B(H);\ x \text{ normal}, \sigma(x) \subset \Omega\} \ni x \mapsto f(x) \ni B(H)$$

is *so*-continuous and also *s*-continuous.

Lemma 1. *The functions* $\mathbb{C} \ni \lambda \mapsto \lambda$, $\mathbb{C} \ni \lambda \mapsto \overline{\lambda}$ *and* $\mathbb{C} \ni \lambda \mapsto (1+|\lambda|^2)^{-1}$ *are operator continuous.*

Proof. Let H be a Hilbert space and $x, y \in B(H)$ be normal. Then for any positive form φ on $B(H)$ we have

$$\begin{aligned}\varphi((x-y)(x-y^*)) &= \varphi(y^*y) - \varphi(x^*x) + \varphi((x-y)x^*) + \overline{\varphi((x-y)x^*)} \\ &\leqslant (\varphi(y^*y)^{\frac{1}{2}} + \varphi(x^*x)^{\frac{1}{2}})(\varphi(y^*y)^{\frac{1}{2}} - \varphi(x^*x)^{\frac{1}{2}}) \\ &\qquad + 2\varphi(1)^{\frac{1}{2}}\varphi(x(x-y)^*(x-y)x^*)^{\frac{1}{2}} \\ &\leqslant (\varphi(y^*y)^{\frac{1}{2}} + \varphi(x^*x)^{\frac{1}{2}})\varphi((y-x)^*(y-x))^{\frac{1}{2}} \\ &\qquad + 2\varphi(1)^{\frac{1}{2}}[\varphi(x\cdot x^*)]((x-y)^*(x-y))^{\frac{1}{2}}\end{aligned}$$

and

$$\begin{aligned}&\varphi([(1+x^*x)^{-1} - (1+y^*y)^{-1}]^2) \\ &\quad = \varphi((1+x^*x)^{-1}[(y-x)^*y + x^*(y-x)](1+y^*y)^{-2}[y^*(y-x) \\ &\qquad + (y-x)^*x](1+x^*x)^{-1}) \\ &\quad \leqslant [\varphi((1+x^*x)^{-1}\cdot(1+x^*x)^{-1}]((y-x)^*(y-x)) \\ &\qquad + 2[\varphi(1+x^*x)^{-1}\cdot(1+x^*x)^{-1}]((y-x)^*(y-x))^{\frac{1}{2}} \\ &\qquad\quad \times [\varphi((1+x^*x)^{-1}x^*\cdot x(1+x^*x)^{-1}]((y-x)(y-x)^*)^{\frac{1}{2}} \\ &\qquad + [\varphi((1+x^*x)^{-1}x^*\cdot x(1+x^*x)^{-1}]((y-x)(y-x)^*).\end{aligned}$$

By the first set of inequalities, $\lambda \mapsto \overline{\lambda}$ is operator continuous. Using this fact and the second set of inequalities it is easy to check that also $\lambda \mapsto (1+|\lambda|^2)^{-1}$ is operator continuous. ∎

By Lemma 1, on the set $\{x \in B(H);\, x \text{ normal}\}$ the *so*-topology coincides with the *so**-topology and the *s*-topology coincides with the s^*-topology.

Lemma 2. *Every continuous function* $f : \mathbb{C} \to \mathbb{C}$ *with*

$$\sup\{|f(\lambda)|(1+|\lambda|)^{-1};\, \lambda \in \mathbb{C}\} < +\infty$$

is operator continuous.

Proof. Denote by C the set of all operator continuous functions on C and by C_b the set of all bounded functions in C. Using Lemma 1, it easy to verify that C is a uniformly closed selfadjoint vector subspace of the $*$-algebra of all continuous complex functions on $\mathbb{C}$ and that $C_bC \subset C$. Hence C_b is a uniformly closed $*$-subalgebra and, again by Lemma 1, it contains the functions $\lambda \mapsto (1+|\lambda|^2)^{-1}$ and $\lambda \mapsto \lambda(1+|\lambda|^2)^{-1}$. By the Stone-Weierstrass theorem we infer that $C_b \supset C_0(\mathbb{C})$.

Now let f be as in the statement. Then $\lambda \mapsto f(\lambda)(1+|\lambda|^2)^{-1}$ belongs to $C_0(\mathbb{C}) \subset C_b$. Since, by Lemma 1, $\lambda \mapsto \overline{\lambda}$ belongs to C, the function $\lambda \mapsto f(\lambda)(1+|\lambda|^2)^{-1}\overline{\lambda}$ belongs to C. Actually, this function is in C_b, so its product with $\lambda \mapsto \lambda$, that is the function $\lambda \mapsto f(\lambda)(1+|\lambda|^2)^{-1}|\lambda|^2$ is again in C. Consequently, the function

$$\lambda \mapsto f(\lambda) = f(\lambda)(1+|\lambda|^2)^{-1} + f(\lambda)(1+|\lambda|^2)^{-1}|\lambda|^2$$

belongs to C. ∎

We now prove the main result of this section (compare with 1.18.(4))

THEOREM. *Let $\Omega \subset \mathbb{C}$ be such that $\overline{(\overline{\Omega} \setminus \Omega)} \cap \Omega = \emptyset$ and let $f : \Omega \to \mathbb{C}$ be a continuous function such that*

$$\sup\{|f(\lambda)|(1 + |\lambda|)^{-1}; \lambda \in \Omega\} < +\infty.$$

Then f is operator continuous.

Proof. Let H be a Hilbert space and $x, y \in B(H)$ be normal operators such that $\sigma(x), \sigma(y) \subset \Omega$.

Since $\overline{(\overline{\Omega} \setminus \Omega)} \cap \Omega = \emptyset$, there exists a compact neighborhood N of $\sigma(x)$ such that $\overline{(\overline{\Omega} \setminus \Omega)} \cap N = \emptyset$. Then $\Omega \cap N = \overline{\Omega} \cap N$, hence Ω is closed in $\Omega \cup N$ and, by the Tietze-Urysohn extension theorem, f can be extended to a continuous function g on $\Omega \cup N$. Let $h : \mathbb{C} \to \mathbb{C}$ be a continuous function such that $\operatorname{supp} h \subset N$ and $h(\lambda) = 1$ for $\lambda \in \sigma(x)$. We obtain a continuous function with compact support $k : \mathbb{C} \to \mathbb{C}$ by putting

$$k(\lambda) = g(\lambda)h(\lambda) \text{ for } \lambda \in \Omega \cup N, \quad k(\lambda) = 0 \text{ for } \lambda \notin \Omega \cup N.$$

Consider also the functions $l : \mathbb{C} \to \mathbb{C}$, $F : \Omega \to \mathbb{C}$ defined by

$$l(\lambda) = (1 + |\lambda|)(1 - h(\lambda)), \quad F(\lambda) = f(\lambda)(1 + |\lambda|)^{-1}.$$

By construction, $l(x) = 0$ and, by assumption, there exists some $\mu_0 \in (0, +\infty)$ such that $|F(\lambda)| \leqslant \mu_0$ for all $\lambda \in \Omega$. Since

$$f(\lambda) = f(\lambda)h(\lambda) + f(\lambda)(1 - h(\lambda)) = k(\lambda) + F(\lambda)l(\lambda), \quad \lambda \in \Omega,$$

it follows that

$$f(x) - f(y) = [k(x) - k(y)] + F(y)[l(x) - l(y)].$$

If $y \xrightarrow{so} x$, then $k(y) \xrightarrow{so} k(x)$ and $l(y) \xrightarrow{so} l(x) = 0$ by Lemma 2, while $\|F(y)\|$ remains bounded by μ_0, so that from the above equality we infer that $f(y) \xrightarrow{so} f(x)$. The same argument applies for the s-topology. Hence f is operator continuous. ∎

7.11. By Theorem 7.9, if A is a $*$-subalgebra of $B(H)$, then the closures of A with respect to the topologies wo, so, so^*, w, s and s^* are all equal and moreover they are equal to the cone with vertex 0 generated by the closures of A_1 in any one of the topologies wo, so, so^*, w, s, s^*.

Clearly, the wo-closure of A is a C^*-subalgebra of $B(H)$. We shall say that A is *non degenerated* if the identity representation $A \ni x \mapsto x \in B(H)$ is non degenerated, i.e. if $\operatorname{lin} AH$ is dense in H.

LEMMA. *The wo-closure of a $*$-subalgebra A of $B(H)$ has a unit element e which is a projection in $B(H)$. Moreover, $e = 1_H$ if and only if A is non degenerated.*

Proof. Let M be the *wo*-closure of A. By Lemma 1/7.8, M_1 is *wo*-compact, thus M_1 contains an extreme point and so, M has a unit element e by Theorem 6.1.

If $e = 1_H$, then there is a net $\{x_\iota\}$ in A, $x_\iota \xrightarrow{so} 1_H$, so that $\xi = \lim_\iota x_\iota \xi$ for each $\xi \in H$, which shows that A is non degenerated. Conversely, if A is non degenerated, then eH contains the dense subspace $eAH = AH$ of H, hence $eH = H$ and $e = 1_H$. ∎

For an arbitrary subset S of $B(H)$, its *commutant* S' is defined by

$$S' = \{x' \in B(H);\ x'x = xx' \text{ for all } x \in S\}$$

and its *bicommutant* S'' is $(S')'$. By induction, the $(n+1)$-commutant of S can be defined as the commutant of the n-commutant of S. Since clearly $(S')'' = S'$, it follows that:

the $(2n-1)$-commutant of $S = S'$, $n \geqslant 1$;
the $2n$-commutant of $S = S''$, $n \geqslant 1$.

Hence the only non trivial problem concerning commutants is the relation between S and S''. We always have $S \subset S''$. Moreover, S'' is *wo*-closed and contains 1_H.

The following theorem is the fundamental result in the spatial theory of operator algebras.

THEOREM (von Neumann density theorem). *Let A be a non degenerate $*$-subalgebra of $B(H)$. Then the wo-closure of A is A''.*

Proof. Let $x \in A''$, $\xi_1, \ldots, \xi_n \in H$ and $\varepsilon > 0$ be arbitrary.

Consider $B(H^{(n)}) = M_n(B(H))$ (see 4.19) and, for each $y \in B(H)$, denote $\mathrm{Diag}(y) = [y_{ij}] \in M_n(B(H))$, where $y_{ij} = \delta_{ij} y$ (Delta Kronecker, $1 \leqslant i,j \leqslant n$). It is easy to see that $\mathrm{Diag}(A)' \subset B(H^{(n)})$ consists of all matrices $[y_{ij}]$ with $y_{ij} \in A' \subset B(H)$ for all $1 \leqslant i,j \leqslant n$. Hence $\mathrm{Diag}(A'') \subset \mathrm{Diag}(A)''$. In particular,

$$\mathrm{Diag}(x) \in \mathrm{Diag}(A)''.$$

Let $\xi = [\xi_k] \in H^{(n)}$ and denote by $p \in B(H^{(n)})$ the orthogonal projection onto $\overline{\mathrm{Diag}(A)\xi}$. As easily verified, $p \in \mathrm{Diag}(A)'$, hence

$$\mathrm{Diag}(x)\, p = p\, \mathrm{Diag}(x).$$

On the other hand, A being non degenerated, 1_H is *wo*-adherent to A, by the above lemma. It follows that $\xi \in \overline{\mathrm{Diag}(A)\xi}$, i.e. $p\xi = \xi$. Consequently,

$$\mathrm{Diag}(x)\xi = \mathrm{Diag}(x)p\xi = p(\mathrm{Diag}(x)\xi) \in pH^{(n)} = \overline{\mathrm{Diag}(A)\xi}$$

and therefore there exists $a \in A$ such that

$$\|\mathrm{Diag}(x)\xi - \mathrm{Diag}(a)\xi\| \leqslant \varepsilon$$

hence

$$\|(x-a)\xi_k\| \leqslant \varepsilon; \quad 1 \leqslant k \leqslant n.$$

This proves that A'' is contained in the *wo*-closure of A. The converse inclusion is trivial. ∎

Let M be a non degenerate $*$-subalgebra of $B(H)$. As we have noted at the begining of this section, the following conditions are equivalent:

1) M is *wo*-closed; 2) M is *so*-closed; 3) M is *so**-closed;
4) M is w-closed; 5) M is s-closed; 6) M is s^*-closed.

By Corollary 7.2, Theorem 7.7, Theorem 7.8 and Theorem 7.4, the above conditions are also equivalent to any of the following ones:

7) M_1 is *wo*-closed; 8) M_1 is *so*-closed; 9) M_1 is *so**-closed;
10) M_1 is w-closed; 11) M_1 is s-closed; 12) M_1 is s^*-closed.

Finally, by the von Neumann density theorem, all above conditions are equivalent to

13) $M = M''$

Note that conditions 4) to 12) are still equivalent for any convex subcone M of $B(H)$ with vertex 0.

By definition, a *von Neumann algebra* is non degenerate $*$-subalgebra M of $B(H)$ which satisfies the above equivalent conditions.

Note that a von Neumann algebra $M \subset B(H)$ always contains the identity operator 1_H.

If $M \subset B(H)$ is a von Neumann algebra, then its commutant M' is also a von Neumann algebra.

For any $S \subset B(H)$ we denote by $R(S)$ the von Neumann algebra generated by S, that is the *wo*-closed $*$-subalgebra of $B(H)$ generated by S and 1_H. If $S = S^*$, then $R(S) = S''$.

By Corollary 1/2.8 and by the von Neumann density theorem, we have the following result:

Corollary. *Let $M \subset B(H)$ be a von Neumann algebra and $x \in B(H)$. Then*

$$x \in M \Leftrightarrow u^*xu = x \text{ for all unitaries } u \in M'. \quad \blacksquare$$

In particular, a projection $e \in B(H)$ belongs to M if and only if $ueH \subset eH$ for all unitaries $u \in M'$.

Let $M \subset B(H)$ be a von Neumann algebra and $\xi \in H$. Then we define the *cyclic projections*

$$p_\xi = p_\xi^M = \text{the orthogonal projection onto } \overline{M'\xi}$$
$$p'_\xi = p_\xi^{M'} = \text{the orthogonal projection onto } \overline{M\xi}.$$

By the above criterion we have $p_\xi^M \in M$, $p_\xi^{M'} \in M'$.

A *vector* $\xi \in H$ is called *cyclic* (respectively *separating) for* M if $p'_\xi = 1_H$ (respectively $p_\xi = 1_H$). More generally, a *set* $S \subset H$ is called *cyclic* (respectively *separating) for* M if the closure of $\operatorname{lin} MS$ (respectively $\operatorname{lin} M'S$) is H. It is easy to see that $S \subset H$ is separating for M if and only if

$$x \in M,\ x\xi = 0 \text{ for all } \xi \in S \Rightarrow x = 0$$

and this justifies the name "separating". Finally, a vector $\xi \in H$ is called *bicyclic* if it is simultaneously cyclic and separating for M, that is if $\overline{M\xi} = \overline{M'\xi} = H$.

Finally, note that if $M_\iota \subset B(H)$, $(\iota \in I)$, are von Neumann algebras, then $M = \bigcap_{\iota \in I} M_\iota \subset B(H)$ is also a von Neumann algebra and

$$R\Big(\bigcup_{\iota \in I} M'\Big)' = \bigcap_{\iota \in I} M_\iota'' = \bigcap_{\iota \in I} M_\iota = M$$

hence $M' = R\Big(\bigcup_{\iota \in I} M_\iota'\Big)$.

After having settled the main problems concerning the natural topologies on $B(H)$, we now examine the "geometry" of elements in $B(H)$. Many problems in operator algebras are succesively reduced from general operators to selfadjoint operators and then to projections. The main tools used in doing so are the polar decomposition for operators and the spectral theorem.

7.12. Let H be a complex Hilbert space and $x \in B(H)$. Put

$$\mathbf{l}(x) = \text{the orthogonal projection onto } \overline{xH};$$
$$\mathbf{r}(x) = \text{the orthogonal projection onto } (\operatorname{Ker} x)^\perp.$$

It is easy to see that

$$\mathbf{r}(x) = \mathbf{l}(x^*). \tag{1}$$

Also, $\mathbf{l}(x)$ (respectively $\mathbf{r}(x)$) is the smallest projection e in $B(H)$ such that $ex = x$ (respectively $xe = e$). We say that $\mathbf{l}(x)$ is the *left support* of x and $\mathbf{r}(x)$ is the *right support* of x.

If x is selfadjoint, then we denote $\mathbf{s}(x) = \mathbf{l}(x) = \mathbf{r}(x)$ and call it simply the *support* of x. Note that

$$\mathbf{l}(x) = \mathbf{s}(xx^*), \quad \mathbf{r}(x) = \mathbf{s}(x^*x); \quad x \in B(H) \tag{2}$$

thus for x normal we still can denote $\mathbf{s}(x) = \mathbf{l}(x) = \mathbf{r}(x)$ and call it the *support* of x.

If x is positive, then $\mathbf{s}(x^\varepsilon) = \mathbf{s}(x)$ for all $\varepsilon > 0$.

If $x = v$ is a partial isometry, then

$$\mathbf{l}(v) = vv^*, \quad \mathbf{r}(v) = v^*v. \tag{3}$$

THEOREM (Polar decomposition for operators). *For every $x \in B(H)$ there exist a positive operator $a \in B(H)$ and a partial isometry $v \in B(H)$, uniquely determined such that*

$$x = va \quad \textit{and} \quad v^*v = \mathbf{s}(a). \tag{4}$$

Moreover, $a = |x|$.

Proof. Let $a = |x| = (x^*x)^{\frac{1}{2}}$ for every $\xi \in H$ we have

$$\|x\xi\|^2 = (x^*x\xi|\xi) = (a^2\xi|\xi) = \|a\xi\|.$$

Therefore, putting $v_0(a\xi) = x\xi$, $(\xi \in H)$, we obtain a linear operator $v_0 : aH \to H$ which can be uniquely extended to an isometry, again denoted by v_0, of $\overline{aH} = \mathbf{s}(a)H$ into H. Furthermore, there is a unique partial isometry $v \in B(H)$ such that $v|\mathbf{s}(a)H = v_0$ and $v|(\mathbf{s}(a)H)^{\perp} = 0$. Then a and v satisfy (4).

If a and v satisfy (4), then $x^*x = av^*va = a\mathbf{s}(a)a = a^2$, hence $a = |x|$, and v acts necessarily as described above. ∎

The relations

$$x = v|x|, \quad v^*v = \mathbf{s}(|x|) \tag{5}$$

are called the *polar decomposition* of x. As easily verified,

$$x^* = v^*(v|x|v^*), \quad (v^*)^*v^* = \mathbf{s}(v|x|v^*)$$

is the polar decomposition of x^*. Thus $|x^*| = v|x|v^*$ and

$$x = |x^*|v, \quad vv^* = \mathbf{s}(|x^*|). \tag{6}$$

Also, note that

$$\mathbf{l}(x) = \mathbf{s}(|x^*|) = vv^*, \quad \mathbf{r}(x) = \mathbf{s}(|x|) = v^*v. \tag{7}$$

If $x \in B(H)$ is selfadjoint with polar decomposition (5), then

$$|x| = x^+ - x^- \quad \text{and} \quad v = \mathbf{s}(x^+) - \mathbf{s}(x^-) \tag{8}$$

in particular $v = v^*$. Note that $\mathbf{s}(x^+)\mathbf{s}(x^-) = 0$.

Now let $x \in B(H)$ and let $M \subset B(H)$ be a von Neumann algebra containing x, for instance $M = R(\{x\})$. For every unitary element $u \in M'$ we have $u^*xu = x$, which entails $(u^*\mathbf{l}(x)u)x = x$, hence $u^*\mathbf{l}(x)u \geqslant \mathbf{l}(x)$ and finally, replacing u by u^*, we get $u^*\mathbf{l}(x)u = \mathbf{l}(x)$. By Corollary 7.11 we infer that

$$\mathbf{l}(x) \in M \quad \text{and} \quad \mathbf{r}(x) = \mathbf{l}(x^*) \in M.$$

Concerning the polar decomposition (5), we have $|x| = (x^*x)^{\frac{1}{2}} \in M$ and for every unitary $u \in M'$ we obtain $u^*xu = x$, $u^*|x|u = |x|$, $u^*\mathbf{s}(|x|)u = \mathbf{s}(|x|)$, hence

$$x = (u^*vu)|x|, \quad (u^*vu)^*(u^*vu) = \mathbf{s}(|x|)$$

so that $v = u^*vu$ by the uniqueness of the polar decomposition. Therefore $v \in M$, again by Corollary 7.11.

Thus, the supports of x and the terms of the polar decomposition of x belong to the same von Neumann algebras as x does.

7.13. Using the same method as for the polar decomposition, we obtain another factorization result (compare with Proposition 3.4).

PROPOSITION. *If $x, x_1, \dots, x_n \in B(H)$ and $x^*x = x_1^*x_1 + \dots + x_n^*x_n$, then there are $z_1, \dots, z_n \in R(\{x, x_1, \dots, x_n\})$ such that*

$$z_1^*z_1 + \dots + z_n^*z_n = \mathbf{s}(xx^*) \quad \textit{and} \quad x_k = z_kx \quad \textit{for all } 1 \leqslant k \leqslant n.$$

Proof. Since $\|x_k\xi\| \leqslant \|x\xi\|$, $(\xi \in H)$, the formulae

$$z_k(x\xi) = x_k\xi \quad \text{and} \quad z_k\eta = 0 \quad \text{for } \eta \in H \ominus \overline{xH}$$

define operators $z_k \in B(H)$, $\|z_k\| \leqslant 1$ such that

$$x_k = z_kx \quad \text{and} \quad z_k(H \ominus \overline{xH}) = 0, \quad (1 \leqslant k \leqslant n).$$

It is easy to check that $z_k \in \{x, x^*, x_k, x_k^*\}'' = R(\{x, x_k\})$, $(1 \leqslant k \leqslant n)$. The positive operator $\Big(\sum\limits_{k=1}^n z_k^*z_k\Big)^{\frac{1}{2}}$ vanishes on $H \ominus \overline{xH}$ and is isometric on $\overline{xH}$, since

$$x^*\Big(\sum_{k=1}^n z_k^*z_k\Big)x = x^*x.$$

It follows that $\sum\limits_{k=1}^n z_k^*z_k = \mathbf{s}(xx^*)$. ∎

COROLLARY. *If $x, y \in B(H)$ and $y^*y \leqslant x^*x$, then there exists $z \in R(\{x, y\})$ such that $z^*z \leqslant \mathbf{s}(xx^*)$ and $y = zx$.* ∎

7.14. *A spectral measure* defined on a locally compact Hausdorff space Ω with values in $B(H)$ is a $B(H)$-valued mapping $e(\cdot)$ defined on the family of all Borel subsets of Ω such that

1) $e(S)$ is a projection for every Borel set $S \subset \Omega$,

$$e(\emptyset) = 0 \quad \text{and} \quad e(\Omega) = 1_H;$$

2) $e(S_1 \cap S_2) = e(S_1)e(S_2)$ for every Borel sets $S_1, S_2 \subset \Omega$;

3) for each $\xi \in H$, the mapping $S \mapsto (e(S)\xi|\xi) = e_\xi(S)$, defined on the family of all Borel subsets of Ω, is a regular Borel measure on Ω.

By 2), $e(S_1)$ commutes with $e(S_2)$ for any S_1, S_2 and

$$S_1 \subset S_2 \Rightarrow e(S_1) \leqslant e(S_2).$$

Using this and 3), it follows that

$$S_1 \subset S_2 \subset \dots \subset S_n \subset \dots \Rightarrow e\Big(\bigcup_{n=1}^\infty S_n\Big) = s^*\text{-}\lim_n e(S_n)$$

hence $e(\cdot)$ is countably additive with respect to the s^*-topology.

Let $\xi, \eta \in H$. Using the polarisation relation

$$(e(S)\xi|\eta) = 4^{-1}\sum_{k=0}^{3} \mathrm{i}^k (e(S)(\xi + \mathrm{i}^k\eta)|\xi + \mathrm{i}^k\eta)$$

we see that $S \mapsto (e(S)\xi|\eta) = e_{\xi,\eta}(S)$ is a bounded regular Borel measure on Ω. For mutually disjoint $S_1, \dots, S_n$ we have

$$\begin{aligned}\sum_{j=1}^{n} |e_{\xi,\eta}(S_j)| &\leqslant \sum_{j=1}^{n} \|e(S_j)\xi\|\,\|e(S_j)\eta\| \leqslant \Big(\sum_{j=1}^{n}\|e(S_j)\xi\|^2\Big)^{\frac{1}{2}}\Big(\sum_{j=1}^{n}\|e(S_j)\eta\|^2\Big)^{\frac{1}{2}} \\ &= \Big(e\Big(\bigcup_{j=1}^{n} S_j\Big)\xi|\xi\Big)^{\frac{1}{2}}\Big(e\Big(\bigcup_{j=1}^{n} S_j\Big)\eta|\eta\Big)^{\frac{1}{2}} \leqslant \|\xi\|\,\|\eta\|.\end{aligned}$$

It follows that the total variation $\|e_{\xi,\eta}\|$ of $e_{\xi,\eta}$ satisfies

$$\|e_{\xi,\eta}\| \leqslant \|\xi\|\,\|\eta\|; \quad \xi, \eta \in H. \tag{1}$$

Denote by $B(\Omega)$ the set of all bounded complex Borel functions on Ω. Then $B(\Omega)$, endowed with the pointwise algebraic operations, conjugation as $*$-operation and with the sup-norm, becomes a C^*-algebra. The complex Borel step-functions form a norm dense $*$-subalgebra of $B(\Omega)$. Using this it is easy to show that for every $f \in B(\Omega)$ there exists a unique element $e(f) \in B(H)$ such that

$$(e(f)\xi|\eta) = \int_{\Omega} f(\omega)\,\mathrm{d}e_{\xi,\eta}(\omega); \quad \xi, \eta \in H. \tag{2}$$

Moreover, the map

$$B(\Omega) \ni f \mapsto e(f) \in B(H)$$

is a $*$-homomorphism, called the *Borel functional calculus associated to the spectral measure* $e(\cdot)$.

For every $f \in B(\Omega)$ and every $\xi \in H$ we have

$$\|e(f)\xi\|^2 = \int_{\Omega} |f(\omega)|^2\,\mathrm{d}e_{\xi,\xi}(\omega). \tag{3}$$

Using (3) and the Lebesgue dominated convergence theorem, it is easy to see that

(4) *if $\{f_n\}$ is a norm bounded sequence in $B(\Omega)$ pointwise convergent to $f \in B(\Omega)$, then* $e(f_n) \xrightarrow{s^*} e(f)$.

For $f \in B(\Omega)$ we have

$$\begin{aligned} e(f) = 0 &\Leftrightarrow e(\overline{f} f) = 0 \\ &\Leftrightarrow e(\{\omega \in \Omega;\ |f(\omega)|^2 > n^{-1}\}) = 0 \text{ for all } n \geqslant 1 \\ &\Leftrightarrow e(\{\omega \in \Omega;\ f(\omega) \neq 0\}) = 0 \\ &\Leftrightarrow e(\{\omega \in \Omega;\ f(\omega) = 0\}) = 1_H, \end{aligned}$$

hence

(5) *the kernel of the Borel functional calculus associated to* $e(\cdot)$ $= \{f \in B(\Omega); f(S) = \{0\}$ *for some Borel set* $S \subset \Omega$, $e(S) = 1_H\}$.

Using (5) and Corollary 1/3.11 it follows that, for every $f \in B(\Omega)$

$$\|e(f)\| = \inf_{S \subset \Omega \text{ Borel}, e(S)=1_H} \sup_{\omega \in S} |f(\omega)| \leqslant \|f\|. \tag{6}$$

Finally we note that, for every $f \in B(\Omega)$

$$\sigma(e(f)) = \bigcap_{\substack{S \subset \Omega \text{ Borel} \\ e(S)=1_H}} \overline{\{f(\omega);\ \omega \in S\}} \subset \overline{f(\Omega)}. \tag{7}$$

Indeed, let $S \subset \Omega$ Borel, $e(S) = 1_H$, denote by χ_S the characteristic function of S and fix $\omega_0 \in S$. Then $e(f) = e(f\chi_S + f(\omega_0)\chi_{\Omega \setminus S})$ by (5), so

$$\sigma(e(f)) \subset \sigma(f\chi_S + f(\omega_0)\chi_{\Omega \setminus S}) = \overline{\{f(\omega);\ \omega \in S\}}.$$

Conversely, suppose that $0 \notin \sigma(e(f))$. Then the positive operator $e(|f|) = |e(f)| \in B(H)$ is invertible, so $e(|f|) \geqslant \varepsilon 1_H$ for some $\varepsilon > 0$. Since $e(1) = 1_H$, we get $e(|f| - \varepsilon) \geqslant 0$. It follows that $e(\{\omega \in \Omega;\ |f(\omega)| - \varepsilon \leqslant -n^{-1}\}) = 0$ for all $n \geqslant 1$ and hence $S = \{\omega \in \Omega;\ |f(\omega)| \geqslant \varepsilon\}$ is a Borel subset of Ω with $e(S) = 1_H$. Clearly, $0 \notin \overline{\{f(\omega);\ \omega \in \Omega\}}$.

In what follows we shall denote

$$e(f) = \int_\Omega f(\omega)\,\mathrm{d}e(\omega); \quad f \in B(\Omega).$$

THEOREM (The spectral theorem). *Let* Ω *be a locally compact Hausdorff space and* $\pi : C_0(\Omega) \to B(H)$ *be a non degenerate* $*$*-representation. There exists a unique spectral measure* $e(\cdot)$ *defined on* Ω *with values in* $B(H)$ *such that*

$$\pi(f) = \int_\Omega f(\omega)\,\mathrm{d}e(\omega); \quad f \in C_0(\Omega). \tag{8}$$

Proof. For every $\xi, \eta \in H$ the map $C_0(\Omega) \ni f \mapsto (\pi(f)\xi|\eta) \in \mathbb{C}$ is a bounded linear functional with norm $\leqslant \|\xi\|\,\|\eta\|$ so, by the Riesz-Kakutani theorem, there exists a unique bounded regular Borel measure $\mu_{\xi,\eta}$ on Ω such that $\|\mu_{\xi,\eta}\| \leqslant \|\xi\|\,\|\eta\|$ and

$$(9) \qquad (\pi(f)\xi|\eta) = \int\limits_{\Omega} f(\omega)\,\mathrm{d}\mu_{\xi,\eta}(\omega); \quad f \in C_0(\Omega).$$

The map $H \times H \ni (\xi, \eta) \mapsto \mu_{\xi,\eta}$ is linear in ξ conjugate linear in η and has a norm $\leqslant 1$. Since, for $f \in C_0(\Omega)$,

$$\int\limits_{\Omega} f(\omega)\,\mathrm{d}\mu_{\xi,\eta}(\omega) = (\pi(f)\xi|\eta) = \overline{(\pi(\overline{f})\eta|\xi)} = \int\limits_{\Omega} f(\omega)\,\mathrm{d}\overline{\mu_{\eta,\xi}}(\omega)$$

it follows that

$$\mu_{\xi,\eta} = \overline{\mu_{\eta,\xi}}; \quad \xi, \eta \in H.$$

On the other hand, for $f, g \in C_0(\Omega)$,

$$\int\limits_{\Omega} f(\omega)\,\mathrm{d}\mu_{\pi(g)\xi,\eta}(\omega) = (\pi(f)\pi(g)\xi|\eta) = (\pi(fg)\xi|\eta) = \int\limits_{\Omega} f(\omega)g(\omega)\,\mathrm{d}\mu_{\xi,\eta}(\omega)$$

hence

$$\mathrm{d}\mu_{\pi(g)\xi,\eta} = g\,\mathrm{d}\mu_{\xi,\eta}; \quad \xi, \eta \in H,\ g \in C_0(\Omega).$$

For each Borel set $S \subset \Omega$ there exists a unique selfadjoint operator $e(S) \in B(H)$, $\|e(S)\| \leqslant 1$ such that

$$(10) \qquad (e(S)\xi|\eta) = \mu_{\xi,\eta}(S); \quad \xi, \eta \in H.$$

For every Borel set $S_1 \subset \Omega$ and every $g \in C_0(\Omega)$ we have

$$\int\limits_{\Omega} g(\omega)\,\mathrm{d}\mu_{\xi,e(S_1)\eta}(\omega) = (e(S_1)\pi(g)\xi|\eta) = \mu_{\pi(g)\xi,\eta}(S_1) = \int\limits_{\Omega} \chi_{S_1}(\omega)g(\omega)\,\mathrm{d}\mu_{\xi,\eta}(\omega),$$

hence

$$\mathrm{d}\mu_{\xi,e(S_1)\eta} = \chi_{S_1}\mathrm{d}\mu_{\xi,\eta}; \quad \xi, \eta \in H.$$

Consequently, if S_2 is another Borel set, then

$$\begin{aligned}(e(S_1)e(S_2)\xi|\eta) &= \mu_{\xi,e(S_1)\eta}(S_2) = \int\limits_{\Omega} \chi_{S_2}(\omega)\chi_{S_1}(\omega)\,\mathrm{d}\mu_{\xi,\eta}(\omega)\\ &= \mu_{\xi,\eta}(S_1 \cap S_2) = (e(S_1 \cap S_2)\xi|\eta); \quad \xi, \eta \in H.\end{aligned}$$

It follows that

$$e(S_1 \cap S_2) = e(S_1)e(S_2).$$

In particular, all $e(S)$ are projection.

Since π is non degenerated, we get $e(\Omega) = 1_H$ and clearly $e(\emptyset) = 0$. Finally, condition 3) follows from (10), hence $e(\cdot)$ is a spectral measure, and relation (8) follows from (9) and (10).

The unicity of $e(\cdot)$ follows from the unicity part of Riesz-Kakutani theorem. ∎

The $e(S)$'s will be called the *spectral projection* of π.

Note that the Borel functional calculus associated to $e(\cdot)$ extends the $*$-representation $\pi : C_0(\Omega) \to B(H)$ to a $*$-representation $B(\Omega) \ni f \mapsto e(f) \in B(H)$ with the properties (3) to (7).

Let $M \subset B(H)$ be a von Neumann algebra and suppose that $\pi(C_0(\Omega)) \subset M$. For every unitary element $u \in M'$, the map $S \mapsto u^* e(S) u$ is a spectral measure and

$$\pi(f) = u^* \pi(f) u = u^* \Big(\int\limits_{\Omega} f(\omega)\, \mathrm{d}e(\omega) \Big) u = \int\limits_{\Omega} f(\omega)\, \mathrm{d}(u^* e(\cdot) u)(\omega)$$

for all $f \in C_0(\Omega)$. By the unicity assertion of the above theorem we infer that $u^* e(S) u = e(S)$ for all Borel subsets S of Ω and all $u \in M'$, unitary. Consequently, all spectral projections $e(S)$ belong to M by Corollary 7.11. Therefore

$$\text{(11)} \qquad \pi(C_0(\Omega)) \subset M \Rightarrow e(f) = \int\limits_{\Omega} f(\omega)\, \mathrm{d}e(\omega) \in M; \quad f \in B(\Omega).$$

Denote by $\mathrm{Baire}(\Omega)$ the smallest class of bounded complex functions on Ω which is closed under taking pointwise limits of uniformly bounded sequences and contains $C_0(\Omega)$. Then $\mathrm{Baire}(\Omega)$ is a C^*-subalgebra of $B(\Omega)$.

By the above theorem and by the properties of the Borel functional calculus associated to a spectral measure we get the following unicity result.

Corollary. *Let Ω be a locally compact Hausdorff space and $\pi : C_0(\Omega) \to B(H)$ be a non degenerate $*$-representation. Then there exists a unique extension of π to a map*

$$\pi_{\mathrm{Baire}} : \mathrm{Baire}(\Omega) \to B(H)$$

such that for every norm bounded sequence $\{f_n\}$ in $\mathrm{Baire}(\Omega)$ *pointwise convergent to $f \in \mathrm{Baire}(\Omega)$ we have $\pi_{\mathrm{Baire}}(f_n) \xrightarrow{wo} \pi_{\mathrm{Baire}}(f)$.* ∎

Of course, $\pi_{\mathrm{Baire}}(f) = e(f)$, $(f \in \mathrm{Baire}(\Omega))$, where $e(\cdot)$ is the spectral measure associated to π by the above theorem, so π_{Baire} is a $*$-homomorphism and the properties (3) to (7) and (11) are valid for π_{Baire} instead of $f \mapsto e(f)$.

Finally, we remark that *if Ω is a locally compact Hausdorff space with a countable basis of open sets, then*

$$\mathrm{Baire}(\Omega) = B(\Omega).$$

Indeed, by the Urysohn-Tikhonov metrization theorem, Ω is metrisable. Let $d(\cdot\,,\cdot)$ be a metric defining the topology of Ω and such that $\sup\{d(\omega,\rho);\, \omega,\rho \in \Omega\} < +\infty$. If $D \subset \Omega$ is open, then putting

$$f_n(\omega) = \min\Big\{ n \inf_{\rho \in \Omega \setminus D} d(\omega,\rho), 1 \Big\}; \quad \omega \in \Omega,\ n \geqslant 1,$$

we get an increasing sequence $\{f_n\}$ of positive continuous functions on Ω which converges pointwise to the characteristic function of D. It follows that the characteristic function of any Borel set belongs to $\mathrm{Baire}(\Omega)$, hence $B(\Omega) \subset \mathrm{Baire}(\Omega)$.

Thus, if Ω has a countable open basis, then Baire(Ω) can be replaced by $B(\Omega)$ in the above Corollary and π_{Baire} may be denoted as π_{Borel}.

7.15. Let $x \in B(H)$ be a normal operator. By 1.16, the continuous functional calculus for x is the unique $*$-representation $C(\sigma(x)) \ni f \mapsto f(x) \in B(H)$ such that

$$f_0(\lambda) \equiv 1 \Rightarrow f_0(x) = 1_H \quad \text{and} \quad f_1(\lambda) = \lambda \Rightarrow f_1(x) = x.$$

Moreover, $f \mapsto f(x)$ is injective. Applying to this $*$-representation the results of 7.14, we get

THEOREM (Borel functional calculus for normal operators). *Let $x \in B(H)$ be a normal operator. There exists a unique $*$-representation*

$$B(\sigma(x)) \ni f \mapsto f(x) \in B(H)$$

such that

(i) $f_0(\lambda) \equiv 1 \Rightarrow f_0(x) = 1_H$ *and* $f_1(\lambda) = \lambda \Rightarrow f_1(x) = x$;

(ii) *if* $\{f_n\}$ *is a norm bounded sequence in* $B(\sigma(x))$ *pointwise convergent to* $f \in B(\sigma(x))$, *then* $f_n(x) \xrightarrow{wo} f(x)$.

Moreover, the map $e(\cdot) : S \mapsto e(S) = \chi_S(x)$ *is a spectral measure on* $\sigma(x)$ *with values in* $B(H)$ *and*

$$f(x) = \int_{\sigma(x)} f(\lambda)\,\mathrm{d}e(\lambda); \quad f \in B(\sigma(x)). \quad ∎ \tag{1}$$

The projections $e(S)$ are called the *spectral projections* of the normal operator $x \in B(H)$.

By 7.14, *the map $f \mapsto f(x)$ has the following properties:*

$$\|f(x)\xi\|^2 = \int_{\sigma(x)} |f(\lambda)|^2\,\mathrm{d}e_{\xi,\xi}(\lambda); \quad f \in B(\sigma(x)),\ \xi \in H \tag{2}$$

(3) *if $\{f_n\}$ is norm bounded sequence in $B(\sigma(x))$ pointwise convergent to $f \in B(\sigma(x))$, then* $f_n(x) \xrightarrow{s^*} f(x)$;

(4) *the kernel of $B(\sigma(x)) \ni f \mapsto f(x) \in B(H)$ $= \{f \in B(\sigma(x));\ f(S) = \{0\}$ for some Borel set $S \subset \sigma(x)$, $e(S) = 1_H\}$*

$$\|f(x)\| = \inf_{S \subset \sigma(x) \text{Borel},\, e(S) = 1_H} \ \sup_{\lambda \in S} |f(\lambda)| \leqslant \|f\| \tag{5}$$

$$\sigma(f(x)) = \bigcap_{\substack{S\subset\sigma(x)\,\text{Borel}\\ e(S)=1_H}} \overline{\{f(\lambda);\,\lambda\in S\}} \subset \overline{f(\sigma(x))} \tag{6}$$

$$f(x) \in R(\{x\}) \quad \textit{for all } f \in B(\sigma(x)). \tag{7}$$

Since, as easily verified, $\{f \in B(\mathbb{R});\, f \,\text{real},\, f(0) = 0\}$ is the smallest class of bounded real functions on $\mathbb{R}$ which is closed under taking pointwise limits of uniformly bounded sequences and contains $\{f \in C_0(\mathbb{R});\, f\,\text{real},\, f(0) = 0\}$, from 1.16.(8) we infer that

$$\begin{gathered} x = x^* \in B(H),\ f \in B(\sigma(x)\cup\{0\}), f\ \textit{real},\ f(0) = 0 \Rightarrow \\ [f|\sigma(x)](x) \in \textit{the } s^*\textit{-closed real subalgebra of } B(H) \textit{ generated by } x. \end{gathered} \tag{8}$$

By the injectivity of $C(\sigma(x)) \ni f \mapsto f(x) \in B(H)$, we get

$$\textit{the support of the spectral measure } e(\cdot) \textit{ is } \sigma(x). \tag{9}$$

If f is a bounded complex Borel function defined on some subset $S \supset \sigma(x)$ of $\mathbb{C}$, then we denote $f(x) = [f|\sigma(x)](x)$.

Using the above theorem and 1.18, it is easy to verify the following statements:

1) *If $x \in B(H)$ is normal, $f \in B(\sigma(x))$ and g is a bounded complex Borel function on $\overline{f(\sigma(x))} \supset \sigma(f(x))$, then*

$$(g \circ f)(x) = g(f(x)).$$

2) *If $x, y \in B(H)$ are commuting normal operators, then for every $f \in B(\sigma(x))$ and every $g \in B(\sigma(y))$, the operators $f(x)$ and $g(y)$ commute.*

3) *If $x \in B(H)$ is normal and $y \in B(H)$ is such that $x = xy = y^*x$, then for every $f \in B(\sigma(x) \cup \{0\})$*

$$f(x) = f(x)y + f(0)(1_H - y) = y^*f(x) + f(0)(1_H - y).$$

In particular, if $f \in B(\sigma(x) \cup \{0\})$ and $f(0) = 0$ then

$$f(x) = f(x)y = y^*f(x).$$

4) *If $x \in B(H)$ is normal, then*

$$\mathbf{s}(x) = \chi_{\mathbb{C}\setminus\{0\}}(x)$$

and, for every $f \in B(\sigma(x) \cup \{0\})$ with $f(0) = 0$,

$$\mathbf{s}(f(x)) \leqslant \mathbf{s}(x).$$

Moreover, if $\{f_n\}$ *is a norm bounded sequence in* $B(\mathbb{C})$*, pointwise convergent on* $\sigma(x)$ *to* $\chi_{\mathbb{C}\setminus\{0\}}$*, then*

$$f_n(x) \xrightarrow{s^*} \mathbf{s}(x).$$

In particular, for $x \geqslant 0$ *we have*

$$nx(1+nx)^{-1} \xrightarrow{s^*} \mathbf{s}(x), \quad x^{\frac{1}{n}} \xrightarrow{s^*} \mathbf{s}(x).$$

5) *If* $x \in B(H)$ *is normal* $f(\lambda) = |\lambda|$ *and* $g(\lambda) = \operatorname{sign}(\lambda)$*, then* $|x| = f(x)$ *and* $x = g(x)|x|$ *is the polar decomposition of* x*.*

6) *Let* M *be a wo-closed* $*$*-subalgebra of* $B(H)$*,* $x \in M$ *normal and* $f \in B(\sigma(x) \cup \{0\})$*,* $f(0) = 0$*. Then* $f(x) \in M$*.*

Indeed by Lemma 7.11, M has a unit element e which is a projection in $B(H)$. Since $x = xe = ex$, we have $f(x) = f(x)e$ by 3). On the other hand, x belongs to the von Neumann algebra $M + \mathbb{C}1_H \subset B(H)$, hence $f(x) \in (M + \mathbb{C}1_H)e = M$.

7) Let $x \in B(H)$ be normal, $S \subset \mathbb{C}$ and $\{f_\alpha\}_{\alpha \in S}$ be a family in $B(\sigma(x))$ such that $\sup\{\|f_\alpha\|;\, \alpha \in S \cap K\} < +\infty$ for each compact set $K \subset \mathbb{C}$ and such the functions

$$S \ni \alpha \mapsto f_\alpha(\lambda) \in \mathbb{C}; \quad \lambda \in \sigma(x),$$

are continuous and their restrictions to the interior of S are analytic. Then the function

$$S \ni \alpha \mapsto f_\alpha(x) \in B(H)$$

is s^*-continuous and its restriction to the interior of S is analytic.

8) For every $x \in B(H)$, $x \geqslant 0$, and every $\alpha \in \mathbb{C}$, $\operatorname{Re}\alpha \geqslant 0$ we define the element

$$x^\alpha = f_\alpha(x) \in B(H)$$

where

$$f_\alpha(\lambda) = \exp(\alpha \ln \lambda) \quad \text{for } \lambda \in (0, +\infty) \quad \text{and} \quad f_\alpha(0) = 0.$$

Then, by 7), *the function*

$$\alpha \mapsto x^\alpha \in B(H)$$

is s^**-continuous on* $\{\alpha \in \mathbb{C};\, \operatorname{Re}\,\alpha \geqslant 0\}$ *and analytic on* $\{\alpha \in \mathbb{C};\, \operatorname{Re}\,\alpha > 0\}$*.*

9) *If* $x \in B(H)$ *is normal, then* $\lambda \in \sigma(x)$ *is an eigenvalue of* x *if and only if* $\chi_{\{\lambda\}}(x) \neq 0$*. Moreover,* $\chi_{\{\lambda\}}(x)$ *is the orthogonal projection onto the eigenspace of* x *corresponding to* λ*.*

10) *Let* $x \in B(H)$ *be compact and normal.* Then for every $\varepsilon > 0$ the range of the spectral projection $\chi_{\{\lambda \in \sigma(x), |\lambda| \geqslant \varepsilon\}}(x)$ is finite dimensional, hence $\{\lambda \in \sigma(x);\, |\lambda| \geqslant \varepsilon\}$ consists of eigenvalues with finite multiplicities. *Let* $\{\lambda_n\}$

be the sequence of all non zero eigenvalues of x, each being repeated as its multiplicity indicates. Then there exists an orthonormal sequence $\{\xi_n\}$ in H such that

$$x = \sum_n \lambda_n(\,\cdot|\xi_n)\xi_n.$$

Note that also for an arbitrary compact $x \in B(H)$ and $\varepsilon > 0$ the set $\{\lambda \in \sigma(x), |\lambda| \geqslant \varepsilon\}$ consists of a finite number of eigenvalues with finite multiplicities (see [81], VII.4).

7.16. In this section we point out some consequences of the Borel functional calculus for von Neumann algebras.

PROPOSITION 1. *Let $x \in B(H)$ be selfadjoint and $\alpha \in \mathbb{R}$. Then there exists a projection e in the commutative von Neumann algebra $R(\{x\})$ such that*

$$xe \geqslant \alpha e \quad \text{and} \quad x(1-e) \leqslant \alpha(1-e).$$

Proof. Take $e = \chi_{[\alpha,+\infty]}(x)$ or $e = \chi_{(\alpha,+\infty)}(x)$. ∎

PROPOSITION 2. *Let $x \in B(H)$, $x \geqslant 0$, and let $\{\alpha_n\} \subset (0,+\infty)$ be a decreasing sequence, $\alpha_n \to 0$. Then there exists an increasing sequence $\{e_n\}$ of projections in $R(\{x\})$ such that*

$$xe_n \geqslant \alpha_n e_n \quad \text{for all } n \quad \text{and} \quad e_n \xrightarrow{s^*} \mathbf{s}(x).$$

Proof. Take $e_n = \chi_{[\alpha_n,+\infty]}(x)$, or $e_n = \chi_{(\alpha_n,+\infty)}(x)$, $(n \in \mathbb{N})$. ∎

PROPOSITION 3. *Let $x \in B(H)$, $0 \leqslant x \leqslant 1_H$. Then there exists a sequence $\{e_n\}$ of projections in $R(\{x\})$ such that*

$$x = \sum_{n=1}^{\infty} 2^{-n} e_n \text{ in the norm topology.}$$

Proof. Using Proposition 1, we can construct inductively a sequence $\{e_n\}$ of projections in $R(\{x\})$ such that

$$\Big(x - \sum_{j=1}^{n-1} 2^{-j} e_j\Big) e_n \geqslant 2^{-n} e_n v$$

$$\Big(x - \sum_{j=1}^{n-1} 2^{-j} e_j\Big)(1 - e_n) \leqslant 2^{-n}(1 - e_n).$$

Then, by induction

$$0 \leqslant x - \sum_{j=1}^{n} 2^{-j} e_j \leqslant 2^{-n}$$

which entails the desired result. ∎

By Proposition 3, every von Neumann algebra is the norm-closed linear span of all projections contained in it.

Proposition 4. *Let $u \in B(H)$ be unitary and $\theta_0 \in R$. Then there exists $x \in R(\{u\})$ selfadjoint such that*

$$u = \exp(\mathrm{i}x) \quad \textit{and} \quad \sigma(x) \subset \{\theta;\, \theta_0 - \pi \leqslant \theta \leqslant \theta_0 + \pi,\, \mathrm{e}^{\mathrm{i}\theta} \in \sigma(u)\}.$$

Proof. Consider the function f defined on the unit circle by

$$f(\mathrm{e}^{\mathrm{i}\theta}) = \theta; \quad \theta_0 - \pi \leqslant \theta < \theta_0 + \pi.$$

Then f is a bounded real Borel function. Putting $x = f(u)$, we get a selfadjoint element $x \in R(\{u\})$ with $u = \exp(\mathrm{i}x)$ and moreover,

$$\sigma(x) \subset \overline{f(\sigma(u))} \subset \{\theta;\, \theta_0 - \pi \leqslant \theta \leqslant \theta_0 + \pi,\, \mathrm{e}^{\mathrm{i}\theta} \in \sigma(u)\}. \quad \blacksquare$$

In particular, if for some $0 < \varepsilon \leqslant \pi$ we have $\sigma(u) \subset \{\mathrm{e}^{\mathrm{i}\theta};\, \theta_0 - \varepsilon \leqslant \theta \leqslant \theta_0 + \varepsilon\}$, then there exists $x \in R(\{u\})$ selfadjoint with $\theta_0 - \varepsilon \leqslant x \leqslant \theta_0 + \varepsilon$, such that $u = \exp(\mathrm{i}x)$. Using this remark and the Kaplansky density theorem, it follows easily the following completion of the Kaplansky density theorem.

Proposition 5 (J. Glimm, R.V. Kadison). *Let A be a non degenerate C^*-subalgebra of $B(H)$. Then every unitary operator $u \in B(H)$ wo-adherent to A is s^*-adherent to the set*

$$\{\exp(\mathrm{i}\,x);\, x \in A \textit{ selfadjoint},\, \|1_H - \exp(\mathrm{i}\,x)\| \leqslant \|1_H - u\|\}. \quad \blacksquare$$

7.17. Let $M \subset B(H)$ be a von Neumann algebra. Then

$$Z = Z_M = M \cap M'$$

is the common *center* of M and M'. Clearly, Z is a von Neumann algebra. Since $Z \subset R(M \cup M')' \subset M' \cap M'' = Z$, we have

$$Z' = R(M \cup M').$$

The elements of Z will be called *central elements* of M.

If $Z_M = \mathbb{C}1_M$, then M is called a *von Neumann factor* or simply a *factor*. For instance, using the von Neumann density theorem (7.11), it follows that $B(H)$ is a factor.

For each $x \in M$ we define its *central support* $\mathbf{z}(x)$ by

$$\mathbf{z}(x) = \text{the orthogonal projection onto } \overline{\mathrm{lin}\,(Mx)H}.$$

It is easy to see that $\mathbf{z}(x)$ is the smallest central projection p of M such that $px = x$. Also,

$$\mathbf{z}(x) = \mathbf{z}(\mathbf{l}(x)) = \mathbf{z}(\mathbf{r}(x)).$$

For $x \in B(H)$, $S \subset B(H)$ and a projection $e \in B(H)$ we denote

$$x_e = ex|_{eH} \in B(eH),$$
$$S_e = \{x_e;\, x \in S\} \subset B(eH).$$

Theorem. *Let $M \subset B(H)$ be a von Neumann algebra with center Z and $e \in M$ be a projection. Then*

(i) *the map $M' \ni x' \mapsto (x')_e \in B(eH)$ is a wo-continuous $*$-homomorphism and its kernel is $M'(1 - \mathbf{z}(e))$;*

(ii) *$M_e \subset B(eH)$ and $(M')_e \subset B(eH)$ are von Neumann algebras;*

(iii) *$(M_e)' = (M')_e$;*

(iv) *the common center of M_e and $(M')_e$ is Z_e.*

Proof. (i) Clearly, $\pi : M' \ni x' \mapsto (x')_e \in B(eH)$ is a *wo*-continuous $*$-homomorphism of M' onto $(M')_e$ and $\operatorname{Ker}\pi \supset M'(1 - \mathbf{z}(e))$. If $x' \in \operatorname{Ker}\pi$, then $x'e = ex'e = 0$ and we infer succesively

$$x'(Me)H = M(x'e)H = \{0\},$$
$$x'\mathbf{z}(e) = 0,$$
$$x' = x'(1 - \mathbf{z}(e)) \in M'(1 - \mathbf{z}(e)).$$

(ii) By Corollary 1/3.11, $(M')_e = \pi(M')$ is a C^*-subalgebra of $B(eH)$ and, by Corollary 1/3.15, $[(M')_e]_1 = \pi((M')_1)$. Using Lemma 1/7.8 and the *wo*-continuity of π, it follows that $[(M')_e]_1$ is *wo*-compact and hence $(M')_e \subset B(eH)$ is a von Neumann algebra (7.11).

Clearly, $M_e \subset ((M')_e)'$. Conversely, let $y \in ((M')_e)'$. There exists $x \in B(H)$, $x = exe$, such that $y = x_e$. For every $x' \in M'$ we have $x_e(x')_e = (x')_e x_e$ in $B(eH)$, hence $xx' = x'x$ in $B(H)$, so $x \in M'' = M$ and $y = x_e \in M_e$. Therefore $M_e = ((M')e)'$.

In particular, M_e is a von Neumann algebra.

(iii) From (1) it follows also that $(M_e)' = ((M')_e)'' = (M')_e$.

(iv) By (i), the map $x' \mapsto (x')_e$ is a $*$-isomorphism of the $*$-algebra $M'\mathbf{z}(e)$ onto $(M')_e$. It is easy to verify that the center of $M'\mathbf{z}(e)$ is $Z\mathbf{z}(e)$, hence the center of $(M')_e$ is Z_e. ∎

The above theorem yields two metods to derive new von Neumann algebras from a given von Neumann algebra.

Let $M \subset B(H)$ be a von Neumann algebra and $e \in M$ be a projection. Then $M_e \subset B(eH)$ is called the *reduced von Neumann algebra* of M by e. Its commutant $(M_e)' = (M')_e$ will be denoted simply by M'_e and its center is Z_e. The map $x \mapsto x_e$ induces a $*$-isomorphism of the *wo*-closed $*$-subalgebra eMe of $B(H)$ onto M_e. Hence the center of eMe is Ze.

Let $M \subset B(H)$ be a von Neumann algebra and $e' \in M'$ be a projection. Then $M_{e'} \subset B(e'H)$ is called the *induced von Neumann* algebra of M by e' and the $*$-homomorphism

$$M \ni x \mapsto x_{e'} \in M_{e'}$$

is called the *canonical induction* defined by e'. If $M \subset B(H)$ is a von Neumann factor, then every induction map defined on M is injective.

7.18. Let $M \subset B(H)$, $N \subset B(K)$ be von Neumann algebras. Then

$$M \otimes N = \Big\{ \sum_{k=1}^{n} a_k \mathbin{\overline{\otimes}} b_k;\ a_k \in M,\ b_k \in N,\ n \in \mathbb{N} \Big\}$$

is a $*$-subalgebra of $B(H \mathbin{\overline{\otimes}} K)$ which contains $1_{H \overline{\otimes} K} = 1_H \mathbin{\overline{\otimes}} 1_K$. The *wo*-closure $M \mathbin{\overline{\otimes}} N$ of $M \mathbin{\overline{\otimes}} N$ in $B(H \mathbin{\overline{\otimes}} K)$ is a von Neumann algebra, called the *tensor product of von Neumann algebras* M and N. Clearly,

$$M \mathbin{\overline{\otimes}} N = R(\{a \mathbin{\overline{\otimes}} b;\ a \in M,\ b \in N\}) = \{a \mathbin{\overline{\otimes}} b;\ a \in M,\ b \in N\}''.$$

The inclusion $M' \mathbin{\overline{\otimes}} N' \subset (M \mathbin{\overline{\otimes}} N)'$ is trivial. Actually, $M' \mathbin{\overline{\otimes}} N' = (M \mathbin{\overline{\otimes}} N)'$, this being a fundamental result due to M. Tomita (see [320], [307]). Here we prove only a very particular case.

Proposition 1. *Let $M \subset B(H)$ be a von Neumann algebra and K be a Hilbert space. Then*

$$(M \mathbin{\overline{\otimes}} B(K))' = M' \otimes \mathbb{C}1_K, \quad Z_{M \overline{\otimes} B(K)} = Z_M \otimes \mathbb{C}1_K.$$

Proof. Let $x' \in (M \mathbin{\overline{\otimes}} B(K))'$. Consider an orthonormal basis $\{\eta_\iota\}_{\iota \in I}$ of K and fix an index $\iota_0 \in I$. For every $\iota, \kappa \in I$ define $u_\iota : H \to H \mathbin{\overline{\otimes}} K$ and $e_{\iota,\kappa} \in B(H)$ by

$$u_\iota(\xi) = \xi \otimes \eta_\iota,\ (\xi \in H); \quad e_{\iota\kappa}(\eta) = (\eta|\eta_\kappa)\eta_\iota,\ (\eta \in K).$$

Then for every $\iota, \kappa \in I$, we have

$$u_\iota^* u_\kappa = \delta_{\iota\kappa} 1_H, \quad u_\iota u_\kappa^* = 1_H \mathbin{\overline{\otimes}} e_{\iota\kappa}.$$

Let

$$a' = u_{\iota_0}^* x' u_{\iota_0} \in B(H).$$

Since $x' \in (M \mathbin{\overline{\otimes}} B(K))'$, we get

$$u_\iota^* x' u_\kappa = u_{\iota_0}^* u_{\iota_0} u_\iota^* x' u_\kappa = u_{\iota_0}^* x' u_{\iota_0} u_\iota^* u_\kappa = \delta_{\iota\kappa} a'; \quad \iota, \kappa \in I.$$

Consequently, for every $\xi, \zeta \in H$ and every $\iota, \kappa \in I$, we obtain

$$\begin{aligned}(x'(\zeta \otimes \eta_\kappa)|\xi \otimes \eta_\iota) &= (x'(1_H \mathbin{\overline{\otimes}} e_{\kappa\kappa})(\zeta \otimes \eta_\kappa)|(1_H \mathbin{\overline{\otimes}} e_{\iota\iota})(\xi \otimes \eta_\iota)) \\ &= (x' u_\kappa u_\kappa^*(\zeta \otimes \eta_\kappa)|u_\iota u_\iota^*(\xi \otimes \eta_\iota)) = (x' u_\kappa \zeta | u_\iota \xi) \\ &= (u_\iota^* x' u_\kappa \zeta|\xi) = \delta_{\iota\kappa}(a'\zeta|\xi) = ((a' \mathbin{\overline{\otimes}} 1_K)(\zeta \otimes \eta_\kappa)|\xi \otimes \eta_\iota)\end{aligned}$$

hence

$$x' = a' \mathbin{\overline{\otimes}} 1_K.$$

Since $x' \in (M \otimes \mathbb{C}1_K)$, it follows that $a' \in M$. Thus, $x' \in M' \otimes \mathbb{C}1_K$. This proves the first equality in the statement.

Let $z \in Z_{M \overline{\otimes} B(K)}$. Then $z \in (M \mathbin{\overline{\otimes}} B(K))'$, hence $z = c \mathbin{\overline{\otimes}} 1_K$ for some $c \in M'$. Also, $z \in M \mathbin{\overline{\otimes}} B(K) = (M \mathbin{\overline{\otimes}} B(K))'' = (M' \otimes \mathbb{C}1_K)'$, hence $c \in M'' = M$. Thus $c \in Z_M$ and $z \in Z_M \otimes \mathbb{C}1_K$. Since the inclusion $Z_M \otimes \mathbb{C}1_K \subset Z_{M \overline{\otimes} B(K)}$ is obvious, this proves the second equality in the statement. ∎

For every von Neumann algebra $M \subset B(H)$ and every Hilbert space K, the map $x \mapsto x \overline{\otimes} 1_K$ is a $*$-isomorphism of M onto $M \otimes \mathbb{C}1_K$ called the *amplification* of M by K.

The above proposition can be restated in a matrix language.

Let H be a Hilbert space and υ be a cardinal number. Consider a set I with $\mathrm{Card}(I) = \upsilon$. Recall that the Hilbert space direct sum $H^{(\upsilon)}$ of υ copies of H consists of all families $\{\xi_\iota\}_{\iota \in I} \subset H$ with $\sum_{\iota \in I} \|\xi_\iota\|^2 < +\infty$ and

$$(\{\xi_\iota\}_{\iota \in I} | \{\zeta_\iota\}_{\iota \in I}) = \sum_{\iota \in I} (\xi_\iota | \zeta_\iota)$$

for all $\{\xi_\iota\}_{\iota \in I}, \{\zeta_\iota\}_{\iota \in I} \in H^{(\upsilon)}$.

To each $x \in B(H^{(\upsilon)})$ we can associate the "matrix elements" $x_{\iota\kappa} \in B(H)$ defined by

$$(x_{\iota\kappa}\xi|\zeta) = (x\{\delta_{\kappa\lambda}\xi\}_{\lambda \in I} \mid \{\delta_{\iota\lambda}\zeta\}_{\lambda \in I}); \quad \xi, \zeta \in H.$$

Then the "matrix" $[x_{\iota\kappa}]$ determines uniquely x, namely

$$x\{\xi_\kappa\}_{\kappa \in I} = \Big\{ \sum_{\kappa \in I} x_{\iota\kappa}\xi_\kappa \Big\}_{\iota \in I}; \quad \{\xi_\kappa\}_{\kappa \in I} \in H^{(\upsilon)}$$

where the sums converge in the norm-topology of H. It is easy to check that

$$\|x_{\iota\kappa}\| \leqslant \|x\| \leqslant \Big(\sum_{\lambda,\mu} \|x_{\lambda\mu}\|^2 \Big)^{\frac{1}{2}}; \quad \iota, \kappa \in I.$$

Also for all $x, y \in B(H^{(\upsilon)})$, $\alpha \in \mathbb{C}$ and $\iota, \kappa \in I$ we have

(1) $(x + y)_{\iota\kappa} = x_{\iota\kappa} + y_{\iota\kappa}$

(2) $(\alpha x)_{\iota\kappa} = \alpha x_{\iota\kappa}$

(3) $(xy)_{\iota\kappa} = \sum_{\lambda \in I} x_{\iota\lambda} y_{\lambda k}$ in the *so*-topology of $B(H)$

(4) $(x^*)_{\iota\kappa} = (x_{\kappa\iota})^*$

(5) the map $B(H^{(\upsilon)}) \ni x \mapsto x_{\iota\kappa} \in B(H)$ is *so*-continuous.

For $S \subset B(H)$ we denote

$$M_\upsilon(S) = \{x \in B(H^{(\upsilon)});\ x_{\iota\kappa} \in S \text{ for all } \iota, \kappa \in I\}.$$
$$\mathrm{Diag}_\upsilon(S) = \{x \in B(H)^{(\upsilon)};\ x_{\iota\kappa} = \delta_{\iota\kappa} x_{\lambda\lambda} \in S \text{ for all } \iota, \kappa, \lambda \in I\}.$$

Using (1)–(5) we see that if S is an *so*-closed $*$-subalgebra of $B(H)$, then $M_\upsilon(S)$ and $\mathrm{Diag}_\upsilon(S)$ are *so*-closed $*$-subalgebras of $B(H^{(\upsilon)})$. Thus, if $S \subset B(H)$ is a von Neumann algebra, then $M_\upsilon(S) \subset B(H^{(\upsilon)})$ and $\mathrm{Diag}_\upsilon(S) \subset B(H^{(\upsilon)})$ are von Neumann algebras. Note that

$$M_\upsilon(B(H)) = B(H^{(\upsilon)}).$$

The map $\mathrm{Diag}_\upsilon(S) \ni x \mapsto x_{\iota\iota} \in S$ does not depend on $\iota \in I$ and is a bijection. Its inverse map is denoted by

$$\mathrm{Diag}_\upsilon(\cdot) : S \mapsto \mathrm{Diag}_\upsilon(S).$$

If S is a $*$-subalgebra of $B(H)$, then $\mathrm{Diag}(\cdot)$ is a $*$-isomorphism.

Let $\{\eta_\iota\}_{\iota \in I}$ be the canonical orthonormal basis of the Hilbert space $\ell^2(I)$. Then the map

$$U : H^{(\upsilon)} \ni \{\xi_\iota\}_{\iota \in I} \mapsto \sum_{\iota \in I} \xi_\iota \otimes \eta_\iota \in H \overline{\otimes} \ell^2(I)$$

is a unitary operator and the map

$$\pi : B(H^{(\upsilon)}) \ni x \mapsto UxU^* \in B(H \overline{\otimes} \ell^2(I))$$

is a $*$-isomorphism. Clearly, π and π^{-1} are *so*-continuous and

$$\pi(\mathrm{Diag}_\upsilon(a)) = a \overline{\otimes} 1_{\ell^2(I)}; \quad a \in B(H).$$

If $M \subset B(H)$ is von Neumann algebra, then

$$\begin{aligned} \pi(\mathrm{Diag}_\upsilon(M')) &= M' \overline{\otimes} \mathbb{C}1_{\ell^2(I)} \\ \pi(M_\upsilon(M)) &\supset M \overline{\otimes} B(\ell^2(I)). \end{aligned} \tag{6}$$

Since $M_\upsilon(M) \subset \mathrm{Diag}_\upsilon(M')'$, by Proposition 1 we obtain

$$\pi(M_\upsilon(M)) \subset (\pi(\mathrm{Diag}_\upsilon(M'))' = (M' \otimes \mathbb{C}1_{\ell^2(I)})' = M \overline{\otimes} B(\ell^2(I)),$$

hence

$$\pi(M_\upsilon(M)) = M \overline{\otimes} B(\ell^2(I)). \tag{7}$$

Thus, Proposition 1 can be rephrased as follows

PROPOSITION 2. *Let $M \subset B(H)$ be a von Neumann algebra and υ be a cardinal number. Then*

$$M_\upsilon(M)' = \mathrm{Diag}_\upsilon(M'), \quad Z_{M_\upsilon(M)} = \mathrm{Diag}_\upsilon(Z_M). \quad \blacksquare$$

Recall that if $n \in \mathbb{N}$ is a finite cardinal number, then *every* matrix $[x_{ik}]$ with elements of $B(H)$ defines an element of $B(H^{(n)})$ (4.19). Thus, for every von Neumann algebra $M \subset B(H)$, the C^*-algebra $M_n(M)$ from 4.19 is $*$-isomorphic to the von Neumann algebra $M_n(M) \subset B(H^{(n)})$. Alternatively, this means that the *algebraic* tensor product

$$M \otimes M_n \subset B(H \overline{\otimes} \ell^2(\{1, \ldots, n\}))$$

is a von Neumann algebra.

Let $\{M_\iota \subset B(H_\iota)\}_{\iota\in I}$ be a family of von Neumann algebras and denote by H the Hilbert space direct sum of the family $\{H_\iota\}_{\iota\in I}$. We identify in the usual way each H_ι with a closed vector subspace of H. Then the set

$$\{x \in B(H);\, xH_\iota \subset H_\iota,\, x^*H_\iota \subset H_\iota,\, x|H_\iota \in M_\iota \text{ for all } \iota \in I\}$$

is a von Neumann algebra $M \subset B(H)$ called the *direct product von Neumann algebra* of the family $\{M_\iota \subset B(H_\iota)\}_{\iota\in I}$.

It is easy to check that the commutant $M' \subset B(H)$ of M is the set

$$\{x' \in B(H);\, x'H_\iota \subset H_\iota,\, x'^*H_\iota \subset H_\iota,\, x'|H_\iota \in M'_\iota \text{ for all } \iota \in I\}$$

that is the direct product of the commutants $M'_\iota \subset B(H_\iota)$.

On the other hand, it is easy to see that M is $*$-isomorphic to the C^*-algebra direct product of the family $\{M_\iota\}_{\iota\in I}$ of C^*-algebras (1.4).

The usual associativity and distributivity properties are valid for tensor products and direct products of von Neumann algebras.

7.19. In this section we record some properties of the tensor product of von Neumann algebras. Let H, K be Hilbert spaces.

(1) For every $S \subset B(H)$, $T \subset B(K)$ we have

$$R(S) \overline{\otimes} R(T) = R((S \otimes 1_K) \cup (1_H \otimes T)).$$

Indeed, we may suppose that S, T are $*$-subalgebras and contain the identity operators. Then $R(S)$ (respectively $R(T)$) is the *so*-closure of S (respectively T) and using Kaplansky density theorem (7.9) it follows that $S \otimes T$ is *so*-dense in $R(S) \otimes R(T)$.

Using (1) it is easy to obtain the following connection between tensor products and reduction (respectively induction):

(2) Let $M \subset B(H)$, $N \subset B(K)$ be von Neumann algebras. If $e \in M$, $f \in M$, $e' \in M'$, $f' \in N'$ are projections, then $e \overline{\otimes} f \in M \overline{\otimes} N$ and $e' \overline{\otimes} f' \in (M \overline{\otimes} N)'$ are projections and

$$(M \overline{\otimes} N)_{e \overline{\otimes} f} = M_e \overline{\otimes} N_f, \quad (M \overline{\otimes} N)_{e' \overline{\otimes} f'} = M_{e'} \overline{\otimes} N_{f'}.$$

(3) Let $M \subset B(H)$, $N \subset B(H)$ be von Neumann algebras. Then

$$(M \overline{\otimes} B(K)) \cap (N \overline{\otimes} B(K)) = (M \cap N) \overline{\otimes} B(K).$$

Indeed, using Proposition 1/7.18 and the last remark in 7.11, we get

$$\begin{aligned}(M \overline{\otimes} B(K)) \cap (N \overline{\otimes} B(K)) &= (M' \otimes \mathbb{C}1_K)' \cap (N' \otimes \mathbb{C}1_K)' \\ &= ((M' \cup N') \otimes \mathbb{C}1_K)' = (R(M' \cup N') \otimes \mathbb{C}1_K)' \\ &= R(M' \cup N')' \overline{\otimes} B(K) = (M \cap N) \overline{\otimes} B(K).\end{aligned}$$

By 7.18, (3) can be restated as follows:

(4) Let $M \subset B(H)$, $N \subset B(H)$ be von Neumann algebras and υ be a cardinal number. Then

$$M_\upsilon(M) \cap M_\upsilon(N) = M_\upsilon(M \cap N).$$

The properties (1)–(3) have obvious reformulations for direct products instead of tensor products.

7.20. The following result is an important tool in studing tensor products.

Proposition. *Let $M \subset B(H)$ be a von Neumann algebra and $n \in \mathbb{N}$. For $x_{ij} \in M$, $x'_{ij} \in M'$, $(1 \leqslant i, j \leqslant n)$, the following statements are equivalent:*

(i) $\sum_{k=1}^{n} x_{ik}x'_{kj} = 0$; $1 \leqslant i, j \leqslant n$.

(ii) *There exist $z_{ij} \in Z_M, (1 \leqslant i, j \leqslant n)$ such that*

$$\sum_{k=1}^{n} x_{ik}z_{kj} = 0; \quad 1 \leqslant i, j \leqslant n,$$
$$\sum_{k=1}^{n} z_{ik}x'_{kj} = x'_{ij}; \quad 1 \leqslant i, j \leqslant n.$$

Proof. It is clear that (ii) implies (i).

Conversely, assume that (i) holds. Consider the elements

$$x = [x_{ij}] \in M_n(M), \quad x' = [x'_{ij}] \in M_n(M').$$

Then $xx' = 0$, hence $\mathbf{r}(x)\mathbf{l}(x') = 0$.

Let p be the orthogonal projection of $H^{(n)}$ onto the closed linear span of

$$\mathrm{Diag}_n(M)\mathbf{r}(x)H^{(n)}.$$

For $y \in \mathrm{Diag}_n(M)$ and $y' \in \mathrm{Diag}_n(M')$ we have

$$y\mathrm{Diag}_n(M)\mathbf{r}(x)H^{(n)} \subset \mathrm{Diag}_n(M)\mathbf{r}(x)H^{(n)},$$
$$y'\mathrm{Diag}_n(M)\mathbf{r}(x)H^{(n)} = \mathrm{Diag}_n(M)\mathbf{r}(x)y'H^{(n)} \subset \mathrm{Diag}_n(M)\mathbf{r}(x)H^{(n)},$$

because by 7.12 $\mathbf{r}(x) \in M_n(M)$ and by 7.18 $y' \in M_n(M)'$. Using again 7.18 and 7.19.(4), we obtain

$$p \in \mathrm{Diag}_n(M)' \cap \mathrm{Diag}_n(M')' = M_n(M') \cap M_n(M) = M_n(M) = M_n(Z_M).$$

On the other hand, $p \geqslant \mathbf{r}(x)$, hence $xp = x$. Also, since $\mathbf{r}(x)\mathbf{l}(x') = 0$ and $\mathbf{l}(x') \in M_n(M') = \mathrm{Diag}_n(M)'$, we have

$$\mathbf{l}(x')\mathrm{Diag}_n(M)\mathbf{r}(x)H^{(n)} = \mathrm{Diag}_n(M)\mathbf{l}(x')\mathbf{r}(x)H^{(n)} = 0$$

so $\mathbf{l}(x')p = 0$, hence $px' = p\mathbf{l}(x')x' = 0$.

Let $z = 1 - p \in M_n(Z_M)$ and let $z_{ij} \in Z_M$, $(1 \leqslant i, j \leqslant n)$, such that $z = [z_{ij}]$. Then the assertion (ii) follows from $xz = 0$ and $zx' = x'$. ∎

In particular, for $x \in M$, $x' \in M'$ it follows that

$$xx' = 0 \Leftrightarrow \mathbf{z}(x)\mathbf{z}(x') = 0. \tag{1}$$

This result can be also obtained by using Theorem 7.17.(i).

Corollary 1. *Let $M \subset B(H)$ be a von Neumann factor. There exist a unique $*$-isomorphism π of the $*$-algebra $M \otimes M'$ onto the $*$-subalgebra of $B(H)$ generated by $M \cup M'$ such that*

$$\pi(x \otimes x') = xx'; \quad x \in M,\, x' \in M'.$$

Proof. If $x_1, \dots, x_n \in M$, $x_1', \dots, x_n' \in M'$ and $\sum\limits_{k=1}^{n} x_k x_k' = 0$ then by the proposition there exist $\lambda_{ij} \in \mathbb{C}$, $(1 \leqslant i, j \leqslant n)$, with

$$\sum_{i=1}^{n} \lambda_{ij} x_i = 0; \quad 1 \leqslant j \leqslant n,$$

$$\sum_{j=1}^{n} \lambda_{ij} x_j' = x_i'; \quad 1 \leqslant i \leqslant n.$$

If moreover $x_1, \dots, x_n$ are lineary independent, then all λ_{ij} are zero, so all x_j' are zero. Consequently, the equation

$$\pi\Big(\sum_{k=1}^{n} x_k \otimes x_k'\Big) = \sum_{k=1}^{n} x_k x_k'$$

$(x_1, \dots, x_n \in M,\, x_1', \dots, x_n' \in M',\, n \in \mathbb{N})$, defines the required $*$-isomorphism. ∎

This result can be applied in the theory of C^*-tensor products:

Corollary 2. *Let $M \subset B(H)$ be a von Neumann factor and $A \subset M, B \subset M'$ be C^*-subalgebras. Assume that A (or B) is nuclear. Then there exists a unique $*$-isomorphism*

$$\rho : A \otimes_{C^*} B \to C^*(A \cup B)$$

such that

$$\rho(a \otimes b) = ab; \quad a \in A,\, b \in B.$$

Proof. By Corollary 1, there exists an injective $*$-homomorphism $\pi : A \otimes B \to B(H)$ such that

$$\pi(a \otimes b) = ab; \quad a \in A,\, b \in B.$$

Then $x \mapsto \|\pi(x)\|$ is a C^*-norm on $A \otimes B$. Since A (or B) is nuclear, it follows that

$$\|\pi(x)\| = \|x\|_{C^*}; \quad x \in A \otimes B.$$

Consequently, π can be extended to a $*$-isomorphism ρ of $A \otimes_{C^*} B$ onto $C^*(A \cup B)$. ∎

7.21. Let be A a *wo*-dense C^*-subalgebra of a von Neumann algebra M and $x \in M$. According to the Kaplansky density theorem (7.9) there exists $a \in A$ with $\|a\| \leqslant \|x\|$ arbitrarily close to x with respect to the s^*-topology. The main goal of this section is to prove a similar result claiming that, for M, A, x as above, a projection $e \in M$ and $\delta > 0$, there exists a projection $e \geqslant f \in M$ arbitrarily close to e with respect to the s^*-topology such that $xf = af$ for some $a \in A$ with $\|a\| \leqslant (1 + \delta)\|x\|$. This is a non-commutative extension of the classical Lusin theorem from the measure theory. For its proof we need, similarly as in the classical situation, a non-commutative extension of the Egorov theorem about "quasi-uniform convergence":

Lemma 1. *Let be $M \subset B(H)$ a von Neumann algebra, $S \subset M$, x an element of the s^*-closure of S in M, $e \in M$ a projection, φ a w-continuous positive form on $B(H)$ and $\varepsilon > 0$. Then there exist a projection $e \geqslant f \in M$ and a sequence $\{x_n\}_{n\geqslant 1}$ in S such that*

$$\varphi(e-f) \leqslant \varepsilon,$$

$$\lim_{n\to\infty} \|xf - x_n f\| = \lim_{n\to\infty} \|fx - fx_n\| = 0.$$

Proof. Let $\{x_\iota\}_{\iota\in I}$ be a net in S such that $x_\iota \xmapsto{s^*} x$.

Denoting

$$a_\iota^{(1)} = e(x-x_\iota)^*(x-x_\iota)e + e(x-x_\iota)(x-x_\iota)^*e,$$

we have $s(a_\iota^{(1)}) \leqslant e$ and $a_\iota^{(1)} \xrightarrow{w} 0$. By Proposition 1/7.16, for every $\iota \in I$ there exists a projection $e \geqslant e_\iota^{(1)} \in \mathcal{R}(\{a_\iota^{(1)}\}) \subset M$ such that

$$a_\iota^{(1)} e_\iota^{(1)} \leqslant \frac{1}{2} e_\iota^1 \quad \text{and} \quad a_\iota^{(1)}(e - e_\iota^{(1)}) \geqslant \frac{1}{2}(e - e_\iota^{(1)}).$$

Since $\varphi(e - e_\iota^{(1)}) \leqslant 2\varphi(a_\iota^{(1)}) \to 0$, there exists $\iota_1 \in I$ with $\varphi(e - e_{\iota_1}^{(1)}) \leqslant \frac{\varepsilon}{2}$.

Now, denoting

$$a_\iota^{(2)} = e_{\iota_1}^{(1)} a_\iota^{(1)} e_{\iota_1}^{(1)},$$

we have $s(a_\iota^{(2)}) \leqslant e_{\iota_1}^{(1)}$ and $a_\iota^{(2)} \xrightarrow{w} 0$. Again by Proposition 1/7.16, for every $\iota \in I$ there exists a projection $e_{\iota_1}^{(1)} \geqslant e_\iota^{(2)} \in \mathcal{R}(\{a_\iota^{(2)}\}) \subset M$ such that

$$a_\iota^{(2)} e_\iota^{(2)} \leqslant \frac{1}{2^2} e_\iota^{(2)} \quad \text{and} \quad a_\iota^{(2)}(e_{\iota_1}^{(1)} - e_\iota^{(2)}) \geqslant \frac{1}{2^2}(e_{\iota_1}^{(1)} - e_\iota^2).$$

Since $\varphi(e_{\iota_1}^{(1)} - e_\iota^{(2)}) \leqslant 2^2\varphi(a_\iota^{(2)}) \to 0$, there exists $\iota_2 \in I$ with $\varphi(e_{\iota_1}^{(1)} - e_{\iota_2}^{(2)}) \leqslant \frac{\varepsilon}{2^2}$.

Using induction, we get a sequence $\{\iota_n\}_{n\geqslant 1}$ in I and a sequence of projections in M

$$e \geqslant e_{\iota_1}^{(1)} \geqslant e_{\iota_2}^{(2)} \geqslant \cdots$$

such that

$$e_{\iota_n}^{(n)} a_{\iota_n}^{(1)} e_{\iota_n}^{(n)} \leqslant \frac{1}{2^n} e_{\iota_n}^{(n)} \quad \text{for all } n \geqslant 1,$$

$$\varphi(e - e_{\iota_1}^{(1)}) \leqslant \frac{\varepsilon}{2} \quad \text{and} \quad \varphi(e_{\iota_{n-1}}^{(n-1)} - e_{\iota_n}^{(n)}) \leqslant \frac{\varepsilon}{2^n}, \quad n > 1.$$

Putting

$$f = \bigwedge_{n\geqslant 1} e_{\iota_n}^{(n)} \leqslant e \quad \text{and} \quad x_n = x_{\iota_n}, \quad n \geqslant 1,$$

we have then

$$\varphi(e - f) \leqslant \varepsilon$$

and

$$\|(x - x_n)f\|^2 \leqslant \frac{1}{2^n}, \quad \|f(x - x_n)\|^2 \leqslant \frac{1}{2^n} \quad \text{for all } n \geqslant 1. \quad \blacksquare$$

For every C^*-algebra A we denote by $\mathcal{U}_o(A)$ the set of all finite products of unitaries of the form $\exp(\mathrm{i}\,x) \in \widetilde{A}$ with $x \in A$ selfadjoint. Clearly, $\mathcal{U}_o(A)$ is a normal subgroup of the group $\mathcal{U}(\widetilde{A})$ of all unitaries in $\widetilde{A}$.

