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A HOMOLOGICAL VIEW IN DILATION THEORY

by

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1. It is well known that in Dilation Theory ([10] , [2], [14] , [5] or [8]) the passage from the theory of one operator to that of several ones is difficult, often impossible and rarely satisfactory (see for instance [14]., Ch.I). Therefore an attempt to find an algebraic background for the difficulties one encounters in this passage, in the framework of which some obstructions could be explicitely determined, may presents some interest. The present note is a preliminary repport aiming to delineate this algebraic approach, to illustrate it by some specific example and to give some particular results.

2. In this sequel we shall denote by $\mathcal{C}_n$ (n = 1,2,...) the category of all ordered n - tuples $\tau=\{T_1, T_2, \ldots, T_n\}$ of mutually commuting contractions on some (arbitrary) complex Hilbert space $H = H_\tau$. If $\tau=\{T_1, \ldots, T_n\}$ and $\tau'=\{T_1', \ldots, T_n'\}$ are objects of $\mathcal{C}_n$, by a morphism $A: \tau \mapsto \tau'$ we mean an operator $A: H_\tau \longrightarrow H_{\tau'}$ such that

$$\|A\| \leq 1, \qquad AT_j = T_j' A \qquad (j = 1,2,\ldots,n)$$

An object $\omega = \{U_1, \ldots, U_n\}$ of $\mathcal{C}_n$ will be called hypo-projective (resp. hypo-injective) if for any diagram

$$(2.1) \qquad \begin{array}{ccc} \tau' & \xrightarrow{P} & \tau \\ & & \uparrow{\scriptstyle A} \\ & & \omega \end{array} \qquad \left(\text{resp.} \quad \begin{array}{ccc} \tau & \xrightarrow{J} & \tau' \\ {\scriptstyle A}\downarrow & & \\ \omega & & \end{array} \right)$$

such that P^* (resp. J) is isometric, there exists a morphism $B: \omega \mapsto \tau'$ (resp. $\tau' \to \omega$) such that the diagram

$$(2.2.) \qquad \begin{tikzcd} \tau' \arrow[r, "P"] & \tau \\ & \omega \arrow[u, "A"'] \arrow[ul, "B"] \end{tikzcd} \qquad \left(\text{resp.} \quad \begin{tikzcd} \tau \arrow[r, "J"] \arrow[d, "A"'] & \tau' \arrow[dl, "B"] \\ \omega & \end{tikzcd} \right)$$

is commutative. A hypo-projective resolution of an object τ in $\mathcal{C}_n$ is a sequence

$$(2.3) \qquad \cdots\; \tau_{m+1} \xrightarrow{P_{m+1}} \tau_m \xrightarrow{P_m} \tau_{m-1} \mapsto \cdots \mapsto \tau_2 \xrightarrow{P_2} \tau_1 \xrightarrow{P_1} \tau_0 \xrightarrow{P_0} \tau$$

of morphisms such that each τ_i ($i = 0,1,2,\ldots$) is hypo-projective, each P_i^* ($i = 0,1,2,\ldots$) is isometric and, moreover, the sequence (2,3) is exact, i.e.

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(2.4) Range P_{n+1} = Kernel P_n $\quad (n = 0,1,2,\ldots)$

If for a given τ there exists a hypo-projective resolution such that $H_{\tau_n} = \{0\}$ for all $n > m$, where m is some integer, we shall say that the projective dimension pd of τ is $\leq m$. Plainly pd (τ) is the smallest possible m in the above definition. For all the remaining τ's we set pd $(\tau) = +\infty$. Hypo-inductive resolutions for τ as well as the inductive dimension id (τ) is defined in an obvious dual manner, so that

(2.5) $\quad \mathrm{pd}(\tau) = \mathrm{id}\,(\tau^*)$ where $\tau^* = \{T_1^*, \ldots T_n^*\}$.

Also we notice that by these definitions , τ is hypo-projective (resp. hypo-injective) if and only if pd $(\tau) = 0$ (resp. id$(\tau) = 0$). Finally we shall put

$$\mathrm{pd}\,(\mathscr{C}_n) = \sup \mathrm{pd}\,(\tau), \qquad \mathrm{id}\,(\mathscr{C}_n) = \sup \mathrm{id}\,(\tau),$$

where τ runs over all the objects of $\mathscr{C}_n$.

