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AUTOMORPHISMS OF MANIFOLDS
(a survey)

by

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PREPRINT SERIES IN MATHEMATICS
No.16/1976

BUCUREȘTI

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november 1976

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§§o Introduction

The geometric topology, in the present stage of his development is mainly concerned with manifolds : differentiable, piece-wise linear and topological. As it is well known the first problem of the geometric topology is to understand the structure of manifolds and their classification ; the next important problem is to find out the structure of the automorphisms of manifolds , and along these lines the understanding of the homotopy type of the space (topological group) of automorphisms seems to be very important. Besides its general mathematical interest which is certainly related with topology , geometry and mechanics , the researches on the homotopy type of the group of automorphisms has suggested and stimulated beautiful mathematical problems and progresses in the field of homotopy theory and algebra.

The present "article" is a sort of survey about the progresses towards the understanding of the homotopy type of the groups of automorphisms , which have grown up from the ideas of Cerf, Morlet , Quinn, Antonelli, Kahn, Hatcher, Wagoner, Lashof-Rothenberg and Burghelea. It is certainly incomplete in references and technical results but we hope not in driving ideeas.

The categories where the objects of the geometric topology live in are $\mathcal{D}iff$, $\mathcal{P}\ell$, $\mathcal{T}op$, which we will denote from now by $\mathcal{A}$, $\mathcal{B}$,.... . We will write $\mathcal{A} \leq \mathcal{B}$ to indicate that $\mathcal{A}$ is finer than $\mathcal{B}$, that is any object respectively morphism of $\mathcal{A}$ has a well defined structure of object respectively morphism of $\mathcal{B}$. The objects of a geometric category are compact manifolds with boundary (possibly empty). For such manifolds we consider the topological group of C^{∞}- diffeomorphisms, $\mathcal{D}iff(M)$ endowed with the C^{∞}-topology if M is a differentiable manifold, $\mathcal{H}omeo\,(M)$ the topological group of homeomorphisms endowed with the compact open topology if M is a topological manifold and the simplicial group of $p.l$-homeomorphisms $\mathcal{P}\ell(M)$ if M is a $p.l$-manifold (which is apparentely less natural as in fact the entire $\mathcal{P}\ell$-category but very useful for the understanding of $\mathcal{D}iff\,M$ and $\mathcal{T}op\,M$).

Since our objective is the homotopy type of these topological groups, it will be convenient to use the semi-simplicial description of these spaces (see [B.L.R]).

For a geometric category $\mathcal{A}$ and a manifold $M \in ob\,\mathcal{A}$ with $\partial M = \partial_+ M \cup \partial_- M$, $\partial_+ M \cap \partial_- M$ a codimension one submanifold of ∂M , we define $\mathcal{A}(M; \partial_- M)$ the s.s group of $\mathcal{A}$ -automorphisms whose k-simplexes are $\mathcal{A}$ -automorphisms

$$\begin{tikzcd} M \times \Delta[k] \arrow[rr, "h"] \arrow[dr, "p_2"'] & & M \times \Delta[k] \arrow[dl, "p_2"] \\ & \Delta[k] & \end{tikzcd}$$

commuting with the projections on $\Delta[k]$ and with $h|\partial_+ M \times \Delta[k] = id$.

In particular , if we take $\partial_- M = \phi$ we denote $\mathcal{A}(M; \phi)$ by $\mathcal{A}(M)$ the s.s group of $\mathcal{A}$ -automorphisms which restrict to identity on ∂M . *We also use the notation $\underline{\mathcal{H}}(X)$ for the s.s associative monoid of the homotopy equivalences of X, X being a topological space.*

The problem we are intersted in can now be easily formulated.

<u>Problem A:</u> <u>Describe the homotopy type of $\mathcal{A}(M)$, i.e. homotopy groups, Postnicov invariants , etc.</u>

Problem B : Describe the difference from the homotopy point of view between $\mathcal{A}(M)$ and $\mathcal{B}(M)$ for $\mathcal{B} \geq \mathcal{A}$ and $M \in$ ob $\mathcal{A}$, i.e. the homotopy type of the s.s. complex $\mathcal{B}(M)/\mathcal{A}(M)$.

Until 1968-69 very litle was known about the homotopy type of $\mathcal{A}(M)$ for manifolds of $\dim M \geq 3$. One knew only:

1) $\mathcal{A}(M)$ have the homotopy type of a countable CW-complex , a general but very weak result,

2) The group of connected components (in fact how to compute this group) of $\mathcal{A}(M)$, a particular but deep and stimulative result [C] (for $\pi_1(M)=0$),

3) Some nontrivial homotopy groups of $\mathcal{D}iff\, S^n$ and $\mathcal{D}iff\, D^n$ computations due to Novicov , Milnor , Munkres see [N] and [Mu], and of course,

4) The contractibility of $\mathcal{P}l(D^n)$, and $\mathcal{T}op(D^n)$ obtained by the "so called" <u>conning construction</u> or the Alexander trick.

5) The homotopy type of $\mathcal{A}(M^n)$ for $n \leq 2$; for manifolds of dimension $n \leq 2$ there is no difference between the clasisification of differentiable , piecewise linear and topological manifolds and automorphisms, and these facts are reflected in the homotopy equivalence of $\mathcal{D}iff(M)$, $\mathcal{P}l(M)$, $\mathcal{T}op(M)$. All these spaces have the homotopy type described in [G], their connected component of the identity being a compact Lie group ($SO(2)$, $SO(3)$, $S^1 \times S^1$, S^1) ; this is a consequence of the pioneering work of Knneser, M.H.Hamstrong, S.Smale, J.Eells and C.Earle, etc.

6) Even for the simplest possible manifolds, namely $M = T^n$, S^n, D^n if $n \geqslant 5$ $\mathcal{D}iff(M^n)$ has the homotopy groups rich enough and do not have the homotopy type of a finite CW-complex [N], $[A, B, K]_2$.

The new developments (after 1968-69), more precisely those which we intend to discuss in this article, have been greately influenced by :

1) Smoothing theory for $p.l$- manifolds as it has been developed by Hirsch-Mazur, Lashof - Rothenberg, and the smoothing theory for topological manifolds respectively the triangulation of topological manifolds as it has been developed by Lashof-Rothenberg, Kirby-Siebenmann,

2) Brouder-Novicov-Sullivan-Wall celebrated work about surgery,

3) Cerf's ideeas about concordances.

We will explain these new developments as follows:
Problem B in Part I

The homotopy type of $\mathcal{A}(M)$ at least in stable ranges and localised to odd primes (the meaning of stable ranges will be explained in Part II §§5) is reconstructed from the homotopy type of the s.s group of block automorphisms $\tilde{\mathcal{A}}(M)$ (which will be described in Part I), and $\mathcal{AS}^+(M)$ or $\mathcal{AS}^-(M)$, ∞-loop spaces associated to M, whose homotopy type are tangential homotopy invariants (there is much evidence to believe they are actually homotopy invariants).

These ∞ -loop spaces are direct factors of the ∞ -loop space $\mathcal{S}^{\mathcal{A}}(M)$ defined using concordances and which is a homotopy invariant ; $\mathcal{S}^{\mathcal{A}}$, ${}^{\mathcal{A}}\mathcal{S}^{+}$, ${}^{\mathcal{A}}\mathcal{S}^{-}$ are new homotopy functors with values in the weak homotopy category of ∞ -loop spaces*) , intimately connected to the algebraic K -theory and Whitehead theory of higher order , and they provide the homotopy theory with new objects to study. The discussion on concordances and of the functors $\mathcal{S}^{\mathcal{A}}$, ${}^{\mathcal{A}}\mathcal{S}^{+}$, ${}^{\mathcal{A}}\mathcal{S}^{-}$ is contained in Part II.

Initially we intended to collect some important applications in Part III about :

1) Homotopy type of $Top(n)$, $Pl(n)$,

2) When $f : M^{n+k} \longrightarrow P^n$, a continous map between two compact manifolds, is homotopic to a locally trivial bundle.

3) What kind of differentiable homeomorphisms from Σ^n, an homotopy sphere, to S^n do exist ?

Since at the last moment we heard about new results which can complete substantially what we know about it, we believe prematurely to discuss these aspects indicating only as references $[BL]_1$, $[BL]_2$, $[BLR]$, $[L]$.

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*) The objects of this category are ∞ -loop spaces and the morphisms homotopy classes of ∞ -loop space maps ; a homotopy f_t between two ∞ -loop space maps f_0 and f_1 requires f_t to be a ∞ -loop space map for any t and we say that f_0 and f_1 are weak homotopic if $[f_0]_k$ and $[f_1]_k$, the Postnikov k-terms of f_0 and f_1 are homotopic for any k.

Before starting our survey we mention some definitions and notations. According to the usual conventions in homotopy theory we say that :

1): two spaces (s.s complexes) X and Y have the same k-homotopy type (or are k-equivalent) if their k-th Postnikov terms are homotopy equivalent.

2): G is a weak group if it is a group in the homotopy category of base pointed s.s-complexes or spaces.

3): f is a k-homotopy equivalence, resp. homotopy injection resp. a homotopy surjection etc, if $f_k: X_k \to Y_k$ (where X_k, Y_k, f_k, are the Postnikov terms of X, Y, f) is a homotopy equivalence resp. a homotopy injection resp. a homotopy surjection ...,

4) f and g are weak homotopic if f_k and g_k, the k-th Postnikov term of f and g are homotopic for any k.

One oberves that if $f: X_k \longrightarrow Y_k$ is a k-homotopy equivalence from X to Y there exists a $(k+1)$-dimensional CW-complex K and the maps $f: K \longrightarrow X$, $g: K \longrightarrow Y$ so that $\pi_i(f) = \pi_i(g)$ if $i \leq k+1$; moreover one can assume that $g_* \circ f_*^{-1} = \sigma_*^{-1} \circ \varphi_* \circ \rho_*$ on $\pi_i(X)$ for $i \leq k$ where σ and ρ are the canonical projections $\rho: X \to X_r$, $\sigma: Y \to Y_r$.

One says that two n-manifolds V and V' have the same tangential k-homotopy type if there exists a k-homotopy equivalence $f: V_k \to V'_k$ as above and for f and g constructed as above $f^*(\tau(V)) = g^*(\tau(V'))$. (More details about these considerations can be found in [B.L.R.]).

We will denote by $\mathcal{P}$ the category of finite CW-complexes and continous maps and by $\mathcal{P}^{\mathcal{A}}$ the category whose objects are pairs (X, ξ) consisting of a finite polyhedron and a stable $\mathcal{A}$-euclidean bundle,

Part I.

§§1 Classical groups of geometric topology and Problem B

Let us denote by $\mathcal{A}(n)$ the s.s group of $\mathcal{A}$-automorphisms of R^n fixing the origin. For $\mathcal{A} = \mathcal{D}iff$, $\mathcal{A}(n)$ is homotopy equivalent to $O(n)$ - the orthogonal group. For $\mathcal{A} = \mathcal{P}l$ or $\mathcal{T}op$ we keep the standard notation, $Pl(n)$, and $Top(n)$.

It is also convenient to denote by $G(n)$ the s.s associative monoid (weak group) of proper homotopy equivalences of R^n which fix the origin.

We have the obvious commutative diagram

$$\begin{tikzcd}
O(n) \arrow[r, hook] \arrow[d] & O(n+1) \arrow[r, hook, dotted] \arrow[d] & O(n+i) \arrow[r, hook] \arrow[d] & O(\infty) = O \arrow[d] \\
Pl(n) \arrow[r, hook] \arrow[d] & Pl(n+1) \arrow[r, hook, dotted] \arrow[d] & Pl(n+i) \arrow[r, hook, dotted] \arrow[d] & Pl(\infty) = Pl \arrow[d] \\
Top(n) \arrow[r, hook] \arrow[d] & Top(n+1) \arrow[r, hook, dotted] \arrow[d] & Top(n+i) \arrow[r, hook, dotted] \arrow[d] & Top(\infty) = Top \arrow[d] \\
G(n) \arrow[r, hook] & G(n+1) \arrow[r, hook, dotted] & G(n+i) \arrow[r, hook, dotted] & G(\infty) = G
\end{tikzcd}$$

where $\mathcal{A}(\infty)$ respectively $G(\infty)$ are $\varinjlim \mathcal{A}(n)$ respectively $\varinjlim G(n)$. The homotopy type of O, Pl, Top are partly understood (and we assume for the purpose of our considerations they are known). For instance one knows O, G/Top, G/Pl, $Top/Pl \sim Top(n)/Pl(n) \sim K(Z_2, 3)$ (for $n \geq 5$);

G is the "so called" spectrum of spheres, its homotopy groups being *the* stable homotopy groups of spheres.

The results which will be discussed in this paper give valuable information about the homotopy types of the groups $\mathcal{A}(n)$ see $[BL]_1$ and $[BL]_2$. The groups $\mathcal{A}(n)$, $\mathcal{A}(\infty)$, are usually called the classical groups of the geometric topology.

There exists a natural map $\psi_n^{\mathcal{A}} : \mathcal{A}(n)/\mathcal{A}(n-1) \longrightarrow \Omega\, \mathcal{A}(n+1)/\mathcal{A}(n)$ called suspension and defined as follows : "For h an $\mathcal{A}$ - automorphism of R^n fixing the origin and $0 \leq t \leq 1$, consider $\psi_n^{\mathcal{A}} : I \times \mathcal{A}(n) \longrightarrow \mathcal{A}(n+1)$ defined by the formula $\psi_n^{\mathcal{A}}(t,h) = R_t \circ \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & h \end{array}\right) \circ R_{-t}$ where R_t is the rotation of angle $t\pi$ in the plane of the first two coordinates of R^{n+1} . Clearly $\psi_n^{\mathcal{A}}$ defines $\psi_n^{\mathcal{A}} : \mathcal{A}(n)/\mathcal{A}(n-1) \longrightarrow \Omega\, \mathcal{A}(n+1)/\mathcal{A}(n)$ (see $[BL]_2$) and the following diagram is commutative

$$\begin{tikzcd}
O(n)/O(n-1) \arrow[r, "\psi_n^{Diff}"] \arrow[d] & \Omega\, O(n+1)/O(n) \arrow[d] \\
Pl(n)/Pl(n-1) \arrow[r, "\psi_n^{Pl}"] \arrow[d] & \Omega\, Pl(n+1)/Pl(n) \arrow[d] \\
Top(n)/Top(n-1) \arrow[r, "\psi_n^{Top}"] \arrow[d] & \Omega\, Top(n+1)/Top(n) \arrow[d] \\
G(n)/G(n-1) \arrow[r, "\psi_n"] & \Omega\, G(n+1)/G(n)
\end{tikzcd}$$

