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QUASIAFFINE TRANSFORMS OF COMPACT PERTURBATIONS OF NORMAL OPERATORS

by

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by

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1. Let H_1 and H_2 be complex Hilbert spaces. If X is a (bounded linear) operator from H_1 into H_2 such that $\ker X = \{0\}$, $\ker X^* = \{0\}$, then X is called a quasiaffinity. If A_1 and A_2 are operators on H_1 and H_2 respectively, and there exists a quasiaffinity $X : H_1 \mapsto H_2$ such that $XA_1 = A_2X$, we say that A_1 is a quasiaffine transform of A_2 and we write $A_1 \prec A_2$. If both $A_1 \prec A_2$ and $A_2 \prec A_1$, then A_1 and A_2 are said to be quasisimilar. These relations proved to be important in the study of the remarkable class of C_0 - contractions, where the relation $A_1 \prec A_2$ implies $A_2 \prec A_1$ (cf. [5], [6]), Furthermore, if H is a separable, infinite dimensional, complex Hilbert space, and Q is any quasinilpotent operator on H, it was shown in [3] that there exist compact quasinilpotent operators K_1 and K_2 on H such that $K_1 \prec Q \prec K_2$. Moreover, it is known [7] that the unilateral shift of infinite multiplicity on H is a quasiaffine transform of any cyclic nonalgebraic strict contraction on H. Aside from these facts, little is known about the relation $\prec$ and its invariants. Thus one might ask whether one can have $A_1 \prec A_2$, where A_1 and A_2 have few properties in common.

The purpose of this paper is to show that this is indeed the case. We prove that every operator on H has a quasiaffine transform belonging to the class (N+ C) consisting of all those operators T on H which can be written in the form T = N + K, where N is a normal operator and K is compact. Furthermore it turns out that K can be made as small as desired. Unfortunately it is not the case that every operator on H is quasisimilar to an operator in (N + C) as we show below (the Section 3). Along the way to the proof of our main result, stated above, we show that if T is any operator on H, whose essential spectrum $\sigma_e(T) = \{0\}$, then there exist compact operators K_1 and K_2 such that $K_1 \prec T \prec K_2$. This generalizes the above mentioned result of [3].

In the remainder of the Note, $\mathcal{L}(H)$ will denote the algebra of all (bounded linear) operators on H. The ideal of compact operators in $\mathcal{L}(H)$ will be denoted by

$\mathcal{LC}(H)$, while the Calkin algebra $\mathcal{L}(H)/\mathcal{LC}(H)$ will be denoted by $\mathcal{A}(H)$. For $T \in \mathcal{L}(H)$, $\tilde{T}$ will denote its image in $\mathcal{A}(H)$. Let us also recall that by the essential spectrum $\sigma_e(T)$ of T we mean the (usual) spectrum of $\tilde{T}$ in $\mathcal{A}(H)$; also by I we shall denote the identity operator on any Hilbert space.

2. We begin with the following :

PROPOSITION 2.1. If $T \in \mathcal{L}(H)$ and $\sigma_e(T) = \{0\}$, then there exist compact operators K_1 and K_2 (on some appropriate Hilbert spaces) such that $K_1 \prec T \prec K_2$.

<u>Proof</u>. We first observe that it suffices to prove that there exist a compact K_2 acting on some Hilbert space H_2 and an operator $X : H \mapsto H_2$ such that $XT = K_2X$ and $\ker X = \{0\}$. (For if $\ker X^* \neq \{0\}$ we can simply replace X by the operator $X' : H \mapsto (\text{range } X)^-$, and the other relation is obtained by the same argument applied to T^* instead of T.) Next recall that by virtue of [8] (or of the main theorem of [1]) we can write $T = T_0 + K_0$, where T_0 is quasinilpotent and K_0 is compact. Furthermore, by virtue of Theorem 1 of [3], there exist an invertible operator $X_0 : H \mapsto H'$ (where H' is a subspace of $H \oplus H \oplus \ldots$) and a quasinilpotent compact operator K on H such that

$$(K \oplus K \oplus \ldots.) H' \subset H'$$

and

$$X_0 T_0 = (K \oplus K \oplus \ldots.) X_0.$$

Now let P_0 denote the (orthogonal) projection of $H \oplus H \oplus \ldots..$ onto the subspace H', and define the compact operator

$$K_0' : H \oplus H \oplus \ldots\ldots. \mapsto H'$$

by $K_0' = X_0 K_0 X_0^{-1} P_0$. Then we can write

$$X_0 T = X_0 (T_0 + K_0) = [(K \oplus K \oplus \ldots\ldots) + K_0'] X_0. \tag{1.1}$$