All the above definitions are obviously modelled after the usual definitions in the theory of the homological dimension in Category Theory (see for instance [3], Ch IV, § 9), with the only difference that instead of projective (resp. injective) objects (*) we used hypo-projective (resp. hypo-injective) objects. [However projective or injective objects essentially do not exist in $\mathscr{C}_n$, indeed if $H_\tau \neq \{0\}$ and I denote the identity on H_τ, then in the diagram

$$\begin{tikzcd} \tau \arrow[r, "\frac{1}{2}I"] \arrow[d, "I"'] & \tau \arrow[dl, "B"] \\ \tau & \end{tikzcd}$$

the operator B is necessarily 2·I and thus it can not be a morphism in $\mathscr{C}_n$.]

3. In this note we shall show that in case n = 1, the basic results in the usual dilation theory as developped in [14] (see [14], Ch.I, §1-4, Ch.II, §1-2) can be restated in the following concise manner :

<u>For any object</u> $\tau = \{T\}$ <u>of</u> $\mathscr{C}_1$ <u>we have</u> pd $(\tau) = 0$ (resp. id$(\tau) = 0$) <u>if and only if</u> T (resp. T^*) <u>is isometric</u> ; <u>moreover</u>

(3.1) $\quad \mathrm{pd}\,(\mathscr{C}_1) \quad (= \mathrm{id}\,(\mathscr{C}_1)) = 1.$

Therefore the final goal of the present approach would be to compute pd$(\mathscr{C}_n)$ for all n = 1,2,..., as well as to find geometrical characterizations for the τ's in $\mathscr{C}_n$ with the property

$$\mathrm{pd}\,(\tau) = k, \ \mathrm{id}\,(\tau) = j \qquad (k, j \leq \mathrm{pd}\,(\mathscr{C}_n)).$$

For the time being, we shall fix our attention only on $\mathscr{C}_2$, showing in particular that

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(*) Let us recall that ω is projective (resp. injective) in $\mathscr{C}_m$ means that in the definitions given by us one replaces the condition that P^* resp. J be isometries with the weaker one $\ker P^* = \{0\}$, $\ker J = \{0\}$.

(3.2) $\quad \mathrm{pd}(\mathcal{C}_2) \ (= \mathrm{id}(\mathcal{C}_2)) = +\infty$

and that if $\tau = \{T_1, T_2\}$ is an object of $\mathcal{C}_2$ such that V_1, V_2 (resp. V_1^*, V_2^*) are isometries, then

(3.3) $\quad \mathrm{pd}(\tau) \in \{0, \infty\}$ (resp. $\mathrm{id}(\tau) \in \{0, \infty\}$),

where both values 0 and ∞ are taken.

Also we shall give a new proof of the Intertwining Dilation Theorem ([13], [14], Ch. II, §2; or equivalently of the Lifting theorem [7]), which, we hope, is more amenable to matrix calculations occuring in concrete problems of extrapolation, as well as some simple corollaries of this theorem, which seem new and useful.

4. We start by characterizing the hypo-projective objects of $\mathcal{C}_1$.

Proposition 4.1. If $\omega = \{U\}$ is hypo-projective (in $\mathcal{C}_1$) then U is an isometry; also if ω is hypo-injective (in $\mathcal{C}_1$), then U^* is an isometry.

Proof Let $\omega = \{U\}$ be a hypo-projective object in $\mathcal{C}_1$ and let V on K be the minimal isometric dilation of U (see [14], Ch I, §4). Let P denote the orthogonal projection of K into H $(= H_\omega)$. In the definition of an hypo-projective object, we take $\tau' = \{V\}$, $\tau = \omega$ and $A = I_{\mathcal{H}}$ (= the identity on H). Thus there exists a contraction $B: H \longmapsto K$ such that

$$PB = I \quad \text{and} \quad BU = VB.$$

Therefore for $h \in H$ we have

$$\|h\| = \|PBh\| \leq \|Bh\| \leq \|h\|,$$

that is $PBh = Bh$, whence $Bh = h$ and consequently

$$Uh = BUh = VBh = Vh.$$

It follows that $U = V/H$ is an isometry. In case ω is hypo-injective, $\omega^* = \{U^*\}$ is hypo-projective and the conclusion follows from the preceding argument.

Proposition 4.2. Let V be an operator on K. Then if V (resp. V^*) is isometric $\omega = \{V\}$ is hypo-projective (resp. hypo-injective) in $\mathcal{C}_1$.

