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LOCAL CONTROLLABILITY ALONG A SINGULAR TRAJECTORY

by

C. VARSAN

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C. VARSAN x)

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x) INSTITUTE OF MATHEMATICS, Str. ACADEMIFI 14, BUCHAREST 1.

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Introduction

Local controllability, in the last years, has been developed by several authors. In [2] and [3] sufficient conditions for local controllability along a singular trajectory in general nonlinear control systems are given. The subject in which we are interested is connected with Harmes' work (see [1]) that contains the same problem as in [2], or [3], but the system and conclusions differ essentially than those stated in [2] and [3].

First of all the control system in [1] is linear in control variable and control range set contains the reference control u=o in its interior. The result is stated in a simpler manner than in [3]. The main goal of this paper is to prove that under hypotheses stated in [1] one obtains the same conclusions as in [1] applying the result given in [3] (Theorem 3).

Also, it is pointed out that it cannot be used the same procedure as in [1] in order to prove local controllability if u=o is not in interior of the control range set ; such problems may be treated using [2] or [3].

§1. Throughout in this paper we shall consider the following control system

$$(1)\quad \frac{dx}{dt} = X(x)+\sum_{1}^{n-1}\alpha_i(t)Y^i(x),\quad x\in R^n,\quad x(o)=p,\quad t\geq o,$$

where $X(x)$, $Y^i(x)$ are analytical vector functions and $\alpha_i(\cdot)$, $i\in \{1,\dots,n-1\}$, are piecewise constant scalar functions with values in the fixed interval $[-1,1]$.

For such a system, sufficient conditions for local controllability along a fixed trajectory $x_o(\cdot)$ corresponding to $\alpha_i(\cdot)=0$ are given

... It is assumed that $x_o(\cdot)$ exists on the entire halfline $[0,\infty)$. Remind that local controllability means $x_o(t) \in \operatorname{int} \mathcal{A}(t,p)$ for all $t > o$, where $\mathcal{A}(t,p)$ is the set of all points in R^n that can be reached by admissible trajectories of (1) at the time t.

As one sees, the control variable in (1) appears linearly and the control range set $\mathcal{U} = \prod_1^{n-1} [-1,1]$ contains the reference control $\mathcal{L}_i = o$, $i=1,\dots n-1$ in its interior. These facts are essentially used in [1] and in what follows.

Also, in [1], the following conditions are supposed to be satisfied

(a_1) the vectors $X(p), Y^1(p), \dots, Y^{n-1}(p)$ are linearly independent

(a_2) the vector-functions $Y^1(x), \dots, Y^{n-1}(x)$ are involutive in a neighbourhood V(p), i.e. the Lie bracket $[Y^i,Y^j](x)$ verifies

$[Y^i,Y^j](x) \in \operatorname{span}\{Y^1(x), \dots, Y^n(x)\}$, for all $i,j=1,\dots,n-1$

In [2] or [3] it is supposed only the property that the vectors $Y^1(p), \dots, Y^{n-1}(p)$ are linearly independent instead of (a_1) and (a_2)

The hypothesis (a_2) contains actually the property that $Y^1(x), \dots, Y^{n-1}(x)$ generate an integral manifold in a neighbourhood of p and enable us with the advantage that one can write the sufficient conditions in [3] in the simpler form in [1] (see lemma 4).

We shall analyse only the singular trajectory $x_o(\cdot)$ because in the nonsingular case the hypotheses and the conclusions are essentially the same in all works dealing with this problem (see for example [3] and [1])

Definition 1

The trajectory $x_o(\cdot)$ is singular for (1) if the dimension of the linear space generated by the vectors $(ad^k X, Y^i)(p)$, $i=1,\dots,n-1$, $k \geq o$, k natural, is not greater then $n-1$, where $(adX,Y)(x) = [X,Y](x) = \frac{\partial Y}{\partial x}(x)\, X(x) - \frac{\partial X}{\partial x}(x)\, Y(x)$.

Proposition.