Lemma 2. *For every C^*-algebra A, $\mathcal{U}_o(A)$ is norm-closed in A.*

Proof. First we notice that $\mathcal{U}_o(\widetilde{A}) = \{\lambda \in \mathbb{C};\, |\lambda| = 1\} \cdot \mathcal{U}_o(A)$ is an open, hence also closed subgroup of $\mathcal{U}(\widetilde{A})$. Indeed, this follows from the fact that every $u \in \mathcal{U}_o(\widetilde{A})$ with $\|1_{\widetilde{A}} - u\| < 2$ is of the form $u = \exp(\mathrm{i}\, y)$ for some selfadjoint $y \in \widetilde{A}$, what is easily seen by using the Gelfand representation of $C^*(\{u\}) \subset \widetilde{A}$.

If A is unital, the above remark completes the proof. Let us therefore assume that A is not unital and let ω denote the linear functional on $\widetilde{A}$ which vanishes on A and carries $1_{\widetilde{A}}$ in 1. Then ω is a $*$-homomorphism.

Now let

$$u \in \overline{\mathcal{U}_o(A)} \subset \overline{\mathcal{U}_o(\widetilde{A})} = \mathcal{U}_o(\widetilde{A}) = \{\lambda \in \mathbb{C};\, |\lambda| = 1\} \cdot \mathcal{U}_o(A)$$

be arbitrary. Then $u = \lambda\, v$ for some $\lambda \in \mathbb{C}$ and $v \in \mathcal{U}_o(A)$. But ω maps $\mathcal{U}_o(A)$, hence also $\overline{\mathcal{U}_o(A)}$ in $\{1\}$ and it follows that $1 = \omega(u) = \lambda\, \omega(v) = \lambda$. ∎

Next we prove that if a unitary element of a unital C^*-algebra M is sufficiently close to 1_M on some projection $f \in M$ then there exists another unitary in M, which is equal with the given one on f and arbitrarily close to 1_M:

Lemma 3. *Let M be a unital C^*-algebra, $u, v \in M$ unitaries, and $f \in M$ a projection such that*

$$\|(u - v)f\| < \frac{1}{2}.$$

Then there exists a partial isometry $w \in M$ such that we have

$$w^*w = 1_M - f, \quad ww^* = 1_M - v^*ufu^*v$$

and, for the unitary $v^*uf + w$,

$$\|1_M - (v^*uf + w)\| \leqslant 3\|(u - v)f\|.$$

Proof. Denoting $x = (1_M - v^*ufu^*v)(1_M - f)$, we have

$$\begin{aligned}\|1_M - f - x\| &= \|v^*ufu^*v(1_M - f)\| = \|(1_M - f)v^*uf\| \\ &= \|(1_M - f)v^*(u - v)f\| \leqslant \|(u - v)f\|.\end{aligned}$$

Similarly,

$$\begin{aligned}\|1_M - v^*ufu^*v - x\| &= \|(1_M - v^*ufu^*v)f\| = \|(u^*v - fu^*v)f\| \\ &= \|(1_M - f)u^*(v - u)f\| \leqslant \|(u - v)f\|.\end{aligned}$$

Since $x^*x \leqslant 1_M - f$, it follows that

$$\begin{aligned}\|1_M - f - |x|\| &\leqslant \|1_M - f - x^*x\| \leqslant \|1_M - f - x\| + \|(1_M - v^*ufu^*v)x - x^*x\| \\ &\leqslant \|(u - v)f\| + \|1_M - v^*ufu^*v - x\| \leqslant 2\|(u - v)f\| < 1,\end{aligned}$$

so, $|x|$ is invertible in $(1_M - f)M(1_M - f)$, having there an inverse b. Let us denote $w = xb$. Then

$$w|x| = xb|x| = x \quad \text{and} \quad w^*w = bx^*xb = b|x|^2b = 1_M - f.$$

In particular, w is a partial isometry.

Now

$$\begin{aligned}\|1_M - v^*ufu^*v - xx^*\| &\leqslant \|1_M - v^*ufu^*v - x\| + \|x(1_M - f) - xx^*\| \\ &\leqslant \|(u-v)f\| + \|1_M - f - x\| \leqslant 2\|(u-v)f\| < 1\end{aligned}$$

implies that xx^* is invertible in $(1_M - v^*ufu^*v)M(1_M - v^*ufu^*v)$ and, taking into account that

$$xx^* = w\|x\|^2w^* \leqslant \|x\|^2ww^* = \|x\|^2xb^2x^* \leqslant \|x\|^2\|b\|^2xx^*,$$

it follows that the projection ww^* is equal to $1_M - v^*ufu^*v$.

Finally,

$$\|1_M - (v^*uf + w)\| \leqslant \|1_M - f - x\| + \|f - v^*uf\| + \|w(|x| - (1_M - f))\| \leqslant 3\|(u-v)f\|. \blacksquare$$

Now we are ready to prove the announced Lusin type theorem:

Theorem (Non-commutative Lusin theorem). *Let $M \subset B(H)$ be a von Neumann algebra, $A \subset M$ a wo-dense C^*-subalgebra, $e \in M$ a projection, φ a w-continuous positive form on $B(H)$, and $\varepsilon, \delta > 0$.*

(i) *If $x \in M$ then there exist*

$$\text{a projection } e \geqslant f \in M \text{ with } \varphi(e - f) \leqslant \varepsilon,$$
$$a \in A \text{ with } \|a\| \leqslant (1+\delta)\|x\|$$

such that

$$af = xf, \quad fa = fx.$$

(ii) *If x in* (i) *is selfadjoint, then a can be chosen selfadjoint.*

(iii) *If $u \in M$ is unitary then there exist*

$$\text{a projection } e \geqslant f \in M \text{ with } \varphi(e - f) \leqslant \varepsilon,$$
$$v \in \mathcal{U}_o(A) \text{ with } \|1_M - v\| \leqslant (1+\delta)\|1_M - u\|$$

such that

$$vf = uf.$$

Proof. Clearly, (i) $\Rightarrow$ (ii), so we have to prove only (i) and (iii). Let us first prove (i).

According to the Kaplansky density theorem, x belongs to the s^*-closure of the closed ball of radius $\|x\|$ in A, so by Lemma 1 there exist

$$\text{a projection } e \geqslant f_1 \in M \text{ with } \varphi(e - f_1) \leqslant \frac{\varepsilon}{2},$$
$$a_1 \in A \text{ with } \|a_1\| \leqslant \|x\|$$

such that

$$\|(x-a_1)f_1\| \leqslant \frac{1}{2}\,\frac{\delta\|x\|}{\sqrt{2}}, \quad \|f_1(x-a_1)\| \leqslant \frac{1}{2}\,\frac{\delta\|x\|}{\sqrt{2}}.$$

Putting $x_1 = (x-a_1)f_1 + f_1(x-a_1) - f_1(x-a_1)f_1$, we have

$$\begin{aligned}\|x_1\| &= \sqrt{\|x_1^* x_1\|}\\ &= \sqrt{\|\left(f_1(x-a_1)^*(1-f_1)+(x-a_1)^*f_1\right)\left((1-f_1)(x-a_1)f_1+f_1(x-a_1)\right)\|}\\ &\leqslant \sqrt{\|(x-a_1)f_1\|^2+\|f_1(x-a_1)\|^2} \leqslant \frac{1}{2}\,\delta\,\|x\|.\end{aligned}$$

A reasoning similar to the above one, with x, e replaced respectively by x_1, f_1, yields the existence of

$$\text{a projection } f_1 \geqslant f_2 \in M \text{ with } \varphi(f_1 - f_2) \leqslant \frac{\varepsilon}{2^2},$$
$$a_2 \in A \text{ with } \|a_2\| \leqslant \|x_1\|$$

such that

$$\|(x_1-a_2)f_2\| \leqslant \frac{1}{2^2}\,\frac{\delta\|x\|}{\sqrt{2}}, \quad \|f_2(x_1-a_2)\| \leqslant \frac{1}{2^2}\,\frac{\delta\|x\|}{\sqrt{2}}.$$

Putting $x_2 = (x_1-a_2)f_2+f_2(x_1-a_2)-f_2(x_1-a_2)f_2$, we have then $\|x_2\| \leqslant \frac{1}{2^2}\,\delta\|x\|$.

Using induction, we get a sequence of projections in M

$$e = f_o \geqslant f_1 \geqslant f_2 \geqslant \cdots$$

and a sequence $\{a_k\}_{k\geqslant 1}$ in A such that, putting

$$x_o = x \quad \text{and} \quad x_k = (x_{k-1}-a_k)f_k + f_k(x_{k-1}-a_k) - f_k(x_{k-1}-a_k)f_k, \quad k \geqslant 1,$$

we have

$$\varphi(f_{k-1}-f_k) \leqslant \frac{\varepsilon}{2^k}, \quad k \geqslant 1,$$
$$\|a_1\| \leqslant \|x\| \quad \text{and} \quad \|a_k\| \leqslant \frac{1}{2^{k-1}}\,\delta\|x\|, \quad k \geqslant 2,$$
$$\|(x_{k-1}-a_k)f_k\| \leqslant \frac{1}{2^k}\,\frac{\delta\|x\|}{\sqrt{2}}, \quad \|f_k(x_{k-1}-a_k)\| \leqslant \frac{1}{2^k}\,\frac{\delta\|x\|}{\sqrt{2}}, \quad k \geqslant 1.$$

Let us denote

$$f = \bigwedge_{k\geqslant 1} f_k \leqslant e \quad \text{and} \quad a = \sum_{k=1}^{\infty} a_k \in A.$$

Then

$$\varphi(e-f)=\sum_{k=1}^{\infty}\varphi(f_{k-1}-f_k)\leqslant\varepsilon \quad\text{and}\quad \|a\|\leqslant\sum_{k=1}^{\infty}\|a_k\|\leqslant(1+\delta)\|x\|.$$

Furthermore, for $1\leqslant k\leqslant n$ we have $x_k f_n=(x_{k-1}-a_k)f_n$, hence $(x_{k-1}-x_k)f_n=a_k f_n$. It follows successively, for every $n\geqslant 1$,

$$(x-x_n)f_n=\sum_{k=1}^{n}a_k f_n,$$

$$\Big(x-\sum_{k=1}^{n}a_k\Big)f_n=x_n f_n=(x_{n-1}-a_n)f_n,$$

$$\Big\|\Big(x-\sum_{k=1}^{n}a_k\Big)f\Big\|\leqslant\Big\|\Big(x-\sum_{k=1}^{n}a_k\Big)f_n\Big\|=\|(x_{n-1}-a_n)f_n\|\leqslant\frac{1}{2^n}\,\frac{\delta\|x\|}{\sqrt{2}}.$$

Passing to limit for $n\to\infty$, we conclude that $\|(x-a)f\|=0$. Similarly we get also $\|f(x-a)\|=0$.

Now we go to prove (iii). The proof will be a multiplicative counterpart of the proof of (i).

According to Proposition 5/7.16, u belongs to the s^*-closure of the set of all $v\in\mathcal{U}_o(A)$ with $\|1_M-v\|\leqslant\|1_M-u\|$, so by Lemma 1 there exist

$$\text{a projection } e\geqslant f_1\in M \text{ with } \varphi(e-f_1)\leqslant\frac{\varepsilon}{2},$$
$$v_1\in\mathcal{U}_o(A) \text{ with } \|1_M-v_1\|\leqslant\|1_M-u\|$$

such that

$$\|(u-v_1)f_1\|\leqslant\frac{\delta}{4\delta+3\cdot 2}\,\|1_M-u\|<\frac{1}{2}.$$

Now Lemma 3 implies the existence of a unitary $u_1\in M$ such that

$$u_1 f_1=v_1^* u f_1 \quad\text{and}\quad \|1_M-u_1\|\leqslant 3\|(u-v_1)f_1\|\leqslant\frac{\delta}{2}\|1_M-u\|.$$

Replacing u,e respectively by u_1,f_1, a reasoning similar to the above one yields first the existence of

$$\text{a projection } f_1\geqslant f_2\in M \text{ with } \varphi(f_1-f_2)\leqslant\frac{\varepsilon}{2^2},$$
$$v_2\in\mathcal{U}_o(A) \text{ with } \|1_M-v_2\|\leqslant\|1_M-u_1\|\leqslant\frac{\delta}{2}\|1_M-u\|$$

such that

$$\|(v_1^*u-v_2)f_2\|=\|(u_1-v_2)f_2\|\leqslant\frac{\delta}{4\delta+3\cdot 2^2}\,\|1_M-u\|<\frac{1}{2},$$

and then the existence of a unitary $u_2 \in M$ such that

$$u_2 f_2 = v_2^* u_1 f_2 = v_2^* v_1^* u f_2 \quad \text{and} \quad \|1_M - u_2\| \leqslant 3\|(u_1 - v_2) f_1\| \leqslant \frac{\delta}{2^2}\|1_M - u\|.$$

Using induction, we get a sequence of projections in M

$$e = f_o \geqslant f_1 \geqslant f_2 \geqslant \cdots$$

and a sequence $\{v_k\}_{k\geqslant 1}$ in $\mathcal{U}_o(A)$ such that

$$\varphi(f_{k-1} - f_k) \leqslant \frac{\varepsilon}{2^k}, \quad k \geqslant 1,$$
$$\|1_M - v_1\| \leqslant \|1_M - u\| \quad \text{and} \quad \|1_M - v_k\| \leqslant \frac{\delta}{2^{k-1}}\,\|1_M - u\|, \quad k \geqslant 2,$$

$$\|(v_{k-1}^* \cdots v_1^* u - v_k) f_k\| \leqslant \frac{\delta}{2\delta + 3 \cdot 2^k}\|1_M - u\|, \quad k \geqslant 1.$$

Denoting

$$f = \bigwedge_{k\geqslant 1} f_k \leqslant e,$$

we have

$$\varphi(e - f) = \sum_{k=1}^{\infty} \varphi(f_{k-1} - f_k) \leqslant \varepsilon.$$

On the other hand, taking into account Lemma 2,

$$\sum_{k=1}^{\infty} \|v_1 \cdots v_{k-1} - v_1 \cdots v_k\| = \sum_{k=1}^{\infty} \|1_M - v_k\| \leqslant \Big(1 + \sum_{k=2}^{\infty} \frac{\delta}{2^{k-1}}\Big) \cdot \|1_M - u\| = (1+\delta)\|1_M - u\|$$

implies that the sequence $\{v_1 \cdot v_k\}_{k\geqslant 1}$ is norm convergent to some $v \in \mathcal{U}_o(A)$ with $\|1_M - v\| \leqslant (1 + \delta)\|1_M - u\|$. Since, for every $k \geqslant 1$,

$$\|(u - v_1 \cdots v_k) f\| \leqslant \|(u - v_1 \cdots v_k) f_k\| = \|(v_{k-1}^* \cdots v_1^* u - v_k) f_k\| \leqslant \frac{\delta}{2\delta + 3 \cdot 2^k}\|1_M - u\|,$$

v satisfies

$$\|(u - v) f\| = 0 \Rightarrow v f = u f. \quad \blacksquare$$

The non-commutative Lusin theorem yields immediately the following form of the Kadison transitivity theorem:

COROLLARY. *Let be $M \subset B(H)$ a von Neumann algebra, $A \subset M$ a wo-dense C^*-subalgebra, $e \in M$ a projection with finite-dimensional range and $\delta > 0$.*

(i) *For every $x \in M$ there exists*

$$a \in A \text{ with } \|a\| \leqslant (1+\delta)\|x\|$$

such that

$$ae = xe, \quad ea = ex.$$

(ii) *If x in* (i) *is selfadjoint, then a can be chosen selfadjoint.*

(iii) *For every unitary $u \in M$ there exists*

$$v \in \mathcal{U}_o(A) \text{ with } \|1_M - v\| \leqslant (1+\delta)\|1_M - u\|$$

such that

$$ve = ue, \quad ev = eu.$$

Proof. For every finite-dimensional linear subspace $K \subset H$ we consider the wo-continuous positive linear form

$$\omega_K = \sum_{k=1}^{\dim(K)} \omega_{\xi_k} : B(H) \to \mathbb{C},$$

where $\xi_1, \ldots, \xi_{\dim(K)}$ stands for an orthonormal basis of K. By the way, ω_K does not depend on the choice of the orthonormal basis of K. Indeed, if $\zeta_1, \ldots, \zeta_{\dim(K)}$ is another orthonormal basis then we have, for every $1 \leqslant j \leqslant \dim(K)$,

$$\zeta_j = \sum_{k=1}^{\dim(K)} (\zeta_j|\xi_k)\xi_k \quad \Rightarrow \quad \omega_{\zeta_j} = \sum_{k_1,k_2=1}^{\dim(K)} (\xi_{k_2}|\zeta_j)(\zeta_j|\xi_{k_1})\omega_{\xi_1,\xi_2},$$

so

$$\sum_{j=1}^{\dim(K)} \omega_{\zeta_j} = \sum_{k_1,k_2=1}^{\dim(K)} \underbrace{\Big(\sum_{j=1}^{\dim(K)} (\xi_{k_2}|\zeta_j)(\zeta_j|\xi_{k_1}) \Big)}_{=(\xi_{k_2}|\xi_{k_1})} \omega_{\xi_1,\xi_2} = \sum_{k=1}^{\dim(K)} \omega_{\xi_k}.$$

Furthermore, if L is any linear subspace of K and p_L denotes the orthogonal projection of H onto L, then $\omega_K(p_L) = \dim(L)$. Indeed, choosing an orthonormal basis $\eta_1, \ldots, \eta_{\dim(L)}$ of L, we have

$$p_L(\xi) = \sum_{j=1}^{\dim(L)} (\xi|\eta_j)\eta_j \quad \text{for every } \xi \in H,$$

so

$$\begin{aligned}\omega_K(p_L) &= \sum_{k=1}^{\dim(K)} \sum_{j=1}^{\dim(L)} (\xi_k|\eta_j)(\eta_j|\xi_k) = \sum_{j=1}^{\dim(L)} \Big(\sum_{k=1}^{\dim(K)} (\eta_j|\xi_k)\xi_k \Big| \eta_j \Big) \\ &= \sum_{j=1}^{\dim(L)} (\eta_j|\eta_j) = \dim(L).\end{aligned}$$

In particular,

$$L \subset K \text{ linear subspace}\ , \quad \omega_K(p_K - p_L) < 1 \quad \Rightarrow \quad L = K.$$

Since (i) $\Rightarrow$ (ii), we have again to prove only (i) and (iii). Taking into account the above remarks, (i) follows immediately by applying statement (i) of the non-commutative Lusin theorem with $\varphi = \omega_{e(H)}$ and $0 < \varepsilon < 1$.

For (iii) let e_u denote the orthogonal projection of H onto the finite-dimensional linear subspace generated by $e(H) \cup u^* e(H)$. By the von Neumann density theorem $e_u \in M$, so we can apply statement (iii) of the non-commutative Lusin theorem with e replaced by e_u, $\varphi = \omega_{e_u(H)}$ and $0 < \varepsilon < 1$, getting some

$$v \in \mathcal{U}_o(A) \text{ with } \|1_M - v\| \leqslant (1+\delta)\|1_M - u\| \quad \text{and} \quad ve_u = ue_u.$$

Since $e \leqslant e_u$ and $u^* eu \leqslant e_u$, it follow

$$ve = ue \quad \text{and} \quad vu^* eu = uu^* eu = eu \Leftrightarrow vu^* e = e \Leftrightarrow euv^* = e.$$

Consequently, $(ev - eu)(ev - eu)^* = e - e(vu^* e) - (euv^*)e + e = e - e - e + e = 0$, hence we have also $ev = eu$. ∎

Usually the Kadison transitivity theorem is formulated in terms of topologically irreducible $*$-representations of C^*-algebras. This will be done in the next section.

7.22. We say that two $*$-representations $\pi_j : A \to B(H_j)$, $j = 1, 2$ of a $*$-algebra A are *topologically disjoint* if there exists no non-zero bounded linear map $T : H_1 \to H_2$ such that

$$T\pi_1(a) = \pi_2(a)T, \quad a \in A.$$

If π_1 and π_2 are unitarily equivalent and non-zero then they are plainly not topologically disjoint. In the case of topologically irreducible π_1 and π_2 also the converse implication holds:

LEMMA 1. *Let $\pi_j : A \to B(H_j)$, $j = 1, 2$ be topologically irreducible $*$-representations of a $*$-algebra A. If π_1 and π_2 are not unitarily equivalent then they are topologically disjoint.*

Proof. Let us assume that π_1 and π_2 are not topologically disjoint, that is there exists a non-zero bounded linear map $T : H_1 \to H_2$ such that

$$T\pi_1(a) = \pi_2(a)T, \quad a \in A.$$

Then we have for every $a \in A$ also

$$T^*\pi_2(a) = (\pi_2(a^*)T)^* = (T\pi_1(a^*))^* = \pi_1(a)T^*,$$

hence

$$T^*T\pi_1(a) = T^*\pi_2(a)T = \pi_1(a)T^*T \quad \text{and} \quad TT^*\pi_2(a) = T\pi_1(a)T^* = \pi_2(a)TT^*.$$

In other words, $T^*T \in \pi_1(A)' = \mathbb{C}1_{H_1}$ and $TT^* \in \pi_2(A)' = \mathbb{C}1_{H_2}$, so $T^*T = \lambda_1 1_{H_1}$ and $TT^* = \lambda_2 1_{H_2}$ for some $\lambda_1, \lambda_2 \geqslant 0$. Since $\|T^*T\| = \|TT^*\| > 0$, we have $\lambda_1 = \lambda_2 > 0$.

Put $U = \frac{1}{\sqrt{\lambda_1}}T : H_1 \to H_2$. Then $U^*U = 1_{H_1}$ and $UU^* = 1_{H_2}$, so U is unitary. Since $U\pi_1(a) = \sqrt{\lambda_1}T\pi_1(a) = \sqrt{\lambda_1}\pi_2(a)T = \pi_2(a)U$ for all $a \in A$, it follows that π_1 and π_2 are unitarily equivalent. ∎

The *wo*-closure of the range of a direct sum of mutually topologically disjoint, non-degenerate $*$-representations is the direct product von Neumann algebra of the *wo*-closures of the ranges of these representations:

LEMMA 2. *Let $\{\pi_\iota : A \to B(H_\iota)\}_{\iota\in I}$ be a family of mutually topologically disjoint, non-degenerate $*$-representations of a $*$-algebra A such that the numerical set $\{\|\pi_\iota(a)\|;\ \iota \in I\}$ is bounded for every $a \in A$. Let us denote*

$$H = \bigoplus_{\iota\in I} H_\iota, \quad \pi = \bigoplus_{\iota\in I} \pi_\iota.$$

Then $\pi(A)$ is wo-dense in the direct product von Neumann algebra of the family $\{\pi_\iota(A)'' \subset B(H_\iota)\}_{\iota\in I}$.

Proof. Let M denote the direct product von Neumann algebra of the family $\{\pi_\iota(A)'' \subset B(H_\iota)\}_{\iota\in I}$. Since π is clearly non-degenerate, by the von Neumann density theorem it is enough to prove that $\pi(A)' \subset M'$.

If $T_\iota : H_\iota \to H$ denotes the canonical imbedding then $T_\iota^* T_\iota = 1_{H_\iota}$ and $T_\iota T_\iota^*$ is the orthogonal projection of H onto $T_\iota H_\iota$. We have seen at the end of 7.18 that

$$\begin{aligned} M' &= \left\{ x' \in B(H);\ \begin{array}{l} x' T_\iota H_\iota \subset T_\iota H_\iota,\ (x')^* T_\iota H_\iota \subset T_\iota H_\iota \text{ and} \\ {T_\iota}^* x' T_\iota \in \pi_\iota(A)' \text{ for all } \iota \in I \end{array} \right\} \\ &= \left\{ x' \in B(H);\ \begin{array}{l} {T_\kappa}^* x' T_\iota = 0 \text{ for all } \iota \neq \kappa \text{ in } I \text{ and} \\ {T_\iota}^* x' T_\iota \in \pi_\iota(A)' \text{ for all } \iota \in I \end{array} \right\}. \end{aligned}$$

In particular, $T_\iota {T_\iota}^* \in M' \subset \pi(A)'$ for all $\iota \in I$.

Let $x' \in \pi(A)'$ be arbitrary. For every $\iota, \kappa \in I$ we have $T_\kappa {T_\kappa}^* x' T_\iota {T_\iota}^* \in \pi(A)'$ and it follows for all $a \in A$

$$T_\kappa {T_\kappa}^* x' T_\iota {T_\iota}^* \pi(a) = \pi(a) T_\kappa {T_\kappa}^* x' T_\iota {T_\iota}^*,$$

$$({T_\kappa}^* x' T_\iota)\underbrace{({T_\iota}^* \pi(a) T_\iota)}_{=\pi_\iota(a)} = \underbrace{({T_\kappa}^* \pi(a) T_\kappa)}_{=\pi_\kappa(a)}({T_\kappa}^* x' T_\iota).$$

Now for $\iota \neq \kappa$ the topological disjointness of π_ι and π_κ yields ${T_\kappa}^* x' T_\iota = 0$, while for $\iota = \kappa$ we have ${T_\iota}^* x' T_\iota \in \pi_\iota(A)'$. Consequently, $x' \in M'$. ∎

The following remarkable theorem enlightens the algebraic character of the topological irreducibility of $*$-representations of C^*-algebras:

THEOREM (Kadison transitivity theorem). *Let be A a C^*-algebra, $\pi_j : A \to B(H_j)$, $1 \leqslant j \leqslant n$ finitely many, mutually not unitarily equivalent, topologically irreducible $*$-representations of A, $K_j \subset H_j$, $1 \leqslant j \leqslant n$ finite-dimensional linear subspaces and $\delta > 0$.*

(i) *For every $x_j \in B(H_j)$, $1 \leqslant j \leqslant n$ there exists*

$$a \in A \text{ with } \|a\| \leqslant (1+\delta) \max_{1\leqslant j\leqslant n} \|x_j\|$$

such that

$$\pi_j(a)|K_j = x_j|K_j, \quad \pi_j(a^*)|K_j = {x_j}^*|K_j \quad \textit{for all } 1 \leqslant j \leqslant n.$$

(ii) *If all operators x_j in* (i) *are selfadjoint, then a can be chosen selfadjoint.*

(iii) *For every unitaries $u_j \in B(H_j)$, $1 \leqslant j \leqslant n$ there exists*

$$v \in \mathcal{U}_o(A) \quad \textit{with} \quad \|1_{\widetilde{A}} - v\| \leqslant (1+\delta) \max_{1 \leqslant j \leqslant n} \|1_{H_j} - u_j\|$$

such that

$$\widetilde{\pi}_j(v)|K_j = u_j|K_j, \quad \widetilde{\pi}_j(v^*)|K_j = {u_j}^*|K_j \quad \textit{for all } 1 \leqslant j \leqslant n.$$

Proof. Using Theorem 4.11 it is easily seen that there exists a family $\{\pi_\iota : A \to B(H_\iota)\}_{\iota \in I}$ of topologically irreducible $*$-representations such that the $*$-representations $\{\pi_j;\, 1 \leqslant j \leqslant n\} \cup \{\pi_\iota;\, \iota \in I\}$ are mutually not unitarily equivalent and the direct sum $*$-representation

$$\pi = \bigoplus_{1 \leqslant j \leqslant n} \pi_j \oplus \bigoplus_{\iota \in I} \pi_\iota : A \to B(H)$$

is injective. By Lemma 1 the $*$-representations $\{\pi_j;\, 1 \leqslant j \leqslant n\} \cup \{\pi_\iota;\, \iota \in I\}$ are mutually topologically disjoint, so Lemma 2 entails that $\pi(A)$ is *wo*-dense in the direct product of the von Neumann algebras $\pi_j(A)'' = (\mathbb{C}1_{H_j})' = B(H_j)$, $1 \leqslant j \leqslant n$ and $\pi_\iota(A)'' = B(H_\iota)$, $\iota \in I$. In particular,

$$\text{the orthogonal projection } e \text{ of } H \text{ onto } \bigoplus_{1 \leqslant j \leqslant n} K_j \oplus \bigoplus_{\iota \in I} \{0\}$$

is a projection in the *wo*-closure of $\pi(A)$ having finite-dimensional range,

$$x = \bigoplus_{1 \leqslant j \leqslant n} x_j \oplus \bigoplus_{\iota \in I} 0_{H_\iota}$$

is an operator in the *wo*-closure of $\pi(A)$ with $\|x\| = \max_{1 \leqslant j \leqslant n} \|x_j\|$, self-adjoint if every x_j is self-adjoint, and

$$u = \bigoplus_{1 \leqslant j \leqslant n} u_j \oplus \bigoplus_{\iota \in I} 1_{H_\iota}$$

is a unitary in the *wo*-closure of $\pi(A)$ with $\|1_H - u\| = \max_{1 \leqslant j \leqslant n} \|1_{H_j} - u_j\|$.

Applying now Corollary 7.21 to the C^*-algebra $\pi(A)$ and the above e, x, u, we deduce immediately all statements of the theorem. ∎

Using the above theorem, we can give several descriptions for the irreducibility of $*$-representations of C^*-algebras, proving for these representations the equivalence of topological irreducibility with the algebraic irreducibility:

COROLLARY 1. *For a $*$-representation $\pi : A \to B(H)$ of a C^*-algebra A the following conditions are equivalent:*

(i) *π is topologically irreducible;*

(ii) *the only closed linear subspaces of H stable under $\pi(A)$ are $\{0\}$ and H;*

(iii) *the only (not necessarily bounded) linear operators $H \to H$, commuting with all $\pi(a)$, $a \in A$, are the scalar multiples of 1_H;*

(iv) *the only (not necessarily closed) linear subspaces of H stable under $\pi(A)$ are $\{0\}$ and H.*

Proof. (i) $\Leftrightarrow$ (ii) follows by noticing that $\pi(A)' \subset B(H)$ is a von Neumann algebra and so, according to Proposition 3/7.16, equal to the norm-closed linear span of all projections contained in it.

To prove (i) $\Rightarrow$ (iii), let us assume that π is topologically irreducible and $T : H \to H$ is a linear operator commuting with every $\pi(a)$.

Then $T\xi \in \mathbb{C}\,\xi$ for every $\xi \in H$. Indeed, $T\xi \notin \mathbb{C}\,\xi$ would imply the linear independence of ξ and $T\xi$ and statement (i) of the Kadison transitivity theorem would entail the existence of some $a \in A$ with

$$\pi(a)\xi = 0 \text{ and } \pi(a)T\xi = \xi \Rightarrow \xi = \pi(a)T\xi = T\pi(a)\xi = 0,$$

in contradiction with $T\xi \notin \mathbb{C}\,\xi$.

Moreover, there exists some $\lambda \in \mathbb{C}$ such that $T\xi = \lambda\xi$ for all $\xi \in H$. Indeed, assuming that there are $\lambda_1 \neq \lambda_2$ in $\mathbb{C}$ and $0 \neq \xi$, $\eta \in H$ with $T\xi = \lambda_1\xi$ and $T\eta = \lambda_2\eta$, ξ and η would be linearly independent and statement (i) of the Kadison transitivity theorem would imply the existence of some $a \in A$ with

$$\pi(a)\xi = \eta \text{ and } \pi(a)\eta = \xi \Rightarrow \lambda_1\eta = \lambda_1\pi(a)\xi = \pi(a)T\xi = T\pi(a)\xi = T\eta = \lambda_2\eta,$$

contradicting $\eta \neq 0$.

To prove (i) $\Rightarrow$ (iv), let us assume that π is topologically irreducible and $K \subset H$ is a non-zero linear subspace stable under $\pi(A)$. Choose some $0 \neq \xi \in K$. According to statement (i) of the Kadison transitivity theorem, for every $\zeta \in H$ there exists $a \in A$ with $\pi(a)\xi = \zeta$ and it follows that $H \subset \pi(A)K \subset K$, hence $K = H$. Finally, the implications (iii) $\Rightarrow$ (i) and (iv) $\Rightarrow$ (ii) are obvious. ∎

For irreducible $*$-representations of C^*-algebras topological disjointness turns out to be equivalent with the algebraic disjointness:

COROLLARY 2. *For two (topologically) irreducible $*$-representations $\pi : A \to B(H_j)$, $j = 1, 2$ of a C^*-algebra A the following conditions are equivalent:*

(i) *π_1 and π_2 are not unitarily equivalent;*

(ii) *π_1 and π_2 are topologically disjoint;*

(iii) *there exists no non-zero (not necessarily bounded) linear map $T : H_1 \to H_2$ such that*

$$T\pi_1(a) = \pi_2(a)T, \quad a \in A.$$

Proof. (i) $\Leftrightarrow$ (ii) follows from Lemma 1 and the implication (iii) $\Rightarrow$ (ii) is trivial.

For (ii) $\Rightarrow$ (iii) let us assume that π_1 and π_2 are topologically disjoint, but there exists a non-zero linear map $T : H_1 \to H_2$ such that

$$T\pi_1(a) = \pi_2(a)T, \quad a \in A.$$

Then there is $\xi \in H_1$ with $T\xi \neq 0$ and statement (i) of the Kadison transitivity theorem entails the existence of some $a \in A$ with $\pi_1(a)\xi = \xi$ and $\pi_2(a)T\xi = 0$, implying the absurdity

$$0 \neq T\xi = T\pi_1(a)\xi = \pi_2(a)T\xi = 0. \quad \blacksquare$$

Let us point out some consequences of the Kadison transitivity theorem for pure states on C^*-algebras:

COROLLARY 3. *Let be A a C^*-algebra and φ a state on A. Let us denote*

$$\mathcal{L}_\varphi = \{a \in A;\, \varphi(a^*a) = 0\} = \{a \in A;\, \pi_\varphi(a)\xi_\varphi = 0\}.$$

Then the following conditions are equivalent:
(i) *φ is a pure state;*
(ii) *the kernel of φ is equal to $\mathcal{L}_\varphi + (\mathcal{L}_\varphi)^*$;*
(iii) *every positive form on A which vanishes on $\mathcal{L}_\varphi$ is scalar multiple of φ.*

Moreover, if φ is a pure state then the quotient vector space $A/\mathcal{L}_\varphi$ is complete with respect to the norm defined by the scalar product

$$(a/\mathcal{L}_\varphi | b/\mathcal{L}_\varphi)_\varphi = \varphi(b^*a).$$

Proof. (i) $\Rightarrow$ (ii). By Theorem 4.5 $\mathcal{L}_\varphi + (\mathcal{L}_\varphi)^*$ is contained in the kernel of φ. Conversely, let a be an arbitrary element of the kernel of φ. Then $\pi_\varphi(a)\xi_\varphi$ is orthogonal to ξ_φ, so $x \in B(H_\varphi)$ defined by

$$x(\xi_\varphi) = 0, \quad x(\pi_\varphi(a)\xi_\varphi) = \pi_\varphi(a)\xi_\varphi, \quad x|H_\varphi \ominus \{\mathbb{C}\cdot\xi_\varphi + \mathbb{C}\cdot\pi_\varphi(a)\xi_\varphi\} \equiv 0$$

is self-adjoint. By statement (ii) of the Kadison transitivity theorem there exists $b^* = b \in A$ such that

$$\pi_\varphi(b)\xi_\varphi = 0, \quad \pi_\varphi(ba)\xi_\varphi = \pi_\varphi(b)\pi_\varphi(a)\xi_\varphi = \pi_\varphi(a)\xi_\varphi,$$

hence

$$\pi_\varphi(a - ba)\xi_\varphi = 0 \Leftrightarrow a - ba \in \mathcal{L}_\varphi$$

and

$$\pi_\varphi((ba)^*)\xi_\varphi = \pi_\varphi(a^*)\pi_\varphi(b)\xi_\varphi = 0 \Leftrightarrow (ba)^* \in \mathcal{L}_\varphi.$$

Consequently, $a = (a - ba) + ba \in \mathcal{L}_\varphi + (\mathcal{L}_\varphi)^*$.

(ii) $\Rightarrow$ (iii). If ψ is a positive form on A which vanishes on $\mathcal{L}_\varphi$ then, according to Theorem 4.5 it vanishes also on $(\mathcal{L}_\varphi)^*$. Therefore ψ vanishes on the kernel of φ and it follows that it is scalar multiple of φ.

(iii) $\Rightarrow$ (i) is a consequence of Proposition 4.7.

Finally, if φ is pure then, according to Corollary 1, the dense, hence non-zero linear subspace $\pi_\varphi(A)\xi_\varphi$ of H_φ, which is stable under $\pi_\varphi(A)$, coincides with H_φ. Therefore it is complete with respect to the scalar product of H_φ. $\blacksquare$

7.23. Notes. For the few general results on functional analysis contained in 7.3–7.6 we refer to [81], [147]. In 7.1, 7.2 we developed some arguments of J.R. Ringrose [259], which allowed a simplification in the proof of the Krein-Shmulyan theorem (7.4), as well as unified proofs for the results in 7.7; [70], [77] and 7.8; [271], [274].

The Kaplansky density theorem (7.9), discovered in [155], [156] is one of the most useful results in the theory of operator algebras. The proof we have presented is the original proof of I. Kaplansky [156] with an ingredient from [274], 1.9. In the same paper I. Kaplansky pointed out some class of operator continuous functions. These functions were further studied by R.V. Kadison [149] who proved Theorem 7.10 and also a converse result. The Kaplansky density theorem is similar to the Goldstine theorem ([81], V.4.5) but, as shown in [274], p. 23, we cannot replace in its statement the $*$-subalgebra by a vector subspace.

The fundamental result of the theory of operator algebras is the von Neumann density theorem (7.11) discovered in [205]. The term "von Neumann algebra" has been introduced by J. Dixmier [77], while F.J. Murray and J. von Neumann [200] called these objects "rings of operators". Also, I.E. Segal and others used the term "W^*-algebras", but we reserved this term for "abstract" von Neumann algebras (Chapter 8). The elementary operations an von Neumann algebras (7.17–7.19) have been considered by many authors: [200], [208], [71], [77], [198], [285], [324], [325], [333]. Our exposition in 7.11, 7.17–7.19 is based on the monograph of J. Dixmier [77], with an improvement in 7.17 (cf. [307]) due to the use of the Kaplansky density theorem.

The material included in 7.12–7.16 concerns mainly standard operator theory, relativized with respect to a von Neumann algebra ([77], [81], [307]). The result in Proposition 3/7.16 was pointed out by G.K. Pedersen and Proposition 5/7.16 is due to J. Glimm and R.V. Kadison [113]. The main result in 7.20 is due to F.J. Murray and J. von Neumann [200] and R.V. Kadison [145].

There is another important operation with von Neumann algebras, namely the crossed product by the action of a locally compact group ([62], [77], [115], [173], [174], [200], [202], [305], [309], [321], [334]).

The non-commutative Lusin theorem (7.21) is due to M.Tomita [326], I (see also [125], [264], [319], [351], I). Our exposition is based on [351], I, §2. The transitivity theorem (7.22), fundamental in the representation theory of C^*-algebras, was originally proved by R.V.Kadison in [146] (see also [113]).

Chapter 8

W^*-ALGEBRAS

As the Gelfand-Naĭmark algebras are concrete realizations of C^*-algebras, the von Neumann algebras are concrete realizations of more special C^*-algebras, called W^*-algebras and the w-topology has an abstract characterization. This section is devoted to a natural introduction of W^*-algebras and to the study of their basic properties related to the w-topology. Also, it is proved that the second dual of a C^*-algebra is a W^*-algebra. This fact will make possible a more detailed study of C^*-algebras by reducing some problems from general C^*-algebras to W^*-algebras.

8.1. Every von Neumann algebra is a dual Banach space.

PROPOSITION. *Let $M \subset B(H)$ be a w-closed vector subspace of $B(H)$ and let M_* be the vector space of all w-continuous linear functionals on M. Then for every $\varphi \in M_*$ and every $\varepsilon > 0$ there exists $\psi \in B(H)_*$ such that $\varphi = \psi|M$ and $\|\psi\| \leqslant \|\varphi\| + \varepsilon$. In particular,*

$$M_* = \{\psi|M;\ \psi \in B(H)_*\}.$$

Moreover, M_* is a norm closed vector subspace in M^* and the map

$$M \ni x \mapsto \Phi(x) \in (M_*)^* \text{ defined by } \Phi(x)\varphi = \varphi(x), \quad (x \in M,\ \varphi \in M_*)$$

is a linear isometry of M onto $(M_*)^*$.

Proof. By the Hahn-Banach theorem there exists $\theta \in B(H)_*$ such that $\varphi = \theta|M$. Let $F = \{\rho \in B(H)_*;\ \rho|M = 0\}$ and $d = \inf\{\|\theta - \rho\|;\ \rho \in F\}$. Again by the Hahn-Banach theorem, there is a linear functional f on $B(H)_*$ such that $\|f\| \leqslant 1$, $f|F = 0$ and $f(\theta) = d$. Using Lemma 1/7.8 and Theorem 7.6 we find $x \in B(H)$, $\|x\| = \|f\| \leqslant 1$, such that $\rho(x) = f(\rho)$ for all $\rho \in B(H)_*$. Since $\rho(x) = 0$ for all $\rho \in F$ and since M is w-closed, using one more time the Hahn-Banach theorem we infer that $x \in M$. It follows that

$$d = f(\theta) = \theta(x) = \varphi(x) \leqslant \|\varphi\|\,\|x\| \leqslant \|\varphi\|,$$

hence there exists $\rho \in F$ such that $\|\theta - \rho\| \leqslant d + \varepsilon \leqslant \|\varphi\| + \varepsilon$. Thus the first assertion in the statement follows with $\psi = \varphi - \rho$.

Clearly, M_* is a norm closed vector subspace of M^*. By the above, the w-topology on M coincides with the M_*-topology. Thus M_1 is M_*-compact (Lemma 1/7.8) and the last assertion in the statement follows applying Theorem 7.6. ∎

Therefore, for a von Neumann algebra $M \subset B(H)$, the set M_* of all w-continuous forms on M is a norm closed vector subspace of M^*, invariant under translations and the $*$-operation and moreover, $(M_*)^*$ is isometrically isomorphic to M. We underline that the w-topology on M coincides with the M_*-topology.

8.2. The envelopping von Neumann algebra of a C^*-algebra. Let A be a C^*-algebra, $H_A = \bigoplus_{\varphi \in S(A)} H_\varphi$ and

$$\pi_A = \bigoplus_{\varphi \in S(A)} \pi_\varphi : A \to B(H_A).$$

Then π_A is an isometric non degenerate $*$-representation of A (see 4.11) called the *universal $*$-representation* of A. The w-closure N_A of $\pi_A(A)$ is a von Neumann subalgebra of $B(H_A)$, called the *envelopping von Neumann algebra* of A.

Let $\varphi \in S(A)$. Denote $\eta_\varphi = \xi_\varphi \in H_\varphi$ and $\eta_\psi = 0 \in H_\psi$ for $\psi \in S(A)$, $\psi \neq \varphi$. Then $\zeta_\varphi = \bigoplus_{\psi \in S(A)} \eta_\psi \in H_A$ and

$$\varphi(x) = \omega_{\zeta_\varphi}(\pi_A(x)); \quad x \in A.$$

Let $\varphi_1, \ldots, \varphi_n \in S(A)$, $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$. Then $\sum_{j=1}^n \lambda_j \omega_{\zeta_{\varphi_j}} \in (N_A)_*$ and, using the Kaplansky density theorem we get

$$\begin{aligned}\Big\| \sum_{j=1}^n \lambda_j \omega_{\zeta_{\varphi_j}} \Big\| &= \sup \Big\{ \Big| \sum_{j=1}^n \lambda_j \omega_{\zeta_{\varphi_j}}(\pi_A(x)) \Big| ; \, x \in A, \|x\| \leqslant 1 \Big\} \\ &= \sup \Big\{ \Big| \sum_{j=1}^n \lambda_j \varphi_j(x) \Big| ; \, x \in A, \|x\| \leqslant 1 \Big\} = \Big\| \sum_{j=1}^n \lambda_j \varphi_j \Big\|.\end{aligned}$$

Since every bounded linear form on A is a linear combination of states (Corollary 1/4.15), we infer that the mapping

$$\sum_{j=1}^n \lambda_j \varphi_j \mapsto \sum_{j=1}^n \lambda_j \omega_{\zeta_{\varphi_j}}$$

is a well defined linear isometry F_A of A^* into $(N_A)_*$. Clearly

$$\psi(x) = F_A(\psi)(\pi_A(x)); \quad x \in A, \, \psi \in A^*.$$

For an arbitrary $\theta \in (N_A)_*$, let $\psi \in A^*$ be defined by $\psi(x) = \theta(\pi_A(x))$, $(x \in A)$. Then $F_A(\psi) \in (N_A)_*$ and $\theta = F_A(\psi)$.

Hence $F_A : A^* \to (N_A)_*$ is a surjective linear isometry.

Since every $F_A(\psi)$, $(\psi \in A^*)$, is wo-continuous, it follows that the topologies w and wo coincide on N_A. Moreover, if $\theta \in (N_A)_*$ is positive, then $\psi = \theta \circ \pi_A$ is a scalar multiple of some state of A, hence $\theta = F_A(\psi)$ is a of the form ω_ζ for some $\zeta \in H_A$.

Consider the map $\Phi_A : N_A \to A^{**}$ defined by

$$\Phi_A(y)(\psi) = F_A(\psi)(y); \quad \psi \in A^*;\, y \in N_A.$$

By Proposition 8.1 we see that Φ_A is the transposed map of F_A *via* the identification of N_A and $((N_A)_*)^*$. Consequently, Φ_A is a linear isometry of N_A onto A^{**} and a homeomorphism with respect to the w (or wo)-topology on N_A and the A^*-topology on A^{**}.

Finally, for every $x \in A$ we have $\Phi_A(\pi_A(x)) \in A^{**}$ and

$$[\Phi_A(\pi_A(x))](\psi) = F_A(\psi)(\pi_A(x)) = \psi(x); \quad \psi \in A^*,$$

hence $\Phi_A \circ \pi_A$ is the canonical embedding of A into A^{**}.

We summarize the above results:

THEOREM. *Let A be a C^*-algebra and $\pi_A : A \to N_A \subset B(H_A)$ be its universal $*$-representation. Then*

(i) *for every $\psi \in A^*$ there exists a unique wo-continuous form $F_A(\psi)$ on N_A such that $\psi = F_A(\psi) \circ \pi_A$ and the map F_A is a linear isometry of A^* onto $(N_A)_*$;*

(ii) *the map $\Phi_A : N_A \to A^{**}$ defined by $\Phi_A(y)(\psi) = F_A(\psi)(y)$, ($\psi \in A^*$, $y \in N_A$), is a surjective linear isometry and a $(w, \sigma(A^{**}, A^*))$-homeomorphism and $\Phi_A \circ \pi_A$ is the canonical embedding of A into A^{**}.* ∎

Let $\psi \in A^*$ and $y \in N_A$. Then $y \cdot F_A(\psi)$, $F_A(\psi) \cdot y$ belong to $(N_A)_*$, hence there exist $y \cdot \psi, \psi \cdot y \in A^*$ such that $F_A(y \cdot \psi) = y \cdot F_A(\psi)$, $F_A(\psi \cdot y) = F_A(\psi) \cdot y$. It is easy to see that for every $x \in A$ we have $\pi_A(x) \cdot \psi = x \cdot \psi = \psi(x\,\cdot)$, $\psi \cdot \pi_A(x) = \psi \cdot x = \psi(\cdot\, x)$.

On the other hand, for every $\psi \in A^*$ we have $F_A(\psi)^* = F_A(\psi^*)$. If $\psi \in A^*$ is positive, then $F_A(\psi) \in (N_A)_*$ is also positive. If $\psi \in A^*$ is positive, then for every $y \in N_A$ the functional $y^* \cdot \psi \cdot y \in A^*$ is positive.

COROLLARY. *Let A, B be C^*-algebras and $\psi : A \to B$ be a bounded linear mapping. There exists a unique w-continuous linear mapping $\Psi_{A,B} : N_A \to N_B$ such that $\|\Psi_{A,B}\| = \|\Psi\|$ and*

$$\Psi_{A,B}(\pi_A(x)) = \pi_B(\Psi(x)); \quad x \in A. \quad ∎$$

8.3. Projections of norm one. This section contains a first application of von Neumann algebras to C^*-algebra theory.

LEMMA. *Let $M \subset B(H)$ be a von Neumann algebra, N be a w-closed $*$-subalgebra of M and Ψ be a linear projection of M onto N with $\|\Psi\| \leqslant 1$. Then Ψ is positive and*

$$\Psi(yxz) = y\Psi(x)z; \quad x \in M,\, y, z \in N.$$

Proof. By Lemma 7.11, N has a unit element e_N which is a projection. As for any element of N, we have $\Psi(e_N) = e_N, \Psi$ being a projection onto N.

Let $\Psi(1-e_N) = a + \mathrm{i}b$ with $a, b \in N$ selfadjoint. Then for $\lambda \in \mathbb{C}$, $|\lambda| \leqslant 1$, we have

$$\|e_N + \lambda a + \mathrm{i}\lambda b\| = \|\Psi(e_N + \lambda(1-e_N))\| \leqslant \|\Psi\| \, \|e_N + \lambda(1-e_N)\| \leqslant 1.$$

If $\varphi \in S(N)$, then $\varphi(e_N) = 1$. Thus, for $\lambda \in \mathbb{R}$, $|\lambda| \leqslant 1$, we get

$$\begin{gathered} |1 + \lambda\varphi(a) + \mathrm{i}\lambda\varphi(b)| = |\varphi(e_N + \lambda a + \mathrm{i}\lambda b)| \leqslant 1, \\ |1 + \lambda\varphi(a)| \leqslant 1. \end{gathered}$$

It follows that $\varphi(a) = 0$ for all $\varphi \in S(N)$ and therefore $a = 0$ (Proposition 4.13). Thus $|1 + \mathrm{i}\varphi(b)| \leqslant 1$, $\varphi(b) = 0$ for all $\varphi \in S(N)$ and $b = 0$. We have proved

$$\Psi(1) = e_N.$$

By Proposition 6.4 we infer that Ψ is positive.

For every $\varphi \in S(N)$ we have

$$|\varphi(\Psi(x) - \Psi(e_N x))| = |(\varphi \circ \Psi)((1-e_N)x)| \leqslant (\varphi \circ \Psi)(1-e_N)^{1/2} (\varphi \circ \Psi)(x^* x)^{1/2} = 0,$$

so that

$$\Psi(e_N x) = \Psi(x); \quad x \in M.$$

Let $e \in N$ be a projection and put $f = e_N - e$. For every $x, y \in M$ we have

$$\|ex + fy\|^2 = \|(ex + fy)^*(ex + fy)\| = \|x^* ex + y^* fy\| \leqslant \|ex\|^2 + \|fy\|^2,$$

hence, for any $\lambda \in \mathbb{R}$ we obtain

$$\begin{aligned} (\lambda + 1)^2 \|f\Psi(ex)\|^2 &= \|f\Psi(ex + \lambda f\Psi(ex))\|^2 \leqslant \|ex + \lambda f\Psi(ex)\|^2 \\ &\leqslant \|ex\|^2 + \|\lambda f\Psi(ex)\|^2 = \|ex\|^2 + \lambda^2 \|f\Psi(ex)\|^2. \end{aligned}$$

Consequently

$$(e_N - e)\Psi(ex) = 0.$$

Thus, for every $x \in M$ and every projection $e \in N$ we get

$$\begin{aligned} e\Psi(x) &= e\Psi(e_N x) = e\Psi(ex) + e\Psi((e_N - e)x) = e\Psi(ex) \\ &= e\Psi(ex) + (e_N - e)\Psi(ex) = e_N \Psi(ex) = \Psi(ex). \end{aligned}$$

Since N is the norm closed linear hull of the projections it contains (Proposition 3/7.16), it follows that

$$\Psi(yx) = y\Psi(x); \quad x \in M, \, y \in N.$$

Using this and the selfadjointness of Ψ we also obtain

$$\Psi(xz) = \Psi(x)z; \quad x \in M, \, z \in N. \quad \blacksquare$$

THEOREM (J. Tomiyama). *Let A be a C^*-algebra, B be a C^*-subalgebra of A and Ψ be a linear projection of norm one of A onto B. Then*

(i) $\Psi(ba) = b\Psi(a)$ *and* $\Psi(ba) = \Psi(a)b$ *for all* $a \in A$, $b \in B$.

(ii) Ψ *is completely positive.*

(iii) Ψ *is a Schwarz map.*

Proof. Consider the universal $*$-representation $\pi_A : A \to N_A$ of A and denote by N the w-closure of $\pi_A(B)$ in N_A. By Corollary 8.2 there exists a w-continuous linear mapping $\Psi_A : N_A \to N_A$, $\|\Psi_A\| = \|\Psi\| \leqslant 1$, such that

$$\Psi_A(\pi_A(a)) = \pi_A(\Psi(a)); \quad a \in A.$$

For every $a \in A$ we have $\Psi_A(\pi_A(a)) = \pi_A(\Psi(a)) \in \pi_A(B)$, hence $\Psi_A(N_A) \subset N$. On the other hand, for every $b \in B$ we have $\Psi_A(\pi_A(b)) = \pi_A(\Psi(b)) = \pi_A(b)$, hence Ψ_A acts identically on N. Consequently, Ψ_A is a linear projection of N_A onto N with $\|\Psi_A\| \leqslant 1$. By the above lemma it follows that Ψ_A is positive and, for all $a \in A$, $b \in B$, we have

$$\pi_A(\Psi(ba)) = \Psi_A(\pi_A(b)\pi_A(a)) = \pi_A(b)\Psi_A(\pi_A(a)) = \pi_A(b\Psi(a))$$

and similarly $\pi_A(\Psi(ab)) = \pi_A(\Psi(a)b)$.

Hence Ψ is positive and satisfies assertion (i). Now the assertions (ii) and (iii) follow by Proposition 5.11 and Corollary 5.10. ∎

8.4. W^*-algebras. By Theorem 7.6, for a Banach space M the following statements are equivalent:

1) M_1 is compact with respect to some locally convex Hausdorff linear topology on M;

2) M is isometrically isomorphic to the dual space of some Banach space.

On the other hand, by Lemma 8.1 or by Lemma 1/7.8, every von Neumann algebra M satisfies the above conditions. In order to obtain an abstract (or space-free) theory of von Neumann algebras, we shall define a particular class of C^*-algebras.

A Banach space M is called a W^*-*algebra* if it is a C^*-algebra and satisfies the above equivalent conditions 1) and 2).

By condition 1) and the Krein-Milman theorem, M_1 has extreme points, hence (Theorem 6.1) every W^*-algebra has a unit element.

Throughout this section, devoted to the identification of W^*-algebras with von Neumann algebras, M will be a W^*-algebra and F will be a norm closed vector subspace of M^* such that the map

$$\Phi : M \to F^* \text{ defined by } \Phi(x)(\psi) = \psi(x); \quad \psi \in F,\, x \in M,$$

is a linear isometry of M onto F^*. We shall identify M and F^* *via* Φ. Consider also the universal $*$-representation

$$\pi_M : M \to N_M \subset B(H_M)$$

where N_M is the envelloping von Neumann algebra of M. As in 8.1 we shall identify $((N_M)_*)^*$ and N_M.

Lemma 1. *There exists a unique central projection p_F of N_M such that the mapping*

$$M \ni x \mapsto \pi_M(x)p_F \in (N_M)p_F \tag{1}$$

is a surjective $*$-isomorphism and $(\sigma(M,F),w)$-homeomorphism.

Moreover, for $\psi \in M^*$ we have

$$\psi \in F \Leftrightarrow \psi = p_F \cdot \psi.$$

Proof. By Theorem 8.2 there is a linear isometry F_M of M^* onto $(N_M)_*$ such that $\psi = F_M(\psi) \circ \pi_M$ for all $\psi \in M^*$. Then $F_M|F$ is a linear isometry of F into $(N_M)_*$ and its transposed map

$$\Psi = (F_M|F)^* : N_M \equiv ((N_M)_*)^* \to F^* \equiv M$$

is a $(w,\sigma(M,F))$-continuous linear contraction of N_M into M, such that $\psi \circ \Psi = F_M(\psi)$ for all $\psi \in F$.

Since $\psi \circ \Psi \circ \pi_M = F_M(\psi) \circ \pi_M = \psi$, $(\psi \in F)$, it follows that $\pi_M \circ \Psi$ is a linear projection of norm one of N_M onto $\pi_M(M)$. By Theorem 8.3 we infer that Ψ is positive and

$$\Psi(\pi_M(y)x\pi_M(z)) = y\Psi(x)z; \quad x \in N_M,\ y,z \in M. \tag{2}$$

Since Ψ is selfadjoint and $(w,\sigma(M,F))$-continuous, $\operatorname{Ker}\Psi$ is a w-closed selfadjoint subspace of N_M. Moreover, by (2),

$$x \in \operatorname{Ker}\Psi,\ y,z \in M \Rightarrow \pi_M(y)x\pi_M(z) \in \operatorname{Ker}\Psi$$

and, since $\pi_M(M)$ is w-dense in N_M, it follows that $\operatorname{Ker}\pi$ is a w-closed selfadjoint two-sided ideal of N_M. By Lemma 7.11, the $*$-algebra $\operatorname{Ker}\pi$ has a unit element q_F which is a projection. Since $\operatorname{Ker}\pi$ is a two-sided ideal, it is easy to check that q_F is a central projection in N_M and

$$\operatorname{Ker}\Psi = (N_M)q_F.$$

For $x,y \in N_M$ we have $x-(\pi_M \circ \Psi)(x) \in \operatorname{Ker}\Psi$, hence $xy-((\pi_M \circ \Psi)(x))y \in \operatorname{Ker}\Psi$, and by (2) this yields $\Psi(xy) = \Psi(x)\Psi(y)$. Thus Ψ is a $*$-homomorphism.

Let $p_F = 1 - q_F$. Then the restriction of Ψ to $(N_M)p_F$ is a $*$-isomorphism onto M and, for every $y \in M$, we have $\Psi(\pi_M(x)p_F) = \Psi(\pi_M(x)) = x$. It follows that the map (1) is a surjective $*$-isomorphism. As $((N_M)p_F)_1$ is w-compact, M_1 is $\sigma(M,F)$-Hausdorff and Ψ is $(w,\sigma(M,F))$-continuous, the map $M_1 \ni x \mapsto \pi_M(x)p_M \in ((N_M)p_F)_1$ is a $(\sigma(M,F),w)$-homeomorphism. Using Proposition 7.2 we infer that the map (1) is also a $(\sigma(M,F),w)$-homeomorphism.

Now let p be an arbitrary central projection in N_M such that the map $M \ni x \mapsto \pi_M(x)p \in (N_M)p$ is a $(\sigma(M,F),w)$-homeomorphic $*$-isomorphism. Then for every $y \in N_M$ there exists a unique $\Theta(y) \in M$ such that $\pi_M(\Theta(y))p = yp$ and $\Theta : N_M \to M$ is a $(w,\sigma(M,F))$-continuous $*$-homomorphism with $\operatorname{Ker}\Theta =$

$(N_M)(1-p)$. Since for all $y \in \pi_M(M)$ we have $\pi_M(\Psi(y)) = y$ hence $\pi_M(\Psi(y))p = yp$, it follows that Θ coincides with Ψ on $\pi_M(M)$. By the w-density of $\pi_M(M)$ in N_M, it follows that $\Theta = \Psi$, $\operatorname{Ker}\Theta = \operatorname{Ker}\Psi$ and $p = p_F$.

Finally, let $\psi \in M^*$. If $\psi \in F$, then

$$(p_F \cdot \psi)(x) = F_M(\psi)(p_F\pi_M(x)) = \psi(\Psi(p_F\pi_M(x))) = \psi(\Psi(\pi_M(x))) = \psi(x)$$

for all $x \in M$, hence $\psi = p_F \cdot \psi$. Conversely, if $\psi = p_F \cdot \psi$, then

$$\psi(x) = (p_F \cdot \psi)(x) = F_M(\psi)(p_F\pi_M(x)); \quad x \in M.$$

Due to the continuity properties of the maps (1) and $F_M(\psi)$ we infer that ψ is F-continuous, i.e. $\psi \in F$ (Proposition 7.2). ∎

By Lemma 1, we have $F = p_F \cdot M^*$. A form $\psi \in M^*$ will be called F-singular if it belongs to $(1 - p_F) \cdot M^*$, that is if $p_F \cdot \psi = 0$.

Let $\varphi, \psi \in M^*$ be positive. Since p_F is a central projection in N_M, it is easy to see that

$$\varphi \leqslant \psi,\ \psi \in F \Rightarrow \varphi \in F,$$

$$\varphi \leqslant \psi,\ \psi \text{ is } F\text{-singular} \Rightarrow \varphi \text{ is } F\text{-singular}.$$

By Corollary 1/4.15 every $\psi \in M^*$ can be written as

$$\psi = \psi_1 - \psi_2 + \mathrm{i}(\psi_3 - \psi_4)$$

with $\psi_k \in M^*$ positive and $\|\psi_k\| \leqslant \|\psi\|$, $(1 \leqslant k \leqslant 4)$. Moreover, "translating" this equality with p_F or with $(1 - p_F)$, it follows that

$$\psi \in F \Rightarrow \text{we can choose } \psi_k \in F, \quad 1 \leqslant k \leqslant 4;$$

$$\psi \text{ is } F\text{-singular} \Rightarrow \text{we can choose } \psi_k\ F\text{-singular}, \quad 1 \leqslant k \leqslant 4.$$

Also, for every $a, b \in N_M$ we have

$$\psi \in F \Rightarrow a \cdot \psi \cdot b \in F \text{ and } \psi^* \in F;$$

$$\psi \text{ is } F\text{-singular } \Rightarrow a \cdot \psi \cdot b \text{ and } \psi^* \text{ are } F\text{-singular}.$$

In particular, for every $a, b \in M$, the mappings

$$M \ni x \mapsto axb \in M, \quad M \ni x \mapsto x^* \in M$$

are F-continuous.

Lemma 1 shows that every W^*-algebra M is $(\sigma(M,F), w)$-homeomorphically $*$-isomorphic to a von Neumann algebra. In particular, since the vector states on a von Neumann algebra are w-continuous, we infer that

$$\begin{aligned} x = x^* &\Leftrightarrow \varphi(x) \in \mathbb{R} &&\text{for all } \varphi \in F,\ \varphi \geqslant 0;\\ x \geqslant 0 &\Leftrightarrow \varphi(x) \geqslant 0 &&\text{for all } \varphi \in F,\ \varphi \geqslant 0;\\ x = 0 &\Leftrightarrow \varphi(x) = 0 &&\text{for all } \varphi \in F,\ \varphi \geqslant 0.\end{aligned}$$

LEMMA 2. *If $\{a_\iota\}_{\iota\in I}$ is a norm bounded increasing net of selfadjoint elements of M, then there exists an element in M, denoted as $\sup\limits_\iota a_\iota$, which is the least upper bounded of $\{a_\iota;\ \iota \in I\}$ with respect to the C^*-algebra order structure on M. Moreover,*

$$a_\iota \xrightarrow{F} \sup_\iota a_\iota .$$

If the a_ι's are projections, then $\sup\limits_\iota a_\iota$ is also a projection.

Proof. Let S_ι be the F-closure of $\{a_\kappa;\ \kappa \geqslant \iota\}$, $(\iota \in I)$. Then $\{S_\iota\}_{\iota\in I}$ is a decreasing net of F-compact sets, hence $\bigcap\limits_{\iota\in I} S_\iota \neq \emptyset$. If $a, a' \in \bigcap\limits_{\iota\in I} S_\iota$, then $\varphi(a) = \sup\limits_{\iota\in I} \varphi(a_\iota) = \varphi(a')$ for all positive $\varphi \in F$, hence $a = a'$. Thus $\bigcap\limits_{\iota\in I} S_\iota$ reduces to a single selfadjoint element $a \in M$.

Clearly, $a_\iota \leqslant a$ for all $\iota \in I$. If $a_\iota \leqslant b$ for all $\iota \in I$, then $\varphi(a) = \sup\limits_\iota \varphi(a_\iota) \leqslant \varphi(b)$ for all positive $\varphi \in F$, hence $a \leqslant b$. Therefore $a = \sup\limits_\iota\ a_\iota$.

Since every $\psi \in F$ is a linear combination of positive forms in F, it follows that $\psi(a_\iota) \to \psi(a)$ for all $\psi \in F$, i.e. $a_\iota \xrightarrow{F} a$.

If the a_ι's are all projections, then $a_\iota = a_\iota a_\kappa$ whenever $\iota \leqslant \kappa$. Taking the F-limit over κ, we obtain $a_\iota = a_\iota a$, $(\iota \in I)$, and now taking the F-limit over ι, we get $a = a^2$, hence a is a projection. ∎

Let $\{e_\iota\}_{\iota\in I}$ be an arbitrary family of mutually orthogonal projections in M and, for every finite set $J \subset I$, denote $e_J = \sum\limits_{\iota\in J} e_\iota$. Then $\{e_J\}_J$ is an increasing net of projections. Using Lemma 2 we define

$$\sum_{\iota\in I} e_\iota = \sup_{J\subset I \text{ finite}} e_J .$$

Note that $e_J \xrightarrow{F} \sum\limits_{\iota\in I} e_\iota$.

Let $\psi \in M^*$ and write $\psi = \psi_1 - \psi_2 + \mathrm{i}(\psi_3 - \psi_4)$ with $\psi_k \in M^*$, $\psi_k \geqslant 0$, $\|\psi_k\| \leqslant \|\psi\|$, $(1 \leqslant k \leqslant 4)$. Since $|\psi(e_\iota)| \leqslant \sum\limits_{k=1}^{4} \psi_k(e_\iota)$, $(\iota \in I)$, it follows that $\sum\limits_{\iota\in I} |\psi(e_\iota)| \leqslant 4\|\psi\| < +\infty$, hence the sum $\sum\limits_{\iota\in I} \psi(e_\iota)$ is legitime. Moreover, we have

$$\psi\Big(\sum_{\iota\in I} e_\iota\Big) = \sum_{\iota\in I} \psi(e_\iota); \quad \psi \in F.$$

LEMMA 3. *Let $\psi \in M^*$. Then*

(i) *$\psi \in F$ if and only if $\psi\Big(\sum\limits_{\iota\in I} e_\iota\Big) = \sum\limits_{\iota\in I} \psi(e_\iota)$ for every family $\{e_\iota\}_{\iota\in I}$ of mutually orthogonal projection in M.*

(ii) *ψ is F-singular if and only if for every non-zero projection $e \in M$ there exists a non-zero projection $f \in M$, $f \leqslant e$, such that $\psi(f) = 0$.*

Proof. (ii) Let ψ be F-singular. Since $\psi = \psi_1 - \psi_2 + \mathrm{i}(\psi_3 - \psi_4)$ with ψ_k positive and F-singular, $(1 \leqslant k \leqslant 4)$, it follows that in proving the "only if" part we may, and shall, assume ψ to be positive.

If $\psi(e) = 0$, then we take $f = e$.

Suppose that $\psi(e) > 0$. Choose a positive form $\theta \in F$ such that $\theta(e) > \psi(e)$ and let P be the set of all projections $p \in M$, $p \leqslant e$, with $\theta(p) \leqslant \psi(p)$. By Lemma 2 and by the Zorn lemma, P has a maximal element denoted by $p_{\max}$. Put $f = e - p_{\max}$.

Since $\theta(f) = \theta(e) - \theta(p_{\max}) > \psi(e) - \psi(p_{\max}) = \psi(f) \geqslant 0$, f is non-zero. By the maximality of $p_{\max}$ it follows that

$$q \in M \text{ projection},\ q \leqslant f \Rightarrow \theta(q) \geqslant \psi(q).$$

As M is $*$-isomorphic to a von Neumann algebra, using Proposition 3/7.16 we infer that

$$a \in M,\ a \geqslant 0,\ a \leqslant f \Rightarrow \theta(a) \geqslant \psi(a).$$

This means that $f \cdot \psi \cdot f \leqslant f \cdot \theta \cdot f$. Since $f \cdot \theta \cdot f \in F$, we have also $f \cdot \psi \cdot f \in F$. But $f \cdot \psi \cdot f$ is F-singular since ψ is F-singular. Therefore $f \cdot \psi \cdot f = 0$, which is equivalent to $\psi(f) = 0$ because ψ is positive.

Conversely, suppose that for every non-zero projection $e \in M$ there is a non-zero projection $f \in M$, $f \leqslant e$, with $\psi(f) = 0$.

Let $e \in M$ be a non-zero projection. Since $(1 - p_F) \cdot \psi$ is F-singular, the above part of the proof shows that there is a non-zero projection $g \in M$, $g \leqslant e$, such that $g \cdot [(1 - p_F) \cdot \psi] \cdot g = 0$. By the assumption, there is a non-zero projection $f \in M$, $f \leqslant g$, with $\psi(f) = 0$. Then

$$(p_F \cdot \psi)(f) = \psi(f) - [(1 - p_F) \cdot \psi](f) = \psi(f) - (g \cdot [(1 - p_F) \cdot \psi] \cdot g)(f) = 0.$$

It follows that for an arbitrary projection $e \in M$ and a maximal family $\{e_\iota\}_{\iota \in I}$ of mutually orthogonal projection $0 \neq e_\iota \in M$, $e_\iota \leqslant e$, with $(p_F \cdot \psi)(e_\iota) = 0$, $(\iota \in I)$, we have $\sum_{\iota \in I} e_\iota = e$. Since $p_F \cdot \psi \in F$, we infer that $(p_F \cdot \psi)(e) = 0$.

As M is the closed linear span of its projections (Proposition 3/7.16), we obtain $p_F \cdot \psi = 0$, so ψ is F-singular.

(i) The "only if" part was proved just before the statement.

Suppose that $\psi\Big(\sum_\iota e_\iota\Big) = \sum_\iota \psi(e_\iota)$ for every family $\{e_\iota\}$ of mutually orthogonal projections in M.

Let $e \in M$ be an arbitrary projection and $\{e_\iota\}$ be a maximal family of mutually orthogonal projection $0 \neq e_\iota \leqslant e$ with $[(1 - p_F) \cdot \psi](e_\iota) = 0$. Since $(1 - p_F) \cdot \psi$ is F-singular, by (ii) it follows that $\sum_\iota e_\iota = e$. By our assumption on ψ and by the "only if" part of (i), we get

$$\begin{aligned} [(1 - p_F) \cdot \psi](e) = \psi(e) - (p_F \cdot \psi)(e) &= \sum_\iota \psi(e_\iota) - \sum_\iota (p_F \cdot \psi)(e_\iota) \\ &= \sum_\iota [(1 - p_F) \cdot \psi](e_\iota) = 0. \end{aligned}$$

As above, we conclude $(1 - p_F) \cdot \psi = 0$, hence $\psi \in F$. ∎

Theorem. *Let M be a W^*-algebra and $\pi_M : M \to N_M \subset B(H_M)$ be its universal $*$-representation.*

(i) *There exists a unique norm-closed vector subspace F of M^* such that the map $\Phi : M \to F^*$ defined by $\Phi(x)\psi = \psi(x)$, $(x \in M,\ \psi \in F)$, be a surjective linear isometry.*

We denote $M_* = F$ and call M_* the predual of M.

(ii) *There exists a unique central projection p_M in N_M such that the map $x \mapsto \pi_M(x)p_M$ be a $*$-isomorphism of M onto the von Neumann algebra $(N_M)p_M \subset B(p_M H_M)$. Moreover, this map is a $(\sigma(M, M_*), w)$-homeomorphism and $M_* = p_M \cdot M^*$.*

(iii) *A linear functional $\psi \in M^*$ satsfies $\psi = p_M \cdot \psi$, that is $\psi \in M_*$, if and only if $\psi\Big(\sum\limits_{\iota \in I} e_\iota\Big) = \sum\limits_{\iota \in I} \psi(e_\iota)$ for every family $\{e_\iota\}_{\iota \in I}$ of mutually orthogonal projections in M.*

In this case ψ is called a normal linear form on M.

(iv) *A linear functional $\psi \in M^*$ satisfies $p_M \cdot \psi = 0$ if and only if for every non-zero projection $e \in M$ there exists a non-zero projection $f \in M$, $f \leqslant e$, with $\psi(f) = 0$. Moreover, f can be choosen such that $f \cdot \psi \cdot f = 0$.*

In this case ψ is called a singular linear form on M.

Proof. (i) follows from Lemma 3, (ii) follows from (i) together with Lemma 1 and (iii), (iv) are a reproduction of Lemma 3. ∎

Due to assertion (ii) in the theorem, the M_*-topology on a W^*-algebra M will be also called the *w-topology*.

By the Krein-Shmulyan theorem (7.4), a convex set $S \subset M$ is w-closed if and only if the sets $S \cap M_\lambda$, $(\lambda > 0)$, are all w-closed.

Also, M is w-sequentially complete, because M_1 is w-compact and every w-Cauchy sequence in M is norm-bounded by the uniform boundedness theorem.

Note that if φ is a normal (respectively singular) linear form on M, that is $\varphi \in p_M \cdot M^*$ (respectively $\varphi \in (1 - p_M) \cdot M^*$), then $a \cdot \varphi \cdot b$, $(a, b \in N_M)$, and φ^* are normal (respectively singular). If $0 \leqslant \psi \leqslant \varphi$ and φ is normal (respectively singular), then ψ is normal (respectively singular). Every normal (respectively singular) linear form on M is a linear combination of normal (respectively singular) positive forms on M. This last assertion will be sharpened in 8.10 and 8.11.

For each $\varphi \in M^*$, $p_M \cdot \varphi$ will be called the *normal part* of φ and $(1 - p_M) \cdot \varphi$ will be called the *singular part* of φ. We shall denote

$$M_*^h = \{\varphi \in M_*;\ \varphi = \varphi^*\}, \quad M_*^+ = \{\varphi \in M_*;\ \varphi \text{ positive}\}.$$

The statement (i) of the theorem entails:

Corollary 1. *Let Φ be a linear isometry of a W^*-algebra M onto a W^*-algebra N. Then Φ is w-continuous.* ∎

Let $\{x_\iota\}_{\iota \in I}$ be a family of elements of the W^*-algebra M such that

$$\sum_{\iota \in I} |\varphi(x_\iota)| < +\infty \quad \text{for all } \varphi \in M_*,\ \varphi \geqslant 0.$$

Since every normal form is a linear combination of positive normal forms, it follows that $\sum\limits_{\iota \in I} |\psi(x_\iota)| < +\infty$ for all $\psi \in M_*$. Put $x_J = \sum\limits_{\iota \in J} x_\iota$ for $J \subset I$, finite. By the uniform boundedness principle (Theorem 7.3), it follows that $\sup\limits_J \|x_J\| < +\infty$. It is now easy to check that $\{x_J\}_J$ is a w-Cauchy net. On the other hand, the set $\{x \in M;\ \|x\| \leqslant \sup\limits_J \|x_J\|\}$ being w-compact, this net has a w-limit point and hence it is w-convergent to some element of M which is denoted by

$$\sum_{\iota \in I} x_\iota.$$

Let M, N be W^*-algebras, $\Phi : M \to N$ be a bounded map and $\{e_\iota\}_{\iota \in I}$ be a family of mutually orthogonal projections in M. Since for every $\psi \in N^*$ we have

$$\sum_{\iota \in I} |\psi(\Phi(e_\iota))| = \sum_{\iota \in I} |(\psi \cdot \Phi)(e_\iota)| \leqslant 4\|\psi \cdot \Phi\| < +\infty,$$

the above discussion shows that there exists an element $\sum\limits_{\iota \in I} \Phi(e_\iota) \in N$ which is the w-limit of the net of partial finite sums.

The statement (iii) of the theorem entails:

COROLLARY 2. *A bounded linear mapping $\Phi : M \to N$ between W^*-algebras is w-continuous if and only if $\Phi\Big(\sum\limits_{\iota \in I} e_\iota\Big) = \sum\limits_{\iota \in I} \Phi(e_\iota)$ for every family $\{e_\iota\}_{\iota \in I}$ of mutually orthogonal projections in M.* ∎

A common consequence of Corollaries 1 and 2 is

COROLLARY 3. *Every $*$-isomorphism between W^*-algebras is w-continuous.* ∎

The w-continuous linear mappings between W^*-algebras will be also called *normal linear mappings*. Thus, if M, N are W^*-algebras, a linear mapping $\Phi : M \to N$ is normal if and only if the transposed map ${}^t\Phi : N^* \to M^*$ has the property ${}^t\Phi(N_*) \subset M_*$. A usual uniform boundedness argument shows that every normal mapping between W^*-algebras is bounded.

On the other hand, $\Phi : M \to N$ is called a *singular linear mapping* if Φ is bounded and ${}^t\Phi(N_*) \subset (1 - p_M) \cdot M^*$.

Note that, contrary to Corollary 3, a $*$-homomorphism between two W^*-algebras is not necessarily normal. Nevertheless, there are important cases when its normality is automatic ([314], [358], [363], [364]).

COROLLARY 4. *Let $\Phi : M \to B(H)$ be a completely positive linear mapping of the W^*-algebra M, with Stinespring dilation $\{\pi, V, K\}$. Then Φ is normal (respectively singular) if and only if π is normal (respectively singular).*

Proof. Recall that (5.3) $\pi : M \to B(K)$ is a $*$-representation and $V : H \to K$ is a bounded linear operator, uniquely determined such that $\Phi(x) = V^*\pi(x)V$, $(x \in M)$, and K is the closed linear span of $\pi(M)VH$. For every $a, b \in M$ and every $\xi, \eta \in H$ we have

$$\begin{aligned}(\omega_{\pi(a)V\xi,\pi(b)V\eta} \circ \pi)(x) &= (\pi(x)\pi(a)V\xi|\pi(b)V\eta) = (\Phi(b^*xa)\xi|\eta) \\ &= (b^* \cdot (\omega_{\xi,\eta} \circ \Phi) \cdot a)(x); \quad x \in M.\end{aligned}$$

If Φ is normal (respectively singular), then $b^* \cdot (\omega_{\xi,\eta} \circ \Phi) \cdot a$ is normal (respectively singular) and hence $\omega_{\pi(a)V\xi,\pi(b)V\eta} \circ \pi$ is normal (respectively singular) for all $a, b \in M$, $\xi, \eta \in H$. Since the linear span of $\{\omega_{\pi(a)V\xi,\, \pi(b)V\eta};\, a, b \in M,\, \xi, \eta \in H\}$ is norm-dense in $B(K)_*$, it follows that π is normal (respectively singular).

Conversely, if π is normal (respectively singular), then the same argument based on the equality

$$\omega_{\xi,\eta} \circ \Phi = \omega_{V\xi,V\eta} \circ \pi; \quad \xi, \eta \in H$$

shows that Φ is normal (respectively singular). ∎

In particular, a positive form φ on M is normal (respectively singular) if and only if the associated GNS representation $\pi_\varphi : M \to B(H_\varphi)$ is normal (respectively singular).

Let M be a W^*-algebra. A W^*-*subalgebra* of M is a w-closed $*$-subalgebra of M. If $S \subset M$, then we denote by $W^*(S)$ the smallest W^*-subalgebra of M containing S. Note that if S is a subset of the W^*-algebra $B(H)$, then $R(S) = W^*(S \cup \{1_H\})$.

COROLLARY 5. *Let M, N be W^*-algebras and $\pi : M \to N$ be a normal $*$-homomorphism. Then $\pi(M)$ is a W^*-subalgebra of N.*

Proof. By Corollary 1/3.11, $\pi(M)$ is a C^*-subalgebra of N. By Corollary 1/3.15, $\pi(M)_1 = \pi(M_1)$. Since M_1 is w-compact and π is w-continuous, we infer that $\pi(M)_1$ is w-compact and hence $\pi(M)$ is w-closed (Theorem 7.4). ∎

Using Theorem 8.2 and the facts established in this section it is easy to verify the following statement:

COROLLARY 6. *Let A be a C^*-algebra and A^{**} its second dual. Then there exists a unique C^*-algebra structure on the Banach space A^{**} such that the canonical image of A in A^{**} is a C^*-subalgebra of A^{**}. Moreover, the C^*-algebra A^{**} is a W^*-algebra.* ∎

On the second dual of a C^*-algebra we shall always consider the W^*-algebra structure guaranted by the above Corollary 6 and we shall identify A to a w-dense C^*-subalgebra of A^{**}. Note that the predual of A^{**} is the dual of A,

$$(A^{**})_* = A^*,$$

that is, every bounded linear form on A extends uniquely to a normal linear form on A^{**}. Moreover, the second dual of a C^*-algebra has the following characteristic universality property:

Corollary 7. *Let A be a C^*-algebra, M be a W^*-algebra and $\Phi : A \to M$ be a bounded linear mapping. Then Φ can be uniquely extended to a normal linear mapping of A^{**} into M.*

Proof. The bitransposed map ${}^{tt}\Phi : A^{**} \to M^{**}$ of Φ is a normal extension of Φ. On the other hand, by the statement (ii) of the theorem, there exists a central projection p_M in the W^*-algebra M^{**} and a $*$-isomorphism $\Theta : M^{**}p_M \to M$ such that $\Theta(yp_M) = y$ for all $y \in M$. Define the normal linear mapping $\Psi : A^{**} \to M$ by

$$\Psi(x) = \Theta({}^{tt}\Phi(x)p_M); \quad x \in A^{**}.$$

Then

$$\Psi(x) = \Theta(\Phi(x)p_M) = \Phi(x); \quad x \in A,$$

hence Ψ is an extension of Φ.

Since A is w-dense in A^{**}, the unicity assertion is obvious. ∎

In particular, if B is a C^*-subalgebra of A, then the inclusion map $B \to A$ extends to an injective normal $*$-homomorphism of B^{**} onto the w-closure of B in A^{**}. This allows us to consider B^{**} as a W^*-subalgebra of A^{**}.

Another particular case of Corollary 7 is

Corollary 8. *Let A be a C^*-algebra and $\pi : A \to B(H)$ be a $*$-representation. Then π can be uniquely extended to a normal $*$-representation $A^{**} \to B(H)$ also denoted by π and*

$$\pi(A^{**}) = \overline{\pi(A)}^w. \quad ∎$$

For instance, the universal $*$ -representation $\pi_A : A \to B(H_A)$ extends to a $*$-isomorphism of A^{**} onto the envelopping von Neumann algebra N_A of A. We shall often identify A^{**} and N_A *via* this $*$-isomorphism.

Corollary 9. *Let M, N be W^*-algebra and $\Phi : M \to N$ be a bounded linear mapping. There exist a normal linear mapping $\Phi_{\rm nor} : M \to N$ and a singular linear mapping $\Phi_{\rm sing} : M \to N$ uniquely determined such that*

$$\Phi(x) = \Phi_{\rm nor}(x) + \Phi_{\rm sing}(x); \quad x \in M.$$

Moreover, if Φ is selfadjoint (respectively positive, respectively a homomorphism), then $\Phi_{\rm nor}$ and $\Phi_{\rm sing}$ are selfadjoint (respectively positive, respectively homomorphisms).

