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Ph. D. Thesis

Summary

Long time behaviour for nonlocal evolution equations

TATIANA IONICA IGNAT

Supervisor

LUCIAN BEZNEA

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Summary

This thesis is devoted to the study of the asymptotic properties of the solutions of some nonlocal diffusion problems. The objective is to characterise the first term in the asymptotic expansion of the solutions for large time. The main strategy used in this thesis is based on the scaling arguments. Mainly this means that we rescale conveniently the solutions and reduce the analysis of the long time behavior to the compactness of the rescaled trajectories. This method as far as we know has been introduced by Kamin and Vazquez [16].

Let us consider the simplest model that appears in our analysis. In Chapter 2 we study the follow ing nonlocal equation:

$$\begin{cases} u_t(x,t) = \displaystyle\int_{\mathbb{R}} J(x-y)(u(y,t)-u(x,t))\,dy, & x\in\mathbb{R},\ t>0,\\ u(x,0)=u_0(x), & x\in\mathbb{R}. \end{cases} \tag{1}$$

We consider $J:\mathbb{R}\to\mathbb{R}$ a nonnegative, smooth, even function with $\int_{\mathbb{R}} J(s)ds = 1$ and the initial data $u_0 \in L^1(\mathbb{R})\cap L^\infty(\mathbb{R})$. Equations like (1) and variations of it, have been recently widely used to model diffusion processes, for example, in biology, dislocations dynamics, etc. We refer [2], [4], [8], [9] and the references therein.

As stated in [8], if $u(x,t)$ is the density of a single population at the point x at time t, and $J(x-y)$ is thought of as the probability distribution of jumping from location y to location x, then $(J*u)(x,t) = \int_{\mathbb{R}} J(x-y)u(y,t)dy$ is the rate at which individuals are arriving to position x from all other places and $-u(x,t) = -\int_{\mathbb{R}} J(y-x)u(x,t)dy$ is the rate at which they are leaving location x to travel to all other sites. This consideration, in the absence of external or internal sources, leads immediately to the fact that the density u satisfies (1). This simple equation is called nonlocal diffusion equation since, in contrast with the classical heat equation $u_t = u_{xx}$, the diffusion of the density u at the time t and point x depend on all the values of u in a neighborhood of x. For a function J supported in the interval $(-1,1)$ we easily can rewrite equation (1) as an integral over the space interval $(x-1, x+1)$. Regarding the well-posedness it is immediately that for any initial data

$u_0 \in L^p(\mathbb{R})$, $1 \leq p \leq \infty$ there exists a unique solution $u \in C([0,\infty), L^p(\mathbb{R}))$. Since the operator $\mathcal{L}u = J * u - u$ is linear and continuous on the $L^p(\mathbb{R})$ spaces the solution is in fact $C^\infty([0,\infty), L^p(\mathbb{R}))$. This is a classical well understood fact about equation of type (1).

The study of the long-time behavior of the solutions to equation (1) has been started in [5] where the authors prove by using the Fourier representation of the solutions that for large time t the solution u is closer and closer to the rescaled heat kernel. In Chapter 2 we prove the same results by a different argument. The main novelty is the method used: scaling arguments. This method is usually used in the case of nonlinear problems to obtain the first term in the asymptotic expansion of the solutions. In order to understand the difficulties of applying this method to nonlinear problems we first analyze the linear case.

The main result in Chapter 2 is the following: For any $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $p \in [1,\infty]$ solution $u(x,t)$ of equation (1) satisfies:

$$\lim_{t\to\infty} t^{\frac{1}{2}(1-\frac{1}{p})} \|u(t) - MG_{At}\|_{L^p(\mathbb{R})} = 0 \tag{2}$$

where

$$G_t(x) = \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{x^2}{4t}\right)$$

is the heat kernel and

$$M = \int_{\mathbb{R}} u_0(x)dx, \quad A = \frac{1}{2}\int_{\mathbb{R}} J(z)z^2 dz.$$

As will be explained in Chapter 2 equation (1) has no regularising effect. So, one cannot expect solutions to be more regular than the initial data is. This is way we impose the initial data to belong to $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ instead of $L^1(\mathbb{R})$ as in the case of the classical heat equation. Without this assumption we cannot guarantee that at positive time t the solution is in $L^p(\mathbb{R})$, $p > 1$, so to estimate these norms will not make sense.

The results in Chapter 2 are based on the paper [15].