Therefore if we denote by P_j the orthogonal projection of $H \oplus H \oplus \ldots.$ onto the subspace formed by all vectors whose first j components are equal to 0, we shall have

$$\epsilon_j = \| K_0' P_j \| \to 0 \qquad (j \to \infty). \tag{1.2}$$

Since K is quasinilpotent, it follows from a theorem of Rota [4] that for every $n = 1, 2, \ldots.$, there exists an invertible operator Q_n acting on H, such that $K_n = Q_n K Q_n^{-1}$ satisfies $\| K_n \| \leq \frac{1}{n}$. Clearly K_n is also compact and Q_n can be chosen such that $\| Q_n \| \leq 1$. We define now by recurrence a sequence $1 \leq j_1 < j_2 < \dots$ of integers such that ~~(1.3) (sup~~

(1.3) $$\Big(\sup_{1 \leq l \leq n} \| Q_l^{-1} \| \Big) \varepsilon_{j_n} < \frac{1}{n^2} \qquad (n = 1, 2, \dots)$$

and subsequently we define the operator Y on $H \oplus H \oplus \dots$ by $Y = Y_1 \oplus Y_2 \oplus \dots$, where

(1.4) $$\begin{cases} Y_i = I & \text{for } 1 \leq i \leq j_1 \\ Y_i = Q_n & \text{for } j_n < i \leq j_{n+1} \end{cases} \qquad (n = 1, 2, \dots).$$

Then Y is a quasiaffinity on $H \oplus H \oplus \dots$, and

$$Y(K \oplus K \oplus \dots) = (T'_1 \oplus T'_2 \oplus \dots)\, Y,$$

where

$$\begin{cases} T'_i = K & \text{for } 1 \leq i \leq j_1 \\ T'_i = K_n & \text{for } j_n < i \leq j_{n+1} \end{cases} \qquad (n = 1, 2, \dots).$$

Since each T'_i is compact and ~~T_1 is compact and~~ $\| T'_i \| \to 0$ $(i \to \infty)$, we infer that $K'_2 = T'_1 \oplus T'_2 \oplus \dots$ is a compact operator. On the other hand, $Y^{-1}(P_{j_{n+1}} - P_{j_{n+1}})$ is a compact operator, and moreover, by (1.3-4) we have

$$\| YK'_o Y^{-1}(P_{j_n} - P_{j_{n+1}}) \| = \| YK'_o P_{j_n} Y^{-1}(P_{j_n} - P_{j_{n+1}}) \| \leq \cancel{\varepsilon_{j_n} \| Q_n^{-1} \|} \leq$$

$$\leq \frac{1}{n^2} \qquad (n = 1, 2, \dots).$$

Therefore we infer that

$K''_2 = Y K'_o Y^{-1}(I - P_{j_1}) + \sum_{j=1}^{\infty} YK'_o Y^{-1}(P_{j_n} - P_{j_{n+1}})$ is a compact operator, that $YX_o T = K_2 YX_o$ and that $\ker YX_o = \{0\}$. This completes the proof.

COROLLARY 2.2. Let T be an algebraic operator on H. Then T is quasisimilar to an operator belonging to the class (N+C).

<u>Proof</u>. Since the spectrum $\sigma(T)$ of T is finite it is easy to see that T is similar to an operator T' of the form

$$T' = (\lambda_1 I + Q_1) \oplus \dots \oplus (\lambda_m I + Q_n),$$

where $\sigma(T) = \{\lambda_1, \ldots, \lambda_n\}$ and $Q_1, \ldots, Q_n$ are nilpotent operators on some suitable Hilbert spaces. By [3], Thm. 4, each Q_j $(1 \leq j \leq n)$ is quasisimilar to a compact operator K_j. It follows that T' (and therefore T also) is quasisimilar with

$$T'' = (\lambda_1 I + K_1) \oplus \ldots\ldots \oplus (\lambda_n I + K_n) = \lambda_1 I \oplus \ldots . \oplus \lambda_n I + $$
$$+ K_1 \oplus \ldots\ldots \oplus K_n,$$

where obviously the first operator is normal, while the second is compact.

3. Unfortunately, the preceding corollary is not valid for every operator T on H. Indeed, in [3], Thm. 5, we exhibited a quasinilpotent operator T on H that does not commute with any nonzero compact operator. That this operator T is not similar to any operator belonging to the class (N + C) follows directly from the following :

PROPOSITION 3.1. If a quasinilpotent operator T on H is quasisimilar to an operator belonging to the class (N+C), then T commutes with a nonzero compact operator.