Proof. By Corollary 7, there is a unique extension of Φ to a normal linear mapping $\Theta : M^{**} \to N$. Then $\Phi_{\rm nor}$ and $\Phi_{\rm sing}$ are defined by

$$\Phi_{\rm nor}(x) = \Theta(xp_M), \quad \Phi_{\rm sing}(x) = \Theta(x(1 - p_M)); \quad x \in M.$$

Since the only linear mapping $M \to N$ which is simultaneously normal and singular is the zero mapping, the unicity of $\Phi_{\rm nor}$ and $\Phi_{\rm sing}$ is obvious. ∎

The mapping Φ_{nor} (respectively Φ_{sing}) is called the *normal part* (respectively the *singular part*) of Φ. Recall that, for $\varphi \in M^*$, we have $\varphi_{\text{nor}} = p_M \cdot \varphi$ and $\varphi_{\text{sing}} = (1 - p_M) \cdot \varphi$.

Note that if $\pi : M \to N$ is a $*$-homomorphism, then $q = \pi(p_M)$ is a central projection in $\overline{\pi(M)}^w$ and

$$\pi_{\text{nor}}(x) = \pi(x)q, \quad \pi_{\text{sing}}(x) = \pi(x)(1-q); \quad x \in M.$$

Using Corollary 9, it is easy to see that a positive linear mapping $\Phi : M \to N$ is singular if and only if there exists no non-zero positive normal mapping $\Psi : M \to N$ such that $\Psi \leqslant \Phi$.

8.5. Topologies on W^*-algebras. As we have seen (8.4, 8.1), every W^*-algebra M is $*$-isomorphic to a von Neumann algebra $R \subset B(H)$ in such a way that the M_*-topology on M corresponds to the R_*-topology on R and the later is the restriction to R of the w-topology on $B(H)$. Consequently, if M is a W^*-subalgebra of the W^*-algebra N, then the w-topology on M is the restriction of the w-topology on N, that is

$$M_* = \{\psi|M;\ \psi \in N_*\}. \tag{1}$$

The following result is similar to Proposition 4.16.

PROPOSITION. *Let M be a W^*-subalgebra of the W^*-algebra N. Then*
(i) *every normal state of M can be extended to a normal state of N;*
(ii) *every singular state of M can be extended to a singular state of N.*

Proof. (i) Let $\varphi \in S(M)$ be normal. By (1) there exists $\theta \in N_*$ with $\theta|M = \varphi$. Replacing θ by $(\theta + \theta^*)/2$, we may assume that θ is selfadjoint. Then, by Corollary 1/4.15, there exist $\rho, \tau \in N^*$ positive such that $\theta = \rho - \tau$. Replacing ρ by $p_N \cdot \rho$ and τ by $p_N \cdot \tau$ we may assume that $\rho, \tau \in N_*$. Consequently, there is $\rho \in N_*$, ρ positive, such that

$$\varphi \leqslant \rho|M.$$

By Corollary 1/4.8 there exists a vector $\eta \in H_{\rho|M}$ such that

$$\varphi = \omega_\eta \circ \pi_{\rho|M}. \tag{2}$$

On the other hand, since M is a C^*-subalgebra of $N, H_{\rho|M}$ can be identified to a closed subspace of H_ρ in such a way that

$$\pi_{\rho|M}(x) = \pi_\rho(x)|H_{\rho|M}; \quad x \in M.$$

With this identification we can define

$$\psi = \omega_\eta \circ \pi_\rho \in N^*.$$

Then ψ is positive and, by (2), $\varphi = \psi|M$. Since $\rho \in N_*$, by Corollary 4/8.4, π_ρ is normal, hence $\psi \in N_*$. Finally, since $\pi_\rho(1_N)$ coincides with $\pi_\rho(1_M)$ on $H_{\rho|M} \ni \eta$, it follows that $\psi(1_N) = \varphi(1_M) = 1$, that is $\psi \in S(N)$.

(ii) Let $\varphi \in S(M)$ be singular. By Proposition 4.16, there exists $\theta \in S(N)$ with $\theta|M = \varphi$. Then

$$\psi = (1 - p_N) \cdot \theta \in N^*$$

is singular, positive and $\|\psi\| \leqslant \|\theta\| = 1$. Since $\psi|M + (p_N \cdot \theta)|M = \theta|M = \varphi$, we have $(p_N \cdot \theta)|M \leqslant \varphi$ and since φ is singular, it follows that $(p_N \cdot \theta)|M$ is singular. But $p_N \cdot \theta \in N_*$ so, by (1), $(p_N \cdot \theta)|M \in M_*$. Consequently $(p_N \cdot \theta)|M = 0$, hence

$$\psi|M = \theta|M = \varphi.$$

Finally, $\psi \in S(N)$ since $\psi(1_M) = \varphi(1_M) = 1$. ∎

Actually, the proof of (ii) gives something more. For instance, if $1_N \in M$, then every linear positive extension on N of a singular state of M is automatically singular on M.

Let M be a W^*-algebra. Besides the w-topology on M, we consider the *s-topology* on M, defined by the seminorms

$$M \ni x \mapsto \varphi(x^*x)^{1/2}; \quad \varphi \in M_*,\ \varphi \text{ positive};$$

and the *s^*-topology* on M, defined by the seminorms

$$M \ni x \mapsto \varphi(x^*x)^{1/2} + \varphi(xx^*)^{1/2}; \quad \varphi \in M_*,\ \varphi \text{ positive}.$$

If $M = B(H)$, then this terminology agrees with that introduced in 7.8.

Note that, by the very definition of these topologies and by Corollary 3/8.4, every $*$-isomorphism between W^*-algebras is s-continuous and s^*-continuous.

If M is a W^*-subalgebra of another W^*-algebra N, then the above proposition shows that the s-topology (respectively s^*-topology) of M is the restriction of the s-topology (respectively s^*-topology) of N.

This holds in particular if M is realized as a von Neumann algebra $M \subset B(H)$. Using Theorem 7.8 and the Hahn-Banach theorem we infer that

Corollary. *Let M be a W^*-algebra. A linear functional on M is w-continuous if and only if it is s^*-continuous.* ∎

Therefore, if τ_w denotes the Mackey topology on M associated to the w-topology, then

$$w \prec s \prec s^* \prec \tau_w.$$

In particular, the closures of a convex subset of M are the same in all these topologies.

By 7.9, the mappings $M \ni x \mapsto xa$, $M \ni x \mapsto ax$, $(a \in M)$, are w, s, and s^*-continuous, the mappings $M_1 \times M \ni (x, y) \mapsto xy$ and $M_1 \times M_1 \ni (x, y) \mapsto xy$ are s-

and respectively s^*-continuous and the map $M \ni x \mapsto x^*$ is w and s^*-continuous. By 7.10, the topologies s and s^* coincide on the set $\{x \in M;\, x \text{ normal}\}$.

By 7.8, the closed unit ball M_1 is complete relative to the uniform structures defined by the s-topology and by the s^*-topology.

By the Kaplansky density teorem, if A is a w-dense $*$-subalgebra of M, then for every $x \in M$ (respectively $x \in M_h$, respectively $x \in M^+$) there is a net $\{x_\iota\}_\iota$ in A (respectively A_h, respectively A^+) such that $\|x_\iota\| \leqslant \|x\|$ for all ι and $x_\iota \xrightarrow{s^*} x$. If additionally A is a C^*-subalgebra of M and $1_M \in A$, then by Proposition 5/7.16, for every unitary $u \in M$ there is a net $\{u_\iota\}_\iota$ of unitaries in A such that $\|1_M - u_\iota\| \leqslant \|1_M - u\|$ for all ι and $u_\iota \xrightarrow{s^*} u$.

Let $\{a_\iota\}_\iota$ be a norm-bounded increasing net of selfadjoint elements of M. By Lemma 2/8.4, there is an element $a = \sup\limits_\iota a_\iota$ in M which is the least upper bound of $\{a_\iota\}_\iota$ with respect to the C^*-algebra order structure of M and $a_\iota \xrightarrow{w} a$. Actually,

$$a_\iota \xrightarrow{s^*} a.$$

Indeed, for every positive $\varphi \in M_*$ we have

$$\varphi((a-a_\iota)^*(a-a_\iota) + (a-a_\iota)(a-a_\iota)^*) = 2\varphi((a-a_\iota)^2) \leqslant 4 \sup_\iota \|a_\iota\|\, \varphi(a-a_\iota) \to 0.$$

The fact that every norm-bounded increasing net $\{a_\iota\}_\iota$ in M_h has a least upper bound a in M_h and $a_\iota \xrightarrow{s^*} a$ is usually called *the Vigier theorem.* In this situation we shall write

$$a_\iota \uparrow a.$$

If $a_\iota \uparrow a$ and $x \in M$, then $x^* a_\iota x \xrightarrow{w} x^* a x$, hence $x^* a_\iota x \uparrow x^* a x$.

If $0 \leqslant a_\iota \uparrow a$, then by 2.6.(3), $\|a_\iota\| \uparrow \|a\|$.

If $a_\iota \uparrow a$ and all a_ι are projections, then a is also a projection.

Let M, N be W^*-algebras and $\pi : M \to N$ be a $*$-homomorphism. By Corollary 2/8.4, π is normal if and only if

$$M_h \ni x_\iota \uparrow x \Rightarrow \pi(x_\iota) \uparrow \pi(x).$$

Using this and the equality $\pi(x)^*\pi(x) = \pi(x^*x)$, $(x \in M)$, it is easy to check that

$$\begin{aligned} \pi \text{ is normal } &\Leftrightarrow \pi \text{ is } s\text{-continuous} \\ &\Leftrightarrow \pi \text{ is } s^*\text{-continuous} \\ &\Leftrightarrow \pi \text{ is continuous with respect to the } s^*\text{-topology on} \\ &\qquad M \text{ and the } w\text{-topology on } N. \end{aligned} \tag{3}$$

8.6. Calculus in W^*-algebras. Since every W^*-algebra can be realized as a von Neumann algebra, all the non-spatial results from 7.10 and 7.12–7.16 can be restated for W^*-algebras.

Thus, let M be a W^*-algebra and $x \in M$. There exists a projection $\mathbf{l}(x) = \mathbf{l}_M(x) \in M$ (respectively $\mathbf{r}(x) = \mathbf{r}_M(x) \in M$), called the left (respectively right) support of x in M, which is the smallest projection $e \in M$ such that $ex = x$ (respectively $xe = x$). If x is normal, then $\mathbf{l}(x) = \mathbf{r}(x)$ is simply called the support of x and is denoted by $\mathbf{s}(x) = \mathbf{s}_M(x)$. The relations 7.12.(1)–(3) are equally valid. Note that if M is a W^*-subalgebra of another W^*-algebra N, then $\mathbf{l}_M(x) = \mathbf{l}_N(x)$, $\mathbf{r}_M(x) = \mathbf{r}_N(x)$ for all $x \in M$.

For every $x \in M$ there exists a unique positive element $|x| \in M$ and a unique partial isometry $v \in M$ such that the polar decomposition (7.12.(5)) holds and then the relations 7.12.(6)–(8) are also valid.

Moreover, the results from Section 7.13 still hold true with obvious reformulations.

Denote by $P(M)$ the set of all projections in M endowed with the natural order structure.

Let $\{e_\iota\}_{\iota \in I}$ be an arbitrary family in $P(M)$. If J is a finite subset of I then, using 2.6.(8) it is easy to check that

$$e_J = \mathbf{s}\Big(\sum_{\iota \in J} e_\iota\Big) \in P(M)$$

is the least upper bound of $\{e_\iota;\ \iota \in J\}$ in $P(M)$. Then $\{e_J\}_{J \subset I,\, J\,\text{finite}}$ is an increasing net of projections and, by the Vigier theorem (8.5),

$$\bigvee_{\iota \in I} e_\iota = \sup_{J \subset I,\, J\,\text{finite}} e_J \in P(M)$$

is the least upper bound of $\{e_\iota;\ \iota \in I\}$ in $P(M)$. Also

$$\bigwedge_{\iota \in I} e_\iota = 1 - \bigvee_{\iota \in I} (1 - e_\iota) \in P(M)$$

is the greatest lower bound of $\{e_\iota;\ \iota \in I\}$ in $P(M)$.

We thus obtain the following result:

PROPOSITION 1. *For every W^*-algebra M, $P(M)$ is an orthocomplemented complete lattice with greatest element* 1, *smallest element* 0 *and orthocomplementation $e \mapsto 1 - e$.* ∎

If $I = \{1, \ldots, n\}$, then we shall write $e_1 \vee \cdots \vee e_n$ or $\bigvee\limits_{k=1}^{n} e_k$ (respectively $e_1 \wedge \cdots \wedge e_n$ or $\bigwedge\limits_{k=1}^{n} e_k$) instead of $\bigvee\limits_{\iota \in I} e_\iota$ (respectively $\bigwedge\limits_{\iota \in I} e_\iota$) Note that

(1) $$e_1 \vee \cdots \vee e_n = \mathbf{s}(e_1 + \cdots + e_n).$$

If the projections e_ι, $(\iota \in I)$, are mutually orthogonal, then

(2) $$\bigvee_{\iota \in I} e_\iota = \sum_{\iota \in I} e_\iota.$$

If $M = B(H)$ and $\{e_\iota\}_{\iota\in I} \subset P(B(H))$, then

$$\bigvee_{\iota\in I} e_\iota = \text{the orthogonal projection onto } \overline{\operatorname{lin}\Big(\bigcup_{\iota\in I} e_\iota H\Big)};$$

$$\bigwedge_{\iota\in I} e_\iota = \text{the orthogonal projection onto } \bigcap_{\iota\in I} e_\iota H.$$

The following result extends Corollary 1.14 in the case of W^*-algebras:

PROPOSITION 2. *Let M be a W^*-algebra and let $\{x_\iota\}_{\iota\in I}$ be a norm-bounded family of elements in M with mutually orthogonal left supports and mutually orthogonal right supports. There exists a unique element $x \in M$ such that*

$$x = s^*\text{-}\lim_{J\subset I,\, J \text{ finite}} \sum_{\iota\in J} x_\iota .$$

Moreover

$$\|x\| = \sup_{\iota\in I} \|x_\iota\|; \tag{3}$$

$$\mathbf{l}(x_\iota)x = x\mathbf{r}(x_\iota) = x_\iota; \quad \iota \in I; \tag{4}$$

$$\mathbf{l}(x) = \sum_{\iota\in I} \mathbf{l}(x_\iota), \quad \mathbf{r}(x) = \sum_{\iota\in I} \mathbf{r}(x_\iota). \tag{5}$$

Proof. Using Corollary 1.14 we obtain $\Big\| \sum_{\iota\in J} x_\iota^* x_\iota \Big\| = \sup_{\iota\in J} \|x_\iota\|^2 \leqslant \Big(\sup_{\iota\in I} \|x_\iota\|\Big)^2$ for each $J \subset I$, J finite. By the Vigier theorem it follows that there is an element $a \in M^+$, $\|a\| = \sup_{\iota\in I} \|x_\iota\|$, such that $\sum_{\iota\in J} x_\iota^* x_\iota \uparrow a$. Then, for every $\varphi \in M_*^+$, we get

$$\begin{aligned}\sum_{\iota\in I} |\varphi(x_\iota)| &= \sum_{\iota\in I} |\varphi(\mathbf{l}(x_\iota)x_\iota)| \leqslant \sum_{\iota\in I} \varphi(\mathbf{l}(x_\iota))^{1/2}\varphi(x_\iota^* x_\iota)^{1/2} \\ &\leqslant \Big(\sum_{\iota\in I} \varphi(\mathbf{l}(x_\iota))\Big)^{1/2}\Big(\sum_{\iota\in I} \varphi(x_\iota^* x_\iota)\Big)^{1/2} \\ &= \varphi\Big(\sum_{\iota\in I} \mathbf{l}(x_\iota)\Big)^{1/2} \varphi(a)^{1/2} < +\infty.\end{aligned}$$

Owing to the remarks preceding Corollary 2/8.4, we infer the existence of a unique element $x \in M$ such that

$$x = w\text{-}\lim_{J\subset I,\, J \text{ finite}} \sum_{\kappa\in J} x_\kappa .$$

Since $\Big(\sum_{\iota\in I} \mathbf{l}(x_\iota)\Big)x_\kappa = x_\kappa$ for all $\kappa \in I$, $\Big(\sum_{\iota\in I} \mathbf{l}(x_\iota)\Big)x = x$ and hence $\mathbf{l}(x) \leqslant \sum_{\iota\in I} \mathbf{l}(x_\iota)$. Since $x_\kappa \mathbf{r}(x_\iota) = 0$ for all $\kappa \neq \iota$, we get $x\mathbf{r}(x_\iota) = x_\iota$. Thus, if $e \in M$ is a projection with $ex = 0$, then $ex_\iota = ex\mathbf{r}(x_\iota) = 0$ so $e\mathbf{l}(x_\iota) = 0$ for all $\iota \in I$, and therefore $e\Big(\sum\limits_{\iota\in I}\mathbf{l}(x_\iota)\Big) = 0$. These and the symmetric arguments prove (4) and (5).

Now

$$x - \sum_{\iota\in J} x_\iota = x(\mathbf{r}(x) - \sum_{\iota\in J}\mathbf{r}(x_\iota)) \xrightarrow[J\subset I,\, J\,\text{finite}]{s^*} 0$$

and

$$x^*x = s^*\text{-}\lim_{J\subset I,\, J\,\text{finite}} \Big(\sum_{\iota\in J} x_\iota\Big)^* \Big(\sum_{\iota\in J} x_\iota\Big) = s^*\text{-}\lim_{J\subset I,\, J\,\text{finite}} \Big(\sum_{\iota\in J} x_\iota^* x_\iota\Big) = a,$$

hence $\|x\| = \|x^*x\|^{1/2} = \|a\|^{1/2} = \sup\limits_{\iota\in I}\|x_\iota\|$. ∎

The element x given by Proposition 2 will be denoted by

$$x = \sum_{\iota\in I} x_\iota.$$

For every W^*-algebra M, its center

$$Z = Z_M = \{z \in M;\ zx = xz \text{ for all } x \in M\}$$

is a W^*-subalgebra of M and for any $x \in M$ there exists a projection $\mathbf{z}(x) \in Z$, called the central support of x, which is the smallest projection $p \in Z$ such that $px = x$ (7.17).

If $e \in M$ is a projection, then

$$\mathbf{z}(e) = \bigvee_{u\in U(M)} u^*eu. \tag{6}$$

Indeed, let $p = \bigvee\limits_{u\in U(M)} u^*eu$. Then $e \leqslant p$ and $v^*pv = p$ for all $v \in U(M)$, hence p is a central projection and $\mathbf{z}(e) \leqslant p$. Conversely, $e \leqslant \mathbf{z}(e)$, so $u^*eu \leqslant u^*\mathbf{z}(e)u = \mathbf{z}(e)$ for all $u \in U(M)$ and therefore $p \leqslant \mathbf{z}(e)$.

For an arbitrary $x \in M$ we have $\mathbf{z}(x) = \mathbf{z}(\mathbf{l}(x)) = \mathbf{z}(\mathbf{r}(x))$ (7.17). It follows that

$$\mathbf{z}(x) = \bigvee_{u\in U(M)} u^*\mathbf{l}(x)u = \bigvee_{u\in U(M)} u^*\mathbf{r}(x)u. \tag{7}$$

On the other hand, Z_h is lattice ordered, because Z is commutative (Corollary 1/4.18), and by the Vigier theorem (8.5) we infer that Z_h is a conditionally complete vector lattice.

Thus, for each $x \in M_h$ we can defined its *central cover* $\mathbf{c}(x) \in Z_h$ as the greatest lower bounded of the set $\{z \in Z_h;\, z \geqslant x\}$. From this definition we infer that

(8) $$x, y \in M_h,\ x \leqslant y \Rightarrow \mathbf{c}(x) \leqslant \mathbf{c}(y);$$

(9) $$x \in M^+ \Rightarrow x \leqslant \mathbf{c}(x) \leqslant \|x\| \Rightarrow \|\mathbf{c}(x)\| = \|x\|;$$

(10) $$x \in M_h \Rightarrow \mathbf{s}(\mathbf{c}(x)) = \mathbf{z}(x).$$

If $e \in M$ is a projection, then $e \leqslant e^2 \leqslant \mathbf{c}(e)^2 \leqslant \mathbf{c}(e)$, because $\|\mathbf{c}(e)\| = \|e\| = 1$, hence $\mathbf{c}(e) = \mathbf{c}(e)^2$ is also a projection. Thus

(11) $$e \in M,\ e \text{ projection} \Rightarrow \mathbf{c}(e) = \mathbf{z}(e).$$

Moreover

(12) $$x \in M_h,\ z \in Z_h \Rightarrow \mathbf{c}(x+z) = \mathbf{c}(x) + z;$$

(13) $$x \in M_h,\ z \in Z^+ \Rightarrow \mathbf{c}(xz) = \mathbf{c}(x)z;$$

(14) $$M_h \ni x_\iota \uparrow x \Rightarrow \mathbf{c}(x_\iota) \uparrow \mathbf{c}(x);$$

(15) $$e_1, \ldots, e_n \in P(M),\ e_1 \leqslant \cdots \leqslant e_n \Rightarrow \mathbf{c}\Big(\sum_{k=1}^n e_k\Big) = \sum_{k=1}^n \mathbf{z}(e_k).$$

Indeed, let $x \in M_h$, $z \in Z_h$. Since $x+z \leqslant \mathbf{c}(x)+z$, we have $\mathbf{c}(x+z) \leqslant \mathbf{c}(x)+z$. Replacing here x by $x+z$ and z by $-z$ we obtain the converse inequality.

If $x \in M_h$ and $z \in Z^+$, then $xz \leqslant \mathbf{c}(x)z$, so $\mathbf{c}(xz) \leqslant \mathbf{c}(x)z$. Replacing here x by $x(z+\varepsilon)$ and z by $(z+\varepsilon)^{-1}$, $(\varepsilon > 0)$, we obtain

$$(z+\varepsilon)\mathbf{c}(x) \leqslant \mathbf{c}(x(z+\varepsilon)) \leqslant \mathbf{c}(xz + \varepsilon\|x\|) \leqslant \mathbf{c}(xz) + \varepsilon\|x\|$$

and letting $\varepsilon \to 0$ we get $\mathbf{c}(x)z \leqslant \mathbf{c}(xz)$.

If $x_\iota \uparrow x$, then $\mathbf{c}(x_\iota) \uparrow z$ for some $z \in Z_h$ by the Vigier theorem. Then $z \geqslant \mathbf{c}(x_\iota) \geqslant x_\iota$ for all ι, so $z \geqslant x$ and consequently $z \geqslant \mathbf{c}(x)$. Conversely, $\mathbf{c}(x_\iota) \leqslant \mathbf{c}(x)$ for all ι, so $z \leqslant \mathbf{c}(x)$.

Finally, let $e_1 \ldots e_n$ be projections in M, $e_0 = 0$ and put $a = \mathbf{c}\Big(\sum\limits_{k=1}^n e_k\Big)$. We have $\mathbf{z}(e_1) \leqslant \cdots \leqslant \mathbf{z}(e_n)$, $\mathbf{z}(e_1 + \cdots + e_n) = \mathbf{z}(e_n)$ and $e_1 + \cdots + e_n \geqslant (n-k+1)e_k$, so $a \geqslant (n-k+1)\mathbf{z}(e_k)$ for all $1 \leqslant k \leqslant n$. It follows that

$$\begin{aligned} a = a\mathbf{z}(e_n) &= \sum_{k=1}^n a(\mathbf{z}(e_k) - \mathbf{z}(e_{k-1})) \geqslant \sum_{k=1}^n (n-k+1)\mathbf{z}(e_k)(\mathbf{z}(e_k) - \mathbf{z}(e_{k-1})) \\ &= \sum_{k=1}^n (n-k+1)(\mathbf{z}(e_k) - \mathbf{z}(e_{k-1})) = \sum_{k=1}^n \mathbf{z}(e_k). \end{aligned}$$

Conversely, $\sum\limits_{k=1}^n e_k \leqslant \sum\limits_{k=1}^n \mathbf{z}(e_k)$, so $a \leqslant \sum\limits_{k=1}^n \mathbf{z}(e_k)$.

Let Ω be a locally compact Hausdorff space. By the Riesz-Kakutani theorem, $C_0(\Omega)^*$ identifies to the Banach space $M(\Omega)$ of all bounded regular Borel measures on Ω. A complex function f on Ω is called *universally measurable* if f is μ-measurable for all $\mu \in M(\Omega)$. The set $\mathrm{Univ}(\Omega)$ of all bounded universally measurable functions on Ω, endowed with pointwise algebraic operations, complex conjugation and sup-norm, is a C^*-algebra. Clearly, $C_0(\Omega)$ and $B(\Omega)$ are C^*-subalgebras of $\mathrm{Univ}(\Omega)$.

A spectral measure on Ω with values in the W^*-algebra M is an M-valued mapping $e(\cdot)$ defined on the family of all Borel subsets of Ω which satisfies the conditions 7.14.1), 7.14.2) and such that, for every $\varphi \in M_*$, the map

$$S \mapsto e_\varphi(S) = \varphi(e(S))$$

is a regular Borel measure on Ω. For every $f \in \mathrm{Univ}(\Omega)$ there exists a unique element $e(f) \in M$ such that

$$\varphi(e(f)) = \int_\Omega f(\omega)\,\mathrm{d}e_\varphi(\omega); \quad \varphi \in M_*,$$

and the map

$$\mathrm{Univ}(\Omega) \ni f \mapsto e(f) \in M$$

is a $*$-homomorphism.

If $\pi : C_0(\Omega) \to M$ is a $*$-homomorphism, then there exists a unique M-valued spectral measure $e^\pi(\cdot)$ on Ω such that

$$\varphi(\pi(f)) = \int_\Omega f(\omega)\,\mathrm{d}e^\pi_\varphi(\omega); \quad f \in C_0(\Omega),\ \varphi \in M_*,$$

and π can be extended to a $*$-homomorphism

$$\pi_{\mathrm{Univ}} : \mathrm{Univ}(\Omega) \to M$$

defined by $\pi_{\mathrm{Univ}}(f) = e^\pi(f)$, $(f \in \mathrm{Univ}(\Omega))$.

In particular, consider the canonical embedding j of $C_0(\Omega)$ in its second dual W^*-algebra $C_0(\Omega)^{**}$. Then j is a $*$-homomorphism. Let $e^j(\cdot)$ be the corresponding $C_0(\Omega)^{**}$-valued spectral measure on Ω and $j_{\mathrm{Univ}} : \mathrm{Univ}(\Omega) \to C_0(\Omega)^{**}$ the corresponding $*$-homomorphism. An easy computation shows that for every $f \in \mathrm{Univ}(\Omega)$ and every $\mu \in M(\Omega) = C_0(\Omega)^* = (C_0(\Omega)^{**})_*$ we have

$$\mu(j_{\mathrm{Univ}}(f)) = \int_\Omega f(\omega)\,\mathrm{d}\mu(\omega).$$

Since $M(\Omega)$ contains the one point supported (or Dirac) measures, we infer that j_{Univ} is injective.

Thus, *via* j_{Univ}, $\mathrm{Univ}(\Omega)$ *can be identified with a* C^**-subalgebra of* $C_0(\Omega)^{**}$.

In the general case, the $*$-homomorphism, $\pi : C_0(\Omega) \to M$ extends by Corollary 6/8.4 to a unique normal $*$-homomorphism, say $\rho^\pi : C_0(\Omega)^{**} \to M$, with $\rho^\pi \circ j = \pi$. Now

$$(\rho^\pi \circ e^j)(\cdot) : S \mapsto \rho^\pi(e^j(S))$$

is a spectral measure on Ω such that

$$\int_\Omega f(\omega)\,\mathrm{d}(\rho^\pi \circ e^j)_\varphi(\omega) = \varphi((\rho^\pi \circ j)(f)) = \varphi(\pi(f)) = \int_\Omega f(\omega)\,\mathrm{d}e^\pi_\varphi(\omega)$$

for all $f \in C_0(\Omega)$ and all $\varphi \in M_*$. It follows that $(\rho^\pi \circ e^j)(\cdot) = e^\pi(\cdot)$ and consequently

$$\rho^\pi \circ j_{\mathrm{Univ}} = \pi_{\mathrm{Univ}}.$$

In other words, regarding $\mathrm{Univ}(\Omega)$ as a C^*-subalgebra of $C_0(\Omega)^{**}, \pi_{\mathrm{Univ}}$ *is the restriction to* $\mathrm{Univ}(\Omega)$ *of the unique normal extension of* π *to* $C_0(\Omega)^{**}$.

Let x be a normal element of the W^*-algebra M. By Theorem 7.15 and 7.15.(7), the continuous functional calculus (1.16) $C(\sigma(x)) \ni f \mapsto f(x) \in M$ extends to a unique map

$$B(\sigma(x)) \ni f \mapsto f(x) \in M,$$

called the Borel functional calculus, such that if $\{f_n\}_n$ is a norm-bounded sequence in $B(\sigma(x))$ which converges pointwise to $f \in B(\sigma(x))$, then $f_n(x) \xrightarrow{w} f(w)$. Moreover, with obvious reformulations, all the results from 7.10, 7.15 and 7.16 are still valid in this setting.

8.7. Ideals in W^*-algebras. The study of w-closed ideals in W^*-algebras leads to a remarkable algebraic property of W^*-algebras, namely every such ideal is a principal ideal, which is almost characteristic for W^*-algebras (see (B_l) and (B_r) in Section 9.35).

First we remark that, being a convex set, every ideal N of a W^*-algebra M has the same closure with respect to the topologies w, s, s^*, i.e. $\overline{N}^w = \overline{N}^s = \overline{N}^{s^*}$, which is again an ideal.

PROPOSITION 1. *Let M be a W^*-algebra and $N \subset M$ be a left ideal. There exists a unique projection $e_N \in M$ such that*

$$\overline{N}^w = Me_N.$$

Every increasing right approximate unit for N is s^*-convergent to e_N. In particular,

$$\overline{(N_1)}^w = (\overline{N}^w)_1.$$

Proof. Let $\{u_\iota\}_{\iota\in I}$ be an increasing right approximate unit for N (Theorem 3.2). By the Vigier theorem (8.5), $u_\iota \uparrow e$ for some $e \in M^+$, hence $e \in \overline{N}^w$ and $Me \subset \overline{N}^w$. For each $x \in N$ we have $\|xu_\iota - x\| \to 0$ and $xu_\iota \xrightarrow{s} xe$, so $x = xe$. It follows that $\overline{N}^w \subset Me$. In particular $e^2 = e$ is a projection.

The uniqueness of e_N is immediate.

If $x \in (\overline{N}^w)_1$, then $xu_\iota \in N_1$ and $xu_\iota \xrightarrow{s} x$. This proves the last assertion in the statement. ∎

COROLLARY. *Let M be a W^*-algebra. Then*

(i) *A subset N of M is a w-closed left (respectively right) ideal if and only if $N = Me$ (respectively $N = eM$) for some projection $e \in M$.*

(ii) *A subset J of M is a w-closed two-sided ideal if and only if $J = Mp$ for some central projection $q \in M$.*

(iii) *A subset A of M is a w-closed facial subalgebra if and only if $A = eMe$ for some projection $e \in M$.*

(iv) *A subset F of M^+ is a w-closed face if and only if $F = eM^+e$ for some projection $e \in M$.*

Proof. (i) is clear.

(ii). Let $J \subset M$ be a w-closed two-sided ideal. Then $J = Mp$ for some projection $p \in M$. For every unitary $u \in M$ we have

$$Mp = J = u^*Ju = M(u^*pu),$$

hence $p = u^*pu$, $up = pu$. It follows that p is central.

(iii) follows from (i) and 3.9.(iii).

(iv) follows from (iii) and 3.9.(ii). ∎

If $M \subset B(H)$ is a von Neumann algebra, then we can also consider the topologies *wo* and *so* on M, with respect to which the convex sets have the same closure (7.7). From the above corollary it follows that *every w-closed left ideal of M (respectively facial subalgebra of M, respectively face of M^+) is also wo-closed*, the converse being trivial since $wo \prec w$.

On the other hand, *all the results contained in* 3.9 *can be restated for W^*-algebras replacing the norm topology by the w-topology.* The proofs are quite similar, the only difference is that we have to approximate an element $x \in \overline{N}^w$ by a norm-bounded net $x_\iota \xrightarrow{s^*} x$ and then $x_\iota^* x_\iota \xrightarrow{w} x^*x$.

Also not necessarily closed ideals in W^*-algebras have some special properties, due to the more refined factorizations which are valid in W^*-algebras (7.13).

PROPOSITION 2. *Let M be a W^*-algebra and N be a left ideal of M. Then $F = (N^*N) \cap M^+$ is a face of M^+ and*

$$N = N_F = \{x \in M; x^*x \in F\}, \quad N^*N = M_F = \operatorname{lin} F.$$

In particular, N^*N is a facial subalgebra of M.

Proof. Clearly, $N \subset \{x \in M; x^*x \in F\}$ and, by the polarization relation 2.8.(1), $N^*N = \operatorname{lin} F$.

Let $x \in M$ such that $x^*x \leqslant b \in F$. Since b is selfadjoint, using again the polarization relation, we can find $x_k, y_k \in N$, $(1 \leqslant k \leqslant n)$, such that

$$x^*x \leqslant b = \sum_{k=1}^n x_k^*x_k - \sum_{k=1}^n y_k^*y_k \leqslant \sum_{k=1}^n x_k^*x_k = a.$$

By Proposition 7.13 there are $z, z_k \in M$, $(1 \leqslant k \leqslant n)$, such that

$$x = za^{1/2}, \quad x_k = z_k a^{1/2} \quad \text{and} \quad \sum_{k=1}^{n} z_k^* z_k = \mathbf{s}(a).$$

It follows that

$$x = za^{1/2} = z\Big(\sum_{k=1}^{n} z_k^* z_k\Big)a^{1/2} = \sum_{k=1}^{n} z z_k^* x_k \in N.$$

This shows that F is a face and $N_F \subset N$. ∎

In particular, $x \in N \Leftrightarrow |x| \in N$. This also follows using the polar decomposition of x.

Proposition 3. *Let M be a W^*-algebra. Then*
(i) *Every invariant face F of M^+ is strongly invariant.*
(ii) *Every two-sided ideal J of M^+ is strongly facial.*

Proof. (i) Let $x \in M$ such that $x^*x \in F$ and let $x = v|x|$, $v \in M$, be the polar decomposition of x. Then

$$xx^* = v|x|^2v^* = v(x^*x)v^* \in F.$$

(ii) Let $F = J \cap M^+$. If $x \in M$, $0 \leqslant x \leqslant a \in F$, then $x^{1/2} = za^{1/2}$ for some $z \in M$ (7.13), hence $x = zaz^* \in F$. Thus F is a face of M^+ and is invariant because J is a two-sided. By (i), F is strongly invariant.

If $x \in J$ with polar decompresition $x = v|x|$, $v \in M$, then $|x| = v^*x \in J$ and $x^* = |x|v^* \in J$. Therefore J is selfadjoint.

If $x \in J$, $x^* = x$, then $x = x^+ - x^-$ with $x^+, x^- \in M^+$ and $\mathbf{s}(x^+)\mathbf{s}(x^-) = 0$ (7.12.(8)), so $x^+ = x\mathbf{s}(x^+) \in J$, $x^- = -x\mathbf{s}(x^-) \in J$.

It follows that $J = \operatorname{lin}(J \cap M^+) = \operatorname{lin} F$. ∎

Thus, Corollary 2.9.(iii) yields a one-to-one correspondence between two-sided ideals of a W^*-algebra M and strongly invariant faces of M^+.

Proposition 4. *Let J be a two-sided ideal of the W^*-algebra M. Then:*
(i) *For every $x \in \overline{J}^w \cap M^+$ there is an increasing net $\{x_\iota\}_\iota$ in $J \cap M^+$ such that $x_\iota \uparrow x$.*
(ii) *For every projection $e \in \overline{J}^w$ there is a family $\{e_\iota\}_\iota$ of mutually orthogonal projections in J such that $e = \sum_\iota e_\iota$.*

Proof. (i) Let $\overline{J}^w = Mp$ for some central projection $p \in M$ and $\{u_\iota\}_\iota$ be an increasing approximate unit for J. Then $\{x^{1/2}u_\iota x^{1/2}\}_\iota$ is an increasing net in $J \cap M^+$ and $x^{1/2}u_\iota x^{1/2} \uparrow x^{1/2}px^{1/2} = x$.

(ii) Let $e \in \overline{J}^w$ be a non-zero projection. By (i), there exists $x \in J \cap M^+$, $0 \neq x \leqslant e$. By Proposition 1/7.16 there exists $\alpha > 0$ and a projection $f \in W^*(\{x\}) \subset M$ such that $0 \neq f \leqslant \alpha^{-1}fxf \in J \subset M^+$. Then $f \leqslant e$ and $f \in J$ because $J \cap M^+$ is a face of M^+, by Proposition 3.

Let $\{e_\iota\}_\iota$ be a maximal family of mutually orthogonal projections in J with $e_\iota \leqslant e$ for all ι. By the above we get $e = \sum_\iota e_\iota$. ∎

8.8. W^*-tensor product. In this section we define a notion of tensor product for W^*-algebras corresponding to the tensor product of von Neumann algebras.

Let M, N be W^*-algebras. Then the set

$$J = \{x \in (M \otimes_{C^*} N)^{**};\ (\varphi \otimes \psi)(x) = 0 \text{ for all } \varphi \in M_*,\ \psi \in N_*\}$$

is a w-closed two-sided ideal of $(M \otimes_{C^*} N)^{**}$.

Indeed, J is clearly a w-closed vector subspace of $(M \otimes_{C^*} N)^{**}$. If $x \in J$, $a \in M$, $b \in N$, $\varphi \in M_*$, $\psi \in N_*$, then $\varphi(a\,\cdot) \in M_*$, $\psi(b\,\cdot) \in N_*$ and we have

$$(\varphi \otimes \psi)((a \otimes b)x) = (\varphi(a\,\cdot) \otimes \psi(b\,\cdot))(x) = 0.$$

So $(M \otimes N)J \subset J$ and similarly $J(M \otimes N) \subset J$. Since $M \otimes N$ is w-dense in $(M \otimes_{C^*} N)^{**}$, it follows that J is a two-sided ideal.

By Corollary 8.7 there exists a unique central projection $p_{M,N}$ in $(M \otimes_{C^*} N)^{**}$ such that

$$J = p_{M,N}(M \otimes_{C^*} N)^{**}.$$

Let $\Phi : M \to B(H)$, $\Psi : N \to B(K)$ be normal completely positive mappings. By Corollary 1/5.3, there exists a unique completely positive linear mapping $\Phi \otimes \Psi : M \otimes_{C^*} N \to B(H \mathbin{\overline{\otimes}} K)$ such that

$$(\Phi \otimes \Psi)(a \otimes b) = \Phi(a) \mathbin{\overline{\otimes}} \Psi(b); \quad a \in M,\ b \in N.$$

Furthermore, by Corollary 7/8.4, there exists a unique extension of $\Phi \otimes \Psi$ to a normal linear mapping

$$\Theta : (M \otimes_{C^*} N)^{**} \to B(H \mathbin{\overline{\otimes}} K).$$

Clearly, Θ is completely positive.

LEMMA. *With the above notations we have*

$$\operatorname{Ker}\Theta \supset J. \tag{1}$$

If in addition Φ and Ψ are both injective, then

$$\operatorname{Ker}\Theta = J. \tag{2}$$

Proof. Let $\xi_1, \xi_2 \in H$ and $\eta_1, \eta_2 \in K$. Then $\omega_{\xi_1,\xi_2} \circ \Phi \in M_*$, $\omega_{\eta_1,\eta_2} \circ \Psi \in N_*$ and

$$(\Theta(x)(\xi_1 \otimes \eta_1)|\xi_2 \otimes \eta_2) = [(\omega_{\xi_1,\xi_2} \circ \Phi) \otimes (\omega_{\eta_1,\eta_2} \circ \Psi)](x) \tag{3}$$

for all $x \in M \otimes N$, hence for all $x \in (M \otimes_{C^*} N)^{**}$.

Thus, if $x \in J$, then $(\Theta(x)(\xi_1 \otimes \eta_1)|\xi_2 \otimes \eta_2) = 0$ for all $\xi_1, \xi_2 \in H$, $\eta_1, \eta_2 \in K$, that is $\Theta(x) = 0$. This proves (1).

Now assume that Φ, Ψ are injective. Then ${}^t\Phi(B(H)_*)$ is norm-dense in M_*, hence

$$\operatorname{lin}\{\omega_{\xi_1,\xi_2} \circ \Phi;\ \xi_1, \xi_2 \in H\} \text{ is norm-dense in } M_*. \tag{4}$$

Similarly,

$$\operatorname{lin}\{\omega_{\eta_1,\eta_2} \circ \Psi;\ \eta_1, \eta_2 \in K\} \text{ is norm-dense in } N_*. \tag{5}$$

If $x \in \operatorname{Ker}\Theta$, then by (3) $[(\omega_{\xi_1,\xi_2} \circ \Phi) \otimes (\omega_{\eta_1,\eta_2} \circ \Psi)](x) = 0$ for all $\xi_1, \xi_2 \in H$, $\eta_1, \eta_2 \in K$, hence by (4), (5) and 4.20.(8), $(\varphi \otimes \psi)(x) = 0$ for all $\varphi \in M_*$, $\psi \in N_*$, that is $x \in J$. ∎

By 8.4 there exist injective non-degenerate normal $*$-representations $\rho : M \to B(H)$, $\sigma : N \to B(K)$. If π is the normal extensions of $\rho \otimes \sigma$ to $(M \otimes_{C^*} N)^{**}$, then by the above lemma

$$\operatorname{Ker} \pi = J.$$

Thus $\pi|(1 - p_{M,N})(M \otimes_{C^*} N)^{**}$ is a $*$-isomorphism of the W^*-subalgebra $(1-p_{M,N})(M \otimes_{C^*} N)^{**}$ of $(M \otimes_{C^*} N)^{**}$ onto $\rho(M) \overline{\otimes} \sigma(N)$.

The W^*-algebra $(1 - p_{M,N})(M \otimes_{C^*} N)^{**}$ is called the *W^*-tensor product* of M and N and is denoted by $M \otimes_{W^*} N$. Clearly, the natural duality between $(M \otimes_{C^*} N)^*$ and $(M \otimes_{C^*} N)^{**}$induces a duality between the norm-closure of $M_* \otimes N_* \subset M^* \otimes N^* \subset (M \otimes_{C^*} N)^*$ and $M \otimes_{W^*} N$. Thus $M \otimes_{W^*} N$ becomes the dual space of the norm-closure of $M_* \otimes N_*$ in $(M \otimes_{C^*} N)^*$. Hence *the predual $(M \otimes_{W^*} N)_*$ of $M \otimes_{W^*} N$ is the norm-closure of $M_* \otimes N_*$ in $(M \otimes_{C^*} N)^*$.*

The C^*-tensor product $M \otimes_{C^*} N$ can be canonically identified with a w-dense C^*-subalgebra of the W^*-tensor product $M \otimes_{W^*} N$.

Indeed, let ρ, σ, π be as above. Then the map

$$M \otimes_{C^*} N \ni x \mapsto (\rho \otimes \sigma)(x) = \pi((1 - p_{M,N})x) \in B(H \overline{\otimes} K)$$

is injective (4.20), hence the map

$$M \otimes_{C^*} N \ni x \mapsto (1 - p_{M,N})x \in M \otimes_{W^*} N$$

is an injective $*$-homomorphism and, clearly, its range is w-dense.

Note also that, for fixed $0 \neq a_0 \in M$, $0 \neq b_0 \in N$, the linear mappings

$$M \ni a \mapsto a \otimes b_0 \in M \otimes_{W^*} N$$
$$M \ni b \mapsto a_0 \otimes b \in M \otimes_{W^*} N$$

are injective and normal. Moreover, if a_0, b_0 are selfadjoint (respectively positive, respectively $1_M, 1_N$), then the above mappings are selfadjoint (respectively completely positive, respectively $*$-homomorphisms).

As we have seen, if $M \subset B(H)$, $N \subset B(K)$ are von Neumann algebras then $M \otimes_{W^*} N$ can be naturally identified with $M \overline{\otimes} N$. For this reason, for arbitrary W^*-algebras M, N we shall denote $M \otimes_{W^*} N$ also by $M \overline{\otimes} N$.

Using the above lemma and 8.4 it is easy to prove:

Proposition. *Let $\Phi : M \to R$, $\Psi : N \to S$ be normal completely positive linear mapping between W^*-algebras. There exists a unique normal completely positive linear mapping $\Phi \overline{\otimes} \Psi : M \overline{\otimes} N \to R \overline{\otimes} S$ such that*

$$(\Phi \overline{\otimes} \Psi)(a \otimes b) = \Phi(a) \otimes \Psi(b); \quad a \in M, b \in N.$$

Moreover, $\Phi \overline{\otimes} \Psi$ is injective if and only if Φ, Ψ are both injective and $\Phi \overline{\otimes} \Psi$ is a $*$-homomorphism if and only if Φ, Ψ are both $*$-homomorphism. ∎

In particular, if M, N are W^*-subalgebras of R, S then the map

$$M \otimes N \ni \sum_{k=1}^{n} a_k \otimes b_k \mapsto \sum_{k=1}^{n} a_k \otimes b_k \in R \otimes S$$

can be uniquely extended to a normal injective $*$-homomorphism of $M \overline{\otimes} N$ into $R \overline{\otimes} S$, that is we can identify $M \overline{\otimes} N$ with the W^*-subalgebra of $R \overline{\otimes} S$ generated by $\{a \otimes b;\, a \in M,\, b \in N\}$.

Note also that for every projections $e \in M$, $f \in N$ we have

$$(e \otimes f)(M \overline{\otimes} N)(e \otimes f) = eMe \overline{\otimes} fNf.$$

If A, B are C^*-algebras, then

$$A \otimes_{C^*} B \subset A^{**} \otimes_{C^*} B^{**} \subset A^{**} \overline{\otimes} B^{**}$$

and, by Corollary 8/8.4, this inclusion map extends to surjective normal $*$-homomorphism

$$(A \otimes_{C^*} B)^{**} \to A^{**} \overline{\otimes} B^{**}.$$

Although $A^* \otimes B^*$ is $(A \otimes_{C^*} B)$-dense in $(A \otimes_{C^*} B)^*$, $A^* \otimes B^*$ is not necessarily norm-dense in $(A \otimes_{C^*} B)^*$, that is the above map is not necessarily a $*$-isomorphism.

On the other hand, let $\{M_\iota\}_{\iota \in I}$ be a family of W^*-algebras and let M be the C^*-direct product of the family $\{M_\iota\}_{\iota \in I}$. Then M is a W^*-algebra.

Indeed, every $\varphi_{\iota_0} \in M_{\iota_0}^*$ defines an element of M^*, still denoted by φ_{ι_0}, by the formula

$$\varphi_{\iota_0}(\{x_\iota\}_{\iota \in I}) = \varphi_{\iota_0}(x_{\iota_0}); \quad \{x_\iota\}_{\iota \in I} \in M$$

and it is easy to check that M is the dual space of the Banach space

$$M_* = \Big\{ \sum_{\iota \in I} \varphi_\iota;\ \varphi_\iota \in (M_\iota)_*,\ \sum_{\iota \in I} \|\varphi_\iota\| < +\infty \Big\} \subset M^*.$$

For this reason M will be also called the W^*-*direct product* of the family $\{M_\iota\}_{\iota \in I}$.

Let $\pi_\iota : M_\iota \to B(H_\iota)$ be injective normal $*$-representations, $(\iota \in I)$, and $H = \bigoplus_{\iota \in I} H_\iota$. For every $x = \{x_\iota\}_{\iota \in I} \in M$ and every $\xi = \{\xi_\iota\}_{\iota \in I} \in H$ define

$$\pi(x)\xi = \{\pi_\iota(x_\iota)\xi_\iota\}_{\iota \in I}.$$

Then the map $\pi : M \to B(H)$ is an injective normal $*$-representation and $\pi(M)$ is the von Neumann algebra direct product of the von Neumann algebras $\pi_\iota(M_\iota) \subset B(H_\iota)$.

If all M_ι are equal to $\mathbb{C}$, then the W^*-direct product of the family $\{M_\iota\}_{\iota \in I}$ is the W^*-algebra $\ell^\infty(I)$ of all bounded complex families $\{\lambda_\iota\}_{\iota \in I}$ and its predual is the Banach space $\ell^1(I)$ of all absolutely summable complex families $\{\alpha_\iota\}_{\iota\in I}$. As a matter of notation, $\ell^\infty(I) = C(I)$ where I is endowed with the discrete topology.

Finally, note that the usual associativity and distributivity properties are valid for tensor products and direct products of W^*-algebras.

8.9. Polar decomposition or linear functionals. Let M be a W^*-algebra and $\varphi \in M_*$. Put

$$L_\varphi = \{x \in M;\ \varphi(ax) = 0 \text{ for all } a \in M\},$$
$$R_\varphi = \{x \in M;\ \varphi(xa) = 0 \text{ for all } a \in M\}.$$

Then L_φ (respectively R_φ) is a w-closed left (respectively right) ideal in M so, by Corollary 8.7, there exists a unique projection $e \in M$ (respectively $f \in M$) with $L_\varphi = Me$ (respectively $R_\varphi = fM$). The projection $\mathbf{r}(\varphi) = \mathbf{r}_M(\varphi) = 1 - e$ (respectively $\mathbf{l}(\varphi) = \mathbf{l}_M(\varphi) = 1 - f$) is called the *right support* (respectively the *left support*) of φ in M. Thus

$$L_\varphi = M(1 - \mathbf{r}(\varphi)), \quad R_\varphi = (1 - \mathbf{l}(\varphi))M. \tag{1}$$

Since $1 - \mathbf{r}(\varphi) \in L_\varphi$, $1 - \mathbf{l}(\varphi) \in R_\varphi$, we get

$$\varphi = \varphi(\cdot\, \mathbf{r}(\varphi)) = \varphi(\mathbf{l}(\varphi)\, \cdot). \tag{2}$$

Also, using the Hahn-Banach theorem, from (1) we infer that

$$\begin{aligned} &\{\varphi(a\,\cdot);\ a \in M\} \text{ is norm-dense in } M_* \cdot \mathbf{r}(\varphi),\\ &\{\varphi(\cdot\, a);\ a \in M\} \text{ is norm-dense in } \mathbf{l}(\varphi) \cdot M_*. \end{aligned} \tag{3}$$

Since $L_\varphi = (L_{\varphi^*})^*$, it follows that

$$\mathbf{r}(\varphi) = \mathbf{l}(\varphi^*). \tag{4}$$

In particular, if $\varphi \in M_*$ is selfadjoint, then $\mathbf{r}(\varphi) = \mathbf{l}(\varphi)$ is called simply the *support* of φ and is denoted by $\mathbf{s}(\varphi) = \mathbf{s}_M(\varphi)$.

If $\varphi \in M_*$ is positive, then the Schwarz inequality shows that

$$\{x \in M;\ \varphi(x^*x) = 0\} = L_\varphi = M(1 - \mathbf{s}(\varphi))$$

hence, for $x \in M$,

$$\varphi(x^*x) = 0 \Leftrightarrow x\mathbf{s}(\varphi) = 0. \tag{5}$$

Thus φ is faithful (4.3) if and only if $\mathbf{s}(\varphi) = 1$.

Remark that, in a certain sense, the singularity of a linear functional means the non existence of support projections.

For instance, if $\varphi \in M^*_+$ is faithful, then the normal part $p_M \cdot \varphi$ of φ is also faithful. Indeed, if $\mathbf{s}(p_M \cdot \varphi) \neq 1$, then there exists a non-zero projection $e \in M$, $e \leqslant 1 - \mathbf{s}(p_M \cdot \varphi)$ with $((1 - p_M) \cdot \varphi)(e) = 0$ and then $\varphi(e) = (p_M \cdot \varphi)(e) = 0$, in contradiction with the faithfulness of φ.

Theorem. *Let M be a W^*-algebra and $\varphi \in M_*$. There exist $\rho \in M_*$, $\rho \geqslant 0$ and a partial isometry $v \in M$, uniquely determined such that*

$$\varphi = \rho(\cdot\, v), \quad v^* v = \mathbf{s}(\rho).$$

Proof. Since M_1 is w-compact and φ is w-continuous, $\{x \in M_1;\, \varphi(x) = \|\varphi\|\}$ is a non-void w-compact and convex subset of M. Let u be an extreme point of $\{x \in M_1;\, \varphi(x) = \|\varphi\|\}$. Then u is an extreme point of M_1 and, by 6.1, u is a partial isometry.

Define $\rho = \varphi(\cdot\, u) \in M_*$. Since $\rho(1) = \varphi(u) = \|\varphi\| \geqslant \|\rho\| \geqslant \rho(1)$, by Proposition 4.6 we infer that $\rho \geqslant 0$.

Define $v = u^*\mathbf{s}(\rho)$. Since u is a partial isometry, we have $\rho(1 - uu^*) = \varphi(u - uu^*u) = 0$ so, by (5), $uu^* \geqslant \mathbf{s}(\rho)$. Consequently

$$\begin{aligned} v^*v &= \mathbf{s}(\rho), \\ \rho(x) = \rho(x\mathbf{s}(\rho)) &= \varphi(x\mathbf{s}(\rho)u) = \varphi(xv^*); \quad x \in M. \end{aligned}$$

Assume that there exists $x \in M_1$ such that

$$\alpha = \varphi(x(1 - vv^*)) > 0.$$

Then, for each integer $n \geqslant 1$ we have

$$\begin{aligned} \|nv^* + x(1 - vv^*)\| &= \|(nv^* + x(1 - vv^*))(nv - (1 - vv^*)x^*)\|^{1/2} \\ &= \|n^2v^*v + x(1 - vv^*)x^*\|^{1/2} \leqslant (n^2 + 1)^{1/2}, \end{aligned}$$

thus

$$\|\varphi\| n + \alpha = \rho(n) + \varphi(x(1 - vv^*)) = \varphi(nv^* + x(1 - vv^*)) \leqslant \|\varphi\|(n^2 + 1)^{1/2},$$

which is not possible for sufficiently large n. It follows that $\varphi(x(1 - vv^*)) = 0$ for all $x \in M$, that is

$$\varphi = \rho(\cdot\, v).$$

Now let ρ, ρ' be normal positive forms on M and $v, v' \in M$ be partial isometries such that $\varphi = \rho(\cdot\, v) = \rho'(\cdot\, v')$ and $v^*v = \mathbf{s}(\rho)$, $v'^*v' = \mathbf{s}(\rho')$. Then $\|\varphi\| = \|\rho\| = \|\rho'\|$. On the other hand

$$\begin{aligned} \rho(1) = \rho(v^*v) = \varphi(v^*) &= \rho'(v^*v') = \rho'(v'^*v'v^*v') \\ &= \varphi(v'^*v'v^*) = \rho(v'^*v'v^*v) = \rho(v'^*v') \end{aligned}$$

so, by (5), $v'^*v' \geqslant \mathbf{s}(\rho) = v^*v$. Similarly, $v^*v \geqslant v'^*v'$. Put

$$e = v^*v = v'^*v'.$$

Since $v'^*v = v'^*v'v'^*vv^*v = ev'^*ve \in eMe$, we can write

$$v'^*v = a + \mathrm{i}b$$

with $a, b \in eMe$ selfadjoint and $\|a\|$, $\|b\| \leqslant 1$. We have

$$\rho(a) + \mathrm{i}\rho(b) = \rho(v'^* v) = \varphi(v'^*) = \rho'(v'^* v') = \|\rho'\| = \|\rho\|$$

so $\rho(a) = \|\rho\|$. It follows that $\rho(e - a) = 0$. Since $e - a \geqslant 0$, we infer that $a = e$. Consequently, $\|e + \mathrm{i}b\| \leqslant 1$, so $b = 0$. Hence

$$v'^* v = e = v^* v'$$

and succesively

$$\begin{aligned} v'v'^*v'v' &= v'ev'^* = v'eev'^* = v'v'^*vv^*v'v'^*, \\ v'v'^*(1 - vv^*)v'v'^* &= 0, \\ (1 - vv^*)v'v'^* &= 0, \\ v'v'^* &\leqslant vv^*. \end{aligned}$$

Similarly, $vv^* \leqslant v'v'^*$. Put

$$f = vv^* = v'v'^*.$$

We conclude

$$v = vv^*v = fv = v'v'^*v = v'e = v'v'^*v' = v',$$

$$\rho(x) = \rho(xv^*v) = \varphi(xv^*) = \rho'(xv^*v') = \rho'(xv'^*v') = \rho'(x), \quad x \in M. \quad \blacksquare$$

If $\varphi \in M_*$ and ρ, v are as in the above theorem, then we denote $|\varphi| = \rho$ and call $|\varphi|$ the *modulus* (or the *absolute value*) of φ. The relations

(6) $$\varphi = |\varphi|(\cdot\, v), \quad v^*v = \mathbf{s}(|\varphi|)$$

are called the *polar decomposition* of φ. Since

$$\varphi^* = (v^* \cdot |\varphi| \cdot v)(\cdot\, v^*), \quad vv^* = \mathbf{s}(v^* \cdot |\varphi| \cdot v),$$

by the unicity of the polar decomposition of φ^* we infer that $|\varphi^*| = v^* \cdot |\varphi| \cdot v$ and

(7) $$\varphi = |\varphi^*|(v\, \cdot), \quad vv^* = \mathbf{s}(|\varphi^*|).$$

Note that $\||\varphi|\| = \|\varphi\|$ and

(8) $$\mathbf{l}(\varphi) = \mathbf{s}(|\varphi|) = v^*v, \quad \mathbf{r}(\varphi) = \mathbf{s}(|\varphi^*|) = vv^*.$$

The following result is a characterization of the modulus of a normal form.

Proposition. *Let A be a w-dense $*$-subalgebra of the W^*-algebra M and $\varphi \in M_*$. Then $|\varphi|$ is the unique normal positive form ρ on M such that*

(9) $$\|\rho\| \leqslant \|\varphi\|, \quad |\varphi(x)|^2 \leqslant \|\varphi\|\, \rho(xx^*) \quad \textit{for all } x \in A.$$

Proof. Let $\varphi = |\varphi|(\cdot\, v)$ be the polar decomposition of φ. Clearly, $\|\,|\varphi|\,\| = \|\varphi\|$ and, for every $x \in M$,

$$|\varphi(x)|^2 = |\,|\varphi|(xv)|^2 \leqslant |\varphi|(xx^*)|\varphi|(v^*v) = \|\varphi\|\,|\varphi|(xx^*).$$

Conversely, suppose that $\rho \in M_*^+$ satisfies (9). Then

$$\|\varphi\|^2 = \sup\{|\varphi(x)|^2;\ x \in M_1\} \leqslant \|\varphi\| \sup\{\rho(xx^*);\ x \in M_1\} \leqslant \|\varphi\|\,\|\rho\|,$$

hence $\|\rho\| = \|\varphi\|$. Using the Kaplansky density theorem we infer that

$$|\varphi(x)|^2 \leqslant \|\varphi\|\,\rho(xx^*) \quad \text{for all } x \in M,$$

hence

$$|\,|\varphi|(x)|^2 = |\varphi(xv^*)|^2 \leqslant \|\rho\|\,\rho(xv^*vx^*) \leqslant \|\rho\|\,\rho(xx^*); \quad x \in M,$$
$$|\,|\varphi|(x)| = |\,|\varphi|(x^*)| \leqslant \|\rho\|^{1/2}\rho(x^*x)^{1/2}; \quad x \in M.$$

The last inequality shows that the map

$$H_\rho \supset \pi_\rho(M)\xi_\rho \ni \pi_\rho(x)\xi_\rho \mapsto |\varphi|(x)$$

is well defined and can be extended to a bounded linear functional on H_ρ with norm $\leqslant \|\rho\|^{1/2}$. Hence there exists $\eta \in H_\rho$,

$$\|\eta\|_\rho \leqslant \|\rho\|^{1/2} = \|\xi_\rho\|_\rho \tag{10}$$

such that

$$|\varphi|(x) = (\pi_\rho(x)\xi_\rho|\eta)_\rho; \quad x \in M.$$

Then

$$(\xi_\rho|\eta)_\rho = |\varphi|(1) = \|\varphi\| = \|\rho\| = (\xi_\rho|\xi_\rho)_\rho. \tag{11}$$

From (10) and (11) it follows that $\eta = \xi_\rho$, hence $\rho = |\varphi|$. ∎

Using the polar decomposition, several problems concerning normal forms can be reduced to normal positive forms. For instance, a weak form of the polar decomposition (Lemma 2/7.8) has been already used in 7.8. As another illustration we prove the following strenghtened form of Proposition 8.5:

COROLLARY. *Let M be a W^*-subalgebra of the W^*-algebra N. Then*

(i) *Every normal form φ on M can be extended to a normal form ψ on N such that $\|\psi\| = \|\varphi\|$.*

(ii) *Every singular form φ on M can be extended to a singular form ψ on N such that $\|\psi\| = \|\varphi\|$.*

Moreover, in both statements, if φ is selfadjoint (respectively positive), then ψ can be chosen also selfadjoint (respectively positive).

Proof. The case of positive forms has been already treated (8.5) and the case of selfadjoint forms is an immediate consequence of the general case.

(i) Let $\varphi \in M_*$ and $\varphi = |\varphi|(\cdot\, v)$ be its polar decomposition. By Proposition 8.5, $|\varphi|$ extends to a normal positive form ρ on N such that $\|\rho\| = \|\,|\varphi|\,\| = \|\varphi\|$. Then $\psi = \rho(\cdot\, v) \in N_*$ extends φ and $\|\psi\| \leqslant \|\rho\| = \|\varphi\|$.

(ii) Let $\varphi \in M^*$ be singular. Then $\varphi \in (M^{**})_*$ has a polar decomposition $\varphi = |\varphi|(\cdot\, v)$, $(v \in M^{**})$, relative to the W^*-algebra M^{**}. Since $|\varphi| = \varphi(\cdot\, v^*)$, $|\varphi| \in M^*$ is singular. By Proposition 8.5, $|\varphi|$ extends to a singular positive form ρ on N such that $\|\rho\| = \|\,|\varphi|\,\| = \|\varphi\|$. Then $\psi = \rho(\cdot\, v) \in N^*$ is singular, $\psi|M = \varphi$ and $\|\psi\| \leqslant \|\rho\| = \|\varphi\|$. ∎

8.10. The Jordan decomposition of linear functionals. In this section we examine the polar decomposition of selfadjoint normal forms.

THEOREM. *Let M be a W^*-algebra and $\varphi \in M_*$, $\varphi = \varphi^*$. There exist $\varphi_1, \varphi_2 \in M_*$, positive, uniquely determined such that*

$$\varphi = \varphi_1 - \varphi_2, \quad \mathbf{s}(\varphi_1)\mathbf{s}(\varphi_2) = 0.$$

Proof. Let $\varphi = |\varphi|(\cdot\, v)$ be the polar decomposition of φ. Since $\varphi = \varphi^*$ and the polar decomposition of φ^* is $\varphi^* = (v^* \cdot |\varphi| \cdot v)(\cdot\, v^*)$, by the unicity part of Theorem 8.9 we infer that

$$v = v^*, \quad |\varphi| = v^* \cdot |\varphi| \cdot v.$$

Since $v^*v = \mathbf{s}(|\varphi|)$, it follows that $e_1 = (\mathbf{s}(|\varphi|) + v)/2$, $e_2 = (\mathbf{s}(|\varphi|) - v)/2$ are projections in M, $e_1 e_2 = 0$, $v = e_1 - e_2$ and

$$|\varphi| = |\varphi|((e_1 - e_2) \cdot (e_1 - e_2)).$$

Since $e_1 + e_2 = \mathbf{s}(|\varphi|)$, we also have

$$|\varphi| = |\varphi|((e_1 + e_2) \cdot (e_1 + e_2)).$$

Consequently,

$$|\varphi| = |\varphi|(e_1 \cdot e_1) + |\varphi|(e_2 \cdot e_2)$$

and, since $\varphi = |\varphi|(\cdot\, v) = |\varphi|(\cdot\,(e_1 - e_2))$,

$$\varphi = |\varphi|(e_1 \cdot e_1) - |\varphi|(e_2 \cdot e_2).$$

Thus, the existence part of the statement is satisfied with

$$\varphi_1 = |\varphi|(e_1 \cdot e_1) = \varphi(\cdot\, e_1), \quad \varphi_2 = |\varphi|(e_2 \cdot e_2) = -\varphi(\cdot\, e_2).$$

Now let φ_1', φ_2' be two arbitrary elements of M_*^+ such that $e_1' = \mathbf{s}(\varphi_1')$, $e_2' = \mathbf{s}(\varphi_2')$ are orthogonal and $\varphi = \varphi_1' - \varphi_2'$. Then

$$\varphi = (\varphi_1' + \varphi_2')(\cdot\,(e_1' - e_2')), \quad (e_1' - e_2')^*(e_1' - e_2') = \mathbf{s}(\varphi_1' + \varphi_2')$$

hence, by the unicity of the polar decomposition of φ,

$$e_1' - e_2' = e_1 - e_2, \quad \varphi_1' + \varphi_2' = |\varphi| = \varphi_1 + \varphi_2.$$

Moreover,

$$e_1' + e_2' = \mathbf{s}(\varphi_1' + \varphi_2') = \mathbf{s}(\varphi_1 + \varphi_2) = e_1 + e_2.$$

It follows that $e_1' = e_1$, $e_2' = e_2$ and

$$\varphi_1' = \varphi(\cdot\, e_1') = \varphi(\cdot\, e_1) = \varphi_1, \quad \varphi_2' = -\varphi(\cdot\, e_2') = -\varphi(\cdot\, e_2) = \varphi_2. \quad \blacksquare$$

If $\varphi \in M_*$ is selfadjoint and φ_1, φ_2 are as in the above theorem, then $\varphi^+ = \varphi_1$ (respectively $\varphi^- = \varphi_2$) is called the *positive* (respectively the *negative*) *part* of φ and the relations

$$\varphi = \varphi^+ - \varphi^-, \quad \mathbf{s}(\varphi^+)\mathbf{s}(\varphi^-) = 0 \tag{1}$$

are usually called the *Jordan decomposition* of φ. Note that

$$|\varphi| = \varphi^+ + \varphi^-, \quad v = \mathbf{s}(\varphi^+) - \mathbf{s}(\varphi^-) \tag{2}$$

are the terms of the polar decomposition of φ. Also, it is easy to see that

$$\begin{aligned} \|\varphi^+\| &= \sup\{\varphi(x);\, x \in M^+,\, \|x\| \leqslant 1\}, \\ \|\varphi^-\| &= \sup\{-\varphi(x);\, x \in M^+,\, \|x\| \leqslant 1\}. \end{aligned} \tag{3}$$

We now characterize the property of two positive normal forms of having orthogonal supports.

PROPOSITION. *Let A be a w-dense $*$-subalgebra of the W^*-algebra M and $\varphi_1, \varphi_2 \in M_*$ be positive. The following conditions are equivalent:*

(i) $\mathbf{s}(\varphi_1)\mathbf{s}(\varphi_2) = 0$;

(ii) *for every $\varepsilon > 0$ there exists $a_\varepsilon \in A \cap M^+$, $\|a_\varepsilon\| \leqslant 1$, such that*

$$\varphi_1(a_\varepsilon) \leqslant \varepsilon, \quad \varphi_2(a_\varepsilon) \geqslant \|\varphi_2\| - \varepsilon;$$

(iii) $\|\varphi_1 - \varphi_2\| = \|\varphi_1\| + \|\varphi_2\|$.

Proof. (i) $\Rightarrow$ (ii). By the Kaplansky density theorem (8.5) there is a net $\{a_\iota\}_\iota$ in $A_1 \cap M^+$ such that $a_\iota \xrightarrow{w} 1 - \mathbf{s}(\varphi_1)$. By assumption, $1 - \mathbf{s}(\varphi_1) \geqslant \mathbf{s}(\varphi_2)$. It follows that $\varphi_1(a_\iota) \to 0$ and $\varphi_2(a_\iota) \to \|\varphi_2\|$.

(ii) $\Rightarrow$ (iii). Clearly, $\|\varphi_1 - \varphi_2\| \leqslant \|\varphi_1\| + \|\varphi_2\|$. Conversely, for each $\varepsilon > 0$ choose a_ε as in (ii). Since

$$\|\varphi_1\|+\|\varphi_2\| \leqslant \varphi_1(1-2a_\varepsilon)+\varphi_2(2a_\varepsilon-1)+4\varepsilon = (\varphi_1-\varphi_2)(1-2a_\varepsilon)+4\varepsilon \leqslant \|\varphi_1-\varphi_2\|+4\varepsilon,$$

it follows that $\|\varphi_1\| + \|\varphi_2\| \leqslant \|\varphi_1 - \varphi_2\|$.

(iii) $\Rightarrow$ (i). Since $\varphi_1 - \varphi_2$ is selfadjoint, w-continuous and since M_1 is w-compact, we have

$$\|\varphi_1 - \varphi_2\| = (\varphi_1 - \varphi_2)(x)$$

for some selfadjoint $x \in M_1$. Using (iii) we infer successively that

$$\begin{gathered}
\varphi_1(x) - \varphi_2(x) = \|\varphi_1 - \varphi_2\| = \|\varphi_1\| + \|\varphi_2\| = \varphi_1(1) + \varphi_2(1), \\
\varphi_1(1-x) + \varphi_2(1+x) = 0, \\
\varphi_1(1-x) = \varphi_2(1+x) = 0, \\
(1-x)\mathbf{s}(\varphi_1) = (1+x)\mathbf{s}(\varphi_2) = 0, \\
\mathbf{s}(\varphi_1) \leqslant \mathbf{s}(x^+), \quad \mathbf{s}(\varphi_2) \leqslant \mathbf{s}(x^-), \\
\mathbf{s}(\varphi_1)\mathbf{s}(\varphi_2) = 0. \quad \blacksquare
\end{gathered}$$

If φ_1, φ_2 satisfy the conditions of the above proposition, the we say that φ_1 and φ_2 are *orthogonal* and we write

$$\varphi_1 \perp \varphi_2.$$

8.11. Let A be a C^*-algebra. Then A^{**} is a W^*-algebra with predual $(A^{**})_* = A^*$, A is a w-dense C^*-subalgebra of A^{**} and every $\varphi \in A^*$ extends to a unique element of $(A^{**})_*$ also denoted by φ (8.2, 8.4).

Using Proposition 8.9 we infer that for every $\varphi \in A^*$ there exists a unique positive form $|\varphi|$ on A, called the *modulus* (or the *absolute value*) of φ such that

(1) $$\| \, |\varphi| \, \| \leqslant \|\varphi\|, \quad |\varphi(x)|^2 \leqslant \|\varphi\| \, |\varphi|(xx^*) \quad \text{for all } x \in A.$$

Moreover, by Theorem 8.9, there exists a unique partial isometry $v \in A^{**}$ such that

$$\varphi = |\varphi| \cdot v, \quad v^* v = s_{A^{**}}(|\varphi|) \tag{2}$$

and (2) is called the *polar decomposition of* φ *in* A^{**}.

Using Theorem 8.10 and Proposition 8.10 we infer that for every selfadjoint $\varphi \in A^*$ there exist unique positive forms φ^+, φ^- on A such that

$$\varphi = \varphi^+ - \varphi^-, \quad \|\varphi^+ - \varphi^-\| = \|\varphi^+\| + \|\varphi^-\| \tag{3}$$

and (3) is called the *Jordan decomposition* of φ. This result strengthens Corollary 1/4.15. The second condition of (3) means the orthogonality of φ^+ and φ^- (8.10). We have

$$|\varphi| = \varphi^+ + \varphi^-, \tag{4}$$

$$\begin{aligned} \|\varphi^+\| &= \sup\{\varphi(x);\, x \in A^+,\, \|x\| \leqslant 1\}, \\ \|\varphi^-\| &= \sup\{-\varphi(x);\, x \in A^+,\, \|x\| \leqslant 1\}. \end{aligned} \tag{5}$$

If A is a W^*-algebra and $\varphi \in A^*$ is singular, then $|\varphi|$ is also singular. If $\varphi \in A^*$ is selfadjoint and singular, then φ^+ and φ^- are also singular.

Let A be a W^*-algebra and $\varphi \in A_*$.

By Proposition 8.9, the present definition of $|\varphi|$ agrees with that given in 8.9, in particular $|\varphi| \in A_*$. If additionally $\varphi \in A_*$ is selfadjoint, then the present definition of φ^+ and φ^- agrees with that given in 8.10, in particular $\varphi^+, \varphi^- \in A_*$ because, by Proposition 8.10, the conditions $\mathbf{s}_A(\varphi^+)\mathbf{s}_A(\varphi^-) = 0$ and $\mathbf{s}_{A^{**}}(\varphi^+)\mathbf{s}_{A^{**}}(\varphi^-) = 0$ are equivalent.

Besides the polar decomposition (2) of $\varphi \in A_*$ in A^{**}, we have also a polar decomposition (8.9.(6)) of φ in A. Then partial isometric terms of these two polar decompositions can be different, that is $\mathbf{s}_A(|\varphi|) \neq \mathbf{s}_{A^{**}}(|\varphi|)$, because A is not a W^*-subalgebra of A^{**}.

For example, consider the C^*-algebra $A = \ell^\infty = C(\mathbb{N})$ of all bounded complex sequences. Then A is a W^*-algebra and $A_* = \ell^1$ is the Banach space of all absolutely summable complex sequences. The map

$$\varphi : A \ni x \mapsto \sum_{n=1}^{\infty} n^{-2} x(n)$$

is a positive normal form on A and $\mathbf{s}_A(\varphi) = 1$.

On the other hand, let $e_n \in A \subset A^{**}$ be defined by

$$e_n(k) = 0 \text{ if } k < n, \quad e_n(k) = 1 \text{ if } k \geqslant n; \quad n \in \mathbb{N},$$

and let $e \in A^{**}$ be the greatest lower bound of the decreasing sequence of projections $\{e_1, \ldots, e_n, \ldots\}$ in A^{**}.

Since $\varphi(e_n) = \sum\limits_{k=n}^{\infty} k^{-2} \to 0$, it is clear that $\varphi(e) = 0$.

The Gelfand spectrum Ω of A is the Stone-Čech compactification of $\mathbb{N}$. If $\omega \in \Omega \setminus \mathbb{N}$, then $\omega(e_n) = 1$ for all $n \in \mathbb{N}$. Now $\omega \in A^* = (A^{**})_*$, so $\omega(e) = 1$ and therefore $e \neq 0$.

It follows that $\mathbf{s}_{A^{**}}(\varphi) \neq 1$, hence the polar decomposition of φ in A^{**} is different from its polar decomposition in A.

PROPOSITION 1. *Let A be a C^*-algebra and $\varphi, \psi \in A^*$. Then*

$$|\,|\varphi+\psi|(x)|^2 \leqslant (\|\varphi\|+\|\psi\|)(|\varphi|(xx^*)+|\psi|(xx^*)); \quad x \in A.$$

Proof. Let $u, v, w \in A^{**}$ be the partial isometric terms of the polar decompositions of $\varphi, \psi, \varphi+\psi$, respectively. Then, using Proposition 8.9 we get

$$\begin{aligned}
|\,|\varphi+\psi|(x)|^2 &= |(\varphi+\psi)(xw^*)|^2 = |\,|\varphi|(xw^*u)+|\psi|(xw^*v)|^2 \\
&\leqslant (|\,|\varphi|(xw^*u)|+|\,|\psi|(xw^*v)|)^2 \\
&\leqslant (|\varphi|(u^*ww^*u)^{1/2}|\varphi|(xx^*)^{1/2}+|\psi|(v^*ww^*v)^{1/2}|\ \psi|(xx^*)^{1/2})^2 \\
&\leqslant (\|\varphi\|^{1/2}|\varphi|(xx^*)^{1/2}+\|\psi\|^{1/2}|\psi|(xx^*)^{1/2})^2 \\
&\leqslant (\|\varphi\|+\|\psi\|)(|\varphi|(xx^*)+|\psi|(xx^*)). \quad \blacksquare
\end{aligned}$$

COROLLARY. *Let A be a C^*-algebra, $\varphi \in A^*$ and p a central projection in A^{**}. Then*

$$|p\cdot\varphi| = p\cdot|\varphi|.$$

Proof. Let $\rho = |p\cdot\varphi|+|(1-p)\cdot\varphi|$. Then clearly

$$\|\rho\| \leqslant \|p\cdot\varphi\|+\|(1-p)\cdot\varphi\| = \|\varphi\|.$$

Let $\varphi = |\varphi|(\cdot\, v)$, $(v \in A^{**})$, be the polar decomposition of φ. Using Proposition 1, we obtain for all $x \in A$,

$$\begin{aligned}
|\varphi(x)|^2 &= |\,|p\cdot\varphi+(1-p)\cdot\varphi|(xv)|^2 \\
&\leqslant (\|p\cdot\varphi\|+\|(1-p)\cdot\varphi\|)\,\rho(xvv^*x^*) \leqslant \|\varphi\|\,\rho(xx^*).
\end{aligned}$$

By Proposition 8.9 it follows that $\rho = |\varphi|$, hence $|p\cdot\varphi| = p\cdot\rho = p\cdot|\varphi|$. $\blacksquare$

In particular, if M is a W^*-algebra and $\varphi \in M^*$, then

$$|p_M\cdot\varphi| = p_M\cdot|\varphi|, \quad |(1-p_M)\cdot\varphi| = (1-p_M)\cdot|\varphi|.$$

PROPOSITION 2. *Let A be a C^*-algebra and $\varphi \in A^*$ be positive. Then*

$$|\varphi\cdot a| \leqslant \|a\|\,\varphi; \quad a \in A^{**}.$$

Proof. Let $\varphi\cdot a = |\varphi\cdot a|\cdot v$, $(v \in A^{**})$, be the polar decomposition of $\varphi\cdot a$ in A^{**}. Then $|\varphi\cdot a| = (\varphi\cdot a)\cdot v^* = \varphi\cdot(v^*\cdot a)$. Thus, replacing a by v^*a, we may suppose that $\varphi\cdot a$ is positive, hence selfadjoit.

Let $a \in A^{**}$ with $\varphi\cdot a \geqslant 0$ and fix $x \in A$, $x \geqslant 0$. Then

$$(\varphi\cdot a^2)(x) = (\varphi\cdot a)(xa) = \overline{(\varphi\cdot a)(a^*x)} = \varphi(a^*xa)$$

and, by the Schwarz inequality,

$$(\varphi\cdot a)(x) = \varphi(x^{1/2}(x^{1/2}a)) \leqslant \varphi(a^*xa)^{1/2}\varphi(x)^{1/2}.$$

Thus, $\varphi\cdot a^2 \geqslant 0$ and

$$(\varphi\cdot a)(x) \leqslant [(\varphi\cdot a^2)(x)]^{1/2}\varphi(x)^{1/2}.$$

The same argument applied recurrently to $\varphi\cdot a^2, \varphi\cdot a^4, \ldots, \varphi\cdot a^{2^{n-1}}$ yields

$$\begin{aligned}
(\varphi\cdot a)(x) &\leqslant [(\varphi\cdot a^{2^n})(x)]^{2^{-n}}\varphi(x)^{2^{-1}+\cdots+2^{-n}} \\
&\leqslant \|\varphi\|^{2^{-n}}\|a\|\,\|x\|^{2^{-n}}\varphi(x)^{2^{-1}+\cdots+2^{-n}}.
\end{aligned}$$

Letting $n \to \infty$, it follows that $(\varphi\cdot a)(x) \leqslant \|a\|\varphi(x)$. $\blacksquare$

PROPOSITION 3. *Let A be a C^*-algebra, $\{\varphi_\iota\}_\iota$ be a net in A^* and $\varphi \in A^*$. Then*

$$\varphi_\iota \overset{\sigma(A^*,A)}{\longrightarrow} \varphi,\ \|\varphi_\iota\| \to \|\varphi\| \Rightarrow |\varphi_\iota| \overset{\sigma(A^*,A)}{\longrightarrow} |\varphi|.$$

Proof. As bounded sets in A^* are relatively $\sigma(A^*, A)$-compact, it suffices to show that $|\varphi|$ is the only possible $\sigma(A^*, A)$-limit point of the net $\{|\varphi_\iota|\}_\iota$. Thus, we may suppose that $|\varphi_\iota| \overset{\sigma(A^*,A)}{\longrightarrow} \rho$ for some $\rho \in A^*$ and then we have to show that $\rho = |\varphi|$.

Clearly, ρ is positive and $\|\rho\| \leqslant \liminf_\iota \| \, |\varphi_\iota| \, \| = \|\varphi\|$. On the other hand, for every $x \in A$,

$$|\varphi(x)|^2 = \lim_\iota |\varphi_\iota(x)|^2 \leqslant \lim_\iota \|\varphi_\iota\| \, |\varphi_\iota|(xx^*) = \|\varphi\| \, \rho(xx^*).$$

By Proposition 8.9 we infer that $\rho = |\varphi|$. ∎

In particular, if A is a W^*-algebra and $A_* \ni \varphi_\iota \overset{\text{norm}}{\longrightarrow} \varphi$, then $|\varphi_\iota| \overset{\sigma(A^*,A)}{\longrightarrow} |\varphi|$.

PROPOSITION 4. *Let A, B be C^*-algebras and $\varphi \in A^*$, $\psi \in B^*$. Then $\varphi \otimes \psi \in (A \otimes_{C^*} B)^*$ and*

$$\|\varphi \otimes \psi\| = \|\varphi\| \, \|\psi\|; \tag{6}$$

$$|\varphi \otimes \psi| = |\varphi| \otimes |\psi|. \tag{7}$$

Proof. For positive φ, ψ the equality (6) was proved in 4.20.(9).

We may consider $A \otimes_{C^*} B$ as a w-dense C^*-subalgebra of $A^{**} \overline{\otimes} B^{**}$. We have $\varphi, |\varphi| \in A^* = (A^{**})_*$ and ψ, $|\psi| \in B^* = (B^{**})_*$. Let $\varphi = |\varphi|(\cdot\, v)$, $(v \in A^{**})$, and $\psi = |\psi|(\cdot\, w)$, $(w \in B^{**})$, be the corresponding polar decompositions. Then $|\varphi| = \varphi(\cdot\, v^*)$ and $|\psi| = \psi(\cdot\, w^*)$. Clearly

$$(\varphi \overline{\otimes} \psi)(x) = (|\varphi| \overline{\otimes} |\psi|)(x(v \otimes w)) \tag{8}$$

$$(|\varphi| \overline{\otimes} |\psi|)(x) = (\varphi \overline{\otimes} \psi)(x(v^* \otimes w^*)) \tag{9}$$

for every $x \in A^{**} \otimes B^{**}$ and hence for all $x \in A^{**} \overline{\otimes} B^{**}$.

Since $\|v \otimes w\| \leqslant 1$, $\|v^* \otimes w^*\| \leqslant 1$, we infer that

$$\|\varphi \otimes \psi\| = \|\varphi \overline{\otimes} \psi\| = \| \, |\varphi| \overline{\otimes} |\psi| \, \| = \| \, |\varphi| \otimes |\psi| \, \| = \| \, |\varphi| \, \| \, \| \, |\psi| \, \| = \|\varphi\| \, \|\psi\|.$$

Using (8), the Schwarz inequality and $\| \, |\varphi| \overline{\otimes} |\psi| \, \| = \|\varphi \otimes \psi\|$, we obtain

$$|(\varphi \otimes \psi)(x)|^2 \leqslant \|\varphi \otimes \psi\| (|\varphi| \otimes |\psi|)(xx^*), \quad x \in A \otimes_{C^*} B,$$

hence $|\varphi \otimes \psi| = |\varphi| \otimes |\psi|$, by Proposition 8.9. ∎

Finally, we analyse the polar decomposition of tensor product functionals on W^*-tensor products.