If we denote by $F_n^{\mathcal{A},\mathcal{B}}$ the homotopy theoretic fibre of $\mathcal{A}(n)/\mathcal{A}(n-1) \longrightarrow \mathcal{B}(n)/\mathcal{B}(n-1)$, $\mathcal{B} \geq \mathcal{A}$, ψ_n induces $\psi_n^{\mathcal{B},\mathcal{A}} : F_n^{\mathcal{B},\mathcal{A}} \longrightarrow \Omega F_{n+1}^{\mathcal{B},\mathcal{A}}$. For any $\mathcal{A}$ -manifold M^n , let $\mathcal{D}^{\mathcal{A}}(M^n)$ be the principal $\mathcal{A}(n)$ bundle associated to the tangent bundle ; clearly $\mathcal{P}^{\mathcal{A}}(M^n)|\partial M^n$ contains $\mathcal{P}^{\mathcal{A}}(\partial M)$ as principal subbundle . If $\mathcal{B} \geq \mathcal{A}$, hence $\mathcal{A}(n) \subseteq \mathcal{B}(n)$, to the principal bundle $\mathcal{P}^{\mathcal{A}}(M^n)$ we associate the bundle $\mathcal{B}(n)/\mathcal{A}(n) \cdots\cdots \mathcal{E}_{\mathcal{B},\mathcal{A}} \longrightarrow M$ (*) whose fibre is $\mathcal{B}(n)/\mathcal{A}(n)$; this bundle $\mathcal{E}_{\mathcal{B},\mathcal{A}}$ has a canonical crossection s. $\mathcal{E}_{\mathcal{B},\mathcal{A}}(M)/\partial M$ contains $\mathcal{E}_{\mathcal{B},\mathcal{A}}(\partial M)$ as subbundle , and the canonical cros-section of $\mathcal{E}_{\mathcal{B},\mathcal{A}}(\partial M)$ over ∂M is the canonical crossection of $\mathcal{E}_{\mathcal{B},\mathcal{A}}(\partial M)$. With the bundle $\mathcal{P}^{\mathcal{A}}(M)$ we can also as-sociate the bundle $\overset{n}{\mathcal{F}}_{\mathcal{B},\mathcal{A}}(M^n)$ with fibre $F_n^{\mathcal{B},\mathcal{A}}$ and the bundle $\overset{n+1}{\mathcal{F}}_{-\mathcal{B},\mathcal{A}}(M^n)$ resp. $\Omega\overset{n+1}{\mathcal{F}}_{-\mathcal{B},\mathcal{A}}(M^n)$ with fibre $F_{n+1}^{\mathcal{B},\mathcal{A}}$ resp. $\Omega F_{n+1}^{\mathcal{B},\mathcal{A}}$. Then the suspen-sion $\psi_n^{\mathcal{B},\mathcal{A}}$ extends to

$$\begin{array}{ccc}
\overset{n}{\mathcal{F}}_{\mathcal{B},\mathcal{A}} & & \\
& \searrow & \\
\downarrow & & M^n \\
& \nearrow & \\
\Omega\overset{n+1}{\mathcal{F}}_{\mathcal{B},\mathcal{A}} & &
\end{array}$$

Let us denote by $\Gamma(\mathcal{E}_{\mathcal{B},\mathcal{A}})$ the s.s complex of cros-sections of (*) which agree to s on ∂M, and by $\Gamma(\mathcal{E}_{\mathcal{B},\mathcal{A}};\partial M)$ the s.s complex of crossections of (*) which agree to s on $\partial_+ M$ and lie in $\mathcal{E}_{\mathcal{B},\mathcal{A}}(\partial M)$ if restricted to $\partial_- M$. The differential of an $\mathcal{B}$ -automorphism defines the map

$$d_{\mathcal{B},\mathcal{A}}:\ \mathcal{B}(M;\partial_- M)/\mathcal{A}(M;\partial_- M) \longrightarrow \Gamma(\mathcal{E}_{\mathcal{B},\mathcal{A}};\partial_- M)$$

The following result has the origin in the pioneering work of Cerf $[C]_2$; it has been stated and sketchely proved by Morlet $[M_o]$ and proved in $[BL]_1$ (We call $f: X \longrightarrow Y$ an I.H.E - map if it induces an injective correspondence for connected components and an homotopy equivalence on any connected component).

Theorem I 1.1 Assume M^n an $\mathcal{A}$-manifold . Then

$$d_{\mathcal{B},\mathcal{A}} : \mathcal{B}(M^n;\partial_- M)/\mathcal{A}(M^n;\partial_- M) \longrightarrow \Gamma(\mathcal{E}_{\mathcal{B},\mathcal{A}};\partial_- M)$$

is an I.H.E. map provided $\mathcal{A}=\mathcal{D}iff$, $\mathcal{B}=\mathcal{P}l$ or $\mathcal{A}=\mathcal{D}iff, \mathcal{P}l$, $\mathcal{B}=\mathcal{T}op$ and $n\neq 4$ $\partial_- M=\phi$ or $\mathcal{A}=\mathcal{D}iff$, $\mathcal{P}l$, $\mathcal{B}=\mathcal{T}op$ and $n\neq 4,5$.

If M^n is paralelizable and $\partial_- M=\phi$ then $\Gamma(\mathcal{E}_{\mathcal{B},\mathcal{A}}) \sim Map(M,\partial M; \mathcal{B}(n)/\mathcal{A}(n))$.

Corollary I 1.2 :

a) $\mathcal{D}iff(D^n) \sim \Omega^{n+1}(\mathcal{P}l(n)/O(n))$

b) $\mathcal{D}iff(D^n) \sim \Omega^{n+1}(Top(n)/O(n))$

Corollary I 1.3 : For $M=N\times I$, $\partial_- M=N\times\{1\}$ the differential $d_{\mathcal{B},\mathcal{A}} : \mathcal{B}(N\times I;\partial_- M=N\times\{1\})/\mathcal{A}(N\times I;\partial_- M) \longrightarrow \Gamma(\mathcal{E}_{\mathcal{B},\mathcal{A}};\partial_- M)=\Gamma(\mathcal{F}^n_{\mathcal{B},\mathcal{A}})$ is an I.H.E map provided $\mathcal{B}=\mathcal{P}l$ or, $\mathcal{B}=\mathcal{T}op$ and $n\neq 4,5$.

Corollary I 1.4:

a) $\mathcal{D}iff(D^n; D^{n-1}_-) \sim \Omega^n F_n^{\mathcal{P}l,\mathcal{D}iff}$

b) $\mathcal{D}iff(D^n; D^{n-1}_-) \sim \Omega^n F_n^{\mathcal{T}op,\mathcal{D}iff}$ $n\neq 4,5$

§§ 2 Block automorphisms

The surgery methods , among many other things give homotopy criteria to decide when an homotopy equivalence is homotopic to an $\mathcal{A}$ -automorphism. They are also able to say how many such automorphisms (up to an equivalence) correspond (i.e. are homotopic) to an homotopy equivalence . The right equivalence of $\mathcal{A}$ -automorphism which follows from the surgery methods turns out to be not "isotopy" but "concordance" . To understand the difference it is very useful to recall their definitions simultaneously :

" $h_1, h_2 : M \longrightarrow M$ are isotopic iff there exists an $\mathcal{A}$ -automorphism $H : M \times I \longrightarrow M \times I$ commuting with the projection on I , with $H/M \times \{0\} = h_1$ and $H/M \times \{1\} = h_2$; they are concordant iff there exists an $\mathcal{A}$ -automorphism $H : M \times I \longrightarrow M \times I$ with $H/M \times \{0\} = h_1$ $H/M \times \{1\} = h_2$".

This definition suggests the introduction of an other s.s group $\tilde{\mathcal{A}}(M)$ (which contains $\mathcal{A}(M)$ as a subgroup) the group of block - automorphisms ; this group is naturally related to the surgery methods and smothing theory methods . It has been defined simultaneously by many authors - Morlet , Quinn , Rourke-Sanderson-Casson Antonelli-Burghelea-Kahn (see [B.L.R])

A k -simplex of $\tilde{\mathcal{A}}(M)$ is an $\mathcal{A}$ -automorphism $h : M \times \Delta[k] \longrightarrow M \times \Delta[k]$ with $h \mid M \times d_i(\Delta[k]) \subset M \times d_i(\Delta[k])$

and $h|\partial M \times \Delta[k] = id$ *). Obviously $\mathcal{Q}(M) \subset \tilde{\mathcal{Q}}(M)$ can be viewed as an approximation of $\mathcal{Q}(M)$. The interest of $\tilde{\mathcal{Q}}(M)$ comes first from the fact that both problems A and B can be, at least theoretically solved via the right parametrised version of surgery theory and smoothing theory. We explain below what we mean by at least theoretically solved and give the results of $[A.B.K]_1$ (which have been also proved in part by Quinn in his unpublished thesis). For this purpose let us denote by $\mathcal{H}(M)$ the space of continous maps which are homotopy equivalences which restrict to identity on ∂M. From tehnical reason we will describe $\mathcal{H}(M)$ as an s.s-associative monoid (or weak-group) whose k-simplexes are homotopy equivalences $h: M \times \Delta[k] \longrightarrow M \times \Delta[k]$ with $h(M \times d_i\Delta[k]) \subset M \times d_i\Delta[k]$, $h|\partial M \times \Delta[k] = id$.

The results of $[A.B.K]_1$ allow to reconstruct $\tilde{\mathcal{Q}}(M)$ from

1) $\mathcal{H}(M)$, 2) $Maps(M, G/_{Top})$ or $Maps(M, G/_{PL})$ or $Maps(M, G/_O)$,

3) the algebraic L-theory of Wall which is a functor L from the category of groups with orientation ; $\mathcal{G}_o$ (**) to the homotopy category of ∞-loop spaces Ω^h so that $\pi_i(L(G,\omega)) = L_i(G,\omega)$

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*) We also assume some tehnical requirements namely to be "product like" near corners (see [BLR]) which allows us to define the degeneracies.

**) A group with orientation means a group G, together with a group-homomorphism $\omega: G \longrightarrow Z_2$ and a morphism of group with orientations $f:(G_1,\omega_1) \longrightarrow (G_2,\omega_2)$ is a group-homomorphism $f: G_1 \longrightarrow G_2$ making commutative the diagram

$$\begin{tikzcd} G_1 \arrow[rr, "f"] \arrow[dr, "\omega_1"'] & & G_2 \arrow[dl, "\omega_2"] \\ & Z_2 & \end{tikzcd}$$

By Alexander trick $\widetilde{\mathcal{P}\ell}(D^n)$ and $\widetilde{Top}(D^n)$ are contractible and by smoothing theory $\widetilde{\mathcal{D}iff}(D^n) \sim \Omega^{n+1}(PL/O)$.

<u>Theorem I. 2.1</u> $[A.B.K]_1$ a) $\widetilde{\mathcal{P}l}(M^n)/\widetilde{\mathcal{D}iff}(M^n) \longrightarrow Maps(M^n, \partial M^n; PL/O, *)$ is an homotopy equivalence

b) $\widetilde{Top}(M^n)/\widetilde{\mathcal{D}iff}(M^n) \longrightarrow Maps(M^n, \partial M^n; Top/O, *)$ is an homotopy equivalence for $n \geq 5$

c) $\widetilde{Top}(M^n)/\widetilde{\mathcal{D}iff}(M^n) \longrightarrow Maps(M^n, \partial M^n; Top/Pl, *)$ is an homotopy equivalence for $n \geq 5$.

<u>Theorem I. 2.2</u> $[A.B.K]_1$: For $n \geq 5$ the natural map $\tilde{d}: \mathcal{H}(M)/\tilde{\mathcal{a}}(M) \longrightarrow Maps(M, \partial M; G/\mathcal{a}, *)$ has as fibre $\Omega^{n+1}(L(\pi_1(M), \omega))$ where the orientation $\omega: \pi_1(M) \longrightarrow \mathbb{Z}_2$ is the first Stiefel-Whitney class.

Theorem I 2.2 makes clear the importance of the algebraic L-theory for our geometric problem. Most often by an algebraic theory we understand a system of functors T_n, defined on an "algebraic category" like, the category of groups, groups with orientations rings, etc, with values in the category of groups, satisfying some naturality-properties which resemble to homology, K-theory etc. All the nice algebraic theories we know can be obtained from functors defined on an "algebraic category" with values in the weak homotopy category of ∞-loop spaces, composed with the homotopy-groups functors. It was Quinn [Q] the first who observed that the algebraic L-theory can be obtained in this way.

An other algebraic theory which relates the two sorts of algebraic L-theory is the algebraic A-theory invented by Rothenberg ; it was proved in $[B]_1$ that this theory can be obtained from a functor $A: \mathcal{G} \longrightarrow \Omega^h$ with $\mathcal{G}$ the category of groups. The geometric significance of the algebraic A-theory becomes deeper when one compairs $\tilde{\mathcal{Q}}(M)$ and $\tilde{\mathcal{Q}}(M\times S^1)$.

Theorem I 2.3 $[B]_1$ For any manifold $M^n \in ob\, \mathcal{Q}$ there exists an obvious map $\mathcal{Q}_\omega: \tilde{\mathcal{Q}}(M)\times\Omega\tilde{\mathcal{Q}}(M) \longrightarrow \tilde{\mathcal{Q}}(M\times S^1)$ if $\partial M\neq\phi$ and $\mathcal{Q}_\omega: \tilde{\mathcal{Q}}(M)\times\Omega\tilde{\mathcal{Q}}(M)\times S^1 \longrightarrow \tilde{\mathcal{Q}}(M\times S^1)$ if $\partial M=\phi$ so that its homotopy theoretical fibre $\mathcal{Q}_{F(M)}$ is homotopy equivalent to $\Omega^n A(\pi_1(M))$ if $n\geqslant 5$. Moreover $\Omega^4 A(G)\sim A(G)$ and if $Wh_1(G)=0$ then $A(G)$ is contractible. (The homotopy groups of $A(G)$ are the so called Rothenberg groups $A_i(G)$).

Observation . From Theorem I 2.3 it is clear that $A(G)_{odd}$ is contractible since $A_i(G)$ are all 2-primary groups, hence $\tilde{\mathcal{Q}}(M\times S^1)_{odd}$ is homotopy equivalent to $(\tilde{\mathcal{Q}}(M)\times\Omega\tilde{\mathcal{Q}}(M))_{odd}$ if $\partial M\neq\phi$ and to $(\tilde{\mathcal{Q}}(M)\times\Omega\tilde{\mathcal{Q}}(M)\times S^1)_{odd}$ if $\partial M=\phi$ $n\geqslant 4$.

We also know that the "localization to odd primes" (which looses the 2-primary information) substantially simplifies the description of the structure of the algebraic L-theory as it was noticed by Novicov , Karoubi etc. We will see in Part II how much we get outside the "prime 2" (i.e. localising to odd primes) about the structure of $\mathcal{A}(M)$.

I should mention that since the appearence of [A B K]$_1$ a lot of work has been done towards the computation of $\pi_i(\tilde{\mathcal{A}}(M))$ for $\mathcal{A} = \mathcal{D}iff$ but it is not in the intention of this article to report about.

Part II

Some hystory about concordances

§§ 3.

The introduction of $\mathcal{C}^{a}(M)$ for the purpose of the understanding of $\pi_0(\mathcal{D}iff(M))$ $(\mathcal{C}^{a}(M) = a(M\times I; \partial_-(M\times I) = M\times\{1\})$ is due to Cerf. He noticed first the isomorphism between the homotopy groups of $B\mathcal{C}^{Diff}(M)$ and the relative homotopy groups of $(\mathcal{F}, \mathcal{F}_0)$ where $\mathcal{F}$ denotes the space of all Morse functions on $M\times I$ and $\mathcal{F}_0$ the space of Morse functions with no critical points (both endowed with the C^∞-topology), and he began the investigation of these relative homotopy groups filtrating $\mathcal{F}$ by codimension. In order to describe the low dimension strata which are necessary for the computation of $\pi_1(\mathcal{F}, \mathcal{F}_0)$ he uses the Thom theory of universal unfoldings. Cerf worked out the case of simply-connected manifolds and much of the geometry involved in the nonsimply-connected case was developed by Chenciner and Laudebach. The full computation of $\pi_1(\mathcal{F}, \mathcal{F}_0)$ have been done in the nonsimply-connected case by Hatcher-Wagoner [H-W] and Volodin. In our considerations the importance of $\mathcal{C}^{a}(M)$ comes from the following resons:

1) "Concordances" allow us to formulate the disjunction lemma as noticed first by Morlet, and this has remarkable consequences as one can see from Theorems II 4.2, 4.3, 4.4, 4.5 (§§4).

2) "Concordances" have good stability properties as was noticed first by Hatcher [H] *for* $\mathcal{A} = \mathcal{P}\ell$

Remarkable connections between "concordances" and the classical groups of geometric topology have been noticed since long time by Kuiper-Lashof [K.L] , and they *are basic ingredients in the proof of stability for* $\mathcal{A} = \mathcal{D}iff$.