Proof. Let $A : H \mapsto H_1$ and $B : H_1 \mapsto H$ be quasiaffinities such that

$$TA = A\,(N+K), \quad BT = (N+K)\,B, \tag{3.1}$$

where H_1 denotes the space on which operate N and K, and where N is normal and K is compact. Clearly we can (and we shall) assume that $H_1 = H$. Let E_n denote the spectral projector of N corresponding to the plane set $\sigma_n = \{\lambda \in \mathbb{C};\, |\lambda| \geq \frac{1}{n}\}$ $(n = 1, 2, \ldots)$. If rank $E_n = +\infty$, there exists an isometry V_n on H such that $E_n = V_n V_n^*$. It is clear that $N_n = V_n^* N V_n$ is normal, that $\sigma(N_n) \subset \sigma_n$ and that

$$TAV_n = AV_n N_n + AKV_n,$$

whence, passing to the Calkin algebra $\mathcal{A}(H)$,

$$\boxed{(3.2.)} \quad \tilde{T}\,\widetilde{AV}_n = \widetilde{AV}_n\, \tilde{N}_n .$$

But since

$\sigma(\tilde{T}) = \sigma_e(T) \subset \sigma(T) = \{0\}$, $\sigma(\tilde{N}_n) = \sigma_e(N_n) \subset \sigma(N_n) \subset \sigma_n$, the relation (3.2) implies $\widetilde{AV}_n = 0$, thus also

$$\boxed{(3.3)} \quad A E_n = 0.$$

Obviously (3.3) is valid also if rank $E_n < +\infty$; thus (3.3) is always valid for all $n = 1, 2, \ldots$. Therefore

$$\|\widetilde{AN}\| \leq \|AN - ANE_n\| + \|\widetilde{ANE_n}\| \leq \|A\| \cdot \|N - NE_n\| + \|\widetilde{AE_n}\widetilde{N_n}\| =$$

$$= \|A\| \cdot \|N - NE_n\| \leq \frac{1}{n} \|A\| \to 0 \quad (n \to \infty),$$

so that AN is compact. It follows that $X = A\,(N+K)\,B$ is also compact and obviously $XT = TX$. If $X = 0$, then $N + K = 0$ and $T = 0$ so the proof is complete.

4. The results of the preceding two sections are supplements to those of [3]. In this section we shall give some useful supplements to the results of [7].

LEMMA 4.1. Let $T \in \mathcal{L}(H)$, $\|T\| < 1$, and let S be a unilateral shift of multiplicity 1. If T is not polynomially compact (i.e. if there is no polynomial $p(\lambda) \neq 0$ such that $p(T)$ is compact) then there exists an operator T_0 on a suitable (separable) Hilbert space, H_0 such that

$$(4.1.) \quad \|T_0\| < 1, \quad T_0 \prec T, \quad T_0 \oplus S \prec T.$$

Proof. We take $1 < \beta < 1/\|T\|$ and set $T_1 = \beta T$. Let $\{f_1, f_2, \ldots\}$ be a dense set in H. Then for every f_j, the map

$$A_j : \varphi \longmapsto \varphi(T_1) f_j \qquad (\varphi \in H^2)$$

is a compact (linear) operator from the classical Hardy space H^2 into H such that $A_j S = T_1 A_j$, where S is identified with the multiplication $\varphi(\lambda) \longmapsto \lambda\varphi(\lambda)$ $(|\lambda| < 1; \varphi \in H^2)$ on H^2. Therefore

$$K(\varphi_1 \oplus \varphi_2 \oplus \ldots) = \sum_{j=1}^{\infty} \frac{1}{2^j} (\|A_j\| + 1)^{-1} A_j \varphi_j \qquad (\varphi_1 \oplus \varphi_2 \oplus \ldots \in H^2 \oplus H^2 \oplus \ldots)$$

is a compact operator from $H^2 \oplus H^2 \oplus \ldots\ldots$ into H, such that

$$T_1 K = K \cdot (S \oplus S \oplus \ldots\ldots)$$

and

$$[(K \mid (H^2 \oplus H^2 \oplus \ldots\ldots)]^- = H$$

(since the range of K contains the set $\{f_1, f_2, \ldots\}$). We set

$$H_o = (H^2 \oplus H^2 \oplus \quad \ldots\ldots) \ominus \ker K,$$

and, denoting by P_o the (orthogonal) projection of $H^2 \oplus H^2 \oplus \quad \ldots\ldots$ onto H_o, we define

$$T_o = \frac{1}{\beta} P_o (S \oplus S \oplus \ldots\ldots) \mid H_o, \quad K_o = K \mid H_o.$$