Let $x_o(.)$ be singular for (1). Then an $\varepsilon_o > o$ and $\lambda_o \in R^n$, $|\lambda_o| = 1$, will exist such that $\langle \psi(t), Y^i(x_o(t)) \rangle = o$, $t \in [o, \varepsilon_o]$, $i = 1, .., n-1$, where $\psi : [o, \varepsilon_o] \longrightarrow R^n$ is defined by $\psi(o) = \lambda_o$, $-\frac{d\psi}{dt} = (\frac{\partial X}{\partial x}(x_o(t)))\psi(t)$. In addition, if there exist $\lambda_o \in R^n$, $|\lambda_o| = 1$, and $\varepsilon_o > o$ such that $\langle \psi(t), Y^i(x_o(t)) \rangle = o$, $i = 1, .., n-1$, for all $t \in [o, \varepsilon_o]$ then $x_o(.)$ is singular for (1), where $\psi(.)$ is defined as before.

Proof.

By hypotheses of analycity of the functions $X(x)$, $Y^i(x)$ if follows that $x_o(.)$ is locally analytical in $t = o$, and therefore there exists $\varepsilon_o > o$ such that $x_o(t)$, $t \in [o, \varepsilon_o]$, is analytical. In order to verify $\varphi_i(t) \stackrel{\Delta}{=} \langle \psi(t), Y^i(x_o(t)) \rangle = o$, $i = 1, .., n-1$, for all $t \in [o, \varepsilon_o]$ it is sufficient to prove that the values of the function $\varphi_i(t)$ and all its derivatives at $t = o$ are equal zero. By singularity we have $\varphi_i(o) = o$, $i = 1, .., n-1$, and taking k-th derivative of $\varphi_i(t)$ at $t = o$ we obtain $\frac{d^k \varphi_i}{dt^k} = \langle \lambda_o, (ad^k X, Y^i)(p) \rangle$. Choosing $\lambda_o \in R^n$, $|\lambda_o| = 1$, such that λ_o is orthogonal to all vectors $(ad^k X, Y^i)(p)$, $k \geq o$, $i = 1, .., n-1$, (see definition of singularity) we obtain $\frac{d^k \varphi_i}{dt^k}(o) = o$ for all $k \geq o$ and the first part is proved.

The second part is obtained in the same way.

§2. A direct consequence of the singularity and condition (a_1) is that the system (1) is locally equivalent with another differential system

$$(2) \quad \frac{dy^1}{dt} = H(t, y, u), \qquad \frac{d\overline{y}}{dt} = \overline{F}(t, y_o(t)) + \sum_{1}^{n-1} u_j \overline{G}_j(t, y_o(t)),$$

$$(y^1(o), \overline{y}(o)) \stackrel{\Delta}{=} y(o) = Mp,$$

where the matrix M is composed by $\lambda_o, \lambda_1, .., \lambda_{n-1}$, $|\lambda_i| = 1$, mutually orthogonal, $\hat{\lambda}_0$ being given by Proposition .

The right hand side in (2) is obtained performing a change of state and control variable by the following formulas

3) $y(t)=ML^{-1}(t)x(t)$, $\frac{dL}{dt}=-\frac{\partial X}{\partial x}(x_0(t))L(t)$, $L(o)=E$ (identity)

4) $F(t,y)=ML^{-1}(t)X(L(t)M^T y)-ML^{-1}(t)\frac{\partial X}{\partial x}(x_0(t))L(t)M^T y$,

$G_i(t,y)=ML^{-1}(t)Y^i(L(t)M^T y)$, $y_0(t)=ML^{-1}(t)x_0(t)$,

5) $\overline{F}(t,y)+\sum_1^{n-1}\alpha_i(t,y,u)\overline{G}_i(t,y)=\overline{F}(t,y_0(t))+\sum_1^{n-1}u_j\overline{G}_j(t,y_0(t))$,

$H(t,y,u)=F^1(t,y)+\sum_1^{n-1}\alpha_i(t,y,u)\,G_i^1(t,y)$,

where $\overline{F}$, $\overline{G}_i$ denotes functions formed with the last (n-1) components of the functions F and respectively G_i, $F=(F^1,\overline{F})$, $G=(G^1,\overline{G})$.

Of course, the condition (5) is true in a neighbourhood of $\alpha=0$ $y=y_0(t)$, $u=o$, and for $t\geqslant o$ in a neighbourhood of zero.