Proposition 5. *Let M, N be W^*-algebras and $\varphi \in M_*$, $\psi \in N_*$ with polar decompositions $\varphi = |\varphi|(\cdot\, v)$, $\psi = |\psi|(\cdot\, w)$ respectively. Then*

$$\varphi \mathbin{\overline{\otimes}} \psi = (|\varphi| \mathbin{\overline{\otimes}} |\psi|)(\,\cdot\,(v \otimes w)) \tag{10}$$

is the polar decomposition of $\varphi \mathbin{\overline{\otimes}} \psi \in (M \mathbin{\overline{\otimes}} N)_*$. In particular,

$$\mathbf{l}(\varphi \mathbin{\overline{\otimes}} \psi) = \mathbf{l}(\varphi) \otimes \mathbf{l}(\psi), \quad \mathbf{r}(\varphi \mathbin{\overline{\otimes}} \psi) = \mathbf{r}(\varphi) \otimes \mathbf{r}(\psi). \tag{11}$$

Proof. We first assume that φ, ψ are both positive and faithful. Then the GNS-representations $\pi_\varphi : M \to \pi_\varphi(M) \subset B(H_\varphi)$ and $\pi_\psi : N \to \pi_\psi(N) \subset B(H_\psi)$ are normal $*$-isomorphisms (4.5; Corollary 4/8.4) and also $\pi_\varphi \mathbin{\overline{\otimes}} \pi_\psi : M \mathbin{\overline{\otimes}} N \to \pi_\varphi(M) \mathbin{\overline{\otimes}} \pi_\psi(N)$ is a $*$-isomorphism (8.8). Note that $\varphi = \omega_{\xi_\varphi} \circ \pi_\varphi$, $\psi = \omega_{\xi_\psi} \circ \pi_\psi$ and $\varphi \mathbin{\overline{\otimes}} \psi = \omega_{\xi_\varphi \otimes \xi_\psi} \circ (\pi_\varphi \mathbin{\overline{\otimes}} \pi_\psi)$. Since φ (respectively ψ) is faithful, the vector $\xi_\varphi \in H_\varphi$ (respectively $\xi_\psi \in H_\psi$) is separating for the von Neumann algebra $\pi_\varphi(M) \subset B(H_\varphi)$ (respectively $\pi_\psi(N) \subset B(H_\psi)$), that is

$$\overline{\pi_\varphi(M)'\xi_\varphi} = H_\varphi \quad (\text{respectively } \overline{\pi_\psi(N)'\xi_\psi} = H_\psi).$$

Since $(\pi_\varphi(M) \mathbin{\overline{\otimes}} \pi_\psi(N))' \supset \pi_\varphi(M)' \mathbin{\overline{\otimes}} \pi_\psi(N)'$, we infer that

$$(\pi_\varphi \mathbin{\overline{\otimes}} \pi_\psi)(M \mathbin{\overline{\otimes}} N)'(\xi_\varphi \otimes \xi_\psi) = (\pi_\varphi(M) \mathbin{\overline{\otimes}} \pi_\psi(N))'(\xi_\varphi \otimes \xi_\psi) = H_\varphi \mathbin{\overline{\otimes}} H_\psi$$

hence $\xi_\varphi \otimes \xi_\psi$ is separating for $(\pi_\varphi \mathbin{\overline{\otimes}} \pi_\psi)(M \mathbin{\overline{\otimes}} N)$. It follows that $\varphi \otimes \psi$ is faithful.

If φ, ψ are arbitrary normal positive forms, then the restriction of φ (respectively ψ) to $\mathbf{s}(\varphi)M\mathbf{s}(\varphi)$ (respectively $\mathbf{s}(\psi)N\mathbf{s}(\psi)$) is faithful so, by the above, the restriction of $\varphi \mathbin{\overline{\otimes}} \psi$ to

$$(\mathbf{s}(\varphi) \mathbin{\overline{\otimes}} \mathbf{s}(\psi))(M \mathbin{\overline{\otimes}} N)(\mathbf{s}(\varphi) \mathbin{\overline{\otimes}} \mathbf{s}(\psi)) = \mathbf{s}(\varphi)M\mathbf{s}(\varphi) \mathbin{\overline{\otimes}} \mathbf{s}(\psi)N\mathbf{s}(\psi)$$

is faithful. This shows that $\mathbf{s}(\varphi \mathbin{\overline{\otimes}} \psi) \geqslant \mathbf{s}(\varphi) \otimes \mathbf{s}(\psi)$. Since

$$(\varphi \mathbin{\overline{\otimes}} \psi)(\mathbf{s}(\varphi) \otimes \mathbf{s}(\psi)) = \varphi(\mathbf{s}(\varphi))\psi(\mathbf{s}(\psi)) = (\varphi \mathbin{\overline{\otimes}} \psi)(1_{M \mathbin{\overline{\otimes}} N}),$$

it follows that

$$\mathbf{s}(\varphi \mathbin{\overline{\otimes}} \psi) = \mathbf{s}(\varphi) \otimes \mathbf{s}(\psi). \tag{12}$$

Now, in the general case we have the equality (10) and, using (12):

$$(v \otimes w)^*(v \otimes w) = v^*v \otimes w^*w = \mathbf{s}(|\varphi|) \otimes \mathbf{s}(|\psi|) = \mathbf{s}(|\varphi| \mathbin{\overline{\otimes}} |\psi|).$$

Thus, by the unicity part of Theorem 8.9, (10) is indeed the polar decomposition of $\varphi \mathbin{\overline{\otimes}} \psi$.

Finally, (11) follows using 8.9.(8). ∎

8.12. Countably decomposable W^*-algebras. Let M be a W^*-algebra. A projection $e \in M$ is called *countably decomposable* if every family $\{e_\iota\}_\iota$ of mutually orthogonal non-zero projections of M majorized by e is at most countable.

PROPOSITION 1. *Let M be a W^*-algebra. A projection $e \in M$ is countably decomposable if and only if there exists a normal positive form φ on M such that $e = \mathbf{s}(\varphi)$.*

Proof. Let $e = \mathbf{s}(\varphi)$ for some $\varphi \in M_*$, $\varphi \geqslant 0$, and $\{e_\iota\}_{\iota \in I}$ be a family of mutually orthogonal projections such that $e = \sum_{\iota \in I} e_\iota$. Then $\sum_{\iota \in I} \varphi(e_\iota) = \varphi(e) < +\infty$ so, the set $J = \{\iota \in I;\ \varphi(e_\iota) \neq 0\}$ is at most countable. If $\iota \notin J$, then $\varphi(e_\iota) = 0$ and, by (8.9.(5)), $e_\iota = e_\iota \mathbf{s}(\varphi) = 0$.

Given any non-zero projection $f \in M$, there exists $\psi \in M_*$, $\psi \geqslant 0$, such that $\psi(f) \neq 0$. Then $\varphi = \psi(f \cdot f)$ is a non-zero normal positive form on M and $0 \neq \mathbf{s}(\varphi) \leqslant f$.

Now let $e \in M$ be a countably decomposable projection and $\{\varphi_\iota\}_{\iota \in I}$ be a maximal family of non-zero normal states on M with mutually orthogonal supports majorized by e. Then $\sum_{\iota \in I} \mathbf{s}(\varphi_\iota) = e$ and I is at most countable so we may suppose $I \subset \mathbb{N}$. Furthermore, $\varphi = \sum_{n \in I} 2^{-n}\varphi_n \in M_*$, $\varphi \geqslant 0$ and $\mathbf{s}(\varphi) = e$. ∎

PROPOSITION 2. *Let M be a W^*-algebra, $\varphi \in M_*^+$, and $\{x_\iota\}_\iota$ be a norm bounded net in M. Then*

$$\varphi(x_\iota^* x_\iota) \to 0 \Leftrightarrow x_\iota \mathbf{s}(\varphi) \xrightarrow{s} 0; \tag{1}$$

$$\varphi(x_\iota^* x_\iota + x_\iota x_\iota^*) \to 0 \Rightarrow \mathbf{s}(\varphi) x_\iota \mathbf{s}(\varphi) \xrightarrow{s^*} 0. \tag{2}$$

Proof. Let $e = \mathbf{s}(\varphi)$ and assume that $\|x_\iota\| \leqslant 1$ for all ι.

If $x_\iota e \xrightarrow{s} 0$, then $\varphi(x_\iota^* x_\iota) = \varphi(e x_\iota^* x_\iota e) = \varphi((x_\iota e)^*(x_\iota e)) \to 0$.

Conversely, suppose that $\varphi(x_\iota^* x_\iota) \to 0$. Let $\psi \in M_*^+$ with $\psi = \psi(\cdot\, e)$ and $\varepsilon > 0$. By 8.9.(3), there is a sequence $\{a_k\}_k$ in M such that $\|\psi - \varphi(a_k \cdot)\| \to 0$. Choose k so that $\|\psi - \varphi(a_k \cdot)\| \leqslant \varepsilon/2$ and then choose L_ε such that $\varphi(x_\iota^* x_\iota)^{1/2} \leqslant \varepsilon/2\|\varphi\|^{1/2}\|a_k\|$, for all $\iota \geqslant \iota_\varepsilon$. Then, for $\iota \geqslant \iota_\varepsilon$ we have

$$\begin{aligned}
\psi(x_\iota^* x_\iota) &\leqslant |(\psi - \varphi(a_k \cdot))(x_\iota^* x_\iota)| + |\varphi(a_k x_\iota^* x_\iota)| \\
&\leqslant \|\psi - \varphi(a_k \cdot)\|\, \|x_\iota\|^2 + \varphi(a_k a_k^*)\varphi((x_\iota^* x_\iota)^2)^{1/2} \\
&\leqslant \|\psi - \varphi(a_k \cdot)\|\, \|x_\iota\|^2 + \|\varphi\|^{1/2}\|a_k\|\, \|x_\iota\| \varphi(x_\iota^* x_\iota)^{1/2} \\
&\leqslant \varepsilon/2 + \varepsilon/2 = \varepsilon.
\end{aligned}$$

Consequently, for every $\psi \in M_*^+$,

$$\psi((x_\iota e)^*(x_\iota e)) = \psi(e x_\iota^* x_\iota e) \to 0,$$

that is, $x_\iota e \xrightarrow{s} 0$.

Finally, if $\varphi(x_\iota^* x_\iota + x_\iota x_\iota^*) \to 0$, then by the above $x_\iota e \xrightarrow{s} 0$ and $x_\iota^* e \xrightarrow{s} 0$, hence $e x_\iota e \xrightarrow{s} 0$ and $e x_\iota^* e \xrightarrow{s} 0$ that is $e x_\iota e \xrightarrow{s^*} 0$. ∎

If the unit 1 of M is countably decomposable, then M itself is called *countably decomposable*. It is not true that every W^*-algebra is countably decomposable, but there is always a family $\{e_\iota\}$ of mutually orthogonal countably decomposable projections in M such that $\sum_\iota e_\iota = 1$.

THEOREM. *Let M be a W^*-algebra. The following statements are equivalent:*

(i) *M is countably decomposable;*
(ii) *There exists a faithful normal state on M;*
(iii) *There exists a faithful state on M;*
(iv) *M_1 is s-metrizable;*
(v) *M_1 is s^*-metrizable.*

Proof. (i) $\Leftrightarrow$ (ii) by Proposition 1.

(ii) $\Leftrightarrow$ (iii) since a positive form on M is faithful if and only if its normal part is faithful (8.9).

(ii) $\Rightarrow$ (iv). Let φ be a faithful normal state on M and define

$$d_\varphi(x,y) = \varphi((x-y)^*(x-y))^{1/2}; \quad x, y \in M_1.$$

Using the Schwarz inequality and the faithfulness of φ it is easy to see that d_φ is a metric on M_1. By Proposition 2, a net $\{x_\iota\}$ in M_1 is s-convergent to $x \in M_1$ if and only if $d_\varphi(x_\iota, x) \to 0$, because $\mathbf{s}(\varphi) = 1$.

(ii) $\Rightarrow$ (v). Similarly, the s^*-topology on M_1 is defined by the metric

$$d^*_\varphi(x,y) = \varphi((x-y)^*(x-y) + (x-y)(x-y)^*)^{1/2}; \quad x, y \in M_1.$$

(iv) $\Rightarrow$ (i). Let $\{e_\iota\}_{\iota \in I}$ be a family of mutually orthogonal projections in M with $\sum\limits_{\iota \in I} e_\iota = 1$. Put $e_J = \sum\limits_{\iota \in J} e_\iota$ for each $J \subset I$, finite. Then net $\{e_J\}_J$ is then s-convergent to 1. Since M_1 is s-metrizable, there exists a sequence $\{J_n\}_n$ of finite subsets of I such that $e_{J_n} \xrightarrow{s} 1$ and we may assume $J_n \subset J_{n+1}$, ($n \in \mathbb{N}$). It follows that $e_\iota = 0$ for $\iota \notin \bigcup\limits_n J_n$.

Similarly, (v) $\Rightarrow$ (i). ∎

Thus, if M is a countably decomposable W^*-algebra, then for every faithful positive normal form φ on M the metric d_φ (respectively d^*_φ) defines the s (respectively s^*) -topology on M_1. Note that the metrics d_φ and d^*_φ on M_1 are complete.

Also, if M is a countably decomposable W^*-algebra and A is a w-dense $*$-subalgebra of M, then the Kaplansky density theorem (see 8.5) can be restated with s^*-convergent *sequences* instead of nets.

A positive linear mapping $\Phi : A \to B$ between C^*-algebras is called *faithful* if

$$x \in A, \quad \Phi(x^*x) = 0 \Rightarrow x = 0.$$

For instance, a $*$-homomorphism is faithful if and only if it is injective.

COROLLARY. *Let M, N be W^*-algebra and $\Phi : M \to N$ be a positive linear mapping. Assume that N is countably decomposable. Then*

(i) *Φ is singular if and only if for every non-zero projection $e \in M$ there exists a non-zero projection $f \leqslant e$ with $\Phi(f) = 0$.*

(ii) *Φ is faithful if and only if Φ_{nor} is faithful.*

Proof. Let ψ be a faithful normal state on N.

(i) Assume that Φ is singular and let $e \in M$ be a non-zero projection. Since $\psi \cdot \Phi$ is a singular functional on M, there exists a non-zero projection $f \in M$, $f \leqslant e$ with $\psi(\Phi(f)) = 0$. Then $\Phi(f) = 0$, because ψ is faithful.

Then converse is obviously true for every bounded linear mapping $\Phi : M \to N$, even if N is not countably decomposable.

(ii) It is easy to see that $\psi \circ \Phi_{\mathrm{nor}} = (\psi \circ \Phi)_{\mathrm{nor}}$ and $\psi \circ \Phi_{\mathrm{sing}} = (\psi \circ \Phi)_{\mathrm{sing}}$. If Φ is faithful, then $\psi \circ \Phi$ is a faithful positive form on M, so $\psi \circ \Phi_{\mathrm{nor}} = (\psi \circ \Phi)_{\mathrm{nor}}$ is also faithful on M (8.9) and hence Φ_{nor} is faithful.

The converse is obvious. ∎

8.13. Countably decomposable von Neumann algebras. Let $M \subset B(H)$ be a von Neumann algebra acting on the Hilbert space H with commutant $M' \subset B(H)$. For $\xi \in H$, the restriction to M of the normal positive form ω_ξ on $B(H)$ will be denoted by ω_ξ^M or still by ω_ξ and its restriction to M' will be denoted by $\omega_\xi^{M'}$ or simply by ω'_ξ. It is easy to check that

$$\mathbf{s}(\omega_\xi^M) = p_\xi^M, \quad \mathbf{s}(\omega_\xi^{M'}) = p_\xi^{M'} \tag{1}$$

where $p_\xi^M (= p_\xi)$, $p_\xi^{M'} (= p'_\xi)$ are the cyclic projections defined in 7.11. Thus, ω_ξ is faithful on M if and only if ξ is separating for M and ω'_ξ is faithful on M' if and only if ξ is cyclic for M.

LEMMA. *Let $M \subset B(H)$ be a von Neumann algebra. If $\{\xi_n\}$ is an orthonormal sequence in H such that the projections p'_{ξ_n} are mutually orthogonal, then*

$$\bigvee_{n=1}^{\infty} p_{\xi_n} = p_\xi \quad \textit{where} \quad \xi = \sum_{n=1}^{\infty} 2^{-n} \xi_n \in H.$$

Proof. Since the projections p'_{ξ_n} are mutually orthogonal and $p'_{\xi_n} \xi_n = \xi_n$, we have $p'_{\xi_n} \xi = 2^{-n} \xi_n$. Then

$$p_\xi H = \overline{M'\xi} \supset \overline{M' p'_{\xi_n} \xi} = \overline{M' \xi_n} = p_{\xi_n} H$$

hence $p_\xi \geqslant \bigvee_{n=1}^{\infty} p_{\xi_n}$. The reverse inequality is immediate. ∎

Proposition. *Let $M \subset B(H)$ be a countably decomposable von Neumann algebra. Then*

(i) *There exists a sequence $\{\xi_n\}_n$ in H separating for M such that $\{p_{\xi_n}\}$ are mutually orthogonal.*

(ii) *There exist central projections $p, q \in M$ with $p + q = 1$ and $\xi, \eta \in H$ such that $p = p_\xi$, $q = q'_\eta$.*

Proof. (i) If $f \in M$ is a non-zero projection, then $f\xi \neq 0$ for some $\xi \in H$, so $0 \neq p_{f\xi} \leqslant f$. Thus, by the same maximality argument as that used in the proof of Proposition 1/8.12, we get a sequence $\{\xi_n\}_n$ in H with $\{p_{\xi_n}\}$ mutually orthogonal and $\sum\limits_{n=1}^{\infty} p_{\xi_n} = 1$. Then the set $\{\xi_n;\, n \in \mathbb{N}\}$ is separating for M. Indeed, if $x \in M$ and $x\xi_n = 0$, then $xp_{\xi_n} = 0$ for all $n \in \mathbb{N}$ hence $x = 0$.

(ii) Let $\{\zeta_n\}_{n\in I} \subset H$, $\|\zeta_n\| = 1$, be a maximal family such that the projections p_{ξ_n}, $(n \in I)$, are mutually orthogonal and also the projections p'_{ζ_n}, $(n \in I)$, are mutually orthogonal. Since M is countably decomposable, we may suppose $I \subset \mathbb{N}$. Put

$$e = \sum_{n\in I} p_{\zeta_n}, \quad e' = \sum_{n\in I} p'_{\zeta_n} \quad \text{and} \quad \zeta = \sum_{n\in I} 2^{-n}\zeta_n.$$

Then $e = p_\zeta$ and $e' = p'_\zeta$ by the lemma.

On the other hand, the maximality of $\{\zeta_n\}$ entails $(1-e)(1-e') = 0$ so, by 7.20.(1), we infer that $\mathbf{z}(1-e)\mathbf{z}(1-e') = 0$. Then

$$p = 1 - \mathbf{z}(1-e) \leqslant e \quad \text{and} \quad q = \mathbf{z}(1-e) \leqslant 1 - \mathbf{z}(1-e') \leqslant e'$$

are central projections and $p = p_\zeta$, $q = q'_\eta$, where $\xi = p\zeta$, $\eta = q\xi$. ∎

Corollary 1. *A von Neumann algebra $M \subset B(H)$ is countably decomposable if and only if there exists a separating sequence $\{\xi_n\}_n \subset H$ for M.*

Proof. If $\{\xi_n\}_n$ is a separating sequence for M, then

$$\varphi = \sum_n 2^{-n}\|\xi_n\|^{-2}\omega_{\xi_n} \in M_*^+$$

is faithful, so M is countably decomposable by Theorem 8.12.

The converse follows from the above proposition. ∎

Corollary 2. *Let $M \subset B(H)$ be a commutative von Neumann algebra. Then M is countably decomposable if and only if there exists a separating vector $\xi \in H$ for M.*

Proof. Assume that M is countably decomposable. By the above proposition there is a separating orthonormal sequence $\{\xi_n\}_n \subset H$ for M such that $\{p_{\xi_n}\}$ are mutually orthogonal. Put

$$\xi = \sum_{n=1}^{\infty} 2^{-n}\xi_n \in H.$$

Since $p_{\xi_n} \in M \subset M'$, we have $\xi_n = 2^n p_{\xi_n}\xi \in M'\xi$, $(n \in N)$. It follows that

$$p_\xi = \sum_{n=1}^{\infty} p_{\xi_n} = 1,$$

hence ξ is separating.

The converse is clear. ∎

Note that every cyclic vector for a commutative von Neumann algebra $M \subset B(H)$ is also separating, because $M \subset M'$.

A commutative $*$-subalgebra C of a C^*-algebra A is called *maximal commutative* in A if it is not contained in any larger commutative $*$-subalgebra of A. In this case, C is a C^*-subalgebra of A and contains the center of A. If A is a W^*-algebra, then C is a W^*-subalgebra of A.

In particular, a commutative von Neumann algebra $M \subset B(H)$ is maximal commutative in $B(H)$ if and only if $M = M'$ or, equivalently if and only if M' is commutative.

Corollary 3. *A commutative countably decomposable von Neumann algebra $M \subset B(H)$ is maximal commutative in $B(H)$ if and only if there exists a cyclic vector $\xi \in H$ for M.*

Proof. If M is maximal commutative in $B(H)$, then by Corollary 2 there exists a separating vector $\xi \in H$ for M and ξ is also cyclic for M since $M = M'$.

Conversely, let $\xi_0 \in H$ be a cyclic vector for M, i.e. $\overline{M\xi_0} = H$. Let $x', y' \in M'$. There is a sequence $\{x_n\}_n$ in M such that $x_n\xi_0 \to x'\xi_0$. For every $x \in M$ we have

$$(x_n^*\xi_0|x\xi_0) = (x^*\xi_0|x_n\xi_0) \to (x^*\xi_0|x'\xi_0) = (x'^*\xi_0|x\xi_0),$$

hence $x_n^*\xi_0 \xrightarrow{\text{weakly}} x'^*\xi_0$. For every $x, y \in M$ we then have

$$\begin{aligned}(x'y'y\xi_0|x\xi_0) &= (y'y\xi_0|xx'^*\xi_0) = \lim_n (y'y\xi_0|xx_n^*\xi_0)\\ &= \lim_n (y'yx_n\xi_0|x\xi_0) = (y'yx'\xi_0|x\xi_0) = (y'x'y\xi_0|x\xi_0),\end{aligned}$$

hence $x'y' = y'x'$. Thus, M' is commutative and M is maximal commutative in $B(H)$. ∎

8.14. W^*-algebras with separable predual. The W^*-algebra $B(H)$ is countably decomposable if and only if H is a separable Hilbert space and in this case every von Neumann algebra acting on H is countably decomposable.

However, a countably decomposable W^*-algebra has not necessarily a faithful normal $*$-representation on a separable Hilbert space.

A W^*-algebra M is called *countably generated* if there is a sequence $S \subset M$ such that $M = W^*(S)$. In this case there is an s-dense sequence in M. Also, M is generated by a sequence of projections.

THEOREM. *Let M be a W^*-algebra. Then the following statements are equivalent:*

(i) *M has a faithful normal $*$-representation on a separable Hilbert space.*
(ii) *M_* is norm-separable.*
(iii) *M_* is $\sigma(M^*, M)$-separable.*
(iv) *M_1 is w-metrizable (and w-separable).*
(v) *M_1 is s-metrizable and s-separable.*
(vi) *M_1 is s^*-metrizable and s^*-separable.*
(vii) *M is countably decomposable and countably generated.*
(viii) *the center of M is countably decomposable and M is countably generated.*

Proof. (i) $\Rightarrow$ (ii). By assumption we may suppose that M is a von Neumann algebra on a separable Hilbert space H. If S is a dense sequence in H, then the linear span of $\{\omega_{\xi,\eta};\ \xi,\eta \in S\}$ is a norm-dense separable subspace of M_*, hence M_* is norm-separable.

(ii) $\Rightarrow$ (iii). By assumption, M_* is norm-separable or equivalently $\sigma(M_*, M)$-separable, because M identifies to $(M_*)^*$. Also, with this identification, M_* is $\sigma(M^*, M)$-dense in its second dual M^*. Hence M^* is $\sigma(M^*, M)$-separable.

(iii) $\Rightarrow$ (ii). Let $\{\psi_n\}_n$ be a $\sigma(M^*, M)$-dense sequence in M^*. By Corollary 1/4.15, $\psi_n = \psi_n^1 - \psi_n^2 + i(\psi_n^3 - \psi_n^4)$ with $\psi_n^j \in M^+$ positive and $\|\psi_n^j\| \leqslant \|\psi_n\|$, $(1 \leqslant j \leqslant 4;\ n \in \mathbb{N})$. Put $\theta_n = \sum_{j=1}^{4} \psi_n^j$, $(n \in \mathbb{N})$, and

$$\theta = \sum_{n=1}^{\infty} (2^n \|\theta_n\|)^{-1} \theta_n \in M_+^*.$$

By Theorem 8.4, there exist a normal positive form φ on M and a singular positive form f on M such that

$$\theta = \varphi + f.$$

Since $\{\psi_n\}_n$ is $\sigma(M^*, M)$-dense in M^*, θ is a faithful positive form on M, hence, by Theorem 8.12, M is countably decomposable. Owing to Theorem 8.4.(iv) we thus get an increasing sequence $\{p_k\}_k$ of projections in M such that

$$\bigvee_{k=1}^{\infty} p_k = 1 \quad \text{and} \quad f(p_k) = 0 \quad \text{for all } k.$$

For every $n, k \in \mathbb{N}$ and $1 \leqslant j \leqslant 4$ we have

$$0 \leqslant p_k \cdot \psi_n^j \cdot p_k \leqslant 2^n \|\theta_n\| (p_k \cdot \theta \cdot p_k) = 2^n \|\theta_n\| (p_k \cdot \varphi \cdot p_k).$$

Since φ is normal, $p_k \cdot \varphi \cdot p_k$ and also $p_k \cdot \psi_n^j \cdot p_k$ are normal (8.4). Hence $p_k \cdot \psi_n \cdot p_k$ is normal, that is

$$p_k \cdot \psi_n \cdot p_k \in M_* \quad \text{for all } n, k.$$

Let $x \in M$ such that $(p_k \cdot \psi_n \cdot p_k)(x) = 0$, i.e. $\psi_n(p_k x p_k) = 0$, for all n, k. Since $\{\psi_n\}_n$ is $\sigma(M^*, M)$-dense in M^*, we obtain $p_k x p_k = 0$ for all k and since $p_k \xrightarrow{s} 1$, we get $x = 0$. It follows that the linear span of $\{p_k \cdot \psi_n \cdot p_k;\, n, k \in \mathbb{N}\}$ is norm-dense in M_*. Hence M_* is norm-separable.

(ii) $\Rightarrow$ (iv). A Banach space X is separable if and only if the closed unit ball X_1^* of its dual space is $\sigma(X^*, X)$-metrizable (see [81], V.5.1). Thus, M_* is separable if and only if M_1 is w-metrizable. In this case M_1 is also w-separable since a metrizable compact space is separable ([81], I.6.19).

(iv) $\Rightarrow$ (v) and (vi). If M_1 is w-metrizable, then, as in the proof of Theorem 8.12, (iv) $\Rightarrow$ (i), we see that M is countably decomposable and, by Theorem 8.12, M_1 is s-metrizable and s^*-metrizable. Since M_1 is w-separable, there is a sequence $S \subset M_1$, w-dense in M_1. Since the convex hull co S is w-dense in M_1, co S is also s-dense and s^*-dense in M_1 (8.5) and consequently M_1 is s-separable and s^*-separable.

(v) $\Rightarrow$ (vii) (respectively (vi) $\Rightarrow$ (vii)). If M_1 is s (respectively s^*)-metrizable, then M is countably decomposable by Theorem 8.12. If S is an s (respectively s^*) - dense sequence in M_1, then M is the s (respectively s^*) -closure of the linear span of S, hence M is countably generated.

(vii) $\Rightarrow$ (viii) Obvious.

(viii) $\Rightarrow$ (i). Let $M \subset B(H)$ be realized as a von Neumann algebra. Denote by Z the center of M. Since Z is countably decomposable, by Corollary 2/8.13 there exists a separating vector $\xi \in H$ for Z. Let $e' = p'_\xi \in M'$ and $K = e'(H) = \overline{M\xi}$. As M is countably generated, there exists an s-dense sequence in M and therefore K is separateble. Since ξ is separating for Z, the central support $\mathbf{z}(e')$ is equal to 1_M so, by Theorem 7.17, the induction map

$$M \ni x \to x_{e'} \in M_{e'} \subset B(K)$$

is a $*$-isomorphism. ∎

In the commutative case, the implication (i) $\Rightarrow$ (vii) can be considerably sharpened.

Proposition. *If M is a commutative von Neumann algebra acting on a separable Hilbert space H, then exists a selfadjoint element $a \in M$ such that*

$$M = \{a\}''.$$

Proof. By the theorem, there exists a sequence $\{e_n\}_n$ of projections in M which generate a w-dense C^*-subalgebra A of M. Put

$$a = \sum_{n=1}^{\infty} 3^{-n}(2e_n - 1) \in A_h.$$

Let Ω be the Gelfand spectrum of A. Then the e_n separate the points of Ω. Thus, if $t, s \in \Omega$, $t \neq s$, then there is a smallest integer $k \geqslant 1$ such that $|e_k(t) - e_k(s)| = 1$ and we have

$$|a(t) - a(s)| = 2\Big|\sum_{n=k}^{\infty} 3^{-n}(e_n(t) - e_n(s))\Big| \geqslant 2 \cdot 3^{-k} - 2\sum_{n=k+1}^{\infty} 3^{-n} = 3^{-k}.$$

Thus $\{a\}$ separates the points of Ω and the Stone-Weierstrass theorem shows that A is the C^*-subalgebra generated by a.

Hence $\{a\}'' = A'' = \overline{A}^w = M$. ∎

Besides commutative W^*-algebras with separable predual, also other important classes of W^*-algebras with separable predual, are singly generated (see [266], [332]), but it is an open problem whether every W^*-algebra with separable predual has this property.

8.15. Let M be a W^*-algebra. For every $\varphi \in M^*$ we have

$$\sup\{|\varphi(e)|;\, e \in P(M)\} \leqslant \|\varphi\| \leqslant 4\sup\{|\varphi(e)|;\, e \in P(M)\}. \tag{1}$$

Indeed, let $\lambda = \sup\{|\varphi(e)|;\, e \in P(M)\}$. Clearly, $\lambda \leqslant \|\varphi\|$. If $x \in M$, $0 \leqslant x \leqslant 1$, then there is a sequence $\{e_n\}_n$ of projections in M such that $x = \sum\limits_{n=1}^{\infty} 2^{-n} e_n$ in the norm topology (Proposition 3/7.16). Thus

$$|\varphi(x)| \leqslant \sum_{n=1}^{\infty} 2^{-n} |\varphi(e_n)| \leqslant \lambda.$$

If $x \in M$ is selfadjoint and $\|x\| \leqslant 1$, then $0 \leqslant x^+$, $x^- \leqslant 1$ hence $|\varphi(x)| \leqslant 2\lambda$. Finally, for an arbitrary $x \in M$, $\|x\| \leqslant 1$, we have $\|\mathrm{Re}\, x\|$, $\|\mathrm{Im}\, x\| \leqslant 1$, hence $|\varphi(x)| \leqslant 4\lambda$.

In this section we prove a strengthened form of the uniform boundedness theorem for normal linear functionals on W^*-algebras.

A non-zero projection $e \in M$ is called *minimal* in M (or an *atom* of M) if every non-zero projection $f \in M$, $f \leqslant e$ is equal to e.

LEMMA. *Let M be a countably decomposable W^*-algebra and φ be a normal positive form on M. Then, for every $\varepsilon > 0$, there exists a finite family $\{e_1, \ldots, e_n\}$ of mutually orthogonal projections in M with $\sum\limits_{k=1}^{n} e_k = 1$ such that each e_k is either an atom or $\varphi(e_k) < \varepsilon$.*

Proof. Let $\{e_\iota\}_{\iota \in I}$ be a maximal family of mutually orthogonal atoms of M with $\varphi(e_\iota) \geqslant \varepsilon$, $(\iota \in I)$. Since $\sum\limits_{\iota \in I} \varphi(e_\iota) \leqslant \varphi(1)$, the set I is finite. By the maximality of $\{e_\iota\}$, any projection $e \leqslant 1 - \sum\limits_{\iota \in I} e_\iota$ with $\varphi(e) \geqslant \varepsilon$ is not an atom of M.

Thus, we may suppose that there are no atoms $e \in M$ with $\varphi(e) \geqslant \varepsilon$. In this case

$$\text{for every } 0 \neq e \in P(M) \text{ there is } 0 \neq f \in P(M),\ f \leqslant e, \text{ with } \varphi(f) < \varepsilon. \tag{2}$$

Indeed, suppose the contrary holds, i.e. there is $0 \neq e \in P(M)$ such that $\varphi(f) \geqslant \varepsilon$ for all $0 \neq f \in P(M)$, $f \leqslant e$. Then e is not an atom, so there exists $0 \neq f_1 \in P(M)$, $f_1 \leqslant e$, with $\varphi(f_1) \leqslant \varphi(e)/2$. Also f_1 is not an atom, so there exists $0 \neq f_2 \in P(M)$, $f_2 \leqslant f_1$ with $\varphi(f_2) \leqslant \varphi(f_1)/2 \leqslant \varphi(e)/2^2$. By induction we find $0 \neq f_n \in P(M)$, $f_n \leqslant e$, with $\varphi(f_n) \leqslant \varphi(e)/2^n < \varepsilon$, a contradiction.

Now let $\{e_n\}_n$ be a maximal family of mutually orthogonal non-zero projections in M with $\varphi(e_n) < \varepsilon$ for all n. Since M is countably decomposable, this family is at most countable. By (2) we have $\sum\limits_n e_n = 1$. Since φ is normal, we have $\varphi\Big(\sum\limits_{n > n_\varepsilon} e_n\Big) < \varepsilon$ for some $n_\varepsilon \in \mathbb{N}$. Then $\{e_1, \ldots, e_{n_\varepsilon}\}$ and $\sum\limits_{n > n_\varepsilon} e_n$ satisfy the requirement of the lemma. ∎

THEOREM. *Let M be a W^*-algebra and $\{\varphi_\iota\}_{\iota\in I}$ be a family of normal linear forms on M such that, for every projection $e \in M$,*

$$\lambda_e = \sup_{\iota\in I} |\varphi_\iota(e)| < +\infty. \tag{3}$$

Then

$$\sup_{\iota\in I} \|\varphi_\iota\| < +\infty.$$

Proof. By the uniform boundedness principle (7.3), it is sufficient to show that $\sup\limits_{\iota\in I} |\varphi_\iota(x)| < +\infty$ for all $x \in M$. Clearly, it is enough to prove this only for selfadjoint $x \in M$. In this case, replacing M by $W^*(\{x\})$, we may suppose that M is commutative.

Thus, assume M is commutative. By (1) we must prove that

$$\sup_{e\in P(M)} \sup_{\iota\in I} |\varphi_\iota(e)| < +\infty. \tag{4}$$

In the contrary case, for each $n \in \mathbb{N}$ there exists $f_n \in P(M)$ and $\varphi_n \in \{\varphi_\iota;\, \iota \in I\}$ such that

$$|\varphi_n(f_n)| \geqslant n; \quad n \in \mathbb{N}. \tag{5}$$

Define

$$\begin{aligned} \theta &= \sum_{n=1}^{\infty} 2^{-n} \|\varphi_n\|^{-1} |\varphi_n| \in M_*^+, \\ X &= \{e \in P(M);\, e \leqslant \mathbf{s}(\theta)\}, \\ X_m &= \{e \in X;\, |\varphi_n(e)| \leqslant m \text{ for all } n \in \mathbb{N}\}; \quad m \in \mathbb{N}. \end{aligned}$$

By 8.12, the s^*-topology on X is defined by the metric

$$d_\theta(e, f) = \theta((e-f)^2)^{1/2}; \quad e, f \in X,$$

and this metric is complete. The sets X_m are s^*-closed and, by the assumption (3), $X = \bigcup\limits_m X_m$. Using the Baire category theorem we infer that

$$\begin{aligned} &\text{there exist } m_0 > 0,\ \varepsilon > 0 \text{ and } e_0 \in X \text{ such that} \\ &e \in X,\ \theta((e-e_0)^2) < \varepsilon \Rightarrow |\varphi_n(e)| \leqslant m_0 \text{ for all } n \in \mathbb{N}. \end{aligned} \tag{6}$$

On the other hand, by the above lemma, there are mutually orthogonal projections $\{e_1, \ldots, e_q,\, e_{q+1}, \ldots, e_p\}$ in M with $\sum\limits_{k=1}^{p} e_k = \mathbf{s}(\theta)$ such that $e_1, \ldots, e_q$ are atoms and

$$\theta(e_j) < \varepsilon \quad \text{for all } q+1 \leqslant j \leqslant p. \tag{7}$$

Let $f \in X$ and fix $1 \leqslant i \leqslant q$, $q+1 \leqslant j \leqslant p$. Since e_i is an atom and M is assumed commutative,

$$|\varphi_n(fe_i)| \leqslant |\varphi_n(e_i)| \leqslant \lambda_{e_i}; \quad n \in \mathbb{N}. \tag{8}$$

Put $g = e_0 + fe_j - e_0 fe_j$, $h = e_0 - e_0 fe_j$. Then $g, h \in X$, $(g - e_0)^2 \leqslant e_j$, $(h - e_0)^2 \leqslant e_j$ so, by (7) and (6),

$$|\varphi_n(g)| \leqslant m_0, \quad |\varphi_n(h)| \leqslant m_0; \quad n \in \mathbb{N}.$$

Since $fe_j = g - h$, it follows that

$$|\varphi_n(fe_j)| \leqslant 2m_0; \quad n \in \mathbb{N}. \tag{9}$$

From (8) and (9) we infer that

$$|\varphi_n(f)| \leqslant \sum_{k=1}^{p} |\varphi_n(fe_k)| \leqslant \lambda_{e_1} + \cdots + \lambda_{e_q} + 2(p-q)m_0. \tag{10}$$

Since $f_n \in X$ for all $n \in \mathbb{N}$, the relations (5) and (10) are contradictory. ∎

The arguments used in the above proof are essentially commutative and in fact they have appeared in measure theory. The similarity with measure theory will be more apparent from the following considerations, which are also necessary for further use.

Let M be a *commutative* W^*-algebra. Let $\rho \in M^*$, positive, and $x \in M$ with polar decomposition $x = v|x|$. Then

$$\begin{aligned}|\rho(x)| &= |\rho(v|x|^{1/2}|x|^{1/2})| = |\rho(|x|^{1/2}(v|x|^{1/2}))| \\ &\leqslant \rho(|x|^{1/2}|x|^{1/2})^{1/2}\rho(|x|^{1/2}v^*v|x|^{1/2})^{1/2} \leqslant \rho(|x|).\end{aligned}$$

Now let $\varphi \in M^*$ with polar decomposition $\varphi = |\varphi|(\cdot\, v)$, ($v \in M^{**}$), and $x \in M$. Since M^{**} is commutative, $|xv|^2 = x^*v^*vx \leqslant x^*x = |x|^2$ so $|xv| \leqslant |x|$, hence $|\varphi(x)| = |\,|\varphi|(xv)| \leqslant |\varphi|(|xv|) \leqslant |\varphi|(|x|)$. Thus

$$|\varphi(x)| \leqslant |\varphi|(|x|); \quad \varphi \in M^*,\ x \in M. \tag{11}$$

Moreover, for every $\varphi \in M^*$ and every projection $e \in M$ we have

$$|\varphi|(e) = \sup\Big\{\sum_{k=1}^{n} |\varphi(e_k)|;\ e_1, \ldots, e_n \in P(M),\ \sum_{k=1}^{n} e_k \leqslant e\Big\}. \tag{12}$$

Indeed, if $e_1, \ldots, e_n$ are mutually orthogonal projections with $\sum\limits_{k=1}^{n} e_k \leqslant e$, then by (11)

$$\sum_{k=1}^{n} |\varphi(e_k)| \leqslant \sum_{k=1}^{n} |\varphi|(e_k) = |\varphi|\Big(\sum_{k=1}^{n} e_k\Big) \leqslant |\varphi|(e).$$

Conversely, let $\varphi = |\varphi|(\cdot\, v)$, $(v \in M^{**})$, be the polar decomposition of φ and $\varepsilon > 0$. Using the Kaplansky density theorem we find $u \in M$, $\|u\| \leqslant \|v^*\| \leqslant 1$, such that

$$|\varphi|(e) = \varphi(ev^*) \leqslant |\varphi(eu)| + \varepsilon/2.$$

Since M is commutative, $eu \in M$ is normal. Using Theorem 7.15 we find mutually orthogonal projections $e_1, \ldots, e_n$ with $\sum\limits_{k=1}^{n} e_k \leqslant e$ and $\lambda_1, \ldots, \lambda_n \in \sigma(eu) \subset \{\lambda \in \mathbb{C};\ |\lambda| \leqslant 1\}$ such that

$$\|eu - \sum_{k=1}^{n} \lambda_k e_k\| \leqslant \varepsilon/2\|\varphi\|.$$

Then

$$|\varphi(eu)| \leqslant \Big|\sum_{k=1}^{n} \lambda_k \varphi(e_k)\Big| + \varepsilon/2 \leqslant \sum_{k=1}^{n} |\varphi(e_k)| + \varepsilon/2,$$

hence

$$|\varphi|(e) \leqslant \sum_{k=1}^{n} |\varphi(e_k)| + \varepsilon.$$

In particular

(13) $\qquad |\varphi|(e) = |\varphi(e)|$ for every minimal projection $e \in M$.

Also, (12) shows that $\|\varphi\| = \|\,|\varphi|\,\| = |\varphi|(1)$ is the "total variation" of φ.

8.16. One more application of the Baire category theorem yields an analogue (in fact an extension) of the Vitali-Hahn-Saks theorem, which is a powerful tool in the study of the topological properties of preduals of W^*-algebras.

THEOREM. *Let M be a W^*-algebra and $\{\varphi_n\}_n$ be a sequence of normal linear forms on M such that $\{\varphi_n(e)\}_n$ is a Cauchy sequence for every projection $e \in M$. Then*

(i) *There exists a normal linear form φ on M such that $\varphi_n(x) \to \varphi(x)$ for all $x \in M$ and $\|\varphi\| \leqslant \sup\limits_n \|\varphi_n\| < +\infty$.*

(ii) *For every norm bounded sequence $\{x_k\}_k$ in M such that $x_k \xrightarrow{s^*} 0$ we have*

$$\varphi_n(x_k) \xrightarrow{k} 0 \textit{ uniformly for } n \in \mathbb{N}.$$

Proof. By Theorem 8.15 we obtain $\lambda = \sup\limits_n \|\varphi_n\| < +\infty$. Using this fact and Proposition 3/7.16, it follows that $\{\varphi_n(x)\}_n$ is a Cauchy sequence for every positive $x \in M$ and hence for all $x \in M$. Put

$$\varphi(x) = \lim_n \varphi_n(x); \quad x \in M.$$

We thus obtain a linear functional φ on M and clearly $\|\varphi\| \leqslant \lambda$, so $\varphi \in M^*$.

We now prove (ii). Let $e = \mathbf{s}\Big(\sum_{n=1}^{\infty} 2^{-n}(|\varphi_n|+|\varphi_n^*|)\Big)$. Then $\varphi_n(x) = \varphi_n(exe)$, $(n \in \mathbb{N})$, and consequently $\varphi(x) = \varphi(exe)$ for all $x \in M$. Thus, replacing M by eMe, we may assume that M is countably decomposable. In this case M_1 is s-metrizable by a complete metric (8.12).

Let $\varepsilon > 0$ be arbitrary but fixed. Then

$$S_n = \{x \in M_1;\, |\varphi_i(x) - \varphi_j(x)| \leqslant \varepsilon \text{ for all } i, j \geqslant n\}, \quad n \in \mathbb{N}$$

are s-closed subset of M_1 and $M_1 = \bigcup_n S_n$. By the Baire category theorem, there exists $n_0 \in \mathbb{N}$, $x_0 \in M_1$ and an s-neighborhood V of x_0 in M_1 such that

$$x \in V \Rightarrow |\varphi_i(x) - \varphi_j(x)| \leqslant \varepsilon \quad \text{for all } i, j \geqslant n_0.$$

By Proposition 1/7.16, for each $k \in \mathbb{N}$ there exists a projection $e_k \in M$, such that

$$e_k(x_k^* x_k + x_k x_k^*)e_k \leqslant \varepsilon e_k, \quad (1 - e_k) \leqslant \varepsilon^{-1}(x_k^* x_k + x_k x_k^*).$$

Since $x_k \xrightarrow{s^*} 0$, we infer that $e_k \xrightarrow{s^*} 1$ and

$$\|x_k e_k\| \leqslant \varepsilon,\ \|e_k x_k\| \leqslant \varepsilon; \quad k \in \mathbb{N}.$$

Then $e_k x_0 e_k \in M_1$, $e_k x_0 e_k + (1 - e_k)x_k(1 - e_k) \in M_1$ and

$$e_k x_0 e_k \xrightarrow{s} x_0, \quad e_k x_0 e_k + (1 - e_k)x_k(1 - e_k) \xrightarrow{s} x_0,$$

so there $k_0 \in \mathbb{N}$ such that for all $k \geqslant k_0$

$$e_k x_0 e_k \in V, \quad e_k x_0 e_k + (1 - e_k)x_k(1 - e_k) \in V.$$

It follows that

$$|\varphi_i - \varphi_j)((1 - e_k)x_k(1 - e_k))| \leqslant 2\varepsilon \quad \text{for all } i, j \geqslant n_0,\ k \geqslant k_0.$$

consequently, for $n \geqslant n_0$ and $k \geqslant k_0$

$$\begin{aligned} |\varphi_n(x_k) - \varphi_{n_0}(x_k)| &\leqslant \|\varphi_n - \varphi_{n_0}\|(\|e_k x_k e_k\| + \|(1 - e_k)x_k e_k\| + \|e_k x_k(1 - e_k)\|) \\ &\qquad + |(\varphi_n - \varphi_{n_0})((1 - e_k)x_k(1 - e_k))| \\ &\leqslant 6\lambda\varepsilon + 2\varepsilon. \end{aligned}$$

Since $\varphi_n(x_k) \xrightarrow{k} 0$ for each $n = 1, 2, \ldots, n_0$, it follows that

$$\varphi_n(x_k) \xrightarrow{k} 0 \quad \text{uniformly for } n \in \mathbb{N}.$$

Finally, it follows that $\varphi(x_k) \to 0$, hence φ is normal and this achieves the proof of (i). ∎

For singular linear forms we note the following result:

Proposition. *Let M be a W^*-algebra. The $\sigma(M^*,M)$-closure of any countable subset of $(1-p_M)\cdot M^*$ is contained in $(1-p_M)\cdot M^*$.*

Proof. Let $\{\varphi_n;\, n\in\mathbb{N}\}$ be an arbitrary countable subset in $(1-p_M)\cdot M^*$ and φ be a $\sigma(M^*,M)$-limit point of $\{\varphi_n;\, n\in\mathbb{N}\}$. Then

$$\theta=\sum_{n=1}^{\infty}2^{-1}\|\varphi_n\|^{-1}|\varphi_n|\in(1-p_M)\cdot M^*.$$

Let $e\in M$ be a non-zero projection. There exists a non-zero projection $f\in M, f\leqslant e$ such that $\theta(f)=0$. Then $\varphi_n(f)=0$ for all n, so $\varphi(f)=0$. Hence $\varphi\in(1-p_M)\cdot M^*$. ∎

As every dual Banach space, the dual M^* of a W^*-algebra M is $\sigma(M^*,M)$-sequentially complete. Furthermore, from the above results it follows that

Corollary 1. *Let M be a W^*-algebra. The $M_*=p_M\cdot M^*$ is $\sigma(M_*,M)$-sequentially complete and $(1-p_M)\cdot M^*$ is $\sigma((1-p_M)\cdot M^*,\ M)$-sequentially complete.* ∎

Since the second dual of a C^*-algebra is a W^*-algebra, we infer that

Corollary 2. *Let A be a C^*-algebra. Then A^* is $\sigma(A^*,A^{**})$-sequentially complete.* ∎

8.17. Compactness in the predual. An important application of Theorem 8.16 is the caracterization of weakly relatively compact subset of the preduals of W^*-algebras by analogy with the Dunford-Pettis theorem.

Theorem. *Let M be a W^*-algebra and $K\subset M_*$. Then the following statements are equivalent:*

(i) *K is relatively $\sigma(M_*,M)$-compact;*

(ii) *K is norm bounded and if $\{e_n\}_n$ is a sequence of mutually orthogonal projections in M, then*

$$\lim_n\varphi(e_n)=0\qquad\textit{uniformly for } \varphi\in K;$$

(iii) *K is norm bounded and there exists $\rho\in M_*^+$ with the property for every $\varepsilon>0$ there exists $\delta>0$ such that*

$$x\in M_1,\quad \rho(x^*x+xx^*)<\delta\Rightarrow|\varphi(x)|<\varepsilon\qquad\textit{for all } \varphi\in K.$$

Proof. (i) $\Rightarrow$ (iii). Suppose that K is relatively $\sigma(M_*,M)$-compact. Then K is $\sigma(M_*,M)$-bounded and hence also norm-bounded (7.3). Thus, we may assume $\|\varphi\|\leqslant 1$ for all $\varphi\in K$.

We first prove the following statement: for every $\varepsilon > 0$ there exists $\delta > 0$ and a finite set $K_\varepsilon \subset K$ such that

$$
\begin{aligned}
(1) \qquad & x \in M_1,\ (|\psi| + |\psi^*|)(x^*x + xx^*) < \delta \text{ for every } \psi \in K_\varepsilon \\
& \Rightarrow |\varphi(x)| < \varepsilon \text{ for every } \varphi \in K.
\end{aligned}
$$

Suppose that the statement is false for some $\varepsilon > 0$. Then, by induction, we can construct a sequence $\{\varphi_k\}$ in K and a sequence $\{x_k\}$ in M_1 such that

$$
(2) \qquad |\varphi_{k+1}(x_k)| \geqslant \varepsilon,
$$

$$
(3) \qquad (|\varphi_n| + |\varphi_n^*|)(x_k^*x_k + x_kx_k^*) < 2^{-k} \quad \text{for } n = 1, 2, \ldots, k.
$$

By the Eberlein-Shmulyan theorem ([81],V.6.1), a subsequence of $\{\varphi_k\}$ is $\sigma(M_*, M)$-covergent. Replacing $\{\varphi_k\}$ by this subsequence, we may suppose that $\{\varphi_k\}$ is $\sigma(M_*, M)$-covergent. Let

$$
\theta = \sum_{n=1}^{\infty} 2^{-n}(|\varphi_n| + |\varphi_n^*|)
$$

and $e = \mathbf{s}(\theta)$. Then, using (3),

$$
\begin{aligned}
\theta(x_k^*x_k + x_kx_k^*) &= \sum_{n=1}^{\infty} 2^{-n}(|\varphi_n| + |\varphi_n^*|)(x_k^*x_k + x_kx_k^*) \\
&\leqslant \sum_{n=1}^{k} 2^{-n}(|\varphi_n| + |\varphi_n^*|)(x_k^*x_k + x_kx_k^*) + \sum_{n=k+1}^{\infty} 2^{n-1}\|\varphi_n\| \\
&\leqslant 2^{-k} + 2^{-k+1} \xrightarrow{k} 0,
\end{aligned}
$$

so, by Proposition 2/8.12, $ex_ke \xrightarrow{s^*} 0$.

By Theorem 8.16 we infer that

$$
\lim_k \varphi_n(x_k) = \lim_k \varphi_n(ex_ke) = 0 \quad \text{uniformly for } n \in \mathbb{N},
$$

is contradiction with (2).

Now, for each $\varepsilon_n = n^{-1}$ choose $\delta_n > 0$ and a finite subset $K_{\varepsilon_n} = \{\varphi_1^n, \ldots, \varphi_{j_n}^n\}$ of K which enjoy the property (1). Then

$$
\rho \sum_{n=1}^{\infty} 2^{-n} \sum_{j=1}^{j_n} 2^{-j}(|\varphi_j^n| + |(\varphi_j^n)^*|) \in M_*^+
$$

satisfies the condition (iii).

(iii) $\Rightarrow$ (ii). Let $\rho \in M_*^+$ satisfies the condition (iii), let $\varepsilon > 0$ and choose $\delta > 0$ as in (iii). By the normality of ρ there exists $n_0 \in \mathbb{N}$ such that

$$\rho(e_n^* e_n + e_n e_n^*) = 2\rho(e_n) < \delta \quad \text{for all } n \geqslant n_0.$$

Then $|\varphi(e_n)| < \varepsilon$ for all $n \geqslant n_0$ and all $\varphi \in K$.

(ii) $\Rightarrow$ (i). Since K is a bounded subset of M^*, by the Alaoglu theorem it follows that the $\sigma(M^*, M)$-closure $\overline{K}$ of K in M^* is $\sigma(M^*, M)$-compact. Thus, it suffices to show that $\overline{K} \subset M_*$.

Let $\varphi \in \overline{K}$. There is a net $\{\varphi_\kappa\}_\kappa$ in K, $\sigma(M^*, M)$-convergente to φ. Let $\{e_\iota\}_{\iota \in I}$ be a family of mutually orthogonal projections in M and $e = \sum\limits_\iota e_\iota$. Then

$$\varphi(e) = \lim_\kappa \varphi_\kappa(e), \tag{4}$$

$$\varphi(e_\iota) = \lim_\kappa \varphi_\kappa(e_\iota), \quad \text{for all } \iota. \tag{5}$$

From (ii) it follows that $\psi(e) = \sum\limits_\iota \psi(e_\iota)$ uniformly for $\psi \in K$. Indeed, in the contrary case there would exist a sequence $\{\psi_n\}_n$ in K, a sequence $\{J_n\}_n$ of mutually disjoint finite subset of I and $\varepsilon > 0$ such that

$$\Big|\sum_{\iota \in J_n} \psi_n(e_\iota)\Big| \geqslant \varepsilon \quad \text{for all } n \in \mathbb{N}.$$

Put $q_n = \sum\limits_{\iota \in J_n} e_\iota$, $(n \in \mathbb{N})$. Then $\{q_n\}_n$ is a sequence of mutually orthogonal projections in M and $|\psi_n(q_n)| \geqslant \varepsilon$, in contradiction with (ii). Thus

$$\varphi_\kappa(e) = \sum_{\iota \in I} \varphi_\kappa(e_\iota), \quad \text{uniformly for } \kappa. \tag{6}$$

From (4), (5), (6) we infer that $\varphi(e) = \sum\limits_{\iota \in I} \varphi(e_\iota)$. Hence $\varphi \in M_*$. ∎

By the above theorem, it follows that a set $K \subset M_*$ is relatively $\sigma(M_*, M)$-compact if and only if the absolutely convex hull of K is relatively $\sigma(M_*, M)$-compact. Thus, the Mackey topology τ_w associated to the topology $w = \sigma(M, M_*)$ on M, originally defined as the topology of uniform convergence on absolutely convex relatively $\sigma(M_*, M)$-compact subset of M_*, is in fact the topology of uniform convergence on relatively $\sigma(M_*, M)$-compact subset of M_*.

The relatively $\sigma(M_*, M)$-compact subset of M_* are also called weakly relatively compact or, shortly, *wrc sets*.

COROLLARY. *Let M be a W^*-algebra. The restriction to M_1 of the Mackey topology τ_w associated to the w-topology coincides with the restriction to M_1 of the s^*-topology.*

Proof. By 8.5, $s^* < \tau_w$. Let $\{x_\iota\} \subset M_1$ be a net such that $x_\iota \xrightarrow{s^*} 0$. Let K be a wrc subset of M_* and let $\rho \in M_*^+$ be as in the statement (iii) of the above theorem. Then $\rho(x_\iota^* x_\iota + x_\iota x_\iota^*) \to 0$, so $\varphi(x_\iota) \to 0$ uniformly for $\varphi \in K$. This shows that $x_\iota \xrightarrow{\tau_w} 0$. ∎

However, *if M is infinite dimensional, then $\tau_w \neq s^*$ on M.*

Indeed, in this case there exists an infinite sequence $\{e_n\}_n$ of mutually orthogonal non-zero projections in M. Let

$$S = \{n^{1/2}e_n;\, n \in \mathbb{N}\} \subset M.$$

Given $\varphi \in M_*^+$ and $\varepsilon > 0$, there exists $n \in \mathbb{N}$ with

$$\varphi((n^{1/2}e_n)^*(n^{1/2}e_n) + (n^{1/2}e_n)(n^{1/2}e_n)^*) = 2n\,\varphi(e_n) < \varepsilon,$$

since otherwise it would follow

$$\sum_n \varepsilon/2n \leqslant \sum_n \varphi(e_n) = \varphi\Big(\sum_n e_n\Big) < +\infty.$$

Hence 0 is s^*-adherent to S. On the other hand, for each n there is an element $\varphi_n \in M_*^+$ with $\mathbf{s}(\varphi_n) \leqslant e_n$ and $\|\varphi_n\| = 1$. Then the set

$$K = \{n^{-1/2}\varphi_n;\, n \in \mathbb{N}\} \cup \{0\} \subset M_*$$

is wrc (in fact K is norm-compact), so $V = \{x \in M;\, |n^{-1/2}\varphi_n(x)| < 1$ for all $n \in \mathbb{N}\}$ is a τ_w-neighborhood of 0. Since $V \cap S = \emptyset$, it follows that 0 is not τ_w-adherent to S.

We now list some other consequences of the above theorem.

Using condition (ii), it follows that

(7) if $K \subset M_*$ is wrc, then also $K^* = \{\varphi^*;\, \varphi \in K\}$ is wrc.

In particular,

(8) the $*$-operation is τ_w-continuous on M,

and, by the corollary and by the properties of the s^*-topology,

(9) the mapping $M_1 \times M_1 \ni (x, y) \mapsto xy \in M_1$ is τ_w-continuous.

Using again condition (ii) together with the Schwarz inequality, it is easy to see that

(10) if $K \subset M_*^+$ is wrc, then also $\{a \cdot \varphi;\, \varphi \in K,\, a \in M_1\}$ and $\{\varphi \cdot a;\, \varphi \in K,\, a \in M_1\}$ are wrc.

In particular, by polar decomposition,

(11) if $|K| = \{|\varphi|;\, \varphi \in K\}$ is wrc, then also K is wrc.

However, the converse of (11) is not true.

For example, let H be an infinite dimensional Hilbert space, let $\{\xi_n\}_n$ be an orthogonal sequence in H and $M = B(H)$. For each $n \in \mathbb{N}$ define $v_n, e_n \in M$ and $\varphi_n, \rho_n \in M_*$ by

$$v_n\xi = (\xi|\xi_n)\xi_1, \quad e_n\xi = (\xi|\xi_n)\xi_n \quad (\xi \in H),$$
$$\varphi_n(x) = (x\xi_1|\xi_n), \quad \rho_n(x) = (x\xi_n|\xi_n) \quad (x \in M).$$

Then $\varphi_n = \rho_n(\cdot\, v_n)$ is the polar decomposition of φ_n and $\varphi_n^* = \rho_1(\cdot\, v_n^*)$ is the polar decomposition of φ_n^*. Thus, $|\varphi_n| = \rho_n$ and $|\varphi_n^*| = \rho_1$ for all $n \in \mathbb{N}$. The one point set $\{|\varphi_n^*|;\, n \in \mathbb{N}\}$ is wrc, hence $\{\varphi_n^*;\, n \in \mathbb{N}\}$ is wrc by (11) and then $\{\varphi_n;\, n \in \mathbb{N}\}$ is wrc by (7). On the other hand, $\{|\varphi_n|;\, n \in \mathbb{N}\}$ is not wrc, because $|\varphi_n|(e_n) = 1$ for all $n \in \mathbb{N}$.

Note also that although the sets $\{\rho_1\} \subset M_*^+$ and $\{\varphi_n^*;\, n \in \mathbb{N}\} \subset M_*$ are wrc, the set

$$\{v_n \cdot \rho_1 \cdot v_n^*;\, n \in \mathbb{N}\} = \{v_n \cdot \varphi_n^*;\, n \in \mathbb{N}\} = \{|\varphi_n|;\, n \in \mathbb{N}\}$$

is not wrc.

Using one more time condition (ii) and Proposition 2/8.11, the result (10) can be improved as follows

(12) if $K \subset M_*^+$ is wrc, then also $\{|a \cdot \varphi|;\, \varphi \in K,\, a \in M_1\}$ and $\{|\varphi \cdot a|;\, \varphi \in K,\, a \in M_1\}$ are wrc.

If in addition M is *commutative*, then

(13) $K \subset M_*$ is wrc if and only if $|K| = \{|\varphi|;\, \varphi \in K\}$ is wrc.

Indeed, suppose $|K|$ is not wrc. Then by the theorem, there exists $\varepsilon > 0$, a sequence $\{e_n\}_n$ of mutually orthogonal projections in M and a sequence $\{\varphi_n\}_n$ in K such that

$$|\varphi_n|(e_n) \geqslant \varepsilon \quad \text{for all } n \in \mathbb{N}. \tag{14}$$

Let $\varphi_n = |\varphi_n|(\cdot\, v_n)$ be the polar decomposition of φ_n and put $x_n e_n v_n^* = v_n^* e_n$, $(n \in \mathbb{N})$. Then $\{x_n\} \subset M_1$ and $x_n \xrightarrow{s^*} 0$, because $e_n \xrightarrow{s^*} 0$. By the corollary, $x_n \xrightarrow{\tau_w} 0$. If K is wrc, then

$$|\varphi_n|(e_n) = \varphi_n(x_n) \to 0,$$

in contradiction with (14). Hence K is not wrc.

Also, if M is *commutative*, then

(15) $K \subset M_*$ is wrc if and only $\{a \cdot \varphi \cdot b; \varphi \in K,\, a, b \in M_1\}$ is wrc.

Indeed, $a \cdot \varphi \cdot b = (abv_\varphi) \cdot |\varphi|$, where $\varphi = |\varphi| \cdot v_\varphi$ is the polar decomposition of φ, so (15) follows using (13) and(10).

The fact that simultaneously with K also $|K|$ or $\{a \cdot \varphi;\, \varphi \in K,\, a \in M_1\}$ si wrc characterizes an important class of W^*-algebras (the "finite" W^*-algebras; see [263]).

A normal linear mapping Φ from the W^*-algebra M into the W^*-algebra N is automatically bounded and τ_w-continuous. Thus, by the corollary,

(16) every normal linear mapping $\Phi : M \to N$ is s^*-continuous on bounded subset of M.

Moreover

(17) every positive normal linear mapping $\Phi : M \to N$ is s^*-continuous on the whole M.

Indeed, let $\{x_\iota\}_\iota$ be a net in M, $x_\iota \xrightarrow{s^*} 0$. we have to show that $\Phi(x_\iota) \xrightarrow{s^*} 0$. Since the $*$-operation is s^*-continuous, we may assume that each x_ι is selfadjoint. Let $\psi \in N_*^+$. Then $\varphi = \psi \circ \Phi \in M_*^+$ and, using the Kadison inequality (5.8), we get

$$\psi(\Phi(x_\iota)^*\Phi(x_\iota))^{1/2} + \psi(\Phi(x_\iota)\Phi(x_\iota)^*)^{1/2} \leqslant \|\Phi\|^{1/2}(\varphi(x_\iota^* x_\iota)^{1/2} + \varphi(x_\iota x_\iota^*)^{1/2}) \to 0.$$

Hence $\Phi(x_\iota) \xrightarrow{s^*} 0$.

8.18. Atomic W^*-algebras. Let M be a W^*-algebra. If e is an atom of M, then the W^*-algebra eMe has only two projections, 0 and e, hence $eMe = Ce$. Thus, if f is a finite sum of mutually orthogonal atoms of M, then fMf is finite dimensional.

A W^*-algebra M is called *atomic* if every non-zero projection of M majorizes an atom of M. In this case, any maximal family $\{e_\iota\}_\iota$ of mutually orthogonal atoms of M has the property $\sum\limits_\iota e_\iota = 1$. In particular, there exists an increasing net of projections $f_\iota \uparrow 1$ such that the W^*-algebras $f_\iota M f_\iota$ are finite dimensional.

Proposition 1. *Let M be an atomic W^*-algebra. If a sequence $\{\varphi_n\}_n$ in M_* is $\sigma(M_*, M)$-convergent to $\varphi \in M_*$ and the sets $\{|\varphi_n|;\, n \in \mathbb{N}\}$, $\{|\varphi_n^*|;\, n \in \mathbb{N}\}$ are both relatively $\sigma(M_*, M)$-compact, then*

$$\|\varphi_n - \varphi\| \longrightarrow 0.$$

Proof. Let $\varepsilon > 0$. By assumption the set

$$K = \{\varphi_n, \varphi_n^*, |\varphi_n|, |\varphi_n^*|, \varphi, \varphi^*, |\varphi|, |\varphi^*|;\, n \in \mathbb{N}\} \subset M_*$$

is relatively $\sigma(M_*, M)$-compact and M contains an increasing net of projections $f_\iota \uparrow 1$ such that the W^*-algebras $f_\iota M f_\iota$ are all finite dimensional. We may suppose that $\|\psi\| \leqslant 1$ for all $\psi \in K$.

Using condition (iii) from Theorem 8.17 we infer that there exists a projection $f \in M$ such that fMf is finite dimensional and

$$\|(1-f)\cdot\psi\cdot(1-f)\| \leqslant \varepsilon^2 \quad \text{for all } \psi \in K.$$

By Proposition 8.9, for every $\psi \in K$ and every $x \in M_1$ we get

$$|\psi((1-f)x)| \leqslant \|\psi\|^{1/2}|\psi|((1-f)xx^*(1-f))^{1/2} \leqslant \varepsilon.$$

Since fMf is finite dimensional, there exists $n_\varepsilon \in \mathbb{N}$ such that

$$\|f\cdot(\varphi_n-\varphi)\cdot f\| \leqslant \varepsilon \quad \text{for all } n \geqslant n_\varepsilon.$$

Then, for every $x \in M_1$ and every $n \geqslant n_\varepsilon$ we have

$$\begin{aligned}|(\varphi_n-\varphi)(x)| &= |(\varphi_n-\varphi)(fxf+fx(1-f)+(1-f)x)|\\ &\leqslant |(\varphi_n-\varphi)(fxf)|+|(\varphi_n^*-\varphi^*)((1-f)x^*f)|+|(\varphi_n-\varphi)((1-f)x)|\\ &\leqslant \varepsilon+2\varepsilon+2\varepsilon=5\varepsilon.\end{aligned}$$

Hence $\|\varphi_n-\varphi\| \leqslant 5\varepsilon$ for all $n \geqslant n_\varepsilon$. ∎

COROLLARY 1. *Let M be an atomic W^*-algebra. A sequence $\{\varphi_n\}_n$ in M_*^+ is $\sigma(M_*, M)$-convergent if and only if it is norm-convergent.* ∎

COROLLARY 2. *Let M be an atomic commutative W^*-algebra. A sequence $\{\varphi_n\}_n$ in M_*^+ is $\sigma(M_*, M)$-convergent if and only if it is norm-convergent.*

Proof. By Proposition 1 and 8.17.(13), (10). ∎

Let M be an *atomic commutative* W^*-algebra and let $\{e_\iota\}_{\iota\in I}$ be the set of all the atoms of M. Then the projections e_ι are mutually orthogonal and $\sum\limits_{\iota\in I} e_\iota = 1$. Moreover, for every $x \in M$ we have $xe_\iota = \lambda_\iota(x)e_\iota$, with $\lambda_\iota(x) \in \mathbb{C}$, $(\iota \in I)$, and

$$x = \sum_{\iota\in I}\lambda_\iota(x)e_\iota \text{ in the } s^*\text{-topology}.$$

The map $x \mapsto \{\lambda_\iota(x)\}_{\iota\in I}$ is a $*$-isomorphism of M onto $\ell^\infty(I)$. Hence M_* is isometrically isomorphic to $\ell^1(I)$.

For every singular form ψ on M we have $\psi(e_\iota) = 0$, $(\iota \in I)$. Hence $\varphi(e_\iota) = (p_M\cdot\varphi)(e_\iota)$, $(\iota \in I)$, for all $\varphi \in M^*$. Using this fact and 8.15.(13), we obtain

(1) $$(p_M\cdot\varphi)(x) = \sum_{\iota\in I}\lambda_\iota(x)\varphi(e_\iota); \quad \varphi \in M^*,\ x \in M;$$

(2) $$|p_M\cdot\varphi|(x) = \sum_{\iota\in I}\lambda_\iota(x)|\varphi(e_\iota)|; \quad \varphi \in M^*,\ x \in M.$$

In particular

(3) $$\|p_M\cdot\varphi\| = \sum_{\iota\in I}|\varphi(e_\iota)|; \quad \varphi \in M^*.$$

Sometimes it is convenient to denote

(4) $$\varphi(J) = \varphi\Big(\sum_{\iota\in J}e_\iota\Big); \quad \varphi \in M^*,\ J \subset I.$$

The following result, known as the "Phillips Lemma", improves Corollary 2.

PROPOSITION 2. *Let M be an atomic commutative W^*-algebra. If a sequence $\{\varphi_n\}_n$ in M^* is ξ is $\sigma(M^*, M)$-convergent to $\varphi \in M^*$, then $\{p_M \cdot \varphi_n\}_n$ is norm-convergent to $p_M \cdot \varphi$.*

Proof. By the uniform boundedness theorem,

$$\lambda = \sup_n \|\varphi_n\| < +\infty.$$

Without loss of generality we may assume that

$$\varphi_n(x) \to 0 \quad \text{for all } x \in M. \tag{5}$$

Let $\{e_\iota\}_{\iota \in I}$ be the family of all the atoms of M. By (3), we have to show that

$$\sum_{\iota \in I} |\varphi_n(e_\iota)| \xrightarrow{n} 0.$$

Assume the contrary holds. Then, replacing $\{\varphi_n\}_n$ by a subsequence, we may suppose that there exists $\varepsilon > 0$ such that

$$\sum_{\iota \in I} |\varphi_n(e_\iota)| \geqslant 4\varepsilon \quad \text{for all } n \in \mathbb{N}. \tag{6}$$

By (5) and (6) we can construct a subsequence $\{\psi_n\}_n$ of $\{\varphi_n\}_n$ and a sequence $\{I_n\}_n$ of mutually disjoint finite subset of I such that

$$\sum_{\iota \in I_n} |\psi_n(e_\iota)| \geqslant 3\varepsilon, \qquad \sum_{\iota \in I_1 \cup \cdots \cup I_{n-1}} |\psi_n(e_\iota)| < \varepsilon.$$

Using 8.15.(12) and the notation (4), we can rewrite these relations as follows

$$|\psi_n|(I_n) \geqslant 3\varepsilon, \tag{7}$$

$$|\psi_n|(I_1 \cup \cdots \cup I_{n-1}) < \varepsilon. \tag{8}$$

Let $F = \{I_1, I_2, \ldots\}$. There exists a decreasing sequence $\{F_n\}_n$ of infinite subset of F such that, putting

$$k_n = \min\{k \in \mathbb{N};\ I_k \in F_n\},$$

we have

$$k_n < k_{n+1}, \tag{9}$$

$$|\psi_{k_n}|(\cup\{I_k; I_k \in F_{n+1}\}) \leqslant \varepsilon. \tag{10}$$

Indeed, let $F_1 = F$ and suppose that $F_2, \ldots, F_n$ have been already chosen. Let $p \in \mathbb{N}$, $p > \lambda/\varepsilon$. There exists mutually disjoint infinite subsets $F^1_{n+1}, \ldots, F^p_{n+1}$ of F such that

$$F^1_{n+1} \cup \cdots \cup F^p_{n+1} = F_n \setminus \{I_{k_n}\}.$$

Since $\| \, |\psi_{k_n}| \, \| \leqslant \lambda$, there exists an index $j, (1 \leqslant j \leqslant p)$, such that (9) and (10) are satisfied with $F_{n+1} = F_{n+1}^j$.

In particular,

$$|\psi_{k_n}|\Big(\bigcup_{m=n+1}^{\infty} I_{k_m}\Big) \leqslant \varepsilon. \tag{11}$$

Put $\theta_n = \psi_{k_n}$, $J_n = I_{k_n}$, $(n \in \mathbb{N})$. By (7), (8) and (11), we have

$$|\theta_n|(J_n) \geqslant 3\varepsilon, \tag{12}$$

$$|\theta_n|\Big(\bigcup_{m \neq n} J_m\Big) \leqslant 2\varepsilon. \tag{13}$$

Now consider

$$x = \sum_{m=1}^{\infty} \sum_{\iota \in J_m,\, \theta_m(e_\iota) \neq 0} (|\theta_m(e_\iota)|/\theta_m(e_\iota)) e_\iota \in M.$$

Using (12), (13) and 8.15.(11), we obtain

$$\begin{aligned} |\theta_n(x)| &\geqslant \Big| \theta_n\Big(\sum_{\iota \in J_n, \theta_n(e_\iota) \neq 0} (|\theta_n(e_\iota)|/\theta_n(e_\iota)) e_\iota \Big) \Big| \\ &\qquad - |\theta_n|\Big(\Big| \sum_{m \neq n} \sum_{\iota \in J_m, \theta_m(e_\iota) \neq 0} (|\theta_m(e_\iota)|/\theta_m(e_\iota)) e_\iota \Big| \Big) \\ &\geqslant |\theta_n|(J_n) - |\theta_n|\Big(\bigcup_{m \neq n} J_m\Big) \\ &\geqslant 3\varepsilon - 2\varepsilon = \varepsilon. \end{aligned}$$

On the other hand, by (5), $\theta_n(x) \to 0$, a contradiction. ∎

8.19. An appropriate extension of the Phillips Lemma to arbitrary W^*-algebras is the following

THEOREM. *Let M be a W^*-algebra. If a sequence $\{\varphi_n\}_n$ in M^* is $\sigma(M^*, M)$-convergent to $\varphi \in M^*$, then $\{p_M \cdot \varphi_n\}_n$ (respectively $\{(1-p_M)\cdot\varphi_n\}_n$) is $\sigma(M^*, M)$-convergent to $p_M \cdot \varphi$ (respectively $(1-p_M)\cdot\varphi$).*

Proof. We may suppose that $\varphi_n \to 0$ in the $\sigma(M^*, M)$-topology and then we have to show that $p_M \cdot \varphi_n \to 0$ in the $\sigma(M_*, M)$-topology.

Put

$$\theta = \sum_n 2^{-n} |(1 - p_M) \cdot \varphi_n| \in (1 - p_M) \cdot M^*.$$

Let $e \in M$ be a non-zero projection. Since θ is singular, there is a family $\{e_\iota\}_{\iota\in I}$ of mutually orthogonal non-zero projections in M with $\sum\limits_{\iota\in I} e_\iota = e$ such that $\theta(e_\iota) = 0$, $(\iota \in I)$. Then

$$\varphi_n(e_\iota) = (p_M \cdot \varphi_n)(e_\iota); \quad n \in \mathbb{N},\, \iota \in I.$$

Let $N = W^*(\{e_\iota; \iota \in I\}) \subset M$. Then N is an atomic commutative W^*-algebra and $\{e_\iota\}_{\iota\in I}$ is the family of all the atoms of N. Since $\varphi_n|N \to 0$ in the $\sigma(N^*, N)$-topology, we have

$$\sum_{\iota\in I} |\varphi_n(e_\iota)| \xrightarrow{n} 0$$

by the Phillips Lemma (Proposition 2/8.18). It follows that

$$(p_M \cdot \varphi_n)(e) = \sum_{\iota\in I}(p_M \cdot \varphi_n)(e_\iota) = \sum_{\iota\in I} \varphi_n(e_\iota) \xrightarrow{n} 0.$$

Using Proposition 3/7.16 we infer that $(p_M \cdot \varphi_n)(x) \to 0$ for all $x \in M$. ∎

We cannot expect to extend Corollary 2/8.18 for more general W^*-algebras.

Indeed, the most usual example of an infinite dimensional non-commutative atomic W^*-algebra is $M = B(H)$ with H an infinite dimensional Hilbert space. Let $\{\xi_n\}_n$ be an orthonormal sequence in H and define $\varphi_n \in M_*$, $u_n \in M$ by

$$\varphi_n(x) = (x\xi_1|\xi_n), \quad (x \in M), \quad u_n\xi = (\xi|\xi_1)\xi_n, \quad (\xi \in H).$$

Then $\sum\limits_n |\varphi_n(x)|^2 = \sum\limits_n |(x\xi_1|\xi_n)|^2 \leqslant \|x\xi_1\|^2 < +\infty$, thus $\varphi_n(x) \to$ for all $x \in M$. However, $\|\varphi_n\| \geqslant |\varphi_n(u_n)| = 1$ for all n, hence $\|\varphi_n\| \not\to 0$.

On the other hand, it is a standing conjecture that a W^*-algebra satisfying the property expressed by Corollary 1/8.18 is necessarily atomic.

8.20. Normal weights. Let M be a W^*-algebra. By Theorem 8.4, a positive form φ on M is w-continuous (or equivalently lower w-semicontinuous on M^+) if and only if

(1) $\quad \varphi\Big(\sup\limits_\iota x_\iota\Big) = \sup\limits_\iota \varphi(x_\iota)$ for every bounded increasing net $\{x_\iota\}_\iota \subset M^*$

and then φ is called a normal positive form.

A weight φ on M^+ will be called a *normal weight* if it satisfies condition (1). Obviously, every lower w-semicontinuous weight on M^+ is normal. In this section we show that every normal weight is lower w-semicontinuous on M^+. Stronger properties of normal weights will be proved later (8.22, 8.24).

Let φ be a normal weight on M^+ and $\pi_\varphi : M \to B(H_\varphi)$ be the associated GNS-representation of M (4.3). We denote by $\pi_\varphi(M)'_*$ the predual of the von Neumann algebra $\pi_\varphi(M)' \subset B(H_\varphi)$.

Lemma 1. *There is a unique linear map* $\Phi : M_\varphi \to \pi_\varphi(M)'_*$ *such that*

$$\Phi(b^*a)(T') = (T'a_\varphi|b_\varphi)_\varphi; \quad T' \in \pi_\varphi(M)';\ a, b \in N_\varphi. \tag{2}$$

Moreover, for every $x \in M_\varphi \cap M_h$,

$$\|\Phi(x)\| = \inf\{\varphi(y) + \varphi(z);\ y, z \in M_\varphi \cap M^+,\ x = y - z\}. \tag{3}$$

Proof. The uniqueness of Φ is clear since $M_\varphi = N_\varphi^* N_\varphi$.

If $a, b, c \in N_\varphi$, $c^* = c$ and $c^*c = a^*a + b^*b$, then by Proposition 7.13 there are elements $x, y \in M$ such that

$$a = xc, \quad b = yc \quad \text{and} \quad x^*x + y^*y = \mathbf{s}(cc^*) = \mathbf{s}(c)$$

and for every $T' \in \pi_\varphi(M)'$ we have

$$\begin{aligned}(T'c_\varphi|c_\varphi)_\varphi &= (T'\pi_\varphi(x^*x + y^*y)c_\varphi|c_\varphi)_\varphi \\ &= (T'\pi_\varphi(x)c_\varphi|\pi_\varphi(x)c_\varphi)_\varphi + (T'\pi_\varphi(y)c_\varphi|\pi_\varphi(y)c_\varphi)_\varphi \\ &= (T'a_\varphi|a_\varphi)_\varphi + (T'b_\varphi|b_\varphi)_\varphi.\end{aligned}$$

It follows that the map

$$\Phi_0 : M_\varphi \cap M^+ \ni a^*a \mapsto \omega_{a_\varphi}|\pi_\varphi(M)' \in \pi_\varphi(M)'_*$$

is well defined and additive. Clearly, Φ_0 is positive homogeneous. Since $M_\varphi = \mathrm{lin}(M_\varphi \cap M^+)$, Φ_0 has a linear extension Φ to M_φ and formula (2) follows using the polarization relation 2.8.(1).

The function ρ defined on $M_\varphi \cap M_h$ by the right hand side of (3) is a seminorm on $M_\varphi \cap M_h$.

If $x \in M_\varphi \cap M^+$, then clearly

$$\|\Phi(x)\| = \Phi(x)(1_H) = \big((x^{1/2})_\varphi|(x^{1/2})_\varphi\big)_\varphi = \varphi(x) = \rho(x).$$

Consequently, for $x = y - z$ with $y, z \in M_\varphi \cap M^+$ we have

$$\|\Phi(x)\| \leqslant \|\Phi(y)\| + \|\Phi(z)\| = \varphi(y) + \varphi(z).$$

Hence $\|\Phi(x)\| \leqslant \rho(x)$ for all $x \in M_\varphi \cap M_h$.

Let $x_0 \in M_\varphi \cap M_h$. By the Hahn-Banach theorem, there is a real linear functional f on $M_\varphi \cap M_h$ such that

$$f(x_0) = \rho(x_0) \quad \text{and} \quad |f(x)| \leqslant \rho(x) \quad \text{for all } x \in M_\varphi \cap M_h.$$

Then f can be extended by linearity to a complex linear functional, also denoted by f, on M_φ. Since $-\varphi(x) \leqslant f(x) \leqslant \varphi(x)$ for all $x \in M_\varphi \cap M^+$, we may consider

$\varphi + f$ and $\varphi - f$ as weights on M^+ so, using the Schwarz inequality (4.3.(6)), for $a, b \in N_\varphi$ we get

$$\begin{aligned}
|f(b^*a)| &\leqslant 2^{-1}[|(\varphi + f)(b^*a)| + |(\varphi - f)(b^*a)|] \\
&\leqslant 2^{-1}[(\varphi + f)(a^*a)^{1/2}(\varphi + f)(b^*b)^{1/2} + (\varphi - f)(a^*a)^{1/2}(\varphi - f)(b^*b)^{1/2}] \\
&\leqslant 2^{-1}[(\varphi + f)(a^*a) + (\varphi - f)(a^*a)]^{1/2}[(\varphi + f)(b^*b) + (\varphi - f)(b^*b)]^{1/2} \\
&= \varphi(a^*a)^{1/2}\varphi(b^*b)^{1/2} \\
&= \|a_\varphi\|_\varphi \|b_\varphi\|_\varphi.
\end{aligned}$$

Hence there exists $T' \in B(H_\varphi)$, $\|T'\| \leqslant 1$, such that

$$f(b^*a) = (T'a_\varphi | b_\varphi)_\varphi; \quad a, b \in N_\varphi.$$

Moreover, $T' \in \pi_\varphi(M)'$ since, for any $x \in M$ and any $a, b \in N_\varphi$,

$$(T'\pi_\varphi(x)a_\varphi | b_\varphi)_\varphi = f(b^*xa) = (\pi_\varphi(x)T'a_\varphi | b_\varphi)_\varphi.$$

It follows that

$$\rho(x_0) = |f(x_0)| = |\Phi(x_0)(T')| \leqslant \|\Phi(x_0)\| \, \|T'\| \leqslant \|\Phi(x_0)\|. \quad \blacksquare$$

LEMMA 2. *Let $\{x_n\}_n$ be a norm-bounded sequence in $M_\varphi \cap M^+$ and assume that $\{\Phi(x_n)\}_n$ is norm-convert in $\pi_\varphi(M)'_*$. Then*

(i) $x_n \xrightarrow{s} x \in M \Rightarrow x \in M_\varphi \cap M^+$;

(ii) $x_n \xrightarrow{s} 0 \Rightarrow \|\Phi(x_n)\| \to 0$.