Once we understood how the scaling arguments work for linear problem the second step is to use the same methods to nonlinear problems. The results in Chapter 3 are the core of this thesis. Let us explain the results in this chapter. We analyze the following nonlocal convection - diffusion equation:

$$\begin{cases} u_t = J * u - u + G * |u|^{q-1}u - |u|^{q-1}u, & x \in \mathbb{R}^d, t > 0, \\ u(0) = \varphi. \end{cases} \tag{3}$$

Let us now be more precise about the assumptions on the kernels J and G. We assume that $J, G : \mathbb{R}^d \to \mathbb{R}$ are non-negative functions with mass one, J being radially symmetric and positive in a neighborhood of the origin. The well-posedness of these problems has been previously considered in [14]. Using a Banach fix point argument it can be easily proved that for any $\varphi \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ there exists a unique solution $u \in C([0,\infty), L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}))$. Easily one can prove that the solution also belong to $C^1([0,\infty), L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}))$. More time regularity issues can be addressed but is out of the scope of this thesis.

The main results of this chapter is the following one: For any $\varphi \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ the solution u of system (3) satisfies

$$\lim_{t\to\infty} t^{\frac{d}{2}(1-\frac{1}{p})} \|u(t) - U_m(t)\|_{L^p(\mathbb{R}^d)} = 0, \quad 1 \le p < \infty, \tag{4}$$

where the m is the mass of the initial data φ and

- for $q > 1 + 1/d$ or $B = 0_{1,d}$, U_m is the rescaled heat kernel solution of

$$\begin{cases} U_t = A\Delta U, & x \in \mathbb{R}^d, t > 0, \\ U(0) = m\delta_0. \end{cases} \tag{5}$$

- for $q = 1 + 1/d$ and $B \neq 0_{1,d}$, U_m is the unique solution of the following equation

$$\begin{cases} U_t = A\Delta U - B \cdot \nabla(|U|^{1/d} U), & x \in \mathbb{R}^d, t > 0, \\ U(0) = m\delta_0. \end{cases} \tag{6}$$

Next, we say a few words about the above asymptotic profile U_m. It is easy to check that

$$U_m(t,x) = t^{-d/2} f_m\Big(\frac{x}{\sqrt{t}}\Big),$$

where the profile f_m is the smooth solution of the equation

$$-A\Delta f_m - \frac{1}{2} x \cdot \nabla f_m = \frac{d}{2} f_m - \alpha B \cdot \nabla(|f_m|^{q-1} f_m) \quad \text{in } \mathbb{R}^d,$$

with $\int_{\mathbb{R}^d} f_m = m$ and

$$\alpha = \begin{cases} 1, & q = 1 + \frac{1}{d}, \\ 0, & q > 1 + \frac{1}{d}. \end{cases}$$

In the case when the nonlinear term is supercritical, i.e. $q > 1 + 1/d$, the first term in the asymptotic behavior has been analyzed in [14] under the additional assumption that

$J \in \mathcal{S}(\mathbb{R}^d)$, the class of rapidly decreasing functions. There the main idea was that the nonlinear part decays faster than the linear semigroup and then the first term in the long time behavior is given by the linear semigroup. This has been already observed in [7] in the case of the classical convection-diffusion system.

The aim of Chapter 3 is to give an answer to the critical case $q = 1+1/d$ even though we give a proof that both treats the critical and super-critical case. The method we employ is the so-called *four step method* (see [16]), that consists in the analysis of some rescaled orbits $\{u_\lambda(t)\}_{\lambda>0}$. Let us be a little more precise about the method we use. We introduce the family $u_\lambda(t,x) = \lambda^d u(\lambda^2 t, \lambda x)$. It follows that u_λ satisfies the following rescaled equation

$$\begin{cases} (u_\lambda)_t = \lambda^2 (J_\lambda * u_\lambda - u_\lambda) + \lambda^{d(1-q)+2}(G_\lambda * u_\lambda^q - u_\lambda^q), & x \in \mathbb{R}^d, t > 0, \\ u_\lambda(0,x) = \varphi_\lambda(x), \end{cases} \tag{7}$$

where $\varphi_\lambda(x) = \lambda^d \varphi(\lambda x)$, $J_\lambda(x) = \lambda^d J(\lambda x)$ and $G_\lambda(x) = \lambda^d G(\lambda x)$.

