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ON INTERTWINNING DILATIONS III

by

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ON INTERTWINING DILATIONS. III

by

Zoia Ceauşescu and Ciprian Foiaş

Abstract. In this Note we give a partial extension of the main uniqueness result of [1]; by virtue of this extension, we also give a characterization of the uniqueness case in the remarkable lifting theorem of [3].

1. Let $\underline{H}_j (j = 1,2)$ be some (complex) Hilbert spaces and let $L(\underline{H}_2,\underline{H}_1)$ denote the space of all (bounded linear) operators from $\underline{H}_2$ into $\underline{H}_1$. If $\underline{H}_1 = \underline{H}_2 = \underline{H}$, then $L(\underline{H}_2,\underline{H}_1)$ will be denote simply $L(\underline{H})$. For $T_j \in L(\underline{H}_j)$ $(j = 1,2)$, $I(T_1,T_2)$ will denote the set of all $A \in L(\underline{H}_2, \underline{H}_1)$ intertwining T_1 with T_2, that is $T_1A=AT_2$. For a contraction $T_j \in L(\underline{H}_j)$, let $U_j \in L(\underline{K}_j)$ denote its minimal isometric dilation, and let P_j denote the orthogonal projection of the Hilbert space $\underline{K}_j$ onto its subspace $\underline{H}_j (j = 1,2)$.
By a contractive intertwining dilation (denoted in the sequel by

CID) of a contraction $A \in I(T_1, T_2)$ we mean a contraction $B \in I(U_1, U_2)$ such that

$$P_1 B = AP_2. \tag{1.1}$$

The fact that there exists at least one CID for any contraction $A \in I(T_1, T_2)$ is known since 1968 ([7]; see also [5], sec.II.2). The characterization of the case in which this CID is unique was recently given in [1] and can be stated as follows:

<u>There exists exactly one CID of a contraction</u> $A \in I(T_1, T_2)$ <u>if and only if any of the factorizations</u> $A.T_2$ <u>and</u> $T_1.A$ <u>of</u> $AT_2 = T_1 A$ <u>is regular</u> (in the sense of [5], sec.VII.3).

In the present note we shall give a partial extension of this result (see Proposition 2.1 below) and we shall apply this extension to the study of the uniqueness of the liftings yielded by a recent interesting theorem of [3] (conjectured in [4]).

We take this opportunity to express our warm thanks to Prof.B.Sz.-Nagy for his encouraging interest in this research and to Prof.T.Ando for his useful remarks on it.

Before finishing this section, let us recall some notations and the definition of a regular factorization. For a contraction $A \in L(\underline{A}, \underline{A}_*)$, D_A will denote the operator $(I-A^*A)^{1/2}$ and $\underline{D}_A$ will denote the closure of the range of D_A. A factorization $A = A_2.A_1$ of a contraction $A \in L(\underline{A}, \underline{A}_*)$ into the product of two contractions $A_2 \in L(\underline{B}, \underline{A}_*)$ and $A_1 \in L(\underline{A}, \underline{B})$ is called <u>regular</u> if (see [5], p. 294).

(1.2) $\{D_{A_2}A_1 a \oplus D_{A_1} a : a \in \underline{A}\}^- = \underline{D}_{A_2} \oplus \underline{D}_{A_1}$.

2. Let $T \in L(\underline{H})$ be a contraction, $U \in L(\underline{K})$ its minimal isometric dilation, $V \in L(\underline{M})$ a unilateral shift such that $\underline{K} \subset \underline{M}$, $U = V|\underline{K}$ and finally, let $Z \in L(\underline{G})$ be an isometry.

P r o p o s i t i o n 2.1. For a contraction $A \in I(T;Z)$ there exists a unique $B \in L(\underline{G},\underline{M})$ such that

(2.1) $B \in I(V;Z)$, $PB = A$, $\|B\| \leq 1$

(where P denotes the orthogonal projection of $\underline{M}$ onto $\underline{H}$) if and only if any of the following conditions holds:

a) The factorization $A.Z$ of AZ $(= TA)$ is regular.

b) The factorization $T.A$ of TA $(= AZ)$ is regular and $\underline{M} \ominus \underline{K}$ does not contain any (closed linear) subspace $\neq \{0\}$ invariant for V.