Proof. Let $\varepsilon > 0$ and put $\psi = \lim_n \Phi(x_n) \in \pi_\varphi(M)'_*$. Without loss of generality we may suppose that, for all n,

$$\|\Phi(x_n) - \psi\| < \varepsilon/2^n \quad \text{ hence } \|\Phi(x_{n+1} - x_n)\| < \varepsilon/2^{n-1}.$$

By Lemma 1 there are sequences $\{y_n\}_n$, $\{z_n\}_n$ in $M_\varphi \cap M^+$ such that

$$x_{n+1} - x_n = y_n - z_n \quad \text{ and } \quad \varphi(y_n) + \varphi(z_n) < \varepsilon/2^{n-1}; \quad n \in \mathbb{N}.$$

Recall that the functions $f_\alpha(t) = t(1 + \alpha t)^{-1}$, $(\alpha > 0)$, from 2.7 are operator monotone, bounded, $f_\alpha \geqslant f_\beta$ for $\alpha \leqslant \beta$ and $\lim_{\alpha \to 0} f_\alpha(t) = t$ uniformly compact subset of R. Also, by Theorem 7.10, each f_α is operator continuous.

(i) Since $x_{n+1} \leqslant x_1 + \sum_{k=1}^n y_k$ and $x_{n+1} \xrightarrow{s} x$, it follows that

$$f_\alpha(x) = s\text{-}\lim_n f_\alpha(x_{n+1}) \leqslant \sup_n f_\alpha\Big(x_1 + \sum_{k=1}^n y_k\Big)$$

and then, by the normality of φ,

$$\varphi(f_\alpha(x)) \leqslant \sup_n \varphi\Big(f_\alpha\Big(x_1 + \sum_{k=1}^n y_k\Big)\Big) \leqslant \sup_n \varphi\Big(x_1 + \sum_{k=1}^n y_k\Big)$$
$$\leqslant \varphi(x_1) + \sum_{k=1}^\infty \varphi(y_k) \leqslant \varphi(x_1) + \sum_{k=1}^\infty \varepsilon/2^{k-1} = \varphi(x_1) + 2\varepsilon.$$

Since $f_\alpha(x) \uparrow x$, using again the normality of φ we get

$$\varphi(x) = \sup_{\alpha>0} \varphi(f_\alpha(x)) \leqslant \varphi(x_1) + 2\varepsilon < +\infty.$$

This shows that $x \in M_\varphi \cap M^+$.

(ii) Since $-\sup_n \|x_n\| \leqslant x_1 - x_{n+1} \leqslant \sum_{k=1}^n z_k$, we have

$$f_\alpha(x_1 - x_{n+1}) \leqslant \sup_n f_\alpha\Big(\sum_{k=1}^n z_k\Big); \quad \alpha > \Big(\sup_n \|x_n\|\Big)^{-1}.$$

Since $x_1 - x_{n+1} \xrightarrow{s} x_1$, we infer that

$$f_\alpha(x_1) = s\text{-}\lim_n f_\alpha(x_1 - x_{n+1}) \leqslant \sup_n f_\alpha\Big(\sum_{k=1}^n z_k\Big).$$

Using the normality of φ we obtain

$$\varphi(x_1) = \sup_{\alpha>0} \varphi(f_\alpha(x_1)) \leqslant \sup_{\alpha>0}\sup_n \varphi\Big(f_\alpha\Big(\sum_{k=1}^n z_k\Big)\Big)$$
$$\leqslant \sup_n \varphi\Big(\sum_{k=1}^n z_k\Big) \leqslant \sum_{k=1}^\infty \varepsilon/2^{k-1} = 2\varepsilon.$$

It follows that

$$\|\psi\| \leqslant \|\psi - \Phi(x_1)\| + \|\Phi(x_1)\| \leqslant \varepsilon/2 + 2\varepsilon = 3\varepsilon/2.$$

Hence $\psi = 0$. ∎

Let $G_\varphi = \{(x, x_\varphi);\, x \in N_\varphi\} \subset M \times H_\varphi$. Recall that every Hilbert space is reflexive as a Banach space, thus $M \times H_\varphi$ is the dual of the Banach space $M_* \times H_\varphi$. For $\lambda, \mu \in \mathbb{R}$, $\lambda, \mu > 0$, we denote

$$M_\lambda = \{x \in M;\, \|x\| \leqslant \lambda\}, \quad (H_\varphi)_\mu = \{\xi \in H_\varphi;\, \|\xi\| \leqslant \mu\}.$$

Lemma 3. *If M is countably decomposable, then $G_\varphi \cap (M_\lambda \times (H_\varphi)_\mu)$ is $\sigma(M \times H_\varphi,\ M_* \times H_\varphi)$-compact for any $\lambda, \mu > 0$.*

Proof. Since $G_\varphi \cap (M_\lambda \times (H_\varphi)_\mu)$ is convex and bounded, it is sufficient to prove that it is closed in the product topology τ on $M \times H_\varphi$ of the s^*-topology on M and the norm-topology on H_φ. Note that M_λ is s^*-metrizable since M is countably decomposable.

If $(x, \xi) \in M \times H_\varphi$ is τ-adherent to $G_\varphi \cap (M_\lambda \times (H_\varphi)_\mu)$, then there is a sequence $\{x_n\}_n$ in M_λ such that

$$x_n \xrightarrow{s^*} x, \quad \|(x_n)_\varphi\|_\varphi \leqslant \beta \quad \text{and} \quad \|(x_n)_\varphi - \xi\|_\varphi \to 0.$$

Then $x_n^* x_n \xrightarrow{s} x^* x$ and $\Phi(x_n^* x_n) = \omega_{(x_n)_\varphi} \to \omega_\xi$ in norm, so $x \in N_\varphi$ by Lemma 2. Furthermore, $(x_n - x)^*(x_n - x) \xrightarrow{s} 0$ and

$$\Phi((x_n - x)^*(x_n - x)) = \omega_{(x_n)_\varphi - x_\varphi} \to \omega_{\xi - x_\varphi}$$

so $\omega_{\xi - x_\varphi} = 0$, again by Lemma 2. Hence $\xi = x_\varphi$ and $(x, \xi) \in C_\varphi$. ∎

If M is not countably decomposable, then let P_0 be the family of countably decomposable projections in M and put

$$M_0 = \bigcup_{p \in P_0} pMp.$$

It is easy to see that M_0 is a selfadjoint two-sided ideal in M. A subset X of $M_0 \cap M^+$ is called *hereditary* in M_0 if

$$x \in X \text{ and } y \in M_0, \quad 0 \leqslant y \leqslant x \Rightarrow y \in X.$$

Lemma 4. *Let E be a convex hereditary subset of $M_0 \cap M^+$. Then E is w-closed relative to M_0 if and only if $E \cap pMp$ is w-closed for all $p \in P_0$.*

Proof. Suppose that $E \cap pMp$ is w-closed for all $p \in P_0$. Let $F = \{x \in M;\ x^*x \in E\}$. If $x, y \in F$ and $\lambda, \mu \geqslant 0$, $\lambda + \mu = 1$, then

$$\begin{aligned}(\lambda x + \mu y)^*(\lambda x + \mu y) &= \lambda^2 x^*x + \mu^2 y^*y + \lambda\mu(x^*y + y^*x) \\ &\leqslant \lambda^2 x^*x + \mu^2 y^*y + \lambda\mu(x^*x + y^*y) = \lambda x^*x + \mu y^*y,\end{aligned}$$

hence $\lambda x + \mu y \in F$. If $x \in F$ and $a \in M$, $\|a\| \leqslant 1$, then

$$(ax)^*(ax) = x^*a^*ax \leqslant x^*x,$$

hence $ax \in F$. It follows that F is convex and $aF \subset F$ for all $a \in M_1$.

We shall show that pF, or equivalently F^*p, is w-closed for any $p \in P_0$. By 8.5 and Theorem 7.4 it is sufficient to show that $F^*p \cap M_\lambda$ is s-closed for all $\lambda > 0$. Let $x \in M$ such that x^* is s-adherent to $F^*p \cap M_\lambda$. Since $M_p \cap M_\lambda$ is

s-metrizable, there is a sequence $\{x_n\}_n$ in pF, $\|x_n\| \leqslant \lambda$, with $x_n^* \xrightarrow{s} x^*$. There exists a projection $q \in P_0$ such that $x_n \in qMq$ for all n. Thus

$$x_n \in F \cap qMq = \{y \in qMq;\, y^*y \in E \cap qMq\}; \quad n \in \mathbb{N}.$$

By assumption, $E \cap qMq$ is w-closed, so $F \cap qMq$ is s-closed, hence w-closed. It follows that $x \in F \cap qMq$. Since $px = x$ and $\|x\| \leqslant \lambda$, we obtain $x^* \in F^*p \cap M_\lambda$. Hence pF is w-closed for all $p \in P_0$.

Now let $x \in M_0$ be w-adherent to E. Then there is a net $\{x_\iota\}_{\iota \in I}$ in E with $x_\iota \xrightarrow{s} x$. If $p = \mathbf{l}(x)$, then $p \in P_0$ and $px_\iota^{1/2} \xrightarrow{s} px^{1/2} = x^{1/2}$. By the above, pF is s-closed so $x^{1/2} \in pF \subset F$, that is $x \in E$. Hence E is w-closed relative to M_0.

Then converse assertion is immediate. ∎

PROPOSITION. *Let φ be a weight on a W^*-algebra M. Then φ is normal if and only if φ is lower w-semicontinuous on M^+.*

Proof. Suppose that φ is normal. We have to show that the set

$$E = \{x \in M^+;\, \varphi(x) \leqslant 1\}$$

is w-closed. Clearly, E is convex and hereditary in M^+. Let $x \in E$. By Proposition 2/7.16, there is an increasing sequence $\{e_n\}_n$ in $W^*(\{x\})$ with $e_n \leqslant nxe_n$ and $e_n \uparrow \mathbf{s}(x)$. Then $\varphi(e_n) \leqslant n$, hence e_n is countably decomposable, $(n \in \mathbb{N})$, so $\mathbf{s}(x) \in P_0$. Thus, $E \subset M_0$.

First assume that M is countably decomposable. As in the last part of the proof of Lemma 4, it is sufficient to prove that

$$F = \{x \in M;\, \varphi(x^*x) \leqslant 1\}$$

is w-closed. Since $F \cap M_\lambda$ is the image $G_\varphi \cap (M_\lambda \times (H_\varphi)_1)$, $(\lambda > 0)$, under the projection map $M \times H_\varphi \ni (x, \xi) \mapsto x \in M$, it follows from Lemma 3 that $F \cap M_\lambda$ is w-compact for every $\lambda > 0$. Since F is convex, we infer that F is w-closed.

Consider now the general case. By the above special case and by Lemma 4, E is w-closed relative to M_0.

Let $x \in M^+$ be w-adherent to E. There is a net $\{x_\iota\}_{\iota \in I}$ in E, $x_\iota \xrightarrow{s} x$. Also, there is an increasing net $\{p_\kappa\}_{\kappa \in K}$ of countably decomposable projections in M with $\bigvee_{\kappa \in K} p_\kappa = 1$. For each $\kappa \in K$ we have

$$e \ni x_\iota^{1/2} p_\kappa x_\iota^{1/2} \xrightarrow[\iota \in I]{w} x^{1/2} p_\kappa x^{1/2} \in M_0,$$

so $x^{1/2} p_\kappa x^{1/2} \in E$, $(\kappa \in K)$. Since $x^{1/2} p_\kappa x^{1/2} \uparrow x$, by the normality of φ we infer that

$$\varphi(x) = \sup_{\kappa \in K} \varphi(x^{1/2} p_\kappa x^{1/2}) \leqslant 1,$$

hence $x \in E$. ∎

Recall that a positive form φ on M is normal if and only if $\varphi\Big(\sum\limits_{\iota\in I} e_\iota\Big) = \sum\limits_{\iota\in I}\varphi(e_\iota)$ for every family $\{e_\iota\}_{\iota\in I}$ of mutually orthogonal projections in M. This cannot be generalized for weights, as the following example shows.

Let ℓ^∞ be the W^*-algebra of all bounded complex sequences. The weight defined on $(\ell^\infty)^+$ by

$$\varphi(\{a_n\}_n) = \begin{cases} \sum\limits_n a_n & \text{if the set } \{n \in \mathbb{N};\, a_n \neq 0\} \text{ is finite} \\ +\infty & \text{otherwise} \end{cases}$$

satisfies the above condition, but is not normal.

8.21. We interrupt the discussion on normal weights in order to prove a general result which is necessary at this point.

Let X be a locally convex Hausdorff real vector space which has a partial ordering defined by a closed convex cone X^+ such that

$$X^+ \cap (-X^+) = \{0\} \quad \text{and} \quad X = \overline{(X^+ - X^+)}.$$

The dual cone $X^*_+ = \{f \in X^*;\, f(x) \geqslant 0 \text{ for all } x \in X^+\}$ defines a partial ordering on X.

A subset E of X^+ is called *hereditary* if

$$x \in B \text{ and } y \in X^+, \quad x - y \in X^+ \Rightarrow y \in E.$$

We shall denote

$$\begin{aligned} E^0 &= \{f \in X^*;\, f(x) \geqslant -1,\, (\forall)x \in E\}, &&\text{for } E \subset X; \\ E^\wedge &= \{f \in X^*_+;\, f(x) \leqslant 1,\, (\forall)x \in E\}, &&\text{for } E \subset X^+; \\ F^0 &= \{x \in X;\, f(x) \geqslant -1,\, (\forall)f \in F\}, &&\text{for } F \subset X^*; \\ F^\wedge &= \{x \in X^+;\, f(x) \leqslant 1,\, (\forall)f \in F\}, &&\text{for } F \subset X^*_+. \end{aligned}$$

PROPOSITION. *Let X be as above. Then the following statements are equivalent:*

(i) $E = \overline{E - X^+} \cap X^+$ *for every closed convex hereditary set* $E \subset X^+$*;*

(ii) $E = E^\wedge$ *for every closed convex hereditary set* $E \subset X^+$*;*

(iii) *Every lower semicontinuous function* $\varphi : X^+ \to [0, +\infty]$*, which is increasing, subadditive and positive homogeneous has the form*

$$\varphi(x) = \sup\{f(x);\, f \in X^*_+,\, f(y) \leqslant \varphi(y)\, (\forall)y \in X^+\}; \quad x \in X^+.$$

Proof. (i) $\Rightarrow$ (ii). The sets $F = E^\wedge$ and

$$F' = -(E - X^+)^0 = \{f \in X^*;\, f(x) \leqslant 1,\, \forall x \in E - X^+\}$$

are equal. Indeed, $F \subset F'$ is obvious. Let $f \in F'$ and $x \in X^+$. Since $0 \in E$, we have $f(-\lambda x) \leqslant 1$ for all $\lambda \geqslant 0$, so $f(x) \geqslant 0$. Thus $F' \subset X^*_+$ and the inclusion $F' \subset F$ is now clear.

By the bipolar theorem if follows that

$$\overline{(E - X^+)} = (E - X^+)^{00} = (-F)^0 = \{x \in X;\, f(x) \leqslant 1,\, (\forall) f \in F\}.$$

Using (i) we thus get

$$E = \overline{(E - X^+)} \cap X^+ = \{x \in X^+;\, f(x) \leqslant 1,\, (\forall) f \in F\} = E^{\wedge\wedge}.$$

(ii) $\Rightarrow$ (iii). If φ satisfies the conditions of (iii), then the set $E = \{x \in X^+;\, \varphi(x) \leqslant 1\}$ is closed, convex and hereditary. Then

$$F = E^\wedge = \{f \in X^*_+;\, f(x) \leqslant \varphi(x),\, (\forall) x \in X^+\}$$

and using (ii) we get

$$\{x \in X^+;\, \varphi(x) \leqslant 1\} = E = F^{\wedge *} = \Big\{x \in X^+;\, \sup_{f \in F} f(x) \leqslant 1\Big\}.$$

It follows that $\varphi(x) = \sup\{f(x);\, f \in F\}$ for all $x \in X^+$.

(iii) $\Rightarrow$ (i). Let $E \subset X^+$ be a closed convex hereditary set and put

$$\varphi(x) = \begin{cases} \inf\{\lambda > 0;\, x \in \lambda E\} & \text{if } \ x \in \bigcup\limits_{\lambda > 0} \lambda E \\ +\infty & \text{otherwise.} \end{cases}$$

Then φ satisfies all conditions of (iii) so, by (iii), $\varphi(x) = \sup\{f(x);\, f \in F\}$, $(x \in X^+)$, where

$$F = \{f \in X^*_+;\, f(x) \leqslant \varphi(x),\, (\forall) x \in X^+\}.$$

It follows that

$$E - X^+ \subset \{x \in X;\, f(x) \leqslant 1,\, (\forall) f \in F\}.$$

Since the later set is closed, we get

$$\overline{(E - X^+)} \cap X^+ \subset \{x \in X^+;\, f(x) \leqslant 1,\, (\forall) f \in F\} \subset E.$$

Hence $\overline{(E - X^+)} \cap X^+ = E$. ∎

8.22. In this section we show that the selfadjoint part of a W^*-algebra equipped with the w-topology satisfies the equivalent condition of Proposition 8.21. More specifically, we prove

LEMMA. *Let M be a W^*-algebra and $E \subset M^+$ be a w-closed hereditary convex set. Then*

$$E = \overline{(E - M^+)}^w \cap M^+.$$

Proof. We shall intensively use the properties of the functions $f_\alpha(t) = t(1+\alpha t)^{-1}$, $(\alpha > 0)$, from (2.7). For $x \in M_h$ we put

$$\alpha_x = \sup\{\alpha > 0;\ -\alpha^{-1} \leqslant x\}.$$

Consider the set

$$S = \{x \in M_h;\ f_\alpha(x) \in E - M^+,\ (\forall)\alpha \in (0,\ \alpha_x)\}.$$

1) $S \cap M_\lambda$ is s-closed for all $\lambda > 0$.

Indeed, let $x \in \overline{(S \cap M_\lambda)}^s$. There exists a net $\{x_\iota\}_{\iota \in I}$ in such that $\|x_\iota\| \leqslant \lambda$, $(\iota \in I)$, and $x_\iota \xrightarrow{s} x$. Then $\alpha_{x_\iota} \geqslant (2\lambda)^{-1}$ for all $\iota \in I$, so $f_\alpha(x_\iota) \in E - M^+$ for all $\alpha \in (0,\ (2\lambda)^{-1})$ and all $\iota \in I$. Fix $\alpha \in (0, (2\lambda)^{-1})$. There is a net $\{y_\iota\}_{\iota \in I}$ in E such that

$$f_\alpha(x_\iota) \leqslant y_\iota; \quad \iota \in I.$$

Since f_α is operator monotone, we get

$$f_{2\alpha}(x_\iota) = f_\alpha(f_\alpha(x_\iota)) \leqslant f_\alpha(y_\iota); \quad \iota \in I.$$

Since $f_{2\alpha}$ is operator continuous on $[-\lambda, \lambda]$, we have

$$f_{2\alpha}(x_\iota) \xrightarrow{s} f_{2\alpha}(x).$$

Since $0 \leqslant f_\alpha(y_\iota) \leqslant \alpha^{-1}$ and M_1 is w-compact, we may suppose that there is $y \in M$ such that

$$f_\alpha(y_\iota) \xrightarrow{w} y.$$

Since $0 \leqslant f_\alpha(y_\iota) \leqslant y_\iota \in E$ and E is hereditary, $f_\alpha(y_\iota)$ belongs to E. Since E is w-closed, it follows that $y \in E$. Furthermore

$$y - f_{2\alpha}(x) = w\text{-}\lim_\iota (f_\alpha(y_\iota) - f_{2\alpha}(x_\iota)) \geqslant 0,$$

hence $f_{2\alpha}(x) \in E - M^+$. We have proved that

$$f_\alpha(x) \in E - M^+ \quad \text{ for all } \alpha \in (0, \lambda^{-1}).$$

Let $\alpha \in [\lambda^{-1}, \alpha_x)$ and choose $\beta \in (0, \lambda^{-1})$. Then $f_\alpha(x) \leqslant f_\beta(x)$, so

$$f_\alpha(x) \in (E - M^+) - M^+ = E - M^+.$$

Hence $x \in S \cap M_\lambda$.

2) S is convex.

Indeed, it is sufficient to show that each $S \cap M_\lambda$ is convex and this will follow from

$$S \cap M_\lambda = \overline{((E - M^+) \cap M_\mu)}^s \cap M_\lambda, \text{ for } \mu > \lambda.$$

If $x \in S \cap M_\lambda$, then $f_\alpha(x) \in E - M^+$ for $\alpha \in (0, \alpha_x)$ and $f_\alpha(x) \in M_\mu$ for sufficiently small $\alpha > 0$, so

$$x = s\text{-}\lim_{\alpha \to 0} f_\alpha(x) \in \overline{((E - M^+) \cap M_\mu)}^s \cap M_\lambda.$$

Conversely, since E is hereditary and $f_\alpha(x) \leqslant x$ for $\alpha \in (0, \alpha_x)$, we have $E - M^+ \subset S$, so $(E - M^+) \cap M_\mu \subset S \cap M_\mu$. Using 1) we get $\overline{((E - M^+) \cap M_\mu)}^s \subset S \cap M_\mu$ and the desired inclusion follows.

3) By 1), 2) and the Krein-Shmulyan theorem (7.4), S is w-closed. As we have seen, $E - M^+ \subset S$. Since $x = w\text{-}\lim\limits_{\alpha \to 0} f_\alpha(x)$, we infer that $S \subset \overline{(E - M^+)}^w$. Thus

$$S = \overline{(E - M^+)}^w.$$

4) Now let $x \in \overline{(E - M^+)}^w \cap M^+ = S \cap M^+$. For all $\alpha > 0$ we have $f_\alpha(x) \in (E - M^+) \cap M^+$, so $f_\alpha(x) \in E$, because E is hereditary. Finally $x = w\text{-}\lim\limits_{\alpha \to 0} f_\alpha(x) \in E$. ∎

Combining Proposition 8.20 and Proposition 8.21 with the above lemma, we obtain the following result:

THEOREM (U. Haagerup). *Let M be a W^*-algebra and φ be a weight on M^+. Then the following statements are equivalent:*

(i) *φ is normal;*

(ii) *φ is lower w-semicontinuous on M^+;*

(iii) *$\varphi(x) = \sup\{f(x);\ f \in M_*^+,\ f \leqslant \varphi\};\ x \in M^+$.* ∎

If φ is a normal weight on M^+ and $a, b \in N_\varphi$, then $b^*xa \in N_\varphi^* N_\varphi = M_\varphi$ for all $x \in M$ and the map

$$\varphi(b^* \cdot a) : x \to \varphi(b^*xa)$$

is a normal linear form on M.

Indeed, if $x_\iota \uparrow x$ in M^+, then also $a^*x_\iota a \uparrow a^*xa$ in M^+, hence $\varphi(a^*x_\iota a) \uparrow \varphi(a^*xa)$ by the normality of φ. Thus $\varphi(a^* \cdot a)$ is a normal positive form and the general case follows by polarization (2.8.(2)).

8.23. Let M be a W^*-algebra and φ be a normal weight on M^+.

Using Proposition 2/7.16 and the normality of φ it is easy to check that

(1) $$x \in M^+, \quad \varphi(x) = 0 \Rightarrow \varphi(\mathbf{s}(x)) = 0.$$

If, $e, f \in M$ are projections and $\varphi(e) = \varphi(f) = 0$, then $\varphi(e \vee f) = 0$, because $e \vee f = \mathbf{s}(e + f)$. It follows that the family

$$E = \{e \in M;\ e \text{ projection},\ \varphi(e) = 0\}$$

is an increasing net. Let $e_0 = \sup E$. Then $\varphi(e_0) = 0$ by the normality of φ. Clearly, e_0 is the greatest projection in M annihilated by φ. The projection

$$\mathbf{s}(\varphi) = 1 - e_0$$

is called the *support* of φ.

Using (1) it follows that

$$(2) \qquad x \in M, \quad \varphi(x^*x) = 0 \Leftrightarrow x\mathbf{s}(\varphi) = 0.$$

In particular we see that φ is faithful (4.3) if and only if $\mathbf{s}(\varphi) = 1$.

On the other hand, the w-closure $\overline{N_\varphi}^w$ of $N_\varphi = \{x \in M; \varphi(x^*x) < +\infty\}$ is a w-closed left ideal of M, therefore $\overline{N}^w = Me$ for some projection $e \in M$ and $\overline{M}^w = eMe$ (8.7). The weight φ is called *w-semifinite* (or simply *semifinite*) if $e = 1$, that is if M_φ, or equivalently N_φ, is w-dense in M. In this case there is an increasing net $\{u_\iota\}_{\iota\in I}$ in $F_\varphi = M_\varphi \cap M^+$ such that $u_\iota \uparrow 1$ (8.7, 3.2).

By *n.s.f. weight* we abreviate the words normal semifinite faithful weight.

Note that *for every W^*-algebra M there is a n.s.f. weight on M^+*. Indeed, if $\{\varphi_\iota\}_{\iota\in I}$ is a maximal family of normal positive forms on M with mutually orthogonal supports, then the formula

$$\varphi(x) = \sum_{\iota\in I} \varphi_\iota(x); \quad x \in M^+$$

defines a n.s.f. weight on M^+.

THEOREM. *Let M be a W^*-algebra and φ be a normal weight on M^+. Then the associated GNS representation $\pi_\varphi : M \to B(H_\varphi)$ is normal and non-degenerated. Moreover, if φ is semifinite, then*

$$(3) \qquad ((M_\varphi)^n)_\varphi \textit{ is dense in } H_\varphi; \quad n \geqslant 1,$$

and if φ is a n.s.f. weight, then π_φ is a $*$-isomorphism of M onto the von Neumann algebra $\pi_\varphi(M) \subset B(H_\varphi)$.

Proof. Clearly, $\pi_\varphi(1) = 1_{H_\varphi}$, hence π_φ is non-degenerated. In order to show that π_φ is normal, i.e. w-continuous, we have to prove that $\omega \circ \pi_\varphi \in M_*$ for every normal form ω on $B(H_\varphi)$. By an obvious approximation, it is sufficient to do this only for $\omega = \omega_{a_\varphi,b_\varphi}$ with $a, b \in N_\varphi$. In this case

$$\omega_{a_\varphi,b_\varphi} \circ \pi_\varphi = \varphi(b^* \cdot a) \in M_*,$$

by the normality of φ (see 8.22).

Since π_φ is normal, $\pi_\varphi(M)$ is a W^*-subalgebra of $B(H_\varphi)$ by Corollary 5/8.4 and, since $1_H \in \pi_\varphi(M)$, $\pi_\varphi(M) \subset B(H_\varphi)$ is a von Neumann algebra.

If φ is semifinite, then there is an icreasing net $\{u_\iota\}_{\iota\in I}$ in $F_\varphi = M_\varphi \cap M^+$ such that $u_\iota \uparrow 1$. For $a \in N_\varphi$ we have

$$(4) \qquad \|a_\varphi - (u_\iota a)_\varphi\|_\varphi^2 = \varphi((a - u_\iota a)^*(a - u_\iota a)) \leqslant 2[\varphi(a^*a) - \varphi(a^*u_\iota a)] \to 0.$$

Since $u_\iota \in N_\varphi \subset N_\varphi^*$ and $a \in N_\varphi$, we have $u_\iota a \in N_\varphi^* N_\varphi = M_\varphi$ and from (4) it follows that $(M_\varphi)_\varphi$ is dense in N_φ, hence also in H_φ. Now (3) follows using inductively (4).

Let φ be a n.s.f. weight. If $x \in M$ and $\pi_\varphi(x) = 0$, then

$$\varphi((xa)^*(xa)) = \|\pi_\varphi(x)a_\varphi\|_\varphi^2 = 0; \quad a \in N_\varphi.$$

Since φ is faithful, it follows that $xN_\varphi = 0$ and since φ is semifinite, we conclude $x = 0$. ∎

Corollary. *Let M be a W^*-algebra, φ be a normal semifinite weight on M^+ and f be a positive form on M. If $f \leqslant \varphi$, then f is normal.*

Proof. By Corollary 2/4.8, there exists a vector $\xi \in H_\varphi$ such that $f(x) = (\omega_\xi \circ \pi_\varphi)(x)$ for all $x \in M_\varphi$. Since φ is semifinite, we infer that $f = \omega_\xi \circ \pi_\varphi$, so the normality of f follows from that of π_φ. ∎

8.24. By Theorem 8.22, every normal weight φ on a W^*-algebra M is the pointwise supremum of the family of normal positive forms

$$F_\varphi = \{f \in M_*^+;\ f \leqslant \varphi\},$$

hence φ is also the supremum of the family

$$\{f \in M^+;\ (1+\varepsilon)f \leqslant \varphi \text{ for some } \varepsilon > 0\}.$$

The next result shows in particular that every normal semifinite weight on a W^*-algebra is the supremum of a directed family of normal positive forms.

Theorem. (F. Combes) *Let M be a W^*-algebra, φ be a normal semifinite weight on M^+. Then the set*

$$\{f \in M_*^+;\ (1+\varepsilon)f \leqslant \varphi \textit{ for some } \varepsilon > 0\}$$

is upward directed.

Proof. We have to show that for every $f_1, f_2 \in F_\varphi$ and every $\varepsilon > 0$, there exists $f \in F_\varphi$ such that $(1-\varepsilon)f_1 \leqslant f$ and $(1-\varepsilon)f_2 \leqslant f$.

Let $f_1, f_2 \in F_\varphi$ and $\varepsilon > 0$. By Corollary 2/4.8, the set T'_φ of all $T' \in \pi_\varphi(M)'$ such that there is $\lambda_{T'} > 0$ with

$$\|T' a_\varphi\|_\varphi \leqslant \lambda_{T'} \|a\|; \quad a \in N_\varphi$$

is a left ideal of the von Neumann algebra $\pi_\varphi(M)'$ and there are T'_1, $T'_2 \in T'_\varphi$ such that $0 \leqslant T'_j \leqslant 1$ and

(1) $$f_j(b^*a) = (T'_j a_\varphi | T'_j b_\varphi)_\varphi; \quad a, b \in N_\varphi,\ j = 1, 2.$$

By Proposition 2/8.7, $(T'_\varphi)^* T'_\varphi$ is a facial subalgebra of $\pi_\varphi(M)'$. Using Proposition 2.10 we obtain an element

$$X' \in (T'_\varphi)^* T'_\varphi,\ (1-\varepsilon){T'}^*_j T'_j \leqslant X' \leqslant 1; \quad j = 1, 2.$$

Let $T' = (X')^{1/2}$. Using again Proposition 2/8.7 we get

(2) $$T' \in T'_\varphi, \quad 0 \leqslant T' \leqslant 1,\ (1-\varepsilon){T'}^*_j T'_j \leqslant {T'}^* T'; \quad j = 1, 2.$$

By Corollary 2/4.8, there is a positive form f on M such that $f \leqslant \varphi$ and

(3) $$f(b^*a) = (T' a_\varphi | T' b_\varphi)_\varphi; \quad a, b \in N_\varphi.$$

Since $f \leqslant \varphi$, we have $f \in M_*^+$ by Corollary 8.23, hence $f \in F_\varphi$. On the other hand, from (1), (2), (3) we infer that

(4) $$(1-\varepsilon) f_j(x) \leqslant f(x); \quad x \in M_\varphi,\ j = 1, 2.$$

Since M_φ is w-dense in M and f, f_1, f_2 are w-continuous, it follows that (4) holds for all $x \in M$, i.e. $(1-\varepsilon)f_j \leqslant f$, $(j = 1, 2)$. ∎

8.25. Tensor product of weights. Let M be a W^*-algebra. If $F \subset M_*^+$ is an upward directed family, then the formula

$$\varphi(x) = \sup\{f(x);\ f \in F\}; \quad x \in M^+$$

defines a normal weight φ on M^+. Then family of projections $\{\mathbf{s}(f);\ f \in F\}$ is also upward directed and, as easily verified

$$\mathbf{s}(\varphi) = \bigvee_{f \in F} \mathbf{s}(f). \tag{1}$$

Now let φ (respectively ψ) be a normal semifinite weight on the W^*-algebra M (respectively N). By the Combes theorem (8.24), the family

$$F = \{f \in M_*^+;\ (1+\varepsilon)f \leqslant \varphi \text{ for some } \varepsilon > 0\} \subset M_*^+$$
$$G = \{g \in N_*^+;\ (1+\varepsilon)g \leqslant \psi \text{ for some } \varepsilon > 0\} \subset N_*^+$$

are upward directed and $\varphi = \sup\limits_{f \in F} f$, $\psi = \sup\limits_{g \in G} g$. Then the family $\{f \mathbin{\overline{\otimes}} g;\ f \in F,\ g \in G\} \subset (M \mathbin{\overline{\otimes}} N)_*^+$ is upward directed, so we can define a normal weight $\varphi \mathbin{\overline{\otimes}} \psi$ on $(M \mathbin{\overline{\otimes}} N)^+$ by

$$(\varphi \mathbin{\overline{\otimes}} \psi)(x) = \sup\{(f \mathbin{\overline{\otimes}} g)(x);\ f \in F,\ g \in G\}; \quad x \in (M \mathbin{\overline{\otimes}} N)^+.$$

It is clear that

$$M_\varphi \otimes M_\psi \subset M_{\varphi \overline{\otimes} \psi}, \quad N_\varphi \otimes N_\psi \subset N_{\varphi \overline{\otimes} \psi}, \tag{2}$$

$$(\varphi \mathbin{\overline{\otimes}} \psi)(a \otimes b) = \varphi(a)\varphi(b); \quad a \in M_\varphi,\ b \in M_\psi. \tag{3}$$

In particular, $\varphi \mathbin{\overline{\otimes}} \psi$ is semifinite.

The weight $\varphi \mathbin{\overline{\otimes}} \psi$ is called the *tensor product* of the normal semifinite weights φ and ψ.

By (1) and by 8.11.(11), we have

$$\mathbf{s}(\varphi \mathbin{\overline{\otimes}} \psi) = \bigvee_{f \in F,\, g \in G} \mathbf{s}(f \mathbin{\overline{\otimes}} g) = \bigvee_{f \in F,\, g \in G} \mathbf{s}(f) \otimes \mathbf{s}(g)$$

and, since $\mathbf{s}(f) \uparrow \mathbf{s}(\varphi)$, $\mathbf{s}(g) \uparrow \mathbf{s}(\psi)$, it follows that

$$\mathbf{s}(\varphi \mathbin{\overline{\otimes}} \psi) = \mathbf{s}(\varphi) \otimes \mathbf{s}(\psi). \tag{4}$$

In particular, the tensor product of two n.s.f. weights is again a n.s.f. weight.

8.26. Notes. The fact that any von Neumann algebra is the dual space of a unique Banach space (8.1 and Lemma 3.(i)/8.4 for $\psi \geqslant 0$) has been discovered by J. Dixmier [70], [75], and the characterization of von Neumann algebras as C^*-algebras which are dual spaces (8.4) is due to S. Sakai [267]. In proving Theorem 8.4 we have not followed the original arguments of S. Sakai ([267]; see also [271], [274]), but the alternative approach (cf. [319]) offered by the result of J. Tomiyama [328] on projections of norm one (8.3) and the remarks of M. Takesaki [314], [316] on singular forms (Lemma 3.(ii)/8.4). Moreover, the proof of Lemma 8.3 we have presented seems to be the simplest possible one (it appears also in a Course of E.C. Lance). Another proof of Lemma 3.(i)/8.4 appeared in [138] (see also [259], [307]).

The second dual of a C^*-algebra and its properties (8.2, 8.4) appearead in the works of S. Sherman [286], Z. Takeda [312] and A. Grothendieck [119]. Our exposition in 8.2, based on the Kaplansky density theorem, follows that in [77]. Another approach to the secound dual of a C^*-algebra is due to M. Tomita ([327]; see also [302], §I.2).

For the results in 8.5 and 8.6, which are mainly reformulations of the corresponding results in the frame of von Neumann algebras (§7), we refer to [77], [267], [268], [269], [270], [271], [274], [319], [328], [340]. T. Okayasu [216] solved a problem of S. Sakai [271] showing that any algebraic isomorphism between W^*-algebras is s-continuous and subsequently [219], following a suggestion of I. Kaplansky [164], obtained a polar decomposition for isomorphisms of C^*-algebras. The central cover (8.6) has been consider by G.K. Pedersen [237], [241]. For the structure and the properties of ideals in W^*-algebras (8.7) we have used [57], [58], [59], [60], [67], [77], [274]. Our exposition of W^*-tensor products is modeled after [271].

The polar decomposition of linear functionals (8.9, 8.11) is due to S. Sakai [270], [271] and M. Tomita [326], II. The example in 8.11 showing that $\mathbf{s}_A(\varphi) \neq \mathbf{s}_{A^{**}}(\varphi)$ appears in [78], 12.5.7. The Jordan decomposition of linear functionals (8.10) is due to A. Grothendieck [119] (see also [312], [313]). The other results in 8.10 and 8.11 are from [84], [233] IV, [273], [319].

For the material included in 8.12–8.14 we have used [6], [77], [145], [332]. The useful result contained in Proposition 8.13.(ii) is known as "the Griffin lemma" ([117], I). The result of Proposition 8.14 (see also Corollary 3/9.33) is due to J. von Neumann [206] and its proof is due to C.E. Rickart [258]. Surveys on the theory of generators of W^*-algebras can be found in [266], [332].

The results from (8.15–8.19), well known in the commutative case [81], [249], appeared with an increasing non-commutative generality in [354], [269], [272], [4] (see also [1], [2], [263], [310], [314], [319], [323]). Our exposition follows the monograph [81], IV.9.7, IV.9.8, the article of C.A. Akemann [4] and the lectures of M. Takasaki [319]. Another approach to a part of Theorem 8.17 appears in [259] (see also [307]). The conjecture mentioned at the end of 8.19 has been solved affirmatively by A. Connes and E. Størmer [64] in the case of factors.

The characterization of normal weights given in Theorem 8.22 in due to U. Haagerup [122], as well as the proof presented in 8.20–8.22. Actually, G.K. Pedersen and M. Takesaki [245] proved that any normal weight can be written as an infinite sum of normal positive functionals (see also [307] and, for a simpler proof, [91]). A variant of Theorem 8.24 for weights on C^*-algebras has been proved for the first time by F. Combes [56], 1.9. For our exposition of Theorem 8.24 we have used [56], [67], [319]. The results of U. Haagerup and F. Combes are important for the theory of standard forms of W^*-algebras ([319], [320], [307], [304]). For more details concerning the tensor product of weights we refer to [63], [303].

Chapter 9

ALGEBRAIC FEATURES OF W^*-ALGEBRAS

As we have seen in the preceding chapter, there is a deep relationship between the w-topology of a W^*-algebra M, the order structure of M_h and the order structure of $P(M)$. Since the existence of the w-topology is characteristic for W^*-algebras, it is only natural to expect that the W^*-algebras M can be characterized among all C^*-algebras in terms of the order structure of M_h or $P(M)$.

This chapter is devoted to a study of several classes of C^*-algebras with particular order structure properties. We examine the cases in which a C^*-algebra belonging to these classes is in fact a W^*-algebra, thus obtaining characterizations of W^*-algebras in terms of the richness and of the order structure of $P(M)$. Also, we describe the w-closure of a C^*-subalgebra A of a W^*-algebra in terms of the "monotone closure" of A_h. A special attention is given to the commutative case.

Many of the above mentioned results involve only the hermitean part of C^*-algebras and hold in fact for Jordan algebras.

9.1. We first introduce an axiom which guarantees the richness of the projection set of a C^*-algebra or of a Jordan algebra.

We shall say that a C^*-algebra A satisfies the *spectral axiom* if for each $a \in A$, $a \geqslant 0$, and $\lambda, \mu \in (0, \infty)$, $\lambda < \mu$, there exists a projection $e \in A$, commuting with a, such that

$$ae \geqslant \lambda e, \quad a(1 - e) \leqslant \mu(1 - e).$$

Now let J be a Jordan algebra in the C^*-algebra A. By 6.6.(6), the images of commuting elements of a Jordan algebra by a Jordan homomorphism are again commuting, whence for $x, y \in J$ the statement "x commutes with y" does not depend on the underlying algebra A.

We shall say that J satisfies the *spectral axiom* if for each $a \in J$, $a \geqslant 0$ and $\lambda, \mu \in (0, \infty)$, there exists a projection $e \in J$, commuting with a, such that

$$ae \geqslant \lambda e, \quad a(1 - e) \leqslant \mu(1 - e).$$

Clearly, a C^*-algebra A satisfies the spectral axiom if and only if the Jordan algebra A_h does.

9.2. Let J be a Jordan algebra in the C^*-algebra A, and let $\widetilde{A}$ be the associate unital C^*-algebra. Denote

$$\widetilde{J} = \begin{cases} J, & \text{if } J \text{ is unital,} \\ J + \mathbb{R} \cdot 1_{\widetilde{A}}, & \text{if } J \text{ is not unital.} \end{cases}$$

Then $\widetilde{J}$ is a unital Jordan algebra in $\widetilde{A}$, generated by J and by its unit element. It is uniquely determined up to Jordan isomorphisms by the following universality property: every injective Jordan homomorphism of J into a unital Jordan algebra K can be extended to an injective Jordan homomorphism of $\widetilde{J}$ into K. Note that J is norm-closed if and only if $\widetilde{J}$ is norm-closed.

If A is a C^*-algebra and $J = A_h$, then $\widetilde{J} = (\widetilde{A})_h$.

The following simple fact is useful for a deeper understanding of the spectral axiom:

PROPOSITION. *Let J be a norm-closed Jordan algebra in the C^*-algebra A. If $0 \leqslant a \in J$, $\lambda \in (0, \infty)$ and $e \in J$ is a projection commuting with a and such that $ae \geqslant \lambda e$, then $e \in aJa$.*

Proof. Since $a \in J \subset \widetilde{J}$ commutes with $e \in \widetilde{J}$, by 6.2 we get $C^*(\{a, e\})_h \subset \widetilde{J}$. Now, since $ae \geqslant \lambda e$, using the Gelfand representation of $C^*(\{a, e\})$ we obtain an element $b \in C^*(\{a, e\})_h \subset \widetilde{J}$ such that $ab = ba = e$. Consequently, $e = ab^2 a \in a\widetilde{J}a = aJa$. ∎

COROLLARY. *Let J be a norm-closed Jordan algebra in the C^*-algebra A. Then J satisfies the spectral axiom if and only if for each $x \in J$ and $\lambda, \mu \in \mathbb{R}$, $\lambda < \mu$, there exists a projection $e \in \widetilde{J}$ commuting with x, such that*

$$xe \geqslant \lambda e, \quad x(1 - e) \leqslant \mu(1 - e).$$

Proof. Assume that J satisfies the spectral axiom and let $x \in J$ and $\lambda, \mu \in \mathbb{R}$, $\lambda < \mu$.

Consider first the case $\mu > 0$ and let $\lambda' \in (\max\{\lambda, 0\}, \mu)$. Since $x^+ \in J$ (6.2.(11)), there exists a projection $e \in J$, commuting with x^+, such that

$$x^+ e \geqslant \lambda' e, \quad x^+(1 - e) \leqslant \mu(1 - e).$$

By the above proposition, we have $e \in x^+ J x^+$, so that $ex^- = x^- e = 0$. Consequently, e commutes with $x = x^+ + x^-$ and

$$\begin{gathered} xe = x^+ e - x^- e = x^+ e \geqslant \lambda' e \geqslant \lambda e, \\ x(1 - e) = x^+(1 - e) - x^-(1 - e) \leqslant x^+(1 - e) \leqslant \mu(1 - e). \end{gathered}$$

Consider now the case $\mu \leqslant 0$. By the above part of the proof, applied to $-x$ and $-\mu < -\lambda$, we get a projection $f \in J$, commuting with x, such that

$$(-x)f \geqslant (-\mu)f, \quad (-x)(1 - f) \leqslant (-\lambda)(1 - f).$$

Then $e = 1_{\widetilde{J}} - f \in \widetilde{J}$ is a projection commuting with x and

$$xe \geqslant \lambda e, \quad a(1 - e) \leqslant \mu(1 - e).$$

Conversely, if the condition of the statement is satisfied, then for each $0 \leqslant a \in J$ and $0 < \lambda < \mu$ there exists a projection $e \in \widetilde{J}$, commuting with a, such that

$$ae \geqslant \lambda e, \quad a(1 - e) \leqslant \mu(1 - e).$$

By the above proposition, we have $e \in aJa \subset J$. Hence J satisfies the spectral axiom. ∎

9.3. We now describe some permanence properties of the spectral axiom. Let J be a norm-closed Jordan algebra in the C^*-algebra A. Then:

(1) *J satisfies the spectral axiom if and only if $\widetilde{J}$ does.*

(2) *If J satisfies the spectral axiom and $p \in \widetilde{J}$ is a projection, then pJp satisfies the spectral axiom.*

(3) *If J satisfies the spectral axiom and K is a real linear subspace of J such that $x\widetilde{J}x \subset K$ for all $x \in K$ then K is a Jordan algebra in A and satisfies the spectral axiom.*

Indeed, (1) can be proved using Corollary 9.2, and (2), (3) are immediate consequences of Proposition 9.2.

Now let B be a C^*-algebra and $\Phi : J \mapsto B_h$ a Jordan homomorphism. By Proposition 6.6, $\Phi(J)$ is a norm-closed Jordan algebra in B. Plainly:

(4) *If J satisfies the spectral axiom, then $\Phi(J)$ does.*

The above statements can be easily formulated for C^*-algebras.

9.4. We now prove some consequences of the spectral axiom. We consider only the Jordan algebra case, the case of C^*-algebras being a consequence of it.

Let J be a norm-closed Jordan algebra in the C^*-algebra A and denote by $P(J)$ the partially ordered set of all projections in J. *We assume that J satisfies the spectral axiom.*

PROPOSITION 1. *If $a \in J$, $0 \leqslant a \leqslant 1$, then there exists a sequence $\{e_k\}_{k \geqslant 1} \subset P(J)$ such that:*

$$0 \leqslant a - 2^{-1} \sum_{j=1}^{k} (2/3)^j e_j \leqslant (2/3)^k; \quad k \geqslant 0.$$

In particular,

$$a = 2^{-1} \sum_{j=1}^{\infty} (2/3)^j e_j.$$

Proof. Using the spectral axiom we can construct a sequence $\{e_k\}_{k \geqslant 1} \subset P(J)$ such that, for each $k \geqslant 1$,

$$e_k \text{ commutes with } a - 2^{-1} \sum_{j=1}^{k-1} (2/3)^j e_j,$$

$$\Big(a - 2^{-1} \sum_{j=1}^{k-1} (2/3)^j e_j\Big) e_k \geqslant 2^{-1} (2/3)^k e_k,$$

$$\Big(a - 2^{-1} \sum_{j=1}^{k-1} (2/3)^j e_j\Big)(1 - e_k) \leqslant (2/3)^k (1 - e_k).$$

Let us assume that the inequalities from the statement hold for $k-1$. Using the equality

$$\begin{aligned}a - 2^{-1}\sum_{j=1}^{k}(2/3)^j e_j \\ = \Big(a - 2^{-1}\sum_{j=1}^{k-1}(2/3)^j e_j\Big)e_k + \Big(a - 2^{-1}\sum_{j=1}^{k-1}(2/3)^j e_j\Big)(1-e_k) - 2^{-1}(2/3)^k e_k\end{aligned}$$

we get

$$a - 2^{-1}\sum_{j=1}^{k}(2/3)^j e_j \geqslant 2^{-1}(2/3)^k e_k - 2^{-1}(2/3)^k e_k = 0,$$
$$a - 2^{-1}\sum_{j=1}^{k-1}(2/3)^j e_j \leqslant (2/3)^{k-1}e_k + (2/3)^k(1-e_k) - 2^{-1}(2/3)^k e_k = (2/3)^k.$$

Hence the inequalities from the above statement hold also for K. Thus, the statement follows by induction. ∎

In particular, *J is the norm-closed real-linear hull of its projections.*

PROPOSITION 2. *If $a \in J$, $a \geqslant 0$, then there exists a sequence $\{e_k\}_{k\geqslant 1} \subset P(J)$ of mutually orthogonal projections, commuting with a, such that, for each $k \geqslant 1$,*

$$ae_k \geqslant 3^{-k}e_k, \quad a\Big(1 - \sum_{j=1}^{k} e_j\Big) \leqslant 2^{-k}\Big(1 - \sum_{j=1}^{k} e_j\Big).$$

Proof. Using 9.3.(2) we can construct a sequence $\{e_k\}_{k\geqslant 1} \subset P(J)$ such that, for every $k \geqslant 1$,

$$e_k \leqslant 1 - \sum_{j=1}^{k-1} e_j, \quad e_k \text{ commutes with } a\Big(1 - \sum_{j=1}^{k-1} e_j\Big), \quad a\Big(1 - \sum_{j=1}^{k-1} e_j\Big) \geqslant 3^{-k}e_k,$$
$$a\Big(1 - \sum_{j=1}^{k-1} e_j\Big)\Big(1 - \sum_{j=1}^{k-1} e_j - e_k\Big) \leqslant 2^{-k}\Big(1 - \sum_{j=1}^{k-1} e_j - e_k\Big).$$

It is easy to check that this sequence satisfies the conditions required in the statement. ∎

Note that *if $a \in J^+$, $a \neq 0$, then there exists a projection $p \in J$, $p \neq 0$, commuting with a, and $\lambda > 0$, such that $ap \geqslant \lambda p$.*

Indeed, assuming the contrary, by the above proposition we should have $\|a\| \leqslant 2^{-k}$ for all $k \geqslant 1$, that is, $a = 0$.

PROPOSITION 3. *For* $\{e_\iota\}_{\iota\in I} \subset P(J)$ *and* $e \in P(J)$*, the following statements are equivalent:*

(i) *e is a minimal upper bound of $\{e_\iota\}_{\iota\in I}$ in $P(J)$;*

(ii) *e is the least upper bound of $\{e_\iota\}_{\iota\in I}$ in $P(\widetilde{J})$.*

Proof. Clearly, (ii) $\Rightarrow$ (i). Conversely, let us assume that (i) holds, but there is some $f \in P(\widetilde{J})$, $f \geqslant e_\iota$ for all $\iota \in I$, such that $f \not\geqslant e$. Then $e(1-f)e \in eJe$, $e(1-f)e \geqslant 0$, $e(1-f)e \neq 0$, so that, by 9.3.(2) and by the above remark, there exists a projection $p \in J$, $0 \neq p \leqslant e$, commuting with $e(1-f)e$, and $\lambda > 0$, such that

$$e(1-f)ep \geqslant \lambda p.$$

Then, for each $\iota \in I$ we have

$$\lambda e_\iota p e_\iota \leqslant e_\iota e(1-f)epe_\iota = 0,$$

hence $pe_\iota = 0$ and $e_\iota \leqslant e - p$. Since e is a minimal upper bound of $\{e_\iota\}$ in $P(J)$, it follows that $p = 0$, a contradiction. Thus, (i) $\Rightarrow$ (ii). ∎

9.5. In this section we prove some particular "lifting" properties for Jordan algebras and C^*-algebras satisfying the spectral axiom.

Let J be a Jordan algebra in the C^*-algebra A, B a C^*-algebra and $\Phi : J \mapsto B_h$ a Jordan homomorphism. It is easy to see that Φ can be extended uniquely to a Jordan homomorphism $\widetilde{\Phi}$ of $\widetilde{J}$ onto $\widetilde{\Phi(J)}$. Clearly $\widetilde{\Phi}(1_{\widetilde{J}}) = 1_{\widetilde{\Phi(J)}}$.

PROPOSITION 1. *Let J be a norm-closed Jordan algebra in the C^*-algebra A, B a C^*-algebra and $\Phi : J \mapsto B_h$ a Jordan homomorphism. Assume that J satisfies the spectral axiom. If $f \in \widetilde{J}$ and $p \in \Phi(J)$ are projections such that*

$$p \leqslant \widetilde{\Phi}(f),$$

then there exists a projection $e \in J$ with

$$e \leqslant f, \quad \Phi(e) = p.$$

Proof. Let $x \in J$ be such that $\Phi(x) = p$. By 9.3.(2) there exists a projection $e \in J$, $e \leqslant f$, commuting with fx^2f, such that

$$fx^2fe \geqslant 3^{-1}e, \quad fx^2f(f-e) \leqslant 2^{-1}(f-e).$$

Then $\Phi(e)$ is a projection, by 6.6.(6) it commutes with $\Phi(fx^2f) = \widetilde{\Phi}(f)p^2\widetilde{\Phi}(f) = p$ and we have

$$\Phi(fx^2fe) \geqslant 3^{-1}\Phi(e), \quad \Phi(fx^2f(f-e)) \leqslant 2^{-1}\widetilde{\Phi}(f-e)),$$

that is,

$$p\Phi(e) \geqslant 3^{-1}\Phi(e), \quad p(1-\Phi(e)) \leqslant 2^{-1}\widetilde{\Phi}(f-e)).$$

By the first equality, $3^{-1}\Phi(e)(1-p) \leqslant 0$, so $\Phi(e) \leqslant p$, while, by the second inequality, $\|p - \Phi(e)\| \leqslant 2^{-1}$. Since $p - \Phi(e)$ is a projection, we conclude that $\Phi(e) = p$. ∎

COROLLARY. *Consider the same situation and assumption as in Proposition 1. If $f \in \widetilde{J}$ is a projection and $\{p_k\}_{k\geqslant 1}$ is a sequence of mutually orthogonal projections in $\Phi(J)$ such that*

$$p_k \leqslant \widetilde{\Phi}(f), \quad k \geqslant 1,$$

then there exists a sequence $\{e_k\}_{k\geqslant 1}$ of mutually orthogonal projections in J with

$$e_k \leqslant f, \quad \Phi(e_k) = p_k; \quad k \geqslant 1.$$

Proof. Using Proposition 1 we construct a sequence $\{e_k\}_{k\geqslant 1} \subset P(J)$ such that, for each $k \geqslant 1$,

$$e_k \leqslant f - \sum_{j=1}^{k-1} e_j, \quad \Phi(e_k) = p_k. \quad \blacksquare$$

PROPOSITION 2. *Let A, B be C^*-algebras and $\pi : A \mapsto B$ a $*$-homomorphism. Assume that A satisfies the spectral axiom. If $e, f \in \widetilde{A}$ are projections and $v \in \pi(A)$ is a partial isometry such that*

$$v^*v \leqslant \widetilde{\pi}(e), \quad vv^* \leqslant \widetilde{\pi}(f),$$

then there exists a partial isometry $u \in A$ with

$$u^*u \leqslant e, \quad uu^* \leqslant f \quad \text{and} \quad \pi(u) = v.$$

Proof. Let $x \in A$ be such that $\pi(x) = v$ and put $y = fxe \in A$. By 9.3.(2) there exists a projection $p \in A$, $p \leqslant e$, commuting with y^*y, such that

$$y^*yp \geqslant 3^{-1}p, \quad y^*y(e-p) \leqslant 2^{-1}(e-p).$$

Then y^*yp is invertible in pAp, so there exists $a \in pAp$, $a \geqslant 0$, with $y^*ya = ay^*y = p$. Consider

$$u = ya^{1/2} = fxea^{1/2} \in A.$$

Since $u^*u = a^{1/2}y^*ya^{1/2} = y^*ya = p$, u is a partial isometry and $u^*u \leqslant e$. On the other hand, since $uu^* = fxeaex^*f$, we have also $uu^* = fuu^* \leqslant f$. Now,

$$\begin{aligned}\pi(y^*y) &= \widetilde{\pi}(e)\pi(x)^*\widetilde{\pi}(f)\pi(x)\widetilde{\pi}(e) = \widetilde{\pi}(e)v^*\widetilde{\pi}(f)v\widetilde{\pi}(e)\\ &= \widetilde{\pi}(e)v^*vv^*\widetilde{\pi}(f)vv^*v\widetilde{\pi}(e) = v^*vv^*vv^*v = v^*v.\end{aligned}$$

Since p commutes with y^*y, and

$$\pi(y^*yp) \geqslant 3^{-1}\pi(p), \quad \pi(y^*y(e-p)) \leqslant 2^{-1}\widetilde{\pi}(e-p),$$

it follows that $\pi(p)$ commutes with v^*v and

$$v^*v\pi(p) \geqslant 3^{-1}\pi(p), \quad v^*v(1-\pi(p)) \leqslant 2^{-1}\widetilde{\pi}(e-p).$$

By the first inequality, $\pi(p) \leqslant v^*v$, hence, by the second inequality, the norm of the projection $v^*v - \pi(p)$ is majorized by 2^{-1}. Consequently, $\pi(p) = v^*v$. Therefore

$$\pi(a) = \pi(ap) = \pi(a)v^*v = \pi(ay^*y) = \pi(p) = v^*v,$$

and we conclude

$$\pi(u) = \widetilde{\pi}(f)\pi(x)\widetilde{\pi}(e)\pi(a)^{1/2} = \widetilde{\pi}(f)v\widetilde{\pi}(e)v^*v = v. \quad \blacksquare$$

9.6. Concerning the "lifting" of positive elements, a careful examination of the proof of Proposition 3.15 shows that the following statement holds:

Let J be a norm-closed Jordan algebra in the C^*-algebra A, B a C^*-algebra and $\Phi : J \mapsto B_h$ a Jordan homomorphism. If $a_0 \in \widetilde{J}$, $a_0 \geqslant 0$, and $b \in \Phi(J)$, $b \geqslant 0$ are such that

$$b \leqslant \widetilde{\Phi}(a_0),$$

then there exists $a \in J$, $a \geqslant 0$, with

$$a \leqslant a_0, \quad \Phi(a) = b.$$

Imitating the proof of Corollary 2/3.15, it follows that

Let J, B, Φ as above. If $\{b_k\}_{k \geqslant 1}$ is a norm bounded increasing sequence of positive elements in $\Phi(J)$, then there exists an increasing sequence $\{a_k\}_{k \geqslant 1}$ of positive elements in J such that

$$\sup_k \|a_k\| = \sup_k \|b_k\| \quad \text{and} \quad \Phi(a_k) = b_k, \quad k \geqslant 1.$$

9.7. Finally, we describe the commutative C^*-algebras satisfying the spectral axiom.

Recall that a locally compact Hausdorff topological space Ω is called *totally disconnected* if every connected component of Ω consists of a single point, and is called 0-*dimensional* if the family of all simultaneously closed and open subsets of Ω is a basis for the topology of Ω.

PROPOSITION. *Let A be a commutative C^*-algebra and Ω be its Gelfand spectrum. Then the following statements are equivalent:*

(i) *A satisfies the spectral axiom;*
(ii) *A is the norm-closed linear hull of its projections;*
(iii) *Ω is totally disconnected;*
(iv) *Ω is 0-dimensional.*

Proof. (i) $\Rightarrow$ (ii) is a consequence of Proposition 1/9.4.

Assume now that (ii) holds, and consider $\omega_1, \omega_2 \in \Omega$, $\omega_1 \neq \omega_2$. By the Urysohn lemma there exists $a \in A$, $a \geqslant 0$, with $\omega_1(a) = 1$ and $\omega_2(a) = 0$. By (ii) there exists $b = \sum_{j=1}^{n} \lambda_j e_j$, with $\lambda_1, \ldots, \lambda_n \geqslant 0$, and $e_1, \ldots, e_n \in A$ mutually orthogonal projections, such that $\|a - b\| \leqslant 1/3$. Then $\omega_1(b) \geqslant 2/3$, $\omega_2(b) \leqslant 1/3$. Since, for $j \neq k$ and $\omega \in \Omega$, we have $\omega(e_j)\omega(e_k) = \omega(e_j e_k) = 0$, there exist $1 \leqslant j_1, j_2 \leqslant n$ such that

$$\begin{aligned} &\omega_1(e_{j_1}) = 1 \quad \text{and} \quad \omega_1(e_j) = 0 \quad \text{for } j \neq j_1, \\ &\omega_2(e_{j_2}) = 1 \quad \text{and} \quad \omega_2(e_j) = 0 \quad \text{for } j \neq j_2. \end{aligned}$$

Then $\lambda_{j_1} = \omega_1(b) \geqslant 2/3$, so $j_2 \neq j_1$. Hence

$$V = \{\omega \in \Omega;\, \omega(e_{j_1}) = 1\} = \{\omega \in \Omega;\, \omega(e_{j_1}) \neq 0\}$$

is a closed and open neighborhood of ω_1 and $\omega_2 \notin V$. Therefore ω_1 and ω_2 belong to distinct connected components of Ω. Consequently, Ω is totally disconnected. Thus, (ii)$\Rightarrow$ (iii).

(iii) $\Rightarrow$ (iv). Every totally disconnected locally compact Hausdorff topological space is 0-dimensional ([130]; 3.5).

(iv) $\Rightarrow$ (i). Assume that Ω is 0-dimensional. Let $a \in A$, $a \geqslant 0$, and $\lambda, \mu \in (0, +\infty)$, $\lambda < \mu$. Then $K = \{\omega \in \Omega; \omega(a) \geqslant \mu\}$ is compact, $D = \{\omega \in \Omega; \omega(a) > \lambda\}$ is open and $K \subset D$. By an easy compactness argument using the 0-dimensionality of Ω, we can get a compact and open set U in Ω such that $K \subset U \subset D$. Denote by $e \in A$ the element corresponding by the Gelfand representation to the characteristic function of U. Then e is a projection and

$$ae \geqslant \lambda e, \quad a(1-e) \leqslant \mu(1-e).$$

Hence A satisfies the spectral axiom. ∎

9.8. Let A be a C^*-algebra and B a C^*-subalgebra of A. We shall say that A is a *Rickart algebra* if A satisfies the spectral axiom and every sequence $\{e_k\}_{k\geqslant 1}$ of mutually orthogonal projections in A has a minimal upper bound e in $P(A)$. By Proposition 3/9.4 if A is a Rickart algebra and $\{e_k\}$, e are as above, then e is the least upper bound of $\{e_k\}$ in $P(A)$. In this case we write

$$e = \sum_{k=1}^{\infty} e_k.$$

Assume that A is a Rickart algebra. We shall say that B is a *Rickart subalgebra* of A if B satisfies the spectral axiom and for every sequence $\{e_k\}_{k\geqslant 1}$ of mutually orthogonal projections in B we have $\sum\limits_{k=1}^{\infty} e_k \in B$.

On the other hand, we shall say that A is *sequentially monotone complete* if every norm-bounded increasing sequence $\{x_k\}_{k\geqslant 1}$ in A_h has a least upper bound x in A_h. In this case we write

$$x = \sup_k x_k \quad \text{or} \quad x_k \uparrow x.$$

Assume that A is sequentially monotone complete. We shall say that B is *sequentially monotone closed* in A if for every norm-bounded increasing sequence $\{x_k\}_{k\geqslant 1}$ in B_h we have $\sup\limits_k x_k \in B$.

Clearly, every Rickart subalgebra of a Rickart algebra is a Rickart algebra, and every sequentially monotone closed C^*-subalgebra of a sequentially monotone complete C^*-algebra is sequentially monotone complete.

The notions of *Rickart-Jordan algebra, Rickart-Jordan subalgebra, sequentially monotone complete Jordan algebra* and *sequentially monotone closed Jordan subalgebra* are now self-explanatory.

9.9. We now prove a characterization of Rickart-Jordan algebras and subalgebras, hence also of Rickart algebras and subalgebras.

LEMMA. *Let J be a norm-closed Jordan algebra in a C^*-algebra A, $a \in J$, $a \geqslant 0$, and $e \in J$ a projection such that $ae = a$ and*

$$b \in J,\ b \geqslant 0,\ ab = 0 \Rightarrow eb = 0.$$

Then e is the greatest lower bound of $\{f \in P(\widetilde{J});\ af = a\}$.

Proof. Obviously, $e \in P = \{f \in P(\widetilde{J});\ af = a\}$. Now let $f \in P$. Then $a(1-f) = 0$, $ae(1-f)e = a(1-e)f = 0$. Since $e(1-f)e \in J$ and $e(1-f)e \geqslant 0$, we get $ee(1-f)e = 0$, so $e = ef \leqslant f$. Consequently, e is the greatest lower bound of P in $P(J)$. ∎

In particular, if J is a norm-closed Jordan algebra in a C^*-algebra A, then for each $a \in J$, $a \geqslant 0$, there exists at most one projection $e \in J$ such that $ae = a$ and

$$b \in J,\ b \geqslant 0,\ ab = 0 \Rightarrow eb = 0.$$

If such a projection e does exist, then we shall say that a *has a support projection* in J, we call e the *support projection of* a in J, and we denote e by $\mathbf{s}_J(a)$ or simply $\mathbf{s}(a)$.

PROPOSITION 1. *Let J be a norm-closed Jordan algebra in the C^*-algebra A. Then the following statements are equivalent:*

(i) *J is a Rickart-Jordan algebra;*

(ii) *every $a \in J$, $a \geqslant 0$, has a support projection in J.*

Moreover, if the above statements are true, then:

(a) *for every $a \in J$, $a \geqslant 0$, and $\lambda \in (0, \infty)$, the projection $e = \mathbf{s}_J((a - \lambda \mathbf{s}_J(a))^+)$ commutes with a and*

$$ae \geqslant \lambda e, \quad a(1-e) \leqslant \lambda(1-e);$$

(b) *for every sequence $\{e_k\}_{k \geqslant 1}$ of projections in J, the projection $\mathbf{s}_J\Big(\sum\limits_{k=1}^{\infty} 2^{-k} e_k\Big)$ is the least upper bound of $\{e_k\}_{k \geqslant 1}$ in $P(J)$.*

Proof. (i) ⇒ (ii). Let $a \in J$, $a \geqslant 0$. By Proposition 2/9.4 there exist a sequence $\{e_k\}_{k \geqslant 1}$ of mutually orthogonal projections in J, commuting with a, such that, for all k,

$$ae_k \geqslant 3^{-k} e_k, \quad a\Big(1 - \sum_{j=1}^{k} e_j\Big) \leqslant 2^{-k}\Big(1 - \sum_{j=1}^{k} e_j\Big).$$

Define

$$e = \sum_{j=1}^{\infty} e_j.$$

For each k we have

$$\|a^{1/2}(1-e)a^{1/2}\| \leqslant \Big\|a^{1/2}\Big(1 - \sum_{j=1}^{k} e_j\Big)a^{1/2}\Big\| = \Big\|a\Big(1 - \sum_{j=1}^{k} e_j\Big)\Big\| \leqslant 2^{-k},$$

so $a = ae = ea$.

Let $b \in J$, $b \geqslant 0$, $ab = 0$. Again by Proposition 2/9.4, there exists a sequence $\{f_k\}_{k\geqslant 1}$ of mutually orthogonal projections in J, commuting with b, such that, for all k,

$$bf_k \geqslant 3^{-k} f_k, \quad b\Big(1 - \sum_{j=1}^{k} f_j\Big) \leqslant 2^{-k}\Big(1 - \sum_{j=1}^{k} f_j\Big).$$

With $f = \sum\limits_{j=1}^{\infty} f_j$, we get $b = bf = fb$ as above. For each $j \geqslant 1$ we have $0 \leqslant 3^{-j} be_j b \leqslant be_j ab = 0$, so $e_j b = 0$. Consequently, for each $j, k \geqslant 1$ we get $0 \leqslant 3^{-k} e_j f_k e_j \leqslant e_j b f_k e_j = 0$, so $e_j f_k = 0$, i.e. $e_j \leqslant 1 - f_k$. Hence $e \leqslant 1 - f_k$, i.e. $f_k \leqslant 1 - e$ for all $k \geqslant 1$, so that $f \leqslant 1 - e$, i.e. ef=0. Therefore $eb = efb = 0$.

(ii) $\Rightarrow$ (a) & (b) $\Rightarrow$ (i). Let $a \in J$, $a \geqslant 0$, and $\lambda \in (0, \infty)$. Denote $e = \mathbf{s}((a - \lambda \mathbf{s}(a))^+)$. Since $\mathbf{s}(a)$ commutes with a, we have $(a - \lambda\mathbf{s}(a))^+ \mathbf{s}(a) = (a - \lambda \mathbf{s}(a))^+$ so, by the above lemma, $e \leqslant \mathbf{s}(a)$.

On the other hand, since $(a - \lambda\mathbf{s}(a))^+ (a - \lambda\mathbf{s}(a))^- = 0$, we have $e(a - \lambda\mathbf{s}(a))^- = 0$, so $e(a - \lambda) = e(a - \lambda\mathbf{s}(a)) = e(a - \lambda\mathbf{s}(a))^+ = (a - \lambda\mathbf{s}(a))^+ \geqslant 0$. In particular, e commutes with a and $ae \geqslant \lambda e$. Since $(1 - e)(a - \lambda\mathbf{s}(a))^+ = 0$, we have also $(1 - e)(a - \lambda\mathbf{s}(a)) = -(1 - e)(a - \lambda\mathbf{s}(a))^- = -(a - \lambda\mathbf{s}(a))^- \leqslant 0$, so $a(1 - e) \leqslant \lambda\mathbf{s}(a)(1 - e) \leqslant \lambda(1 - e)$.

Thus, the statement (a) is proved.

Now let $\{e_k\}_{k\geqslant 1}$ be a sequence of projections in J and put

$$a = \sum_{k=1}^{\infty} 2^{-k} e_k \in J, \quad a \geqslant 0,$$
$$e = \mathbf{s}(a).$$

Since $0 \leqslant a \leqslant 1$ and $ea = a$, by 2.6.(8) we get $a \leqslant e$. Hence, for all k, we have $2^{-k} e_k \leqslant a \leqslant e$, so e is an upper bound of $\{e_k\}$ in $P(J)$. If f is another bound of $\{e_k\}$ in $P(J)$, then

$$ae(1 - f)e = \sum_{k=1}^{\infty} 2^{-k} e_k (1 - f)e = 0.$$

Since $e(1-f)e \in J$ and $e(1-f)e \geqslant 0$ it follows that $ee(1-f)e = 0$, so $e = ef \leqslant f$. Consequently, e is the least upper bound of $\{e_k\}$ in $P(J)$.

Thus, also the statement (b) is proved. ∎

PROPOSITION 2. *Let $K \subset J$ be norm-closed Jordan algebras in the C^*-algebra A. Assume that J is a Rickart-Jordan algebra. Then the following statements are equivalent:*

(i) *K is a Rickart-Jordan subalgebra of J;*

(ii) *for every $a \in K$, $a \geqslant 0$, we have $\mathbf{s}_J(a) \in K$.*

Proof. (i) $\Rightarrow$ (ii). Let $a \in K$. Since K satisfies the spectral axiom, by Proposition 2/9.4 we obtain a sequence $\{e_k\}_{k\geqslant 1}$ of mutually orthogonal projections in K, commuting with a, such that for all k we have

$$ae_k \geqslant 3^{-k}e_k, \quad a\Big(1-\sum_{j=1}^{k} e_j\Big) \leqslant 2^{-k}\Big(1-\sum_{j=1}^{k} e_j\Big).$$

As we have seen in the proof of Proposition 1, $\mathbf{s}_J(a) = \sum\limits_{j=1}^{\infty} e_j$. Since K is a Rickart-Jordan subalgebra of J and the projections e_j belong to K, it follows that $\mathbf{s}_J(a) \in K$.

(ii) $\Rightarrow$ (i). This is a consequence of the statements (a) and (b) from Proposition 1. ∎

Let $K \subset J$ be norm-closed Jordan algebras in the C^*-algebra A. Assume that J is a Rickart-Jordan algebra and K is a Rickart-Jordan subalgebra of J. By Proposition 1, $P(J)$ is a *sequentially complete lattice:* if $\{e_k\}_{k\geqslant 1}$ is an arbitrary sequence in $P(J)$, then

$$\bigvee_{k=1}^{\infty} e_k = \mathbf{s}_J\Big(\sum_{k=1}^{\infty} 2^{-k}e_k\Big)$$

is the least upper bound of $\{e_k\}$ in $P(J)$ and

$$\bigwedge_{k=1}^{\infty} e_k = \Big(\bigvee_{j=1}^{\infty} e_j\Big) - \bigvee_{k=1}^{\infty}\Big(\Big(\bigvee_{j=1}^{\infty} e_j\Big) - e_k\Big)$$

is the greatest lower bound of $\{e_k\}$ in $P(J)$. If the projections e_k belong to K, then, by Proposition 2, $\bigvee\limits_{k=1}^{\infty} e_k \in K$, $\bigwedge\limits_{k=1}^{\infty} e_k \in K$.

9.10. In this section we consider some basic properties of sequentially monotone complete Jordan algebras.

We begin with three general lemmas.

Lemma 1. *Let J be a norm-closed Jordan algebra in the C^*-algebra A. If $\{x_\iota\}_{\iota\in I}$ and $\{y_\iota\}_{\iota\in I}$ are upward directed families in J which have least upper bounds in J, then also $\{x_\iota + y_\iota\}_{\iota\in I}$ has a least upper bound in J, and*

$$\sup_{\iota\in I}(x_\iota + y_\iota) = \sup_{\iota\in I} x_\iota + \sup_{\iota\in I} y_\iota.$$

Proof. Let $x = \sup\limits_{\iota} x_\iota$, $y = \sup\limits_{\iota} y_\iota$. Clearly, $x_\iota + y_\iota \leqslant x + y$ for all $\iota \in I$. If $z \in J$ is such that $x_\iota + y_\iota \leqslant z$ for all $\iota \in I$, then $x_\kappa + y_\iota \leqslant x_\kappa + y_\kappa \leqslant z$, i.e. $x_\kappa \leqslant z - y_\iota$, for all $\iota, \kappa \in I$ with $\iota \leqslant \kappa$, hence $x \leqslant z - y_\iota$, i.e. $y_\iota \leqslant z - x$, for all $\iota \in I$, and consequently $y \leqslant z - x$, i.e. $x + y \leqslant z$. Thus, $x + y$ is the least upper bound of $\{x_\iota + y_\iota\}$ in J. ∎

LEMMA 2. *Let J be a norm-closed Jordan algebra in the C^*-algebra A. If $\{x_\iota\}_{\iota\in I}$ is an upward directed family in J which has a least upper bound in J and $y \in J$, then also $\{yx_\iota y\}_{\iota\in I}$ has a least upper bound in J and*

$$\sup_{\iota\in I} yx_\iota y = y\Big(\sup_{\iota\in I} x_\iota\Big)y.$$

Proof. Without restricting the generality, we may assume that $x_\iota \geqslant 0$ for all $\iota \in I$. Let $x = \sup_{\iota\in I} x_\iota$. Clearly, $yx_\iota y \leqslant yxy$ for all $\iota \in I$, so it remains to prove that if $yx_\iota y \leqslant z \in J$ for all $\iota \in I$, then $yxy \leqslant z$.

We first assume that $y \geqslant 0$. Let $\varepsilon > 0$ be arbitrary. We recall that $(\varepsilon+y)^{-1} \in \widetilde{J}$ by 6.2.(12). For each $\iota \in I$ we have

$$\begin{aligned}(\varepsilon + y)x_\iota(\varepsilon + y) &= \varepsilon^2 x_\iota + \varepsilon(x_\iota y + yx_\iota) + yx_\iota y\\ &\leqslant \varepsilon^2 x + \varepsilon(x_\iota + y)^2 + z\\ &\leqslant \varepsilon^2 x + \varepsilon\|x_\iota + y\|(x_\iota + y) + z\\ &\leqslant \varepsilon^2 x + \varepsilon\|x + y\|(x + y) + z,\end{aligned}$$

so

$$x_\iota \leqslant (\varepsilon + y)^{-1}\big(\varepsilon^2 x + \varepsilon\|x + y\|(x + y) + z\big)(\varepsilon + y)^{-1} \in J.$$

Thus

$$x \leqslant (\varepsilon + y)^{-1}\big(\varepsilon^2 x + \varepsilon\|x + y\|(x + y) + z\big)(\varepsilon + y)^{-1}$$

and we successively get

$$\begin{gathered}(\varepsilon + y)x(\varepsilon + y) \leqslant \varepsilon^2 x + \varepsilon\|x + y\|(x + y) + z,\\ \varepsilon\big(xy + yx - \|x + y\|(x + y)\big) \leqslant z - yxy.\end{gathered}$$

Letting here $\varepsilon \to 0$, we get $0 \leqslant z - yxy$, i.e. $yxy \leqslant z$.

Now let $y \in J$ be arbitrary. By the above part of the proof we have

$$\sup_\iota y^+(x_\iota - x)y^+ = 0, \quad \sup_\iota y^-(x_\iota - x)y^- = 0$$

so, by Lemma 1,

$$\sup_\iota \big(y^+(x_\iota - x)y^+ + y^-(x_\iota - x)y^-\big) = 0.$$

Since for all $\iota \in I$ we have

$$\begin{aligned}2\big(y^+(x_\iota - x)y^+ + y^-(x_\iota - x)y^-\big) &\leqslant (y^+ - y^-)(x_\iota - x)(y^+ - y^-)\\ &= yx_\iota y - yxy \leqslant z - yxy,\end{aligned}$$

it follows that $0 \leqslant z - yxy$, that is $yxy \leqslant z$. ∎

Lemma 3. *Let J be a norm-closed Jordan algebra in the C^*-algebra A. If $\{x_\iota\}_{\iota\in I}$ is an upward directed family in J which has a least upper bound in J and which is norm-convergent, then*

$$\sup_{\iota\in I} x_\iota = \operatorname{norm-lim}_{\iota\in I} x_\iota .$$

Proof. Let $x = \sup_{\iota\in I} x_\iota$ and $y = \operatorname{norm-lim}_{\iota\in I} x_\iota$. Then the family $\{x - x_\iota\}$ is downward directed, consists of positive elements and it is norm-convergent to $x-y$. Hence $x-y \geqslant 0$, i.e. $x \geqslant y$, and $x - y \leqslant x - x_\iota$, i.e. $x_\iota \leqslant y$ for all $\iota \in I$, so $x \leqslant y$. ∎

The main result of this section is the following

Proposition. *Let $K \subset J$ be norm-closed Jordan algebras in the C^*-algebra A. Assume that J is sequentially monotone complete and K is sequentially monotone closed in J. Then J is a Rickart-Jordan algebra and K is a Rickart-Jordan subalgebra of J.*

Moreover, for any $a \in J$, $a \geqslant 0$, and any increasing sequence $\{f_k\}_{k\geqslant 1}$ of positive functions from $C(\sigma(a))$ which is pointwise convergent to the characteristic function $\chi_{\sigma(a)\setminus\{0\}}$, we have

$$\sup_k f_k(a) = \mathbf{s}_J(a).$$

Proof. Let $a \in J$, $a \geqslant 0$, and $\{f_k\}$ as in the statement. Denote

$$a_k = f_k(a) \quad \text{for all } k \geqslant 1,$$
$$e = \sup_k a_k,$$

and consider also the function $g \in C(\sigma(a))$ defined by $g(\lambda) = \lambda$.

Since $0 \leqslant a_k \leqslant 1$, we have $0 \leqslant a_k^2 \leqslant a_k \leqslant e$, so by Proposition 2.7, $0 \leqslant a_k \leqslant e^{1/2}$ for all $k \geqslant 1$. Thus $0 \leqslant e \leqslant e^{1/2}$, that is $0 \leqslant e \leqslant 1$.

By Lemma 2, $aea = \sup_k aa_ka = \sup_k (gf_kg)(a)$. By the Dini theorem, the sequence $\{gf_kg\}$ converges uniformly to g^2, so $a^2 = g^2(a) = \operatorname{norm-lim}(gf_kg) = \sup_k (gf_kg)(a) = aea$, by Lemma 3. It follows that $a(1-e)a = 0$ and, since $1-e \geqslant 0$, we infer that $(1-e)a = 0$, hence $a = ea = ae$. Consequently, by 1.18.(3), $a_k = ea_k = a_ke$ for all $k \geqslant 1$ and using again Lemma 2 we get

$$e^3 = e\Big(\sup_k a_k\Big)e = \sup_k ea_ke = \sup_k a_k = e.$$

Hence e is a projection and $ae = a$.

Let $b \in J$, $b \geqslant 0$, $ab = 0$. Then, by 1.18.(3), $a_kb = 0$ for all k, so that, by Lemma 2, $beb = b\big(\sup_k a_k\big)b = \sup_k ba_kb = 0$, that is $eb = 0$.

We conclude that a has a support in J and $\mathbf{s}_J(a) = e$.

Note that the sequences $\{f_k\}$ as in the statement do exist, for instance, $f_k(\lambda) = (k^{-1} + \lambda)^{-1}\lambda$. Therefore, by the above part of the proof, by Proposition 1/9.9 and Proposition 2/9.9, J is a Rickart-Jordan algebra and K is a Rickart-Jordan subalgebra of J. ∎

Thus, in sequentially monotone complete norm-closed Jordan algebras the spectral axiom holds and every positive element has a support. Moreover, if J is such an algebra, then

$$\begin{gathered}(k^{-1}+a)^{-1}a \uparrow \mathbf{s}_J(a); \quad a \in J,\, a \geqslant 0;\\ a^{1/k} \uparrow \mathbf{s}_J(a); \quad a \in J,\, 0 \leqslant a \leqslant 1.\end{gathered}$$

Also, *if $\{a_k\}_{k\geqslant 1}$ is a norm-bounded increasing sequence of positive elements in J, then*

$$\sup_k \mathbf{s}_J(a_k) = \mathbf{s}_J\Big(\sup_k a_k\Big).$$

Indeed, let $e = \sup\limits_k \mathbf{s}(a_k)$, $a = \sup\limits_k a_k$. Using Proposition 2.7 we get $0 \leqslant \mathbf{s}(a_k) = \mathbf{s}(a_k)^{1/2} \leqslant e^{1/2}$ for all $k \geqslant 1$, so $0 \leqslant e \leqslant e^{1/2}$, i.e. $0 \leqslant e \leqslant 1$, and using 2.6.(8) we further obtain $\mathbf{s}(a_k) = e\mathbf{s}(a_k) = \mathbf{s}(a_k)e$ for all $k \geqslant 1$. Now, by Lemma 2 we have

$$e^3 = e\Big(\sup_k \mathbf{s}(a_k)\Big)e = \sup_k e\mathbf{s}(a_k)e = \sup_k \mathbf{s}(a_k) = e$$

so e is a projection. Since $a_k = a_k\mathbf{s}(a_k) = a_k\mathbf{s}(a_k)e = a_ke = ea_k$ for all $k \geqslant 1$, using again Lemma 2 we obtain $eae = a$. Hence $ae = a$, so that $\mathbf{s}(a) \leqslant e$ by Lemma 9.9. Conversely, it is clear that $e \leqslant \mathbf{s}(a)$, since $\mathbf{s}(a_k) \leqslant \mathbf{s}(a)$ for all $k \geqslant 1$.