Recall that for a manifold $M \in ob\, \mathcal{A}$ we denote by $\mathcal{C}^{\mathcal{A}}(M)$ the s.s group $\mathcal{A}(M \times I\, ;\, M \times \{1\})$. Analogously we define $\tilde{\mathcal{C}}^{\mathcal{A}}(M)$ as the s.s complex whose k-simplexes are $\mathcal{A}$-automorphisms $h: \Delta[k] \times I \times M \longrightarrow \Delta[k] \times I \times M$ with $h(d_i \Delta[k] \times I \times M) \subset d_i \Delta[k] \times I \times M$ $h \mid \Delta[k] \times (I \times \partial M \cup \{0\} \times M) = id$. plus some extraproperties (near corners) which allow us to define the degeneracies. As one easy notice , $\tilde{\mathcal{C}}^{\mathcal{A}}(M)$ is contractible but this s.sgroup is still interesting since it permits a simple geometric description of $B\mathcal{C}^{\mathcal{A}}(M)$ the classifying space of $\mathcal{C}^{\mathcal{A}}(M)$ as the quotion $\tilde{\mathcal{C}}^{\mathcal{A}}(M) / \mathcal{C}^{\mathcal{A}}(M)$. Having defined $\mathcal{C}^{\mathcal{A}}(M)$ and $B\mathcal{C}^{\mathcal{A}}(M)$ we can formulate for them Problem A and Problem B. The homotopy reduction of Problem B has been already discussed in Part I since $\mathcal{C}^{\mathcal{A}}(M) = \mathcal{A}(M \times I\, ;\, M \times \{0\})$ but we will come back to it in this Part , in connection with "stability".

§§4. Morlet's disjunction lemma and its applications

As we have mentioned above , one of the main reasons of our interest for concordances comes from the possibility to formulate the "disjunction lemma". This lemma , a beautiful piece of geometric topology has been stated and sketchely proved by Morlet [M_σ] . A complete proof following Morlet's ideeas has been given in [BLR] and a different proof has been also produced by K.Millet [Mi] .

Let $(V^n, \partial V^n)$ be a compact manifold with boundary and $f:(D^p,\partial D^p) \longrightarrow (V^n, \partial V^n)$ $g:(D^q,\partial D^q) \longrightarrow (V^n,\partial V^n)$ be two proper embeddings i.e. they intersect transversaly ∂V and $f^{-1}(\partial V)$ resp. $g^{-1}(\partial V)$ are ∂D^p resp. ∂D^q . Assume that $f(D^p) \cap g(D^q) = \phi$. For an embedding f as above we define $\mathcal{C}^{\mathcal{a}}_{emb}(D^p, V; f)$ the s.s complex whose k-simplexes are embeddings $h: \Delta[k] \times I \times D^p \longrightarrow \Delta[k] \times I \times V^n$ commuting with the projection on $\Delta[k]$ and satisfying the following supplementary properties :

1) $h_t:(I \times D^p) \longrightarrow (I \times V^n)$ intersects transversally with $h^{-1}(\partial(I \times V^n)) = \partial(I \times D^p)$.

2) $h_t \mid I \times \partial D^p = id_I \times f$ and $h_t \mid \{0\} \times D^p = f$.

Theorem II 4.1. (Morlet's disjunction Lemma) . If $n-q \geq 3$ and $n-p \geq 3$ then $\pi_i(\mathcal{C}^{\mathcal{a}}_{emb}(D^p, V; f), \mathcal{C}^{\mathcal{a}}_{emb}(D^p, V \setminus g(D^q); f)) = 0$ for $i \leq 2n-p-q-5$ (If $\mathcal{a} = \mathcal{Top}$ one requires $n \geq 5$).

We may say that Theorem 4.1 has been stated by Morlet only for $\mathcal{a} = \mathcal{Diff}$ and $\mathcal{Pl}$. For $\mathcal{a} = \mathcal{Top}$ this Theorem is due to Erick Pedersen. (see Appendix in [B L R]).

The handlebody-structure of a manifold , the obvious relation of spaces of "concordences of embeddings" associated to a handlebdy decomposition and the relations between spaces of embeddings and concordances of embeddings (as for instance the fibration $\mathcal{E}mb^{\mathcal{A}}(D^{p+1}, M^n \times I; f\times id) \longrightarrow \mathcal{C}^{\mathcal{A}_p}_{emb}(D^p, M^n; f) \longrightarrow \mathcal{E}mb^{\mathcal{A}}(D^p, M; f)$ for $n-p \geq 3$ permits the proof of the following theorems ([B.L.R.]):

<u>Theorem II 4.2.</u> Let V^n be a compact k-connected manifold, $n \geq 5$ and $D^n \subset \mathrm{Int}\, V^n$. Then

a) $\mathcal{A} = \mathcal{D}iff$; $\pi_j(\mathcal{A}(V^n)/\mathcal{A}(D^n)) \longrightarrow \pi_j(\tilde{\mathcal{A}}(V^n)/\tilde{\mathcal{A}}(D^n))$ is an isomorphism if 1) $j \leq 2k-3$ ($k \leq n-4$ if ∂V not 1-connected or $n=5$), $j \leq 2k-2$ if $(k+1) < n/2$ and either $k \equiv 2,4,5,6 \pmod 8$ or TV is trivial over the $(k+1)$-skeleton.

b) $\mathcal{A} = \mathcal{P}l$ or $\mathcal{T}op$; $\pi_j(\mathcal{A}(V)) \longrightarrow \pi_j(\tilde{\mathcal{A}}(V))$ is an isomorphism if

1) $j \leq \inf(2k-3, k+2)$ ($k \leq n-4$ if ∂V not 1-connected or $n=5$)

2) $j \leq 2k-2$ if $(k+1) < n/2$ and either $k = 2,4$ or $k=3$ and TV is trivial over the 4- skeleton.

<u>Theorem II 4.3.</u> Let V^n be a compact k-connected manifold $n \geq 5$ and $D^n \subset \mathrm{Int}\, V^n$. Then : a) $\mathcal{A} = \mathcal{D}iff$ $\pi_j(\mathcal{C}^{\mathcal{A}}(D^n)) \to \pi_j(\mathcal{C}^{\mathcal{A}}(V))$ is an isomorphism for

1) $j \leq 2k-4$ ($k \leq n-4$ if ∂V not 1-connected or $n=5$)

2) $j \leq 2k-2$ if $k \leq n-4$ ($(k+1) < n/2$ and either $k = 2,4,5,6 \pmod 8$

or TV is trivial over the $(k+1)$-skeleton

b) $\mathcal{A} = \mathcal{P}\ell$ or $\mathcal{T}op$ $\pi_i(\mathcal{C}^{\mathcal{A}}(V)) = 0$ if

1) $j \leq \inf(2k-3, k+2)$ $k \leq n-4$ if ∂V not 1-connected or

2) $j \leq 2k-2$ if $(k+1) < n/2$ and either $k = 2, 4$ or $k = 3$ and TV is trivial over the 4-skeleton of V.

<u>Theorem II 4.4</u>. Suppose V^n, V'^n are k-connected compact manifolds of the same r-tangential homotopy type $n/2 > r+1 \geq k$. Then $\tilde{\mathcal{A}}(V)/\mathcal{A}(V)$ and $\tilde{\mathcal{A}}(V')/\mathcal{A}(V')$ have the same j-homotopy type for

a) $\mathcal{A} = \mathcal{D}iff$ if $j \leq \inf(2r-1, r+k-1)$

b) $\mathcal{A} = \mathcal{P}\ell$ or $\mathcal{T}op$ if $j \leq \inf(2r-1, r+k-1, r+3)$.

<u>Theorem II 4.5</u> Suppose V^n and V'^n are k-connected compact manifolds of $\dim n \geq 5$ and V and V' of the same r-tangential homotopy type $n/2 > r+1 \geq k$. Then $\mathcal{C}^{\mathcal{A}}(V)$ and $\mathcal{C}^{\mathcal{A}}(V')$ have the same j-homotopy type for

a) $\mathcal{A} = \mathcal{D}iff$ $j \leq \inf(2r-2, r+k-2)$

b) $\mathcal{A} = \mathcal{P}\ell, \mathcal{T}op$ $j \leq \inf(2r-2, r+k-2, r+2)$

These theorems shows that for both $\mathcal{C}^{\mathcal{A}}(M)$ and $\tilde{\mathcal{A}}(M)/\mathcal{A}(M)$ the Postnicov $\lambda^{\mathcal{A}}(n)$-th term depends only or the tangential homotopy type of M, where $\lambda^{\mathcal{A}} : Z_+ \longrightarrow Z_+$ is an increasing function with $\lambda^{\mathcal{A}}(n) \geq n-2$.

§§ 5. Stability

The next remarkable fact about concordances are their stability properties. The "stability" has been discovered for $\mathcal{A} = \mathcal{P\ell}$ by A.Hatcher [H] and then proved for $\mathcal{A} = \mathcal{D}iff$ by Burghelea Lashof $[BL]_2$. The proof for $\mathcal{A} = \mathcal{T}op$ is a trivial consequence of the relationship between $\mathcal{P\ell}$ -concordances and $\mathcal{T}op$ -concordances for a $p\text{-}\ell$ - manifold. Unfortunately for a moment the proof of the "stability property" of concordances for $\mathcal{A} = \mathcal{D}iff$, $\mathcal{T}op$ depends on the proof for $\mathcal{A} = \mathcal{P\ell}$; we don't know a proof simultaneously true in all geometric catgeories. To explain the stability property we describe the "transfer map " for concordances as we will briefly sketch below.

Let $E \xrightarrow{\xi} B$ be a locally trivial $\mathcal{A}$ -bundle , $E, B \in ob\,\mathcal{A}$, called for short an $\mathcal{A}$ -bundle. There exists a well defined homotopy class $\mathcal{A}\ell^{\xi} : \tilde{\mathcal{C}}^{\mathcal{A}}(B)/\mathcal{C}^{\mathcal{A}}(B) \longrightarrow \tilde{\mathcal{C}}^{\mathcal{A}}(E)/\mathcal{C}^{\mathcal{A}}(E)$ called the "transfer map" satisfying the following properties :

1). If $E_1 \xrightarrow{\xi_1} E_2 \xrightarrow{\xi_2} E_3$ are two $\mathcal{A}$ -bundles then $\ell^{\xi_2 \cdot \xi_1} = \ell^{\xi_1} \circ \ell^{\xi_2}$.

2) If $N^n \subset B^n$, $\xi : E \to B$ is an $\mathcal{A}$ bundle , $\xi' = \xi : \xi^{-1}(N) \to N$ is the restriction of ξ to N and

$$i_{N,B} : \tilde{\mathcal{C}}^{\mathcal{A}}(N)/\mathcal{C}^{\mathcal{A}}(N) \longrightarrow \tilde{\mathcal{C}}^{\mathcal{A}}(B)/\mathcal{C}^{\mathcal{A}}(B)$$

$$i_{E',E} : \tilde{\mathcal{C}}^{\mathcal{A}}(E')/\mathcal{C}^{\mathcal{A}}(E') \longrightarrow \tilde{\mathcal{C}}^{\mathcal{A}}(E)/\mathcal{C}^{\mathcal{A}}(E)$$

$E' = \xi^{-1}(N)$, are the natural s.s maps induced by the inclusions $N'^n \subset B^n$ and $E' \subset E$, then $\ell^\xi \cdot i_{N,B} \sim i_{E',E} \cdot \ell^{\xi'}$.

3) If $\mathcal{B} \geq \mathcal{A}$ and $r^B_{\mathcal{B},\mathcal{A}} : \tilde{\mathcal{C}}^{\mathcal{A}}_{(B)} / \mathcal{C}^{\mathcal{A}}_{(B)} \longrightarrow \tilde{\mathcal{C}}^{\mathcal{B}}(B) / \mathcal{C}^{\mathcal{B}}_{(B)}$

are the s.s-maps defined by regarding $\mathcal{A}$ -concordances as $\mathcal{B}$ -concordances , then ${}^{\mathcal{B}}\ell^\xi \cdot r^B_{\mathcal{B},\mathcal{A}} \sim r^E_{\mathcal{B},\mathcal{A}} \cdot \ell^\xi$.

If ξ is the trivial bundle and K its fibre we will write ℓ^K instead of ℓ^ξ .

The reason we call ℓ^ξ "transfer map" comes from the fact that for $\mathcal{B} \geq \mathcal{A}$ ℓ^ξ induces

$$\ell^\xi_{\mathcal{B},\mathcal{A}} : \mathcal{C}^{\mathcal{B}}(M) / \mathcal{C}^{\mathcal{A}}(M) \longrightarrow \mathcal{C}^{\mathcal{B}}(E) / \mathcal{C}^{\mathcal{A}}(E) \quad \text{and} \quad \mathcal{C}^{\mathcal{B}}(\dots) / \mathcal{C}^{\mathcal{A}}(\dots)$$

in "stable ranges" behaves like a generalised homology theory while $\ell^\xi_{\mathcal{B},\mathcal{A}}$ like the transfer morphism associated the fibration $E \longrightarrow M$.

Since we are able to prove the expected algebraic properties of the "transfer map" only for the trivial bundle, we will define ℓ^ξ only in this particular case. If K is an $\mathcal{A}$ -manifold with empty boundary define $\Omega\ell^K : \mathcal{C}^{\mathcal{A}}(M) \to \mathcal{C}^{\mathcal{A}}(M \times K)$ by $\Omega\ell^K(h) = h \times id_K$. A similar definition works for $\tilde{\mathcal{C}}^{\mathcal{A}}(\dots)$ and therefore it induces $\ell^K : \tilde{\mathcal{C}}^{\mathcal{A}}(M) / \mathcal{C}^{\mathcal{A}}(M) \longrightarrow \tilde{\mathcal{C}}^{\mathcal{A}}(M \times K) / \mathcal{C}^{\mathcal{A}}(M \times K)$.

Assume now $\partial K \neq \phi$ and in this case regard K as $K_0 \cup \partial K \times [0,1]$ i.e. we specify a particular collar of the neighborhood which we identify with $\partial K \times [0,1]$ and denote by K_0 the closure of the complement of this collar. We define again the group homomorphism $\Omega\ell^K : \mathcal{C}^{\mathcal{A}}(M) \longrightarrow \mathcal{C}^{\mathcal{A}}(M \times K)$ as follows :

For a concordance $h \in \mathcal{C}(M)$ i.e. $h: M\times I \longrightarrow M\times I$, $I=[0,1]$ we consider the $\mathcal{A}$ -automorphism $h\times id_K: M\times K_0\times I \longrightarrow M\times K_0\times I$; $h\times id_K$ is not yet a concordance since $h\times id_{K_0}$ is not identity if restricted to $M\times\partial K_0\times I$, but $h\times id_{\partial K_0}$ is a concordance. We construct a concordance $\Omega \ell^K(h)$ taking $\Omega \ell^K(h) = h\times id_{K_0}$ on $M\times K_0\times I$ and rotating the concordance $h\times id_{\partial K_0}$ around $M\times\partial K_0$ inside $M\times\partial K_0\times[0,1]\times I$ as indicated in Fig. 1

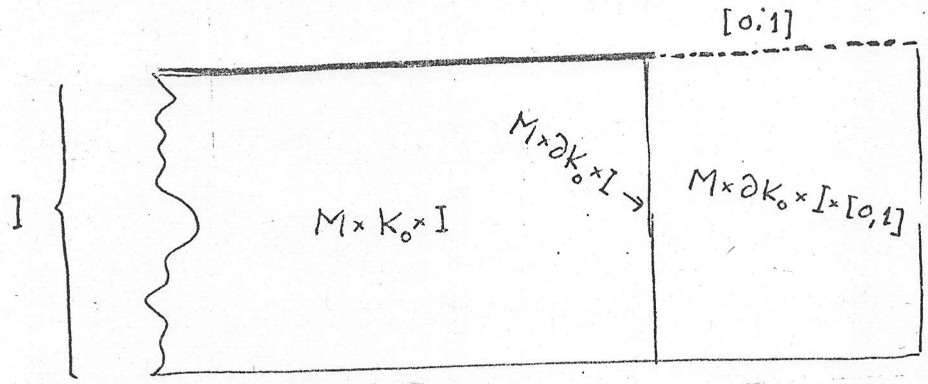

Fig. 1.