Then K_o is a quasiaffinity and

$$(4.2) \qquad \|T_o\| < 1, \quad TK_o = K_o T_o .$$

This proves the first two assertions of (4.1). Now for an arbitrary element $f \neq 0$ of H we consider again the (compact linear) operator

$$A_f : \varphi \mapsto \varphi(T) f$$

from H^2 into H, which satisfies $A_f S = TA_f$. If there exists $f \in H$, $f \neq 0$, such that

$$(4.3) \qquad A_f H^2 \cap K_o H_o = \{0\}, \ \ker A_f = \{0\},$$ then upon setting

$$X(h_o \oplus \varphi) = K_o h_o + A_f \varphi \qquad (h_o \oplus \varphi \in H_o \oplus H^2),$$

we obtain a quasiaffinity $X : H_o \oplus H^2 \mapsto H$ such that $TX = X(T_o \oplus S)$, i.e. the third assertion in (4.1.) Hence it remains to consider the case when (4.3) fails for every $f \in H$, $f \neq 0$. This means that for every nonzero $f \in H$, there exist $\varphi_f \in H^2$, $\varphi_f \neq 0$, and $h_f \in H_o$, such that

$$\boxed{(4.4)} \qquad \varphi_f(T) f = K_o h_f .$$

Let

$$\varphi_f(\lambda) = p_f(\lambda) q_f(\lambda) \qquad (|\lambda| < 1),$$

where $p_f(\lambda)$ is a polynomial and $q_f \in H^2$ satisfies $q_f(\lambda) \neq 0$ for all complex λ, $|\lambda| < (1/2) \max\{\|T\|, \|T_o\|\} + (1/2)$.

Then $q_f(T)^{-1}$ and $q_f(T_o)^{-1}$ exist and consequently

$$p_f(T) f = q_f(T)^{-1} K_o h_f = K_o q(T_o)^{-1} h_f .$$

Thus in (4.4) we can assume that φ_f is a polynomial. Let $F_n \subset H$ $(n = 1, 2, \ldots.)$ be the set of those $f \in H$, $\|f\| = 1$, for which there exists a polynomial

$\varphi_f(\lambda) = \sum_{j=0}^{n} a_j \lambda^j$ in (4.4) such that

$$\boxed{1 \leq \sum_{j=0}^{n} |a_j| \leq n} \longrightarrow \}, \quad \|h_f\| \leq n.$$

Obviously F_n is closed, and by our negative assumption

$$\bigcup_{n=1}^{\infty} F_n = \{ f \in H, \ \|f\| = 1 \}.$$

By virtue of the Baire Theorem there exists a "ball" $B = \{ f \in H; \ \|f\| = 1, \ \|f - f_o\| < \rho \} \subset F_n$, where $f_o \in H$, $\|f_o\| = 1$, $0 < \rho < 1$ and n are suitably chosen. Let now $g \in H$, $g \neq 0$. Then

$$f = \frac{f_o + \frac{\rho}{3} \|g\|^{-1} g}{\left\| f_o + \frac{\rho}{3} \|g\|^{-1} g \right\|} \in B,$$

so that by chosing conveniently the polynomials φ_f and φ_{f_o}, the polynomial $\psi(\lambda) \equiv \varphi_f(\lambda)\varphi_{f_o}(\lambda)$ is of degree $\leq 2n$, and

$$\psi(T) g = K_o \left\{ \frac{3}{\rho} \|g\| \left[\|f_o + \frac{\rho}{3} \|g\|^{-1} g\| \varphi_{f_o}(T_o) h_f - \varphi_f(T_o) h_{f_o} \right] \right\}.$$

Thus we can assume also that the polynomial $\varphi_f(\lambda) \not\equiv 0$ occuring in (4.4) can always be chosen of degree $\leq N = 2n$. Let $\mathring{H}$ denote the linear quotient space $H/K_o H_o$ and let $\mathring{T}$ denote the linear transformation on $\mathring{H}$ induced by T. Obviously