By definition we have

6) $\frac{\partial \overline{F}}{\partial y}(t,y_0(t))\equiv o$, $\frac{\partial\alpha_i}{\partial y}(t,y_0(t),o)\equiv o$, $i=1,..,n-1$,

7) $\frac{\partial\alpha_i}{\partial u_j}(t,y_0(t),o)=\begin{cases}1 & i=j\\ o & i\neq j\end{cases}$, $\frac{\partial^k\alpha_i}{\partial u_j^k}(t,y_0(t),o)\equiv o$

for all $k\geqslant 2$, $i,j=1,..,n-1$.

In order to prove that (1) is locally controllable along the singular trajectory $x_0(.)$ it is sufficient to show that (2) is locally controllable along the fixed trajectory $y_0(.)$ corresponding to the control $u_i=o$, $i=1,..,n-1$.

Remark. By definition the vectors $G_i(o,y)$, $i=1,..,n-1$, are linearly independent and therefore $\overline{G}_i(o,y)$, $i=1,..,n-1$, are linearly independent.

For the general nonlinear system

(#) $\frac{dz}{dt}=f(t,z,u)$, $t>t_0$ $z\in R^n$, $u\in U\subseteq R^r$

the definition of singularity of a fixed trajectory $z_o(.)$ corresponding to $u_o(.)$ is the following (see [3]).

Definition 2.

The trajectory $z_o(.)$ is singular for (#) if there exist $\tilde{\varepsilon}_o > o$ and $\tilde{\lambda}_o \in R^n$, $|\tilde{\lambda}_o| = 1$, such that $\langle\tilde{\Psi}(t), f(t,z_o(t),u) - f(t, z_o(t), u_o(t))\rangle = o$ for all $u \in U$, $t \in [o, \tilde{\varepsilon}_o]$ where $\tilde{\Psi}(o) = \tilde{\lambda}_o$, $-\dfrac{d\tilde{\Psi}}{dt} = (\dfrac{\partial f}{\partial z}(t, z_o(t), u_o(t)))^T \tilde{\Psi}(t)$.

Definition 2 of singularity is the same with definition 1 when the nonlinear system (#) is replaced by (1).

Lemma 1

Let $x_o(.)$ be singular for (1). Then $y_o(t) = L^{-1}(t)x_o(t)$, $t \geq o$ is singular for (2) with $\tilde{\lambda}_o = (1,o,\dots,o) \equiv \tilde{\Psi}(t)$

Proof.

Choosing $\tilde{\lambda}_o^T = (1,o,..,o) \in R^n$, and $\varepsilon_o = \tilde{\varepsilon}_o > o$ given by singularity of $x_o(.)$ (see Proposition) we have $\langle\tilde{\Psi}(t), f(t,z_o(t),u) - f(t,z_o(t),u_o(t))\rangle =$

$$\sum_1^{n-1} (\alpha_i(t,y_o(t),u) - \alpha_i(t,y_o(t),o))\, G_i^1(t,y(t)) =$$

$$\sum_1^{n-1} \alpha_i(t,y_o(t),u)\, G_i(t,y_o(t)) .$$

By definition $G_i^1(t,y_o(t)) = \lambda_o^T L^{-1}(t) \; Y^{i+1}(x_o(t))$,

$i=1,\dots n-1$, where λ_o is given by singularity of $x_o(.)$.

Since $\lambda_o^T L^{-1}(t) \triangleq \Psi(t)$ verifies $-\dfrac{d\Psi}{dt} = (\dfrac{\partial X}{\partial x}(x_o(t)))^T \Psi(t)$ and $\Psi(o) = \lambda_o$ it follows $G_i^1(t,y_o(t)) = o$ for all $t \in [o, \tilde{\varepsilon}_o]$ and the proof is finished.

Using Remark 1 and Lemma 1 one obtains that the trajectory $y_o(.)$ and the system (2) verify the general hypotheses in [3] .