To be precised represent a point u in $[0,1]\times I$ (Fig. 2) by its polar coordinates (r,θ), r being the distance from u to A and $\theta = \measuredangle BAu \in [0,\frac{\pi}{2}]$;

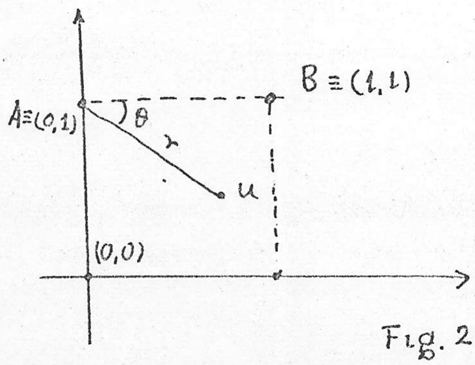

Fig. 2

write $h(m,t) = (h_1(m,t), h_2(m,t)) \in M \times I$ and define $\Omega \ell^K(h)$ by the formula:

$$\Omega \ell^K(h)(m,k,t) = (h_1(m,t), k, h_2(m,t)) \qquad \text{for } k \in K_0$$

$$\Omega \ell^K(h)(m,k,t) = \begin{cases} h_1(m, 1-r), k', 1-h_2(m, 1-r), \theta) & \text{for } 0 \leq r \leq 1 \\ (m, k', r, \theta) & \text{if } r \geq 1,\ m \in M,\ k' \in \partial K_0 \\ (r,\theta) \in [0,1] \times I & . \end{cases}$$

$\Omega \ell^K$ defined in this way is an s.s group homomorphism and it extends (by a similar formula) to a group homomorphism $\Omega \ell^K : \tilde{\mathcal{C}}^{\mathcal{A}}(M) \longrightarrow \tilde{\mathcal{C}}^{\mathcal{A}}(M \times K)$, hence it induces the s.s-map $\ell^K : \tilde{\mathcal{C}}^{\mathcal{A}}(M)/\mathcal{C}^{\mathcal{A}}(M) \longrightarrow \tilde{\mathcal{C}}^{\mathcal{A}}(M\times K)/\mathcal{C}^{\mathcal{A}}(M\times K)$. It is pretty obvious that ℓ^K (up to a homotopy) does not depends on the choosen collar.

A particular case of this construction is the case when $K = I$ and $K = D^r$. From now on we denote ℓ^I by Σ and ℓ^{D^r} by Σ_r; It is easy to observe that $(\Sigma)^r \sim \Sigma_r$ ~~by a preci-sed~~ homotopy because of Property a) of the "transfer map".

We are ready now to state the stability theorems.

<u>Theorem II 5.1</u> : For any geometric category $\mathcal{A}$, there exists an increasing function $\omega^{\mathcal{A}} : Z_+ \longrightarrow Z_+$ with $\lim_{n\to\infty} \omega^{\mathcal{A}}(n) \to \infty$ so that the transfer map, Σ called also „suspension", $\Sigma : \tilde{\mathcal{C}}^{\mathcal{A}}(M)/\mathcal{C}^{\mathcal{A}}(M) \longrightarrow \tilde{\mathcal{C}}^{\mathcal{A}}(M\times I)/\mathcal{C}^{\mathcal{A}}(M\times I)$ induces an $\omega^{\mathcal{A}}(\dim)$ - homotopy equivalences ; moreover $\omega^{\mathcal{A}}(n) \geq n - 10/3$ for and $\mathcal{A} = \mathcal{P}\ell$ and $\mathcal{T}op$ and $\omega^{\mathcal{A}}(n) \geq \frac{n-31}{6}$ for $\mathcal{A} = \mathcal{D}iff$.

For $\mathcal{A} = \mathcal{P}\ell$ this theorem was proved by Hatcher [H] and need the construction of the simply-homotopy type-space $\mathcal{S}(X)$ which Hatcher associates to any finite polyhedron. For $\mathcal{A} = \mathcal{T}op$ the proof is a simple consequence of the case $\mathcal{A} = \mathcal{P}\ell$, Theorem I 1.1 and of the homotopy equivalence of Top/Pl with $Top(n)/Pl(n)$ for $n \geq 5$. The last two facts imply

$$\tilde{\mathcal{C}}^{\mathcal{P}\ell}(M)/\mathcal{C}^{\mathcal{P}\ell}(M) \sim \tilde{\mathcal{C}}^{\mathcal{T}op}(M)/\mathcal{C}^{\mathcal{T}op}(M) \text{ for } \dim M \geq 5 .$$

For $\mathcal{A} = \mathcal{D}iff$ Theorem II 5.1 is due to $[BL]_2$ and its proof is based on :

-i) The compatibility between the "suspension" - map for classical groups and the suspension "Σ" .

-ii) The Kuiper-Lashof's results $[K\text{-}L]$ which connect the topological concordances (resp. $p.\ell$ - concordances) to $Top(n+1)/Top(n)$ (resp. $Pl(n+1)/Pl(n)$).

-iii) The truth of Theorem II 5.1 for $\mathcal{A} = \mathcal{P}\ell$; the proof goes on the following lines.

With the notations of Part.I §§ 1 we can identity $\Gamma(\mathcal{F}^{n}_{\mathcal{B}\mathcal{A}}(M))$ to $\Gamma(\mathcal{E}^{(M\times I)}_{\mathcal{B}\mathcal{A}})$ and i) means the commutativity of the following diagram whose horisontal lines are I.H.E.- maps .

$$\begin{tikzcd}
\mathcal{C}^{\mathcal{B}}(M)/\mathcal{C}^{\mathcal{A}}(M) \arrow[r, "d"] \arrow[dd, "\Sigma"] & \Gamma(\mathcal{F}^{n}_{\mathcal{B},\mathcal{A}}(M), s) \arrow[d, "\Gamma(\tilde{\varphi}_n^{\mathcal{B},\mathcal{A}})"] \\
 & \Gamma(\Omega \mathcal{F}^{n+1}_{\mathcal{B},\mathcal{A}}(M)) \arrow[d, phantom, "\wr"] \\
\mathcal{C}^{\mathcal{B}}(M\times I)/\mathcal{C}^{\mathcal{A}}(M\times I) \arrow[r] & \Gamma(\mathcal{F}^{n+1}_{\mathcal{B},\mathcal{A}}(M\times I))
\end{tikzcd}$$

Fig 3

If M has a trivial tangent bundle then the diagram above becomes

$$\begin{tikzcd}
\mathcal{C}^{\mathcal{B}}(M)/\mathcal{C}^{\mathcal{A}}(M) \arrow[r] \arrow[d, "\Sigma"] & \mathrm{Maps}(M, \partial M; F^{n}_{\mathcal{B},\mathcal{A}}, *) \arrow[d] \\
\mathcal{C}^{\mathcal{B}}(M\times I)/\mathcal{C}^{\mathcal{A}}(M\times I) \arrow[r] & \mathrm{Maps}(M, \partial M; \Omega F^{n+1}_{\mathcal{B},\mathcal{A}}, *) \\
 & \cong \mathrm{Maps}(M\times I, \partial(M\times I); F^{n+1}_{\mathcal{B},\mathcal{A}}, *)
\end{tikzcd}$$

Fig. 4

A carefuly analysis (for $M = S^n \times D^k$) combined with Theorem II 4.5 converts the diagram above (Fig. 4) in to the following diagram

$$\begin{tikzcd}
\pi_i(\mathcal{C}^{Top}(S^n \times D^k)) \arrow[r, "\pi_i(\Sigma)"] \arrow[d, "\pi_i(d)"] & \pi_i(\mathcal{C}^{Top}(S^n \times D^{k+1})) \arrow[d, "\pi_i(d)"] \\
\pi_i(\Omega^k F^{n+k}_{Top,Diff}) \arrow[r, "\pi_i(\Omega^k \Psi)"] & \pi_i(\Omega^{k+1} F^{n+k+1}_{Top,Diff})
\end{tikzcd}$$

with $\pi_i(d)$ an isomorphism for $i \leq 2n-4$ and an epimorphism for $i \leq 2n-3$. Since by Kuiper-Lashof's results $\mathcal{C}^{Top}(S^n)$ and $F^{n}_{Top,Diff}$ are homotopy equivalent by a homotopy equivalence d which induces for homotopy groups the homomorphism $\pi_i(d)$, we conclude that $\Psi^{r}_{Top,Diff} : F^{r}_{Top,Diff} \longrightarrow F^{r+1}_{Top,Diff}$ is $r+s-1$ connected for $2s+1 \leq \omega^{Top}(r)$. Using this fact together with the truth of Theorem II 5.1 for $\mathcal{A} = Top$ one concludes Theorem II 5.1 for $\mathcal{A} = Diff$.

Corollary II. 5.2. If $\xi : E \longrightarrow M$ is an $\mathcal{A}$ -bundle with fibre a disc , then $\iota^{\xi} : \tilde{\mathcal{C}}^{\mathcal{A}}(M)/\mathcal{C}^{\mathcal{A}}(M) \longrightarrow \tilde{\mathcal{C}}^{\mathcal{A}}(E)/\mathcal{C}^{\mathcal{A}}(E)$ induces a $\omega^{\mathcal{A}}(\dim M)$ -homotopy equivalence.

Let $D^r \subset K^r$, $K^r \in ob\,\mathcal{A}$ and let us consider :

1) $\tilde{\mathcal{E}}^{\mathcal{A}}(M)/\mathcal{E}^{\mathcal{A}}(M) \xrightarrow{\Sigma^r} \tilde{\mathcal{E}}^{\mathcal{A}}(M\times D^r)/\mathcal{E}^{\mathcal{A}}(M\times D^r) \xrightarrow{i} \tilde{\mathcal{E}}^{\mathcal{A}}(M\times K)/\mathcal{E}^{\mathcal{A}}(M\times K)$

2) $\tilde{\mathcal{E}}^{\mathcal{A}}(M)/\mathcal{E}^{\mathcal{A}}(M) \xrightarrow{\ell^K} \tilde{\mathcal{E}}^{\mathcal{A}}(M\times K)/\mathcal{E}^{\mathcal{A}}(M\times K)$

3) $\varphi : \tilde{\mathcal{E}}^{\mathcal{A}}(M)/\mathcal{E}^{\mathcal{A}}(M) \longrightarrow G$

with G a weak commutative weak group. and i induced by the inclusion $M\times D^r \subset M\times K^r$

<u>Theorem II. 5.3.</u> $\Sigma^2 \circ \ell^K \sim \chi(K)(\varphi\circ i\circ \Sigma^r)$ if $K' = K\times I$.

This last theorem was proved for $\mathcal{A} = \mathcal{P}\ell$ by Hatcher (see [H]) ; a proof simultaneously valid for all geometric categories is given in $[BL]_3$.

§§ 6. Topological nilpotencies

For an $\mathcal{A}$ -manifold M^n and a base pointed $\mathcal{A}$ -manifold (V, v_0) (for instance $V = R$, $v_0 = 0$ or $V = S^1$, $v_0 = e$) we define $\mathcal{E}mb^{\mathcal{A}}(M, M\times V)$ respectively $\mathcal{E}mb^{\mathcal{A}}_{m_0}(M, M\times V)$ the s.s-complex of $\mathcal{A}$- embeddings which restrict on ∂M respectively $\partial M \cup m_0$ to the canonical embedding i, $i(m) = (m, v_0)$; we denote by $\check{\mathcal{E}mb}^{\mathcal{A}}(M, M\times V)$ respectively $\check{\mathcal{E}mb}^{\mathcal{A}}_{m_0}(M; M\times V)$ the union of those connected components of $\mathcal{E}mb^{\mathcal{A}}(M; M\times V)$ respectively $\mathcal{E}mb^{\mathcal{A}}_{m_0}(M, M\times V)$ whose embeddings composed by the canonical projection on V is homotopic to the constant map.

Let $j^{\mathcal{A}}(M): \check{\mathcal{E}mb}^{\mathcal{A}}(M, M\times S^1) \longrightarrow \mathcal{E}mb^{\mathcal{A}}(M, M\times R)$ if $\partial M \neq \phi$ respectively $j^{\mathcal{A}}_{m_0}(M): \check{\mathcal{E}mb}^{\mathcal{A}}_{m_0}(M, M\times S^1) \longrightarrow \mathcal{E}mb^{\mathcal{A}}_{m_0}(M, M\times R)$ if $\partial M = \phi$, be the obvious s.s maps induced by the lifting in the covering $M\times R \longrightarrow M\times S^1$, and $\mathcal{N}^{\mathcal{A}}(M)$ the homotopy theoretic fibre of $j^{\mathcal{A}}(M)$ respectively $j^{\mathcal{A}}_{m_0}(M)$ (since the homotopy type of the homotopy theoretic fibre of $j^{\mathcal{A}}_{m_0}(M)$ if $\partial M = \phi$ does not depend on m_0, we delete m_0 from our notation). Analogously we have $\mathcal{C}j^{\mathcal{A}}(M): \mathcal{C}^{\mathcal{A}}_{emb}(M, M\times S^1) \to \mathcal{C}^{\mathcal{A}}_{emb}(M, M\times R)$ and $\mathcal{CN}^{\mathcal{A}}(M)$ the homotopy theoretic fibre of $\mathcal{C}j^{\mathcal{A}}(M)$. Theorem II 6.1. describes the properties of $\mathcal{N}(M)$ and $\mathcal{CN}(M)$, and justifies the name of topological nilpotency respectively topological concordance nilpotency as a topological analogous for the algebraic nilpotency. The algebraic nilpotency defined for a group G is connected to the algebraic K-theory of G by the following formula: $K_i(G\times Z) \simeq K_i(G) \oplus K_{i-1}(G) \oplus Nil^+_i(G) \oplus Nil^-_i(G))$.

Theorem II. 6.1: There exists a homotopy equivalence $\mathcal{C}r^{\mathcal{A}}(M): \mathcal{C}^{\mathcal{A}}(M\times S^1) \longrightarrow \mathcal{C}^{\mathcal{A}}(M\times I)\times B\mathcal{C}^{\mathcal{A}}(M\times I)\times \mathcal{C}\mathcal{N}^{\mathcal{A}}(M)$ which is natural with respect to inclusions $M^n \subset N^n$ and the category $\mathcal{A}$.

2) There exists an homotopy equivalence $r^{\mathcal{A}}(M): \mathcal{A}(M\times S^1) \longrightarrow \mathcal{A}(M\times I)^{\circ}\times B\mathcal{A}(M\times I)\times \mathcal{N}^{\mathcal{A}}(M)^{\circ}\times T$ with $T = S^1$ if $\partial M = \phi$ and $T = pt$ if $\partial M \neq \phi$; G° denotes the base point connected component of G. $r^{\mathcal{A}}(M)$ is natural with respect to inclusions $M^n \subset N^n$ and the category $\mathcal{A}$.

3) $\mathcal{N}^{\mathcal{A}}(M) \longrightarrow \mathcal{N}^{\mathcal{B}}(M)$ and $\mathcal{C}\mathcal{N}^{\mathcal{A}}(M) \longrightarrow \mathcal{C}\mathcal{N}^{\mathcal{B}}(M)$ are homotopy equivalences for M an $\mathcal{A}$-manifold $\mathcal{B} \geq \mathcal{A}$, and $\dim M \geq 4$ if $\mathcal{B} = Top$.

4) If M^n and N^n are κ-tangential homotopy equivalent then $\mathcal{N}(M^n)$ and $\mathcal{N}(N^n)$ respectively $\mathcal{C}\mathcal{N}(M^n)$ and $\mathcal{C}\mathcal{N}(N^n)$ are $\inf(\kappa-2, \frac{n-7}{2})$ - homotopy equivalent respectively $\{\inf(\kappa-2, \frac{n-7}{2})-1\}$ -homotopy equivalent.