(4.5) $\qquad \varphi(\mathring{T}) f = 0$

for all $f \in H$, $f \neq 0$ and some adequate polynomial $\varphi = \varphi_f \neq 0$ of degree $\leq N$. For any $\mathring{f} \in \mathring{H}$, $\mathring{f} \neq \mathring{0}$, among all nonzero polynomials occuring in (4.5), there exists a unique polynomial $\varphi_{\mathring{f}}$, with leading coefficient equal to 1 and dividing all the others. Clearly the degree of $\varphi_{\mathring{f}}$ is $\leq N$. Let N_o $(\leq N)$ be the greatest degree of these polynomials $\varphi_{\mathring{f}}$ and let $\varphi_{\mathring{f}_o}$ be of degree N_o. Let now $\mathring{f} \in \mathring{H}$, $\mathring{f} \neq \mathring{0}$ be arbitrary. Then the linear space $\mathfrak{M}$ spanned by the elements $\mathring{f}_o$, $\mathring{T}\mathring{f}_o$, $\mathring{T}^2 \mathring{f}_o$, ..., $\mathring{f}$, $\mathring{T}\mathring{f}$, $\mathring{T}^2\mathring{f}$, ... is of (linear) dimension $\leq 2N_o$. Therefore applying classical linear algebra arguments to $\mathring{T}/\mathfrak{M}$ we easily obtain that $\varphi_{\mathring{f}}$ divides $\varphi_{\mathring{f}_o}$, so that $\varphi_{\mathring{f}_o}(\mathring{T})\mathring{f} = \mathring{0}$. This means that for the polynomial $\varphi_o = \varphi_{\mathring{f}_o}$ $(\neq 0)$ we have

(4.6) $\qquad \varphi_0(T)\, H \subset K_0\, H_0.$

By the closed graph theorem, we infer from (4.6) that there exists an operator Z mapping H into H_0, such that $\varphi_0(T) = K_0 Z$, hence that T is polynomially compact. The proof is thus complete.

PROPOSITION 4.2. If $T \in \mathcal{L}(H)$, $\|T\| < 1$ and T is not polynomially compact, then T has a quasiaffine affine transform which is a unilateral shift of infinite multiplicity.

Proof. Let S be as in the proof of the preceding lemma. We apply first this lemma to βT, where $\beta = \frac{1}{2}(1 + \|T\|)$. We obtain thus an operator T_0 acting on a suitable Hilbert space H_0 such that $\|T_0\| < \beta$ and such that

$$T_0 \oplus \beta S \prec T.$$

By virtue of [8], Lemma, p. 31, we have

$$S \oplus S \oplus \ldots\ldots \prec \beta S$$

thus also

$$T_0 \oplus (S \oplus S \oplus \ldots\ldots\ldots) \prec T. \tag{4.7}$$

We next choose a sequence $\{f_1, f_2, \ldots\}$ dense in H_0 and denote by A_j $(j = 1, 2, \ldots)$ the operator from H^2 into H_0 defined by

$$A_j : \varphi \longmapsto \varphi(T_0) f_j \qquad (\varphi \in H^2);$$

finally we define $B_j = (1 + \|A_j\|)^{-1} \, 2^{-j} A_j$ $(j = 1, 2, \ldots)$. For convenience in what follows, we shall write $S \oplus S \oplus \ldots$ as

$$S_\infty = \bigoplus_{j,k=1}^{\infty} S_{jk} \qquad \text{(where } S_{jk} = S \text{ for all } j, k)$$

acting on

$$H_\infty = \bigoplus_{j,k=1}^{\infty} H_{jk} \qquad \text{(where } H_{jk} = H^2 \text{ for all } j, k).$$

For $\bigoplus_{j,k=1}^{\infty} \varphi_{jk} \in H_\infty$, we set

$$X\Big(\bigoplus_{j,k=1}^{\infty} \varphi_{jk}\Big) = \Big(\sum_{j,k=1}^{\infty} \frac{1}{jk} B_j \varphi_{jk}\Big) \oplus \bigoplus_{j,k=1}^{\infty} \frac{1}{jk} \varphi_{jk}.$$