Let $m \geq 1$ and $k \geq o$ natural numbers. One defines the m-linear forms

$\varphi_m, \varphi_{m,k} : R^{n-1} \longrightarrow R$,

$$(8)\quad \varphi_m(u) = \langle\lambda_o, \sum_{i_1+\dots+i_{n-1}=m} \frac{u_1^{i_1} \cdots u_{n-1}^{i_{n-1}}}{(i_1)! \cdots (i_{n-1})!} (ad^{i_1} Y^1, \dots (ad^{i_{n-1}} Y^{n-1}, X)\dots)$$

(9) $$\varphi_{m,k}(u) = \langle \lambda_0, \sum_{i_1+\dots+i_{n-1}=m} \frac{u_1^{i_1}\dots u_{n-1}^{i_{n-1}}}{(i_1)!\dots(i_{n-1})!} [ad^k X, [ad^{i_1} Y^1, \dots [ad^{i_{n-1}} Y^{n-1}, X]]] (x_0) \rangle$$

where $\lambda_0 \in R^n$, $|\lambda_0| = 1$, is given by singulartiy (see Proposition).

The main result in [1] regarding local controllability along the trajectory $x_o(.)$ is stated in terms of m-linear forms $\varphi_{m,k}(u)$ and has the following containt

Theorem 1

Let the hypotheses (a_1) and (a_2) be verified.

Let $x_o(.)$ be a singular trajectory for (1). Then (1) is locally controllable along $x_o(.)$ if there exist $m^* \geq 2$, $k^* \geq o$, natural numbers such that $\varphi_{m,k}(u) \equiv o$ for all $1 \leq m < m^*$, $o \leq k$, $\varphi_{m^*,k}(u) \equiv o$ for all $o \leq k < k^*$, and $\varphi_{m^*,k^*}(u)$ changes sign on R^{n-1}

$(\varphi_{m^*,k^*}(u^1) \cdot \varphi_{m^*,k^*}(u^2) < o)$.

The final result in [3] (see Theorem 3) is expressed in terms of changing sign of a certain m-linear forms also, but the forms have different structures

Remark 2.

If (a_1) is verified then the hypothesis H in [3] (Theorem 3) for the system (2) is satisfied. It follows because $\frac{\partial H}{\partial y}(t, y_o(t), o) = o$, $\frac{\partial F}{\partial v}(t, y_o(t)) = o$, and $G_j(o,p)$, $j=1,\dots,n-1$, are linearly independent.

Remark 3.

The condition of conjugacy in [3] (see definition 3) has a simpler form for (2) because the second part of the system (2) doesn't depend on the state variable y. So, choosing as λ_θ, $\theta = 1,\dots,n$, the canonical base in R^n we have to verify the conjugacy condition only for $\theta = 1$.

Therefore, in order to verify all the hypotheses in [3] (Theorem 3) we have to prove that (i_1) and (i_2) in definition 3 in [3] ([illegible]) are satisfied for the $H_1(t,y,u) \equiv H(t,y,u)$ from the system (2).

In order to finish the analysis of the controllability between [1] and [3] (or [2]) we have to establish the connection of the n-linear form $\varphi_{m,k}(u)$ and conjugacy conditions for $H(t,y,u)$.

__Lemma 2.__

Let the condition (a_2) be verified. The for $t \geq 0$, t sufficiently small we have

$$T^i_j(t) \overset{\Delta}{=} \frac{\partial Y^i}{\partial x}(x_0(t))Y^j(x_0(t)) = \frac{\partial Y^j}{\partial x}(x_0(t))Y^i(x_0(t)) + z^1(t),$$

$$\frac{\partial^2 Y^i}{\partial x^2}(x_0(t))Y^j(x_0(t))Y^k(x_0(t)) = \frac{\partial^2 Y^j}{\partial x^2}(x_0(t))Y^i(x_0(t))Y^k(x_0(t)) + z^2(t)$$

$$\text{if } T^i_j(t) \in \text{span}\{Y^1(x_0(t)),\dots,Y^{n-1}(x_0(t))\},\ i,j = 1,\dots,n-1;$$

$$T^{i_0}_{i_1\dots i_l}(t) \overset{\Delta}{=} \frac{\partial^l Y^{i_0}}{\partial x^l}(x_0(t))Y^{i_1}(x_0(t))\dots Y^{i_l}(x_0(t)) = T^{i_1}_{i_0 i_2 \dots i_l}(t) + z^l(t)$$