The proof of 1) 2) 3) can be found in [B.L.R] ch VI, or $[B]_2$ and of 4) in $[B]_1$. For $M = M'\times I$ 1) and 2) follow immediately since we can find an homotopy inverse $K: \mathcal{A}(M\times S^1) \longrightarrow \mathcal{A}(M\times I)$ for the inclusion $\mathcal{A}(M\times I) \subseteq \mathcal{A}(M\times S^1)$. The proof of 3) follows from Theorem I 1.1 while of 4) from Theorems II 4.4. and 4.5. and Theorem I.2.3.

One should notice that one can define a transfer map $\mathcal{N}\ell^{\xi}: \mathcal{C}\mathcal{N}(M) \to \mathcal{C}\mathcal{N}(E)$ which satisfies the properties 1) and 2) (of the transfer map ℓ^{ξ}) and $\ell^{\xi\times id_{S^1}} \sim \Omega\ell^{\xi}\times \ell^{\xi}\times \mathcal{N}\ell^{\xi}$.

§§ 7. The homotopy functors $\mathcal{S}^{\mathcal{A}}$ and $\mathcal{S}^{\mathcal{N}}$

The considerations of §§5. imply Theorem II 7.1 which beside its interest in the study of the homotopy type of automorphisms directs the attention of the homotopy theory towards new kind of functors. Before stating Theorem II. 7.1. , recall that we have denoted by Ω^{wh} the category whose objects are ∞-loop spaces and morphisms weak homotopy classes of ∞-loop space maps ; we say $f,g: X \longrightarrow Y$ are week homotopic if their k-th Postnikov terms $[f]_k$ and $[g]_k$ are homotopic for any k.

Theorem II. 7.1 : For any geometric category $\mathcal{A}$ there exists a functor $\mathcal{S}^{\mathcal{A}}: \mathcal{P} \to \Omega^h$ with the following properties:

a) $\mathcal{S}^{\mathcal{A}}$ are homotopy functors , i.e. if $f,g: X \longrightarrow Y$ are homotopic then $\mathcal{S}^{\mathcal{A}}(f), \mathcal{S}^{\mathcal{A}}(g)$ are weak homotopic ∞-loop space maps.

b) If $f: X \longrightarrow Y$ is k-connected then $\mathcal{S}^{\mathcal{A}}(f)$ is k-connected.

c) If $E \xrightarrow{\xi} B$ is a locally trivial $\mathcal{A}$-bundle E, B compact $\mathcal{A}$-manifolds there exists a "transfer map" $\ell^{\xi}: \mathcal{S}^{\mathcal{A}}(B) \longrightarrow \mathcal{S}^{\mathcal{A}}(E)$, a morphism in Ω^{wh} so that

i) $E_1 \xrightarrow{\xi_1} E_2 \xrightarrow{\xi_2} E_3$ are two bundles as above then $\ell^{\xi_2 \circ \xi_1} \sim \ell^{\xi_1} \circ \ell^{\xi_2}$.

ii) If
$$\begin{array}{ccc} E_1 & \xrightarrow{\xi_1} & B_1 \\ \downarrow f_E & & \downarrow f_B \\ E_2 & \xrightarrow{\xi_2} & B_2 \end{array}$$
is a cartesian diagram with

bundles as above then $\ell^{\xi_2} \circ \mathcal{S}^{\mathcal{A}}(f_B) \sim \mathcal{S}^{\mathcal{A}}(f_E) \circ \ell^{\xi_1}$.

d) If ξ is the trivial bundle , i.e. $\xi : B \times K \longrightarrow B$ is the projection on B with K a compact connected $\mathcal{A}$ -manifold and $i : B \longrightarrow B \times K$ a crossection then ℓ^K is homotopic to $\chi(K)(\mathcal{S}^{\mathcal{A}}_{(i)})$.

e) If $\mathcal{B} \geq \mathcal{A}$ then for any X there exists a ∞ -loop space map $r^{(X)}_{\mathcal{B}\mathcal{A}} : \mathcal{S}^{\mathcal{A}}(X) \longrightarrow \mathcal{S}^{\mathcal{B}}(X)$, and $r_{\mathcal{B}\mathcal{A}}(X)$ defines a natural transformation of functors whose homotopy theoretic fibre $\mathcal{F}_{\mathcal{B}\mathcal{A}}(X)$ is a ∞ · loop space whose homotopy groups $\pi_i(\mathcal{F}_{\mathcal{B}\mathcal{A}}(X)) = H_i(X : \mathcal{W}(\mathcal{B}, \mathcal{A})$

f) For any $X \in ob$ there exists a well defined homotopy class $\mathcal{A}_{\beta(X)} : \underline{\mathcal{H}}(X) \to \mathcal{S}^{\mathcal{A}}_{(X)}$ natural with respect to $\mathcal{A}$ and homotopy equivalences in the sens that if $f : X \longrightarrow Y$ is an homotopy equivalence which induces $f_{\#} : \underline{\mathcal{H}}(X) \to \underline{\mathcal{H}}(Y)$ the diagram

$$\begin{tikzcd}
 & \mathcal{S}^{\mathcal{A}}_{(X)} \arrow[dr, "r_{\mathcal{B}\mathcal{A}}(X)"] \arrow[dd, "\mathcal{S}^{\mathcal{A}}_{(f)}"] & \\
\underline{\mathcal{H}}(X) \arrow[ur, "\mathcal{A}_{\beta(X)}"] \arrow[rr, "\mathcal{B}_{\beta(X)}"] \arrow[d, "f_{\#}"] & & \mathcal{S}^{\mathcal{B}}_{(X)} \arrow[d, "\mathcal{S}^{\mathcal{B}}_{(f)}"] \\
\underline{\mathcal{H}}(Y) \arrow[rr, "\mathcal{B}_{\beta(Y)}"] \arrow[dr, "\mathcal{A}_{\beta(Y)}"'] & & \mathcal{S}^{\mathcal{B}}_{(Y)} \\
 & \mathcal{S}^{\mathcal{A}}_{(Y)} \arrow[ur, "r_{\mathcal{B}\mathcal{A}}(Y)"'] &
\end{tikzcd}$$

is commutative;

moreover β is compatible with the transfer map , i.e. the diagram

$$\begin{tikzcd} \underline{\mathcal{H}}(X) \arrow[r, "\beta(X)"] \arrow[d, "\times id_K"] & \mathcal{S}^{\mathcal{A}}(X) \arrow[d, "i^K"] \\ \underline{\mathcal{H}}(X \times K) \arrow[r, "\beta(X\times K)"] & \mathcal{S}^{\mathcal{A}}(X \times K) \end{tikzcd}$$

is commutative.

g) There exists an increasing function $\omega^{\mathcal{A}} : Z_+ \longrightarrow Z_+$ called "the stable range" $\omega^{\mathcal{A}}(n) \geq \begin{cases} \frac{n-10}{3} \text{ if } \mathcal{A} = \mathcal{P}\ell, \mathcal{T}op \\ \frac{n-31}{6} \text{ if } \mathcal{A} = \mathcal{D}iff \end{cases}$

and a natural map (with respect to inclusions $N^n \subset M^n$ and $\mathcal{B} \geq \mathcal{A}$) $a_i(M) : \tilde{\mathcal{B}}^{\mathcal{A}}(M)/\mathcal{B}^{\mathcal{A}}(M) \approx B\mathcal{B}^{\mathcal{A}}(M) \longrightarrow \mathcal{S}^{\mathcal{A}}(M)$ which is a $\omega^{\mathcal{A}}(\dim M)$ - homotopy equivalence.

h) $r_{\mathcal{T}op, \mathcal{P}\ell}$ is an isomorphism, $\mathcal{S}^{\mathcal{P}\ell}(pl) \sim *$, $\mathcal{S}^{\mathcal{D}iff}(pl)$ is the (-1)- component of the Ω -spectrum associated to $\mathcal{W}_{\mathcal{P}\ell, \mathcal{D}iff}$.

Theorem II 72; There exists a functor $\mathcal{SN} : \mathcal{P} \longrightarrow \Omega^{wh}$ which satisfies :

a) b) c) d) from Theorem A with $\mathcal{S}^{\mathcal{A}}$ replaced by $\mathcal{SN}$.

e) $\mathcal{S}^{\mathcal{A}}(X \times S^1)$ is naturaly homotopy equivalent to $\mathcal{S}^{\mathcal{A}}(X) \times B\mathcal{S}^{\mathcal{A}}(X) \times B\mathcal{SN}(X)$.

f) For any $\mathcal{A}$ -manifold M^n there exists a map (natural with respect to inclusions $N^n \subset M^n$ and the category $\mathcal{A}$), $i^{\mathcal{A}}(M) : \mathcal{BN}^{\mathcal{A}}(M) \longrightarrow \mathcal{SN}(M)$ which is a $\omega^{\mathcal{A}}(n)$ -homotopy equivalence.

Comments on Theorems II 7.1 and II 7.2 :

i) : g) Theorem II 7.1 and f) Theorem II 7.2 imply the unicity of $\mathcal{S}^{\mathcal{A}}$ and $\mathcal{S}^{\mathcal{N}}$ up to an isomorphism of functors.

ii): Hatcher-Wagnoner , Hatcher and Volodin have computed $\pi_0(\mathcal{S}^{\mathcal{A}}(X))$, $\pi_1(\mathcal{S}^{\mathcal{A}}(X))$ and partly $\pi_2(\mathcal{S}^{\mathcal{A}}(X))$ in terms of an algebraic higher order Whitehead theory (see Hatcher - this volume).

iii) : Hatcher [H] discovered a way to describe $\mathcal{S}^{Diff}$ in terms of $\mathcal{S}^{Pl}$.

iv) : The map $\beta^{\mathcal{A}}(X) : \underline{\mathcal{H}}(X) \longrightarrow \mathcal{S}^{\mathcal{A}}(X)$ induces the principal fibration $\Omega\, \mathcal{S}^{\mathcal{A}}(X) \to \mathcal{E}^{\mathcal{A}}(X) \to \underline{\mathcal{H}}(X)$ (denoted by $*^{\mathcal{A}}(X)$) which is a homotopy invariant of X , and will play an important role in what follows.

v): The functors $\mathcal{S}^{\mathcal{A}}$ and $\mathcal{S}^{\mathcal{N}}$ can be extended to the category of countable CW -complexes whose skeleta are finite CW -complexes ; in particular if restricted to $K(G,1)$ for finitely generated and presented group , $\mathcal{S}^{\mathcal{A}}$ and $\mathcal{S}^{\mathcal{N}}$ define functors from the category of these groups to the homotopy category of ∞ -loop spaces and composing with π_i they define algebraic theories.

<u>About the proof of Theorems II 7.1 and II 7.2</u> : To prove theorems II 7.1 and II 7.2 we need :

1) Theorem II 4.4
2) Corollary II 5.2
3) Theorem II 5.3
4) Corollary II 7.3 a corollary of Corollary II 5.2
5) Proposition II 7.4 which indicates a general procedure to construct functors with values in Ω^{wh}.

<u>Corollary II 7.3</u> : The natural inclusion

$$\theta_{k-1} : \tilde{\mathcal{C}}\mathcal{A}(M\times D^{k-1}) / \mathcal{C}\mathcal{A}(M\times D^{k-1}) \longrightarrow \underline{\tilde{\mathcal{A}}}(M\times D^{k}) / \underline{\mathcal{A}}(M\times D^{k})$$

is a $\inf\left(k-4, \frac{n+k-6}{2}\right)$ - homotopy equivalence.

This corollary follows from the commutative diagram

$$\begin{tikzcd}
\tilde{\mathcal{A}}(M\times D^k)/\mathcal{A}(M\times D^k) \arrow[r] & \underline{\tilde{\mathcal{A}}}(M\times D^k)/\underline{\mathcal{A}}(M\times D^k) \arrow[r] & \tilde{\mathcal{A}}(\partial(M\times D^k))/\mathcal{A}(\partial(M\times D^k)) \\
\tilde{\mathcal{A}}(M\times D^k)/\mathcal{A}(M\times D^k) \arrow[r] \arrow[u] & \tilde{\mathcal{C}}\mathcal{A}(M\times D^{k-1})/\mathcal{C}\mathcal{A}(M\times D^{k-1}) \arrow[r] \arrow[u, "\theta_{k-1}"] & \tilde{\mathcal{A}}(M\times D^{k-1})/\mathcal{A}(M\times D^{k-1}) \arrow[u]
\end{tikzcd}$$ (*

where horizontal lines are fibrations, and from Theorem II 4.4 applied to the inclusion $M\times D^{k-1} \subset \partial(M\times D^k)$ which is $(k-2)$-connected.

---

*) We have shorted the notation of Part I $\tilde{\mathcal{A}}(M\times D^n;\partial(M\times D^n))$
respectively $\mathcal{A}(M\times D^n;\partial(M\times D^n))$ to $\underline{\tilde{\mathcal{A}}}(M\times D^n)$
respectively $\underline{\mathcal{A}}(M\times D^n)$.

There exists an alternative way to describe the category Ω^{wh} up to an isomorphism , which will be very adequated for our presentation. To explain this way let us introduce the notion of Postnikov tower in Ω^{h} as a sequence $\{X_n , \bar{p}^X_{n+1} | \bar{p}^X_{n+1} : X_{n+1} \longrightarrow X_n$, morphism in $\Omega^h \}$ which satisfies:

:

i) $\pi_i(X_n) = 0. \quad i \geqslant n+1$

ii) $\pi_i(\bar{p}^X_{n+1})$ is an isomorphism for $i \leq n$.

A morphism of Postnicov towers $\{\bar{f}_n\} : \{X_n, \bar{p}^X_{n+1}\} \to \{Y_n, \bar{p}^Y_{n+1}\}$ is a sequence $\bar{f}_n : X_n \longrightarrow Y_n$ of morphisms in Ω^h so that $\bar{p}^Y_n \cdot \bar{f}_n = \bar{f}_{n-1} \cdot \bar{p}^X_{n-1}$. If $\mathcal{P}\Omega^h$ denotes the category of Postnikov towers in Ω^h ,it is not difficult to see that:

<u>Proposition II 7.4</u> : There exists an isomorphism of categories $\Omega^{wh} \rightsquigarrow \mathcal{P}\Omega^h$.

Consequentely , in order to construct a functor F from a category $\mathcal{C}$ to Ω^{wh} it sufficies to construct a functor from $\mathcal{C}$ to $\mathcal{P}\Omega^h$.

<u>Proof of II 7.1</u> (ideas) :Observe first ~~it~~ that, suffices to construct $\mathcal{f}^{\mathcal{a}}$ on the category $\underset{\sim}{\mathcal{a}}$ of $\mathcal{a}$ -manifolds and continous maps , since the homotopy category of $\underset{\sim}{\mathcal{a}}$ and of $\mathcal{P}$ are isomorphic. The reader will guess that $\mathcal{f}^{\mathcal{a}}(M)$ is defined as $\varinjlim_{k} \tilde{\mathcal{C}}^{\mathcal{a}}(M\times D^k)/\mathcal{C}^{\mathcal{a}}(M\times D^k)$ for $\Sigma : \tilde{\mathcal{C}}^{\mathcal{a}}(M\times D^k)/\mathcal{C}^{\mathcal{a}}(M\times D^k) \longrightarrow \tilde{\mathcal{C}}^{\mathcal{a}}(M\times D^{k+1})/\mathcal{C}^{\mathcal{a}}(M\times D^{k+1})$ and $\mathcal{f}^{\mathcal{a}}(f)$ will be constructed using the transfer map. The details of the proof appeal to 1) 2) 3) and 5) . In order to prove f)

we notice that because of the homotopy commutativity of the diagram

$$\begin{tikzcd}
\tilde{\mathcal{C}}^{\mathcal{A}}_{(M)}/\mathcal{C}^{\mathcal{A}}_{(M)} \arrow[r] \arrow[d] & \cdots \arrow[r] & \tilde{\mathcal{C}}^{\mathcal{A}}_{(M\times I^k)}/\mathcal{C}^{\mathcal{A}}_{(M\times I^k)} \arrow[r, "\Sigma"] \arrow[d, "\theta_k"] & \tilde{\mathcal{C}}^{\mathcal{A}}_{(M\times I^{k+1})}/\mathcal{C}^{\mathcal{A}}_{(M\times I^{k+1})} \arrow[r] \arrow[d, "\theta_{k+1}"] & \cdots \\
\tilde{\underline{\mathcal{A}}}(M\times I)/\underline{\mathcal{A}}(M\times I) \arrow[r] & \cdots \arrow[r] & \underline{\mathcal{A}}(M\times I^{k+1})/\underline{\mathcal{A}}(M\times I^{k+1}) \arrow[r, "\times id_I"] & \underline{\mathcal{A}}(M\times I^{k+2})/\underline{\mathcal{A}}(M\times I^{k+2}) \arrow[r] & \cdots
\end{tikzcd}$$

there exists a well defined weak homotopy equivalence (isomorphism in Ω^{wh}) $\theta : \mathcal{S}^{\mathcal{A}}(M) \longrightarrow \lim\limits_{\overrightarrow{k}} \tilde{\underline{\mathcal{A}}}(M\times D^k)/\underline{\mathcal{A}}(M\times D^k)$, consequentely if suffices to construct $\alpha_{\beta(M)} : \underline{\mathcal{H}}(M) \longrightarrow \lim\limits_{\overrightarrow{k}} \tilde{\underline{\mathcal{A}}}(M\times D^k)/\underline{\mathcal{A}}(M\times D^k)$ for any $\mathcal{A}$ -manifold M .