Plainly it is sufficient to consider the caze when V^* is an isometry. Then an straightforward reformulation of the hypo-injectivity of $\omega = \{V\}$ shows that the corresponding statement in Proposition 4.2. is equivalent to the following:

Proposition 4.3. Let T be a contraction on H and V an isometry on K. Let H_0 be an invariant subspace for T and $A: H_0 \longmapsto K$ be a contraction such that $A(T/H_0) = V^*A$. Then there exists a contraction $B: H \mapsto K$ such that $BT = V^*B$ and $B/H_0 = A$.

Proof : The proof will be accomplished in several steps one of which is inspired by the previous inductive constructions [13], [7].

Let

(4.1.) $$T = \begin{pmatrix} T & X \\ 0 & S \end{pmatrix}$$

denote the matrix of T with respect to the decomposition $H = H_o \oplus H_o^{\perp}$. We shall say that $(\overset{*}{*})$ holds for S if the conclusion of the statement is valid for any A, V and T of the form (4.1.).

<u>1^{st} step</u> : If S is of the form αI_1, where I_1 denotes the identity on $H_1 = H_o^{\perp}$ and $|\alpha|<1$, then $(\overset{*}{*})$ holds for S. To this purpose, we seek a contraction $B : H_o \oplus H_1 \longmapsto K$ of the form

$$B = (A \quad A')$$

such that

$$(A \quad A')\begin{pmatrix} T_o & X \\ 0 & \alpha I_1 \end{pmatrix} = V^* \ (A \quad A'),$$

i.e.

$$(4.2.) \qquad AX = V^*A' - \alpha A' = (V^* - \alpha)A'.$$

Now, by[], in (4.1.), X is of the form

$$(4.3.) \qquad X = D_{o*} L (1-|\alpha|^2)^{1/2} I_1 = (1-|\alpha|^2)^{1/2} D_{o*} L,$$

where $D_{o*} = (1-T_oT_o^*)^{1/2}$ and $L: H_1 \longmapsto (D_{o*}H_o)^-$ is a contraction ; also B is contraction if and only if $A'^* = CD_{A^*}$ (*) for some convenient contraction $C: (D_{A^*}K)^- \longmapsto H$. Thus, by (4.2.) and (4.3.), we must find such a contraction satisfying

$$(4.4.) \qquad C D_{A^*} (V - \bar{\alpha}) = (1-|\alpha|^2)^{1/2} L^* D_{o*} A^*.$$

Since $\|L\| \leq 1$, $AT_o = V^*A$ and V is an isometry, we have

$$(V-\bar{\alpha})^* D_{A^*}^2 (V-\bar{\alpha}) - (1-|\alpha|^2) A D_{o*} LL^* D_{o*} A^* \geq$$

$$\geq (V-\bar{\alpha})^* D_{A^*}^2 (V-\bar{\alpha}) - (1-|\alpha|^2) A D_{o*}^2 A^* =$$

$$= (V-\bar{\alpha})^* D_{A^*}^2 (V-\bar{\alpha}) - (1-|\alpha|^2)(V^* D_{A^*}^2 V - D_{A^*}^2) =$$

In general, for any contraction $A : H \longmapsto K$ we denote by D_A the operator $(1 - A^* A)^{1/2}$; thus in particular $D_{o*} = D_{T_o^*}$.

$$= |\alpha|^2 V^* D_{A^*} V - \bar{\alpha} V^* D_{A^*}^2 - \alpha D_{A^*}^2 V + D_{A^*}^2 =$$

$$= (1 - \alpha V)^* D_{A^*}^2 (1 - \alpha V) \geq 0 ;$$

thus it is possible to find an adequate C (see [4]).

2nd step: If $(\overset{*}{*})$ holds for the contractions $S_1, S_2, \ldots S_n$, then $(\overset{*}{*})$ holds also for the any contraction of the form

$$S = \begin{pmatrix} S_1 & * & \cdots & * \\ 0 & S_2 & * & \vdots \\ \vdots & & \ddots & * \\ 0 & \cdots & 0 & S_m \end{pmatrix}$$

Indeed applying $(\overset{*}{*})$ to the compression to $H_0 \oplus H_1$ $\begin{pmatrix} T_0 & * \\ 0 & S_1 \end{pmatrix}$ of T we obtain a contraction $B_1 : H_0 \oplus H_1$ (when H_i denotes the space on which operates S_i, $i = 1, 2, \ldots, n$) such that $B_1 / H_0 = A$ and

$$B_1 \begin{pmatrix} T_0 & * \\ 0 & S_1 \end{pmatrix} = V^* B_1 .$$

Repeating the same procedure (n - 1)-times we finally find $B : H_0 \oplus H_1 \oplus \ldots \oplus H_n$ with the desired properties.

