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ESSENTIAL PARAMETERS IN PREDICTION

by

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PREPRINT SERIES IN MATHEMATICS
No. 17/1976

BUCUREȘTI

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November 1976

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ESSENTIAL PARAMETERS IN PREDICTION

by

Ion Suciu and Ilie Valuşescu

A time domain analysis of a certain stationary processes considered as time evolutions in the state space of a correlated action is given. Using a factorization theorem for L^2-contractive analytic functions by means of the evaluations functions, in some conditions, a reduction of parameters in the estimation of the prediction-error operator is obtained.

Introduction

In this paper we present a time domain analysis for a certain infinite variate stationary processes whose corresponding spectral analysis can be done in a geometrical model as in [3]. For this, we use the context of a correlated action analogous to Wiener - Masani schema for the (finite) multivariate processes [5], [6], the stochastic processes being assimilated as a time evolution in the state space of the action.

In section 1 we attach to any correlated action $\{\mathcal{E}, \mathcal{H}, \Gamma\}$ its measuring space as the Aronszajn reproducing kernel Hilbert space relative to Γ. In section 2, for any Γ-stationary discrete process, the shift operator is constructed and a geometrical model for prediction (in sense of [3]) is attached. Section 3 is devoted to prediction-error operator. We obtain in Theorem 2 evaluations of this operator both in terms of correlation and in terms of analytic function which factorizes the spectral distribution of the process. In section 4 we establish some relations between the prediction-error operator and the white noises contained in the process. In section 5, using a factorization theorem for the L^2-contractive analytic functions by means of a contractive analytic functions and an evaluation function, we show that, in certain cases, it is possible to reduce some parameters ~~in~~ for the evaluation of the prediction-error operator.

1. Correlated actions. The Aronszajn space

Let $\mathcal{E}$ be a separable Hilbert space and $\mathcal{H}$ be an $\mathcal{L}(\mathcal{E})$ - module. The map from $\mathcal{L}(\mathcal{E}) \times \mathcal{H}$ into $\mathcal{H}$ given by

$$(A,\ h) \longrightarrow Ah$$

will be called an action of $\mathcal{L}(\mathcal{E})$ onto $\mathcal{H}$.

We call a correlation of the action of $\mathcal{L}(\mathcal{E})$ onto $\mathcal{H}$ a map $\mathcal{H} \times \mathcal{H} \longrightarrow \mathcal{L}(\mathcal{E})$ given by

$$(f,\ g) \longrightarrow \Gamma[f,\ g]$$

which verifies :

(i) $\Gamma[h,\ g] = \Gamma[g,\ h]^*$, $\Gamma[h,\ h] \geqslant 0$, and $\Gamma[h,\ h] = 0$ implies $h = 0$.

(ii) $\Gamma[\sum_i A_i h_i, \sum_j B_j g_j] = \sum_{i,j} A_i^* \Gamma[h_i, g_j] B_j$

The triplet $\{\mathcal{E}, \mathcal{H}, \Gamma\}$ will be called the correlated action of $\mathcal{L}(\mathcal{E})$ onto H, The space $\mathcal{E}$ will be called the space of parameters and $\mathcal{H}$ the state space.

Denote $\Lambda = \mathcal{E} \times \mathcal{H}$. If we consider $\lambda_1 = (a_1, h_1)$, $\lambda_2 = (a_2, h_2)$, we shall define a positive definite hernel on $\Lambda \times \Lambda$ to $\mathbb{C}$ by the equality

(1.1) $$\gamma[\lambda_1, \lambda_2] = (\Gamma[h_2, h_1] a_1, a_2)_{\mathcal{E}} .$$

We have

$$\gamma[\lambda_1, \lambda_2] = (\Gamma[h_2, h_1] a_1, a_2) = (a_1, \Gamma[h_1, h_2] a_2) =$$
$$= \overline{(\Gamma[h_1, h_2] a_2, a_1)} = \overline{\gamma[\lambda_2, \lambda_1]}$$

Hence

(1.2) $$\gamma[\lambda_1, \lambda_2] = \overline{\gamma[\lambda_2, \lambda_1]} .$$

We also have

(1.3) $$\gamma[\lambda, \lambda] = (\Gamma[h, h] a, a) \geqslant 0 .$$

For any system $(\lambda_1, ---, \lambda_n)$ in Λ and $(c_1, ---, c_n)$ in $\mathbb{C}$ we have

$$\sum_{i,j} \gamma[\lambda_i,\lambda_j]\, c_i\bar{c}_j = \sum_{i,j} \left(\Gamma[h_j,h_i]\, a_i, a_j\right) c_i\bar{c}_j =$$

$$(1.4) \qquad = \sum_{i,j} \left(\Gamma[h_j,h_i]\, c_i a_i, c_j a_j\right) = \sum_{i,j} \left(\Gamma[h_j,h_i]\, A_i a, A_j a\right) =$$

$$= \left(\Gamma\left[\sum_j A_j h_j, \sum_i A_i h_i\right] a, a\right) \geq 0$$

where we have denoted by A_i an operator in $\mathcal{L}(\mathcal{E})$ with the property $A_i a = c_i a$, for a fixed a in $\mathcal{E}$.