Let us consider $\underline{\mathcal{A}}_0^R(M\times I^k) \subseteq \underline{\mathcal{A}}(M\times I^k)$ the s.s-group of $\mathcal{A}$ -isomorphisms of the trivial disc bundle $M\times D^k \to M$ and the commutative diagram

$$\begin{tikzcd}
\tilde{\underline{\mathcal{A}}}(M\times I^k)/\underline{\mathcal{A}}_0^R(M\times I^k) \arrow[r, "\beta_k"] \arrow[d, "\ldots\times id_I"] & \tilde{\underline{\mathcal{A}}}(M\times I^k)/\underline{\mathcal{A}}(M\times I^k) \arrow[d, "\ldots\times id_I"] \\
\tilde{\underline{\mathcal{A}}}(M\times I^k)/\underline{\mathcal{A}}_0^R(M\times I^k) \arrow[r, "\beta_{k+1}"] & \tilde{\underline{\mathcal{A}}}(M\times I^{k+1})/\underline{\mathcal{A}}(M\times I^{k+1})
\end{tikzcd}$$

which by passing to $\lim\limits_{\overrightarrow{k}}$ induce $^{\mathcal{A}}\beta(M) : \lim\limits_{\overrightarrow{k}} \tilde{\underline{\mathcal{A}}}(M\times I^k)/\underline{\mathcal{A}}_0^R(M\times I^k) \longrightarrow \lim \tilde{\underline{\mathcal{A}}}(M\times I^k)/\underline{\mathcal{A}}(M\times I^k)$
The first limit identifies (naturally) up to homotopy to $\underline{\mathcal{H}}(M$ while the second to $\mathcal{S}^{\mathcal{A}}(M)$ via θ .

The proof of Theorem II. 7.2 goes on the same lines. One defines the transfer map for $\mathcal{C}\mathcal{N}$, and using the decomposition stated by Theorem II 6.1 one checks that all the properties of the transfer map proved for $\tilde{\mathcal{C}}\ldots/\mathcal{C}\ldots$ hold for $\mathcal{C}\mathcal{N}\ldots$ and also that 1) 2) 3) 4) have analogous for $\mathcal{C}\mathcal{N}\ldots$.

§§ 8. A natural involution and its applications

We begin this section with some algebraic and homotopy theoretic considerations.

Properties of $Z(\frac{1}{2})$-modules. Let (M,τ) be a commutative $Z(\frac{1}{2})$-module M with involution $\tau: M \longrightarrow M$, i.e. $\tau^2 = id$ and τ is a $Z(\frac{1}{2})$-morphism. $f: (M_1,\tau_1) \longrightarrow (M_2,\tau_2)$ is called a morphism of modules with involutions if f is a $Z(\frac{1}{2})$-morphism and $\tau_2 \circ f = f \circ \tau_1$. We have

a) $M = M^s \oplus M^a$

b) $f(M_1^s) \subset M_2^s$, $f(M_1^a) \subset M_2^a$ and $f = f^s \oplus f^a$

with $f^s = f|M_1^s$ and $f^a = f|M_1^a$.

c) If $\ldots \longrightarrow (M_i, \tau_i) \xrightarrow{f_i} (M_{i+1}, \tau_{i+1}) \xrightarrow{f_{i+1}} (M_{i+2}, \tau_{i+2}) \rightarrow$

is an exact sequence of modules with involutions then

$$\ldots \longrightarrow M_i^s \xrightarrow{f_i^s} M_{i+1}^s \xrightarrow{f_{i+1}^s} M_{i+2}^s \xrightarrow{f_{i+2}^s} M_{i+3}^s \longrightarrow \ldots$$

and

$$\ldots \longrightarrow M_i^a \xrightarrow{f_i^a} M_{i+1}^a \xrightarrow{f_{i+1}^a} M_{i+2}^a \xrightarrow{f_{i+2}^a} M_{i+3}^a$$

are exact sequences.

The algebraic splitting we just described coresponds to a geometric splitting we will describe below. A weak commutative group with involution (X, τ) is an H-space X, whose multiplication satisfies up to homotopy the axioms of "commutative groups" and τ is an H-map with τ^2 homotopic to identity. $f : (X_1, \tau_1) \longrightarrow (X_2, \tau_2)$ is a morphism of a weak commutative groups with involutions if f is an H-map with $\tau_2 \cdot f$ homotopic to $f \cdot \tau_1$. Observe also that for any weak commutative group we have two canonical involutions id_X and $\nu : X \to X$; ν represents the "inverse" with respect to the multiplication on X.

Properties of weak commutative groups with involutions

A based pointed space (s.s complex) (X, x) is called a odd-space or a $Z(\frac{1}{2})$-space if $\pi_1(X, x)$ is abelian and all homotopy groups $\pi_i(X, x)$ are $Z(\frac{1}{2})$-modules. An odd-weak commutative group with involution (X, τ) is a $Z(\frac{1}{2})$-weak commutative group with involution if the space (X, x) is a $Z(\frac{1}{2})$-space. By localisation to odd primes we pass functorially from weak commutative groups with involution to $Z(\frac{1}{2})$-weak commutative group with involution.

Using "general homotopy theory" one can prove that :

a) For any (X, τ), a $Z(\frac{1}{2})$-weak commutative group with involution, there exists two $Z(\frac{1}{2})$-weak commutative groups X^s and X^a together with an homotopy equivalence of weak commutative groups with involution
$h(X,\tau) : (X,\tau) \longrightarrow (X^s \times X^a, id_{X^s} \times \nu_{X^a})$.

b) If $f : (X_1, \tau_1) \longrightarrow (X_2, \tau_2)$ is a morphism of $Z(\frac{1}{2})$-weak commutative groups with involution, then there exists the morphisms $f^s : (X_1^s, id_{X_1^s}) \longrightarrow (X_2^s, id_{X_2^s})$ and $f^a : (X_1^a, \nu_{X_1^a}) \longrightarrow (X_2^a, \nu_{X_2^a})$ of $Z(\frac{1}{2})$-weak commutative groups with involution so that the following diagram is (homotopy) commutative.

$$\begin{tikzcd} X_1 \arrow[r, "f"] \arrow[d, "h(X_1,\tau_1)"] & X_2 \arrow[d, "h(X_2,\tau_2)"] \\ X_1^s \times X_1^a \arrow[r, "f^s \times f^a"] & X_2^s \times X_2^a \end{tikzcd}$$

c) Let $(X_1, \tau_1) \xrightarrow{f} (X_2, \tau_2) \xrightarrow{g} (X_3, \tau_3)$ be a fibration with (X_i, τ_i) $Z(\frac{1}{2})$-weak commutative groups with involutions and f, g morphisms. Then $X_1^s \xrightarrow{f^s} X_2^s \xrightarrow{g^s} X_3^s$ and $X_1^a \xrightarrow{f^a} X_2^a \xrightarrow{g^a} X_3^a$ are fibrations. This geometric splitting corresponds entirely to the algebraic splitting in the following sense; if K is a space (s.s complex) then the set $[K, X]$ of the homotopy classes of maps has a natural structure of

$Z(\frac{1}{2})$ - module with involution and $[K,X]^s = [K,X^s]$ respectively $[K,X]^a = [K,X^a]$.

Let $\tau:[0,1]\to[0,1]$ be the involution given by $\tau(\alpha)=1-\alpha$. The conjugation with $id_M\times\tau$, induces on $\mathcal{a}(M\times I)$, $\tilde{\mathcal{a}}(M\times I)$, $\Omega\,\tilde{\mathcal{a}}(M\times I)/\mathcal{a}(M\times I)$, $\mathcal{C}^{\mathcal{a}}(M\times I)$, $\tilde{\mathcal{C}}^{\mathcal{a}}(M\times I)/\mathcal{C}^{\mathcal{a}}(M\times I)$ an involution $\underset{\sim}{\tau}$ so that all natural s.s maps and group-homomorphisms which will be involved in our considerations are equivariant with respect to $\underset{\sim}{\tau}$. Moreover all the s.s-complexes except the last one are weak commutative groups with involution while the last one is a weak group with involution which becomes a weak commutative group with involution if we assume $M^n \sim N^{n-1}\times I$. All our knowledge about the structure of $\mathcal{a}(M)$ comes from a partial understanding of the fibrations

$$1)_r\quad \Omega\,\tilde{\mathcal{a}}(M\times I^r)/\mathcal{a}(M\times I^r) \longrightarrow \mathcal{a}(M\times I^r) \longrightarrow \tilde{\mathcal{a}}(M\times I^r) \quad \text{and}$$

$$2)_{r-1}\quad \tilde{\mathcal{a}}(M\times I^r)/\mathcal{a}(M\times I^r) \xrightarrow{i_r} \tilde{\mathcal{C}}^{\mathcal{a}}(M\times I^{r-1})/\mathcal{C}^{\mathcal{a}}(M\times I^{r-1}) \xrightarrow{\pi_{r-1}} \tilde{\mathcal{a}}(M\times I^{r-1})/\mathcal{a}(M\times I^{r-1})$$

which will be done using the algebraic and the geometric decomposition we have discussed above and Theorem II 5.1 ; one obtains (see $[BL]_2$ and $[BL]_3$ §§ 4) the following:

<u>Proposition 8.1.</u> (All s.s complexes and maps in this statement are assumed to be localised to odd primes.)

a) $\tilde{\mathcal{a}}(M\times I) \longrightarrow \tilde{\mathcal{a}}^a(M\times I)$ is a homotopy equivalence

b) $\{\pi_i(\tilde{\mathcal{C}}^{\mathcal{a}}(M\times I)/\mathcal{C}^{\mathcal{a}}(M\times I))\}^a = 0$ if $i \le \omega^{\mathcal{a}}(\dim M+1)$

c) $\pi_1\cdot\Sigma\cdot i_1$ induces the isomorphism $2\cdot id$

for $\pi_r^s\left(\tilde{\mathcal{a}}(M\times I)/\mathcal{a}(M\times I)\right)$.

d) $\mathcal{a}^d(M\times I)\longrightarrow\tilde{\mathcal{a}}(M\times I)$ induces for homotopy groups an isomorphism in dimension $i\leq\omega^{\mathcal{a}}(\dim M+1)$ and an epiomorphism in dimension $i\leq\omega^{\mathcal{a}}(\dim M+1)+1$.

e) The fibration $\Omega\,\tilde{\mathcal{a}}(M\times I)/\mathcal{a}(M\times I)\longrightarrow\mathcal{a}(M\times I)\longrightarrow\tilde{\mathcal{a}}(M\times I)$ which is classified by $\tilde{\mathcal{a}}(M\times I)\longrightarrow\tilde{\mathcal{a}}(M\times I)/\mathcal{a}(M\times I)$ is $\omega^{\mathcal{a}}(\dim M+1)$-trivial in the sense that the classifying map is $\omega^{\mathcal{a}}(\dim M+1)$-trivial.

c') $\pi_1\circ\Sigma\circ i_1$ and $2\,id$ are $\omega^{\mathcal{a}}(\dim M+1)$-homotopic, and $\pi_2\circ\Sigma^2\circ i_1$ is $\omega^{\mathcal{a}}(\dim M+1)$-trivial.

c") $\pi_{2s-1}\circ\Sigma^{2s-1}\circ i_1$ is a $\omega^{\mathcal{a}}(\dim M+1)$- homotopy equivalence and $\pi_{2s}\circ\Sigma^{2s}\circ i_1$ is a $\omega^{\mathcal{a}}(\dim M+1)$-trivial.

For any G, a $Z(\frac{1}{2})$-weak group, and K an arbitrary s.s complex, the set of homotopy classes $[K,G]$ is a group whose elements are all uniquely divisible by 2, hence a $Z(\frac{1}{2})$-module; consequentely for any $f\in[K,G]$ and $\alpha\in Z(\frac{1}{2})$, αf is a well defined element of $[K,G]$. As for $k\geq 1$ $\tilde{\mathcal{a}}(M\times I^k)/\mathcal{a}(M\times I^k)$, $\tilde{\mathcal{C}}^{\mathcal{a}}(M\times I^k)/\mathcal{C}^{\mathcal{a}}(M\times I^k)$ are $Z(\frac{1}{2})$-weak groups (since we have assumed all these spaces localised to odd primes) we define the homotopy classes p_{k+1}, j_k

$$p_{k+1}:\tilde{\mathcal{C}}^{\mathcal{a}}(M\times I^k)/\mathcal{C}^{\mathcal{a}}(M\times I^k)\longrightarrow\tilde{\mathcal{a}}(M\times I^{k+1})/\mathcal{a}(M\times I^{k+1})$$

$$j_k:\tilde{\mathcal{a}}(M\times I^k)/\mathcal{a}(M\times I^k)\longrightarrow\tilde{\mathcal{C}}^{\mathcal{a}}(M\times I^k)/\mathcal{C}^{\mathcal{a}}(M\times I^k)$$

by $p_{k+1}=\frac{1}{2}\pi_{k+1}\circ\Sigma$, $j_k=\frac{1}{2}\Sigma\circ i_k$ for $k\geq 1$

and the homotopy class

$$\omega^u_k : \tilde{\mathcal{A}}(M\times I^k)/\mathcal{A}(M\times I^k) \longrightarrow \tilde{\mathcal{A}}(M\times I^{k+2u})/\mathcal{A}(M\times I^{k+2u}) \text{ by } \omega^u_k = \tfrac{1}{2}\pi_{k+2u}\circ\Sigma^{2u+1}\circ i_k$$

(j_k can be defined even for $k=0$ at the $\omega^{\mathcal{A}}(\dim M)$-th Postnikov term level), with i_{k+1} and π_k the maps in the fibration

$$(k)\qquad \tilde{\mathcal{A}}(M\times I^{k+1})/\mathcal{A}(M\times I^{k+1}) \xrightarrow{i_{k+1}} \tilde{\mathcal{B}}^{\mathcal{A}}(M\times I^k)/\mathcal{B}^{\mathcal{A}}(M\times I^k) \xrightarrow{\pi_k} \tilde{\mathcal{A}}(M\times I^k)/\mathcal{A}(M\times I^k)$$

Theorem 8.2. If $\ell \leq \omega^{\mathcal{A}}(\dim M + k)$ then the Postnikov ℓ-th term of the sequence (k) is a fibration which satisfies :

1) $p_{k+1}\, i_{k+1} \sim id$, $\pi_k\, j_k \sim id$

2) $\pi_k\, i_{k+1} \sim 0$, $p_{k+1}\, j_k \sim 0$

3) $j_k \pi_k + i_{k+1} p_{k+1} \sim id$ (the ℓ-th Postnikov term of $\tilde{\mathcal{B}}^{\mathcal{A}}(M\times I^k)/\mathcal{B}^{\mathcal{A}}(M\times I^k)$ is a weak commutative group since it is homotopy equivalent by ${}^{\mathcal{A}}\beta(M\times I^k)$ to $[\mathcal{S}^{\mathcal{A}}(M\times I^k)]_\ell$).