It is clear that X is an injective (bounded linear) operator from H_∞ into $H_o \oplus H_\infty$, and that

$$(T_o \oplus S_\infty)X = X S_\infty .$$

Thus, in order to complete the proof it remains only to prove that

$$(XH_\infty)^- = H_o \oplus H_\infty .$$

For this purpose, let $u = h_o \oplus \bigoplus_{j,k=1}^{\infty} \varphi_{jk} \in H_o \oplus H_\infty$ be orthogonal to XH_∞. Then, in particular,

$$0 = (h_o, B_j \varphi) + (\varphi_{jk}, \varphi)$$

for all $\varphi \in H^2$ and $j,k = 1,2,\ldots.$, whence

$$\varphi_{jk} = - B_j^* h_o \qquad (j,k = 1,2,\ldots.) . \tag{4.8}$$

Since

$$\sum_{k=1}^{\infty} \|\varphi_{jk}\|^2 < \infty \qquad (j = 1,2,\ldots.),$$

we infer, by virtue of (4.8), that

$$B_j^* h_o = 0 \qquad (j = 1,2,\ldots.). \tag{4.9}$$

In particular, (4.9) implies that

$(h_o, f_j) = (h_o, B_j \mathbb{1})(1 + \|A_j\|) = (B_j h_o, \mathbb{1})(1 + \|A_j\|) = 0$ for all $j = 1,2,\ldots.$ and therefore $h_o = 0$. Thus, by (4.8), we have also $u = 0$. The proof is now complete.

REMARK 4.3. It appears likely that the conclusion of Proposition 4.2 remains valid if we replace the assumption that the operator T is not polynomially compact with the weaker assumption that T is nonalgebraic (i.e. that there is no polynomial $p(\lambda) \neq 0$ such that $p(T) = 0$).

5. We pass now to our main result.

THEOREM 5.1. For any $T \in \mathcal{L}(H)$ and any $\varepsilon > 0$ there exists a quasi-affine transform $N + K$ of T, where N is a normal operator and K is a compact operator such that $\|K\| \leq \varepsilon$.

Proof. We begin by proving first the theorem for the particular case in which T is polynomially compact. In this case, since $\sigma_e(T)$ is finite, T is similar to an operator of the form

$$(\lambda_1 I + Q_1) \oplus \ldots\ldots \oplus (\lambda_n I + Q_n)$$

where the λ_j's are mutually distinct complex numbers, while the operators Q_j satisfy $\sigma_e(Q_j) = \{0\}$ $(j = 1, \ldots\ldots, n)$. Thus it is sufficient to prove the statement for each operator Q_j $(j = 1, \ldots, n)$. By virtue of Proposition 2.1 any such operator has a compact quasiaffine transform. Therefore the case we are now considering is reduced to that of a compact operator. Using the spectral decomposition of T we easily obtain that T is similar to an operator of the form $T_1 \oplus T_2$ where T_1 acts on a finite dimensional space while the spectral radius of T_2 is $\leq \frac{\varepsilon}{2}$. By virtue of a theorem of Rota [4], T_2 is similar to an operator T'_2 of norm $\leq \varepsilon$ and obviously T'_2 is also compact. On the other hand, since T_1 acts on a finite dimensional space, it is clear that there exists a normal operator N_1 and an operator T'_1 similar to T_1 such that $\|T'_1 - N_1\| \leq \varepsilon$. Therefore T is quasisimilar (even similar) to N + K where $N = N_1 \oplus 0$ is normal and $K = (T'_1 - N_1) \oplus T'_2$ is compact with norm $\leq \varepsilon$.

Turning now to the case in which T is not polynomially compact, the theorem reduces, by Proposition 4.2, to the case $T = S \oplus S \oplus \ldots\ldots$, where S is a shift of multiplicity 1. If the theorem is true for S, then for every $n = 1, 2, \ldots.$, there exist a quasiaffinity X_n from a suitable Hilbert space H_n into H^2, and an operator $T_n = N_n + K_n$ on H_n, where N_n is normal and K_n is compact, such that

$$\|K_n\| \leq \frac{\varepsilon}{2^n} \qquad , \; SX_n = X_n T_n .$$

Setting

$$N = \bigoplus_{n=1}^{\infty} N_n, \quad K = \bigoplus_{n=1}^{\infty} K_n \quad \text{and} \quad X = \bigoplus_{n=1}^{\infty} (1 + \|X_n\|)^{-1} X_n,$$

we obtain $TX = X(N + K)$, where X is a quasiaffinity, N is normal and K is a compact operator satisfying $\|K\| < \varepsilon$. Thus it remains to prove the theorem for the unilateral shift of multiplicity one.