We first emphasize that property (4) is equivalent with the fact that for example at $t = t_0$ the rescaled family $u_\lambda(t_0)$ converges to some function $U(t_0)$ in any $L^p(\mathbb{R}^d)$-norm, $1 \leq p < \infty$. There are various problems that appear in this type of approach. We first have to prove that the family $\{u_\lambda\}_{\lambda>0}$ is compact and hence, up to a subsequence, converges to some function U. Secondly we have to characterize the limit function U. The main difficulty in proving the compactness of the trajectories $\{u_\lambda\}_{\lambda>0}$ is the lack of any information about the derivatives of function u. Let us recall the following *energy estimate*: for any $0 < t_1 < t_2 < \infty$ the following holds

$$\|u_\lambda(t_2)\|^2_{L^2(\mathbb{R}^d)} + \lambda^2 \int_{t_1}^{t_2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} J_\lambda(x-y)(u_\lambda(t,x) - u_\lambda(t,y))^2 \, dxdydt = \|u_\lambda(t_1)\|^2_{L^2(\mathbb{R}^d)} \leq C(t_1). \tag{8}$$

This estimate is the nonlocal analogous of the classical energy estimate

$$\|u(t_2)\|^2_{L^2(\mathbb{R}^d)} + \int_{t_1}^{t_2} \int_{\mathbb{R}^d} |\nabla u|^2 dxdt = \|u(t_1)\|^2_{L^2(\mathbb{R}^d)}$$

that holds for the classical convection-diffusion problem

$$u_t = \Delta u + a \cdot \nabla(|u|^{q-1} u).$$

We prove that estimate (8) is sufficient in order to obtain the compactness of the trajectories $\{u_\lambda\}_{\lambda>0}$. This requires a new version of the classical compactness arguments in the space $L^p((0,T) \times \Omega)$, one which can be adapted to nonlocal evolution equations. Let us now recall a classical compactness result in the spaces $L^p((0,T),B)$, with B a Banach space. Aubin-Lions-Simon Lemma [18, Th. 5] assumes that we have three Banach spaces $X \hookrightarrow B \hookrightarrow Y$ where the embedding $X \hookrightarrow B$ is compact. A sequence $\{f_n\}_{n\geq 1}$ is relatively compact in $L^p((0,T),B)$ (and in $C([0,T],B)$ if $p=\infty$) if we can guarantee that $\{f_n\}_{n\geq 1}$ is bounded in $L^p((0,T),X)$ and $\|\tau_h f_n - f_n\|_{L^p((0,T-h),Y)} \to 0$ as $h\to 0$ uniformly in n.

There are situations where we cannot bound uniformly a sequence $\{g_n\}_{n\geq 1}$ in a space that is compactly embedded in $L^p(\Omega)$. Instead of that we have estimates on some Dirichlet forms that vary with n, estimates that allow us to obtain the compactness of the sequence $\{g_n\}_{n\geq 1}$ (see for example [3], [17] and [1, Th. 6.11, p. 128]). Let us now be more precise. We choose $1<p<\infty$ and $\Omega \subset \mathbb{R}^d$ a smooth domain. Function $\rho : \mathbb{R}^d \to \mathbb{R}$ is a nonnegative smooth radial function with compact support, non identically zero, satisfying $\rho(x)\geq \rho(y)$ if $|x|\leq |y|$. Set $\rho_n(x) = n^d\rho(nx)$. Let $\{g_n\}_{n\geq 1}$ be a bounded sequence in $L^p(\Omega)$ such that

$$n^p \int_\Omega \int_\Omega \rho_n(x-y)|g_n(x)-g_n(y)|^p dxdy \leq M.$$

Then as proved in [3], [17], [1, Th. 6.11, p. 128], sequence $\{g_n\}_{n\geq 1}$ is relatively compact in $L^p(\Omega)$. Our main contribution is to use this compactness argument instead of the compact embedding $X \hookrightarrow B$ in the Aubin-Lions-Simon Lemma and to obtain a new compactness criterion in $L^p((0,T)\times\Omega)$. The main compactness tool that we prove and use in Chapter 3 is given in Theorem 3.2.1. We prove that if $\{f_n\}_{n\geq 1}$ is a bounded sequence in $L^p((0,T)\times\Omega)$, $1<p<\infty$, that satisfies

$$n^p \int_0^T \int_\Omega \int_\Omega \rho_n(x-y)|f_n(t,x)-f_n(t,y)|^p dxdydt \leq M \tag{9}$$

and

$$\|\partial_t f_n\|_{L^p((0,T),W^{-1,p}(\Omega))} \leq M \tag{10}$$

then $\{f_n\}_{n\geq 1}$ is relatively compact in $L^p((0,T)\times\Omega)$.

The results in Chapter 3 are based on the paper [12].