P r o o f. Let us denote $\underline{\tilde{H}} = \underline{H} \oplus (\underline{M} \ominus \underline{K})$ and $\tilde{T} = \tilde{P}V|\underline{\tilde{H}}$ (where $\tilde{P}$ denotes the orthogonal projection of $\underline{M}$ onto $\underline{\tilde{H}}$). Since the space

(2.2) $\underline{M} \ominus \underline{\tilde{H}} = \bigvee_{n=0}^{\infty} U^n((U-T)\underline{H})^- = \bigvee_{n=0}^{\infty} V^n((U-T)\underline{H})^-$

is invariant for V, it follows at once that V is the minimal isometric dilation of $\tilde{T}$. Now, notice that for any $B \in L(\underline{G},\underline{M})$ satisfying (2.1), the operator $\tilde{A} = \tilde{P}B \in L(\underline{G},\underline{\tilde{H}})$ will satisfy

2.3) $\tilde{A} \in I(\tilde{T};Z)$, $P\tilde{A} = A$, $\|\tilde{A}\| \leq 1$.

Also, for any operator $\tilde{A}$ satisfying (2.3), any contractive intertwining dilation B of $\tilde{A}$ will satisfy (2.1). Thus, in order that (2.1) should uniquely determine B it is necessary that: c) A be the unique operator $\tilde{A}$ satisfying (2.3). Also, it follows directly from the uniqueness result, stated in section 1, that for the uniqueness of an operator $B \in L(\underline{G},\underline{K})(\subset L(\underline{G},\underline{M}))$ satisfying (2.1), it is necessary that either one of the following conditions hold: a') the factorization $A \cdot Z$ be regular; b') the factorization $T.A$ be regular. Conversely, the condition c) together with any of the conditions a') and b') is sufficient for the uniqueness of B. Indeed, for any operator $B_i (i = 1,2)$ satisfying (7.1) the condition (c) implies $\tilde{P}B_1 = A = \tilde{P}B_2$ while any of conditions (a') and (b') does imply $P'B_1 = P'B_2$ (again by the uniqueness result stated in section 1), where P' is the orthogonal projection of $\underline{M}$ onto $\underline{K}$. Since $\underline{H}$ and $\underline{K}$ span $\underline{M}$, we can conclude $B_1 = B_2$. Consequently, in order to show that any of a) and b) is a sufficient condition it is enough to prove that each of them implies the condition c). First, notice that $\tilde{T}|\underline{H} = T$. Thus, with respect to the decomposition $\underline{\tilde{H}} = \underline{H} \oplus (\underline{M} \ominus \underline{K})$ the contraction T will have the matrix form (see [6], Theorem 1).

$$(2.4) \qquad \tilde{T} = \begin{pmatrix} T & D_{T^*} L D_S \\ 0 & S \end{pmatrix}$$

where L is a contraction from $\underline{D}_S$ to $\underline{D}_{T^*}$.

We shall show now that actually L is an isometry. Indeed, since, (2.2) implies

$$((V-\tilde{T})\underline{\tilde{H}})^- = ((U-T)\underline{H})^-,$$

it follows in particular that for any $m \in \underline{M} \ominus \underline{K}$ there exists $(h_j)_{j=1}^{\infty} \subset \underline{H}$ such that $(U-T)h_j \longrightarrow (V-\tilde{T})m$. Using [5], § II, 1 and [6], Formula (2.1), we infer that

$$\| D_T h_j - T^* L D_S m \|^2 + \| D_L D_S m \|^2 \longrightarrow 0 \qquad (j \to \infty)$$

and from this, that $D_L D_S m = 0$ (for any $m \in \underline{M} \ominus \underline{K}$). Thus, L is an isometry. Now, let $\tilde{A} \in L(\underline{G}, \underline{\tilde{H}})$ satisfy (2.3). It has, with respect to the decomposition $\underline{\tilde{H}} = \underline{H} \oplus (\underline{M} \ominus \underline{K})$, the matrix form