4) The diagram (fig. 5) is (homotopy) commutative

$$\begin{tikzcd}
\left[\tilde{\mathcal{A}}(M\times I^{k+1})/\mathcal{A}(M\times I^{k+1})\right]_\ell \arrow[r, shift left, "i_{k+1}"] \arrow[d, "2\,id"] & \left[\tilde{\mathcal{B}}^{\mathcal{A}}(M\times I^k)/\mathcal{B}^{\mathcal{A}}(M\times I^k)\right]_\ell \arrow[l, shift left, "p_{k+1}"] \arrow[r, shift left, "\pi_k"] \arrow[d, "\Sigma"] & \left[\tilde{\mathcal{A}}(M\times I^k)/\mathcal{A}(M\times I^k)\right]_\ell \arrow[l, shift left, "j_k"] \arrow[d, "\frac{1}{2}\omega^1_k"] \\
\left[\tilde{\mathcal{A}}(M\times I^{k+1})/\mathcal{A}(M\times I^{k+1})\right]_\ell \arrow[r, shift right, "j_{k+1}"'] & \left[\tilde{\mathcal{B}}^{\mathcal{A}}(M\times I^{k+1})/\mathcal{B}^{\mathcal{A}}(M\times I^{k+1})\right]_\ell \arrow[l, shift right, "\pi_{k+1}"'] \arrow[r, shift right, "p_{k+2}"'] & \left[\tilde{\mathcal{A}}(M\times I^{k+2})/\mathcal{A}(M\times I^{k+2})\right]_\ell \arrow[l, shift right, "i_{k+2}"']
\end{tikzcd}$$

Fig. 5

5) $\omega^{u}_{k} = \omega^{1}_{k+2u-2} \circ \omega^{1}_{k+2u-4} \circ \dots \dots \circ \omega^{1}_{k+2} \circ \omega^{1}_{k}$

6) The following diagram (fig. 6) is (homotopy) commutative with ω^{u}_{k+1} homotopy equivalence , and natural (up to homotopy) with respect to inclusions $M^n \subset V^n$.

$$\begin{tikzcd}
\left[\mathcal{A}(M\times I^{k+1}) / \mathcal{A}(M\times I^{k+1})\right]_{\ell} \arrow[r, shift left, "i_{k+1}"] \arrow[d, "\omega^{s}_{k+1}"] & \left[\tilde{\mathcal{B}}^{\mathcal{A}}(M\times I^{k}) / \mathcal{B}^{\mathcal{A}}(M\times I^{k})\right]_{\ell} \arrow[l, shift left, "p_{k+1}"] \arrow[r, shift left, "\pi_{k}"] \arrow[d, "\Sigma^{2s}"] & \left[\tilde{\mathcal{A}}(M\times I^{k}) / \mathcal{A}(M\times I^{k})\right]_{\ell} \arrow[l, shift left, "j_{k}"] \arrow[d, "\omega^{s}_{k}"] \\
\left[\mathcal{A}(M\times I^{k+2s+1}) / \mathcal{A}(M\times I^{k+2s+1})\right]_{\ell} \arrow[r, shift right, "p_{k+2s+1}"'] & \left[\mathcal{B}^{\mathcal{A}}(M\times I^{k+2s}) / \mathcal{B}^{\mathcal{A}}(M\times I^{k+2s})\right]_{\ell} \arrow[l, shift right, "i_{k+2s+1}"'] \arrow[r, shift left, "\pi_{k+2s}"] & \left[\tilde{\mathcal{A}}(M\times I^{k+2s}) / \mathcal{A}(M\times I^{k+2s})\right]_{\ell} \arrow[l, shift left, "j_{k+2s}"]
\end{tikzcd}$$

Fig. 6 .

§§ 9. *The functors* ${}^{\mathcal{A}}\mathcal{S}^{\pm}$ *and the structure theorem*

<u>Theorem II 9.1</u> : For any geometric category $\mathcal{A}$, there exist two functors ${}^{\mathcal{A}}\mathcal{S}^{\pm}: \mathcal{P}^{\mathcal{A}} \longrightarrow \Omega^{wh}$ *(see §§0 for the definition of* $\mathcal{P}$*) so that:*

a) ${}^{\mathcal{A}}\mathcal{S}^{\pm}$ are homotopy functors

b) If $f: (X,\xi) \longrightarrow (Y,\eta)$ is a morphism with f κ-connected, ${}^{\mathcal{A}}\mathcal{S}^{\pm}(f)$ are κ-connected.

c) There exists a functorial isomorphism between the functors $\mathcal{S}^{\mathcal{A}}_{odd} \circ \pi^{\mathcal{A}}$ ($\mathcal{S}^{\mathcal{A}}_{odd}(X) = (\mathcal{S}^{\mathcal{A}}(X))_{odd}$) and ${}^{\mathcal{A}}\mathcal{S}^{+} \times {}^{\mathcal{A}}\mathcal{S}^{-}$ defined by ${}^{\mathcal{A}}p^{\pm}(X,\xi): \mathcal{S}^{\mathcal{A}}_{odd}(X) \longrightarrow {}^{\mathcal{A}}\mathcal{S}^{\pm}(X,\xi)$ *with* $\pi^{\mathcal{A}}: \mathcal{P}^{\mathcal{A}} \rightarrow \mathcal{P}$ *the forgetful functor.*

d) For any $(K,\eta) \in ob\,\mathcal{P}^{\mathcal{A}}$ there exists a "transfer map" ${}^{\pm}\ell^{(K,\eta)}: {}^{\mathcal{A}}\mathcal{S}^{\pm}(X,\xi) \longrightarrow {}^{\mathcal{A}}\mathcal{S}^{\pm}(X \times K; \xi \times \eta)$ which is a natural transformation of functors from ${}^{\mathcal{A}}\mathcal{S}^{\pm}$ to ${}^{\mathcal{A}}\mathcal{S}^{\pm} \circ (\ldots \times (K,\eta))$

so that :

i) ${}^{\pm}\ell^{(K_1 \times K_2,\, \eta_1 \times \eta_2)} \sim {}^{\pm}\ell^{(K_2,\eta_2)} \circ {}^{\pm}\ell^{(K_1,\eta_1)} \sim {}^{\pm}\ell^{(K_1,\eta_1)} \circ {}^{\pm}\ell^{(K_2,\eta_2)}$

ii) ${}^{\pm}\ell^{(K,\eta)} \sim \chi(K) \cdot {}^{\mathcal{A}}\mathcal{S}^{\pm}(i)$ with i given by $i(x) = (x, k_0) \in X \times K$

iii) Via the identification of $\mathcal{S}^{\mathcal{A}}_{odd}(X)$ with ${}^{\mathcal{A}}\mathcal{S}^{+}(X,\xi) \times {}^{\mathcal{A}}\mathcal{S}^{-}(X,\xi)$ $(\ell^{K})_{odd} \sim {}^{+}\ell^{(K,\xi)} \times {}^{-}\ell^{(K,\xi)}$

e) If $\mathcal{B} \supset \mathcal{A}$ for any $(X,\xi) \in ob\,\mathcal{P}^{\mathcal{A}}$ there exists a morphism $r^{\pm}_{\mathcal{B},\mathcal{A}}(X,\xi): {}^{\mathcal{A}}\mathcal{S}^{\pm}(X,\xi) \longrightarrow {}^{\mathcal{B}}\mathcal{S}^{\pm}(X,\xi)$, so that $r^{\pm}_{\mathcal{B},\mathcal{A}}$ defines a natural transformation between ${}^{\mathcal{A}}\mathcal{S}^{\pm}$ and ${}^{\mathcal{B}}\mathcal{S}^{\pm} \circ \pi^{\mathcal{B},\mathcal{A}}$ *with* $\pi^{\mathcal{B},\mathcal{A}}: \mathcal{P}^{\mathcal{B}} \rightarrow \mathcal{P}^{\mathcal{A}}$ *the forgetful functor.*

f) For any $(X,\xi) \in ob\,\mathcal{P}^{\mathcal{A}}$ there exists a map ${}^{\mathcal{A}}\beta^{\pm}(X,\xi): \underline{\mathcal{H}}(X) \longrightarrow {}^{\mathcal{A}}\mathcal{S}^{\pm}(X,\xi)$ so that ${}^{\mathcal{A}}\beta^{\pm}(X,\xi) = {}^{\mathcal{A}}p^{\pm}(X,\xi) \circ {}^{\mathcal{A}}\beta(X)$.

g) For any manifold M^k and n with $2n \geqslant k$ there exists the homotopy class

$$i^{+}_{2n}(M) : \left(\tilde{\mathcal{a}}(M^k \times I^{2n-k}) / \mathcal{a}(M^k \times I^{2n-k})\right)_{odd} \longrightarrow \mathcal{f}^{\mathcal{a}}_{+}(M; \tau(M)) \text{ respectively}$$

$$i^{-}_{2n}(M) : \left(\tilde{\mathcal{a}}(M^k \times I^{2n+1-k}) / \mathcal{a}(M^k \times I^{2n+1-k})\right)_{odd} \longrightarrow \mathcal{f}^{\mathcal{a}}_{-}(M; \tau(M))$$

natural with respect to inclusions $N^k \subset M^k$ and the category $\mathcal{a}$, which is $\omega^{\mathcal{a}}_{(2n)}$ respectively $\omega^{\mathcal{a}}(2n+1)$ -homotopy equivalence.

Theorem II 9.2 : There exist the functors $\mathcal{a}_{\mathcal{N}}^{\pm} : \mathcal{P}^{\mathcal{a}} \longrightarrow \Omega^{wh}$ which satisfy a) b) c) d) of Theorem II 9.1 with $\mathcal{a}_{\mathcal{f}}^{\pm}$ replaced by $\mathcal{a}_{\mathcal{N}}^{\pm}$ and $\mathcal{f}^{\mathcal{a}}$ replaced by $\mathcal{f}\mathcal{N}$ as well as :

e) $\mathcal{a}_{\mathcal{N}}^{\pm}$ and $\mathcal{B}_{\mathcal{N}}^{\pm} \circ \pi^{\mathcal{B}\mathcal{a}}$ are naturally isomorphic.

f) For any $M^k \in ob\,\mathcal{a}$ and n so that $2n \geqslant k$ there exists

$$\mathcal{a}_{i_{2n}}(M^k) : \mathcal{N}(M^k \times I^{2n-k})_{odd} \longrightarrow \mathcal{a}_{\mathcal{N}}^{+}(M, \tau(M)) \quad \text{respectively}$$

$$\mathcal{a}_{i_{2n+1}}(M^k) : \mathcal{N}(M^k \times I^{2n+1-k})_{odd} \longrightarrow \mathcal{a}_{\mathcal{N}}^{-}(M, \tau(M)) \quad \text{which}$$

is natural with respect to inclusions $N^k \subset M^k$ and the category $\mathcal{a}$, commutes with $\omega_r^{s'}$, and is a $\omega^{\mathcal{a}}_{(2n)}$ resp. $\omega^{\mathcal{a}}_{(2n+1)}$-homotopy equivalence.

Comments about Theorems II 9.1 and 9.2 :

i) h) of Theorem C and g) of Theorem D imply the unicity of $\mathcal{S}^{\pm}_{\mathcal{A}}$ and $\mathcal{N}^{\pm}_{\mathcal{A}}$ up to an isomorphism of functors.

ii) The maps $^{\mathcal{A}}\beta^{\pm}(X) : \underline{\mathcal{H}}(X) \longrightarrow {}^{\mathcal{A}}\mathcal{S}^{\pm}(X,\xi)$ induce the principal fibrations $\Omega\, {}^{\mathcal{A}}\mathcal{S}^{\pm}(X,\xi) \to {}^{\mathcal{A}}\mathcal{E}^{\pm}(X,\xi) \longrightarrow \underline{\mathcal{H}}(X)$ denoted by $^{\mathcal{A}}{}_{*}^{\pm}(M)$, which are homotopy invariants of (X,ξ) and consequently for any manifold M^n the fibrations $^{\mathcal{A}}{}_{*}^{\pm}(M) : \Omega\, {}^{\mathcal{A}}\mathcal{S}^{\pm}(M,\tau(M)) \to {}^{\mathcal{A}}\mathcal{E}^{\pm}(M,\tau(M)) \to \underline{\mathcal{H}}(M)$ are tangential homotopy invariants of M.

iii) Clearly f) implies that $^{\mathcal{A}}{}_{*}(M)$ fibrewise localised to odd primes is the Whitney sum of $^{\mathcal{A}}{}_{*}^{+}(M)$ and $^{\mathcal{A}}{}_{*}^{-}(M)$ for any ξ, hence we have the commutative diagram

$$\begin{array}{ccccc}
{}^{\mathcal{A}}\mathcal{S}^{\pm}(X,\xi) & \longrightarrow & {}^{\mathcal{A}}\mathcal{E}^{\pm}(X,\xi) & \longrightarrow & \underline{\mathcal{H}}(X) \\
\uparrow {}^{\mathcal{A}}\beta^{\pm}(X,\xi) & & \uparrow & & \uparrow \approx \\
\mathcal{S}^{\mathcal{A}}(X) & \longrightarrow & \mathcal{E}^{\mathcal{A}}(X) & \longrightarrow & \underline{\mathcal{H}}(X)
\end{array}$$

hence the triviality of $^{\mathcal{A}}{}_{*}(X)$ implies the triviality of $^{\mathcal{A}}{}_{*}^{\pm}(X,\xi)$ for any ξ.

<u>Theorem II 9.3</u> : (The main theorem) : Let M^n be an $\mathcal{A}$-manifold and let $**(M)$ be the pull back of $^{\mathcal{A}}{}_{*}^{\varepsilon(n)}(M)$ by the natural map $\tilde{\mathcal{A}}(M) \longrightarrow \underline{\mathcal{H}}(M)$

$$**(M) : \Omega\, {}^{\mathcal{A}}\mathcal{S}^{\varepsilon(n)}(M,\tau(M)) \longrightarrow \mathbb{E}^{\mathcal{A}}(M) \longrightarrow \tilde{\mathcal{A}}(M)$$

where

$$\varepsilon(n) = \begin{cases} + & \text{if} \quad n \text{ is even} \\ - & \text{if} \quad n \text{ is odd} \end{cases}$$

Then the fibrewise odd localisation of $\Omega\tilde{\mathcal{Q}}(M^n)/\mathcal{Q}(M^n) \longrightarrow \mathcal{Q}(M^n) \longrightarrow \tilde{\mathcal{Q}}(M^n)$ and $**(M)$ are $\omega^{\mathcal{Q}}(n)$-isomorphic in particular $[E^{\mathcal{Q}}(M^n)_{odd}]_{\omega^{\mathcal{Q}}(n)}$ and $[\mathcal{Q}(M^n)_{odd}]_{\omega^{\mathcal{Q}}(n)}$ are homotopy equivalent.

<u>Corollary II 9.4</u> : 1) Let M^n be an $\mathcal{Q}$ -manifold with nonempty boundary and $p : M \longrightarrow \partial M$ so that $i \circ p \sim id$. Then $**(M)$ is trivial and consequentely $[\mathcal{Q}(M^n)_{odd}]_\ell$ and $[\Omega^{\mathcal{Q}}\mathcal{J}^{\varepsilon(n)}(M, \tau(M^n)) \times \tilde{\mathcal{Q}}(M)_{odd}]_\ell$, $\ell \leq \omega^{\mathcal{Q}}(n)$ are homotopy equivalent.

2) If M^n is $\mathcal{Q}$ -isomorphic to $N^{n-1} \times S^1$, then the restriction of $**(M)$ to the connected component of the identity of $\tilde{\mathcal{Q}}(M^n)$ is trivial hence $(\mathcal{Q}(M^n))_{odd} \longrightarrow \tilde{\mathcal{Q}}(M)_{odd}$ is split surjective.