9.11. In this section we prove some permanence properties. Let J be a norm-closed Jordan algebra in the C^*-algebra A and assume that J *is a Rickart-Jordan algebra*. Then:

(1) *$\widetilde{J}$ is a Rickart-Jordan algebra and J is a Rickart-Jordan subalgebra of $\widetilde{J}$;*

(2) *for any projection $p \in \widetilde{J}$, pJp is a Rickart-Jordan subalgebra of $\widetilde{J}$;*

(3) *if $\{K_\iota\}_{\iota\in I}$ is any family of norm-closed Rickart-Jordan subalgebras of J, then $\bigcap\limits_{\iota\in I} K_\iota$ is a Rickart-Jordan subalgebra of J.*

To prove (1), first note that, by 9.3.(1), $\widetilde{J}$ satisfies the spectral axiom. Then let $\{\widetilde{e}_k\}_{k\geqslant 1}$ be a sequence of mutually orthogonal projections in $\widetilde{J}$. We have either $\widetilde{e}_k \in J$ or $1-\widetilde{e}_k \in J$ for al $k \geqslant 1$. If all $\widetilde{e}_k$ belong to J, then, by Proposition 3/9.4, the least upper bound of $\{\widetilde{e}_k\}$ in $P(J)$ is also the least upper bound in $P(\widetilde{J})$. On the other hand, if some $1-\widetilde{e}_j \in J$, then $\widetilde{e}_k = \widetilde{e}_k(1-\widetilde{e}_j)\widetilde{e}_k \in J$ for all $k \neq j$. With $e =$ the least upper bound of $\{\widetilde{e}_k\}_{k\neq j}$ in $P(J)$, we have $e \leqslant 1-\widetilde{e}_j$ and it is easy to see that $e+\widetilde{e}_j$ is the least upper bound of $\{\widetilde{e}_k\}_{k\neq 1}$ in $P(\widetilde{J})$. Thus, $\widetilde{J}$ is a Rickart-Jordan algebra and, by Proposition 3/9.4, J is a Rickart-Jordan subalgebra of $\widetilde{J}$.

Now (2) is a consequence of 9.3.(2) and Proposition 3/9.4, while (3) follows easily using Proposition 2/9.9. ∎

Let K be another norm-closed Jordan algebra in the C^*-algebra B, and assume that K is a Rickart-Jordan algebra.

We shall say that a Jordan homomorphism $\Phi : J \to K$ is *countably additive* if for every sequence $\{e_k\}_{k\geqslant 1}$ of mutually orthogonal projections in J we have

$$\sum_{k=1}^{\infty} \Phi(e_k) = \Phi\Big(\sum_{k=1}^{\infty} e_k\Big).$$

Then:

(4) *if* $\varphi : J \to K$ *is a countably additive Jordan homomorphism then* $\operatorname{Ker}\Phi$ *is a Rickart-Jordan subalgebra of* J *and* $\varphi(J)$ *is a Rickart-Jordan subalgebra of* K.

Indeed, the assertion concerning $\operatorname{Ker}\Phi$ follows easily using 9.3.(3) and the countable additivity of Φ. Now, by 9.3.(4), $\Phi(J)$ satisfies the spectral axiom. Let $\{f_k\}_{k\geqslant 1}$ be a sequence of mutually orthogonal projections in $\Phi(J)$. By Corollary of Proposition 1/9.5 there exists a sequence $\{e_k\}_{k\geqslant 1}$ of mutually orthogonal projections in J such that $\Phi(e_k) = f_k$ for all k. Then, by the countable additivity of Φ,

$$\sum_{k=1}^{\infty} f_k = \sum_{k=1}^{\infty} \Phi(e_k) = \Phi\Big(\sum_{k=1}^{\infty} e_k\Big) \in \Phi(J).$$

Thus, $\Phi(J)$ is a Rickart-Jordan subalgebra of K.

We now assume that J *is sequentially monotone complete*. Then:

(1′) $\widetilde{J}$ *is sequentially monotone complete and* J *is sequentially monotone closed in* $\widetilde{J}$;

(2′) *for any projection* $p \in \widetilde{J}$, pJp *is sequentially monotone closed in* $\widetilde{J}$;

(3′) *if* $\{K_i\}_{i\in I}$ *is any family of sequentially monotone closed Jordan subalgebras of* J, *then* $\bigcap_{i\in I} K_i$ *is a sequentially monotone closed Jordan subalgebra of* J.

It is enough to prove (1′) only in the case $J \neq \widetilde{J}$. Let $\{\widetilde{x}_k\}_{k\geqslant 1}$ be a norm-bounded increasing sequence in $\widetilde{J}$. Then for all k,

$$\widetilde{x}_k = \lambda_k + x_k \quad \text{with} \quad \lambda_k \in \mathbb{R},\ x_k \in J.$$

The sequence $\{\lambda_k\}$ is bounded and increasing, so $\lambda_k \uparrow \lambda \in \mathbb{R}$. Let $e = \bigvee_{k=1}^{\infty} \mathbf{s}_J(x_k^2) \in J$. Then, for each $k \geqslant 1$, $x_k = x_k e = e x_k$, so x_k commutes with e. Hence $\{\widetilde{x}_k e\}$ is a norm bounded increasing sequence in J. Consider $x = \sup_k \widetilde{x}_k e = \sup_k(\lambda_k e + x_k) \in J$. If $\widetilde{y} \in \widetilde{J}$ is such that $\lambda_k e + x_k \leqslant \widetilde{y}$ for all k and $\widetilde{y} = \mu + y$ with $\mu \in \mathbb{R}$ and $y \in J$, then $\mu \geqslant 0$. Also, putting $f = e \vee \mathbf{s}_J(y^2)$ we have $\lambda_k e + x_k \leqslant \mu f + y$ for all $k \geqslant 1$, so $x \leqslant \mu f + y \leqslant \mu + y = \widetilde{y}$. Hence x is the least upper bound of $\{\lambda_k e + x_k\}$ also in $\widetilde{J}$. By Lemma 1/9.10 we conclude that $\widetilde{x} = \lambda(1-e) + x$ is the least upper bound of $\{\lambda_k(1-e) + \lambda_k e + x_k\} = \{\widetilde{x}_k\}$ in $\widetilde{J}$. Note that if all $\widetilde{x}_k$ belong to J, then $\lambda = 0$, so $\widetilde{x} = x \in J$. Hence $\widetilde{J}$ is sequentially monotone complete and J is sequentially monotone closed in $\widetilde{J}$.

Now, $(2')$ is a consequence of $(1')$ and $(3')$ is obvious.

Let K be another sequentially monotone complete Jordan algebra in the C^*-algebra B.

We shall say that Jordan homomorphism $\Phi : J \to K$ is *sequentially normal* if for every norm-bounded increasing sequence $\{x_k\}_{k \geqslant 1}$ in J we have

$$\sup_k \Phi(x_k) = \Phi\Big(\sup_k x_k\Big).$$

Then:

$(4')$ *if* $\Phi : J \to K$ *is a sequentially normal Jordan homomorphism, then* $\operatorname{Ker}\Phi$ *is sequentially monotone closed in* J *and* $\Phi(J)$ *is sequentially monotone closed in* K.

Indeed, the statement concerning $\operatorname{Ker}\Phi$ is immediate and the statement concerning $\Phi(J)$ follows easily using 9.6.

Finally, we prove a "mixture" of (4) and $(4')$:

$(4'')$ *if* $J \subset A$ *is a sequentially monotone complete norm-closed Jordan algebra,* $K \subset B$ *is a norm-closed Rickart-Jordan algebra and* $\Phi : J \to K$ *is a countably additive Jordan homomorphism, then* $\Phi(J)$ *is a sequentially monotone complete Jordan algebra and* $\Phi : J \to \Phi(J)$ *is a sequentially normal.*

Taking into account 9.6, it is enough to prove that if $\{x_k\}_{k \geqslant 1}$ is norm-bounded increasing sequence in J, $x = \sup_k x_k$ and if $y \in \Phi(J)$, $y \geqslant \Phi(x_k)$ for all k, then $y \geqslant \Phi(x)$. Clearly, for each $k \geqslant 1$ there exists $z_k \in J$, $z_k \geqslant x_k$, such that $\Phi(z_k) = y$. By (4), $\operatorname{Ker}\Phi$ is a Rickart-Jordan subalgebra of J so, since $z_k - z_1 \in \operatorname{Ker}\Phi$ for all $k \geqslant 1$, we have $e = \bigvee_{k=1}^{\infty} \mathbf{s}_J((z_k - z_1)^2) \in \operatorname{Ker}\Phi$. Now, $(z_k - z_1)(1 - e) = 0$ for all $k \geqslant 1$, so

$$(1-e)z_1(1-e) = (1-e)z_k(1-e) \geqslant (1-e)x_k(1-e).$$

Using $(1')$ and Lemma 2/9.10, we infer that $(1-e)z_1(1-e) \geqslant (1-e)x(1-e)$, and since $e \in \operatorname{Ker}\Phi$, we conclude

$$y = \Phi(z_1) = \Phi((1-e)z_1(1-e)) \geqslant \Phi((1-e)x(1-e)) = \Phi(x).$$

9.12. Let A be a Rickart algebra. For each $a \in A$, $a \geqslant 0$, we denote by $\mathbf{s}_A(a)$, or simply by $\mathbf{s}(a)$, the support projection of a in the Rickart-Jordan algebra A_h. If $x \in A$ is arbitrary, then we define its *left support* $\mathbf{l}(x) = \mathbf{l}_A(x)$ in A by

$$\mathbf{l}_A(x) = \mathbf{s}_A(xx^*)$$

and its *right support* $\mathbf{r}(x) = \mathbf{r}_A(x)$ in A by

$$\mathbf{r}_A(x) = \mathbf{s}_A(x^*x) = \mathbf{l}_A(x^*).$$

If x is normal, then we call $\mathbf{l}_A(x) = \mathbf{r}_A(x)$ simply the *support* of x in A and we denote it by $\mathbf{s}_A(x)$ or $\mathbf{s}(x)$.

Clearly, if B is a Rickart subalgebra of A, then

$$x \in B \Rightarrow \mathbf{l}_A(x), \quad \mathbf{r}_A(x) \in B.$$

For $S \subset \widetilde{A}$ we define

$$S' \cap A = \{x \in A;\ xy = yx \text{ for all } y \in S\}.$$

If $S = S^* = \{y^*;\ y \in S\}$, then $S' \cap A$ is a C^*-subalgebra of A. Since $S' \cap A = C^*(S \cup \{1_{\widetilde{A}}\})' \cap A$ and $C^*(S \cup \{1_{\widetilde{A}}\})$ is the linear span of its unitaries, for $x \in A$ we have

$$x \in S' \cap A \Leftrightarrow u^*xu = x \quad \text{ for all unitaries } u \in C^*(S \cup \{1_{\widetilde{A}}\}).$$

Using this remark, it follows that

$$a \in S' \cap A,\ a \geqslant 0 \Rightarrow \mathbf{s}_A(a) \in S' \cap A.$$

From Proposition 2/9.9 we infer that

(1) *if $S \subset A$, $S = S^*$, then $S' \cap A$ is a Rickart subalgebra of A.*

Since the center of A is $A' \cap A$, it follows that

(2) *the center of A is a Rickart subalgebra of A*

If B is a maximal commutative $*$-subalgebra of A, then $B = B' \cap A$ hence

(3) *every maximal commutative $*$-subalgebra of A is a Rickart subalgebra of A.*

If $x \in A$ is normal, then, by the Fuglede-Putnam theorem (Theorem 1.1) we have $\{x\}' \cap A = \{x, x^*\}' \cap A$, hence

(4) *if $x \in A$ is normal, then $\{x\}' \cap A$ is a Rickart subalgebra of A.*

Now let A be a sequentially monotone complete C^*-algebra. With similar arguments we obtain the following conclusions:

(1′) *if $S \subset A$, $S = S^*$, then $S' \cap A$ is a sequentially monotone closed C^*-subalgebra of A;*
(2′) *the center of A is sequentially monotone closed C^*-subalgebra;*
(3′) *every maximal commutative $*$-subalgebra of A is a sequentially monotone closed C^*-subalgebra of A;*
(4′) *if $x \in A$ is normal, then $\{x\}' \cap A$ is a sequentially monotone closed C^*-subalgebra of A.*

9.13. Rickart algebras can be characterised in terms of their maximal commutative $*$-subalgebras:

Proposition 1. *Let A be a C^*-algebra. The following statements are equivalent:*
(i) *A is a Rickart algebra;*
(ii) *every maximal commutative $*$-subalgebra of A is a Rickart algebra.*

Proof. (i) $\Rightarrow$ (ii) is a consequence of 9.12.(3).

Conversely, assume that (ii) holds. Then A satisfies the spectral axiom. Let $\{e_k\}_{k\geqslant 1}$ be a sequence of mutually orthogonal projections in A. By the Zorn lemma, there exists a maximal totally ordered family F of upper bounds of $\{e_k\}$ in $P(A)$. Again by the Zorn lemma there exists a maximal commutative $*$-subalgebra B of A containing $\{e_k\}$ and F. By (ii), B is a Rickart algebra, so there exists a least upper bound e of $\{e_k\}$ in $P(B)$. Then $e \leqslant f$ for all $f \in F$. If $e' \leqslant e$, $e' \neq e$, would be an upper bound of $\{e_k\}$ in $P(A)$, then $F \cup \{e'\}$ would be a totally ordered family of upper bounds of $\{e_k\}$ in $P(A)$, in contradiction with the maximality of F. Hence e is a minimal upper bound of $\{e_k\}$ in $P(A)$. ∎

Also Rickart subalgebras can be characterised in terms of their maximal commutative $*$-subalgebras:

Proposition 2. *Let A be a Rickart algebra and B a C^*-subalgebra of A. The following statements are equivalent:*
(i) *B is a Rickart subalgebra of A;*
(ii) *every maximal commutative $*$-subalgebra of B is a Rickart subalgebra of A.*

Proof. (i) $\Rightarrow$ (ii) is again a consequence of 9.12.(3) and (ii) $\Rightarrow$ (i) follows by a routine verification. ∎

9.14. In this section we study some connections between the order relation and multiplication in sequentially monotone complete C^*-algebras.

Lemma. *Let A be a sequentially monotone complete C^*-algebra, $a \in A$, $a \geqslant 0$, and $x \in A$ such that $\mathbf{s}_A(a) \geqslant \mathbf{l}_A(x)$. Then:*

$$x^*ax \leqslant x^*x \Leftrightarrow a \leqslant \mathbf{l}_A(x)$$
$$x^*ax = x^*x \Leftrightarrow a = \mathbf{l}_A(x).$$

Proof. We assume that $x^*ax \leqslant x^*x$. By Proposition 2/9.4 there exists an increasing sequence $\{e_k\}_{k\geqslant 1}$ of projections in A, commuting with xx^*, such that, for all $k \geqslant 1$,

$$xx^*e_k \geqslant 3^{-k}e_k, \quad \|xx^*(1-e_k)\| \leqslant 2^{-k}.$$

Then

$$e_k \uparrow \mathbf{s}(xx^*) = \mathbf{l}(x).$$

Indeed, by 9.10, $e = \sup\limits_k e_k$ is a projection. Since, for all $k \geqslant 1$,

$$\|(xx^*)^{1/2}(1-e)(xx^*)^{1/2}\| \leqslant \|xx^*(1-e_k)\| \leqslant 2^{-k},$$

we have $xx^* = exx^* = xx^*e$. If $b \in A$, $b \geqslant 0$, $xx^*b = 0$, then, for all $k \geqslant 1$, $0 \leqslant 3^{-1}be_kb \leqslant be_kxx^*b = 0$, so $be_kb = 0$ and hence, by Lemma 2/9, $beb = 0$, $eb = 0$.

For each $k \geqslant 1$, xx^*e_k is invertible in e_kAe_k so there exists $a_k \in A$, $a_k = e_ka_ke_k \geqslant 0$, such that $xx^*a_k = a_kxx^* = e_k$ and we successively obtain:

$$0 \leqslant e_kae_k = a_kx(x^*ax)x^*a_k \leqslant a_kx(x^*x)x^*a_k = e_k,$$

$$\|a^{1/2}e_ka^{1/2}\| = \|(e_ka^{1/2})^*(e_ka^{1/2})\| = \|(e_ka^{1/2})(e_ka^{1/2})^*\| = \|e_kae_k\| \leqslant 1$$

$$a^{1/2}e_ka^{1/2} \leqslant 1,$$

$$a^{1/2}e_ka^{1/2} = \mathbf{s}(a)(a^{1/2}e_ka^{1/2})\mathbf{s}(a) \leqslant \mathbf{s}(a).$$

Using Lemma 2/9.10 we infer that $a^{1/2}\mathbf{l}(x)a^{1/2} \leqslant \mathbf{s}(a)$, and since $\mathbf{s}(a) \leqslant \mathbf{l}(x)$ we conclude that $a \leqslant \mathbf{l}(x)$.

Now we assume that $x^*ax = x^*x$. By the above part of the proof $\mathbf{l}(x) - a \geqslant 0$, so we have succesively: $x^*(\mathbf{l}(x) - a)x = 0$, $(\mathbf{l}(x) - a)xx^* = 0$, $(\mathbf{l}(x) - a)\mathbf{l}(x) = 0$ and $\mathbf{l}(x) = a\mathbf{l}(x) = a$. ∎

PROPOSITION 1. *Let A be a sequentially monotone complete C^*-algebra and $x, y \in A$ such that $\mathbf{l}_A(y) \leqslant \mathbf{l}_A(x)$, $\mathbf{r}_A(y) \leqslant \mathbf{l}_A(x)$. Then:*

$$x^*yx \geqslant 0 \Leftrightarrow y \geqslant 0,$$
$$x^*yx = 0 \Leftrightarrow y = 0.$$

Proof. We assume that $x^*yx \geqslant 0$. If $a, b \in A_h$ are defined by $y = a + ib$, then $\mathbf{s}(a) \leqslant \mathbf{l}(x)$, $x^*ax \geqslant 0$ and $\mathbf{s}(b) \leqslant \mathbf{l}(x)$, $x^*bx = 0$, since $\mathbf{l}(x) - \|a\|^{-1}a \geqslant 0$, $\mathbf{s}(\mathbf{l}(x) - \|a\|^{-1}a) \leqslant \mathbf{l}(x)$, $x^*(\mathbf{l}(x) - \|a\|^{-1}a)xx^*x$, by the above lemma we have $\mathbf{l}(x) - \|a\|^{-1}a \leqslant \mathbf{l}(x)$, so $a \geqslant 0$. Similarly we obtain $b = 0$, hence $y \geqslant 0$.

If $x^*yx = 0$, then by the above part of the proof we have $y \geqslant 0$ and $-y \geqslant 0$, so $y = 0$. ∎

PROPOSITION 2. *Let A be a sequentially monotone complete C^*-algebra and $x, y \in A$. If $y^*y \geqslant x^*x$, then there exists a unique $v \in A$ such that*

$$y = vx, \quad v^*v \geqslant \mathbf{l}_A(x)$$

and we have $vv^* \geqslant \mathbf{l}_A(y)$.

If $y^*y = x^*x$, then the above defined v satisfies the equalities $v^*v = \mathbf{l}_A(x)$, $vv^* = \mathbf{l}_A(y)$.

Proof. Assume that $y^*y \geqslant x^*x$ and denote $u_k = y(k^{-1} + x^*x)^{-1}x$ for each $k \geqslant 1$. By the polarization formula 2.8.(2) we have

$$u_k = 4^{-1}\sum_{n=0}^{3} \mathrm{i}^n u_{n,k}; \quad u_{n,k} = (x^* + \mathrm{i}^n y^*)^*(k^{-1} + x^*x)^{-1}(x^* + \mathrm{i}^n y^*).$$

Since

$$\begin{aligned}\|u_{n,k}\| &= \|(k^{-1}+x^*x)^{-1/2}(x^*+\mathrm{i}^n y^*)(x^*+\mathrm{i}^n y^*)^*(k^{-1}+x^*x)^{-1/2}\| \\ &\leqslant \|(k^{-1}+x^*x)^{-1/2}[2(x^*x+y^*y)](k^{-1}+x^*x)^{-1/2}\| \\ &\leqslant \|(k^{-1}+x^*x)^{-1/2}[4x^*x](k^{-1}+x^*x)^{-1/2}\| \leqslant 4,\end{aligned}$$

the least upper bounds $u_{n,\infty} = \sup\limits_k u_{n,k}$ do exists and $0 \leqslant u_{n,\infty} \leqslant 4$. Consider

$$u_\infty = 4^{-1}\sum_{n=0}^{3} \mathrm{i}^n u_{n,\infty}.$$

By Proposition 3.4, y belongs to the smallest closed left ideal of A containing x^*x s, by Proposition 3.3, $\lim\limits_{k\to\infty} \|y - u_k x\| = 0$. Therefore for any $\varepsilon > 0$ there exists $k_\varepsilon \geqslant 1$, an integer, such that, for $k \geqslant k_\varepsilon$,

$$\begin{aligned}(y-u_\infty x)^*(y-u_\infty x) &\leqslant \varepsilon + (u_k x - u_\infty x)^*(u_k x - u_\infty x) \\ &= \varepsilon + x^*\Big[4^{-1}\sum_{n=0}^{3}\mathrm{i}^n(u_{n,k}-u_{n,\infty})\Big]^*\Big[4^{-1}\sum_{n=0}^{3}\mathrm{i}^n(u_{n,k}-u_{n,\infty})\Big]x \\ &\leqslant \varepsilon + x^*\Big[4^{-1}\sum_{n=0}^{3}(u_{n,k}-u_{n,\infty})^2\Big]x \\ &\leqslant \varepsilon + \sum_{n=0}^{3} x^*(u_{n,\infty}-u_{n,k})x.\end{aligned}$$

By Lemma 2/9.10 and Lemma 1/9.10, 0 is the greatest lower bound decreasing sequence $\Big\{\sum\limits_{n=0}^{3} x^*(u_{n,\infty}-u_{n.k})x\Big\}_{k\geqslant 1}$, so

$$0 \leqslant (y-u_\infty x)^*(y-u_\infty x) \leqslant \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, it follows that $y = u_\infty x$. Now, with

$$v = u_\infty \mathbf{l}(x),$$

we have $y = vx$ and $\mathbf{s}(v^*v) \leqslant \mathbf{l}(x)$. Since $x^*v^*vx = y^*y \leqslant x^*x$, by the above Lemma we get $v^*v \leqslant \mathbf{l}(x)$.

We have thus proved the existence of the required element v. The uniqueness of v is a consequence of Proposition 1. Since $y = vx$, we have $(1-\mathbf{l}(y))vx = 0$, $x^*v^*(1-\mathbf{l}(y))vx = 0$ so, by Proposition 1, $v^*(1-\mathbf{l}(y))v = 0$, $(1-\mathbf{l}(y))v = 0$ and finally $vv^* = \mathbf{l}(y)vv^* = \mathbf{l}(y)vv^*\mathbf{l}(y) \leqslant \mathbf{l}(y)$.

Now assume that $y^*y = x^*x$ and let v be the above defined element. By the Lemma, $v^*v = \mathbf{l}(x)$. In particular, v is a partial isometry, so vv^* is a projection and $vv^*y = vv^*vx = vx = y$. Using Lemma 9.9 it follows that $\mathbf{l}(y) \leqslant vv^*$, hence $vv^* = \mathbf{l}(y)$. ∎

In particular, in sequentially monotone complete C^*-algebras the polar decomposition theorem holds:

Corollary. *Let A be a sequentially monotone complete C^*-algebra. For every $x \in A$ there exists a unique $v \in A$ such that*

$$x = v|x|, \quad v^*v \leqslant \mathbf{s}_A(|x|).$$

Moreover

$$v^*v = \mathbf{s}_A(|x|) = \mathbf{r}_A(x), \quad vv^* = \mathbf{l}_A(x).$$

The above propositions allow us to prove the following completion of Lemma 2/9.10:

Proposition 3. *Let A be a sequentially monotone complete C^*-algebra, $\{a_\iota\}_{\iota\in I}$ a norm bounded upward directed family in A_h and $x \in A$ such that*

$$\mathbf{s}_A(a_\iota) \leqslant \mathbf{l}_A(x), \quad \iota \in I.$$

Then $\{a_\iota\}_{\iota\in I}$ has a least upper bound in A_h if and only if $\{x^*a_\iota x\}_{\iota\in I}$ has a least upper bound in A_h and in this case

$$\sup_\iota x^*a_\iota x = x^*\Big(\sup_\iota a_\iota\Big)x.$$

Proof. Without loss of generality, we may assume that

$$0 \leqslant a_\iota \leqslant \mathbf{l}(x); \quad \iota \in I.$$

Suppose that $a = \sup_\iota a_\iota$ does exist in A_h. Clearly $x^*a_\iota x \leqslant x^*ax$ for all $\iota \in I$. Let $z \in A_h$ be such that $x^*a_\iota x \leqslant z$ for all $\iota \in I$. Defining $x_1, x_2 \in A_h$ by $x = x_1 + \mathrm{i}x_2$ and using Lemma 2/9.10 and Lemma 1/9.10, we infer that 0 is the least upper bound of $\{x_1(a_\iota - a)x_1 + x_2(a_\iota - a)x_2\}_{\iota\in I}$ in A_h. Since for all $\iota \in I$,

$$\begin{aligned} 2[x_1(a_\iota - a)x_1 + x_2(a_\iota - a)x_2] &\leqslant (x_1 - \mathrm{i}x_2)(a_\iota - a)(x_1 + \mathrm{i}x_2) \\ &= x^*a_\iota x - x^*ax \leqslant z - x^*ax, \end{aligned}$$

we get $0 \leqslant z - x^*ax$, that is $x^*ax \leqslant z$. Hence $x^*ax = \sup_\iota x^*a_\iota x$.

Conversely, assume that $b = \sup_\iota x^*a_\iota x$ does exists in A_h. Since $0 \leqslant x^*a_\iota x \leqslant x^*x$ for all $\iota \in I$, we have $0 \leqslant b \leqslant x^*x$. By Proposition 2 there exists $v \in A$ such that $b^{1/2} = vx$ and $v^*v \leqslant \mathbf{l}(x)$. We shall show that $a = v^*v$ is the least upper bound of $\{a_\iota\}$ in A_h. Since $x^*a_\iota x \leqslant b = x^*ax$ for all $\iota \in I$, by Proposition 1 it follows that $a_\iota \leqslant a$ for all $\iota \in I$. Let $y \in A_h$ be such that $a_\iota \leqslant y$ for all $\iota \in I$. Denote

$$e = \mathbf{l}(x) \vee \mathbf{s}(y) \quad \text{and} \quad x_0 = xx^* + e - \mathbf{l}(x) \geqslant 0.$$

Since $x^*ax = b$ is the least upper bound of $\{x^*a_\iota x\}$ in A_h, by the first part of the proof it follows that $x_0ax_0 = x(x^*ax)x^*$ is the least upper bound of $\{x(x^*a_\iota x)x^*\} = \{x_0a_\iota x_0\}$ in A_h. But $x_0a_\iota x_0 \leqslant x_0yx_0$ for all $\iota \in I$, so $x_0ax_0 \leqslant x_0yx_0$. Since $\mathbf{s}(x_0) = e$, using Proposition 1 we get $a \leqslant y$. ∎

Note that *if* $\{a_\iota\}_{\iota\in I}$ *is an upward directed family of selfadjoint elements in an arbitrary* C^**-algebra* A*, wich has a least upper bound* a *in* A_h*, then* x^*ax *is the least upper bound of* $\{x^*a_\iota x\}_{\iota\in I}$ *in* A_h*, for every* $x \in A$.

Indeed, in the first part of the proof of Proposition 3 we used only Lemma 2/9.10 and Lemma 1/9.10, which are valid in arbitrary C^*-algebras.

9.15. Let A be a C^*-algebra and $\{a_\iota\}_{\iota\in I} \subset A$, $a_\iota \geqslant 1$ for all $\iota \in I$, and $a \in A$, $a \geqslant 0$. By $\sum\limits_{\iota\in I} a_\iota$ we shall denote the least upper bound of $\Big\{ \sum\limits_{\iota\in F} a_\iota \Big\}_{F\subset I,\,\text{finite}}$ in A_h, if it does exist. Even if $\sum\limits_{\iota\in F} a_\iota$ doesn't exist, we shall write

$$\sum_{\iota\in I} a_\iota \leqslant a$$

in order to shorten the statement "$\sum\limits_{\iota\in F} a_\iota \leqslant a$ for all finite subsets F of I".

The following result extends Proposition 7.13:

PROPOSITION. *Let* A *be a sequentially monotone complete* C^**-algebra,* $x \in A$ *and* $\{y_\iota\}_{\iota\in I} \subset A$*. If* $\sum\limits_{\iota\in I} y_\iota^* y_\iota \leqslant x^*x$*, then there exist* $v_\iota \in A$ *for all* $\iota \in I$*, uniquely determined such that*

$$y_\iota = v_\iota x \quad \textit{and} \quad v_\iota^* v_\iota \leqslant \mathbf{l}_A(x) \quad \textit{for all } \iota \in I,$$

and we have $\sum\limits_{\iota\in I} v_\iota^* v_\iota \leqslant \mathbf{l}_A(x)$ *and* $v_\iota v_\iota^* \leqslant \mathbf{l}_A(y_\iota)$ *for all* $\iota \in I$.

If $\sum\limits_{\iota\in I} y_\iota^* y_\iota = x^*x$*, then the above defined elements* v_ι*,* $\iota \in I$*, satisfy also the equality* $\sum\limits_{\iota\in I} v_\iota^* v_\iota = \mathbf{l}_A(x)$.

Proof. The existence and the uniqueness of $\{v_\iota\}_{\iota\in I}$ is a consequence of Proposition 2/9.14. Again by Proposition 2/9.14 we have $v_\iota v_\iota^* \leqslant \mathbf{l}(y_\iota)$, $\iota \in I$. For every finite $F \subset I$ we have

$$\mathbf{s}\Big(\sum_{\iota\in F} v_\iota^* v_\iota\Big) \leqslant \mathbf{l}(x), \quad x^*\Big(\sum_{\iota\in F} v_\iota^* v_\iota\Big)x = \sum_{\iota\in F} y_\iota^* y_\iota \leqslant x^*x,$$

so, by Lemma 9.14, $\sum\limits_{\iota\in F} v_\iota^* v_\iota \leqslant \mathbf{l}(x)$. Hence $\sum\limits_{\iota\in I} v_\iota^* v_\iota \leqslant \mathbf{l}(x)$.

If $\sum\limits_{\iota\in I} x^* v_\iota^* v_\iota x = \sum\limits_{\iota\in I} y_\iota^* y_\iota = x^*x$, then by Proposition 3/9.14, $a = \sum\limits_{\iota\in I} v_\iota^* v_\iota$ does exist and $x^*ax = x^*x$. Since $0 \leqslant a \leqslant \mathbf{l}(x)$, using Lemma 9.14 we infer that $a = \mathbf{l}(x)$. ∎

As an application, we prove an extension of Proposition 3.13 for sequentially monotone complete C^*-algebras:

Corollary. *Let A be a sequentially monotone complete C^*-algebra $\{x_\iota\}_{\iota\in I} \subset A$ and $\{y_\kappa\}_{\kappa\in K} \subset A$. if $\sum_{\kappa\in K} y_\kappa^* y_\kappa$ does exist and*

$$\sum_{\iota\in I} x_\iota^* x_\iota \geqslant \sum_{\kappa\in K} y_\kappa^* y_\kappa$$

then there exist $z_{\iota\kappa} \in \mathbf{l}_A(x_\iota) A \mathbf{l}_A(y_\kappa)$, $(\iota \in I, \kappa \in K)$, such that

$$x_\iota x_\iota^* = \sum_{\kappa\in K} z_{\iota\kappa} z_{\iota\kappa}^*, \ (\iota \in I); \quad y_\kappa y_\kappa^* \geqslant \sum_{\iota\in I} z_{\iota\kappa}^* z_{\iota\kappa}, \ (\kappa \in K).$$

If both $\sum_{\kappa\in K} y_\kappa^* y_\kappa$ and $\sum_{\iota\in I} x_\iota^* x_\iota$ do exist and are equal, then the above elements $z_{\iota\kappa}$, $(\iota \in I,\ \kappa \in K)$, can be choosed such that they satisfy also the equalities

$$y_\kappa y_\kappa^* = \sum_{\iota\in I} z_{\iota\kappa}^* z_{\iota\kappa}, \quad \kappa \in K.$$

Proof. Assume that $a = \sum_{\kappa\in K} y_\kappa^* y_\kappa$ does exist and $\sum_{\iota\in I} x_\iota^* x_\iota \leqslant a$. Then by the above proposition, there exist $u_\iota \in \mathbf{l}(x_\iota)A$, $(\iota \in I)$, such that $x_\iota = u_\iota a^{1/2}$, $(\iota \in I)$, and there exists $v_\kappa \in \mathbf{l}(y_\kappa)A$, $(\kappa \in K)$, such that $y_\kappa = v_\kappa a^{1/2}$, $(\kappa \in K)$. We define

$$z_{\iota\kappa} = u_\iota a^{1/2} v_\kappa^* = x_\iota v_\kappa^* = u_\iota y_\kappa^* \in \mathbf{l}(x_\iota) A\, \mathbf{l}(y_\kappa), \quad \iota \in I,\ \kappa \in K.$$

By the last remark in 9.14, for every $\iota \in I$ we have

$$x_\iota^* x_\iota = u_\iota a u_\iota^* = \sum_{\kappa\in K} u_\iota y_\kappa^* y_\kappa u_\iota^* = \sum_{\kappa\in K} z_{\iota\kappa} z_{\iota\kappa}^*.$$

On the other hand for every $\kappa \in K$ we have

$$y_\kappa y_\kappa^* = v_\kappa a v_\kappa^* \geqslant \sum_{\iota\in I} v_\kappa x_\iota^* x_\iota v_\kappa^*$$

so $y_\kappa y_\kappa^* \geqslant \sum_{\iota\in I} z_{\iota\kappa}^* z_{\iota\kappa}$.

If we assume additionally that also $\sum_{\iota\in I} x_\iota^* x_\iota$ does exist and is equal to a, then using again the last remark in 9.14, we obtain, for every $\kappa \in K$,

$$y_\kappa y_\kappa^* = v_\kappa a v_\kappa^* = \sum_{\iota\in I} v_\kappa x_\iota^* x_\iota v_\kappa^* = \sum_{\iota\in I} z_{\iota\kappa}^* z_{\iota\kappa}. \quad \blacksquare$$

9.16. We now describe the commutative Rickart algebras and the commutative sequentially monotone complete C^*-algebras.

PROPOSITION 1. *Let A be a commutative C^*-algebra and Ω its Gelfand spectrum. Then the following statements are equivalent:*

(i) *A is a Rickart algebra;*

(ii) *A is sequentially monotone complete;*

(iii) *Ω is 0-dimensional and, for any sequence $\{U_k\}_{k\geqslant 1}$ of simultaneously compact and open subsets of Ω, the closure of $\bigcup\limits_{k\geqslant 1} U_k$ is compact and open.*

Proof. (i) $\Rightarrow$ (iii). Since A satisfies the spectral axiom, Ω is 0-dimensional by Proposition 9.7. Let $\{U_k\}_{k\geqslant 1}$ be a sequence of compact and open subsets of Ω, and denote by e_k the projection in A whose Gelfand transform is the characteristic function of U_k, $k \geqslant 1$. Then the Gelfand transform of $e = \sum\limits_{k=1}^{\infty} e_k$ is the characteristic function of some compact and open set U containing $\bigcup\limits_{k=1}^{\infty} U_k$. If $U \setminus \overline{\bigcup\limits_{k=1}^{\infty} U_k}$ would be nonempty, then, by the 0-dimensionality of Ω, we could find a nonempty compact and open set $V \subset U \setminus \overline{\bigcup\limits_{k=1}^{\infty} U_k}$ and, with $f \in A$ the projection whose Gelfand transform is the characteristic function of V, we should have $e_k \leqslant e - f$ for all $k \geqslant 1$, hence $e \leqslant e - f$ in contradiction with $f \neq 0$. Consequently, $U = \overline{\bigcup\limits_{k=1}^{\infty} U_k}$.

(iii) $\Rightarrow$ (ii). Let $\{e_k\}_{k\geqslant 1}$ be a norm-bounded sequence of positive elements in A. We define

$$U_\lambda = \bigcup_{k=1}^{\infty} \{\omega \in \Omega;\, \omega(a_k) > \lambda\}; \quad \lambda \geqslant 0.$$

Each U_λ is the union of a sequence of compact and open sets. Indeed, by the 0-dimensionality of Ω and by a standard compactness argument, for every integers $k \geqslant 1$ and $n \geqslant 1$ there exists a compact open set $V_{\lambda,k,n}$ such that

$$\{\omega \in \Omega;\, \omega(a_k) \geqslant \lambda + n^{-1}\} \subset V_{\lambda,k,n} \subset U_\lambda$$

and we have $U_\lambda = \bigcup\limits_{k,n} V_{\lambda,k,n}$. Hence $\overline{U}_\lambda$ is compact and open, so its characteristic function is the Gelfand transform of some projection $e_\lambda \in A$. Clearly,

$$\lambda \geqslant \mu \Rightarrow e_\lambda \leqslant e_\mu; \quad \lambda > \alpha = \sup_k \|a_k\| \Rightarrow e_\lambda = 0.$$

Hence the net $\sum\limits_{j=1}^{n} t_j(e_{\lambda_j} - e_{\lambda_{j-1}})_{0=\lambda_0 \leqslant t_1 \leqslant \lambda_1 \leqslant \cdots \leqslant \lambda_n > \alpha}$ converges in the norm topology of A when $\max\limits_{1\leqslant j\leqslant n} \|\lambda_j - \lambda_{j-1}\| \to 0$ that is, the Riemann-Stieltjes integral

$$\int_0^{\infty} \lambda \,\mathrm{d}e_\lambda \in A$$

does exist. Then

$$a = -\int_0^\infty \lambda \, \mathrm{d}e_\lambda \geqslant 0.$$

For each $\omega \in \Omega$ we have

$$\omega(a) = -\int_0^\infty \lambda \, \mathrm{d}\omega(e_\lambda)$$

so, for $\lambda \in (0, \infty)$,

$$\text{(1)} \qquad \{\omega \in \Omega;\ \omega(a) \geqslant \lambda\} = \{\omega \in \Omega;\ \omega(e_\mu) = 1 \text{ for } \mu < \lambda\} = \bigcap_{\mu<\lambda} \overline{U}_\mu.$$

It follows that $\omega(a_k) \leqslant \omega(a)$ for all $k \geqslant 1$, $\omega \in \Omega$, hence

$$a_k \leqslant a \quad \text{for all } k \geqslant 1.$$

Now let $b \in A_h$ be such that $a_k \leqslant b$ for all $k \geqslant 1$ and assume that $a \nleqslant b$. Then there exist $\omega_0 \in \Omega$ and $\varepsilon_0 > 0$ such that $\omega_0(a) > \omega_0(b) + 2\varepsilon_0$. By the continuity of the functions $\omega \mapsto \omega(a)$ and $\omega \mapsto \omega(b)$, there exists an open neighborhood V of ω_0 with

$$\omega(a) \geqslant \omega_0(b) + 2\varepsilon_0, \quad \omega_b \leqslant \omega_0(b) + \varepsilon_0; \quad \omega \in V.$$

Using (1) we infer that

$$V \subset \bigcap_{\mu<\omega_0(b)+2\varepsilon_0} \overline{U}_\mu \subset U_{\omega_0(b)+\varepsilon_0}.$$

Since V is open, there exists $\omega_1 \in V \cap U_{\omega_0(b)+\varepsilon_0} \subset U_{\omega_1(b)}$. So, for some $k \geqslant 1$, we have $\omega_1(a_k) > \omega_1(b)$, in contradiction with $a_k \leqslant b$. Consequently, $a \geqslant b$.

(ii) $\Rightarrow$ (i). This is a consequence of Proposition 9.10. ∎

Concerning commutative Rickart subalgebras, we have:

PROPOSITION 2. *Let A be a sequentially monotone closed C^*-algebra. Then every commutative Rickart subalgebra of A is sequentially monotone closed in A.*

Proof. Let B be a commutative Rickart subalgebra of A. If B_0 is a maximal commutative $*$-subalgebra of A containing B, then, by 9.12.(3′), B_0 is sequentially monotone closed in A and B is Rickart subalgebra of B_0. Hence we may assume, without loos of generality, that A is commutative.

Let $\{a_k\}_{k\geqslant 1}$ be a norm-bounded increasing sequence of positive elements in B and a its least upper bound in A. By the last remark in 9.10, the least upper bound of $\{\mathbf{s}_A(a_k)\} \subset P(B)$ in $P(A)$ is $\mathbf{s}_A(a)$, so $\mathbf{s}_A(a) \in B$.

Using the Gelfand representation of A it is easy to see that for every $\lambda \in (0, +\infty)$ the sequence $\{(a_k - \lambda\mathbf{s}_A(a))\}^+ \subset B^+$ is increasing and its least upper bound in A_h is $(a - \lambda\mathbf{s}_A(a))^+$. Again by 9.10, the least upper bound of $\{\mathbf{s}_A(a_k - \lambda\mathbf{s}_A(a))^+\} \subset P(B)$ in $P(A)$ is $\mathbf{s}_A((a - \lambda\mathbf{s}_A(a))^+)$, so

$$e_\lambda = \mathbf{s}_A((a - \lambda\mathbf{s}_A(a))^+) \in B; \quad \lambda \in (0, +\infty).$$

Now, by Proposition 1/9.9, we have $ae_\lambda \geqslant \lambda e_\lambda$ and $a(1-e_\lambda) \leqslant 1-e_\lambda$ for all $\lambda \in (0,+\infty)$. Consequently, for every integer $n \geqslant 1$ we have

$$0 \leqslant a - \|a\|n^{-1}\sum_{j=1}^{n-1} e_{j\|a\|/n} \leqslant \|a\|n^{-1}.$$

Hence a belongs to the norm-closed linear span of $\{e_\lambda\}_{\lambda>0}$, and we conclude that $a \in B$. ∎

9.17. By the Vigier theorem (8.5), every w-closed Jordan algebra J in the W^*-algebra M is sequentially monotone complete and the least upper bound of any norm-bounded increasing sequence $\{x_\kappa\} \subset J$ in J coincides with the limit of $\{x_\kappa\}$ in the s-topology of M. In this section we examine the Rickart-Jordan subalgebras of w-closed Jordan algebras, in particular the Rickart subalgebras of W^*-algebras.

If J is a Jordan algebra in the W^*-algebra M, then its w-closure $\overline{J}^w \subset M$ is again a Jordan algebra. Indeed, if $\{x_\iota\} \subset J$, $x_\iota \xrightarrow{w} x$ and $\{y_\kappa\} \subset J$, $y_\kappa \xrightarrow{w} y$, then for each,

$$J \ni 2^{-1}(x_\iota y_\kappa + y_\kappa x_\iota) \xrightarrow{w} 2^{-1}(x_\iota y + y x_\iota),$$

so

$$\overline{J}^w \ni 2^{-1}(x_\iota y + y x_\iota) \xrightarrow{w} 2^{-1}(xy + yx).$$

Now, let a J be *w-closed Jordan algebra* in the W^*-algebra M. Then $J_1 = \{x \in J;\ \|x\| \leqslant 1\}$ is a w-compact convex set so, by the Krein-Milman theorem, it has extreme points. Using Theorem 6.2, it follows that *J is unital.* Also, note that if S is a family of mutually commuting elements of J, then by 6.2,

$$W^*(S)_h \subset J.$$

The Kaplansky density theorem (Theorem 7.9) can be also formulated for Jordan algebras:

LEMMA 1. *Let J be a Jordan algebra in the W^*-algebra M. If $x \in M$ is w-adherent to J, then there exists ε net $\{x_\iota\}_{\iota\in I}$ in J such that $\|x_\iota\| \leqslant \|x\|$, $(\iota \in I)$ and $x_\iota \xrightarrow{s} x$. Moreover, if x is positive then the x_ι's can be choosen positive too.*

Proof. Clearly, we may assume, without reducing the generality, that J is norm-closed.

It is sufficient to show that the closed unit ball J_1 of J is s-dense in the closed unit ball $(\overline{J}^w)_1$ of the w-closure $\overline{J}^w$ of J. Since $(\overline{J}^w)_1$ is w-compact, by the Krein-Milman theorem and by Corollary 8.5, it is the s-closed convex hull of its extreme points. Hence it is sufficient to prove that every extreme point v of $(\overline{J}^w)_1$ is s-adherent to J_1.

By Corollary 8.5, v is s-adherent to J, that is there exists a net $\{y_\iota\}_{\iota\in I}$ in J such that $y_\iota \xrightarrow{s} v$. Using Theorem 7.10 we deduce

$$J_1 \ni 2y_\iota(1+y_\iota^2)^{-1} \xrightarrow{s} 2v(1+v^2)^{-1}.$$

Since, by Theorem 6.2, v^2 is the unit of $\overline{J}^w$, we conclude that $v = 2v(1+v^2)^{-1}$ is s-adherent to J_1. ∎

Let J be a *w-closed Jordan algebra* in the W^*-algebra M.

For every w-continuos positive linear functional φ on J we can define its *support* $\mathbf{s}_J(\varphi) \in P(j)$ as in 8.23: $1 - \mathbf{s}_J(\varphi)$ is the least upper bound of $\{e \in P(J);\, \varphi(e) = 0\}$ in $P(J)$. Then, for each $a \in J$, $a \geqslant 0$, we have

$$\varphi(a) = 0 \Leftrightarrow a\mathbf{s}_J(\varphi) = 0.$$

If ψ is w-continuous positive linear functional on M, we denote $\mathbf{s}_J(\psi|J)$ simply by $\mathbf{s}_J(\psi)$.

A projection $e \in J$ is called *countably decomposable* in J if any family of mutually orthogonal non-zero projections of J majorized by e is at most countable. By the arguments in the proof of Proposition 1/8.12, a projection $e \in J$ is countably decomposable in J if and only if there exists a w-continuous positive linear functional ψ on M with $e = \mathbf{s}_J(\psi)$.

We shall say that J is *countably decomposable* if its unit is countably decomposable in J.

The following lemma, essentially due to G.K. Pedersen, is the main technical result in this section:

Lemma 2. *Let M be a W^*-algebra, J a norm-closed Rickart-Jordan subalgebra of M_h and J^w the w-closure of J. Then, for any two orthogonal projections $e, f \in \overline{J}^w$ wich are countably decomposable in J^w, there exists a projection $p \in J$ such that*

$$e \leqslant p \leqslant 1_{\overline{J}^w} - f.$$

Proof. Let φ, ψ be w-continuous positive linear functionals on M with $e = \mathbf{s}_{\overline{J}^w}(\varphi)$ and $f = \mathbf{s}_{\overline{J}^w}(\psi)$. We denote $\theta = \varphi + \psi$, and we consider a fixed $\varepsilon > 0$.

By Lemma 1 there exists a net $\{x_\iota\}_{\iota \in I}$ in J such that $0 \leqslant x_\iota \leqslant 1$ for all $\iota \in I$ and $x_\iota \xrightarrow{s} e$.

Putting

$$q_0 = 1_{J^w}; \quad y_0 = 0,$$

we shall construct by recurrence a sequence $\{y_k\}_{k \geqslant 1}$ contained in $\{x_\iota;\, \iota \in I\}$ and sequence $\{q_k\}_{k \geqslant 1}$ of projections in J such that, for every $k \geqslant 1$,

$$q_k \leqslant q_{k-1}, \quad \theta(q_{k-1} - q_k) \leqslant 2^{-k+1}\varepsilon, \tag{1}$$

$$q_k(y_k - y_{k-1})^2 q_k \leqslant 2^{-k+1}, \tag{2}$$

$$\theta((e - y_k)^2) \leqslant k^{-1}, \tag{3}$$

$$\theta(q_k(e - y_k)^2 q_k) < 2^{-1}4^{-k}\varepsilon. \tag{4}$$

For $k = 1$ we put $q_1 = q_0$ and we choose y_1 such that (3) and (4) be satisfied.

Assume that the elements q_k and y_k were already defined for $1 \leqslant k \leqslant n$, where $n \geqslant 1$ is an integer.

There exists a continuous function $g : [0, +\infty) \to [0, 1]$ such that $g(\lambda) = 1$ for λ in some non-empty open interval Ω contained in $[2^{-n-1}, 2^{-n}]$ and

$$\theta(g(q_n(e - y_n)^2 q_n)) < 8^{-1}2^{-1}4^{-n-1}\varepsilon. \tag{5}$$

Indeed, otherwise $\theta(1)$ would be greater than any integral multiple of $8^{-1}2^{-1}4^{-n-1}\varepsilon$.

Let $\lambda_0 \in \Omega$ and χ_0 denote the characteristic function of $[0, \lambda_0)$. There exists a function $h : [0, +\infty) \to [0, 1]$, whose support is contained in Ω, such that $\chi_0 + h$ is continuous.

For each $\iota \in I$ we denote

$$u_\iota = q_n(x_\iota - y_n)^2 q_n, \quad v_\iota = q_n(x_\iota - e)^2 q_n.$$

Since $0 \leqslant v_\iota \leqslant 4$, $(\iota \in I)$, and $h^2 \leqslant g$, for any $\iota \in I$ we have

$$\begin{aligned}\chi_0(u_\iota)v_\iota\chi_0(u_\iota) &\leqslant 2(\chi_0 + h)(u_\iota)v_\iota(\chi_0 + h)(u_\iota) + 2h(u_\iota)v_\iota h(u_\iota)\\ &\leqslant 2(\chi_0 + h)(u_\iota)v_\iota(\chi_0 + h)(u_\iota) + 8g(u_\iota).\end{aligned}$$

Now, $u_\iota \xrightarrow{s} q_n(e - y_n)^2 q_n$, $v_\iota \xrightarrow{s} 0$ and the fonctions $\chi_0 + h$ and g are continuous so, by (5) and Theorem 7.10, we get

$$\overline{\lim_\iota}\, \theta(\chi_0(u_\iota)v_\iota\chi_0(u_\iota)) \leqslant 8\theta(g(q_n(e - y_n)q_n)) < 2^{-1}4^{-n-1}\varepsilon.$$

Hence there exists $\iota_1 \in I$ such that

(6) $$\theta(\chi_0(u_\iota)v_\iota\chi_0(u_\iota)) < 2^{-1}4^{-n-1}\varepsilon; \quad \iota \geqslant \iota_1.$$

On the other hand, since $x_\iota \xrightarrow{s} e$, using (4) for $k = n$, we can find $\iota_2 \in I$ such that

(7) $$\theta(q_n(x_\iota - y_n)^2 q_n) < 2^{-1}4^{-n-1}\varepsilon; \quad \iota \geqslant \iota_2,$$

and

(8) $$\theta((e - x_\iota)^2) \geqslant (n + 1)^{-1}; \quad \iota \geqslant \iota_2.$$

We choose $\iota_3 \in I$ such that $\iota_3 \geqslant \iota_1$, $\iota_3 \geqslant \iota_2$, and we define

$$q_{n+1} = \chi_0(u_{\iota_3})q_n, \quad y_{n+1} = x_{\iota_3}.$$

Using (1) and (4) from 7.15 it is easy to check that

$$q_{n+1} = \mathbf{s}_M((\lambda_0\mathbf{s}_M(u_{\iota_3}) - u_{\iota_3})^+).$$

Since $u_{\iota_3} \in J$, using Proposition 2/9.9 we infer that q_{n+1} is a projection in J.

Clearly, $q_{n+1} \leqslant q_n$. Since $\chi_0(\lambda) = 1$ for $\lambda \in [0, 2^{-n-1}]$, we have

$$\begin{aligned}(1 - \chi_0)(u_{\iota_3}) &\leqslant 2^{n+1}u_{\iota_3} = 2^{n+1}q_n(y_{n+1} - y_n)^2 q_n,\\ q_n - q_{n+1} &\leqslant 2^{n+1}q_n(y_{n+1} - y_n)^2 q_n,\end{aligned}$$

so, using (7) we get $\theta(q_n - q_{n+1}) \leqslant 2^{-n}$, that is (1) is satisfied for $k = n+1$.

Since $\chi_0(\lambda) = 0$ for $\lambda \in [2^{-n}, +\infty)$, we have

$$q_{n+1}(y_{n+1} - y_n)^2 q_{n+1} = \chi_0(u_{\iota_3}) u_{\iota_3} \chi_0(u_{\iota_3}) \leqslant 2^{-n},$$

so (2) is satisfied for $k = n+1$

Finally, by (8) and (6), (3) and (4) are also satisfied for $k = n+1$.

Let q be the greatest lower bound of the descreasing sequence $\{q_k\}_{k\geqslant 1} \subset P(J)$ in $P(M)$. Then $q_k \xrightarrow{s} q$ and, since J is a Rickart-Jordan subalgebra of M_h, we have $q \in P(J)$. By (2),

$$\|(y_{k+1} - y_k)q\|^2 \leqslant \|(y_{k+1} - y_k)q_{k+1}\|^2 \leqslant 2^{-k}; \quad k \geqslant 1,$$

so the sequence $\{y_k q\}_{k\geqslant 1}$ converges in the norm-topology. Consequently, the sequence $\{y_k q y_k\}_{k\geqslant 1} \subset J$ is norm-convergent. Denote

$$a_\varepsilon = \text{norm-}\lim_k y_k q y_k \in J, \quad p_\varepsilon = \mathbf{s}_M(a_\varepsilon) = \mathbf{s}_J(a_\varepsilon) \in P(J).$$

Since $e \leqslant 1_{\overline{J}^w} - f = 1_{\overline{J}^w} - \mathbf{s}_{\overline{J}^w}(\varphi)$, from (3) we infer that

$$0 \leqslant \psi(a_\varepsilon) = \lim_k \psi(y_k q y_k) \leqslant \overline{\lim_k}\, \psi(y_k^2) = \overline{\lim_k}\, \psi((e - y_k)^2) = 0.$$

Hence $a_\varepsilon \mathbf{s}_{\overline{J}^w}(\psi) = 0$ and

$$p_\varepsilon \leqslant 1_{\overline{J}^w} - \mathbf{s}_{\overline{J}^w}(\psi) = 1_{\overline{J}^w} - f. \tag{9}$$

On the other hand, since $q_k \xrightarrow{s} q$, from (1) we infer that

$$\varphi(1_{\overline{J}^w} - q) \leqslant \theta(1_{\overline{J}^w} - q) = \sum_{k=1}^{\infty} \theta(q_k - q_{k+1}) \leqslant \sum_{k=1}^{\infty} 2^{-k}\varepsilon = \varepsilon,$$

hence $\varphi(q) \geqslant \varphi(1_{\overline{J}^w}) - \varepsilon$. Furthermore, since $\mathbf{s}_{\overline{J}^w}(\varphi) = e$, from (3) we infer that, for all $k \geqslant 1$,

$$\begin{aligned}
|\varphi(q) - \varphi(y_k q y_k)| &\leqslant |\varphi((1_{\overline{J}^w} - y_k)q)| + |\varphi(y_k q(1_{\overline{J}^w} - y_k))| \\
&\leqslant 2\|\varphi\|^{1/2} \varphi((1_{\overline{J}^w} - y_k)^2)^{1/2} = 2\|\varphi\|^{1/2}\varphi((e - y_k)^2)^{1/2} \\
&\leqslant \|\varphi\|^{1/2} k^{-1/2}.
\end{aligned}$$

hence

$$\varphi(p_\varepsilon) \geqslant \varphi(a_\varepsilon) = \lim_k \varphi(y_k q y_k) = \varphi(q) \geqslant \varphi(1_{\overline{J}^w}) - \varepsilon. \tag{10}$$

Till now, $\varepsilon > 0$ was fixed. Now, denote by p the least upper bound of the sequence $\{p_{\frac{1}{n}}\}_{n\geqslant 1} \subset P(J)$ in $P(M)$. Since J is a norm-closed Rickart-Jordan subalgebra of M_h, we have $p \in P(J)$. By (9), we have $p \leqslant 1_{\overline{J}^w} - f$ and, by (10), $\varphi(p) \geqslant \varphi(p_{\frac{1}{n}}) \geqslant \varphi(1_{\overline{J}^w}) - n^{-1}$ for all $n \geqslant 1$, hence $\varphi(p) \geqslant \varphi(1_{\overline{J}^w})$, so $\varphi(1_{\overline{J}^w} - p) = 0$ and finally $e = \mathbf{s}_{\overline{J}^w}(\varphi) \leqslant p$. ∎

We are now able to prove that norm-closed Rickart-Jordan subalgebras of countably decomposable w-closed Jordan algebras are w-closed.

THEOREM 1. *Let M be a W^*-algebra and J a norm-closed Rickart-Jordan subalgebra of M_h such that w-closure of J is countably decomposable. Then J is w-closed.*

Proof. Indeed, if e is an arbitrary projection in the w-closure $\overline{J}^w$ of J, then both e and $1_{\overline{J}^w} - e$ are countably decomposable projections so, by Lemma 2, $e \in J$. Since, by Proposition 1/9.4, $\overline{J}^w$ is the norm-closed linear span of its projections, we conclude that $\overline{J}^w = J$. ∎

The next lemma is similar to 9.12.(1):

LEMMA 3. *Let M be a W^*-algebra, J a norm-closed Rickart-Jordan subalgebra of M_h and $S \subset M_h$. Then*

$$S' \cap J = \{x \in J;\, xy = yx \text{ for all } y \in S\}$$

is a norm-closed Rickart-Jordan subalgebra of M_h. Moreover, if $p \in (S' \cap J)' \cap M_h$ is any projection, then the mapping

$$S' \cap J \ni x \mapsto xp \in M_h$$

is a countably additive Jordan homomorphism.

Proof. Clearly, $S' \cap J$ is norm-closed Jordan algebra in M. Let $a \in S' \cap J$, $a \geqslant 0$. By 9.10, the least upper bound of $\{(k^{-1} + a)^{-1}a\}_{k \geqslant 1}$ in M_h is $\mathbf{s}_M(a)$ so, by the Vigier theorem (8.5)

$$(k^{-1} + a)^{-1}a \xrightarrow{s} \mathbf{s}_M(a).$$

Since $a \in J$ and J is a norm-closed Rickart-Jordan subalgebra of M_h, we have $\mathbf{s}_M(a) \in J$. On the other hand, for each $k \geqslant 1$, the element $(k^{-1} + a)^{-1}a \in S' \cap J$ commutes with all $y \in S$, hence also $\mathbf{s}_M(a)$ commutes with all $y \in S$. Thus, $\mathbf{s}_M(a) \in S' \cap J$.

Consequently, by Proposition 2/9.9, $S' \cap J$ is a Rickart-Jordan subalgebra of M_h.

Now let $p \in (S' \cap J)' \cap M_h$ be a projection. Clearly, the mapping

$$\Phi : S' \cap J \ni x \mapsto xp \in M_h$$

is a Jordan homomorphism. If $\{e_k\}_{k \geqslant 1}$ is an increasing sequence of projections in $S' \cap J$ and e is its least upper bound in $P(S' \cap J)$, then, since $S' \cap J$ is a Rickart-Jordan subalgebra of M_h, we have $e_k \xrightarrow{s} e$, so $e_k p \xrightarrow{s} ep$. Therefore ep is the least upper bound of $\{e_k p\}$ in M_h and we conclude that Φ is countably additive. ∎

Theorem 2. *Let M be a W^*-algebra and J a norm-closed Rickart-Jordan subalgebra of M_h which is sequentially monotone complete. Then J is sequantially monotone closed in M_h.*

Proof. We have to prove that if $\{a_k\}_{k\geqslant 1}$ is a norm-bounded increasing sequence of positive elements of J and a is its least upper bound in J, then $a_k \xrightarrow{w} a$, that is, for every w-continuous positive linear functional φ on M we have $\varphi(a_k) \to \varphi(a)$.

By Proposition 1/9.4 there exists a sequence $\{e_n\}_{n\geqslant 1} \subset P(J)$ such that the norm-closed linear span of $\{e_n\}$ contains the sequence $\{a_k\}$ and a. Denote by p the least upper bound of the countable family

$$\{(1-2e_{n_j})\cdots(1-2e_{n_1})\mathbf{s}_M(\varphi)(1-2e_{n_1})\cdots(1-2e_{n_j});\, n_1,\dots,n_j \geqslant 1,\, j\geqslant 0\}$$

of countably decomposable projections of M, in $P(M)$. Then p is countably decomposable and for any $n \geqslant 1$ we have

$$(1-2e_n)p(1-2e_n) = p \quad \text{so} \quad e_n p = p e_n,$$

hence $\{p\}' \cap J$ contains $\{a_k\}$ and a.

Now, by Lemma 3, $\{p\}' \cap J$ is a norm-closed Rickart-Jordan subalgebra of M_h and the mapping $\Phi : \{p\}' \cap J \ni x \mapsto xp \in M_h$ is a countable additive Jordan homomorphism. Since p is countably decomposable in M_h, it follows by 9.11.(4) and by Theorem 1 that $\Phi(\{p\}' \cap J)$ is a w-closed Jordan algebra in M.

Finally, since $a \in \{p\}' \cap J$ is the least upper bound of $\{a_k\} \subset \{p\}' \cap J$ in J, hence also in $\{p\}' \cap J$, using the arguments of the proof of 9.11.(4$''$) we infer that $\Phi(a)$ is the least upper bound of $\{\Phi(a_k)\}$ in $\Phi(\{p\}' \cap J)$. Thus, $a_k p = \Phi(a_k) \xrightarrow{s} \Phi(a) = ap$. Since $\mathbf{s}_M(\varphi) \leqslant p$, we conclude that

$$\varphi(a_k) = \varphi(a_k p) \to \varphi(ap) = \varphi(a). \quad \blacksquare$$

Using 9.11.(4$''$) and Theorem 2, we obtain:

Corollary. *Let J be a sequentially monotone complete norm-closed Jordan algebra in the C^*-algebra A, M a W^*-algebra and*

$$\Phi : J \to M_h$$

a countably additive Jordan homomorphism. Then Φ is sequentially normal.

9.18. It is clear that countable additivity and sequential normality can be defined for arbitrary positive linear maps of a Rickart algebra, respectively of a sequentially monotone complete C^*-algebra into a W^*-algebra. In this section we prove that these two notions are essentially the same.

LEMMA 1. *Let A be a Rickart algebra and $\Phi : A \to B(H)$ a completely positive linear mapping, with Stinespring dilation $\{\pi, V, K\}$. Then Φ is countably additive if and only if π is countably additive.*

Proof. Clearly, if π is countably additive then Φ is too. Assume that Φ is countably additive. Let $\{e_k\}$ be a sequence of mutually orthogonal projections in A and $e = \sum_k e_k$. Then $p = \pi(e) - \sum_k \pi(e_k)$ is a projection. For every $\xi \in H$ and every unitary $u \in \widetilde{A}$ we have

$$\begin{aligned}(\pi(e)\widetilde{\pi}(u)V\xi|\widetilde{\pi}(u)V\xi) = (\Phi(u^*au)\xi|\xi) &= \sum_k (\Phi(u^*e_ku)\xi|\xi)\\ &= \Big(\Big[\sum_k \pi(e_k)\Big]\widetilde{\pi}(u)V\xi|\widetilde{\pi}(u)V\xi\Big),\end{aligned}$$

so $\|p\widetilde{\pi}(u)V\xi\|^2 = 0$. Since K is the closed linear span of $\pi(A)VH$ we conclude that $p = 0$. ∎

By a similar argument we obtain:

LEMMA 2. *Let A be a sequentially monotone complete C^*-algebra and $\Phi : A \to B(H)$ a completely positive linear mapping. Then Φ is sequentially normal if and only if π is sequentially normal.* ∎

The following result should be compared with Corollary 2/8.4.

PROPOSITION. *Let A be a sequentially monotone complete C^*-algebra, M a W^*-algebra and $\Phi : A \to M$ a positive linear mapping. Then Φ is countably additive if and only if A is sequentially normal.*

Proof. Clearly, if Φ is sequentially normal, then it is countably additive.

Assume that Φ is countably additive and let φ be an arbitrary w-continuous positive linear functional on H. Then the positive linear functional $\psi = \varphi \circ \Phi$ on A is countably additive. By Lemma 1, the GNS representation π_ψ associated to ψ is countably additive and hence (by Corollary 9.17) sequentially normal so that, by Lemma 2, ψ is sequentially normal. Consequently, if $\{x_k\}$ is a norm-bounded increasing sequence in A_h, then $\Phi(x_k)$ is w-convergent to $\Phi\big(\sup_k x_k\big)$, so $\sup_k \Phi(x_k) = \Phi\big(\sup_k x_k\big)$. ∎

9.19. Let A be a C^*-algebra and B a C^*-subalgebra of A. We shall say that A is an AW^*-*algebra* if A satisfies the spectral axiom and every family $\{e_\iota\}_{\iota \in I}$ of mutually orthogonal projections in A has a minimal upper bound e in $P(A)$.

By Proposition 3/9.4, if A is an AW^*-algebra and $\{e_\iota\}$, e are as above, then e is the least upper bound of $\{e_\iota\}$ in $P(A)$. We shall denote $e = \sum_{\iota \in I} e_\iota$.

Assume that A is an AW^*-algebra. We shall say that B is an AW^*-*subalgebra* of A if B satisfies the spectral axiom and, for every family $\{e_\iota\}_{\iota \in I}$ of mutually orthogonal projections in B, we have $\sum_{\iota \in I} e_\iota \in B$.

We shall say that A is *monotone complete* if every norm-bounded upward directed net $\{x_\iota\}_{\iota\in I}$ in A_h has a least upper bound x in A_h. We shall denote $x = \sup_{\iota\in I} x_\iota$ or $x_\iota \uparrow x$.

Assume that A is monotone complete. We shall say that, B is *monotone closed* in A if for every norm-bonded upward directed $\{x_\iota\}_{\iota\in I}$ in B_h we have $\sup_{\iota\in I} x_\iota \in B$.

Clearly, every AW^*-subalgebra of an AW^*-algebra is an AW^*-algebra and every monotone closed C^*-subalgebra of a monotone complete C^*-algebra is monotone complete.

The notions of *AW-Jordan algebra, AW-Jordan subalgebra, monotone complete Jordan algebra and monotone closed Jordan subalgebra* are self-explanatories.

It is obvious that every AW^*-algebra is a Rickart algebra, every AW-Jordan algebra is a Rickart algebra, and every monotone complete C^*-algebra or Jordan algebra is sequentially monotone complete. It is also obvious that a "countably decomposable" C^*-algebra is an AW^*-algebra if and only if it is a Rickart algebra, and a similary assertion holds for Jordan algebras.

Thus, many (but not all) results and proofs in what follows will be very similar to those already given for Rickart(-Jordan) algebras.

9.20. In this section we prove characterizations of AW-Jordan algebras and AW-Jordan subalgebras, similar to the results in 9.9.

Lemma. *Let J be a norm-closed Jordan algebra in the C^*-algebra A, $S \subset J^+$ and $e \in J$ a projection such that $ae = a$ and*

$$b \in J,\ b \geqslant 0,\ ab = 0 \text{ for all } a \in S \Rightarrow eb = 0.$$

Then e is the greatest lower bound of $\{f \in P(\widetilde{J});\ af = a \text{ for all } a \in S\}$ in $P(\widetilde{J})$.

Proof. The proof is an obvious extension of the proof of Lemma 9.9. ∎

In particular, if J is a norm-closed Jordan algebra in the C^*-algebra A, then for each $S \subset J^+$ there exists at most one projection $e \in J$ such that $ae = a$ for all $a \in S$ and

$$b \in J,\ b \geqslant 0,\ ab = 0 \text{ for all } a \in S \Rightarrow eb = 0.$$

If such a projection e does exist, then we shall any that *S has a support projection in J*, we call e the *support projection of S in J*, and we denote e by $\mathbf{s}_J(S)$, or simply $\mathbf{s}(S)$.

Proposition 1. *Let J be a norm-closed Jordan algebra in the C^*-algebra A. Then the following statements are equivalent:*

(i) *J is an AW-Jordan algebra;*

(ii) *every $S \subset J^+$ has a support projection in J.*

Moreover, if the above statements are true, then:

(a) *J is unital;*

(b) *for every family $\{e_\iota\}_{\iota\in I}$ of projections in J the projection $\mathbf{s}_J(\{e_\iota; \iota \in I\})$ is the least upper bound of $\{e_\iota\}_{\iota\in I}$ in $P(J)$.*

Proof. (i) $\Rightarrow$ (ii)&(a). Let $S \subset J^+$ be arbitrary. By the Zorn lemma there exists a maximal family $\{f_\iota\}_{\iota \in I}$ of mutually orthogonal projections of J such that $af_\iota = 0$ for all $a \in S$, $\iota \in I$. Put

$$f = \sum_\iota f_\iota.$$

Assume that $faf \neq 0$ for some $a \in S$. Then, by 9.3.(2) and by the remark preceding Proposition 3/9.4, there exists a projection $p \in J$, $0 \neq p \leqslant f$, commuting with faf, and $\lambda > 0$, such that $fafp \geqslant \lambda p$. By Proposition 9.2 we have $p \in fafJfaf$, so for all $\iota \in I$

$$pf_\iota \in fafJfaff_\iota = fafJf(af_\iota) = \{0\}, \quad \text{i.e. } f_\iota \leqslant f - p,$$

hence $f \leqslant f - p$ and $p = 0$, in contradiction with the choice of the projection p. Thus

$$af = 0 \quad \text{for all } a \in S. \tag{1}$$

Now let $b \in J^+$ be such that $ab = 0$ for all $a \in S$. Suppose that $(1-f)b^2(1-f) \neq 0$. As above, we can find a projection $q \in J$, $0 \neq q \leqslant 1 - f$, commuting with $(1-f)b^2(1-f)$, and $\mu > 0$, such that $(1-f)b^2(1-f)q \geqslant \mu q$. By Proposition 9.2,

$$q \in (1-f)b^2(1-f)J(1-f)b^2(1-f),$$

so, for all $a \in S$,

$$aq \in a(1-f)b^2(1-f)J(1-f)b^2(1-f) = (ab - afb)b(1-f)J(1-f)b^2(1-f) = \{0\}$$

in contradiction with the maximality of $\{f_\iota\}$. Consequently,

$$b \in J,\ b \geqslant 0,\ ab = 0 \text{ for all } a \in S \Rightarrow b = bf = fb. \tag{2}$$

We conclude that for every $S \subset J^+$ there exists $f_S \in P(J)$ satisfying (1) and (2).

Taking $S = \{0\}$ from (1) we infer that $f_{\{0\}}$ is a unit element of J. Hence J is unital.

If $S \subset J^+$ is arbitrary, then from (1) and (2) it follows that $1_J - f_S$ is the support of S in J.

(ii) $\Rightarrow$ (i)&(b). Let $\{e_\iota\}_{\iota \in I}$ be an arbitrary family in $P(J)$ and $e = \mathbf{s}_J(\{e_\iota;\ \iota \in I\})$. By the definition of the support e, $e_\iota = e_\iota e \leqslant e$, $(\iota \in I)$, and for each $f \in P(J)$ with $e_\iota \leqslant f$, $(\iota \in I)$, we have $e_\iota(e - f) = 0$, $(\iota \in I)$, so $e(e-f) = 0$ and $e = ef \leqslant f$. Hence e is the least upper bound of $\{e_\iota\}$ in $P(J)$.

On the other hand, by Proposition 1/9.9, J is a Rickart-Jordan algebra and hence it satisfies the spectral axiom.

Thus J is an AW-Jordan algebra. ∎

PROPOSITION 2. *Let $K \subset J$ be norm-closed Jordan algebras in the C^*-algebra A. Assume that J is an AW-Jordan algebra. Then the following statements are equivalent:*
(i) *K is an AW-Jordan subalgebra of J;*
(ii) *for any $S \subset K^+$ we have $\mathbf{s}_J(S) \in K$.*

Proof. (i) $\Rightarrow$ (ii). Let $S \subset K^+$. By the Zorn lemma there exists a maximal family $\{e_\iota\}_{\iota \in I}$ of mutually orthogonal projections in K such that $e_\iota \leqslant \mathbf{s}_J(S)$, $(\iota \in I)$. Let e be the least upper bound of $\{e_\iota\}$ in $P(J)$. Then $e \leqslant \mathbf{s}_J(S)$ and, by (i), $e \in P(K)$.

Assume that $a \neq ae$ for some $a \in S$, that is $a(1_K - e) \neq 0$. By 9.3.(2) and by the remark preceding Proposition 3/9.4, there exists $p \in P(K)$, $0 \neq p \leqslant 1_K - e$, commuting with $(1_K - e)a^2(1_K - e)$, and $\lambda > 0$, such that $(1_K - e)a^2(1_K - e)p \geqslant \lambda p$. By Proposition 9.2,

$$p \in (1_K - e)a^2(1_K - e)K(1_K - e)a^2(1_K - e).$$

Since $e \leqslant \mathbf{s}_J(S)$, we have $a(1_K - e) = a\mathbf{s}_J(S)(1_K - e) = a(1_K - e)\mathbf{s}_J(S)$, so $p = p\mathbf{s}_J(S) \leqslant \mathbf{s}_J(S)$, in contradiction with the maximality of $\{e_\iota\}$. Thus, $a = ae$ for all $a \in S$ and, by the above Lemma, $\mathbf{s}_J(S) \leqslant e$. Hence $\mathbf{s}_J(S) = e \in K$.

(ii) $\Rightarrow$ (i). This is a consequence of Proposition 2/9.9 and of the statement (b) from Proposition 1. ∎

Let $K \subset J$ be norm-closed Jordan algebras in the C^*-algebra A. Assume that J is an AW-Jordan algebra and K is an AW-Jordan subalgebra of J.

By Proposition 1, $P(J)$ *is an orthocomplemented complete lattice with greatest element* 1_J *and smallest element* 0 *and with orthocomplementation* $e \mapsto 1_J - e$: if $\{e_\iota\}_{\iota \in I}$ is an arbitrary family in $P(J)$, then

$$\bigvee_{\iota \in I} e_\iota = \mathbf{s}_J(\{e_\iota;\ \iota \in I\})$$

is the least upper bound of $\{e_\iota\}_{\iota \in I}$ in $P(J)$ and

$$\bigwedge_{\iota \in I} e_\iota = 1_J - \bigvee_{\iota \in I}(1_J - e_\iota) = \Big(\bigvee_{\kappa \in I} e_\kappa\Big) - \bigvee_{\iota \in I}\Big(\Big(\bigvee_{\kappa \in I} e_\kappa\Big) - e_\iota\Big)$$

is the greatest lower bound of $\{e_\iota\}_{\iota \in I}$ in $P(J)$. If the projections e_ι belong to K, then, by Proposition 2, $\bigvee_{\iota \in I} e_\iota \in K$ and $\bigwedge_{\iota \in I} e_\iota \in K$.

Note also that if $S \subset J^+$, then

$$\mathbf{s}_J(S) = \bigvee_{a \in S} \mathbf{s}_J(a).$$

9.21. We now prove a statement without correspondent for Rickart-Jordan algebras.

PROPOSITION. *Let J be a norm-closed AW-Jordan algebra in the C^*-algebra A and K a real linear subspace of J such that*

$$xJx \subset K \quad \text{for all } x \in K$$

and for every family $\{e_\iota\}_{\iota \in I}$ of mutually orthogonal projections in K

$$\sum_{\iota \in I} e_\iota \in K,$$

then there exists a projection e in J such that $K = eJe$.

Proof. Let $\{e_\iota\}$ be a maximal family of mutually orthogonal projections in K. Then $e = \sum_\iota e_\iota \in K$, so $eJe \subset K$.

Assume that $x \neq xe$ for some $x \in K$, that is $(1_J - e)x^2(1_J - e) \neq 0$. By 9.3.(2) and by the remark preceding Proposition 3/9.4, there exists a projection $f \in J$, $0 \neq f \leqslant 1_J - e$, commuting with $(1_J - e)x^2(1_J - e)$ and $\lambda > 0$, such that $(1_J - e)x^2(1_J - e) \geqslant \lambda f$. Since K is a Jordan algebra in A, we have $(1_J - e)x^2(1_J - e) \in K$. Hence, using Proposition 9.2, we get $f \in (1_J - e)x^2(1_J - e)J(1_J - e)x^2(1_J - e) \subset K$, in contradiction with the maximality of $\{e_\iota\}$. Thus, $x = xe = ex = exe$ for all $x \in K$, that is $K \subset eJe$. ∎

Let J be a Jordan algebra in the C^*-algebra A. We shall say that $z \in J$ is a *central element* of J if z commutes with every $x \in J$. The set of all central elements of J is denoted by Z_J and is called the *center* of J. Clearly, Z_J is a Jordan algebra in A and Z_J is norm-closed whenever J is norm-closed.

COROLLARY. *Let J be a norm-closed AW-Jordan algebra in the C^*-algebra A and K a real linear subspace of J such that*

$$yKy \subset K \quad \text{for all } y \in J$$

and, for every family $\{e_\iota\}_{\iota \in I}$ of mutually orthogonal projections in K,

$$\sum_{\iota \in I} e_\iota \in K.$$

Then there exists a central projection p in J such that $K = Jp$.

Proof. From the assumptions on K we succesively obtain:

$$xy + yx = (y + 1_J)x(y + 1_J) - yxy - x \in K; \quad x \in K,\ y \in J,$$
$$2x^2 = xx + xx \in K; \quad x \in K,$$
$$2xyx = x(xy + yx) + (xy + yx)x - (x^2y + yx^2) \in K; \quad x \in K,\ y \in J.$$

Hence $xJx \subset K$ for all $x \in K$ so that, by the above proposition, there exists a projection p in J with $K = pJp$.

Let $e \in P(J)$ be arbitrary. Since $p \in K$, we have

$$(1_J - 2e)p(1_J - 2e) \in K = pJp, \quad (1_J - 2e)p(1_J - 2e) = (1_J - 2e)p(1_J - 2e)p$$
$$p(1_J - 2e) = p(1_J - 2e)p, \quad pe = pep.$$

Hence p commutes with e.

By Proposition 1/9.4 J is the norm-closed linear span of $P(J)$, so p commutes with every $y \in J$. ∎

In particular, *if A is an AW^*-algebra and I is a two-sided ideal in A such that for every family $\{e_\iota\}$ of mutually orthogonal projections in I we have $\sum_\iota e_\iota \in I$, than $I = Ap$ for some central projections p of A.*

9.22. The following proposition is similar to Proposition 9.10.

Proposition. *Let $K \subset J$ be norm-closed Jordan algebras in the C^*-algebra A. Assume that J is monotone complete and K is monotone closed in J. Then J is an AW-Jordan algebra and K is an AW-Jordan subalgebra of J.*

Moreover, for every $S \subset J^+$ and every increasing sequence $\{f_k\}_{k\geqslant 1}$ of operator monotone continuous positive functions on $[0, +\infty)$ which is pointwise convergent to the characteristic function $\chi_{(0,+\infty)}$ we have

$$\sup_{F\subset S\,\text{finite},\, k\geqslant 1} f_k\Big(\sum_{a\in F} a\Big) = \mathbf{s}_J(S).$$

Proof. Let $S \subset J^+$ and $\{f_k\}_{k\geqslant 1}$ be as in the statement. Denote

$$a_{F,k} = f_k\Big(\sum_{a\in F} a\Big) \text{ for } F \subset S \text{ finite and } k \geqslant 1$$
$$e = \sup_{F,k} a_{F,k}.$$

By Proposition 9.10, J is a Rickart-Jordan algebra and, for each $F \subset S$ finite, we have

$$\sup_k a_{F,k} = \mathbf{s}_J\Big(\sum_{a\in F} a\Big).$$

Hence, for $e_F = \mathbf{s}_J\Big(\sum_{a\in F} a\Big) \in P(J)$ we have $0 \leqslant a_{F,k} \leqslant e_F \leqslant e \leqslant 1_J$, so that

$$e = \sup_F e_F,$$

and, by 2.6.(8), $e_F = ee_F = e_Fe$. Using Lemma 2/9.10 we obtain

$$e^3 = e\Big(\sup_F e_F\Big)e = \sup_F ee_Fe = \sup_F e_F = e;$$

hence e is a projection.

Now, for each $a \in S$ we have $0 \leqslant a(1_J - e)a \leqslant a(1_J - e_{\{a\}})a = 0$, that is $a = ae$. On the other hand, if $b \in J^+$ and $ab = 0$ for all $a \in S$, then for each finite $F \subset S$ we have $\Big(\sum_{a\in F} a\Big)b = 0$, $e_Fb = 0$, so $0 \leqslant beb = b\Big(\sup_F e_F\Big)b = \sup_F be_Fb = 0$ and $eb = 0$. We conclude that S has support in J and $\mathbf{s}_J(S) = e$.

Note that sequences $\{f_k\}$ as in the statement do exist, for instance $f_k(\lambda) = (k^{-1} + \lambda)^{-1}\lambda$. Thus, by the above part of the proof, by Proposition 1/9.20 and by Proposition 2/9.20, J is an AW-Jordan algebra and K is an AW-Jordan subalgebra of J. ∎

We mention explicitely that if J is a monotone complete norm-closed Jordan algebra in the C^*-algebra A, then *for any* $S \subset J^+$

$$\mathbf{s}_J(S) = \sup_{F \subset S \text{ finite}, \, k \geqslant 1} \Big(k^{-1} + \sum_{a \in F} a\Big)^{-1} \sum_{a \in F} a.$$

Also, *if* $\{a_\iota\}_{\iota \in I}$ *is a norm-bounded upward directed net of positive elements in* J, *then*

$$\sup_\iota \mathbf{s}_J(a_\iota) = \mathbf{s}_J\Big(\sup_\iota a_\iota\Big).$$

The proof is similar to that of the similar assertion in 9.10.