$\tilde{A} = \begin{pmatrix} A \\ CD_A \end{pmatrix}$, where C is a contraction from $\underline{D}_A$ to $\underline{M} \ominus \underline{K}$, and

$$(2.5) \qquad D_{T^*} L D_S C D_A = 0, \qquad S C D_A = C D_A Z .$$

Since L is an isometry, the first relation of (2.5) implies $D_S C D_A = 0$; while the second implies $S^n C D_A = C D_A Z^n$ $(n=1,2,\ldots)$, thus

$$(2.6) \qquad D_S S^n C D_A = 0 \qquad (n = 0,1,\ldots)$$

Since $S = (I - P')V | \underline{M} \ominus \underline{K}$ the relation (2.6) shows that the range of C lies in the subspace

$$\underline{M}_o = \{ m \in \underline{M} \ominus \underline{K} : \quad V^n m \in \underline{M} \ominus \underline{K}, \quad n = 0,1,\ldots \} ,$$

which is invariant for V.

From (2.5) it follows

$$(2.7) \qquad V_o C D_A = C D_A Z \qquad (\text{where } V_o = V | \underline{M}_o) .$$

Now, assuming that a)($\equiv$a')) holds (i.e. $\underline{D}_A = (D_A Z \underline{G})^-$) we can define a contraction $X \in L(\underline{D}_A)$ by $X D_A Z = D_A$, which satisfies

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(by virtue of (2.7)) $V_0CX = C$. Whence, since V_0 is a unilateral shift, $C^* = X^*C^*V_0^* = \ldots = X^{*n}C^*V_0^{*n} \to 0$ (strongly, for $n \to \infty$). Thus, $C = 0$, and consequently $\tilde{A} = A$. Also, if b) holds, then $\underline{M}_0 = \{0\}$ and thus $\tilde{A} = A$, again. Thus the sufficiency has been proved.

Now, for the necessity it remains only to show that if b') holds but neither a) nor b) is valid, then c) is not valid. To this aim let us define a partial isometry W on $\underline{N} = \underline{D}_T \oplus \underline{D}_A$ (where $\{0\} \oplus \underline{D}_A$ will be identified with $\underline{D}_A$) by

(2.8) $W(\underline{N} \ominus (\{0\} \oplus (D_A Z\underline{G})^-)) = \{0\}$, $W(0 \oplus D_A Zg) = D_T Ag \oplus D_A g$ $(g \in \underline{G})$.

By virtue of b'), W^* is an isometry and, since a) is not valid we have

(2.9) $$W^*\underline{N} \subsetneqq Q\underline{N},$$

where Q denotes the orthogonal projection of $\underline{N}$ onto its second component. Moreover, by (2.8) and b'), (2.7) is equivalent to

(2.10) $$V_0CQ = CW^* = (CQ)W^*$$

Now, it is plain, by (2.9) (implying that W^* "contains" a unilateral shift) and by the non-validity of b) that there exists a contraction $C = CQ \neq 0$ from $\underline{D}_A$ to $\underline{M}_0$ satisfying (2.10) (and thus (2.7) too). Consequently, there exists $\tilde{A} \neq A$ satisfying (2.3). This achieves the proof.

We shall use, for a subsequent application, Proposition

2.1 in the following form

C o r o l l a r y 2.1. *Let $V_i \in L(\underline{H}_i)$ be a unilateral shift (i=1,2), $\underline{H}_o$ a (closed linear) subspace of $\underline{H}_2$ invariant for V_2^* and let $B_o \in I(V_1^* ; V_2^* | \underline{H}_o)$ be a contraction. There exists a unique operator $B' \in L(\underline{H}_2, \underline{H}_1)$ satisfying*

$$(2.11) \qquad B' \in I(V_1^*, V_2^*), B' | \underline{H}_o = B_o \quad , \|B'\| \leq 1$$

if and only if any of the following conditions holds:

a) *The factorization* $V_1^* \cdot B_o$ *of* $V_1^* B_o (= B_o(V_2^* | \underline{H}_o))$ *is regular.*

b) *The factorization* $B_o.(V_2^* | \underline{H}_o)$ *of* $B_o(V_2^* | \underline{H}_o)$ $(= V_1^* B_o)$ *is regular and*

$$(2.12) \qquad \underline{H}_2 = \bigvee_{n=o}^{\infty} V_2^n \underline{H}_o \ .$$

This corollary can be obtained at once by passing to the adjoints, using [5] , Prop. I. 3.2. a), and remarking that (2.12) is equivalent to the following direct translation of the second property in the condition b) of Proposition 2.1: $\underline{H}_2' = \underline{H}_2 \ominus \bigvee_{n=o}^{\infty} V_2^n \underline{H}_o$ does not contain any subspace $\neq \{0\}$ invariant for V_2.