3) Let K be an $\mathcal{Q}$ -manifold with and $\chi(K) = 0$ and let us consider the diagram :

$$\begin{array}{ccc} & \mathcal{Q}(M \times K)_{odd} & \xrightarrow{e} & \tilde{\mathcal{Q}}(M \times K)_{odd} \\ {\scriptstyle \ldots \times id_K} \nearrow & & & \nearrow {\scriptstyle \ldots \times id_K} \\ \mathcal{Q}(M)_{odd} & \xrightarrow{e} & \tilde{\mathcal{Q}}(M)_{odd} & \end{array}$$

Then for any L which has the homotopy type of a CW-complex of dimension $\leq \omega^{\mathcal{Q}}(n)$ and any $f : L \longrightarrow \tilde{\mathcal{Q}}(M)_{odd}$

the composition $f : L \xrightarrow{f} \tilde{\mathcal{A}}(M)_{odd} \xrightarrow{\dots \times id_K} \tilde{\mathcal{A}}(M \times K)_{odd}$

has a lifting $\underline{f} : L \longrightarrow \mathcal{A}(M \times K)_{odd}$ with $e \cdot \underline{f} \sim f$.

The proof of Theorems 9.1 ; 9.3 and Corollary 9.4 can be found in $[BL]_3$ §§ 5, however we find instructive to point out some steps.

About the proof of Theorem 9.1 : One construct the functors $\mathcal{A}_{\mathcal{S}}^{\pm}$ as functors defined on the category $\underset{\sim}{\mathcal{A}_o}$ of compact $\mathcal{A}$ - manifolds and continous maps $f : M^n \longrightarrow N^n$ with $f_*(\tau(N)) \overset{s}{\sim} \tau(M)$ with values in the category of Postnikov towers of Ω^h according to Proposition II. 7.4 . For any n one choose an increasing sequence $r_1^n < r_2^n < r_3^n < \dots r_{i+1}^n \dots$ with $2 r_1^n \geqslant n$ and $\omega^{\mathcal{A}}(r_i^n) \geqslant i$ for any geometric category $\mathcal{A}$.

For a manifold $M \in ob\,\mathcal{A}$ one defines $\mathcal{A}_{\mathcal{S}}^{\varepsilon}(M, \tau(M))_i =$ $= [\tilde{\mathcal{A}}(M \times D^{\varepsilon_i^n - n}) / \mathcal{A}(M \times D^{\varepsilon_i^n - n})]_i$ where $\varepsilon_i^n = 2 r_i^n$ if $\varepsilon = +$ and $\varepsilon_i^n = 2 r_i^n + 1$ if $\varepsilon = -$. As one follows from Theorem II. 7.2 one can construct the homotopy equivalence ${}^{\varepsilon}\Sigma : \mathcal{A}_{\mathcal{S}}^{\pm}(M^n, \tau(M^n)) \longrightarrow \mathcal{A}_{\mathcal{S}}^{\pm}(M^n \times I, \tau(M^n) \times \tau(I))$, ${}^{\varepsilon}\Sigma_i = [\omega^{r_i^n}_{\varepsilon_i^{n+1} - n}]_i^{-1} \cdot [\omega^{r_i^{n+1}}_{\varepsilon_i^n - n}]_i$ where $[f]_i$ denotes the i-th Postnikov term of f . If $f : M^n \subset N^n$ is a 0 -codimensional embedding $\mathcal{A}_{\mathcal{S}}^{\pm}(f)$ is obviously defined ; if f is a continous map $f : M^n \longrightarrow N^k$ so that $f_*(\tau(N^k)) \overset{s}{\sim} \tau(M)$, choose an embedding $\tilde{f} : M^n \times D^{2s-n} \longrightarrow N^k \times D^{2s-k}$ (for a very big s) so that $\tilde{f} \mid M^n \times \{0\}$ is homotopic to $i \circ f$ when $i(x) = (x, 0)$, and define $\mathcal{A}_{\mathcal{S}}^{\varepsilon}(f) = {}^{\varepsilon}\Sigma^{-1} \circ \mathcal{A}_{\mathcal{S}}^{\varepsilon}(\tilde{f}) \circ {}^{\varepsilon}\Sigma^{2s-n}$. Proposition II 7.1 and Theorem II 7.2 allow us to verify all the statements of $\underset{II\ 9.1}{\text{Theorem}}$ except f). f) will follow inspecting the diagram (fig. 7).

$$\begin{tikzcd}
\mathcal{H}(M) & \tilde{\underline{\mathcal{A}}}(M\times D^{2N+1}) / \underline{\mathcal{A}}_0^R(M\times D^{2N+1}) \arrow[l, "j_1"'] \arrow[rr, "d"] \arrow[dr, "\pi"] & & \tilde{\mathcal{A}}(\partial(M\times D^{2N+1})) / \mathcal{A}(\partial(M\times D^{2N+1})) \\
 & & \tilde{\underline{\mathcal{A}}}(M\times D^{2N+1}) / \underline{\mathcal{A}}(M\times D^{2N+1}) \arrow[ur, "d_1'"] & \\
 & \tilde{\underline{\underline{\mathcal{A}}}}(M\times D^{2N+1}) / \underline{\underline{\mathcal{A}}}_0^R(M\times D^{2N+1}) \arrow[uu, "i"] \arrow[uul, "j_2"'] \arrow[d, "d"] & & \\
\tilde{\mathcal{A}}(M\times S^{2N}) \arrow[uuu, "j"] \arrow[r, "\pi'"] & \tilde{\mathcal{A}}(M\times S^{2N}) / \mathcal{A}_0(M\times S^{2N}) \arrow[uuul, "j_3"] \arrow[rr, "\pi''"] & & \mathcal{A}(M\times S^{2N}) / \mathcal{A}(M\times S^{2N}) \arrow[uuu, "i'"']
\end{tikzcd}$$

Fig. 7.

where $\tilde{\underline{\underline{\mathcal{A}}}}(M\times D^{2N+1})$ is the subgroup of $\tilde{\underline{\mathcal{A}}}(M\times D^{2N+1})$ consisting of those block automorphisms which are identity on $\partial M \times D^{2N+1}$,

$$\underline{\underline{\mathcal{A}}}(M\times D^{2N+1}) = \tilde{\underline{\underline{\mathcal{A}}}}(M\times D^{2N+1}) \cap \underline{\mathcal{A}}(M\times D^{2N+1})$$

$$\underline{\underline{\mathcal{A}}}_0^R(M\times D^{2N+1}) = \tilde{\underline{\underline{\mathcal{A}}}}(M\times D^{2N+1}) \cap \underline{\mathcal{A}}_0^R(M\times D^{2N+1}) \text{ (for the definition of } \mathcal{A}_0^R\dots \text{ see §§ 7)}$$

where the arrows are defined as follows :

j, j_1, j_2, j_3 associate to any block automorphism of $M\times D^{2N+1}$ or $M\times S^{2N}$ the homotopy equivalence which one obtains composing on the left by the canonical inclusion $M \longrightarrow M\times D^{2N+1}$ or $M \longrightarrow M\times S^{2N}$ and on the right by the projection $M\times D^{2N+1} \longrightarrow M$ or $M\times S^{2N} \longrightarrow M$ (N very big). π, π', π'' are induced by the obvious group-factorisations, i is given by the inclusion $\tilde{\underline{\underline{\mathcal{A}}}}\dots \subset \tilde{\underline{\mathcal{A}}}\dots$ and i' is induced by the inclusion of $M\times S^{2N}$ in $\partial(M\times D^{2N+1})$; d and d' are induced by restriction to the boundary $\partial(M\times D^{2N+1})$.

For the proof of Theorem II. 9.2 one repeat the arguments in the proof of Theorem II. 9.1 with $\tilde{\mathcal{A}}(M\times D^k)/\mathcal{A}(M\times D^k)$ replaced by $\mathcal{N}(M\times D^k)$.

<u>Proof of Theorem II. 9.3</u> : In order to prove Theorem II. 9.3 we consider the diagram

$$\begin{tikzcd}
\Omega\, \mathcal{S}^{\mathcal{A}\,\varepsilon(n)}(M,\tau(M)) \arrow[r] & \mathcal{E}^{\mathcal{A}\,\varepsilon(n)}(M,\tau(M)) \arrow[r] & \mathcal{H}(M) \arrow[r] & \mathcal{S}^{\mathcal{A}\,\varepsilon(n)}(M,\tau(M)) \\
\Omega\, \tilde{\mathcal{A}}(M\times S^{2N})/\mathcal{A}(M\times S^{2N}) \arrow[r] & \mathcal{A}(M\times S^{2N}) \arrow[r] & \tilde{\mathcal{A}}(M\times S^{2N}) \arrow[u] & \\
\Omega\, \tilde{\mathcal{A}}(M)/\mathcal{A}(M) \arrow[r] \arrow[u, "\Omega l^{S^{2N}}"] & \mathcal{A}(M) \arrow[r] \arrow[u, "l^{S^{2N}}"] & \tilde{\mathcal{A}}(M) \arrow[u, "l^{S^{2N}}"] &
\end{tikzcd}$$

Fig 8

We first observe that $\Omega l^{S^{2N}}$ is a $\omega^{\mathcal{A}}(\dim M)+1$ -homotopy equivalence since, considering the diagram

$$\begin{tikzcd}
(1)\ \tilde{\mathcal{A}}(M)/\mathcal{A}(M) \arrow[d] & \tilde{\mathcal{B}}^{\mathcal{A}}(M)/\mathcal{B}^{\mathcal{A}}(M) \arrow[l] \arrow[d] & \tilde{\mathcal{A}}(M\times I)/\mathcal{A}(M\times I) \arrow[l] \arrow[d] \\
(2)\ \tilde{\mathcal{A}}(M\times K)/\mathcal{A}(M\times K) & \tilde{\mathcal{B}}^{\mathcal{A}}(M\times K)/\mathcal{B}^{\mathcal{A}}(M\times K) \arrow[l] & \tilde{\mathcal{A}}(M\times K\times I)/\mathcal{A}(M\times K\times I) \arrow[l]
\end{tikzcd}$$

the fibrations (1) and (2) are $\omega^{\mathcal{A}}(\dim M)$- trivial by Theorem 4.2 and because $K = S^{2N}$ and $\chi(S^{2N}) = 2$, $l^{S^{2N}}$ is a $\omega^{\mathcal{A}}(\dim M)$ - homotopy equivalence hence $\Omega l^{S^{2N}}$ is a $\omega^{\mathcal{A}}(\dim M)$- homotopy equivalence.

From diagram Fig. 7 we observe that there exists a $(\omega^{\mathcal{A}}(\dim M)+2)$ -homotopy equivalence $\pi: [\tilde{\mathcal{A}}(M\times S^{2N})/\mathcal{A}(M\times S^{2N})]_l \longrightarrow [\mathcal{S}^{\mathcal{A}\,\varepsilon(n)}(M,\tau(M))]_l$ $l \leq \omega^{\mathcal{A}}(\dim M)+2$ making commutative the diagram

$$\begin{tikzcd}
{[\mathcal{H}(M)]_l} \arrow[r] & {[\mathcal{S}^{\mathcal{A}\,\varepsilon(n)}(M^n)]_l} \\
{[\tilde{\mathcal{A}}(M\times S^{2N})]_l} \arrow[r] \arrow[u, "{[j]_l}"] & {[\tilde{\mathcal{A}}(M\times S^{2N})/\mathcal{A}(M\times S^{2N})]_l} \arrow[u, dashed, "\pi"]
\end{tikzcd}$$

Since any 3 consecutive terms of the first 2 lines in diagram Fig. 8 form a fibration, j and π induces π' so that the following diagram is commutative

$$\begin{tikzcd}
\left[\Omega\, \mathcal{A}_{\mathcal{S}}^{\varepsilon(n)}(M,\tau(M))\right]_u \arrow[r] & \left[\mathcal{A}_{\mathcal{E}}^{\varepsilon(n)}(M,\tau(M))\right] \arrow[r] & \left[\underline{\mathcal{H}}(M)\right] \\
\left[\Omega\, \tilde{\mathcal{A}}(M\times S^{2N}) / \mathcal{A}(M\times S^{2N})\right]_u \arrow[u, "{[\Omega\pi]_u}"] \arrow[r] & \left[\mathcal{A}(M\times S^{2N})\right] \arrow[u, dashed, "\pi'"] \arrow[r] & \left[\tilde{\mathcal{A}}(M\times S^{2N})\right] \arrow[u, "{[j]_u}"] \\
\left[\Omega\, \tilde{\mathcal{A}}(M) / \mathcal{A}(M)\right]_u \arrow[u, "{[\Omega\, \rho^{S^{2N}}]_u}"] \arrow[r] & \left[\mathcal{A}(M)\right]_u \arrow[u] \arrow[r] & \left[\tilde{\mathcal{A}}(M)\right] \arrow[u]
\end{tikzcd}$$

Since $\Omega\pi$ is an u-homotopy equivalence for $u \leq \omega^{\mathcal{A}}(\dim M)+2$ the assertion follows.

About Corollary 9.4 : 1) follows immediately since $\tilde{\mathcal{A}}(M) \longrightarrow \underline{\mathcal{H}}(M)$ factors through $\mathcal{H}(M)$ and because of the hypothesis, $\mathcal{H}(M) \longrightarrow \underline{\mathcal{H}}(M)$ is homotopic to the constant map.

2) follows from 1) and from the decomposition given by Theorem I 2.3.

$\mathcal{S}^{\mathcal{A}}(M)_{odd} \longrightarrow \mathcal{S}^{\mathcal{A}}(M\times K)_{odd}$ 3) follows from the fact that is homotopic to the constant map since $\chi(K)=0$ (by Theorem II 7.1).

§§ 10. Problems

We end up this survey with a few problems whose solution will give a deeper understanding of the functors $\mathcal{S}^{\mathcal{A}}$, ${}^{\mathcal{A}}\mathcal{S}^{\pm}$, $\mathcal{SN}$ and $\mathcal{N}^{\pm}$ and at the same time will push a litle further our knowledge about the structure of $\mathcal{A}(M)$.

P.1) The estimate of $\omega^{\mathrm{Diff}}(n)$ is definitely susceptible to improvements; find the right values of $\omega^{\mathcal{A}}(n)$.

P.2) Does d) remain true for $\mathcal{A}$-locally trivial bundle with a crossection ? (important consequences will follow even this happens only for sphere bundles associated with vector bundles; for instance this will imply that ${}^{\mathcal{A}}\mathcal{S}^{\pm}$ are actually defined on $\mathcal{P}$).

P.3) What can be said about $\mathcal{S}^{\xi}$ when ξ is a finite cover?

P.4) Study the fibration $*\,{}^{\mathcal{A}}(M)$

P.5) Compute $\pi_i(\mathcal{S}^{\mathcal{A}}(X))$, $\pi_i(\mathcal{W}_{Pl.\mathrm{Diff}})$, $\pi_i(\mathcal{SN}(X))$

P.6) Which is the relationship between $\pi_i(\mathcal{S}^{\mathcal{A}}(K(G,1)))$ and the higher order K-theory respectively Whitehead-theory (with various possible definitions).

P.7) Which is the relationship between $\pi_i(\mathcal{SN}(K(G,1))$ and the algebraic nilpotency (eventualy higher order algebraic nilpotency).

P.8) Is $\pi_i(\mathcal{SN}(K(G,1))) \otimes \mathbb{Z}(\frac{1}{2}) = 0$ if G is a finitely generated free abelian group.

P.9) Does ${}^{\mathcal{A}}\beta(X): \underline{\mathcal{H}}(X) \longrightarrow \mathcal{S}^{\mathcal{A}}(X)$ deloaps to a map $B({}^{\mathcal{A}}\beta(X)): B\underline{\mathcal{H}}(X) \longrightarrow B\mathcal{S}^{\mathcal{A}}(X)$ which satisfies the same properties as ${}^{\mathcal{A}}\beta(X)$?

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