3rd. Statement $(\overset{*}{*})$ holds for any finite rank strict contraction. Indeed such an operator S can be put in the form

$$S = \begin{pmatrix} \alpha_1 & * & \cdots & * \\ & \ddots & & \vdots \\ 0 & & \alpha_m & * \\ & & & 0 \end{pmatrix}$$

where all α_j's are numbers, $|\alpha_j| < 1$, while 0 represents the 0-operator on some (finite or infinite dimensional) Hilbert space. Then we apply the first two steps and finally again the 1st step.

4th step: Now let T have the form (4.1.) with S an arbitrary contraction. Let $S = WR$ be the polar decomposition of S, when $R \geq 0$. Then there exists a sequence $\{W_n\}_{n=1}^{\infty}$ of finite rank strict contractions such that $W_n \longrightarrow W$, $W_n^* \longrightarrow W^*$ (strongly, for $n \to \infty$). We set $S_n = W_n R$ $(n = 1, 2, \ldots)$. Then

$$S_n^* S_n = R W_n^* W_n R \longrightarrow R^2 \quad (\text{strongly}, \text{ for } n \to \infty)$$

thus also

$$D_n = (I - S_n^* S_n)^{1/2} \to (I - R^2)^{1/2} = D_S \quad (\text{strongly, for } n \to \infty),$$

therefore

$$T_n = \begin{pmatrix} T_0 & D_{0*} L P D_n \\ 0 & S_n \end{pmatrix}$$

where P denotes the orthogonal projection of $H_1 = H_0^{\perp}$ into $(D_S H_1)^-$, strongly converges (for $n \to \infty$) to T. By the preceding step, there exist contradictions B_n ($n = 1, 2, \ldots$) such that

$$(4.5) \qquad B_n T_n = V^* B_n \quad \text{and} \quad B_n / H_0 = A \qquad (n = 1, 2, \ldots)$$

Clearly, we can assume that $B_n \to B$ weakly for some contraction B ; therefore we can infer that $B_n T_n \longrightarrow BT$ and $V^* B_n \longrightarrow V^* B$ (weakly, for $n \to \infty$), thus, by (4.5.), B has all desired properties.

This completes the proof.

Corollary 4.4. Let T_1 and T_2 be some contractions on H_1 and H_2 respectively. Let U_1 and U_2 be some isometries on K_1 and K_2, respectively, where K_1 and K_2 contains H_1 and H_2 respectively. Assume that

$$(4.6.) \qquad P_1 U_1 = T_1 P_1 , \quad P_2 U_2 = T_2 P_2 ,$$

where P_1 and P_2 denote the orthogonal projections of K_1 onto H_1 and K_2 onto H_2, respectively Let *moreover* A: $H_1 \longrightarrow H_2$ be a contraction such that $AT_1 = T_2 A$. Then there exists a contraction B : $K_1 \longrightarrow K_2$ such that

$$(4.7) \qquad BU_1 = U_2 B \quad \underline{\text{and}} \quad P_2 B = AP_1$$

Proof . We pass to the adjoints and consider A^* as valued in K_1 . Because of (4.6.), H_1 and H_2 are invariant to U_1^* and U_2^* , respectively thus $U_1^* A^* = A^* (U_2^*/H_2)$. Then we apply Proposition 4.3. and take once again the adjoints.

Remark Corollary 4.4. is essentially the Intertwining Dilation Theorem (see [14] , Ch. II, §2). Proposition 4.3. cam be also deduced from this theorem and even in an easy way (which we let to the reader to find). However we believe that the direct proof given above, making no use of the special structure of the minimal isometric dilation of a contraction, is instructive and may be useful .