3. Now, we shall give an application concerning the lifting of operators commuting with shifts. Let $S_j \in L(\underline{G}_j)$ be a unilateral shift (j=1,2), $\underline{G}_o$ a (closed linear) subspace of $\underline{G}_2$ invariant for S_2 and let us set $S_o = S_2 | \underline{G}_o$. A *contractive*

intertwining lifting (denoted in the sequel by CIL) of a contraction $A \in I(S_1 ; S_o)$ is by definition any contraction $B \in I(S_1 ; S_2)$ satisfying $B|\underline{G}_o = A$. By [3], Th. 2, the existence of a CIL of a contraction $A \in I(S_1; S_o)$ is equivalent to the condition

$$(3.1) \qquad \|(I-S_1^k S_1^{*k}) A g_o\| \leq \|(I-S_2^k S_2^{*k}) g_o\| \qquad (g_o \in G_o \ , \ k = 1,2,\ldots)$$

By virtue of (3.1), for each $k = 1,2,\ldots$ we can (and shall) define the contraction $C_k : ((I-S_2^k S_2^{*k}) \underline{G}_o)^- \quad ((I-S_1^k S_1^{*k}) \underline{G}_1)^-$ by the formula

$$(3.2) \qquad C_k (I-S_2^k S_2^{*k}) g_o = (I-S_1^k S_1^{*k}) A g_o \qquad (g_o \in \underline{G}_o).$$

For convenience we shall take C_o as the 0 operator from $\underline{G}_o$ to $\underline{G}_1$.

P r o p o s i t i o n 3.1. Let $A \in I(S_1 ; S_o)$ be a contraction satisfying (3.1). Then A has a unique CIL if and only if any of the following conditions holds:

(i) For every $g_1 \in \underline{G}_1$ there exist $(g_{ok})_{k=1} \subset \underline{G}_o$ and $(n_k)_{k=1}^{\infty} \subset \mathbb{N}$ ($= \{1,2,\ldots\}$) such that

$$(3.3) \qquad (I-S_1 S_1^*) S_1^{*\,n_k - 1} A g_{ok} \longrightarrow (I-S_1 S_1^*) g_1 , \quad D_{C_{n_k}} (I-S_2^{n_k} S_2^{*n_k}) g_{ok} \longrightarrow 0,$$

strongly, for $k \rightarrow \infty$.

(ii) For every $g_2 \in \underline{G}_2$ there exist $(g_{ok})_{k=1}^{\infty} \subset \underline{G}_o$ and $(n_k)_{k=1}^{\infty} \subset \mathbb{N}$ sucht that

(3.4) $\quad (I-S_2S_2^*)S_2^{*n_k-1}g_o \to (I-S_2S_2^*)g_2, \quad D_{C_{n_k-1}}(I-S_2^{n_k-1}S_2^{*n_k-1})g_{ok} \to 0$

strongly, for $k \to \infty$.

Before proving this proposition, we shall sketch the construction of such a CIL, along the proof of [3].

Let $U_j \in L(\underline{K}_j)$ be the minimal unitary dilation of S_j $(j=0,1,2)$; plainly $\underline{K}_o$ can be and will be identified with $\bigvee_{n=o}^{\infty} U_2^{-n}\underline{G}_o$ and U_o with $U_2|\underline{K}_o$. Also, let us denote

$\underline{H}_j = \underline{K}_j \ominus \underline{G}_j \quad (= (I-P_j)\underline{K}_j)$, $V_j = U_j^{-1}|\underline{H}_j$ $\quad (j=1,2)$ and $\underline{H}_o = ((I-P_2)\underline{K}_o)^-$,

where P_j denotes the orthogonal projection of $\underline{K}_j$ onto $\underline{G}_j$ $(j=1,2)$. Obviously U_j^{-1} is the minimal unitary dilation of V_j $(j=1,2)$ and $\underline{H}_o$ is invariant for V_2^*. We define a contraction $A_o \in L(\underline{K}_o, \underline{K}_1)$ by