$$\text{if } T^{i_0}_{i_1 \dots i_k \dots i_l}(t) \in \text{span}\{Y^1(x_0(t)),\dots,Y^{n-1}(x_0(t))\},\ 1 \leq i_k \leq n-1,$$

where $z^l(t) \in \text{span}\{Y^1(x_0(t)),\dots,Y^{n-1}(x_0(t))\}$,

and $\frac{\partial^l Y^{i_0}}{\partial x^l}(x)$ means the differential Frechét of order l of $Y^{i_0}: R^n \longrightarrow R^n$.

__Proof .__

By hypothesis $(ad\ Y^i, Y^j)(x) = \sum_1^{n-1} c_k(x)Y^k(x)$, for x in a neighbouhood V of p. Choosing $\varepsilon_0 > 0$ sufficiently small such that $x_0(t) \in V$ for $t \in [0, \varepsilon_0]$, then it is proved the first conclusion with

$$z^1(t) = -\sum_1^{n-1} c_k(x_0(t))\, Y^k(x_0(t)).$$

Using involutive hypothesis (a_2) again we obtain

$$(ad\ Y^k, (ad\ Y^i, Y^j))(x) = \sum_1^{n-1} c_p(x)Y^p(x)$$

~~$(ad\ Y^i, Y^j)(x) = -(ad\ Y^j, Y^i)(x)$ we obtain~~

~~$(ad\ Y^k, (ad\ Y^i, Y^j))(x)$ [illegible] $(ad\ Y^k, (ad\ Y^j, Y^i))(x) =$ [illegible] $(x)Y^p(x)$~~

and performing the calculus in the left hand side we obtain the second conclusion taking into account the property

$T^i_j(t) \in \text{span}\{Y^1(x_0(t)),\dots,Y^{n-1}(x_0(t))\}$

In order to obtain the last conclusion we use again (a_2) and it follows

$$(\mathrm{ad}\, Y^{i_l}, \mathrm{ad}\, Y^{i_{l-1}}, \ldots (\mathrm{ad}\, Y^{i_1}, Y^{i_0}) \ldots)(x) = \sum_1^{n-1} d_l(x)\, Y^l(x)$$

Taking into account the hypothesis $T^{i_0}_{i_1 \ldots i_l}(t) \in \mathrm{span}\{Y^1(x_0(t)), \ldots, Y^{n-1}(x_0(t))\}$ and performing the calculus in the left hand side we obtain the desired conclusion and the proof is finished.

<u>Lemma 3</u>.

Let (a_1) and (a_2) be verified. Then for every $t_1 > o$ sufficiently small and for every $u \in R^{n-1}$ in a neighbourhood of zero

a) $$j^2(y(.)) \overset{\Delta}{=} \int_0^{t_1} \Big\{ d_1^2 H(t, y_o(t), o; y(t)) + 2 \sum_1^{n-1} \bar{u}_i(t) \Big[d_1 H(t, y_o(t), e_i; y(t)) - d_1 H(t, y_o(t), o; y(t)) \Big] \Big\}\, dt = \int_0^{t_1} \sum_0^{\infty} \frac{t^k}{k!} \varphi_{2,k}(\mathcal{V}(t))\, dt$$

b) $$j^m(y(.)) \overset{\Delta}{=} \int_0^{t_1} \Big\{ d_1^m H(t, y_o(t), o; y(t)) + m! \sum_1^{n-1} \bar{u}_i(t) \Big[d_1^{m-1} H(t, y_o(t), e_i; y(t)) - d_1^{m-1} H(t, y_o(t), o; y(t)) \Big] \Big\}\, dt = \int_0^{t_1} \sum_0^{\infty} \frac{t^k}{k!} \varphi_{m,k}(\mathcal{V}(t))\, dt$$

if $j^l(y(.)) \equiv o$, $l = 2, \ldots, m-1$,

where " d_1" means differential with respect to state variable y,

$$y(t) = \int_0^t \Big(\sum_1^{n-1} \bar{u}_i(s)\, ML^{-1}(s) Y^i(x_o(s))\Big) ds, \quad \bar{u}(o) = u, \quad y(t_1) = o,$$

c) $L(t) M^T y(t) = \sum_1^{n-1} \mathcal{V}_i(t)\, Y^i(x_o(t))$, and

$e_1, \ldots e_{n-1}$ is the canonical base in R^{n-1}.