Concerning the unilateral shift of multiplicity one, we shall work from now on with its canonical realization on l^2, namely

$$Sx = (0, \alpha_1, \alpha_2 \ldots) \text{ for all } x = (\alpha_1, \alpha_2, \alpha_3, \ldots\ldots) \in l^2.$$

Also we shall consider operators T and X on l^2 defined by

$$Tx = (0, w_1\alpha_1, w_2\alpha_2, \ldots\ldots) \text{ and } Xx = (\xi_1\alpha_1, \xi_2\alpha_2, \ldots\ldots)$$

for all $x = (\alpha_1, \alpha_2, \alpha_3, \ldots) \in \ell^2$, where $\{w_n\}_{n=1}^{\infty}$ and $\{\xi_n\}_{n=1}^{\infty}$ are some bounded sequence of positive members, which will be chosen latter. Obviously X is a quasiaffinity; moreover SX = XT if and only if $\xi_n = \xi_{n+1} w_n$ for all $n = 1, 2, \ldots$, that is

$$\xi_{n+1} = \frac{\xi_1}{w_1 w_2 \cdots w_n} \qquad (n = 1, 2, \ldots).$$

Consequently, T is a quasiaffine transform of S whenever

$$\text{(5.1)} \qquad \inf \{ w_1 w_2 \ldots\ldots w_n \,;\, n = 1, 2, \ldots \} > 0.$$

Let $\varepsilon > 0$ be fixed. It is sufficient to construct the bounded sequence $\{w_n\}_{n=1}^{\infty}$ such that (5.1) holds and that T = N + K, where N and K are normal and respectively compact operators on ℓ^2, and where $\|K\| \leq \varepsilon$. For this purpose let $\{p_n\}_{n=1}^{\infty}$ be any (strictly) increasing sequence of integers divisible by 8 and such that

$$\text{(5.2)} \qquad p_1 \geqslant \frac{40401}{\varepsilon^2} + 256 .$$

Obviously we can chose by recurrence another sequence $\{q_n\}_{n=1}^{\infty}$ of integers > 0 such that

$$\text{(5.3)} \qquad \left(\frac{p_1 + 1}{p_1}\right)^{q_1} \left(\begin{array}{c} \dfrac{p_1!}{p_1} \\ p_1 \end{array}\right)^2 \cdots\cdots \left(\frac{p_k + 1}{p_k}\right)^{q_1} \left(\begin{array}{c} \dfrac{p_k!}{p_k} \\ p_k \end{array}\right)^2 \geqslant 1$$

for all $k = 1, 2, \ldots\ldots$ Also we denote $m_o = 0$ and $m_k = 2\,(p_1 + \ldots + p_k) + q_1 + \ldots + q_k$ $(k = 1, 2, \ldots)$. Finally we define

$$w_j = \begin{cases} \dfrac{j - m_k}{p_{k+1}} & \text{for } m_k + 1 \leqslant j \leqslant m_k + p_{k+1} \\[2ex] \dfrac{p_{k+1} + 1}{p_{k+1}} & \text{for } m_k + p_{k+1} + 1 \leqslant j \leqslant m_k + p_{k+1} + q_{k+1} \\[2ex] \dfrac{m_{k+1} + 1 - j}{p_{k+1}} & \text{for } m_k + p_{k+1} + q_{k+1} + 1 \leqslant j \leqslant m_{k+1}, \end{cases}$$

where $k = 0,1,2,\ldots.$ Then the infimum in (5.1) coincides with

$$\inf \left\{ \left(\frac{p_1+1}{p_1}\right)^{q_1} \binom{\frac{p_1}{p_1}}{p_1}^2 \cdots\cdots \left(\frac{p_k+1}{p_k}\right)^{q_k} \binom{\frac{q_k}{p_k}}{p_k}^2 ; \quad \boxed{k = 1,2,\ldots\ \}} \right. ,$$

so that, by (5.3), it is $\geqslant 1 > 0$. Thus (5.1) is fulfilled. Let now K' be the operator defined by

$$K'(\alpha_1, \alpha_2, \ldots) = (0, k_1\alpha_1, k_2\alpha_2, \ldots) \text{ for all } (\alpha_1, \alpha_2, \ldots) \in \ell^2,$$

where

$$\begin{cases} k_j = 0 & \text{for } j \notin \{m_k\}_{k=1}^{\infty} \\ k_{m_k} = w_k & \text{for } k = 1,2,\ldots\ldots. \end{cases}$$

Obviously K' is a compact operator on ℓ^2 such that

(5.4) $\quad \|K'\| \leq \dfrac{1}{p_1}$.