(3.5) $\quad A_o k_o = s - \lim_{n\to\infty} U_1^{-n} A P_o U_2^n k_o \qquad (k_o \in \underline{K}_o),$

where P_o denotes the orthogonal projection of $\underline{K}_2$ onto $\underline{G}_o$.
As a consequence of (3.1) it follows that the operator B_o defined by

(3.6) $\quad B_o(I-P_2)k_o = (I-P_1)A_o k_o \qquad (k_o \in \underline{K}_o)$,

is a contraction and $B_o \in I(V_1^* ; V_2^*|\underline{H}_o)$. Now, if B' is any contraction $\in I(V_1^* ; V_2^*)$ such that $B'|\underline{H}_o = B_o$ and

$\hat{B}$ is its CID, uniquely determined by virtue of [5], Prop. VII. 3.2. b) and by the uniqueness theorem stated in sec. 1, then

$B = P_1\hat{B}|\underline{G}_2 = \hat{B}|\underline{G}_2$ is a CIL of A. On the other hand, if for any CIL B of A we consider its uniquely determined extension $\hat{B} \in I(U_1;U_2)$, then (since $\hat{B}\underline{G}_2 \subset \underline{G}_1$) it is easy to verify that the contraction

$$B' = (I - P_1)\hat{B}|(I - P_2)\underline{K}_2 \in L(\underline{H}_2, \underline{H}_1)$$

lies in $I(V_1^*, V_2^*)$ and $B'|\underline{H}_o = B_o$ (where B_o is defined by (3.6)); obviously $\hat{B}$ is a CID of B'. Consequently, by virtue of this one-to-one correspondence between the CIL's of A and the operators B' satisfying (2.11) (where V_i , $\underline{H}_i$,... etc. have the present meanings), A has a unique CIL if and only if any of the conditions a) and b) given in Corollary 2.1 holds.

We shall conclude this discussion with the following

L e m m a 3.1. <u>Let</u> $A \in I(S_1, S_o)$ <u>be a contraction</u> satisfying (3.1) <u>and let</u> $B_o : \underline{H}_o \longrightarrow \underline{H}_1$ <u>be the contraction defined by</u> (3.5) <u>and</u> (3.6). <u>Then</u> A <u>has a unique</u> CIL <u>if and only if any of the following conditions holds</u>:

$$(3.7)_1 \quad \underline{D}_{V_1^*} \oplus \{0\} \subset \{D_{V_1^*} B_o(1-P_2)k_o \oplus D_{B_o}(1-P_2)k_o : k_o \in \underline{K}_o\}^-$$

$$(3.7)_2 \quad \{0\} \oplus \underline{D}_{V_2^*} \subset \{D_{B_o}(I-P_2)k_o \oplus D_{V_2^*}(I-P_2)U_2^{-1}k_o : k_o \in \underline{K}_o\}^- .$$

P r o o f. The condition a) in Corollary 2.1 is obviously equivalent to

$$\underline{D}_{V_1^*} \oplus \{0\} \subset \{D_{V_1^*} B_o h_o \oplus D_{B_o} h_o : h_o \in \underline{H}_o\}^-,$$

which, by virtue of the definition of $\underline{H}_o$, is equivalent to the inclusion $(3.7)_1$. We shall show now that the condition b) in Corollary 2.1 is equivalent to the inclusion $(3.7)_2$.

First, we remark that the map: $D_{V_2^*}|\underline{H}_o h_o \longrightarrow D_{V_2^*} h_o$ $(h_o \in \underline{H}_o)$ extends, by continuity, to a unitary operator $L(\underline{D}_{V_2^*}|\underline{H}_o, (D_{V_2^*}\underline{H}_o)^-)$.