Corollary 4.5. Let T_0 and T_1 be contractions on H and H_1 respectively, and V be an isometry on K. Let $A : H_o \longmapsto K$ and $X : H_o \longmapsto H_1$ be contractions satisfying $AT_o = V^*A$ and $XT_o = T_1X$. Then there exists a contraction $B : H_1 \longmapsto K$ satisfying $BT_1 = V^*B$ and $A = BX$ if and only if $A^*A \leq X^*X$.

Proof The "only if" statement is obvious. Thus assume $A^*A \leq XX^x$ and set $H_o' = \overline{XH_o}$ and $T_o' = T_1/H_o'$. Then setting $A'Xf = Af$ for $f \in H_o$, we define a contraction from XH_o to K which can be extended by continuity to all of H_2. Moreover, for f in H_o we have

$$A'T_o'Xf = A'XT_of = AT_of = V^*Af = V^*A'Xf.$$

Therefore $A'T_o' = V^*A'$ and we apply Proposition 4.3. to A' obtaining an adequate con $B' : H_1 \mapsto K$. Finally we set $B = B'X$.

Corollary 4.6. Set $A : H_o \longmapsto G$ and T_o on H_o, and S on be some contractions such that $AT_o = SA$. Then in order that for any contraction T on some Hilbert space $H \supset H_o$, such that $T/H_o = T_o$, there should exist a contraction $B : H \mapsto G$ such that

$$(4.8) \qquad BT = SB \quad \text{and} \quad B/H_o = A$$

it is necessary and sufficient that

$$(4.9) \qquad AA^* \leq S^m S^{*m} \quad (\text{for all } m = 1, 2, \dots).$$

Proof If an adequate B always exists, let us take for T the minimal coisometric extension of T_o (i.e. T^* is the minimal isometric dilation of T_o^*) and let $B : H \longmapsto G$ be the contraction satisfying (4.8.). Then, for every $h_o \in H_o$ and $n = 1, 2, \dots$,

$$\|A^*h_o\| \leq \|B^*h_o\| = \|T^{*m}B^*h_o\| = \|B^*S^{*m}h_o\| \leq \|S^{*m}h_o\|,$$

thus (4.9) is valid.

Conversely let T, T_o, A and S be as in th statement and let assume that (4.9.) is valid. Let U_t on K_t be the minimal isometric dilation of S, let

$$\mathcal{R} = \bigcap_{m \geq 0} U_+^m K_+$$

and let P_t, denote the orthogonal profection of K_t onto G and Q that onto R. Then (see [14], Ch. II, §3),

$$\|Qg\| = \lim_{m \to \infty} \|S^{*m} g\| \qquad (g \in G),$$

thus there exists a contraction $A' : H_o \to (QH)^-$ such that $A = P_+ A'$. Since $U_+^*(Q(G)) = QS^*$, it follows that $V' = U_+^* / H'$ is an isometry in H' such that $A'T_o = V^* A'$. Indeed, this follows from the fact that

$$P_+ U_+ / (\overline{QH}) = P_+ V'^*, \qquad P_+ (A'T_o - V'^* A') = AT_o - SA = 0$$

and

$$\ker\, (P_+ / (\overline{QH})) = \{0\}$$

therefore by virtue Proposition 4.3 there exists a contraction $B' : H \to (QH)^-$ such that $V^*B' = B'T$, $B'/H_o = A'$. Finally we set $B = P_+ B'$. Then

$$SB = P_+ U_+ P_+ B' = P_+ U_+ B' = P_+ V'^* B' = P_+ A' T = AT.$$

The others desired properties of B are obvious.

5. Let us return to the approach considered in sections 2 and 3. Let $\tau = \{T\}$ be an object in $\mathcal{C}_1$. If it is not a hypo-projective (i.e. by proposition 4.1 and 4.2 if T is not an isometry) let U on K be the minimalisometric delation of T. Let P denote the orthogonal projections of K onto $H = H_\tau$. Then (see [9] or [14], Ch. II, §1-2) $K \ominus H$ is invariant to U and thus

$$\tau_1 = \{ U/_{K\ominus H} \} \quad \text{and} \quad \tau_0 = \{U\}$$

are, by Proposition 4.2 hypo-projective objects in $\mathcal{C}_1$. We set $P_0 = P$ and $P_1 = I_{K\ominus H}$ (= the identitiy on $K\ominus H$). Set moreover $\tau_n = \{0\}$ where $H_{\tau_n} = \{0\}$ for $n > 1$. Then

$$\dots \mapsto \{0\} \mapsto \{0\} \mapsto \tau_1 \xrightarrow{I_{K\ominus H}} \tau_0 \xrightarrow{P} \tau \to 0$$

is obviously a hypo-projective resolution, thus $\mathrm{pd}(\tau) \le 1$. (Obviously since τ is not hypo-projective we must have necessarily $\mathrm{pd}(\tau) = 1$). This finishes the proof of the assertion on $\mathcal{C}_1$, underlined in section 3.