Moreover T-K' is an orthogonal sum of operators T_k $(k = 1,2,\ldots)$ acting on $\mathbb{C}^{m_k - m_{k-1}}$ by the matrix (where obviously the entry $\dfrac{p_k+1}{p_k}$ occurs q_k-times).

$$\begin{pmatrix}
0 \\
\frac{1}{p_k} & 0 \\
& \frac{2}{p_k} & 0 \\
& & \ddots & \ddots \\
& & & \ddots & 0 \\
& & & & 1 & 0 & & & & & & \text{\huge 0} \\
& & & & & \frac{p_k+1}{p_k} & 0 \\
& & & & & & \frac{p_k+1}{p_k} & 0 \\
& & & & & & & \ddots & \ddots \\
& & & & & & & & \ddots & 0 \\
& & & & & & & & & \frac{p_k+1}{p_k} & 0 \\
& & \text{\huge 0} & & & & & & & & 1 & 0 \\
& & & & & & & & & & & \frac{p_k-1}{p_k} & 0 \\
& & & & & & & & & & & & \ddots & \ddots \\
& & & & & & & & & & & & & \frac{3}{p_k} & 0 \\
& & & & & & & & & & & & & & \frac{2}{p_k} & 0
\end{pmatrix}$$

~~q_k - times~~ By virtue of a remarkable theorem of Berg ([2], Thm. 1.1), there exists a normal operator N_k on $\mathbb{C}^{m_k - m_{k-1}}$ such that

$$\|T_k - N_k\| \leq \frac{200}{\sqrt{p_k}} \longrightarrow 0 \qquad (k \to \infty).$$

Thus setting $K_k = T_k - N_k$ and $K'' = K_1 \oplus K_2 \oplus \ldots\ldots$ we obtain that K'' is compact and

$$(5.5) \qquad \|K''\| \leq \frac{200}{\sqrt{p_1}}.$$

It follows that $T = (N_1 \oplus N_2 \oplus \ldots.) + K' + K''$, where the explicit orthogonal sum is normal, and $K = K' + K''$ is a compact operator which by virtue of (5.2), (5.4) and (5.5), is of norm $\leq \varepsilon$. The proof is now complete.

REMARK 5.2. In Theorem 5.1 we cannot get rid of the compact operator K. Indeed, if S is any unilateral shift and N any normal operator, then there is no non-zero operator X satisfying SX = XN. This follows at once from the fact that if X satisfies the preceding relation, then for any element h of the space on which acts S, ~~K and X~~ we have

$$\|XX^*h\| = \|S^n XX^*h\| = \|XN^nX^*h\| \leq \|X\| \cdot \|N^nX^*h\| = \|X\| \cdot \|N^{*n}X^*h\| =$$
$$\cancel{\quad N^{*n} X^* h \quad} = \|X\| \cdot \|X^*S^{*n}h\| \leq \|X\| \cdot \|X^*\| \cdot \|S^{*n}h\| \to 0$$

$(n \to \infty)$.

REMARK 5.3. Let T be the weighted shift constructed during the proof of Theorem 5.1. Then T = N + K, N is normal, K is compact and $\|K\| \leq \varepsilon$. Moreover if $TX_1 = X_1N_1$ for some operators X_1 and N_1, where N_1 is normal, then $X_1 = 0$. Indeed, there exists a quasiaffinity X such that SX = XT, where S is the canonical unilateral shift on ℓ^2 (see the final part of the proof of Theorem 5.1). Therefore, by virtue of Remark 5.2, $XX_1 = 0$, whence $X_1 = 0$. Finally let us remark also that

$$(5.6) \qquad \sigma_e(T) = \{\lambda \in \mathbb{C} ;\ |\lambda| \leq 1\}.$$

To prove (5.6) we notice that $T^*X^* = X^*S^*$ and consequently that the point spectrum $\sigma_p(T^*)$ of T^* contains that of S^*, that is

$$\sigma_p(T^*) \supset \{\lambda \in \mathbb{C} ; |\lambda| < 1\}.$$

Since $SX = XT$, T has no point spectrum, so that (T being in the class (N+C)), $\{\lambda \in \mathbb{C}; |\lambda| < 1\}$ must necessarily be included in $\sigma_e(T)$.

Therefore in order to prove (5. 6) it will be sufficient to verify that $\|\tilde{T}\| \leq 1$. Or if T' denotes the weighted shift whose weights w'_n are defined by $w'_n = \min\{w_n, 1\}$ (n = 1,2,....), where w_n are the weights of T, then $\|T'\| \leq 1$ and obviously $T - T'$ is compact, being the orthogonal sum of finite rank operators of norms $\longrightarrow 0$. Thus $\|\tilde{T}\| = \|\tilde{T}'\| \leq \| T'\| \leq 1$.

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