<u>Proof.</u>

By definition of $\mathcal{V}_i(.)$ it follows $\mathcal{V}_i(o) = \mathcal{V}_i(t_1) = o$ and taking first derivative from the equations defining $\mathcal{V}_i(.)$ we obtain

$$\sum_1^{n-1} \mathcal{V}_i(t) \frac{\partial X}{\partial x}(x_o(t))\, Y^i(x_o(t)) + \sum_1^{n-1} \bar{u}_i(t)\, Y^i(x_o(t)) =$$
$$\sum_1^{n-1} \mathcal{V}_i(t) \frac{\partial Y^i}{\partial x}(x_o(t))\, X(x_o(t)) + \sum_1^{n-1} \frac{d\mathcal{V}_i(t)}{dt}\, Y^i(x_o(t)),$$

and it follows

10) $$\sum_1^{n-1} \bar{u}_i(t)\, Y^i(x_o(t)) = \sum_1^{n-1} \frac{d\mathcal{V}_i}{dt}\, Y^i(x_o(t)) + \sum_1^{n-1} \mathcal{V}_i(t)\, (\mathrm{ad}\, X, Y^i)(x_o(t))$$

By definition of $\alpha_i(t,y,u)$ and $H(t,y,u)$ (see (5),(6),and (7)) using (c) we obtain

$$11)\quad d_1^2 H(t,y_0(t);o;y(t)) = \sum_{i,j=1}^{n-1}\gamma_i(t)\gamma_j(t)\,\psi^T(t)\frac{\partial^2 X}{\partial x^2}(x_0(t))Y^i(x_0(t))Y^j(x_0(t))$$

$$12)\quad d_1 H(t,y_0(t),e_i;y(t)) - d_1 H(t,y_0(t),o;\,y(t)) = \sum_{j=1}^{n-1}\gamma_j(t)\,\psi^T(t)\,\frac{\partial Y^i}{\partial x}(x_0(t))\,Y^j(x_0(t))$$

where $\psi^T(t) = \lambda_0^T L^{-1}(t)$.

Using lemma 2 from (11) and (12) we obtain

$$13)\quad E_1 \overset{\Delta}{=} 2\sum_{i=1}^{n-1}\int_0^{t_1}\bar{u}_i(t)\left[d_1H(t,\bar{y}_0(t),e_i;\,y(t)) - d_1H(t,y_0(t),o;\,y(t))\right]dt =$$
$$2\sum_{i,j=1}^{n-1}\int_0^{t_1}\gamma_i(t)\gamma_j(t)\psi^T(t)\frac{\partial Y^i}{\partial x}(x_0(t))(ad\,X,Y^j)(x_0(t))dt + \sum_{i,j=1}^{n-1}\int_0^{t_1}\left[\frac{d}{dt}(\gamma_i(t)\gamma_j(t))\right]\psi^T(t)T^i_j(t)\,dt$$

Integrating by parts in the last term of (13) it follows

$$14)\quad E_1 = 2\sum_{i,j=1}^{n-1}\int_0^{t_1}\gamma_i(t)\gamma_j(t)\psi^T(t)\frac{\partial Y^i}{\partial x}(x_0(t))(ad\,X,Y^j)(x_0(t))\,dt\;-$$
$$-\sum_{i,j=1}^{n-1}\int_0^{t_1}\gamma_i(t)\gamma_j(t)\left[-\psi^T(t)\frac{\partial X}{\partial x}(x_0(t))\frac{\partial Y^i}{\partial x}(x_0(t))Y^j(x_0(t)) +\right.$$
$$\psi^T(t)\frac{\partial^2 Y^i}{\partial x^2}(x_0(t))X(x_0(t))Y^j(x_0(t)) +$$
$$\left.+\psi^T(t)\frac{\partial Y^i}{\partial x}(x_0(t))\frac{\partial Y^j}{\partial x}(x_0(t))X(x_0(t))\right]dt$$