6. We start now studying the category $\mathcal{C}_2$.

Lemma 6.1. If $\omega = \{V_1, V_2\}$ is a hypo-projective (resp. hypo-injective) object in $\mathcal{C}_2$, then V_1 and V_2 (resp. V_1^* and V_2^*) are isometries.

Proof. Let U_1 on K_1 be the minimal isometric dilation of V_1 and let V_2' be the contraction B obtained by virtue of Corollary 4.4. where we set $A = V_2$, $T_1 = T_2 = V_1$. Then there exists a contraction $Q : H_\omega \mapsto K_1$ such that

$$(6.1) \qquad P_1 Q = I \quad \text{and} \quad U_1 Q = Q V_1, \quad V_2' Q = Q V_2,$$

where P_1 denotes the orthogonal projection of K_1 onto $H = H_\omega$ and I denotes the identity on H. Again as in the proof of Proposition 4.1., from (6.1.) it follows that $Q = I$, thus $V_1 = U_1/H$ is an isometry. By symmetry, V_2 must be also an isometry. For the second statement of the Lemma we apply the preceding argument to ω^*.

Lema 6.2. Let $\omega = \{V_1, V_2\}$ be a hypo-projective (resp. hypo-injective) object in $\mathcal{C}_2$ and let $H' \subset H$ ($= H_\omega$) reduce both V_1 and V_2. Then $\omega' = \{V_1/H', V_2/H'\}$ is also hypo-projective (resp. hypo-injective.).

Proof. Obvious.

Corollary 6.3. For any object $\tau = \{T_1, T_2\}$ of $\mathcal{C}_2$ such that T_1, T_2 (resp. T_1^*, T_2^*) are isometries

$$(6.2.) \qquad \mathrm{pd}(\tau) \in \{0, \infty\} \quad (\text{resp. } \mathrm{id}(\tau) \in \{0, \infty\})$$

holds.

Proof . Obviously it suffices to proof the statement concerning pd (τ). Also it is obvious that what we have to prove is that $n = 0$ if pd $(\tau) = n < \infty$. In this case, anyway we have a hypo-projective object $\omega = \{U_1, U_2\}$ and a morphism $P : \omega \mapsto \tau$ such that P^* is an isometry. Without loss of generality we can assume that $K (= H_\omega) \supset H (= H_\tau)$ and that P is the orthogonal projection of K onto H. We have $PU_1 = T_1P_1$, $PU_2 = T_2P$ whence, since T_1, T_2 are isometries, by assumption and U_1, U_2 by Lemma 6.1., we have

$$\|PU_1h\| = \|U_1h\| \quad , \quad \|PU_2h\| = \|U_2h\| , \quad (h \in H)$$

and therefore $U_1 h = PU_1 h = T_1 h$, $\quad U_2 h = PU_2 h = T_2 h$

for all $h \in H$, i.e. H is invariant for U_1 and U_2. Since $\ker P = K \ominus H$ is also invariant to U_1 and U_2, it follows that H reduces U_1 and U_2. Since $\tau = \{U_1/H , U_2/H\}$, the conclusion follows from Lemma 6.2.

7. In order to obtain a strong *property* of hypo-projective objects in $\mathcal{C}_2$ let us recall that if $A_1 : \mathcal{A} \to \mathcal{B}$ and $A_2 : \mathcal{B} \to \mathcal{A}_*$ are contractions and if $A = A_2 A_1$, there this factorization is called regular if

$$(7.1) \qquad \{D_{A_2} A_1 a \oplus D_{A_1} a \; ; \; a \in \mathcal{A}\}^- = (D_{A_2} \mathcal{B})^- \oplus (D_{A_1} \mathcal{A})^-$$

(see [14], ch. VII, §3).

Lemma 7.1. Let ω be a hypo-projective object in $\mathcal{C}_2$ and let $P : \omega \mapsto \tau = \{T_1, T_2\}$ be a morphism (in $\mathcal{C}_2$) such that P^* is isometric. Then at least one of the factorizations $T_1 \cdot T_2$ and $T_2 \cdot T_1$ of $T_1T_2 = T_2T_1$ is regular.