Therefore the condition b) in Corollary 2.1 can be reformulated in the form

$$(3.8) \qquad \begin{cases} \{D_{B_o} V_2^* h_o \oplus D_{V_2^*} h_o = h_o \in \underline{H}_o\}^- = \underline{D}_{B_o} \oplus (D_{V_2^*}\underline{H}_o)^- \\ \underline{H}_2 = \bigvee_{n=o}^{\infty} V_2^n \underline{H}_o \ . \end{cases}$$

Or

$$\underline{H}_2 = \bigvee_{n=o}^{\infty} V_2^n D_{V_2^*} \underline{H}_2 \qquad (*)$$

and, by virtue of the structure of the isometric dilation of $(V_2^*|\underline{H}_o)^*$ (see [5], § II.1)

$$\bigvee_{n=o}^{\infty} V_2^n \underline{H}_o = \bigvee_{n=o}^{\infty} V_2^n ((I - V_2V_2^*)\underline{H}_o)^- \ ,$$

where the spaces, occuring in each of the right parts of the preceding two relations, are mutually orthogonal. Therefore the second relation (3.8) holds if and only if

$$\underline{D}_{V_2^*} = D_{V_2^*}\underline{H}_o = ((I-V_2V_2^*)\underline{H}_o)^- = (D_{V_2^*}\underline{H}_o)^- \ .$$

It follows that (3.8) is equivalent to

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*) Here we used the fact that for an isometry $V(=V_2)$, $D_{V^*} = I - VV^*$ is an orthogonal projection.

$$(3.9)\qquad \{D_{B_o} V_2^* h_o \oplus D_{V_2^*} h_o \,:\, h_o \in \underline{H}_o\}^- = \underline{D}_{B_o} \oplus \underline{D}_{V_2^*} .$$

since

$$\{D_{B_o} V_2^* h_o \oplus D_{V_2^*} h_o : h_o \in \underline{H}_o\}^- = (\bigcup_{n=1}^{\infty} \{D_{B_o} V_2^*(I-P_2)U^{-n} g_o \oplus D_{V_2^*}(1-P_2)U^{-n} g_o :$$

$$: g_o \in \underline{G}_o\})^- = (\bigcup_{n=1}^{\infty} \{D_{B_o}(1-P_2)U^{-n+1} g_o \oplus D_{V_2^*}(1-P_2)U^{-n} g_o \,:\, g_o \in \underline{G}_o\})^- =$$

$$= \{D_{B_o}(1-P_2)k_o \oplus D_{V_2^*}(1-P_2)U_2^{-1} k_o : k_o \in \underline{K}_o\}^- ,$$

the relation (3.9) is equivalent to

$$\{D_{B_o}(1-P_2)k_o \oplus D_{V_2^*} \; (I-P_2)U_2^{-1} k_o \,:\, k_o \in \underline{K}_o\}^- = \underline{D}_{B_o} \oplus \underline{D}_{V_2^*} ,$$

which, in its turn, is equivalent to $(3.7)_2$.

4. In order to interprete the conditions $(3.7)_{1-2}$, let us note that for $n \geqslant 1$ and

$$k_j = U_j^{-n} g_j \quad (g_j \in \underline{G}_j) \;,\; k_o = U_2^{-n} g_o \quad (g_o \in \underline{G}_o)$$

we have

$$(4.1)_j \qquad \| D_{V_j^*} (I-P_j)k_j \| = \| D_{V_j^*}(I-P_j)U_j^{-n} g_j \| = \| D_{S_j^*} S_j^{*\,n-1} g_j \| \; (j=1,2) ,$$

and (by (3.6) and (3.5))

$$\| D_{B_o}(I-P_2)k_o \|^2 = \| (I-P_2)k_o \|^2 - \| (I-P_1) A_o k_o \|^2 =$$

$$\| (I-P_2)U_2^{-n} g_o \|^2 - \| (I-P_1)U_1^{-n} A g_o \|^2 =$$

$$\|(I-S_2^n S_2^{*n})g_o\|^2 - \|(I-S_1^n S_1^{*n}) Ag_o\|^2 ,$$

or, on account of (3.2),

$$(4.2) \quad \|D_{B_o}(I-P_2)k_o\| = \|D_{C_n}(I-S_2^n S^{*n})g_o\| \qquad (n = 1,2,\ldots).$$

It is useful to notice that (4.2) holds also if $n = 0$. From $(4.1)_j$ we infer that there exist unitary operators ψ_j $(j=1,2)$, from $\underline{D}_{V_j^*}$ onto $\underline{D}_{S_j^*}$,

defined by

$$(4.3)_j \quad \psi_j D_{V_j^*}(I-P_j)U_j^{-n}g_j = D_{S_j^*} S_j^{*n-1} g_j \quad (g_j \in \underline{G}_j;\ j=1,2;\quad n = 1,2,\ldots).$$