9.23. We now consider some permanence properties, as in 9.11. Let J be a norm-closed *AW-Jordan algebra* in the C^*-algebra A. Then:

(1) *the center* Z_J *of* J *is an AW-Jordan subalgebra of* J;
(2) *for any* $p \in P(J)$, pJp *is an AW-Jordan subalgebra of* J;
(3) *if* $\{K_\iota\}_{\iota \in I}$ *is any family of norm-closed AW-Jordan subalgebras of* J, *then* $\bigcap_{\iota \in I} K_\iota$ *is an AW-Jordan subalgebra of* J;

To prove (1), consider $S \subset Z_J^+$. For every $e \in P(J)$, the mapping $\Phi_e : x \mapsto (1_J - 2e)x(1_J - 2e)$ is a Jordan isomorphism of J onto J and $\Phi_e(S) = S$. Therefore, $\Phi_e(\mathbf{s}_J(S)) = \mathbf{s}_J(\Phi_e(S)) = \mathbf{s}_J(S)$, so that $\mathbf{s}_J(S)e = e\mathbf{s}_J(S)$. Using Proposition 1/9.4, we infer that $\mathbf{s}_J(S) \in Z_J$. Thus, by Proposition 2/9.20, Z_J is an *AW*-Jordan subalgebra of J.

Now (2) is a consequence of 9.3.(2), while (3) can be easily proved using Proposition 2/9.20.

Let K be another norm-closed *AW*-Jordan algebra in the C^*-algebra B. We shall say that Jordan homomorphism $\Phi : J \to K$ is *completely additive* if for every family $\{e_\iota\}_{\iota \in I}$ of mutually orthogonal projections in J we have

$$\sum_{\iota \in I} \Phi(e_\iota) = \Phi\Big(\sum_{\iota \in I} e_\iota\Big).$$

Then, using Corollary 9.21 and 9.3.(4), it is easy to show that:

(4) *If* $\Phi : J \to K$ *is a completely additive Jordan homomorphism, then there is a central projection* p *of* J *with* $\operatorname{Ker}\Phi = Jp$, $\Phi(J)$ *is an AW-Jordan subalgebra of* K *and* $\Phi | J(1_J - p)$, *is a Jordan isomorphism of* $J(1_J - p)$ *onto* $\Phi(J)$.

We now assume that J *is monotone complete*. Then:

(1$'$) *the center* Z_J *of* J *is monotone complete;*
(2$'$) *for any* $p \in P(J)$, pJp *is monotone closed in* J;
(3$'$) *if* $\{K_\iota\}_{\iota \in I}$ *is any family of monotone closed Jordan subalgebras of* J, *then* $\bigcap_{\iota \in I} K_\iota$ *is a monotone closed Jordan subalgebra of* J.

The proof (1$'$) is similar to that of (1), while (2$'$) and (3$'$) are obvious.

Let K be another monotone complete Jordan algebra in the C^*-algebra B. We shall say that a Jordan homomorphism $\Phi : J \to K$ is *normal* if for every norm-bounded upward directed net $\{x_\iota\}_{\iota\in I}$ in J we have

$$\sup_\iota \Phi(x_\iota) = \Phi\Big(\sup_\iota x_\iota\Big).$$

Then, using Proposition 9.22 and Corollary 9.21, it is easy to show that:

(4′) *if $\Phi : J \to K$ is a normal Jordan homorphism, then there is a central projection p in J with* $\operatorname{Ker}\Phi = Jp$*, $\Phi(J)$ is monotone closed in K and $\Phi|J(1_J - p)$ is a Jordan isomorphism of $J(1_J - p)$ onto $\Phi(J)$.*

Remark that statements similar to (1) and (1′) hold also for Rickart-Jordan algebras, respectively for sequentially monotone complete Jordan algebras, but these statements are not so important, because most of the consistent results concerning the center, for instance Corollary 9.21, are no more valid in the "sequential setting".

9.24. Let A be an *AW^*-algebra*. For $S \subset A^+$ we denote by $\mathbf{s}_A(S)$, or simply by $\mathbf{s}(S)$, the support projection of S in A_h. If $S \subset A$ is arbitrary, then we define its *left support* $\mathbf{l}(S) = \mathbf{l}_A(S)$ in A by

$$\mathbf{l}_A(S) = \mathbf{s}_A(\{xx^*;\, x \in S\})$$

and its *right support* $\mathbf{r}(S) = \mathbf{r}_A(S)$ in A by

$$\mathbf{r}_A(S) = \mathbf{s}_A(\{x^*x;\, x \in S\}) = \mathbf{l}_A(S).$$

If S consists of normal elements, then $\mathbf{l}_A(S) = \mathbf{r}_A(S)$ is called simply the *support* of S in A and is denoted by $\mathbf{s}_A(S)$ or $\mathbf{s}(S)$.

Clearly, if B is an AW^*-subalgebra of A, then

$$S \subset B \Rightarrow \mathbf{l}_A(S), \mathbf{r}_A(S) \in B.$$

With the same arguments as in 9.12 and using Proposition 2/9.20, we obtain:

(1) *if $Q \subset A$, $Q = Q^*$, then $Q' \cap A$ is an AW^*-subalgebra of A;*
(2) *the center of A is an AW^*-subalgebra of A* (see 9.23.(1))*;*
(3) *every maximal commutative $*$-subalgebra of A is an AW^*-subalgebra of A;*
(4) *if $x \in A$ is normal, then $\{x\}' \cap A$ is an AW^*-subalgebra of A.*

Now let A be a *monotone complete C^*-algebra*. Then we similarly obtain:

(1′) *if $Q \subset A$, $Q = Q^*$, then $Q' \cap A$ is a monotone closed C^*-subalgebra of A;*
(2′) *the center of A is a monotone closed C^*-subalgebra of A;*
(3′) *every maximal commutative $*$-subalgebra of A is a monotone closed C^*-subalgebra of A;*
(4′) *if $x \in A$ is normal, then $\{x\}' \cap A$ is a monotone closed C^*-subalgebra of A.*

9.25. Using 9.24.(3) instead of 9.12.(3), the proofs of Proposition 1/9.13 and Proposition 2/9.13 can be repeated almost verbatium in order to get the following statements:

Proposition 1. *Let A be a C^*-algebra. Then the following statements are equivalent:*

(i) *A is so AW^*-algebra;*

(ii) *every maximal commutative $*$-subalgebra of A is an AW^*-algebra.* ∎

Proposition 2. *Let A be an AW^*-algebra and B a C^*-subalgebra of A. Then the following statements are equivalent:*

(i) *B is an AW^*-subalgebra of A;*

(ii) *every maximal commutative $*$-subalgebra of B is an AW-subalgebra of A.* ∎

9.26. We now describe the commutative AW^*-algebras and the commutative monotone complete C^*-algebras.

We recall that a *Stone space* is a compact Hausdorff topological space Ω such that the closure of any closure of any open subset of Ω is again open.

Proposition 1. *Let A be a commutative C^*-algebra and Ω a Gelfand spectrum. The following statements are equivalent:*

(i) *A is an AW^*-algebra;*

(ii) *A is monotone complete;*

(iii) *Ω is a Stone space.*

Proof. (i) $\Rightarrow$ (iii). By Proposition 1/9.20 A is unital, so Ω a compact. Since A satisfies the spectral axiom, by Proposition 9.7, Ω is 0-dimensional.

Now let $U \subset \Omega$ be open. Consider a maximal family $\{V_\iota\}_{\iota\in I}$ of mutually disjoint compact and open subsets of U. Let e_ι be the projection in A whose Gelfand transform is the characteristic function of V_ι, $(\iota \in I)$. Then the Gelfand transform of $e = \sum\limits_{\iota\in I} e_\iota$ is the characteristic function of some compact and open set V containing $\bigcup\limits_{\iota\in I} V_\iota$.

If $V\setminus\overline{\bigcup\limits_{\iota\in I} V_\iota}$ would be non-empty, then, by the 0-dimensionality of Ω, we should find a non-empty compact and open set W contained in $V\setminus\overline{\bigcup\limits_{\iota\in I} V_\iota}$ and, with $f \neq 0$ the projection in A whose Gelfand transform is the characteristic function of W, we should have $e_\iota \leqslant e - f$ for all $\iota \in I$, hence $e \leqslant e - f$, in contradiction with $f \neq 0$. Hence $V\setminus\overline{\bigcup\limits_{\iota\in I} V_\iota} \subset \overline{U}$.

On the other hand, if $U\setminus V$ would be non-empty, then, again by the 0-dimensionality of Ω, we should find a non-empty compact and open set contained in $U\setminus V$, in contradiction with the maximality of $\{V_\iota\}$. Therefore $U \subset V$, so $\overline{U} \subset V$.

We conclude that $\overline{U} = V$ is open.

(iii) $\Rightarrow$ (ii). Let $\{a_\iota\}_{\iota\in I}$ be a norm-bounded family of positive elements in A, and define $U_\lambda = \bigcup\limits_{\iota\in I}\{\omega \in \Omega;\, \omega(a_\iota) > \lambda\}$, for each $\lambda \geqslant 0$. Then each U_λ is open, so $\overline{U}_\lambda$ is compact and open. Therefore, there is a projection $e_\lambda \in A$ whose Gelfand transform is the characteristic function of $\overline{U}_\lambda$. Clearly,

$$\lambda \geqslant \mu \Rightarrow e_\lambda \leqslant e_\mu; \quad \lambda > \sup_\iota \|a_\iota\| \Rightarrow e_\lambda = 0,$$

so the Riemann-Stieltjes integral

$$\int_0^{\infty} \lambda \, \mathrm{d}e_\lambda$$

does exist in the norm-topology of A and $a = -\int_0^{\infty} \lambda \, \mathrm{d}e_\lambda \in A^+$.

Now, with arguments similar to those used in proving the implication (iii) $\Rightarrow$ (ii) of Proposition 1/9.16, it is easy to show that a is the least upper bound of $\{a_\iota\}$ in A_h.

(ii) $\Rightarrow$ (i). This is a consequence of Proposition 9.22. ∎

The proof of the following statement is completely similar to that of Proposition 2/9.16:

PROPOSITION 2. *Let A be a monotone complete C^*-algebra. Then every commutative AW^*-subalgebra of A is monotone closed in A.* ∎

9.27. In this section we prove a remarkable property of commutative AW^*-algebras, which characterizes them among all commutative C^*-algebras.

We begin with some preliminaries. Let Z be a unital commutative C^*-algebra and X be a Z-module. A subset K of X is called *Z-convex* if

$$x_1, x_2 \in K,\ z \in Z,\ 0 \leqslant z \leqslant 1 \Rightarrow zx_1 + (1-z)x_2 \in K.$$

Clearly, Z-convex subsets of X are convex. A *Z-extreme point* of a Z-convex subset K of X is an element $x \in K$ such that

$$x_1, x_2 \in K,\ z \in Z,\ 0 \leqslant z \leqslant 1,\ z \text{ and } 1-z \text{ invertible},\ x = zx_1 + (1-z)x_2$$

$$\Rightarrow x_1 = x_2 = x.$$

The following result is useful in several situations:

PROPOSITION. *Let Z be a unital commutative C^*-algebra, X a Z-module, K a Z-convex subset of X and $x \in X$. Then the following statements are equivalent:*
(i) *x is a Z-extreme point of K;*
(ii) *x is an extreme point of K;*

Proof. Clearly, (i) $\Rightarrow$ (ii). Assume now that (ii) holds and let $x_1, x_2 \in K$, and $z \in Z$, $0 \leqslant z \leqslant 1$, with z and $1-z$ invertible, be such that $x = zx_1 + (1-z)x_2$. There are scalars α and β such that $0 < \alpha \leqslant z \leqslant \beta < 1$, so, putting $z_1 = (2z-1)^+$ and $z_2 = (2z-1)^-$, we have $0 \leqslant z_1 + z_2 \leqslant \max\{|2\alpha - 1|, |2\beta - 1|\} < 1$. It follows that $0 \leqslant z_1 \leqslant 1$, $0 \leqslant z_2 \leqslant 1$ and that $1 - z_1 - z_2$ is invertible. Since K is Z-convex, $y_1 = z_1x_1 + (1-z_1)x_2 \in K$ and $y_2 = z_2x_2 + (1-z_2)x_2 \in K$ and since

$$x = 2^{-1}y_1 + 2^{-1}y_2$$

by (ii) it follows that $y_1 = y_2$, i.e. $(1 - z_1 - z_2)x_1 = (1 - z_1 - z_2)x_2$, hence $x_1 = x_2$ because $1 - z_1 - z_2$ is invertible. ∎

COROLLARY 1. *Let A, Z be unital commutative C^*-algebras and $\Phi : A \to Z$ a positive linear map with $\Phi(1_A) = 1_Z$. Then Φ is an extreme point of*

$$\{\Psi \in B(A, Z);\ \Psi \geqslant 0, \Psi(1_A) = 1_Z\}$$

if and only if it is a $*$-homomorphism.

Proof. Let Φ be an extreme point of

$$S = \{\Psi \in B(A, Z);\ \Psi \geqslant 0, \Psi(1_A) = 1_Z\}.$$

If $a \in A$ and $(1/3)1_A \leqslant a \leqslant (2/3)1_A$, then $(1/3)1_Z \leqslant \Phi(a) \leqslant (2/3)1_Z$ so $\Phi(a)$ and $1_Z - \Phi(a)$ are invertible. Then $\Phi_1 = \Phi(a)^{-1}\Phi(a\,\cdot\,)$ and $\Phi_2 = (1_Z - \Phi(a))^{-1}\Phi((1_A - a)\,\cdot\,)$ belong to S and

$$\Phi = \Phi(a)\Phi_1 + (1_Z - \Phi(a))\Phi_2.$$

Since S is a Z-convex subset of the Z-module $B(A, Z)$, it follows that $\Phi_1 = \Phi_2 = \Phi$, that is

$$\Phi(ax) = \Phi(a)\Phi_1(x) = \Phi(a)\Phi(x); \quad x \in A.$$

Since A is the linear span of $\{a \in A; (1/3)1_A \leqslant a \leqslant (2/3)1_A\}$, we conclude that Φ is multiplicative.

Conversely, if Φ is a $*$-homomorphism, then, by the last remarks in 4.9, for each pure state φ of Z the state $\varphi \circ \Phi$ of A is again a pure state, hence Φ is an extreme point of S. ∎

We now prove the main result of this section :

THEOREM. *Let Z be a commutative C^*-algebra. The following statements are equivalent:*

(i) *Z is an AW^*-algebra;*

(ii) *if A is any commutative C^*-algebra and $\sigma : Z \to A$ is an injective $*$-homomorphism, then there exists a $*$-homomorphism $\pi : A \to Z$ such that $\pi(\sigma(z)) = z$ for all $z \in Z$;*

(iii) *if A is any commutative C^*-algebra, B a C^*-subalgebra of A and $\rho : B \to Z$ a $*$-homomorphism, then there exists a $*$-homomorphism $\pi : A \to Z$ such that $\pi|B = \rho$.*

Proof. (i) $\Rightarrow$ (iii). Let A, B, ρ be as in (iii). Replacing A by the C^*-algebra with adjoined unit $A \oplus \mathbb{C}$ defined in 1.5, B by $B \oplus \mathbb{C}$ and ρ by $B \oplus \mathbb{C} \ni x \oplus \lambda \mapsto \rho(x) + \lambda \cdot 1_Z$, we may assume, without restricting the generality, that A is unital, B contains 1_A and $\rho(1_A) = 1_Z$.

Let F be a family of all pairs (X, Φ), where X is any real linear subspace of A_h containing B_h, and Φ is an extreme point of

$$S_X = \{\Psi : X \to Z_h \text{ real linear; } \Psi(x) \geqslant 0 \text{ for } x \in X \cap A^+ \text{ and } \Psi|B_h = \rho|B_h\}.$$

We consider on F the partial ordering defined by

$$(X_1, \Phi_1 \leqslant (X_2, \Phi_2)) \Leftrightarrow X_1 \subset X_2 \text{ and } \Phi_1 = \Phi_2 | X_1.$$

Let $\{(X_\iota, \Phi_\iota)\}_{\iota \in I}$ be totally ordered subset of F. Then $X = \bigcup_{\iota \in I} X_\iota$ is a real linear subspace of A_h containing B_h and the real linear map $\Phi : X \to Z_h$ defined by $\Phi(x) = \Phi_\iota(x)$ whenever $x \in X_\iota$, $(\iota \in I)$, is an extreme point of S_X, i.e. $(X, \Phi) \in F$. Moreover, (X, Φ) is the least upper bound of $\{(X_\iota, \Phi_\iota)\}$ in F.

By the Zorn lemma it follows that there exists a maximal element (X_0, Φ_0) in F.

Assume that there exists some $x_0 \in A_h \setminus X_0$ and denote by Y_0 the real linear span of $X_0 \cup \{x_0\}$. By Corollary 1/4.18 and Proposition 1/9.26, Z_h is a conditionally complete vector lattice, so the family $\{\Phi_0(x);\, x \in X, x \leqslant x_0\} \subset Z_h$, which is bounded above by $\|x_0\| 1_Z$, has a least upper bound z_0 in Z_h. We define a real linear map $\Psi_0 : Y_0 \to Z_h$ by

$$\Psi_0(x + \lambda x_0) = \Phi_0(x) + \lambda z_0; \quad x \in X_0,\ \lambda \in \mathbb{R}.$$

Then $\Psi_0 \in S_{Y_0}$. Indeed if $x + x_0 \geqslant 0$, then $-x \leqslant x_0$, $\Phi_0(-x) \leqslant z_0$, $\Phi_0(x) + z_0 \geqslant 0$ and, if $x - x_0 \geqslant 0$, then $x' \leqslant x$ for all $x' \in X_0$, $x' \leqslant x_0$, hence $\Phi_0(x') \leqslant \Phi_0(x)$ for all $x' \in X_0$, $x' \leqslant x_0$, so that $z_0 \leqslant \Phi_0(x)$, $\Phi_0(x) - z_0 \geqslant 0$.

Moreover, Ψ_0 is an extreme point of S_{Y_0}. Indeed, if $\Psi_1, \Psi_2 \in S_{Y_0}$ are such that $\Psi_0 = 2^{-1}\Psi_1 + 2^{-1}\Psi_2$, then $\Phi_0 = 2^{-1}(\Psi_1 | X_0) + 2^{-1}(\Psi_2 | X_0)$, so

$$\Psi_1 | X_0 = \Psi_2 | X_0 = \Phi_0. \tag{1}$$

Therefore $\Psi_1(x_0) \geqslant \Psi_1(x) = \Phi_0(x)$ for all $x \in X_0$, $x \leqslant x_0$, so that $\Psi_1(x_0) \geqslant z_0 = \Psi_0(x_0)$ and, similary, $\Psi_2(x_0) \geqslant \Psi_0(x_0)$. Since $\Psi_1(x_0) + \Psi_2(x_0) = 2\Psi_0(x_0)$, it follows that

$$\Psi_1(x_0) = \Psi_2(x_0) = 2\Psi_0(x_0). \tag{2}$$

By (1) and (2) we conclude that $\Psi_1 = \Psi_2 = \Psi_0$.

Thus, $(Y_0, \Psi_0) \in F$, in contradiction with the maximality of (X_0, Φ_0). Consenquently $X_0 = A_h$.

Let π be the complex linear extension of Φ_0 to the whole A. Then π is an extreme point of $\{\sigma \in B(A, Z);\, \sigma \geqslant 0,\, \sigma | B = \rho\}$. By the above corollary, π is also an extreme point of $\{\sigma \in B(A, Z);\, \sigma \geqslant 0,\, \sigma(1_A) = 1_Z\}$, so again by the above corollary, π is a $*$-homomorphism.

(iii) $\Rightarrow$ (ii). Let A, σ be as in (ii). By applying (iii) to A, $B = \sigma(Z)$ and $\rho = \sigma^{-1}$ we get a $*$-homomorphism $\pi : A \to Z$ such that $\pi(\sigma(z)) = \sigma^{-1}(\sigma(z)) = z$ for all $z \in Z$.

(ii) $\Rightarrow$ (i). By Corollary 6/3.4, the second dual Z^{**} of Z is a commutative W^*-algebra and the canonical imbedding σ of Z in Z^{**} is a $*$-homomorphism. By (ii), there exists a $*$-homomorphism $\pi : Z^{**} \to Z$ such that $\pi(\sigma(z)) = z$ for all $z \in Z$.

Let $\{z_\iota\}$ be a norm-bounded upward directed family in Z_h. By the Vigier theorem (8.5), $\{\sigma(z_\iota)\}$ has a least upper bound x_0 in $(Z^{**})_h$ and it is easy to see that $\pi(x_0)$ is the least upper bound of $\{z_\iota\}$ in Z_h.

Hence Z is monotone complete and, by Proposition 9.22, it is an AW^*-algebra. ∎

By the above theorem, the commutative AW^*-algebras are exactly the injective objects in the category of all commutative C^*-algebras and $*$-homomorphisms.

COROLLARY 2. *Let Ω be a compact Hausdorff topological space. The following statements are equivalent:*

(i) *Ω is a Stone space;*

(ii) *if X is a compact Hausdorff topological space and $f : X \to \Omega$ is a surjective continuous map, then there is a continuous map $s : \Omega \to X$ such that $f(\mathbf{s}(\omega)) = \omega$ for all $\omega \in \Omega$;*

(iii) *if X is a compact Hausdorff topological space and F is a map of Ω into the family of all non-empty closed subsets of X such that*

$$K \subset X \text{ closed} \Rightarrow \{\omega \in \Omega; K \cap F(\omega) \neq \emptyset\} \subset \Omega \text{ closed;}$$

then there exists a continuous map $s : \Omega \to X$ such that $\mathbf{s}(\omega) \in F(\omega)$ for all $\omega \in \Omega$.

Proof. (i) $\Rightarrow$ (ii). By Proposition 1/9.26, this is an immediate consequence of the corresponding implication from the above Theorem.

(ii) $\Rightarrow$ (iii). Let X, F be as in (iii). Then

$$\Gamma = \{(\omega, x) \in \Omega \times X;\, x \in F(\omega)\}$$

is a closed subset of $\Omega \times X$, hence it is compact. Since the map $\Gamma \ni (\omega, x) \mapsto \omega \in \Omega$ is surjective and continuous, by (ii) it follows that there exists a continuous map $\Omega \ni \omega \mapsto (\omega, \mathbf{s}(\omega)) \in \Gamma$. Then $s : \Omega \to X$ is continuous and $\mathbf{s}(\omega) \in F(\omega)$ for all $\omega \in \Omega$.

(iii) $\Rightarrow$ (i). Let A be a commutative C^*-algebra and consider $\sigma : C(\Omega) \to A$ an injective $*$-homomorphism. Identifying $\omega \in \Omega$ with the corresponding character on $C(\Omega)$, we denote

$$F(\omega) = \{\gamma \in \Omega_{\widetilde{A}}; \gamma \circ \sigma = \omega\} \subset \Omega_{\widetilde{A}}.$$

By the last remark in 4.9 and by Proposition 4.16, all $F(\omega)$ are non-empty. Clearly, $F(\omega)$ are closed sets and

$$K \subset \Omega_{\widetilde{A}} \text{ closed} \Rightarrow \{\omega \in \Omega; K \cap F(\omega) \neq \emptyset\} \subset \Omega \text{ closed.}$$

Hence by (iii), there exists a continuous map $s : \Omega \to \Omega_{\widetilde{A}}$ such that $s(\omega) \circ \sigma = \omega$, for all $\omega \in \Omega$. Let $\pi : A \to C(\Omega)$ be the $*$-homomorphism defined by

$$\pi(x)(\omega) = s(\omega)(x); \quad x \in A,\, \omega \in \Omega.$$

Then

$$\pi(\sigma(z))(\omega) = s(\omega)(\sigma(z)) = z(\omega); \quad z \in C(\Omega),\, \omega \in \Omega.$$

Thus, the C^*-algebra $C(\Omega)$ satisfies the statement (ii) from the above theorem, so that $C(\Omega)$ is an AW^*-algebra and, by Proposition 1/9.26, Ω is a Stone space. ∎

By the above corollary, the Stone spaces are exactly the projective objects in the category of compact Hausdorff spaces and continuous maps.

9.28. In this section we define and use central supports.

PROPOSITION. *Let J be norm-closed AW-Jordan algebra in the C^*-algebra A and Z_J its center. For any $x \in J$ the set*

$$\{q \in P(Z_J);\, xq = x\}$$

contains its greatest lower bound p in $P(J)$ and

$$(1)\quad p = \bigvee_{e_1,\ldots,e_n \in P(J), n \geqslant 0} (1_J - 2e_n)\cdots(1_J - 2e_1)\mathbf{s}_J(x^2)(1_J - 2e_1)\cdots(1_J - 2e_n).$$

Proof. Let p be the projection of J defined by (1). Then for every $e \in P(J)$, we have $(1_J - 2e)p(1_J - 2e) = p$, hence $pe = ep$, so that $p \in P(Z_J)$. Since $\mathbf{s}_J(x^2) \leqslant p$, we have $xq = x$.

Now let $q \in P(Z_J)$ be such that $xq = x$. By Lemma 9.20 we have $\mathbf{s}_J(x^2) \leqslant q$, whence, for any $e_1, \ldots, e_n \in P(J)$ and $n \geqslant 0$,

$$(1_J - 2e_n)\cdots(1_J - 2e_1)\mathbf{s}_J(x^2)(1_J - 2e_1)\cdots(1_J - 2e_n) \leqslant q$$

so that $p \leqslant q$. ∎

If J is norm-closed AW-Jordan algebra and $x \in J$, then the central projection p of J, defined in the above proposition, is called the *central support of x in J* and is denoted by $\mathbf{z}_J(x)$.

If $e \in J$ is a projection, then $\mathbf{z}_J(e)$ is the greatest lower bound of $\{q \in P(Z_J);\, e \leqslant q\}$ in $P(J)$.

COROLLARY 1. *Let J be a norm-closed AW-Jordan algebra in the C^*-algebra A and $e \in J$ a projection. Then*

$$f \in P(Z_{eJe}) \Rightarrow f = \mathbf{z}_J(f)e,$$

so the mapping

$$Z_J \ni z \mapsto ze \in Z_{eJe}$$

is a surjective Jordan homomorphism, whose kernel is $Z_J(1_J - \mathbf{z}_J(e))$.

Proof. Let $f \in P(Z_{eJe})$. Since $f \leqslant e$, we have $f \leqslant \mathbf{z}_J(f)e$. For every $x \in J$,

$$(\mathbf{z}_J(f)e - f)xf = \mathbf{z}_J(f)(exe)f - fxf = \mathbf{z}_J(f)f(exe)f - fxf = fxf - fxf = 0$$

so for all $e_1, \ldots, e_n \in P(J)$, $n \geqslant 0$,

$$(\mathbf{z}_J(f)e - f)(1_J - 2e_n)\cdots(1_J - 2e_1)f(1_J - 2e_1)\cdots(1_J - 2e_n) = 0,$$
$$(1_J - 2e_n)\cdots(1_J - 2e_1)f(1_J - 2e_1)\cdots(1_J - 2e_n) \leqslant 1_J - (\mathbf{z}_J(f)e - f).$$

Using (1) we get $\mathbf{z}_J(f) \leqslant 1_J - (\mathbf{z}_J(f)e - f)$, $\mathbf{z}_J(f)(\mathbf{z}_J(f)e - f) = 0$ whence $f = \mathbf{z}_J(f)e$.

By 9.23.(2) and 9.23.(1), Z_{eJe} is an AW-Jordan subalgebra of J and hence, by Proposition 1/9.4, it is the norm-closed linear span of its projectios. Using the above part of the proof it follows that

$$\Phi : Z_J \ni z \mapsto ze \in Z_{eJe}$$

is a surjective Jordan homomorphism. By 9.23.(1), Z_J is an AW-Jordan subalgebra of J, so, for each $z \in Z_J$ we have $\mathbf{s}_J\left(z^2\right) \in Z_J$ and

$$z \in \operatorname{Ker}\Phi \Leftrightarrow ze = 0 \Leftrightarrow \mathbf{s}_J\left(z^2\right)e = 0 \Leftrightarrow \mathbf{s}_J\left(z^2\right) \leqslant 1_J - \mathbf{z}_J(e).$$

Thus, $\operatorname{Ker}\Phi = Z_J(1_J - \mathbf{z}_J(e))$. ∎

Let J be a norm-closed AW-Jordan algebra. A projection $e \in J$ is called *abelian* if eJe is comutative, that is if $eJe = Z_{eJe}$.

By the above corollary, $e \in P(J)$ is abelian if and only if

$$f \in P(J),\ f \leqslant e \ \Rightarrow f = \mathbf{z}_J(f)e.$$

Clearly, if $e \in P(J)$ is abelian and $f \in P(J)$, $f \leqslant e$, then also f is abelian.

COROLLARY 2. *Let J be a norm-closed AW-Jordan algebra in the C^*-algebra A and e_1, e_2 be orthogonal projections in J. Then*

$$\mathbf{z}_J(e_1 + e_2) \leqslant \mathbf{z}_J(e_1) + \mathbf{z}_J(e_2)$$

and, if $e_1 + e_2$ is abelian,

$$\mathbf{z}_J(e_1 + e_2) = \mathbf{z}_J(e_1) + \mathbf{z}_J(e_2).$$

Proof. Let $e = e_1 + e_2 \in P(J)$. Since

$$\begin{aligned} e_1 &\leqslant \mathbf{z}_J(e_1) \leqslant \mathbf{z}_J(e_1) + \mathbf{z}_J(e_2) - \mathbf{z}_J(e_1)\mathbf{z}_J(e_2) \in P(Z_J), \\ e_2 &\leqslant \mathbf{z}_J(e_2) \leqslant \mathbf{z}_J(e_1) + \mathbf{z}_J(e_2) - \mathbf{z}_J(e_1)\mathbf{z}_J(e_2) \in P(Z_J), \end{aligned}$$

we have

$$\begin{gathered} e \leqslant \mathbf{z}_J(e_1) + \mathbf{z}_J(e_2) - \mathbf{z}_J(e_1)\mathbf{z}_J(e_2) \in P(Z_J), \\ \mathbf{z}_J(e) \leqslant \mathbf{z}_J(e_1) + \mathbf{z}_J(e_2) - \mathbf{z}_J(e_1)\mathbf{z}_J(e_2) \leqslant \mathbf{z}_J(e_1) + \mathbf{z}_J(e_2). \end{gathered}$$

Now assume that e is abelian. By Corollary 1, the mapping $Z_J\mathbf{z}_J(e) \ni z \mapsto ze \in eJe$ is a Jordan isomorphism, hence there are orthogonal projections $p_1, p_2 \in Z_J\mathbf{z}_J(e)$ such that $e_1 = p_1e$ and $e_2 = p_2e$. Then $\mathbf{z}_J(e_1) \leqslant p_1$, $\mathbf{z}_J(e_2) \leqslant p_2$ and $p_1 + p_2 \in Z_J\mathbf{z}_J(e)$, so that $\mathbf{z}_J(e_1) + \mathbf{z}_J(e_2) \leqslant p_1 + p_2 \leqslant \mathbf{z}_J(e)$. ∎

9.29. We now prove some weak versions of Proposition 2/8.6 for AW^*-algebras.

Let A be an AW^*-algebra. Then the center of the Jordan algebra A_h is the hermitian part of the center Z_A of A. For every $x \in A$ we define its *central support in* A by

$$\mathbf{z}_A(x) = \mathbf{z}_{A_h}(xx^*) = \mathbf{z}_{A_h}(x^*x) = \bigwedge_{p \in P(Z_A),\, xp = x} p.$$

Using 9.28.(1), it is easy to check that for any $x \in A$

$$\text{(1)} \qquad \mathbf{z}_A(x) = \bigvee\{u^*\mathbf{l}_A(x)u;\ u \in A, \text{ unitary}\} = \bigvee\{u^*\mathbf{r}_A(x)u;\ u \in A, \text{ unitary}\}.$$

PROPOSITION. *Let A be an AW^*-algebra, $\{p_\iota\}_{\iota \in I}$ a family of mutually orthogonal central projections of A and $\{x_\iota\}_{\iota \in I} \subset A$ such that*

$$\mathbf{z}_A(x_\iota) \leqslant p_\iota \quad \textit{for all } \iota \in I \quad \textit{and} \quad \sup_{\iota \in I} \|x_\iota\| < +\infty.$$

Then there exists a unique $x \in A$ such that

$$\mathbf{z}_A(x) \leqslant \sum_{\iota \in I} p_\iota \quad \textit{and} \quad xp_\iota = x_\iota \quad \textit{for all } \iota \in I.$$

Proof. We first prove the existence of x. Without restricting the generality, we may assume that $x_\iota \geqslant 0$ for all $\iota \in I$.

By 9.24.(1), the set

$$B = (\{x_\iota;\ \iota \in I\}' \cap A)' \cap A \supset \{x_\iota;\ \iota \in I\} \cup Z_A$$

is an AW^*-subalgebra of A. Since $\{x_\iota\}' \cap A \supset \{x_\iota\}$, we have that $B \subset \{x_\iota\}' \cap A$, so $B' \cap A \supset B$, hence B is commutative. Consequently, by Proposition 1/9.26, B is monotone complete. Let x be the least upper bound in B_h of the norm-bounded upward directed family

$$\Big\{\sum_{\iota \in F} x_\iota\Big\}_{F \subset I \text{ finite}} \subset B^+.$$

By the last remark in 9.22, $\mathbf{s}_A(x) = \mathbf{s}_B(x)$ is the least upper bound in $P(B)$, hence also in $P(A)$, of the family

$$\Big\{\mathbf{s}_B\Big(\sum_{\iota \in F} x_\iota\Big) = \mathbf{s}_A\Big(\sum_{\iota \in F} x_\iota\Big)\Big\}_{F \subset I \text{ finite}}.$$

Since $\mathbf{s}_A\Big(\sum_{\iota \in F} x_\iota\Big) \leqslant \sum_{\iota \in F} p_\iota \leqslant \sum_{\iota \in I} p_\iota$, it follows that $\mathbf{s}_A(x) \leqslant \sum_{\iota \in I} p_\iota$, hence

$$\mathbf{z}_A(x) \leqslant \sum_{\iota \in F} p_\iota.$$

On the other hand, by Lemma 2/9.10, for each $\iota \in I$, the element $p_\iota x p_\iota$ is the least upper bound in B_h of $\Big\{ \sum\limits_{\kappa \in F} p_\iota x_\kappa p_\iota \Big\}_{F \subset I \text{ finite}} = \{0, x_\iota\}$ so that

$$x p_\iota = p_\iota x p_\iota = x_\iota; \quad \iota \in I.$$

We now prove the uniqueness of x. If both x, $x' \in A$ satisfy the required conditions, then $a = (x - x')^*(x - x') \in A$, $a \geqslant 0$, and $\mathbf{z}_A(a) \leqslant \sum\limits_{\iota \in I} p_\iota$, $a p_\iota = 0$, $(\iota \in I)$. We shall prove that $a = 0$.

By 9.24.(1), the set $C = (\{a\}' \cap A)' \cap A \supset \{a\} \cup Z_A$ is an AW^*-subalgebra of A. As in the first part of the proof we see that C is comutative and hence monotone complete. Also, using the last remark in 9.22 and Lemma 2/9.10, we succesfuly deduce:

$$\sum_\iota p_\iota \text{ is the least upper bound of } \{p_\iota\} \text{ in } P(C),$$

$$\sum_\iota p_\iota \text{ is the least least upper bound of } \Big\{ \sum_{\iota \in F} p_\iota \Big\}_{F \subset I \text{ finite}} \text{ in } C_h,$$

$$a\Big(\sum_\iota p_\iota\Big)a \text{ is the least upper bound of } \Big\{ \sum_{\iota \in F} a p_\iota a \Big\}_F = \{0\} \text{ in } C_h,$$

$$a\Big(\sum_\iota p_\iota\Big)a = 0,$$

hence $a\Big(\sum\limits_\iota p_\iota\Big) = 0$. Since $\mathbf{z}_A(a) \leqslant \sum\limits_\iota p_\iota$, we conclude that $a = 0$. ∎

Note that if J is a non-closed AW-Jordan algebra and $\{e_\iota\}_{\iota \in I} \subset P(J)$, then

$$\{e_\iota;\, \iota \in I\}' \cap J = \{x \in J;\, (1_J - 2e_\iota)x(1_J - 2e_\iota) = x,\, \iota \in I\}$$

is an AW-Jordan subalgebra of J. In particular, if K is a norm-closed AW-Jordan subalgebra of J, then $K' \cap J$ is an AW-Jordan subalgebrea of J. Using these remarks, it is easy to extend the above proposition for norm-closed AW-Jordan algebras.

COROLLARY. *Let A be an AW^*-algebra and $\{x_\iota\}_{\iota \in I} \subset A$ be such that the projection $\{\mathbf{l}_A(x_\iota), \mathbf{r}_A(x_k)\}_{\iota, k \in I}$ are mutually orthogonal and $\sup\limits_{\iota \in I} \|x_\iota\| < +\infty$. Then there exists a unique $x \in A$ such that*

$$\mathbf{l}_A(x) \leqslant \sum_{\iota \in I} \mathbf{l}_A(x_\iota), \quad \mathbf{r}_A(x) \leqslant \sum_{\iota \in I} \mathbf{r}_A(x_\iota), \quad \mathbf{l}_A(x_\iota)x = x\mathbf{r}_A(x_\iota) = x.$$

Proof. Denote $p_\iota = \mathbf{l}_A(x_\iota) + \mathbf{r}_A(x_\iota) \in P(A)$, $\iota \in I$. Then

$$x_\iota p_\iota = x_\iota \mathbf{r}_A(x_\iota) p_\iota = x_\iota \mathbf{r}_A(x_\iota) = x_\iota, \quad p_\iota x_\iota = p_\iota \mathbf{l}_A(x_\iota) x_\iota = \mathbf{l}_A(x_\iota) x_\iota = x_\iota,$$

so

$$x_\iota p_\iota = p_\iota x_\iota = x_\iota; \quad \iota \in I. \tag{2}$$

By 9.24.(1),

$$B = \{p_\iota;\ \iota \in I\}' \cap A \supset \{x_\iota;\ \iota \in I\}$$

is an AW^*-subalgebra of A and, clearly, $\{p_\iota;\ \iota \in I\}$ is contained in the center of B. By (2), $\mathbf{z}_B(x_\iota) \leqslant p_\iota$, $(\iota \in I)$, hence, using the above proposition, there exists a unique $x \in B$ such that

$$x \sum_{\iota \in I} p_\iota = x \quad \text{and} \quad xp_\iota = x_\iota \quad \text{for all } \iota \in I. \tag{3}$$

Thus, to complete the proof, we have only to show that $x \in A$ satisfies the conditions required in the statement if and only if $x \in B$ and it satisfies (3).

Assume that $x \in B$ and (3) holds. Then also the element

$$\Big(\sum_{\iota \in I} \mathbf{l}_A(x_\iota)\Big) x \Big(\sum_{\iota \in I} \mathbf{r}_A(x_\iota)\Big) \in B$$

satisfies (3), so

$$x = \Big(\sum_{\iota \in I} \mathbf{l}_A(x_\iota)\Big) x \Big(\sum_{\iota \in I} \mathbf{r}_A(x_\iota)\Big).$$

Therefore, $\mathbf{l}_A(x) \leqslant \sum_{\iota \in I} \mathbf{l}_A(x_\iota)$, $\mathbf{r}_A(x) \leqslant \sum_{\iota \in I} \mathbf{r}_A(x_\iota)$. On the other hand, from (3) it follows that for all $\iota \in I$

$$\mathbf{l}_A(x_\iota)x = \mathbf{l}_A(x_\iota)p_\iota x = \mathbf{l}_A(x_\iota)x_\iota = x_\iota, \quad x\mathbf{r}_A(x_\iota) = xp_\iota \mathbf{r}_A(x_\iota) = x_\iota \mathbf{r}_A(x_\iota) = x_\iota.$$

Assume now that $x \in A$ satisfies the conditions required in the statement. Then for each $\iota \in I$,

$$xp_\iota = x\Big(\sum_{\kappa \in I} \mathbf{r}_A(x_\kappa)\Big)p_\iota = x\mathbf{r}_A(x_\iota) = x_\iota,$$
$$p_\iota x = p_\iota\Big(\sum_{\kappa \in I} \mathbf{l}_A(x_\kappa)\Big)x = \mathbf{l}_A(x_\iota)x = x_\iota,$$

so $x \in B$ and $xp_\iota = x_\iota$, $(\iota \in I)$. Since $\mathbf{r}_A(x) \leqslant \sum_{\iota \in I} p_\iota$, also the first condition in (3) holds. ∎

9.30. Let A be a $*$-algebra. We shall say that $e, f \in P(A)$ are *equivalent*, and we shall write $e \sim f$, if there exists a partial isometry $v \in A$ such that $v^*v = e$ and $vv^* = f$. We shall say that $e \in P(A)$ is *dominated* by $f \in P(A)$, and we shall write $e \prec f$, if e is equivalent with some $f_0 \in P(A)$, $f_0 \leqslant f$, that is, if there exists a partial isometry $v \in A$ such that $v^*v = e$ and $vv^* \leqslant f$.

If A is an AW^*-algebra, then for $e, f \in P(A)$ we have

$$e \sim f \Rightarrow \mathbf{z}_A(e) = \mathbf{z}_A(f), \tag{1}$$
$$e \prec f \Rightarrow \mathbf{z}_A(e) \leqslant \mathbf{z}_A(f). \tag{2}$$

Indeed, (1) is a consequence of (2) and, if $v \in A$ is a partial isometry such that $v^*v = e$, $vv^* \leqslant f$, then we have succesively $(1_A - \mathbf{z}_A(f))vv^*(1_A - \mathbf{z}_A(f)) = 0$, $(1_A - \mathbf{z}_A(f))v = 0$, $e = v^*v = v^*v\mathbf{z}_A(f) \leqslant \mathbf{z}_A(f)$, $\mathbf{z}_A(e) \leqslant \mathbf{z}_A(f)$.

PROPOSITION. *Let A be an AW^*-algebra and $\{e_\iota\}_{\iota\in I}$, $\{f_\iota\}_{\iota\in I}$ be two families of mutually orthogonal projections in A such that*

$$e_\iota \sim f_\iota \quad \textit{for all } \iota \in I.$$

If either

$$\{\mathbf{z}_A(e_\iota) = \mathbf{z}_A(f_\iota)\}_{\iota\in I} \textit{ are mutually orthogonal}$$

or

$$\sum_{\iota\in I} e_\iota \textit{ and } \sum_{\iota\in I} f_\iota \textit{ are orthogonal.}$$

then

$$\sum_{\iota\in I} e_\iota \sim \sum_{\iota\in I} f_\iota.$$

Proof. For each $\iota \in I$, let $v_\iota \in A$ be a partial isometry such that $v_\iota^* v_\iota = \iota$ and $v_\iota v_\iota^* = \iota$.

Assume first that $p_\iota = \mathbf{z}_A(e_\iota) = \mathbf{z}_A(f_\iota) = \mathbf{z}_A(v_\iota)$, $(\iota \in I)$, are mutually orthogonal. Then, by Proposition 9.29, there exists $v \in A$ such that $\mathbf{z}_A(v) \leqslant \sum_{\iota\in I} p_\iota$ and $vp_\iota = v_\iota$ for all $\iota \in I$. Again by Proposition 9.29, there is a unique $x \in A$ such that $\mathbf{z}_A(x) \leqslant \sum_{\iota\in I} p_\iota$ and $xp_\iota = e_\iota$ for all $\iota \in I$. Since both $x = v^*v$ and $x = \sum_{\iota\in I} e_\iota$ satisfy these conditions, we get $v^*v = \sum_{\iota\in I} e_\iota$. Similary we obtain $v^*v = \sum_{\iota\in I} f_\iota$.

Assume now that $\sum_{\iota\in I} e_\iota$ and $\sum_{\iota\in I} f_\iota$ are orthogonal. Then, by Corollary 9.29, there exists $w \in A$ such that $\mathbf{l}_A(w) \leqslant \sum_{\iota\in I} f_\iota$, $\mathbf{r}_A(w) \leqslant \sum_{\iota\in I} e_\iota$ and $f_\iota w = we_\iota = v_\iota$ for all $\iota \in I$. Let

$$B = \{e_\iota;\, \iota \in I\}' \cap A.$$

By 9.24.(1), B is an AW^*-subalgebra of A and, clearly, $\{e_\iota;\, \iota \in I\}$ is contained in the center of B. Using Proposition 9.29, we infer that there is a unique $y \in B$ such that

$$\mathbf{z}_B(y) \leqslant \sum_{\iota\in I} e_\iota \quad \text{and} \quad ye_\iota = e_\iota \quad \text{for all } \iota \in I. \tag{3}$$

Since

$$w^*we_\iota = w^*v_\iota e_\iota = w^*f_\iota we_\iota = (f_\iota w)^* we_\iota = v_\iota^* v_\iota = e_\iota; \quad \iota \in I,$$
$$\mathbf{s}_B(w^*w) = \mathbf{s}_A(w^*w) = \mathbf{r}_A(w) \leqslant \sum_{\iota\in I} e_\iota,$$

w^*w belongs to B and $y = w^*w$ satisfies (3). Clearly, $\sum_{\iota\in I} e_\iota$ belongs to B and $y = \sum_{\iota\in I} e_\iota$ satisfies (3). Consequently,

$$w^*w = \sum_{\iota\in I} e_\iota.$$

In particular, w is a partial isometry, so ww^* is a projection. Since $f_\iota = v_\iota v_\iota^* = we_\iota v_\iota^* = wv_\iota^* = ww^* f_\iota \leqslant ww^*$, $(\iota \in I)$, we get

$$\sum_{\iota \in I} f_\iota \leqslant ww^* = \mathbf{l}_A(w) \leqslant \sum_{\iota \in I} f_\iota. \quad \blacksquare$$

As a consequence, for abelian projections we prove the converse implications in (1) and (2):

Corollary 1. *Let A be an AW^*-algebra, e an abelian projection in A and f an arbitrary projection in A. Then*

$$\mathbf{z}_A(e) \leqslant \mathbf{z}_A(f) \Rightarrow e \prec f$$

and, if f is also abelian,

$$\mathbf{z}_A(e) = \mathbf{z}_A(f) \Rightarrow e \sim f.$$

Proof. Assume that $\mathbf{z}_A(e) \leqslant \mathbf{z}_A(f)$. Let $\{(e_\iota, f_\iota)\}_{\iota \in I}$ be a maximal family of pairs of projections in A such that

$$\begin{aligned} &\{e_\iota\} \text{ are mutually orthogonal;} \\ &\{f_\iota\} \text{ are mutually orthogonal;} \\ &e \geqslant e_\iota \sim f_\iota \leqslant f, \text{ for all } \iota \in I. \end{aligned} \tag{4}$$

By Corollary 2/9.29, $\{\mathbf{z}_A(e_\iota)\}$ are mutually orthogonal so, by the above proposition, $e \geqslant \sum\limits_\iota e_\iota \sim \sum\limits_\iota f_\iota \leqslant f$. In order to prove that $e \prec f$ we have only to show that $e_0 = e - \sum\limits_\iota e_\iota = 0$.

Suppose that $e_0 \neq 0$. Then, with $f_0 = f - \sum\limits_\iota f_\iota$, we have

$$e_0 A f_0 \neq \{0\}. \tag{5}$$

Indeed, using 9.29.(1), from $e_0 A f_0 = \{0\}$ we should infer that $u^* e_0 u f_0 = 0$, i.e. $u^* e_0 u \leqslant 1_A - f_0$ for all unitaries $u \in A$, hence $\mathbf{z}_A(e_0) \leqslant 1_A - f_0$, $f_0 \leqslant 1_A - \mathbf{z}_A(e_0)$, $\mathbf{z}_A(f_0) \leqslant 1_A - \mathbf{z}_A(e_0)$ and $\mathbf{z}_A(e_0) \leqslant 1_A - \mathbf{z}_A(f_0)$. But, by Corollary 2/9.29 and by (1),

$$\mathbf{z}_A(e_0) = \mathbf{z}_A(e) - \mathbf{z}_A\Big(\sum_\iota e_\iota\Big) \leqslant \mathbf{z}_A(f) - \mathbf{z}_A\Big(\sum_\iota f_\iota\Big) \leqslant \mathbf{z}_A(f_0),$$

so we should have $\mathbf{z}_A(e_0) = 0$, in contradiction with $e_0 \neq 0$.

Now, by (5), there is $0 \neq x \in e_0 A f_0$. Since $0 \neq x^* x \in f_0 A f_0$, using 9.3.(2) and the remark preceding Proposition 3/9.4, we get a projection $0 \neq f_1 \leqslant f_0$, commuting with $x^* x$, and $\lambda > 0$, such that $x^* x f_1 \geqslant \lambda f_1$. Then $x^* x f_1 \geqslant 0$ is invertible in $f_1 A f_1$, so there exists $a \in f_1 A f_1$, $a \geqslant 0$, such that $x^* x a = a x^* x = f_1$. Consider $v = x a^{1/2}$. Then $v^* v = a^{1/2} x^* x a^{1/2} = x^* x a = f_1 \leqslant f_0$, in particular v is a partial isometry. Since $vv^* = x a x^* \in e_0 A e_0$, it follows that $e_1 = vv^* \leqslant e_0$.

Therefore, we have found a pair (e_1, f_1) of non-zero equivalent projections such that $e_1 \leqslant e_0 = e - \sum_\iota e_\iota$ and $f_1 \leqslant f_0 = f - \sum_\iota f_\iota$. Hence (4) holds for $\{(e_\iota, f_\iota)\}_\iota \cup \{(e_1, f_1)\}$ instead of $\{(e_\iota, f_\iota)\}_\iota$, in contradiction with the maximality of $\{(e_\iota, f_\iota)\}_\iota$.

We conclude that $e_0 = 0$.

Assume now that f is also abelian and $\mathbf{z}_A(e) = \mathbf{z}_A(f)$. Then, by the first part of the proof, there is a projection $p \leqslant f$ such that $e \sim p$. Using Corollary 2/9.28 and (1) we obtain

$$\mathbf{z}_A(f - p) = \mathbf{z}_A(f) - \mathbf{z}_A(p) = \mathbf{z}_A(e) - \mathbf{z}_A(e) = 0,$$

so $p = f$ and $e \sim f$. ∎

Finally, we prove a decomposition result for AW^*-algebras:

Corollary 2. *Let A be an AW^*-algebra. There exists a sequence $\{p_n\}_{n \geqslant 1}$ of mutually orthogonal central projections of A such that $\sum_{n=1}^{\infty} p_n = 1$, $Ap_1 = Z_A p_1$ and, for each $n \geqslant 2$, the projection p_n is the sum of n mutually orthogonal equivalent projections.*

Proof. Let $\{q_\iota\}_{\iota \in I}$ be a maximal family of mutually orthogonal central projections in A such that, for each $\iota \in I$, either $Aq_\iota = Z_A q_\iota$, or q_ι is the sum of a finite number $n_\iota \geqslant 2$ of mutually orthogonal equivalent projections. We prove that $\sum_{\iota \in I} q_\iota = 1_A$.

Assume first that $q_0 = 1_A - \sum_{\iota \in I} q_\iota \neq 0$ and that q_0 majorizes some abelian projection $e_0 \neq 0$. Let $\{e_\kappa\}_{\kappa \in K}$ be a maximal family of mutually orthogonal equivalent projections in A, containing e_0. Then each e_κ is abelian, because e_0 is abelian, and, if $v_\kappa \in A$ is a partial isometry with $v_\kappa^* v_\kappa = e_\kappa$, $v_\kappa v_\kappa^* = e_0$, then the mapping $e_0 A e_0 \ni x \mapsto v_\kappa^* x v_\kappa \in e_\kappa A e_\kappa$ is a $*$-isomorphism. By (1) we have $e_\kappa \leqslant \mathbf{z}_A(e_\kappa) = \mathbf{z}_A(e_0) \leqslant q_0$, $(\kappa \in K)$, so $\sum_{\kappa \in K} e_\kappa \leqslant \mathbf{z}_A(e_0)$. By the maximality of $\{e_\kappa\}$ and by Corollary 1, we cannot have $\mathbf{z}_A\Big(\mathbf{z}_A(e_0) - \sum_\kappa e_\kappa\Big) = \mathbf{z}_A(e_0)$, hence

$$q_1 = \mathbf{z}_A(e_0) - \mathbf{z}_A\Big(\mathbf{z}_A(e_0) - \sum_\kappa e_\kappa\Big) \leqslant q_0$$

is a non-zero central projection. We have $q_1\Big(\mathbf{z}_A(e_0) - \sum_\kappa e_\kappa\Big) = 0$, $q_1 = \sum_\kappa q_1 e_\kappa$ and the projections $\{q_1 e_\kappa\}$ are mutually orthogonal, equivalent and abelian. If K is finite, this is in contradiction with the maximality of $\{q_\iota\}$. If K is infinite and $K = K_1 \cup K_2$, $K_1 \cap K_2 = \emptyset$, card(K_1) =card(K_2), then

$$q_1 = \sum_{\kappa \in K_1} q_1 e_\kappa + \sum_{\kappa \in K_2} q_1 e_\kappa$$

and, by the above proposition, $\sum_{\kappa\in K_1} q_1e_\kappa \sim \sum_{\kappa\in K_2} q_1e_\kappa$, which is again in contradiction with the maximality of $\{q_\iota\}$.

Assume now that $q_0 \neq 0$ and that q_0 does not majorize any non-zero abelian projection. Let $\{(f_\lambda, g_\lambda)\}_{\lambda\in L}$ be a maximal family of pairs of projections in A such that $\{f_\lambda, g_\mu\}_{\lambda,\mu\in L}$ are mutually orthogonal and $f_\lambda \leqslant q_0$, $g_\lambda \leqslant q_0$, $f_\lambda \sim g_\lambda$ for all $\lambda \in L$. Suppose that

$$h_0 = q_0 - \sum_\lambda f_\lambda - \sum_\lambda g_\lambda \neq 0.$$

Since h_0 is not abelian, by Theorem 4.18 there exists $0 \neq x \in h_0Ah_0$ such that $x^2 = 0$. By 9.3.(2) and by the remark preceding Proposition 3/9.4, there is a projection $0 \neq f_0 \leqslant h_0$, commuting with x^*x, such that $x^*xf_0 \geqslant 0$ is invertible in f_0Af_0. If $a \geqslant 0$ is the inverse of x^*xf_0 in f_0Af_0 and $v = xa^{1/2}$, then

$$v^*v = a^{1/2}x^*xa^{1/2} = a^{1/2}(x^*xf_0)a^{1/2} = f_0 \leqslant h_0$$

hence

$$g_0 = vv^* = xax^* \in h_0Ah_0 \text{ is a projection}, \quad vv^* = g_0 \leqslant h_0.$$

Moreover,

$$f_0g_0 = (a^{1/2}x^*xa^{1/2})(xax^*) = (ax^*x)(xax^*) = ax^*x^2ax^* = 0.$$

Thus, we have found a pair (f_0, g_0) of non-zero orthogonal equivalent projections, majorized by h_0, in contradiction with the maximality of $\{(f_\lambda, g_\lambda)\}_{\lambda\in L}$. Therefore $h_0 = 0$, that is

$$q_0 = \sum_{\lambda\in L} f_\lambda + \sum_{\lambda\in L} g_\lambda,$$

and, by the above proposition, $\sum_{\lambda\in L} f_\lambda \sim \sum_{\lambda\in L} g_\lambda$, in contradiction with the maximality of $\{q_\iota\}$.

We conclude that $q_0 = 0$, that is $\sum_{\iota\in I} q_\iota = 1_A$. Putting

$$p_1 = \sum_{Aq_\iota = Z_Aq_\iota} q_\iota \quad \text{and} \quad p_n = \sum_{n_\iota = n} q_\iota \quad \text{for } n \geqslant 2,$$

and using Proposition 9.29 and the above proposition, it is easy to see that the sequence $\{p_n\}_{n\geqslant 1}$ satisfies the conditions required in the statement. ∎

9.31. In this section we prove a structure theorem for Jordan $*$-isomorphisms between AW^*-algebras.

LEMMA. *Let A be a unital C^*-algebra such that for some integer $n \geqslant 2$ the unit element 1_A is the sum of n mutually orthogonal equivalent projections. Then, for every C^*-algebra B and every Jordan $*$-homomorphism $\Phi : A \to B$, there exists a central projection q in $C^*(\Phi(A))$ such that the mapping*

$$A \ni s \mapsto \Phi(x)q \in B$$

is a $*$-homomorphism and the mapping

$$A \ni s \mapsto \Phi(x) - \Phi(x)q \in B$$

is a $*$-antihomomorphism.

Proof. Let $e_1, \ldots, e_n \in A$ be mutually orthogonal equivalent projections in A with $\sum_{j=1}^{n} e_j = 1_A$ and let $v_1, \ldots, v_n \in A$ be partial isometries such that

$$v_j^* v_j = e_n,\ v_j v_j^* = e_j; \quad 1 \leqslant j \leqslant n.$$

Clearly,

$$e_{jk} = v_j v_k^*; \quad 1 \leqslant j,\ k \leqslant n$$

are matrix units in the sense of 2.12:

$$e_{jk} e_{lm} = \delta_{kl} e_{jm},\ e_{jk}^* = e_{kj} \quad \text{and} \quad \sum_{j=1}^{n} e_{jj} = \sum_{j=1}^{n} e_j = 1_A.$$

The set

$$D = \{e_{jk};\ 1 \leqslant j,\ k \leqslant n\}' \cap A$$

is a C^*-subalgebra of A and every $x \in A$ can be uniquely written under the form

$$x = \sum_{j,k=1}^{n} d_{jk} e_{jk}; \quad d_{jk} \in D, \tag{1}$$

namely

$$d_{jk} = \sum_{l=1}^{n} e_{lj} x e_{kl}.$$

By 6.6.(7), we have

$$\Phi\Big(\sum_{j,k=1}^{n} d_{jk} e_{jk} \Big) = \sum_{j,k=1}^{n} \Phi(d_{jk}) \Phi(e_{jk}); \quad d_{jk} \in D. \tag{2}$$

Define, for $j \neq k$,

$$g_{jk} = \Phi(e_{jj}) \Phi(e_{jk}) \Phi(e_{kk}), \quad h_{jk} = \Phi(e_{jj}) \Phi(e_{kj}) \Phi(e_{kk}).$$

Since $e_{jk} = e_{jj}e_{jk}e_{kk} + e_{kk}e_{jk}e_{jj}$, $(j \neq k)$, by 6.6.(3) we get

$$\Phi(e_{jk}) = g_{jk} + h_{kj}; \quad j \neq k. \tag{3}$$

Using 6.6.(7), for $j \neq k$ we obtain

$$\begin{aligned}\Phi(e_{jj})g_{jk} &= \Phi(e_{jj})\Phi(e_{jj})\Phi(e_{jk})\Phi(e_{kk}) = \Phi(e_{jj})\Phi(e_{jk})\Phi(e_{kk}) = g_{jk},\\ \Phi(e_{jj})h_{kj} &= \Phi(e_{jj})\Phi(e_{kk})\Phi(e_{jk})\Phi(e_{jj}) = 0,\end{aligned}$$

and similarly

$$g_{jk}\Phi(e_{kk}) = g_{jk}, \quad h_{kj}\Phi(e_{kk}) = 0.$$

Using (3) it follows that

$$g_{jk} = \Phi(e_{jj})\Phi(e_{jk}) = \Phi(e_{jk})\Phi(e_{kk}); \quad j \neq k. \tag{4}$$

By similar computations we get

$$h_{jk} = \Phi(e_{jj})\Phi(e_{kj}) = \Phi(e_{kj})\Phi(e_{kk}); \quad j \neq k. \tag{5}$$

Now, for $j \neq k \neq m \neq j$, using (4) and 6.6.(7), we obtain

$$\begin{aligned}g_{jk}g_{km} &= \Phi(e_{jj})\Phi(e_{jk})\Phi(e_{kk})\Phi(e_{km}) = \Phi(e_{jj})\Phi(e_{jk})\Phi(e_{km})\\ &= \Phi(e_{jj})\Phi(e_{jk})\Phi(e_{km}) + \Phi(e_{jj})\Phi(e_{km})\Phi(e_{jk})\\ &= \Phi(e_{jj})\Phi(e_{jk}e_{km} + e_{km}e_{jk}) = \Phi(e_{jj})\Phi(e_{jm}) = g_{jm}.\end{aligned}$$

On the other hand, for $j \neq k \neq l \neq m$, using 6.6.(7) we obtain $g_{jk}g_{lm} = 0$. Hence

$$g_{jk}g_{lm} = \begin{cases} g_{jm} & \text{if } j \neq k = l \neq m \neq j,\\ 0 & \text{if } j \neq k \neq l \neq m.\end{cases} \tag{6}$$

Similarly

$$h_{jk}h_{lm} = \begin{cases} h_{jm} & \text{if } j \neq k = l \neq m \neq j,\\ 0 & \text{if } j \neq k \neq l \neq m.\end{cases} \tag{7}$$

For $j \neq k \neq l \neq j$, by (6) we have $g_{jk}g_{kj} = g_{jk}g_{kl}g_{lj} = g_{jl}g_{lj}$, so the element

$$g_{jj} = g_{jk}g_{kj}$$

does not depend on $k \neq j$. Similarly, by (7), the element

$$h_{jj} = h_{jk}h_{kj}$$

does not depend on $k \neq j$.

For $j \neq k$, by (4) and by 6.6.(2), we have

$$\begin{aligned} g_{jk}g_{kj}g_{jk} &= \Phi(e_{jj})\Phi(e_{jk})\Phi(e_{kj})\Phi(e_{jj})\Phi(e_{jj})(e_{jk}) \\ &= \Phi(e_{jj})\Phi(e_{jk})\Phi(e_{kj})\Phi(e_{jk})\Phi(e_{kk}) \\ &= \Phi(e_{jj})\Phi(e_{jk}e_{kj}e_{jk})\Phi(e_{kk}) = g_{jk}, \end{aligned}$$

and, by (5) and by 6.6.(2),

$$h_{jk}h_{kj}h_{jk} = h_{jk}.$$

Using (6), (7) and the above two equalities, it is easy to check that $\{g_{jk}\}$ and $\{h_{jk}\}$ are matrix unit systems, i.e.:

$$(8) \quad g_{jk}g_{lm} = \delta_{kl}g_{jm},\ g^*_{jk} = g_{kj};\ h_{jk}h_{lm} = \delta_{kl}h_{jm},\ h^*_{jk} = h_{kj}; \quad 1 \leqslant j,k,l,m \leqslant n.$$

We prove that

$$(9) \qquad g_{jk}h_{lm} = 0; \quad 1 \leqslant j,k,l,m \leqslant n.$$

Indeed, if $k \neq l$, then (9) is a consequence of (4), (5) and 6.6.(7). If $j \neq k = l$, then, by (8), (4), (5) and 6.6.(2), we have

$$g_{jk}h_{km} = g_{jk}h_{kj}h_{jm} = \Phi(e_{jk})\Phi(e_{kk})\Phi(e_{kk})\Phi(e_{jk})h_{jm} = \Phi(e_{jk}e_{kk}e_{jk})h_{jm} = 0.$$

Finally, if $j = k = l$, then, for some $i \neq j$, using the above computation, we get $g_{jj}g_{jm} = g_{ji}g_{ij}g_{jm} = 0$.

We remark that (3) holds also for $j = k$:

$$(10) \qquad \Phi(e_{jk}) = g_{jk} + h_{kj}; \quad 1 \leqslant j,k \leqslant n.$$

Indeed, for each $1 \leqslant j \leqslant n$ and some $i \neq j$, using 6.6.(2) and (4), (5), we obtain

$$\begin{aligned} \Phi(e_{jj}) &= \Phi(e_{jj})\Phi(e_{jj} + e_{ii})\Phi(e_{jj}) = \Phi(e_{jj})\Phi(e_{ji}e_{ij} + e_{ij}e_{ji})\Phi(e_{jj}) \\ &= \Phi(e_{jj})\Phi(e_{ji})\Phi(e_{ij})\Phi(e_{jj}) + \Phi(e_{jj})\Phi(e_{ij})\Phi(e_{ji})\Phi(e_{jj}) \\ &= g_{ji}g_{ij} + h_{ji}h_{ij} = g_{jj} + h_{jj}. \end{aligned}$$

Let $a, b \in D$ and $i \leqslant j \leqslant n$. Choose some $k \neq j$. Since

$$abe_{jj} + bae_{kk} = (ae_{jk} + be_{kj})^2,$$

using 6.6.(7) we infer that

$$\begin{aligned} \Phi(ab)\Phi(e_{jj}) + \Phi(ba)\Phi(e_{kk}) &= (\Phi(a)\Phi(e_{jk}) + \Phi(b)\Phi(e_{kj}))^2 \\ &= \Phi(a)\Phi(b)\Phi(e_{jk})\Phi(e_{kj}) + \Phi(b)\Phi(a)\Phi(e_{kj})\Phi(e_{jk}). \end{aligned}$$

Multiplying on the left and on the right by $\Phi(e_{jj})$ and using (4) and (5), we get

$$\Phi(ab)\Phi(e_{jj}) = \Phi(a)\Phi(b)g_{jk}g_{kj} + \Phi(b)\Phi(a)h_{jk}h_{kj} + \Phi(a)\Phi(b)g_{jj} + \Phi(b)\Phi(a)h_{jj}.$$

Multiplying the above equality on the right by g_{jk} and using (8) and (9), we deduce

$$(11)\quad \Phi(ab)g_{jk} = \Phi(a)\Phi(b)g_{jk};\ \Phi(ab)h_{jk} = \Phi(b)\Phi(a)h_{jk};\quad a, b \in D,\ 1 \leqslant j, k \leqslant n.$$

Consider

$$q = \sum_{j=1}^{n} g_{jj}.$$

Then, by (10),

$$\Phi(1_A) - q = \sum_{j=1}^{n} (\Phi(e_{jj}) - g_{jj}) = \sum_{j=1}^{n} h_{jj}.$$

Using (8) and (9), it is easy to see that q and $\Phi(1_A) - q$ are orthogonal projections in $C^*(\Phi(A))$.

By 6.6.(6), q commutes with every element of $\Phi(D)$. On the other hand, by (10) and (9),

$$q\Phi(e_{jk}) = q(g_{jk} + h_{kj}) = g_{jk} = (g_{jk} + h_{kj})q = \Phi(e_{jk})q$$

for all $1 \leqslant j,\ k \leqslant n$, so q commutes also with the elements $\Phi(e_{jk})$, $1 \leqslant j,\ k \leqslant n$. Since each $x \in A$ is of the form (1) and since (2) holds it follows that q is a central projection of $C^*(\Phi(A))$. By 6.6.(7), $\Phi(1_A)$ is the unit element of $C^*(\Phi(A))$, so also $\Phi(1_A) - q$ is a central projection of $C^*(\Phi(A))$.

Finally, since each $x \in A$ can be written in the form (1), using 6.6.(7) and (10), (8), (9) and (11), it is easy to check that

$$A \ni x \mapsto \Phi(x)q \in B$$

is a $*$-homomorphism and

$$A \ni x \mapsto \Phi(x)(\Phi(1_A) - q) = \Phi(x) - \Phi(x)q \in B$$

is a $*$-antihomomorphism. ∎

THEOREM. *Let A be an AW^*-algebra, B a C^*-algebra and $\Phi : A \to B$ a linear bijection. The following statements are equivalent:*

(i) *Φ is a Jordan $*$-isomorphism;*

(ii) *B is an AW^*-algebra and there exist central projections p in A and q in B such that $\Phi|Ap$ is a $*$-isomorphism of Ap onto Bq and $\Phi|A(1_A - p)$ is a $*$-antiisomorphism of $A(1_A - p)$ onto $B(1_B - q)$.*

Proof. Clearly, (ii) $\Rightarrow$ (i). Assume that (i) holds. Since $\Phi|A_h$ is a Jordan isomorphism of A_h onto B_h, B_h is an AW^*-Jordan algebra in B, so B is an AW^*-algebra. Let $\{p_n\}_{n\geqslant 1}$ be a sequence of mutually orthogonal central projections in A satisfying the conditions form Corollary 2/9.30. Consider

$$q_n = \Phi(p_n);\quad n \geqslant 1.$$

Clearly,

$$Ap_1 \ni x \mapsto \Phi(x) \in Bq_1$$

is simultaneously a $*$-isomorphism and a $*$-antiisomorphism.

Let $n \leqslant 2$. By the above lemma, there exists a central projection $q_{1n} \leqslant q_n$ of B such that

$$Aq_n \ni x \mapsto \Phi(x)q_{1n} \in Bq_n$$

is a $*$-homomorphism and

$$Ap_n \ni x \mapsto \Phi(x)(q_n - q_{1n}) \in Bq_n$$

is a $*$-antihomomorphism. Then $p_{1n} = \Phi^{-1}(q_{1n})$ is a central projection of A,

$$Ap_{1n} \ni x \mapsto \Phi(x) \in Bq_{1n}$$

is a $*$-isomorphism and

$$A(p_n - p_{1n}) \ni x \mapsto \Phi(x) \in B(q_n - q_{1n})$$

is a $*$-antiisomorphism.

Putting

$$p = p_1 + \sum_{n=2}^{\infty} p_{1n},$$

and using Proposition 9.29, it is easy to check that (ii) holds. ∎

By Proposition 1/7.16 and Proposition 1/8.6, every W^*-algebra is an AW^*-algebra, so the above theorem holds for A and B W^*-algebras.

COROLLARY. *Let A, B be C^*-algebras and $\Phi : A \to B$ a linear bijection. Then the following statements are equivalent:*

(i) *Φ is a Jordan $*$-isomorphism;*

(ii) *there are norm-closed two-sided ideals $\mathfrak{M}$ of A and $\mathfrak{N}$ of B such that*

$$\Phi(\mathfrak{M}) = \mathfrak{N} \quad \textit{and} \quad \Phi(x^*) = \Phi(x)^*, \quad x \in A,$$
$$\Phi(xy) = \Phi(x)\,\Phi(y) \quad \textit{for} \quad x, y \in A, x \textit{ or } y \textit{ or } \mathfrak{M},$$
$$A/\mathfrak{M} \ni x/\mathfrak{M} \mapsto \Phi(x)/\mathfrak{N} \in B/\mathfrak{N} \quad \textit{is a } *\textit{-antiisomorphism.}$$

Proof. (ii) $\Rightarrow$ (i). Let $x \in A$. Since $A \ni z \mapsto \Phi(z)/N \in B/N$ is a $*$-antihomomorphism, we have $\Phi(x^2) - \Phi(x)^2 \in N$. Hence there exists $y \in M$ such that $\Phi(x^2) - \Phi(x)^2 = \Phi(y)$ and we conclude

$$\begin{aligned}\Phi(y)^*\Phi(y) &= \Phi(y^*)\Phi(y) = \Phi(y^*)\Phi(x^2) - \Phi(y^*)\Phi(x)\Phi(x)\\ &= \Phi(y^*x^2) - \Phi(y^*x)\Phi(x) = \Phi(y^*x^2) - \Phi(y^*x^2) = 0,\end{aligned}$$

so $\Phi(y) = 0$, that is $\Phi(x^2) = \Phi(x)^2$.

(i) $\Rightarrow$ (ii). Recall that, by Corollary 6/8.4, A can be identified with a w-dense C^*-subalgebra of the W^*-algebra A^{**} and B can be identified with a w-dense C^*-subalgebra of the W^*-algebra B^{**}. By Proposition 6.6, Φ is bounded so, with the above identifications, the bitransposed map $\Psi =^{tt} \Phi : A^{**} \to B^{**}$ is a normal linear extension of Φ. Since Φ is bijective, also Ψ is bijective and, since A is w-dense in A^{**}, Φ is a Jordan $*$-homomorphism. Hence Ψ is a Jordan $*$-isomorphism.

By the above theorem, there are central projections p of A^{**} and q of B^{**} such that $\Psi|A^{**}p$ is a $*$-isomorphism of $A^{**}p$ onto $B^{**}q$ and $\Psi|A^{**}(1-p)$ is a $*$-antiisomorphism of $A^{**}(1-p)$ onto $B^{**}(1-q)$.

Then

$$M = A \cap A^{**}p \quad \text{and} \quad N = B \cap B^{**}q \tag{12}$$

are norm-closed two-sided ideals of A and B, respectively, and $\Phi(M) = \Psi(M) = N$. If $x \in A$ and $y \in M$, then, using 6.6.(7), we obtain

$$\Phi(xy) = \Psi(xpy) = \Psi(xp)\Psi(y) = \Psi(x)\Psi(p)\Psi(y) = \Psi(x)\Psi(y) = \Phi(x)\Phi(y).$$

Similarly, if $x \in M$ and $y \in A$, then $\Phi(xy) = \Phi(x)\Phi(y)$.

Thus, in order to complete the proof we have only to show that the map $\widetilde{\Phi} : A/M \to B/N$, induced by Φ, is a $*$-antiisomorphism. By (12), we may consider two well defined $*$-isomorphisms.

$$\begin{aligned}
&\pi_1 : A/M \ni x/M \mapsto x(1-p) \in A(1-p),\\
&\pi_2 : B/N \ni y/N \mapsto y(1-q) \in B(1-q).
\end{aligned}$$

Using 6.6.(7), for all $x \in A$ we obtain

$$\begin{aligned}
\widetilde{\Phi}(x/M) = \Phi(x)/N &= \pi_2^{-1}(\Phi(x)(1-q)) = \pi_2^{-1}(\Psi(x(1-p)))\\
&= \pi_2^{-1} \circ [\Psi|A(1-p)] \circ \pi_1(x/M).
\end{aligned}$$

Hence $\widetilde{\Phi} = \pi_2^{-1} \circ [\Psi|A(1-p)] \circ \pi_1$ is a $*$-antiisomorphism. ∎

In particular, if A is simple, that is the only closed two-sided ideals of A are $\{0\}$ and A, then every Jordan $*$-homomorphism of A onto another C^*-algebra is either a $*$-isomorphism or a $*$-antiisomorphism.

9.32. We now interrupt the treatment of AW-Jordan and AW^*-algebras in order to complete the results given in 6.7 and 6.8.

We first consider the case of Jordan algebras. Here some preliminaries on the second dual of a Jordan algebra are necessary.

Let J be a Jordan algebra in the C^*-algebra A. By Corollary 6/8.4, J is a Jordan algebra in the W^*-algebra A^{**} so, by the remarks made at the beginning of 9.17, the w-closure $\overline{J}^w$ of J in A is a unital Jordan algebra in A^{**}.

Let J^* be the dual space of the real normed space J and $\varphi \in J^*$. By the Hahn-Banach theorem, there exists $\psi \in A^*$ with $\psi|J = \varphi$. Denote by $G(\varphi)$ the restriction of ψ to $\overline{J}^w \subset A^{**}$. Since J is w-dense in $\overline{J}^w$, $G(\varphi)$ is the unique w-continuous real linear functional on $\overline{J}^w$ satisfying $G(\varphi)|J = \varphi$. Using Lemma 1/9.17, it is easy to see that $\|G(\varphi)\| = \|\varphi\|$.

For each $x \in \overline{J}^w$ we define an element $F(x)$ of the second dual J^{**} of the normed space J by $F(x)(\varphi) = G(\varphi)(x)$, $(\varphi \in J^*)$. Clearly, $F : \overline{J}^w \to J^{**}$ is a real linear map, which is $(w, \sigma(J^{**}, J^*))$-continuous and is an extension of the canonical imbedding of J into J^{**}. We have

$$\|F(x) = \sup\{|G(\varphi)(x)|;\ \|\varphi\| \leqslant 1\} \leqslant \sup\{\|G(\varphi)\|\,\|x\|;\ \|\varphi\| \leqslant 1\} \leqslant \|x\|$$

for all $x \in \overline{J}^w$. On the other hand, if $x \in \overline{J}^w$ and $\varepsilon > 0$, then there exists $\psi \in A^{**}$, $\psi = \psi^*$, $\|\psi\| \leqslant 1$, such that $\|x\| \leqslant \psi(x) + \varepsilon$ so that, denoting by $\varphi \in J^*$ the restriction of ψ to J, we get

$$\|x\| \leqslant \psi(x) + \varepsilon = G(\varphi)(x) + \varepsilon = F(x)(\varphi) + \varepsilon \leqslant \|F(x)\| + \varepsilon.$$

Therefore F is isometric. Using the w-compactness of the closed unital ball of $\overline{J}^w$, the $(w, \sigma(J^{**}, J^*))$-continuity of F and Theorem 7.4, we infer that $F(\overline{J}^w)$ is $\sigma(J^{**}, J^*))$-closed. Finally, if $\varphi \in J^*$ vanishes on $F(\overline{J}^w)$, then $G(\varphi)$ vanishes of $\overline{J}^w$ so φ vanishes on J, that is $\varphi = 0$. By the Hahn-Banach theorem it follows that $F(\overline{J}^w)$ is $\sigma(J^{**}, J^*))$-dense in J^{**}, so $F(\overline{J}^w) = J^{**}$.

We conclude that the canonical imbedding of J into J^{**} can be extended to a $(w, \sigma(J^{**}, J^*))$-continuous linear isometry $F = F_J$ of $\overline{J}^w$ onto J^{**}.

It follows that if $J \subset A$, $K \subset B$ are norm-closed Jordan algebras, $\overline{J}^w \subset A^{**}$, $\overline{K}^w \subset B^{**}$ are their w-closures and $T : J \to K$ is a bounded real linear mapping, then

$$S = F_K^{-1} \circ^{tt} T \circ F_J : \overline{J}^w \to \overline{K}^w$$

is a w-continuous real linear extension of T. Moreover, if T is bijective, then S is bijective and, if T is isometric, then S is isometric.

THEOREM 1. *Let J be a norm-closed Jordan algebra in the C^*-algebra A, K be a norm-closed Jordan algebra in the C^*-algebra B and $T : J \to K$ be a real linear bijection. Then the following statements are equivalent:*

(i) *T is an isometry;*

(ii) *there exist a unitary central element $v \in \overline{K}^w \subset B$, $vK = K$, and a Jordan isomorphism $\Phi : J \to K$ such that*

$$T(x) = v\Phi(x); \quad x \in J.$$

Proof. (ii) $\Rightarrow$ (i). By Proposition 6.6, $\|\Phi\| \leqslant 1$ and $\|\Phi^{-1}\| \leqslant 1$, so Φ is an isometry. Thus also T is an isometry.

(i) $\Rightarrow$ (ii). By the remarks preceding the statement, T can be extended to a w-continuous linear isometry S of $\overline{J}^w$ onto $\overline{K}^w$. Since $\overline{J}^w$ is unital, by Theorem 6.7 there exist a unitary central element v of $\overline{K}^w$ and a Jordan isomorphism Ψ of $\overline{J}^w$ onto $\overline{K}^w$ such that

$$S(x) = v\Psi(x); \quad x \in \overline{J}^w. \tag{1}$$

Let $b \in K^+$ be arbitrary. Since $b^{1/2} \in K$, there exists $x \in J$ such that $T(x) = b^{1/2}$ and, using (1) we obtain

$$vb = vT(x)^2 = vS(x)^2 = v\Psi(x)^2 = v\Psi(x^2) = S(x^2) = T(x^2) \in K.$$

Therefore, we succesively get

$$vK = v(K^+ - K^-) \subset K, \quad vK \subset K = v(vK) \subset vK, \quad vK = K$$

and, consequently,

$$\Psi(x) = vS(x) = vT(x) \in K; \quad x \in J,$$
$$\Psi^{-1}(y) = S^{-1}(vy) = T^{-1}(vy) \in J; \quad y \in K,$$

so that $\Phi = \Psi|J$ is a Jordan isomorphism of J onto K and, by (1), $T(x) = v\Phi(x)$ for all $x \in J$. ∎

Combining the above theorem with Proposition 6.5 we get the following extension of Corollary 6.7:

COROLLARY 1. *Let J be a norm-closed Jordan algebra in the C^*-algebra A and K be a norm-closed Jordan algebra in the C^*-algebra B. The following statements are equivalent:*
(i) *there exists a Jordan isomorphism of J onto K;*
(ii) *there exists an affine homeomorphism of $Q(J)$ (with the J-topology) onto $Q(K)$ (with the K-topology);*
(iii) *there exists a real linear isometry of J onto K.* ∎

We now consider the case of C^*-algebras.

THEOREM 2. *Let A, B be C^*-algebras and $T : A \to B$ be a linear bijection. The following statements are equivalent:*
(i) *T is an isometry;*
(ii) *there exist a unitary $v \in B^{**}$ with $vB = Bv = B$ (v is "multiplier" of B) and a Jordan $*$-isomorphism $\Phi : A \to B$ such that*

$$T(x) = v\Phi(x); \quad x \in A.$$

Proof. (ii) $\Rightarrow$ (i). By Proposition 6.6, Φ is bounded, so it can be extended to a w-continuous linear bijection $\Psi : A^{**} \to B^{**}$. Since A is w-dense in A^{**}, Ψ is a Jordan $*$-isomorphism. By Theorem 6.8, Ψ is an isometry, so Ψ is also an isometry. Consequently, T is an isometry.

(i) $\Rightarrow$ (ii). Since T can be extended to a w-continuous linear isometry S of A^{**} onto B^{**} and A^{**} is unital, by Theorem 6.8 there exits a unitary $v \in B^{**}$ and a Jordan $*$-isomorphism $\Psi : A^{**} \to B^{**}$ such that

$$S(x) = v\Psi(x); \quad x \in A. \tag{2}$$

Using (2), for every $a \in A^+$ we obtain

$$\begin{aligned}\Psi(a) &= \Psi(a^{1/2})\Psi(a^{1/2}) = \Psi(a^{1/2})v^*S(a^{1/2})\\ &= S(a^{1/2})^*S(a^{1/2}) = T(a^{1/2})^*T(a^{1/2}) \in B.\end{aligned}$$

Since A is the linear span of A^+, it follows that $\Psi(A) \subset B$.

Now let $b \in B^+$. There is $x \in A$ with $b^{1/2} = T(x) = S(x) = v\Psi(x) = \Psi(x^*)v^*$, so $b = \Psi(x^*)v^*v\Psi(x) = \Psi(x^*)\Psi(x) = v^*S(x^*)\Psi(x)$. Since $\Psi(A) \subset B$, we have $S(x^*)\Psi(x) = T(x^*)\Psi(x) \in B$, so there is $y \in A$ with $T(y) = S(x^*)\Psi(x)$. Hence

$$b = v^*T(y) = v^*S(y) = \Psi(y) \in \Psi(A).$$

Since B is the linear span of B^+, it follows that $B \subset \Psi(A)$, hence $\Psi(A) = B$.

We conclude that $\Phi = \Psi|A$ is a Jordan $*$-isomorphism of A onto B and, by (2), $T(x) = v\Phi(x)$ for all $x \in A$. Note that $vB = v\Phi(A) = T(A) = B$. Since v is unitary, it follows further that also $v^*B = v^*(vB) = B$, so $Bv = B$. ∎

COROLLARY 2. *Let A, B be C^*-algebras and $\Phi : A \to B$ be a linear bijection. The following statements are equivalent:*

(i) Φ *is a Jordan $*$-isomorphism;*

(ii) Φ *is a positive isometry;*

(iii) Φ *is an order isomorphism and Φ maps some increasing approximate unit for A onto an increasing approximate unit for B.*

Proof. (i) $\Rightarrow$ (iii). Cleary, Φ is an order isomorphism. By the above theorem, Φ is an isometry so

$$\Phi(\{a \in A^+;\ \|a\| < 1\}) = \{b \in B^+;\ \|b\| < 1\}.$$

By Theorem 3.2, $\{a \in A^+;\ \|a\| < 1\}$ is an increasing approximate unit for A and $\{b \in B^+;\ \|b\| < 1\}$ is an increasing approximate unit for B.