Proof Without loss of generality we can assume that P is the orthogonal projection of $K (= H_\omega)$ onto $H (= H_\tau)$. Let $\omega = \{V_1, V_2\}$ and let

$$(7.2.) \qquad K_1 = \bigvee_{n \geq 0} V_1^n H.$$

Plainly, $U_1 = V_1/K_1$ is the minimal isometric dilation of T_1, thus $P_1 U_1 = T_1 P_1$, where P_1 denotes the orthogonal projection of K_1 onto H. Therefore (for instance by virtue of Corollary 4.4) there exists a contraction U_2 on K_1 such that $U_1U_2 = U_2U_1$ and $P_1U_2 = T_2T_1$. We set $\sigma = \{U_1, U_2\}$ and since $P_1 : \sigma \mapsto \tau$ is a morphism (in $\mathcal{C}_2$) and P_1^* (as operator from H into K_1) as isometric, there exists a contraction $Q : K \mapsto K_1$ such that

$$(7.3) \qquad P_1 Q = P \quad \text{and} \quad QV_1 = U_1 Q , \quad QV_2 = U_2 Q .$$

From the first relation (7.3.) we easily infer that $Q/H = I_H$ (= the identity on H). Thus from the second relation (7.3) it follows

$$QU_1^n h = QV_1^n h = U_1^n h \qquad (\text{for all } n \geq 0 , \; h \in H),$$

whence, by (7.2), $Q/K_1 = I_{K_1}$ (= the identity on K_1). But Q is the contraction and $QK \subset K_1$.

Therefore Q must be the orthogonal projection of K onto K_1. Now, from the last relation (7.3) we infer that

$$U_2 = QV_2 / K_1$$

is uniquely determined by H, T_1 and T_2. Taking into account the main theorem of [2] we can conclude that at least one of the factorizations $T_1 . T_2 = T_2 . T_1$ is regular.

Corollary 7.2. There exist an object $\sigma = \{U_1, U_2\}$ in $\mathcal{C}_2$ such that U_1 and U_2 are doubly commuting isometries and such that pd $(\sigma) = \infty$.

Proof. Let S be the unilateral shift of multiplicity one on some Hilbert space G. We set $U_1 = S \otimes I$, $U_2 = I \otimes S$, where I denotes the identity on G. Let Ker $S^* = \mathbb{C}\, e_0$ $\|e_0\| = 1$ and $e_1 = Se_0$. Moreover let P denote the orthogonal projection of $K = G \otimes G$ onto $H = \mathbb{C} e_0 \otimes e_0 + \mathbb{C} e_0 \otimes e_1 + \mathbb{C} e_1 \otimes e_0 + \mathbb{C} e_1 \otimes e_1$. Then $P : \sigma \to \tau = \{T_1, T_2\}$ is a morphism, where T_1 and T_2 are defined by $T_1 = PU_1 / H$, $T_2 = PU_2 / H$. Then

$$(D_{T_1} H)^- = \mathbb{C}\, e_1 \otimes e_0 + \mathbb{C}\, e_1 \otimes e_1$$

$$(D_{T_2} H)^- = \mathbb{C}\, e_0 \otimes e_1 + \mathbb{C}\, e_1 \otimes e_1$$

and

$$\{D_{T_1} T_2 h \oplus D_{T_2} h : h \in H\}^- \subset (\mathbb{C} e_1 \otimes e_0) \oplus (D_{T_2} H)^- \neq (D_{T_1} H)^- \oplus (D_{T_2} H)^-$$

thus $T_1 . T_2$ is not regular. By symmetry neither is $T_2 . T_1$ regular and thus by virtue of Lemma 7.1, $\sigma = \{U_1, U_2\}$ is not hypo-projective.

Corollary 7.3. We have

$$(7.4) \qquad pd(\mathcal{C}_2)\ (= id(\mathcal{C}_2)) = \infty.$$

Proof. Obvious.

Lemma 7.4 If $\omega = \{V_1, V_2\}$ is an object of $\mathcal{C}_2$ such that both V_1, V_2 are isometric and at least one of V_1, V_2 is unitary, then ω is hypo-projective.