Also, by (4.2), we can define for each $n = 1,2,\ldots$ an isometric operator θ_n: $(D_{C_n}(I-S_2^n S_2^{*n})\underline{G}_o)^- \longrightarrow \underline{D}_{B_o}$ by

$$(4.4) \quad \theta_n D_{C_n}(I-S_2^n S_2^{*n})g_o = D_{B_o}(I-P_2)U_2^{-n}g_o \qquad (g_o \in \underline{G}_o, n = 0,1,2,\ldots).$$

After these preliminaries we are able to pass to the

P r o o f of P r o p o s i t i o n 3.1. By virtue of $(4.3)_1$, (3.6) and (3.5), the condition $(3.7)_1$ in Lemma 3.1 means that

$$\Big(\bigcup_{n=1}^{\infty} \{D_{S_1^*} S_1^{*n-1} Ag_o \oplus D_{B_o} (I-P_2)U_2^{-n} g_o : g_o \in \underline{G}_o\} \Big)^- =$$

$$(\psi_1 + I_{\underline{D}_{B_o}})\{D_{V_1^*} B_o(1-P_2)k_o \oplus D_{B_o}(1-P_2)k_o : k_o \in \underline{K}_o\}^- \supset$$

$$\supset (\psi_1 \oplus I_{\underline{D}_{B_o}})(\underline{D}_{V_1^*} \oplus \{0\}) = \underline{D}_{S_1^*} \oplus \{0\}.$$

Obviously this condition is equivalent to the fact that for every $g_1 \in \underline{G}_1$ there exist $(g_{ok})_{k=1}^{\infty} \subset \underline{G}_o$ and $(n_k)_{k=1}^{\infty} \subset \mathbb{N}$ such that

$$(4.5) \qquad D_{S_1^*} S_1^{*\,n_k-1} A g_{ok} \longrightarrow D_{S_1^*} g_1 \;,\; D_{B_o}(I-P_2)U_2^{-n_k} g_{ok} \longrightarrow 0$$

strongly, for $k \to \infty$. By virtue of (4.4), the relation (4.5) coincides with the relation (3.3), therefore (3.7) is equivalent to the condition (i) in Proposition 3.1.

Analogously, by $(4.1)_2$, the condition $(3.7)_2$ becomes

$$\Big(\bigcup_{n=1}^{\infty}\{D_{B_o}(I-P_2)U_2^{-n+1} g_o \oplus D_{S_2^*} S_2^{*\,n-1} g_o : g_o \in \underline{G}_o\}\Big)^- =$$

$$(I_{\underline{D}_{B_o}} \oplus \psi_2)\{D_{B_o}(1-P_2)k_o \oplus D_{V_2^*}(I-P_2)U_2^{-1} k_o : k_o \in \underline{K}_o\}^- \supset$$

$$(I_{\underline{D}_{B_o}} \oplus \psi_2)(\{0\} \oplus \underline{D}_{V_2^*}) = \{0\} \oplus \underline{D}_{S_2^*} .$$

As in the preceding case it is easy to see (using again (4.4)) that this last condition is fulfilled if and only if so is the condition (ii) in Proposition 3.1.

This finishes the proof.

C o r o l l a r y 4.1. <u>In order that every contraction</u>

$A \in I(S_1, S_o)$ satisfying (3.1) should have a unique CIL it is necessary and sufficient that the following condition holds:

(iii) For every $g_2 \in G_2$ there exist $(g_{ok})_{k=1}^{\infty} \subset G_o$ and $(n_k)_{k=1}^{\infty} \subset \mathbb{N}$ such that

$$(4.6) \qquad (1-S_2S_2^*)S_2^{*n_k-1} g_{ok} \to (1-S_2S_2^*)g_2 \ , \ (I-S_2^{n_k-1} S_2^{*n_k-1}) g_{ok} \to 0$$

strongly, for $k \to \infty$.

P r o o f. If the condition (iii) above is fulfilled, so is the condition (ii) in Proposition 3.1, for any contraction $A \in I(S_1, S_o)$ satisfying (3.1). Conversely if we take $A = 0$, then the condition (i) in Proposition 3.1 is never fulfilled, while the condition (ii) in the same proposition coincides with the condition (iii) above.