(iii) $\Rightarrow$ (ii). By Proposition 5.2, Φ is bounded, so it can be extended to a w-continuous linear bijection $\Psi : A^{**} \to B^{**}$. Since A is w-dense in A^{**} and B in B^{**}, Ψ is an order isomorphism.

Let $\{u_\iota\}$ be an increasing approximate unit for A such that $\{\Phi(u_\iota)\}$ is an increasing approximate unit for B. By the arguments used in the proof of Proposition 1/8.7, we have $w\text{-}\lim\limits_{\iota} u_\iota = 1$ and $w\text{-}\lim\limits_{\iota} \Psi(u_\iota) = w\text{-}\lim\limits_{\iota} \Phi(u_\iota) = 1$, so, by the w-continuity of Ψ, $\Psi(1) = 1$. Using Proposition 6.4 we deduce that $\|\Psi\| = 1$ and $\|\Psi^{-1}\| = 1$, so that Ψ is an isometry. Therefore Φ is an isometry.

(ii) $\Rightarrow$ (i). By the above theorem, there exists a unitary $v \in B^{**}$, $vB = B$, and a Jordan $*$-isomorphism $\Psi : A \to B$ such that

$$\Phi(x) = v\Psi(x); \quad x \in A. \tag{3}$$

If $b \in B^+$, then $\Psi^{-1}(b) \in A^+$, so $vb = \Phi(\Psi^{-1}(b)) \in B^+$. Hence $vB^+ \subset B^+$, so $v(B^{**})^+ \subset (B^{**})^+$. In particular, $v = v \cdot 1 \geqslant 0$, so $v = 1$. Using (3) it follows that $\Phi = \Psi$ is a Jordan $*$-isomorphism. ∎

Remark that another condition equivalent with the above three conditions is given in Corollary 9.31.

By the above Theorems 1 and 2, each real linear isometry of A_h onto B_h can be extended (by complexification) to a complex linear isometry of A onto B.

9.33. By the Vigier theorem (8.5), every w-closed Jordan algebra J in the W^*-algebra M is monotone complete and the least upper bound of a norm-bounded upward directed net $\{x_\iota\}_\iota \subset J$ in J coincides with the limit of $\{x_\iota\}$ in the s-topology on M. As we have seen in 9.17, a projection $f \in J$ is countably decomposable in J if and only if there exists a w-continuous positive linear functional ψ on M with $f = \mathbf{s}_J(\psi)$. Using this remark it is easy to see that for every projection $e \in J$ there exists a family $\{e_\iota\}$ of mutually orthogonal countably decomposable projections in J such that $e = \sum_\iota e_\iota$.

Now, we prove the following remarkable result:

THEOREM (G.K. Pedersen). *Let M be a W^*-algebra and J a norm-closed AW-Jordan subalgebra of M_h. Then J is w-closed.*

Proof. Let e be an arbitrary projection in the w-closure $\overline{J}^w$ of J. By the remarks preceding the statement, there are two families $\{e_\iota\}$ and $\{f_\kappa\}$ of mutually orthogonal countably decomposable projections in $\overline{J}^w$ such that $e = \sum_\iota e_\iota$ and $1_{\overline{J}^w} - e = \sum_\kappa f_\kappa$. For each ι and κ we have $e_\iota f_\kappa = 0$ so, by Lemma 2/9.17, there exists a projection $p_{\iota\kappa} \in J$ such that $e_\iota \leqslant p_{\iota\kappa} \leqslant 1_{\overline{J}^w} - f_\kappa$. Then

$$e = \bigwedge_\kappa \bigvee_\iota p_{\iota\kappa} \in J.$$

By Proposition 1/9.4, $\overline{J}^w$ is the norm-closed linear span of its projections and, consequently, $\overline{J}^w = J$. ∎

In particular, *every AW^*-subalgebra of a W^*-algebra M is a W^*-subalgebra of M.*

Using 9.23.(4) and the above theorem we obtain:

COROLLARY 1. *Let J be a norm-closed AW-Jordan algebra in the C^*-algebra A, M a W^*-algebra and $\Phi : J \to M_h$ a completely additive Jordan homomorphism. Then $\Phi(J)$ is a w-closed Jordan algebra in M.* ∎

The following result corresponds to Corollary 9.17:

COROLLARY 2. *Let J be a monotone complete norm-closed Jordan algebra in the C^*-algebra A, M a W^*-algebra and $\Phi : J \to M_h$ a completely additive Jordan homomorphism. Then Φ is normal.*

Proof. Indeed, by 9.23.(4) and by Corollary 1, there exists a central projection p of J with $\operatorname{Ker}\Phi = Jp$ and $\Phi|J(1_J - p)$ is a Jordan isomorphism of $J(1_J - p)$ onto the w-closed Jordan subalgebra $\Phi(J)$ of M_h. Since the mappings $J \ni x \mapsto x(1_J - p) \in J(1_J - p)$ and $J(1_J - p) \ni y \mapsto \Phi(y) \in \Phi(J) \subset M_h$ are normal, it follows that Φ is normal. ∎

Let H be a Hilbert space. Recall that (7.11) a von Neumann algebra $M \subset B(H)$ is a non-degenerate *wo*-closed $*$-subalgebra M of $B(H)$ or, equivalently, M is a unital W^*-subalgebra of the W^*-algebra $B(H)$. The above theorem, as well as Theorem 1/9.17, give in particular several useful characterizations of von Neumann algebras.

Thus, using the above theorem, Proposition 2/9.25, Proposition 9.22 and Proposition 2/9.20, we see that *for a non-degenerate Gelfand-Naĭmark algebra $M \subset B(H)$ the following statements are equivalent:*

(i) *M is a von Neumann algebra;*
(ii) *every maximal commutative $*$-subalgebra of M is wo-closed;*
(iii) *M contains the wo-limit of any norm-bounded upward directe family of mutually commuting elements from M_h;*
(iv) *M contains the support projection of any element of M^+ and the least upper bound in $P(B(H))$ of any family of mutually orthogonal projections from M.*

Moreover, using Theorem 1/9.17, Proposition 2/9.13, Proposition 9.10 and Proposition 2/9.9, we see that *if the Hilbert space H is separable, then the following statements are also equivalent with the above statements* (i)–(iv)*:*

(iii′) *M contains the wo-limit of any norm-bounded increasing sequence of mutually commuting elements from M_h;*
(iv′) *M contains the support projection of any element of M^+.*

Consider now a locally compact Hausdorff topological space Ω. Then the C^*-algebra $B(\Omega)$ of all bounded complex Borel functions defined on Ω is clearly sequentially monotone complete and the C^*-subalgebra Baire(Ω) (7.14) is sequentially monotone closed in $B(\Omega)$. Recall that (7.14) if Ω has a countable basis of open sets, then Baire(Ω)$= B(\Omega)$.

By Corollary 7.14, for every non-degenerate $*$-representation

$$\pi : C_0(\Omega) \to B(H),$$

the canonical extension

$$\pi_{\mathrm{Baire}} : \mathrm{Baire}(\Omega) \to B(H)$$

is sequentially normal. Using 9.11.(4′) and the above equivalence (i) $\Leftrightarrow$ (iii′), it follows that if H is separable, then

$$\pi_{\mathrm{Baire}}(\mathrm{Baire}(\Omega)) = \pi(C_0(\Omega))'' \subset B(H).$$

In particular, *if H is separable Hilbert space, then for every normal operator $x \in B(H)$ we have*

$$\{x\}'' = \{f(x);\, f \in B(\sigma(x))\}.$$

Taking into account Proposition 8.14 we obtain:

Corollary 3. *If M is a comutative W^*-algebra with separable predual M_*, then there exists a selfadjoint element $a \in M$ such that*

$$M = \{f(a);\, f \in B(\sigma(a))\}. \quad \blacksquare$$

9.34. It is clear that complete additivity and normality can be defined for arbitrary positive linear maps of an AW^*-algebra, respectively of a monotone complete C^*-algebra, into a W^*-algebra.

Using Corollary 2/9.33 instead of Corollary 9.17, the arguments used in the proofs of Lemma 1, Lemma 2/9.18 and Proposition 9.18 can be repeated almost verbatium in order to prove the next statements:

Lemma 1. *Let A be an AW^*-algebra and $\Phi : A \to B(H)$ a completely positive linear mapping, with Stinespring dilation $\{\pi, V, K\}$. Then Φ is completely additive if and only if π is completely additive.* $\blacksquare$

Lemma 2. *Let A be a monotone complete C^*-algebra and $\Phi : A \to B(H)$ a completely positive linear mapping, with Stinespring dilation $\{\pi, V, K\}$. Then Φ is normal if and only if π is normal.* $\blacksquare$

Proposition. *Let A be a monotone complete C^*-algebra and $\Phi : A \to M$ a positive linear mapping. Then Φ is completely additive if and only if it is normal.* $\blacksquare$

9.35. It is easy to see that, for a C^*-algebra A, the condition (ii) from Proposition 1/9.20 with $J = A_h$ is equivalent with each one of the following conditions:

(B_l) for every $S \subset A$ there is a projection $e \in A$ with

$$\{x \in A;\, xy = 0 \text{ for all } y \in S\} = Ae,$$

(B_r) for every $S \subset A$ there is a projection $f \in A$ with

$$\{x \in A;\, yx = 0 \text{ for all } y \in S\} = fA.$$

The condition (B_l) and (B_r) are purely algebric. Indeed, they can be formulated for any $*$-algebra A, and a $*$-algebra which satisfies these conditions is called a *Baer $*$-algebra*. Thus, by Proposition 1/9.20, a C^*-algebra is an AW^*-algebra if and only if it is a Baer $*$-algebra.

Note that the algebraic nature of (B_l) and (B_r) justifies the name "AW^*-algebra" = "algebraic W^*-algebra".

Also, the order structure on the projection set $P(A)$ of a C^*-algebra A is of a purely algebraic character: for $e, f \in P(A)$ we have $e \leqslant f$ if and only if $e = ef = fe$. Hence, for any $*$-algebra A we can introduce a natural order relation on $P(A)$, we can formulate completness conditions for the ordered space $P(A)$ and, assuming that $P(A)$ is a complete lattice, we can define the notion of a completely additive positive linear functional on A.

Therefore, the following theorem is an algebric characterization of W^*-algebras among all C^*-algebras:

THEOREM. (G.K. Pedersen) *Let M be a C^*-algebra. Then the following statements are equivalent:*

(i) *M is a W^*-algebra;*

(ii) *M is an AW^*-algebra and there exists a sufficient family of completely additive, positive linear functionals on M.*

Proof. (i) $\Rightarrow$ (ii). By the Vigier theorem (8.5), M is monotone complete, hence it is an AW^*-algebra. On the other hand, by 8.4, every w-continuous linear functional on M is a linear combination of completely additive positive linear functionals, hence the family of all completely additive positive linear functionals on M is sufficient.

(ii) $\Rightarrow$ (i). Let F be a sufficient family of completely additive positive linear functionals on M. As we have seen in 4.3, the sufficiency of F means that the direct sum $*$-representation

$$\pi = \bigoplus_{\varphi \in F} \pi_\varphi : M \to B(H)$$

is injective. By Lemma 1/9.34, each π_φ is completely additive, so π is complete additive. Now, by Corollary 1/9.33, $\pi(M)$ is a W^*-subalgebra of $B(H)$. Since π is a $*$-isomorphism of M onto $\pi(M)$, we conclude that M is a W^*-algebra. ∎

In particular, *a C^*-algebra M is a W^*-algebra if and only if M is monotone complete and there exists a sufficient family of normal positive linear functional on M.*

9.36. Now, we characterize the commutative W^*-algebras in terms of their Gelfand spectrum.

Let Ω be a Stone space. Then, by Proposition 1/9.26, $C(\Omega)$ is an AW^*-algebra.

A *normal positive measure* on Ω is a regular positive (finite) Borel measure μ on Ω such that the following implication holds:

$$F \subset \Omega \text{ closed and with empty interior } \Rightarrow \mu(F) = 0. \tag{1}$$

A regular positive Borel measure μ on Ω is normal if and only if the following implication holds:

$$E \subset \Omega \text{ Borel set } \Rightarrow \mu(E) = \mu(\overline{E}). \tag{2}$$

Indeed, if (1) holds and $E \subset \Omega$ is a Borel set, then by the regularity of μ, we have

$$\mu(E) = \inf\{\mu(U);\, U \supset E,\ \text{open}\ \} = \inf\{\mu(\overline{U});\, U \supset E,\ \text{open}\} \geqslant \mu(\overline{E}) \geqslant \mu(E).$$

Conversely, if (2) holds and $F \subset \Omega$ is a closed set with empty interior, then

$$\mu(\Omega \backslash F) = \mu(\overline{\Omega \backslash F}) = \mu(\Omega).$$

By the Riesz-Kakutani theorem, there exist a correspondence between the positive linear functionals φ on $C(\Omega)$ and the regular positive Borel measures μ on Ω, defined by the formula

$$\varphi(x) = \int_{\Omega} x(\omega)\,\mathrm{d}\mu(\omega); \quad x \in C(\Omega).$$

Note that

(3) *a positive linear functional φ on $C(\Omega)$ is completely additive $\Leftrightarrow$ $\Leftrightarrow$ the corresponding regular positive Borel measure μ on Ω is normal.*

Indeed, if μ is normal, $\{e_\iota\}$ is a family of mutually orthogonal projections in $C(\Omega)$ and $e = \sum_\iota e_\iota$, then each e_ι is the characteristic function of some compact and open set $U_\iota \subset \Omega$, the sets $\{U_\iota\}$ are mutually disjoint and e is the characteristic function of $\overline{\bigcup_\iota U_\iota}$ so, by (2) and by the regularity of μ, we have

$$\varphi(e) = \mu\Big(\overline{\bigcup_\iota U_\iota}\Big) = \mu\Big(\bigcup_\iota U_\iota\Big) = \sum_\iota \mu(U_\iota) = \sum_\iota \varphi(e_\iota).$$

Conversely, if φ is completely additive and $F \subset \Omega$ is a closed set with empty interior, then, choosing a maximal family $\{U_\iota\}$ of mutually disjoint compact and open subsets of $\Omega\backslash F$ and denoting by e_ι the characteristic function of U_ι, it is easy to see that $\{e_\iota\}$ are mutually orthogonal projections in $C(\Omega)$ and $\sum_\iota e_\iota = 1_{C(\Omega)}$, so

$$\mu(\Omega) = \varphi\Big(\sum_\iota e_\iota\Big) = \sum_\iota \varphi(e_\iota) = \sum_\iota \mu(U_\iota) \leqslant \mu(\Omega\backslash F)$$

and hence $\mu(F) = 0$.

Using (2) it is easy to see that

(4) *the support of a normal measure μ on Ω is compact and open.*

Therefore, if φ is the completely additive positive linear functional on $C(\Omega)$ corresponding to a normal positive measure μ on Ω, then φ "has a support projection in $C(\Omega)$", namely, if $e \in C(\Omega)$ is the characteristic function of the support of μ, then for any $x \in C(\Omega)$, $x \geqslant 0$ we have

$$\varphi(x) = 0 \Leftrightarrow xe = 0.$$

A *hyperstonean space* is a Stone space Ω such that the union of the supports of all normal positive measure on Ω is dense in Ω.

Clearly, a Stone space Ω is hyperstonean if and only if the family of all completely additive linear functionals on the AW^*-algebra $C(\Omega)$ is sufficient.

Therefore, using Proposition 1/9.26 and Theorem 9.35, we get:

PROPOSITION. *Let M be a commutative C^*-algebra and Ω be its Gelfand spectrum. The following statements are equivalent:*
(i) *M is a W^*-algebra;*
(ii) *Ω is a hyperstonean space.* ∎

Let Ω be a hyperstonean space. Using (1) and (2) it is easy to see that for a Borel set $E \subset \Omega$ we have

$$\begin{aligned}&\mu(E) = 0 \textit{ for every normal positive measure } \mu \textit{ on } \Omega \Leftrightarrow \\ &\Leftrightarrow \textit{the closure } \overline{E} \textit{ of } E \textit{ has an empty interior.}\end{aligned} \tag{5}$$

Note also the following particular property of hyperstonean space:

$$\begin{aligned}&\textit{if } \{\Omega_k\}_{k\geqslant 1} \textit{ is a sequence of open dense subsets of } \Omega, \\ &\textit{then } \bigcap_{k\geqslant 1} \Omega_k \textit{ contains an open dense subset of } \Omega.\end{aligned} \tag{6}$$

Indeed, for every normal positive measure μ on Ω, we have

$$\mu(\Omega\backslash\Omega_k) = 0; \quad k \geqslant 1,$$

by (1), hence, using (2),

$$0 \leqslant \mu\Big(\overline{\bigcup_k(\Omega\backslash\Omega_k)}\Big) = \mu\Big(\bigcup_k(\Omega\backslash\Omega_k)\Big) \leqslant \sum_k \mu(\Omega\backslash\Omega_k) = 0.$$

By (5) it follows that the interior of $\overline{\bigcup_k(\Omega\backslash\Omega_k)}$ is empty, that is, the interior of $\bigcap_k \Omega_k$ is dense in Ω.

9.37. In this section we describe some canonical forms for commutative W^*-algebras.

We begin with a short review of some well known definitions from measure theory.

Let (X, Σ, μ) be a positive mesure space, i.e. Σ is a σ-algebra of subsets of X and $\mu : \Sigma \to [0, +\infty]$ is a countably additive set function. Recall that a subset S of X is called *μ-negligible* if $S \subset E$ for some $E \in \Sigma$ with $\mu(E) = 0$ and is called *locally μ-negligible* if $S \cap E$ is μ-negligible for all $E \in \Sigma$ with $\mu(E) < +\infty$. Also, two complex functions f, g on X are called *μ-equivalent* (respectively, *locally μ-equivalent*) if the set $\{t \in X;\ f(t) \neq g(t)\}$ is μ-negligible (respectively, locally μ-negligible), and a complex function f on X is called *μ-essentially bounded* if

$$\|f\|_{L^\infty} = \inf\{\lambda > 0;\ \{t \in X;\ |f(t)| > \lambda\} \text{ locally } \mu\text{-negligible}\} < +\infty.$$

Then the set $L^\infty(\mu) = L^\infty(X, \Sigma, \mu)$ of all locally μ-equivalence classes of μ-essentially bounded Σ-measurable complex functions on X, endowed with pointwise algebraic operations and $*$-operation and with the norm $\|\cdot\|_{L^\infty}$, is a commutative C^*-algebra.

PROPOSITION 1. *Every commutative W^*-algebra is $*$-isomorphic with an $L^\infty(\mu)$ for some regular positive Borel measure μ on some locally compact Hausdorff space.*

Every commutative countably decomposable W^*-algebra is $*$-isomorphic with an $L^\infty(\mu)$ for some regular positive (finite) Borel measure μ on some compact Hausdorff space.

Every commutative W^*-algebra with separable predual is $*$-isomorphic with an $L^\infty(\mu)$ for some regular positive (finite) Borel measure μ on some metrizable compact space.

Proof. Let M be a commutative W^*-algebra. By Proposition 9.36, M is $*$-isomorphic with $C(\Omega)$ for some hyperstonean space Ω. We shall identify M with $C(\Omega)$.

If M is countably decomposable, then there exists a faithful normal state φ on M, which yields a normal positive measure μ on Ω with the support equal to the whole Ω. Using 9.36.(5), we see that the canonical imbedding of $C(\Omega)$ into $L^\infty(\mu)$ is an isometric $*$-homomorphism. Since $C(\Omega)$ is norm-dense in $L^\infty(\mu)$, we thus get a $*$-isomorphism of M onto $L^\infty(\mu)$.

In the general case, let $\{e_\iota\}_{\iota\in I}$ be a maximal family of mutually orthogonal countable decomposable non-zero projections in M. Then $\sum_\iota e_\iota = 1_M$. For each $\iota \in I$, put $\Omega_\iota = \{\omega \in \Omega;\ e_\iota(\omega) = 1\}$ and let μ_ι be a normal positive measure on Ω_ι with the support equal to Ω_ι. Then union $\Omega_0 = \bigcup_\iota \Omega_\iota$ is an open dense subset of Ω and hence a locally compact Hausdorff space with the relative topology. By the Riesz-Kakutani theorem, the mapping

$$x \mapsto \sum_\iota \mu_\iota(x),$$

defined for continuous functions x on Ω_0 with compact support, gives rise to a regular positive Borel measure μ on Ω_0 such that the restriction of μ to Ω_ι is μ_ι, for all $\iota \in I$. Using again 9.36.(5), it is easy to check that the mapping

$$M = C(\Omega) \ni x \mapsto x|\Omega_0 \in L^\infty(\mu)$$

is a $*$-isomorphism of M onto $L^\infty(\mu)$.

Assume now that the predual of M is separable and let φ be a faithful normal state on M. By Corollary 3/9.33, there exists a selfadjoint element $a \in M$ such that the Borel functional calculus

$$\pi : B(\sigma(a)) \ni f \mapsto f(a) \in M$$

is a surjective $*$-homomorphism. Then $A = C^*(\{a\})$ is a w-dense C^*-subalgebra of M and the functional calculus $f \mapsto f(a)$ is a $*$-isomorphism of $C(\sigma(a))$ onto A. By the Riesz-Kakutani theorem, the restriction of φ to A gives rise to a unique regular positive Borel measure μ on $\sigma(a)$ such that

$$\mu(f) = \varphi(f(a)) \quad \text{for all } f \in C(\sigma(a)). \tag{1}$$

Since φ is faithful, the support of μ is equal to the whole $\sigma(a)$. Using the Lebesgue dominated convergence theorem, the normality of μ, the property 7.15.(3) of the Borel functional calculus and the last remark in 7.14, we infer that the equality (1) holds for all $f \in B(\sigma(a))$, that is

$$\mu(f) = \varphi(\pi(f)) \quad \text{for all } f \in B(\sigma(a)). \tag{2}$$

It follows that a function $f \in B(\sigma(a))$ belongs to $\operatorname{Ker}\pi$ if and only if f is μ-negligible. Consequently, π factorizes to a $*$-isomorphism of $L^\infty(\mu)$ onto M. ∎

We now examine the converses of the statements in Proposition 1.

Let (X, Σ, μ) be a positive measure space. Then the vector space $L^1(\mu) = L^1(X, \Sigma, \mu)$ of all μ-equivalence classes of μ-integrable complex functions on X with the norm $\|\varphi\|_{L^1} = \int |\varphi| \mathrm{d}\mu$, $(\varphi \in L^1(\mu))$ is a Banach space.

For $f \in L^\infty(\mu)$ and $\varphi \in L^1(\mu)$ we have obviously:

$$\Big| \int f\varphi \,\mathrm{d}\mu \Big| \leqslant \|f\|_{L^\infty} \|\varphi\|_{L^1}. \tag{3}$$

We shall say that $L^\infty(\mu)$ is the dual space of $L^1(\mu)$ and we shall write $L^\infty(\mu) = L^1(\mu)^*$ if $L^\infty(\mu)$ is isometrically isomorphic with the dual space $L^1(\mu)^*$ of $L^1(\mu)$ via the canonical pairing

$$L^1(\mu) \times L^\infty(\mu) \ni (\varphi, f) \mapsto \int f\varphi \,\mathrm{d}\mu \in \mathbb{C}.$$

Thus, the C^*-algebra $L^\infty(\mu)$ is a W^*-algebra whenever $L^\infty(\mu) = L^1(\mu)^*$.

The positive measure μ is called *decomposable* if there exists a family $\{X_\iota\}_{\iota \in I}$ of mutually disjoint μ-integrable subsets of X with $X = \bigcup_{\iota \in I} X_\iota$ such that

a) $\mu(E) = \sum_{\iota \in I} \mu(E \cap X_\iota)$ for each μ-integrable set $E \subset X$, and

b) if S is a subset of X such that $S \cap X_\iota$ is μ-measurable for all $\iota \in I$, then S is μ-measurable ([131], 19.25).

Every regular positive Borel measure on a locally compact Hausdorff space is decomposable ([131], 19.31) and for every decomposable measure μ, $L^\infty(\mu) = L^1(\mu)^*$ ([131], 20.20), so $L^\infty(\mu)$ is a W^*-algebra.

If μ is σ-finite, then μ is clearly decomposable and, moreover, there exist a finite positive measure ν such that $L^\infty(\mu) = L^\infty(\nu)$ and the mapping $f \mapsto \int f \,\mathrm{d}\nu$ is then a faithful positive linear functional on $L^\infty(\mu)$ so that $L^\infty(\mu)$ is a countably decomposable W^*-algebra.

If μ is a regular positive Borel measure on some metrizable compact space, then $L^1(\mu)$ is separable and hence $L^\infty(\mu)$ is a W-algebra with a separable predual.

Thus, the converses of all statements in Proposition 1 are true.

In the general case we have:

PROPOSITION 2. *For an arbitrary positive measure space* (X, Σ, μ) *the following statements are equivalent:*

(i) $L^\infty(\mu)$ *is a* W^**-algebra;*

(ii) $L^\infty(\mu)$ *is an* AW^**-algebra;*

(iii) *the projection lattice of* $L^\infty(\mu)$ *is complete;*

(iv) $L^\infty(\mu)$ *is the dual space of* $L^1(\mu)$*;*

(v) *there exists a collection* $\{(X_\iota, \Sigma_\iota, \mu_\iota)\}$ *of finite positive measure spaces such that the* C^**-algebra* $L^\infty(\mu)$ *is* $*$*-isomorphic to the direct product of the* C^**-algebras* $L^\infty(\mu_\iota)$, $\iota \in I$.

Proof. It is clear that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii), (iv) $\Rightarrow$ (i) and (v) $\Rightarrow$ (i). Also, (i) $\Rightarrow$ (v) follows easily from Proposition 1. So, it remains to prove (iii) $\Rightarrow$ (iv).

Assume that the projection lattice of $L^\infty(\mu)$ is complete. Since $L^\infty(\mu)$ satisfies obviously the spectral axiom, $L^\infty(\mu)$ is then an AW^*-algebra.

Let $\varphi \in L^1(\mu)$. By (3), the formula

$$F_\varphi(f) = \int f\varphi \, \mathrm{d}\mu; \quad f \in L^\infty(\mu) \tag{4}$$

defines a bounded linear functional F_φ on $L^\infty(\mu)$. If φ is a positive function, then also linear functional F_φ is positive so $\|F_\varphi\| = F_\varphi(1) = \|\varphi\|_{L^1}$. If φ is arbitrary, then, using (4), it is easy to see that $\|F_\varphi\| = \|F_{|\varphi|}\|$ and hence

$$\|F_\varphi\| = \|\varphi\|_{L^1}; \quad \varphi \in L^1(\mu). \tag{5}$$

Let $\varphi \in L^1(\mu)$ be positive and denote by $p \in L^\infty(\mu)$ the element defined by characteristic function of $\{t \in X;\, \varphi(t) \neq 0\}$. Then $F_\varphi \in L^\infty(\mu)^*$ is positive and, since the set $\{t \in X;\, \varphi(t) \neq 0\}$ is a countable union of μ-integrable sets, for $f \in L^\infty(\mu)$, $f \geqslant 0$ we have

$$F_\varphi(f) = 0 \Leftrightarrow fp = 0 \text{ in } L^\infty(\mu). \tag{6}$$

Using the Lebesgue dominated convergence theorem, it is casy to check that F_φ is countably additive. We show that F_φ is in fact completely additive. Indeed, let $\{e_\iota\}_{\iota\in I}$ be an upward directed net of projections in $L^\infty(\mu)$ with least upper bound $e = \bigvee\limits_\iota e_\iota \in L^\infty(\mu)$ and put $\lambda = \sup\limits_\iota f_\varphi(e_\iota)$. Then $F_\varphi(e) \geqslant \lambda$ and there exists an increasing sequence $\{e_n\}_{n\geqslant 1}$ contained in $\{e_\iota;\, \iota \in I\}$ such that $\sup\limits_n F_\varphi(e_n) = \lambda$. Let $e_0 = \bigvee\limits_n e_n \in L^\infty(\mu)$. Then $e_0 \leqslant e$ and, since F_φ is countably additive, we have

$$F_\varphi(e_0) = \sup_n F_\varphi(e_n) = \lambda.$$

For each $\iota \in I$ and every $n \geqslant 1$ there exists some $\kappa \in I$ such that $e_\iota \vee e_n \leqslant e_\kappa$, so

$$F_\varphi(e_\iota \vee e_n) \leqslant F_\varphi(e_\kappa) \leqslant \lambda = F_\varphi(e_0).$$

Since $e_\iota \vee e_0$ is the least upper bound of $\{e_\iota \vee e_n\}_{n\geqslant 1}$ and F_φ is countably additive, it follows that $F_\varphi(e_\iota \vee e_0) \leqslant F_\varphi(e_0)$, hence, by (6), $(e_\iota \vee e_0 - e_0)p = 0$, that is $e_\iota p \leqslant e_0 p$. Since ep is the least upper bound of $\{e_\iota p\}_{\iota\in I}$, we infer that $ep \leqslant e_0 p$ and hence $ep = e_0 p$. Consequently, using again (6), we conclude

$$F_\varphi(e) = F_\varphi(ep) = F_\varphi(e_0 p) = F_\varphi(e_0) = \lambda = \sup_\iota f_\varphi(e_\iota).$$

Now let $\{\varphi_\kappa\}_{\kappa\in K} \subset L^1(\mu)$ be the family of all characteristic functions of μ-integrable subsets of X. Each φ_κ defines a projection $p_\kappa \in L^\infty(\mu)$ and obviously

$$\bigvee_{\kappa\in K} p_\kappa = 1 \text{ in } L^\infty(\mu). \tag{7}$$

If $f \in L^\infty(\mu)$, $f \geqslant 0$, and $F_{\varphi_\iota}(f) = 0$ for all $\kappa \in K$, then, by (6), $fp_\kappa = 0$ for all $\kappa \in K$. Since $L^\infty(\mu)$ is an AW^*-algebra, using (7) we infer that $f = 0$.

Thus, $L^\infty(\mu)$ is an AW^*-algebra and $\{F_\varphi;\, \varphi \in L^1(\mu),\, \varphi \geqslant 0\}$ is a sufficient family of completely additive positive linear functionals on $L^\infty(\mu)$. By Theorem 9.35 we infer that $L^\infty(\mu)$ is a W^*-algebra. Moreover, by Theorem 8.4, $\{F_\varphi;\, \varphi \in L^1(\mu),\, \varphi \geqslant 0\} \subset L^\infty(\mu)_*$.

Since every μ-integrable function is a linear combiation of positive μ-integrable functions, it follows that $\{F_\varphi;\, \varphi \in L^1(\mu)\}$ is a separating norm-closed vector subspace of $L^\infty(\mu)_*$ and hence $L^\infty(\mu)_* = \{F_\varphi;\, \varphi \in L^1(\mu)\}$. Therefore, the dual space of $L^1(\mu)$ is identical with the dual space of $L^\infty(\mu)_*$, i.e. $L^1(\mu)^* = L^\infty(\mu)$. ∎

Thus, whenever $L^\infty(\mu)$ is a W^*-algebra, its predual is $L^1(\mu)$.

A positive measure μ is called *localizable* if it satisfies the equivalent conditions from Proposition 2. Note that there exist non-localizable measures ([131]; 20.17, 20.21).

If μ is a localizable positive measure, then $L^\infty(\mu)$ is a W^*-algebra and μ defines a normal semifinite faithful weight on $L^\infty(\mu)^+$, still denoted by μ, namely:

$$\mu(f) = \int f\, \mathrm{d}\mu; \quad f \in L^\infty(\mu)^+.$$

It is easy to see that

$$\mathfrak{M}_\mu = L^\infty(\mu) \cap L^1(\mu), \quad \mathfrak{N}_\mu = \mathfrak{N}_\mu^* = L^\infty(\mu) \cap L^2(\mu), \quad \mathcal{H}_\mu = L^2(\mu)$$

and the GNS-representation $\pi_\mu : L^\infty(\mu) \to B(L^2(\mu))$ is defined by

$$\pi_\mu(f)\xi = f\xi; \quad \xi \in L^2(\mu),\, f \in L^\infty(\mu).$$

Thus, $\pi_\mu(L^\infty(\mu))$ consists of all multiplication operators by functions from $L^\infty(\mu)$.

PROPOSITION 3. *For every localizable positive measure μ, π_μ is a $*$-isomorphism of $L^\infty(\mu)$ onto $\pi_\mu(L^\infty(\mu))$ and $\pi_\mu(L^\infty(\mu))$ is a maximal commutative $*$-subalgebra of $B(L^2(\mu))$.*

Proof. By Theorem 8.23, π_μ is a $*$-isomorphism and $\pi_\mu(L^\infty(\mu)) \subset B(L^2(\mu))$ is a von Neumann algebra.

Denote by A the set of all functions $f \in L^2(\mu)$ such that the linear mapping

$$T_f^0 : L^2(\mu) \cap L^\infty(\mu) \ni \xi \mapsto f\xi \in L^2(\mu)$$

is bounded with respect to the L^2-norm. For every $f \in A$, T_f^0 can be extended to a unique bounded operator T_f on $L^2(\mu)$. Since the complex conjugation is isometric on $L^2(\mu)$, it is clear that

$$f \in A \to \overline{f} \in A \quad \text{and} \quad T_{\overline{f}} = T_f^*.$$

If $f, g \in A$, then $T_f T_g = T_g T_f$ since for any $\xi, \eta \in L^2(\mu) \cap L^\infty(\mu)$ we have

$$(T_f T_g \xi | \eta)_\mu = (T_g \xi | T_{\overline{f}} \eta)_\mu = \int f g \xi \overline{\eta} \, \mathrm{d}\mu = (T_f \xi | T_{\overline{g}} \eta)_\mu = (T_g T_f \xi | \eta)_\mu.$$

Let $x', y' \in (\pi_\mu(L^\infty(\mu))'$ and $\eta \in L^2(\mu) \cap L^\infty(\mu)$. Put $f = x'\eta \in L^2(\mu)$, $g = y'\eta \in L^2(\mu)$. For every $\xi \in L^2(\mu) \cap L^\infty(\mu)$ we have

$$f\xi = \pi_\mu(\xi) x' \eta = x' \pi_\mu(\xi) \eta = x' \pi_\mu(\eta) \xi,$$

hence $f \in A$ and $T_f = x' \pi_\mu(\eta)$. Similarly, $g \in A$ and $T_g = y' \pi_\mu(\eta)$. It follows that

$$\pi_\mu(\eta) x' y' \pi_\mu(\eta) = T_f T_g = T_g T_f = \pi_\mu(\eta) y' x' \pi_\mu(\eta). \tag{8}$$

Since $N_\mu = L^\infty(\mu) \cap L^2(\mu)$ is a w-dense ideal of the W^*-algebra $L^\infty(\mu)$, there exists an upward directed net $\{\eta_\iota\}_{\iota \in I} \subset N_\mu$, s-convergent to 1. Then $\{\eta_\iota\} \subset L^2(\mu) \cap L^\infty(\mu)$ and $\pi_\mu(\eta_\iota) \xrightarrow{s} 1$ so that, by (8),

$$x'y' = y'x'.$$

This shows that $(\pi_\mu(L^\infty(\mu)))'$ is commutative, hence $\pi_\mu(L^\infty(\mu))$ is maximal commutative in $B(L^2(\mu))$. ∎

From Propositions 1 and 3 it follows that *every commutative W^*-algebra is $*$-isomorphic to some maximal commutative von Neumann algebra*. Actually, the proof of Proposition 3 works replacing $L^\infty(\mu)$ by an arbitrary commutative W^*-algebra M and μ by an arbitrary n.s.f. weight on M^+.

Note that if μ is a finite positive measure, then the GNS-representation π_μ is cyclic, namely the cyclic vector $\xi_\mu = 1_\mu \in L^2(\mu)$ is the function identically equal to one. In this case the maximal commutativity of $\pi_\mu(L^\infty(\mu))$ follows also from Corollary 3/8.13.

Both Proposition 3 and Corollary 3/8.13, as well as their proofs, are very particular cases of a general fundamental theorem for arbitrary W^*-algebras ([307], Corollary 10.15). The main particularity here is the L^2-isometric character of the complex conjugation rather than the commutativity itself.

9.38. Finally, we describe the w-closure of a C^*-subalgebra A of a W^*-algebra in terms of the "monotone closure" of A_h. Since the results hold in the more general frame-work of norm-closed Jordan algebras in W^*-algebras, we shall consider this setting.

We begin the "sequential case".

Let M be a W^*-algebra. For every $S \subset M_h$ we denote:

$$S^\sigma = \{a \in M_h;\ \text{there exists an increasing sequence } \{a_k\} \subset S \text{ such that } a = \sup_k a_k\},$$

$$S_\delta = \{a \in M_h;\ \text{there exists a decreasing sequence } \{a_k\} \subset S \text{ such that } a = \inf_k a_k\}.$$

If J is a norm-closed Jordan algebra in M, then we denote by J_1 the closed unit ball of J and by J_1^+ the intersection $J_1 \cap J^+$.

The following lemma is similar to Lemma 2/9.17:

LEMMA. *Let J be a norm-closed Jordan algebra in the W^*-algebra M and $\overline{J}^w$ its w-closure. Then, for any two orthogonal projections $e, f \in \overline{J}^w$ which are countably decomposable in $\overline{J}^w$, there exists $a \in ((J_1^+)^\sigma)_\delta$ such that $ae = e$ and $af = 0$.*

Proof. Let φ, ψ be w-continuous positive linear functionals on M with $e = \mathbf{s}_{\overline{J}^w}(\varphi)$ and $f = \mathbf{s}_{\overline{J}^w}(\psi)$. By Lemma 1/9.17, for every integer $k \geqslant 1$ there is some $b_k \in J_1^+$ such that

$$\varphi(1_{\overline{J}^w} - b_k) \leqslant k^{-1} \quad \text{and} \quad \psi(b_k) \leqslant k^{-1}2^{-k}.$$

For $1 \leqslant m < n$ we denote

$$a_{m,n} = \Big(m + \sum_{k=m+1}^{n} kb_k\Big)^{-1} \sum_{k=m+1}^{n} kb_k \in J_1^+.$$

By 2.6.(7), we have

$$a_{m,n_1} \leqslant a_{m,n_2} \quad \text{for } 1 \leqslant m < n_1 \leqslant n_2,$$
$$a_{m_1,n} \geqslant a_{m_2,n} \quad \text{for } 1 \leqslant m_1 \leqslant m_2 < n.$$

Hence, putting

$$a_m = \sup_{n>m} a_{m,n} \in (J_1^+)^\sigma; \quad m \geqslant 1,$$

the sequence $\{a_m\}$ is decreasing. Thus,

$$a = \inf_{m \geqslant 1} a_m \in ((J_1^+)^\sigma)_\delta.$$

Using 2.6.(7), for all $1 \leqslant m < n$ we obtain

$$\begin{aligned}
0 \leqslant \varphi(1_{\overline{J}^w} - a_{m,n}) &= m\,\varphi\Big(\Big(m1_{\overline{J}^w} + \sum_{k=m+1}^{n} kb_k\Big)^{-1}\Big) \leqslant m\,\varphi\big((m1_{\overline{J}^w} + nb_n)^{-1}\big) \\
&= \frac{m}{m+n}\,\varphi\Big(\Big(1_{\overline{J}^w} - \frac{n}{m+n}(1_{\overline{J}^w} - b_n)\Big)^{-1}\Big) \\
&= \frac{m}{m+n}\sum_{j=0}^{\infty}\Big(\frac{n}{m+n}\Big)^{j}\varphi((1_{\overline{J}^w} - b_n)^j) \\
&\leqslant \frac{m}{m+n} + \frac{n}{m+n}\,\varphi(1_{\overline{J}^w} - b_n) \leqslant \frac{m+1}{m+n}
\end{aligned}$$

and

$$0 \leqslant \psi(a_{m,n}) \leqslant \frac{1}{m}\sum_{k=m+1}^{n} k\,\psi(b_k) \leqslant \frac{1}{m}.$$

Consequently, $\varphi(1_{\overline{J}^w} - a) = 0$, $\psi(a) = 0$, that is $ae = e$, $af = 0$. ∎

THEOREM. *Let J be a norm-closed Jordan algebra in the W^*-algebra M such that the w-closure $\overline{J}^w$ of J is countably decomposable. Then*

$$((J_1^+)^\sigma)_\delta = (\overline{J}^w)_1^+.$$

Proof. Clearly, $((J_1^+)^\sigma)_\delta \subset (\overline{J}^w)_1^+$ and, by the above lemma, $((J_1^+)^\sigma)_\delta$ contains all the projections of $\overline{J}^w$.

Let $a \in (\overline{J}^w)_1^+$ be arbitrary. In 9.17 we have seen that $W^*(\{a\})_h \subset \overline{J}^w$ so, by Proposition 3/7.16, there exists a sequence $\{e_k\}_{k \geqslant 1}$ of projections in $\overline{J}^w$ such that

$$a = \sum_{k=1}^{\infty} 2^{-k} e_k.$$

For each $k \geqslant 1$, $e_k \in ((J_1^+)^\sigma)_\delta$, so there exists a decreasing sequence $\{a_{k,n}\}_{n \geqslant 1}$ in $(J_1^+)^\sigma$ with $e_k = \inf_n a_{k,n}$. Since $1_{\overline{J}^w}$ is the largest element of $(\overline{J}^w)_1^+$ and $1_{\overline{J}^w} \in ((J_1^+)^\sigma)_\delta$, we have $1_{\overline{J}^w} \in (J_1^+)^\sigma$. Hence

$$a_n = \sum_{k=1}^{n} 2^{-k} a_{k,n} + 2^{-n} 1_{\overline{J}^w} \in (J_1^+)^\sigma; \quad n \geqslant 1.$$

Since the sequence $\{a_n\}$ is decreasing and $a = \inf_n a_n$, we conclude that $a \in ((J_1^+)^\sigma)_\delta$. ∎

Sometimes, the above theorem is called suggestively: the "up-down" theorem.

9.39. We now turn to the general case.

Let M be a W^*-algebra. For any $S \subset M_h$ we denote:

$$S^m = \{a \in M_h;\ \text{there exists an upward directed net } \{a_\iota\} \subset S \text{ such that } a = \sup_\iota a_\iota\},$$

$$S_m = \{a \in M_h;\ \text{there exists a downward directed net } \{a_\iota\} \subset S \text{ such that } a = \inf_\iota a_\iota\}.$$

LEMMA 1. *Let J be a norm-closed Jordan algebra in a W^*-algebra M such that $W^*(J) = M$. Then*

$$a \in (\widetilde{J}_1^+)^m,\ \varepsilon > 0 \Rightarrow (1+\varepsilon)^{-1}(a + \varepsilon 1_M) \in (J_1^+)^m.$$

Proof. If J is unital, then the statement is obvious, so we assume that J is not unital. We denote $1_M = 1_{\overline{J}^w}$ simply by 1.

Let $a \in (\widetilde{J}_1^+)^m$ and $\varepsilon > 0$ be arbitrary.

There exists an upward directed net $\{\widetilde{a}_\iota\}_{\iota \in I}$ in $\widetilde{J}_1^+$ such that

$$a = \sup_\iota \widetilde{a}_\iota.$$

For each $\iota \in I$ we have

$$\widetilde{a}_\iota = b_\iota + \lambda_\iota \quad \text{ with } b_\iota \in J,\ \lambda_\iota \in [0,1].$$

Then $\{\lambda_\iota\}_{\iota \in I}$ is an upward directed net in $[0,1]$.

Now, by the proof of Proposition 2.10, the family

$$S = \{x \in J^+;\ \|x\| < 1\}$$

is upward directed. Indeed, if $x, y \in S$, then, by 6.2.(11), we have $u = x(1-x)^{-1} \in J^+$, $v = y(1-y)^{-1} \in J^+$, $z = (u+v)(1+u+v)^{-1} \in J^+$, and, by 2.7, $x = u(1+u)^{-1} \leqslant (u+v)(1+u+v)^{-1} = z$, similarly $y \leqslant z$ and $z \leqslant (\|u\|+\|v\|)(1+\|u\|+\|v\|)^{-1} < 1$. Clearly, $\sup\limits_{x \in S} x \leqslant 1$. On the other hand, by 9.22,

$$\sup_{F \subset S,\, \text{finite};\, k \geqslant 1} \Big(k^{-1} + \sum_{x \in F} x\Big)^{-1} \sum_{x \in F} x = \mathbf{s}_M(S) = 1,$$

so $1 \leqslant \sup\limits_{x \in S} x$. Therefore

$$\sup_{x \in S} x = 1.$$

For each $\iota \in I$ we have $-b_\iota \leqslant \lambda_\iota$, so $b_\iota^- \leqslant \lambda_\iota$. Hence

$$(\lambda_\iota + \lambda)^{-1} b_\iota^- \in S; \quad \iota \in I,\ \lambda > 0. \tag{1}$$

Putting

$$S_{\iota,\lambda} = \{x \in S; x \geqslant -(\lambda_\iota + \lambda)^{-1} b_\iota\}; \quad \iota \in I,\ \lambda > 0,$$

we have

$$S_{\iota,\lambda} \supset \{x \in S;\, x \geqslant (\lambda_\iota + \lambda)^{-1} b_\iota^-\}; \quad \iota \in I,\, \lambda > 0,$$

so, by (1) each $S_{\iota,\lambda}$ is an upward directed cofinal subfamily of the upward directed family S. Consequently,

$$\sup_{x \in S_{\iota,\lambda}} x = 1; \quad \iota \in I,\, \lambda > 0. \tag{2}$$

Denote $K = \{(\iota,\lambda,x);\, \iota \in I,\, \lambda \in (0,\varepsilon),\, x \in S_{\iota,\lambda}\}$ and

$$a_\kappa = b_\iota + (\lambda_\iota + \lambda)x; \quad \kappa = (\iota,\lambda,x) \in K.$$

Then $\{a_\kappa\}_{\kappa \in K}$ is an upward directed family in J^+.

Indeed, by the definition of K, we have $a_\kappa \in J^+$ for all $\kappa \in K$. Let $\kappa' = (\iota',\lambda',x') \in K$ and $\kappa'' = (\iota'',\lambda'',x'') \in K$ be arbitrary and choose $\iota \in I$, $\lambda \in (0,\varepsilon)$, $x \in S$ such that $\widetilde{a}_\iota \geqslant \widetilde{a}_{\iota'}$, $\widetilde{a}_\iota \geqslant \widetilde{a}_{\iota''}$, $\lambda > \lambda'$, $\lambda > \lambda''$, $x \geqslant x'$, $x \geqslant x''$ and

$$\begin{aligned} &x \geqslant |b_\iota - b_{\iota'}|(\lambda - \lambda' + |b_\iota - b_{\iota'}|)^{-1},\\ &x \geqslant |b_\iota - b_{\iota''}|(\lambda - \lambda'' + |b_\iota - b_{\iota''}|)^{-1},\\ &x \geqslant (\lambda_\iota + \lambda)^{-1} b_\iota^-. \end{aligned}$$

Then $\kappa = (\iota,\lambda,x) \in K$. We have

$$\begin{aligned} a_\kappa - a_{\kappa'} &= b_\iota + (\lambda_\iota + \lambda)x - b_{\iota'} - (\lambda_{\iota'} + \lambda')x' \\ &\geqslant b_\iota - b_{\iota'} + (\lambda_\iota - \lambda_{\iota'} + \lambda - \lambda')x \\ &\geqslant b_\iota - b_{\iota'} + (\lambda_\iota - \lambda_{\iota'} + \lambda - \lambda')|b_\iota - b_{\iota'}|(\lambda - \lambda' + |b_\iota - b_{\iota'}|)^{-1} \\ &= (\lambda - \lambda' + |b_\iota - b_{\iota'}|)^{-1}[(\lambda - \lambda')(b_\iota - b_{\iota'}) \\ &\qquad + (b_\iota - b_{\iota'} + \lambda_\iota - \lambda_{\iota'} + \lambda - \lambda')|b_\iota - b_{\iota'}|] \\ &= (\lambda - \lambda' + |b_\iota - b_{\iota'}|)^{-1}[(\lambda - \lambda')(b_\iota - b_{\iota'}) \\ &\qquad + |b_\iota - b_{\iota'}| + (\widetilde{a}_\iota - \widetilde{a}_{\iota'})|b_\iota - b_{\iota'}|] \geqslant 0 \end{aligned}$$

and, similarly, $a_\kappa - a_{\kappa''} \geqslant 0$.

Since

$$a_\kappa \leqslant b_\iota + \lambda_\iota + \lambda = \widetilde{a}_\iota + \lambda \leqslant a + \varepsilon; \quad \kappa \in K,$$

the upward directed family $\{a_\kappa\}$ in J^+ is also norm-bounded and

$$\sup_\kappa a_\kappa \leqslant a + \varepsilon.$$

On the other hand, using (2), for all $\iota \in I$ and $\lambda \in (0,\varepsilon)$ we get

$$\widetilde{a}_\iota + \lambda = b_\iota + (\lambda_\iota + \lambda) = \sup_{x \in S_{\iota,\lambda}} (b_\iota + (\lambda_\iota + \lambda)x) \leqslant \sup_\kappa a_\kappa,$$

so

$$a+\varepsilon \leqslant \sup_{\kappa} a_\kappa .$$

We conclude that

$$a+\varepsilon = \sup_{\kappa} a_\kappa$$

and, consequently,

$$(1+\varepsilon)^{-1}(a+\varepsilon) = \sup_{\kappa}(1+\varepsilon)^{-1}a_\kappa \in (J_1^+)^m. \quad \blacksquare$$

LEMMA 2. *Let J be a norm-closed Jordan algebra in the W^*-algebra M such that $W^*(J) = M$. Then*

$$((\widetilde{J}_1^+)^m)_m = ((J_1^+)^m)_m.$$

Proof. Clearly, $((\widetilde{J}_1^+)^m)_m \supset ((J_1^+)^m)_m$. Let $a \in ((\widetilde{J}_1^+)^m)_m$ and let $\{a_\iota\}_{\iota\in I}$ be a downward directed net in $(\widetilde{J}_1^+)^m$ with $a = \inf_{\iota} a_\iota$. By Lemma 1 we have

$$a_{\iota,n} = (1+n^{-1})^{-1}(a_\iota + n^{-1}\cdot 1_M) \in (J_1^+)^m; \quad \iota\in I,\, n\geqslant 1.$$

Since the net $\{a_{\iota,n}\}_{\iota\in I,\, n\geqslant 1}$ is downward directed and its greatest lower bound is a, it follows that $a \in ((J_1^+)^m)_m$. $\blacksquare$

We shall now prove an "invariance" result for $((J_1^+)^m)_m$:

LEMMA 3. *Let J be a norm-closed Jordan algebra in the W^*-algebra M and $\overline{J}^w$ its w-closure. Then, for any family $\{a_\iota\}_{\iota\in I} \subset ((J_1^+)^m)_m$, we have*

$$1_{\overline{J}^w} - \bigvee_{\iota\in I} \mathbf{s}_M(1_{\overline{J}^w} - a_\iota) \in ((J_1^+)^m)_m.$$

Proof. Without restricting the generality, we may assume that $W^*(J) = M$. We shall denote $1_M = 1_{\overline{J}^w}$ simply by 1.

For each $\iota \in I$, there exists a downward directed net

$$\{a_{\iota,\kappa_\iota}\}_{\kappa_\iota\in K_\iota} \subset (J_1^+)^m$$

such that

$$a_\iota = \inf_{\kappa_\iota\in K_\iota} a_{\iota,\kappa_\iota}.$$

For convenience, we denote

$$K = \prod_{\iota\in I} K_\iota,$$
$$a_{\iota,\kappa} = a_{\iota,\kappa_\iota}; \quad \iota\in I,\ \kappa\in K,\ \kappa_\iota = \text{the } \iota\text{-coordinate of } \kappa.$$

Then, for each $\iota \in I$, $\{a_{\iota,\kappa}\}_{\kappa\in K}$ is a downward directed family in $(J_1^+)^m$ and

$$a_\iota = \inf_{\kappa\in K} a_{\iota,\kappa}.$$

Consider the elements

$$a_{F,\kappa,n} = \Big(1 + n\sum_{\iota\in F}(1-a_{\iota,\kappa})\Big)^{-1}; \quad F \subset I \text{ finite}, \kappa \in K, n \geqslant 1.$$

Since each $a_{\iota,\kappa}$ belongs to $(J_1^+)^m \subset (\widetilde{J}_1^+)^m$, using 2.6.(7) and 6.2.(12), we obtain

$$a_{F,\kappa,n} \in (\widetilde{J}_1^+)^m; \quad F \subset I \text{ finite}, \kappa \in K, n \geqslant 1.$$

Again by 2.6.(7), the family $\{a_{F,\kappa,n}\}$ is downward directed so, using Lemma 2, we get

$$\text{(3)} \qquad a = \inf_{F,\kappa,n} a_{F,\kappa,n} \in ((\widetilde{J}_1^+)^m)_m = ((J_1^+)^m)_m.$$

Since

$$1 - a_{F,\kappa,n} = \Big(n^{-1} + \sum_{\iota\in F}(1-a_{\iota,\kappa})\Big)^{-1}\sum_{\iota\in F}(1-a_{\iota,\kappa}),$$
$$1 - a = \sup_{F,\kappa,n}(1 - a_{F,\kappa,n}),$$

by 9.22, we have

$$1 - a \geqslant \sup_{F,n}(1 - a_{F,\kappa,n}) = \mathbf{s}_M(\{1 - a_{\iota,\kappa};\, \iota \in I\}); \quad \kappa \in K.$$

Furthermore, since $\{\mathbf{s}_M(\{1 - a_{\iota,\kappa};\, \iota \in I\})\}_{\kappa\in K}$ is an upward directed family of projections in M and, for each $\iota \in I$,

$$1 - a_{\iota,\kappa} \uparrow 1 - a_\iota,$$

using again 9.22 we deduce

$$\begin{aligned} 1 - a &\geqslant \sup_{\kappa\in K}\mathbf{s}_M(\{1 - a_{\iota,\kappa};\, \iota\in I\}) = \bigvee_{\kappa\in K}\mathbf{s}_M(\{1 - a_{\iota,\kappa};\, \iota \in I\}) \\ &= \bigvee_{\iota\in I}\bigvee_{\kappa\in K}\mathbf{s}_M(1 - a_{\iota,\kappa}) = \bigvee_{\iota\in I}\mathbf{s}_M(1 - a_\iota). \end{aligned}$$

Conversely, for each $F \subset I$ finite, $\kappa \in K$ and $n \geqslant 1$, we have

$$1 - a_{F,\kappa,n} \leqslant \mathbf{s}_M\Big(\sum_{\iota\in F}(1 - a_{\iota,\kappa})\Big) \leqslant \mathbf{s}_M\Big(\sum_{\iota\in F}(1-a_\iota)\Big) \leqslant \bigvee_{\iota\in I}\mathbf{s}_M(1-a_\iota),$$

so

$$1 - a \leqslant \bigvee_{\iota\in I}\mathbf{s}_M(1 - a_\iota).$$

Consequently,

$$1 - a = \bigvee_{\iota\in I} \mathbf{s}_M(1 - a_\iota), \quad a = 1 - \bigvee_{\iota\in I} \mathbf{s}_M(1 - a_\iota). \tag{4}$$

By (3) and (4), we conclude

$$1 - \bigvee_{\iota\in I} \mathbf{s}_M(1 - a_\iota) \in ((J_1^+)^m)_m. \quad \blacksquare$$

LEMMA 4. *Let J be a norm-closed Jordan algebra in the W^*-algebra M and $\overline{J}^w$ its closure. Then $((J_1^+)^m)_m$ contains every countably decomposable projection in $\overline{J}^w$.*

Proof. Let $e \in \overline{J}^w$ be a countably decomposable projection, and choose a family $\{e_\iota\}_{\iota\in I}$ of mutually orthogonal countably decomposable projections in $\overline{J}^w$ such that

$$1_{\overline{J}^w} - e = \sum_{\iota\in I} e_\iota.$$

By Lemma 9.38, for each $\iota \in I$ there exists $a_\iota \in ((J_1^+)^\sigma)_\delta \subset ((J_1^+)^m)$, such that $a_\iota e = e$ and $a_\iota e_\iota = 0$. Then

$$(1_{\overline{J}^w} - a_\iota)e = 0,\ (1_{\overline{J}^w} - a_\iota)e_\iota = e_\iota; \quad \iota \in I,$$

so we succesively get

$$\begin{gathered} e_\iota \leqslant \mathbf{s}_M(1_{\overline{J}^w} - a_\iota) \leqslant 1_{\overline{J}^w} - e; \quad \iota \in I, \\ 1_{\overline{J}^w} - e = \sum_\iota e_\iota \leqslant \bigvee_\iota \mathbf{s}_M(1_{\overline{J}^w} - a_\iota) \leqslant 1_{\overline{J}^w} - e, \\ 1_{\overline{J}^w} - e = \bigvee_\iota \mathbf{s}_M(1_{\overline{J}^w} - a_\iota), \\ e = 1_{\overline{J}^w} - \bigvee_\iota \mathbf{s}_M(1_{\overline{J}^w} - a_\iota). \end{gathered}$$

Using Lemma 3, we conclude that $e \in ((J_1^+)^m)_m$. $\blacksquare$

THEOREM. *Let J be a norm-closed Jordan algebra in the W^*-algebra M and $\overline{J}^w$ its w-closure. Then*

$$(((J_1^+)^m)_m)^m = (\overline{J}^w)_1^+.$$

Proof. Clearly, $(((J_1^+)^m)_m)^m \subset (\overline{J}^w)_1^+$. Let $a \in (\overline{J}^w)_1^+$ be arbitrary. As we have seen in 9.17, we have $W^*(\{a\})_h \subset \overline{J}^w$ so, by Proposition 3/7.16, there exists a sequence $\{e_k\}_{k\geqslant 1}$ of projections in $\overline{J}^w$ such that

$$a = \sum_{k=1}^{\infty} 2^{-k} e_k.$$

We denote

$$I = \left\{ (f_1, \dots, f_n);\quad \begin{array}{l} f_1, \dots, f_n \text{ countably decomposable projections in } \overline{J}^w, \\ f_1 \leqslant e_1, \dots, f_n \leqslant e_n,\ n \geqslant 1 \text{ integer} \end{array} \right\}$$

$$a_\iota = \sum_{k=1}^{n} 2^{-k} f_k; \quad \iota = (f_1, \dots, f_n) \in I.$$

By Lemma 4, we have $a_\iota \in ((J_1^+)^m)_m$ for all $\iota \in I$. Since the family $\{a_\iota\}_{\iota \in I}$ is upward directed and its least upper bound is a, we conclude that $a \in (((J_1^+)^m)_m)^m$. ∎

Sometimes, the above theorem is called suggestively: the "up-down-up" theorem.

Note that an alternative proof of the last remark in 9.35 can be obtained using Lemma 2/9.34, 9.23.(4′) and the above theorem, without making use of Theorem 9.35.

9.40. In this section we prove that the "up-down" theorem does not hold in the general case, so that the "up-down-up" theorem is the best possible general result.

Let M be an AW^*-algebra.

If e is a *minimal projection* in M, that is $e \neq 0$ and $eMe = \mathbb{C}e$, and if $f \in P(M)$, $f \sim e$, then also f is minimal. Indeed, if $v \in M$ is a partial isometry such that $v^*v = e$, $vv^* = f$, then $\mathbb{C}e = eMe \ni x \mapsto vxv^* \in fMf$ is a $*$-isomorphism, so $f \neq 0$ and $fMf = \mathbb{C}f$. Clearly, if $e \in P(M)$ is minimal, then e is abelian.

We denote

$$q_M = \bigvee \{e \in P(M);\ e \text{ minimal}\}.$$

If $e \in P(M)$ is minimal and $\{e_\iota\}$ is a maximal family of mutually orthogonal equivalent projections in M, containing e, then, by 9.30.(1), $\sum_\iota e_\iota \leqslant \mathbf{z}_M(e)$ and, by Corollary 1/9.30 and by the maximality of $\{e_\iota\}$, $\mathbf{z}_M\Big(\mathbf{z}_M(e) - \sum_\iota e_\iota\Big) \neq \mathbf{z}_M(e)$. Thus

$$\mathbf{z}_M\Big(\mathbf{z}_M(e) - \sum_\iota e_\iota\Big) e \neq e,$$

and by minimality of e we succesively get

$$\mathbf{z}_M\Big(\mathbf{z}_M(e) - \sum_\iota e_\iota\Big) e = 0,$$

$$\mathbf{z}_M\Big(\mathbf{z}_M(e) - \sum_\iota e_\iota\Big) \mathbf{z}_M(e) = 0,$$

$$\Big(\mathbf{z}_M(e) - \sum_\iota e_\iota\Big) \mathbf{z}_M(e) = 0,$$

$$\mathbf{z}_M(e) = \sum_\iota e_\iota.$$

Consequently,

$$\mathbf{z}_M(e) \leqslant q_M.$$

It follows that

$$q_M = \bigvee\{\mathbf{z}_M(e);\ e \in P(M) \text{ minimal}\}, \tag{1}$$

so q_M is a central projection of M.

By analogy with 8.18, we shall say that M is *atomic* if every non-zero projection of M majorizes a minimal projection. Note that

$$M \text{ is atomic} \Leftrightarrow q_M = 1_M. \tag{2}$$

Indeed, if M is atomic and $1_M - q_M \neq 0$, then $1_M - q_M$ would majorize a minimal projection, in contradiction with the definition of q_M. Conversely, if $q_M = 1_M$ and $0 \neq e \in P(M)$, then there is a minimal projection f in M with $\mathbf{z}_M(e)f \neq 0$. By the minimality of f we have $f = \mathbf{z}_M(e)f \leqslant \mathbf{z}_M(e)$, $\mathbf{z}_M(f) \leqslant \mathbf{z}_M(e)$ so, by Corollary 1/9.30, there exists a projection $f_0 \leqslant e$ equivalent with f. Thus, e majorizes the minimal projection f_0. Also, from (2) it follows that

(3) *M is atomic $\Leftrightarrow$ every non-zero central projection of M majorizes a minimal projection.*

Now, let A be a C^*-algebra. Using Proposition 4.7, it is easy to see that a projection e in A^{**} is minimal if and only if there exists a pure state φ on A with $e = \mathbf{s}_{A^{**}}(\varphi)$. Therefore:

$$q_{A^{**}} = \bigvee\{\mathbf{s}_{A^{**}}(\varphi);\ \varphi \text{ pure state on } A\}. \tag{4}$$

PROPOSITION. *Let A be a C^*-algebra. If*

$$((A_1^+)^m)_m = (A^{**})_1^+$$

then A^{**} is atomic.

Proof. Denote $M = A^{**}$. By the assumption, $q_M \in ((A_1^+)^m)_m$. Let $\{a_\iota\}_{\iota \in I}$ be a downward directed net in $(A_1^+)^m$ such that

$$q_M = \inf_\iota a_\iota.$$

Let $\iota \in I$ be fixed. There exists an upward directed net $\{a_{\iota,\kappa}\}_{k \in K}$ in A_1^+ such that

$$a_\iota = \sup_\kappa a_{\iota,\kappa}.$$

If φ is an arbitrary pure state on A, then, by (4), we have

$$\mathbf{s}_M(\varphi) = \mathbf{s}_M(\varphi) q_M \mathbf{s}_M(\varphi) \leqslant \mathbf{s}_M(\varphi) a_\iota \mathbf{s}_M(\varphi) \leqslant \mathbf{s}_M(\varphi),$$
$$\mathbf{s}_M(\varphi) a_\iota \mathbf{s}_M(\varphi) = \mathbf{s}_M(\varphi),$$

so

$$\sup_\kappa \varphi(a_{\iota,\kappa}) = \varphi(a_\iota) = \varphi(\mathbf{s}_M(\varphi) a_\iota \mathbf{s}_M(\varphi)) = \varphi(\mathbf{s}_M(\varphi)) = 1.$$

Using Corollary 3/4.16, it follows that $\{a_{\iota,\kappa}\}_{k\in K}$ is an increasing approximate unit of A, thus, with an argument similar to the one used in the proof of Proposition 1/8.7,

$$a_\iota = \sup_\kappa a_{\iota,\kappa} = 1_M.$$

Consequently,

$$q_M = \inf_\iota a_\iota = 1_M,$$

that is, M is atomic. ∎

Note that also the converse implication is true: if A^{**} is atomic, then $((A_1^+)^m)_m = (A^{**})_1^+$; but we do not need it for the purpose of this section.

COROLLARY. *If* $A = C([0,1])$, *then* $((A_1^+)^m)_m \neq (A^{**})_1^+$.

Proof. Let ψ be the state on A corresponding to the Lebesgue measure:

$$\psi(a) = \int_0^1 a(t)\,\mathrm{d}t; \quad a \in A.$$

Then

$$e = \mathbf{s}_{A^{**}}(\psi) \neq 0.$$

Let φ be an arbitrary pure state on A. By the last remarks in (4.9), there is some $t \in [0,1]$ such that

$$\varphi(a) = a(t); \quad a \in A.$$

Let $\{a_k\}_{k\geqslant 1}$ be a decreasing sequence in A_1^+ pointwise convergent to the characteristic function of $\{t\}$, and let a be the greatest lower bound of $\{a_k\}$ in A^{**}. Then, by the Lebesgue dominated convergence theorem,

$$\psi(a) = \inf_k \psi(a_k) = 0$$

and

$$1 \geqslant \varphi(\mathbf{s}_{A^{**}}(a)) \geqslant \varphi(a) = \inf_k \varphi(a_k) = \inf_k a_k(t) = 1,$$
$$\varphi(\mathbf{s}_{A^{**}}(a)) = 1.$$

Hence

$$ae = 0, \quad \mathbf{s}_{A^{**}}(a)e = 0, \quad \mathbf{s}_{A^{**}}(a) \leqslant 1_{A^{**}} - e$$

and

$$\mathbf{s}_{A^{**}}(\varphi) \leqslant \mathbf{s}_{A^{**}}(a)$$

so that

$$\mathbf{s}_{A^{**}}(\varphi) \leqslant 1_{A^{**}} - e.$$

Using (4), it follows that

$$q_{A^{**}} \leqslant 1_{A^{**}} - e$$

so A^{**} is not atomic. By the above proposition we conclude that

$$((A_1^+)^m)_m \neq (A^{**})_1^+. \quad \blacksquare$$

9.41. In this last section we shall show that, given a C^*-subalgebra A of an arbitrary W^*-algebra M, the smallest "sequentially monotone closed" subset of M_h containing A_h is again the real part of some C^*-subalgebra of M.

Let M be a W^*-algebra.

A subset S of M_h is called *sequentially monotone closed* (respectively *monotone closed*) if $S^\sigma = S = S_\delta$ (respectively $S^m = S = S_m$). Since M_h is monotone closed and the intersection of any family of sequentially monotone closed (respectively monotone closed) subset of M_h is again sequentially monotone closed (respectively monotone closed), for every $S \subset M_h$ there exists a smallest sequentially monotone closed (respectively monotone closed)subset of M_h containing S, which is called the *sequential monotone closure* (respectively *monotone closure*) of S in M_h.

By Theorem 9.39, if J is a norm-closed Jordan algebra in M, then the monotone closure of J coincides with its w-closure $\overline{J}^w$ which is again a norm-closed Jordan algebra in M. Moreover, if J is the real part A_h of some C^*-subalgebra A of M, then its monotone closure is the real part of the C^*-subalgebra $\overline{A}^w$ of M; note that $\overline{A}^w$ is actually the smallest monotone closed C^*-subalgebra of M containing A.

Here we are concerned with similar statements in the "sequential case", without assuming the W^*-algebra M to be countably decomposable.

We begin with two simple remarks. First,

(1) *every countable subset of the sequential monotone closure of an arbitrary set $S \subset M_h$ is contained in the sequential monotone closure of a norm-separable subset of S.*

Indeed, let Q be the sequential monotone closure of S in M_h and denote by Q_0 the set of all $x \in Q$ such that x belongs to the sequential monotone closure of some norm-separable subset of S. It is sufficient to show that $Q_0 = Q$. But this is clear, since $Q_0 \supset S$ and Q_0 is sequentially monotone closed.

The second remark is that

(2) *if $S \subset M_h$ is a real vector subspace with $1_M \in S$, then the norm-closure of S is contained in $S^\sigma \cap S_\delta$.*

Indeed, let $x \in M$ be norm-adherent to S. Then there is a sequence $\{x_k\}_{k\geqslant 1} \subset S$ norm-convergent to x and such that $\|x_{k+1} - x_k\| \leqslant 2^{-k}$ for all $k \geqslant 1$. Moreover,

$$y_n = -1_M + x_1 + \sum_{k=1}^{n}(x_{k+1} - x_k + 2^{-k}) \in S; \quad n \geqslant 1,$$

is an increasing sequence, norm-convergent to x, hence $x = \sup_n y_n \in S^\sigma$ (see Lemma 3/9.10). The same argument applied to $-x$ shows that $x \in S_\delta$.

Lemma. *Let J be a norm-closed Jordan algebra in the W^*-algebra M with $1_M \in J$. Then the sequentially monotone closure K of J in M_h is again a norm-closed Jordan algebra in M.*

Proof. The set $K_1 = \{a \in M_h;\, a + x \in K \text{ for all } x \in J\}$ contains J and is sequentially monotone closed, so $K_1 \supset K$. Then, the set $K_2 = \{a \in M_h;\, a + x \in K \text{ for all } x \in K\}$ contains J and is sequentially monotone closed, hence $K_2 \supset K$, that is $K + K \subset K$. Similarly, $\lambda K \subset K$ for every $\lambda \in \mathbb{R}$. Thus, K is a real vector subspace of M_h.

By (2), it follows that K is norm-closed.

The set $K_3 = \{x \in K;\ x^k \in K \text{ for all integers } k \geqslant 1\}$ contains J. Let $\{x_n\}_{n\geqslant 1}$ be an increasing sequence in K_3 with $\|x_n\| \leqslant 1$ for all $n \geqslant 1$. Then $x = \sup\limits_n x_n \in K$ and $\|x\| \leqslant 1$. For each $0 \leqslant t < 1$ we have

$$(1_M - tx_n)^{-1} = \sum_{k=0}^{\infty} t^k x_n^k \in K,$$

because K is norm-closed. Since $(1_M - tx_n)^{-1} \uparrow (1_M - tx_n)^{-1}$ as $n \to \infty$, we get $(1_M - tx_n)^{-1} \in K$. Since for $t \to 0$ we have $t^{-n-1}\Big((1_M - tx_n)^{-1} - \sum\limits_{k=0}^{n} t^k x^k\Big) = x^{n+1}(1_M - tx_n)^{-1} \xrightarrow{\text{norm}} x^{n+1}$, by induction over n we see that $x^n \in K$ for all $n \geqslant 1$, that is $x \in K_3$. Thus, K_3 is sequentially monotone closed, so that $K_3 = K$. This shows, in particular, that $x \in K \Rightarrow x^2 \in K$.

Hence K is a norm-closed Jordan algebra in M. ∎

Proposition. *Let A be a C^*-subalgebra M. Then the sequential monotone closure of A_h in M_h is the real part if the smallest sequentially monotone closed C^*-subalgebra of M containing A.*

Proof. Without restricting the generality, we may suppose that A a w-dense in M.

Let K be the sequential monotone closure of A_h in M_h. If $B = K + \mathrm{i}K$ is a C^*-subalgebra of M, then it is clear that B is the smallest sequentially monotone closed C^*-subalgebra of M containing A.

Thus, all we have to show is that K is the real part of some C^*-subalgebra of M, i.e. (6.2.(13)) that K is a norm-closed Jordan algebra in M and

(3) $$x, y \in K \Rightarrow \mathrm{i}(xy - yx) \in K.$$

Taking into account the remark (1), we see that it is sufficient to prove the statement under the supplimentary assumption that A is norm-separable. In this case, A has a countable increasing approximate unit (3.2) with least upper bound 1_M in M_h (8.7), so that $1_M \in K$ and we may further assume that $1_M \in A$. Then, by the above lemma it follows that K is a norm-closed Jordan algebra in M.

It remains to provs that (3) holds. First note that

(4) $$\mathrm{i}(xy - yx) = (1_M - \mathrm{i}y)x(1_M + \mathrm{i}y) - x - yxy; \quad x, y \in M_h.$$

The set $\{x \in K;\, \mathrm{i}(xy - yx) \in K \text{ for all } y \in A_h\}$ contains A_h and, using (4), it is sequentially monotone closed, so it coincides with K, that is

$$x \in K,\ y \in A_h \Rightarrow \mathrm{i}(xy - yx) \in K.$$

Now, the set $\{x \in K;\, \mathrm{i}(xy - yx) \in K \text{ for all } y \in K\}$ contains A_h and, as above, it is sequentially monotone closed, hence it coincides with K, that is (3) holds. ∎

Thus, the sequential monotone closure of a C^*-algebra A in its second dual A^{**} is a C^*-subalgebra of A^{**}, denoted by $Ba(A)$ and called *the Baire envelope of A*.

9.42. Notes. The main ideas contained in the material presented in this Chapter 9 are due to C.E. Rickart [255], I. Kaplansky [158], [160], [162], R.V. Kadison [144], [145], [150], and the main results concerning W^*-algebras, in their strongest forms (Lemma 2 and Theorem 1/9.17; Theorem 9.33; Theorem 9.38; Theorem 9.39; Proposition and Corollary 9.40), are due to G.K. Pedersen [236], [238], [239].

Earlier, Theorem 9.35 has been proved in the commutative case (i.e., Proposition 9.36) by J. Dixmier [72], has been conjectured in general and proved in the "finite case" by J. Feldman [101], then extended to the "semifinite case" by K. Saitô [265], and gifted with an elegant proof in this case by D. Laison [170]; namely this last proof has been refined up to Lemma 2/9.17 by G.K. Pedersen [239] in order to obtain the general result.

On the other hand, R.V. Kadison already proved [144] that every monotone complete C^*-algebra with a sufficient family of normal states is a W^*-algebra. Moreover, R.V. Kadison made several detalied analyses [145], [151] of the w-closure of an operator algebra from the point of view of monotone convergence, further studied by G.K. Pedersen [235], and also R.V. Kadison [150] conjectured weak forms of Theorems 9.38 and 9.39, namely that the w-closure should be attained in a finite number of "up-down" steps. The proof of Theorem 9.38 is the original proof from [236], [238], Lemma 1/9.39 is a variant of [8]; Theorem 2.1, (iii) $\Rightarrow$ (iv) and the proof of Theorem 9.39 is a combination of the original proof from [238] with that from [351], I, §1. The results included in 9.41 are due to R.V. Kadison ([145], p. 317–318) and G.K. Pedersen [235]. In [235] it is proved that for a type I C^*-algebra A its Baire envelope $B(A)$ is equal to the smallest sequentially w-closed C^*-subalgebra of A containing A (see also [233], [355], [356]).

That not every AW^*-algebra is a W^*-algebra has been proved even in the commutative case (J. Dixmier [72] gave an example of a Stone space which is not hyperstonean), but the following problem remained: is it true that if A is an AW^*-algebra whose center Z_A is a W^*-algebra, then is A a W^*-algebra? The answer to this problem is again negative, as announced by J.A. Dyer [82] and also proved by J.D. Maitland-Wright [192], [189] who constructed very natural "wild AW^*-factors". However, J.D. Maitland-Wright [190] proved that if A is a finite AW^*-algebra with a faithful state and if Z_A is a W^*-algebra, then A is a W^*-algebra, and a similar result for monotone complete C^*-algebras has been obtained by R.V. Kadison and G.K. Pedersen [152]. Some "intrinsic" conditions which insure that a "semifinite" AW^*-algebra satisfying them is in fact a W^*-algebra appeared in [190], [191].

The first axiom insuring the existence of enough many projections in a C^*-algebra A concerned the existence of support projections of positive elements of A ([255]). Then several other axioms were introduced, such as $(B_1),(B_r)$ from 9.35 ([162]), or the "EP-axiom": "for every non-zero element $x \in A$ there exists a selfadjoint element $y \in \{\{x\}' \cap A\}' \cap A$ with x^*xy^2 a non-zero projection" ([162], §13), or the requirement that every maximal commutative $*$-subalgebra of A be the norm-closed linear hull of its projections ([158]). In our exposition we used spectral axiom (9.1) which is a slightly modified version of the axiom introduced in ([351], III, §1), modified in order to obtain Proposition 9.7. This axiom has some advantages, namely it is preserved under passage to quotiente algebras, it allows the lifting of projections and of partial isometries (9.5; cf. [351], III.2.4; for AW^*-algebras this is due to F.B. Wright [345], see also [306], II), and offers unified smooth proofs in the theory of Rickart and AW^*-algebras.

For the various characterizations and permanence properties of Rickart and AW^*-algebras (9.9, 9.11, 9.12, 9.13, 9.20, 9.21, 9.23, 9.24, 9.25), as well as for the few results related to the geometry of projections (9.28, 9.29, 9.30), we have used [255], [158], [160], [161], [162], [25], to which we also refer for further results concerning AW^*-algebras and

Baer $*$-rings (see also [136], [254]). Note that the statements from 9.29 and 9.30 are just particular cases of the unrestricted additivity of equivalence which holds true in any AW^*-algebra ([25], §20; [158], Theorem 5.5; [162], Theorem 64). Also, the polar decomposition theorem holds in any AW^*-algebra ([25], §21; [162], Theorem 65). We mention that the C^*-tensor product of two C^*-algebras is an AW^*-algebra (respectively a W^*-algebra) if and only if both are AW^*-algebras (respectively W^*-algebras) and at least one of them is finite dimensional ([24], [315]).

The structure theorem for Jordan $*$-homomorphisms (9.31) is due to N. Jacobson and C.E. Rickart [133] and to R.V. Kadison ([145], Theorem 10; [352], Theorem 2.6; see also [297], Theorem 3.3; [301] §5). In completing (9.32) the results from 6.7, 6.8 we have used [89], [227], [347] (see also [232]). Another equivalent condition in Corollary of Theorem 2/9.32 is that Φ maps the quasi-unitary elements of A onto the quasi-unitary elements of B ([227], Theorem 3, Theorem 4).

The terminology "Stone space" seems to be introduced by J. Dixmier [72] in connection with the contributions of M.H. Stone [294], [295]; Stone spaces are also called "extremally disconnected spaces". Together with Stone spaces, J. Dixmier [72] also introduced and studied hyperstonean spaces, giving several illuminating examples. The characteristic properties of Stone spaces (Theorem & Corollary 9.27) are due to A.M. Gleason [109] and M. Hasumi [127], [128]. In our proof we used Proposition 9.27 (cf. [351], II.5.2) and its Corollary ([296], Cor. 3.6; [301], §6; [351], II.5.7; [357]); for another proof see [185]. The continuous functions on a Stone space, that is the commutative AW^*-algebras, have several interesting Banach space properties. Thus (compare with Theorem 9.27), a Banach space A is isometrically isomorphic with a commutative AW^*-algebra if and only if, whenever A is imbedded as a closed vector subspace of some other Banach space X, there exists a linear projection of norm one of X onto A ([116], [128], [167], [201]). Also, if A is a commutative AW^*-algebra, then every $\sigma(A^*, A)$-convergent sequence from A is already $\sigma(A^*, A^{**})$-convergent (compare with Theorem 8.19), every bounded linear operator from A to any separable Banach space is weakly compact ([118]) and the closed unit ball of A is the closed convex hull of its extreme points ([116]). Some other related results are contained in [21], [203], [317].

For the representation of every commutative W^*-algebra as an $L^\infty(\mu)$ (Proposition 1/9.37) we refer to [72], [77], [285]. The localizable measure were introduced and studied by I.E. Segal [284], [285] who proved, among other results, the statements contained in Proposition 3/9.37.

As we already mentioned, the (sequentially) monotone complete C^*-algebras were considered by R.V. Kadison [144], [145]. For the various results concerning them (9.6, 9.10, 9.11, 9.12, 9.14, 9.15, 9.17, 9.18, 9.22, 9.23, 9.24) we have used [50], [145], [152], [166], [233], [235], [236], [237], [238], [239], [302], [330], [355], [356], to which we refer also for further results; other related results are contained in [186], [187], [188]. Some results which were proved here for AW^*-algebras and monotone complete C^*-algebras can be extended to the "sequential case" with the supplimentary assumption of "countable generation". For instance, a Rickart algebra A is called countaby generated if there is a sequence $\{x_n\} \subset A$ such that A is the smallest Rickart subalgebra of A containing $\{x_n\}$. If A is a countably generated Rickart algebra, then A is unital, every $x \in A$ has a central support $\mathbf{z}_A(x) \in Z_A$, for every projection $e \in A$ we have $Z_{eAe} = eZ_A$, and every countably additive $*$-homomorphism of A onto a W^*-algebra M maps Z_A onto Z_M (see [356], p. 352, 353, 360).

In the present exposition we stated the results for norm-closed Jordan algebras of selfadjoint operators ([35], [89], [297], [298], [300], [330], [331]) instead of C^*-algebras, because, applying these results for the real parts of C^*-algebras, we immediately get the corresponding statements for C^*-algebras and, usually, the proofs are essentially the same. Besides the abstract Jordan algebras ([132], [139], [140], [207]) one considers also some normed Jordan algebras, namely JB-algebras ([11], [12], [13]) and JBW-algebras ([289]). As shown by E.M. Alfsen, F.W. Shultz and E. Störmer [13], the study of JB-algebras can be reduced to that of Jordan algebras of selfadjoint operators and the exceptional Jordan algebra M_3^8 ([140]). F.W. Shultz [289] proved that the "envelopping algebra" of a JB-algebra can be identified with its second dual and used this to show that a JB-algebra J is a dual space, i.e. a JBW-algebra, if and only if it is monotone complete and admits a separating set of normal states, in which case the predual is unique and J splits into the direct sum of a "special part", isomorphic to a w-closed Jordan algebra of selfadjoint operators, and a "purely exceptional part", isomorphic to $C(\Omega, M_3^8)$. On the other hand, I. Kaplansky, in his final lecture to the 1976 St. Andrews Colloquium of the Edinburgh Mathematical Society, introduced the concept of a Jordan C^*-algebra ([193], [194], [195]), J.D. Maitland-Wright [193] showed that each JB-algebra is the selfadjoint part of a unique Jordan C^*-algebra and obtained a structure theorem for Jordan C^*-algebras and, together with M.A. Youngson [194], [195], showed that surjective unital linear isometries between Jordan C^*-algebras are Jordan $*$-isomorphisms and extended the Russo-Dye theorem. Also the Vidav-Palmer theorem (see 1.19) can be generalized to Jordan algebras ([29]).

The statements: Corollary 9.2, 9.3.(1), Lemma 2/9.10, Proposition 3/9.14, Theorem 2/9.17 in the Jordan case, Corollary 9.31, Theorem 1/9.32 were worked out by the second author (L.Z.) in October 1978, in Rome; some of these may be new.

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