Computation gives

$$15)\quad \sum_{i,j=1}^{n-1}\gamma_i(t)\gamma_j(t)\psi^T(t)\frac{\partial Y^i}{\partial x}(x_0(t))\left[(ad\,X,Y^j)(x_0(t)) - \frac{\partial Y^j}{\partial x}(x_0(t))X(x_0(t))\right] =$$
$$-\sum_{i,j=1}^{n-1}\gamma_i(t)\gamma_j(t)\psi^T(t)\frac{\partial Y^i}{\partial x}(x_0(t))\frac{\partial X}{\partial x}(x_0(t))Y^j(x_0(t))$$

and (14) becomes

$$14')\quad E_1 = \sum_{i,j=1}^{n-1}\int_0^{t_1}\gamma_i(t)\gamma_j(t)\,\psi^T(t)\left[\frac{\partial Y^i}{\partial x}(ad\,X,Y^j) + \frac{\partial X}{\partial x}\frac{\partial Y^i}{\partial x}Y^j -\right.$$

$$-\frac{\partial^2 Y^i}{\partial x^2} X Y^j - \frac{\partial Y^i}{\partial x}\frac{\partial X}{\partial x} Y^j \big](x_o(t))\,dt .$$

Using (11) and (14') we obtain

16) $$j^2(y(\cdot)) = \sum_{i,j=1}^{n-1} \int_0^{t_1} \alpha_i(t)\,\alpha_j(t)\,\psi^T(t)\,(ad\,Y^j,(ad\,Y^i,X))(x_o(t))\,dt$$

and developing in series the analytical scalar function

$\psi^T(t)\,(ad\,Y^j,(ad\,Y^i,X))(x_o(t))$ it follows

17) $$j^2(y(\cdot)) = \int_0^{t_1}\Big[\sum_0^{\infty} \frac{t^k}{k!}\,\varphi_{2,k}(\alpha(t))\Big]dt .$$

This form for $j^2(y(\cdot))$ was possible because $\frac{\partial \alpha_i}{\partial y}(t, y_o(t), o) \equiv o$ $i=1,\ldots,n-1$.

In order to obtain (b) we proceed by induction making use of the same facts that were involved in establishing (a).

For (b) we need not only $\frac{\partial \alpha_i}{\partial y}(t, y_o(t), o)) \equiv o$ but

18) $$\psi^T(t)\,\frac{\partial^l Y^{i_0}}{\partial x^l}(x_o(t))\,Y^{i_1}(x_o(t)),\ldots,Y^{i_l}(x_o(t)) \equiv o \quad \text{on } [o, t_1]$$

also, for all $l=2,\ldots,m-2$, $1 \le i_k \le n-1$, $k=0,1,\ldots,l$.

This last condition is a consequence of the fact that $j^l(y(\cdot)) \equiv o$ for all $l=2,\ldots m-1$.

The property (18) allows us to neglect higher order derivatives of α_i up to order $m-2$ with respect to state variable y and $j^m(y(\cdot))$ takes the form

$$j^m(y(\cdot)) = \int_0^{t_1} \sum_{i_1,\ldots,i_m=1}^{n-1} \alpha_{i_1}(t)\ldots\alpha_{i_m}(t)\,\psi^T(t)\,\frac{\partial^m X}{\partial x^m}(x_o(t))\,Y^{i_1}(x_o(t))\ldots Y^{i_m}(x_o(t))\,dt +$$

$$m! \int_0^{t_1} \sum_{i_1,\ldots,i_{m-1}=1}^{n-1} \alpha_{i_1}(t)\ldots\alpha_{i_{m-1}}(t) \sum_{i_0=1}^{n-1} u_{i_0}(t)\,\psi^T(t)\,\frac{\partial^{m-1} Y^{i_0}}{\partial x^{m-1}}(x_o(t))\,Y^{i_1}(x_o(t))\ldots Y^{i_{m-1}}(x_o(t))\,dt$$

Now using again Lemma 2 and (10) a similar computation as in case m=2 allow us to obtain conclusion (b) and the proof is finished.

Now we are in position to state the final result.