Proof. Let $P : \sigma = \{U_1, U_2\} \mapsto \tau = \{T_1, T_2\}$ be a morphism (in $\mathcal{C}_2$) such that P^* is an isometry. Again without loss of generality we can assume that K (= H_σ) contains H (= H_τ) and that P is the orthogonal projection of K onto H . We can also assume that V_1 is unitary. Replacing U_1 by its minimal isometric dilation and U_2 by any operator obtained as in Corollary 4.4 (with an obvious change of notations), we can also asume that U_1 is isometric. Since $PU_1 = T_1 P$ we have also

$$(7.5) \qquad T_1^* = U_1^* / H.$$

Thus from

$$(7.6.)\qquad \|A^* h\| = \|V_1^{*n} A_1^* h\| = \|A^* T_1^{*m} h\| \le \|T_1^{*n} h\|.$$

($n = 1,2, \dots; h \in H$) we infer (as in the proof of Corollary 4.6.), that there exists a contraction $A': G (= H_\omega) \mapsto H' = (QH)^-$ such that $A^* = A'^*/(Q/H)$ where Q denotes the orthogonal projection of K onto

$$R = \bigcap_{m \geq 0} U_1^n K.$$

Since, if $V_1 = (U_1/H)^*$, we have

$$V_1^* A'^* (Q/H) = V_1^* A^* = A^* T_1^* = A'^* (Q U_1^*/H) = A'^* V_1'^* (Q/H),$$

we infer that $A'V_1 = V_1'A'$. Also, for $h \in H$,

$$\| Q T_2^* h\| = \lim_{m\to\infty} \| U_1^{*m} T_2^* h\| = \lim_{m\to\infty} \| U_1^{*m} U_2^* h\| = \lim_{m\to\infty} \| U_2^* U_1^{*m} h\| \le$$

$$\le \lim_{m\to\infty} \| U_1^{*m} h\| = \|Qh\|,$$

there exists a contraction V_2' on H' such that $V_2'^* (Q/H) = Q T_2^*$ and

$$V_1'^* V_2'^* (Q/H) = V_1'^* Q T_2^* = U_1^* Q T_2^* = Q T_1^* T_2^* = Q T_2^* T_1^* =$$

$$= V_2'^* V_1'^* (Q/H);$$

Consequently $\sigma' = \{V_1, V_2\}$ is an object of $\mathcal{L}_2$. Also $A': G \to K'$ is a morphism $\omega \leftrightarrow \sigma'$. For this it remains to show that $A'V_2 = V_2'A'$. This follows from

$$V_2^* A'^* (Q/H) = V_2^* A^* = A^* T_2^* = A' Q T_2^* = A' V_2'^* (Q/H).$$

Now let Q' denote the orthogonal projection of K onto H'.

and let

$$(7.6)\qquad H'' = \bigvee_{m \geq 0} U_1^n H', \quad V_1'' = U_1 / H''.$$

Since for $h'' = U_1^m h'$ ($m = 0,1,2, \dots; h' \in H'$), we have, for any $n \geq m$,

$$V_1''^n V_2'^* Q' V_1''^{-m} h'' = V_1''^m V_2'^* h'$$

and

$$V_1''^n A'^* Q' V_1''^{-m} h'' = V_1^m A'^* h',$$

the strong limits A''^* and $V_2''^*$ of the sequences

$\left\{ V_1''^m A'^* Q' V_1''^{-m} \right\}_{m=1}^{\infty}$ and $\left\{ V_1''^m V_2'^* Q' V_1''^{-m} \right\}_{m=1}^{\infty}$,

respectively, exist and satisfy

$$A'' V_1 = V_2'' A'', \quad V_2'' V_1'' = V_1'' V_2''.$$

(For these constructions see [6]). Setting now $B = A''$, we obtain the morphism $B : \omega \mapsto \sigma$ with all the desired properties.

It is clear that all the preceding results can be stated in an equivalent form involving hypo-injectivity instead hypo-projectivity. In particular we have the following.

Corollary 7. 5 . We have

$$(7.7) \quad \{ pd(\omega) : \omega = \{U_1, U_2\},\ U_1, U_2 \text{ isometries} \} = \{0, \infty\}$$

$$(7.8) \quad \{ id(\omega) : \omega = \{U_1, U_2\},\ U_1^*, U_2^* \text{ isometries} \} = \{0, \infty\}$$

Proof. Obvious.

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