R e m a r k 4.1. The condition (iii) in corollary 3.1. is fulfilled if for instance

$$(4.7) \qquad \underline{D}_{S_2^*} = (D_{S_2^*} \underline{G}_o)^- .$$

Indeed, in this case, the condition (iii) is satisfied by taking always $n_i = 1$ $(i = 1,2,\ldots)$.

R e m a r k 4.2. It is perhaps instructive to give an explicit functional interpretation of Corollary 3.1. To this purpose let $\{\underline{F}, \underline{E}, \Theta(\lambda)\}$ be an inner analytic function (for the definition see [5] , § V.2). Let moreover $\underline{E}_*$ be any complex Hilbert space, $\underline{E}_* \neq \{0\}$. We shall denote by S_1 and S_2 the multiplication operators by the polynomial $p(\lambda) \equiv \lambda$ on the Hardy spaces $\underline{G}_1 = H^2(\underline{E}_*)$ and $\underline{G}_2 = H^2(\underline{E})$, respectively. Also we set $\underline{G}_o = \Theta H^2(\underline{F})$. Then it is easy to verify that the uniqueness of the CIL of any contraction $A \in I(S_1, S_o)$ satisfying (3.1) coincides with the following property:

(iv) If $\{E, E_*, \Theta_*(\lambda)\}$ is any contractive analytic function (in the sense of [5], Ch.V) then $\Theta_*(\lambda)\Theta(\lambda) \equiv 0$ (for $|\lambda| < 1$) implies $\Theta_*(\lambda) \equiv 0$.

Also it is easy to verify that the condition (iii) can be restated as follows:

(v) For every $e \in \underline{E}$ there exists a sequence $\{f_n\}_{n=1}^{\infty}$ such that

$$f_n = \begin{bmatrix} f_{n,o} \\ \cdot \\ \cdot \\ \cdot \\ f_{n,n} \end{bmatrix} \in \underline{F} \oplus \underline{F} \oplus \ldots \oplus \underline{F} \quad ((n+1) - \text{copies})$$

and that for $n \to \infty$

$$\left\| \begin{bmatrix} \Theta_o & & O \\ \Theta_1 & \Theta_o & \\ \vdots & \ddots & \ddots \\ \Theta_n & \ldots \Theta_1 & \Theta_o \end{bmatrix} \begin{bmatrix} f_{n,o} \\ f_{n,1} \\ \vdots \\ f_{n,n-1} \\ f_{n,n} \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ e \end{bmatrix} \right\|_{\underline{E} \oplus \underline{E} \oplus \ldots \oplus \underline{E}} \longrightarrow 0,$$

((n+1) - copies)

where

$$\Theta(\lambda) = \theta_o + \theta_1 + \lambda^2 \theta_2 + \dots + \lambda^n \theta_n + \dots .$$

Thus Corollary 3.1 asserts that the conditions (iv) and (v) are equivalent.

Finally let us also remark that (4.7) means that $(\theta_o \underline{F})^{-} = \underline{E}$.

5. We shall finish with an open intriguing question. Namely in [2] it was proved that if $A, A' \in I(T_1, T_2)$ (where we use the notation from the section 1) <u>are two contractions and if</u> A <u>Harnack-dominates</u> A', <u>then if</u> A <u>has a unique</u> CID, <u>so has</u> A'. We recall that for two contractions $A, A' \in L(\underline{H}_2, \underline{H}_1)$ we say (according to [2]) that A <u>Harnack-dominates</u> A' if there exists an operator

$$K = \begin{bmatrix} I & X \\ O & Y \end{bmatrix} : \begin{matrix} \underline{H}_1 \\ \oplus \\ \underline{D}_A \end{matrix} \longrightarrow \begin{matrix} \underline{H}_1 \\ \oplus \\ \underline{D}_{A'} \end{matrix}$$

such that

$$K . \begin{bmatrix} A \\ D_A \end{bmatrix} = \begin{bmatrix} A' \\ D_{A'} \end{bmatrix} .$$

Does the above underlined fact have an analogue for the case considered in sec. 3? More precisely, with the notations of sec.3, $A, A' \in I(S_1, S_o)$ are two contractions satisfying (3.1) and if A Harnack-dominates A', is it true that if A has a unique CIL, so has A'?

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