Theorem 2

Let the hypotheses in theorem 1 are verified. Then the assumptions in theorem 3 (see [3]) are satisfied for the system (2); therefore the system (1) is locally controllable along $x_0(.)$.

Proof.

What we have to verify (see remark 3) are the condition (i_1) and (i_2) from definition 3 in [3]

Choosing $\tilde{\lambda}_0, \dots, \tilde{\lambda}_{n-1}$ the canonical base in R^n the proving of (i_1) and (i_2) is necessary to make only for first component $H(t,y,u)$ of the system (2). By hypothesis the conclusions of Lemma 3 are satisfied.

Since $\varphi_{m,k}(u) \equiv 0$ for all $1 \leq m < m^*$ and all $0 \leq k$ (see theorem 1) it follows $j^{l}(y(.)) \equiv 0$, $l=2,\dots,m-1$.

Since $\varphi_{m^*,k}(u) \equiv 0$ $0 \leq k < k^*$, and $\varphi_{m^*,k^*}(u)$ changes sign we obtain that there exist $y^1(.), y^2(.)$ with $y^1(t_1)=y^2(t_1)=0$ such that $j^m(y^1(.)) \cdot j^m(y(.)) < 0$, where $y^1(.)$ and $y^2(.)$ correspond to u^1, $u^2 \in R^{n-1}$ such that $\varphi_{m^*,k^*}(u^1) . \varphi_{m^*,k^*}(u^2) < 0$ and are defined as in Lemma 3. The proof is finisched.

Final remarks.

Hypotheses used in [1] for local controllability along a singular trajectory are too restrictive (see the conditions $0 \in \text{int}\,\mathcal{U}$ and $Y^1,\dots,Y^{n-1}$ are involutive). Choosing $n=2$, and $\mathcal{U} = \{-1,0,1\}$, then $0 \notin \text{int}\,\mathcal{U}$ and we cannot use the same procedure as before in order to obtain local controllability. In such cases we need to verify hypotheses and to use the result given in [3] or [2].

Also, if we consider as an example, the particular control system

$$\frac{dx_1}{dt} = u_1 x_3, \quad \frac{dx_2}{dt} = u_1 + u_2 x_2^2, \quad \frac{dx_3}{dt} = u_2 + u_1 x_3^2, \quad p=(0,0,0),$$

$|u_i| \leq 1, \quad i=1,2$

with reference control $u_0(.)=(0,0)$ and corresponding singular trajectory $x_0(.)=(0,0,0)$ we obtain that

$X^T=(0,0,0)$, $(Y^1)^T=(x_3,1,x_3^2)$, $(Y^2)^T=(0,x_2^2,1)$

are linearly dependent and Y^1, Y^2 are not involutive.

Theorem 1 in [1] is not applicable but Theorem 3 in [3] provide local controllability of the system considered. Choosing $\overset{1}{u}=(1,0)$, $\overset{2}{u}=(-1,1)$, $\overset{3}{u}=(-1,-1)$ and defining

$H_\theta(t,x,u)=\langle\lambda_\theta, f(x,u)\rangle$, $\tilde{H}_\theta(t,x,\alpha)=\langle\lambda_\theta, \sum \alpha_i(f(x,u_i)-f(x,u_0))\rangle$

for $\theta = 1,2,3$,

where $\lambda_1^T = (1,0,0)$, $\lambda_2^T=(0,1,0)$, $\lambda_3^T=(0,0,1)$,

$f^T(x,u)=(u_1x_3, u_1+u_2x_2^2, u_2+u_1x_3^2)$,

then we have

$$B_2^1(t)\alpha = d_x^1\tilde{H}_1(t,0,\alpha;\bar{x}(t,\alpha))=(\alpha_1-\alpha_2-\alpha_3)(\alpha_2-\alpha_3)$$

$$A_j^j(t)\alpha = d_x^j H_1(t,0,0;\bar{x}(t,\alpha))\equiv 0 \quad \text{for all } j \geq 2;$$

and it follows that $B_2^1(t)\alpha$ changes sign on R_+^3 and conditions in Theorem 3 via Theorem 4 in [3] are verified for m=2 and k=0.

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