In the last chapter of the thesis we consider a different model of convection-diffusion where the two involved kernels that appear in the diffusive/convective part compete. We analyze the following equation

$$\begin{cases} u_t(t,x) &= \displaystyle\int_{\mathbb{R}} K(x-y)(u(t,y)-u(t,x))dy \\ & \qquad + \displaystyle\int_{\mathbb{R}} G(x-y) f\Big(\frac{u(t,y)+u(t,x)}{2}\Big)dy, t>0, x\in\mathbb{R}, \\ u(0) &= \varphi, \end{cases} \tag{11}$$

in the particular case when $f(u) = u^2$. This model has been proposed in [6] as a regularization of the following nonlocal advection equation inspired by the peridynamic theory

$$u_t(t,x) = \int_{\mathbb{R}} G(x-y) f\Big(\frac{u(t,y)+u(t,x)}{2}\Big) dy.$$

The general model in [6] assumes that K is a nonnegative even function and G is an odd function. We consider here kernels K and G that are integrable. For simplicity we assume that K has mass one. We will analyze the well-posedness of system (11) and the long time behaviour of its solutions. The results presented here hold under the assumption that kernel K dominates G, i.e. for some positive constant $C = C_{GK}$ the following holds

$$|G(x)| \leq C_{GK} K(x), \quad \forall\, x \in \mathbb{R}. \tag{12}$$

For any $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ we prove that there exists a unique local solution $u \in C([0, T_{max}], L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}))$ of equation (11), a solution that conserves the mass of the initial data. Moreover, under the smallness assumption on the initial data $\|\varphi\|_{L^\infty(\mathbb{R})} < 1/C_{GK}$ the solution is global, preserves the sign of the initial data and satisfies

$$\|u(t)\|_{L^1(\mathbb{R})} \leq \|\varphi\|_{L^1(\mathbb{R})}, \|u(t)\|_{L^\infty(\mathbb{R})} \leq \|\varphi\|_{L^\infty(\mathbb{R})}.$$

Once the well-posedness of the global solutions has been established we obtain the decay of the solutions of system (11). We prove that

$$\|u(t)\|_{L^2(\mathbb{R})} \leq C(\|\varphi\|_{L^1(\mathbb{R})}, \|\varphi\|_{L^\infty(\mathbb{R})}) t^{-\frac{1}{4}}, \quad \forall t > 0. \tag{13}$$

Moreover, if the initial data satisfy $\|\varphi\|_{L^\infty(\mathbb{R})} \leq 1/(2C_{GK})$ the following estimate holds for any $2 \leq p < \infty$

$$\|u(t)\|_{L^p(\mathbb{R})} \leq C(\|\varphi\|_{L^1(\mathbb{R})}, \|\varphi\|_{L^\infty(\mathbb{R})}) t^{-\frac{1}{2}(1-\frac{1}{p})}, \quad \forall t > 0. \tag{14}$$

The above condition on the smallness of the $L^\infty(\mathbb{R})$-norm of the solution is similar to the Courant–Friedrichs–Lewy (CFL) condition that appears in the study of the stability of the numerical approximations for conservation laws, see [10, Ch. 3]. This guarantees that the diffusive part controls the nonlinear convective term. The difference with the previous works on nonlocal convection-diffusion equations [14, 13, 12] is that in this case the convective term

$$Tu = \int_{\mathbb{R}} G(y-x) (\frac{u(t,y)+u(t,x)}{2})^2 dy$$

does not satisfy the dissipative condition $(Tu, u)_{L^2(\mathbb{R})} \leq 0$. Thus, extra assumptions should be imposed to the initial data, so on the solutions, to control this term by the diffusive part.

The last result of this chapter concerns the first term in the asymptotic expansion of the solution u. We introduce the following quantities

$$A = \frac{1}{2} \int_{\mathbb{R}} K(z) z^2 dz \quad \text{and} \quad B = \int_{\mathbb{R}} G(z) z dz.$$

The main result concerning the asymptotic expansion for the solutions of system (11) is the following one. Let us assume that $K \in L^1(\mathbb{R}, 1+x^2)$ is positive in a neighborhood of the origin. For any $\varphi \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ with $\|\varphi\|_{L^\infty(\mathbb{R})} \leq 1/(2C_{GK})$ solution u of system (11) satisfies

$$\lim_{t\to\infty} t^{\frac{d}{2}(1-\frac{1}{p})} \|u(t) - U(t)\|_{L^p(\mathbb{R}^d)} = 0, \quad 1 \leq p < \infty,$$

where U is the solution of the viscous Burgers' equation

$$\begin{cases} U_t = AU_{xx} - \frac{B}{2}(U^2)_x, & t > 0, x \in \mathbb{R}^d, \\ U(0) = m\delta_0, \end{cases}$$

and m is the mass of the initial data φ. In this case U can be computed explicitly by using Hoph-Cole transformation.

The results in Chapter 4 are based on paper [11].

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