From (1.1) - (1.4) it follows that $\gamma[\lambda_1,\lambda_2]$ is a complex valued positive definite kernel on $\Lambda \times \Lambda$.

To the positive definite kernel $\gamma[\lambda_1,\lambda_2]$ we can attach the Aronszajn reproducing kernel Hilbert space $\mathcal{K}$. Let us recall the construction of this space. In the space $\mathbb{C}^\Lambda$ of all functions defined on Λ with values in $\mathbb{C}$ we consider the subspace $\mathcal{F}$ spanned by $(\gamma_\mu)_{\mu\in\Lambda}$, where for any μ in Λ, γ_μ is the function from $\mathbb{C}^\Lambda$ given by

$$(1.5) \qquad \gamma_\mu(\lambda) = \gamma[\mu,\lambda] \qquad \lambda \in \Lambda .$$

We define on $\mathcal{F}$ a sesqui-linear form, as follows

$$(1.6) \qquad \left\langle \sum_i c_i \gamma_{\mu_i}, \sum_j d_j \gamma_{\lambda_j} \right\rangle_\gamma = \sum_{i,j} c_i \bar{d}_j\, \gamma[\mu_i,\lambda_j] .$$

Factorizing $\mathcal{F}$ by $\langle\cdot,\cdot\rangle = 0$, we obtain a prehilbertian space. By completion we obtain a Hilbert space $\mathcal{K}$ which is the Aronszajn reproducing kernel Hilbert space, with reproducing kernel γ. We shall call $\mathcal{K}$ the Aronszajn space attached to the correlated action $\{\mathcal{E}, \mathcal{H}, \Gamma\}$, or the measuring space of the action.

Let us remark that for the elements with representants in $\mathcal{F}$, the scalar product has the form :

$$(1.7) \qquad \left\langle \sum_i c_i \gamma_{\mu_i}, \sum_j d_j \gamma_{\lambda_j} \right\rangle_\gamma = \sum_{i,j} c_i \bar{d}_j \left(\Gamma[g_j,h_i]\, a_i, b_j\right)_{\mathcal{E}}$$

where $\mu_i = (a_i,h_i)$, $\lambda_j = (b_j,g_j)$.

2. Γ - stationary processes

Let $\{\mathcal{E}, \mathcal{H}, \Gamma\}$ be a correlated action and $\mathcal{K}$ be the attached Aronszajn space. By a discrete Γ-stationary process we shall mean a sequence $\{f_n\}_{-\infty}^{+\infty}$ of elements in $\mathcal{H}$ such that $\Gamma[f_n, f_m]$ depends only on the difference m-n and not on m and n separately.

We define the map $\Gamma : \mathbb{Z} \to \mathcal{L}(\mathcal{E})$ by

(2.1.) $$\Gamma(n) = \Gamma[f_0, f_n]$$.

For any $n_1, \ldots\ldots, n_p \in \mathbb{Z}$ and $a_1, \ldots\ldots, a_p \in \mathcal{E}$ we have :

$$\sum_{i,j} \left(\Gamma(n_i - n_j) a_i, a_j \right)_{\mathcal{E}} = \sum_{i,j} \left(\Gamma[f_{n_j}, f_{n_i}] a_i, a_j \right) =$$
$$= \left(\Gamma\left[\sum_j A_j f_{n_j}, \sum_i A_i f_{n_i} \right] a, a \right) \geq 0$$

where $A_i \in \mathcal{L}(\mathcal{E})$, $A_i a = a_i$ for a fixed $a \in \mathcal{E}$.

Hence Γ is an $\mathcal{L}(\mathcal{E})$-valued positive definite function on $\mathbb{Z}$. The map $n \to \Gamma(n)$ from $\mathbb{Z}$ into $\mathcal{L}(\mathcal{E})$ is called the correlation function of the Γ-stationary process $\{f_n\}_{-\infty}^{+\infty}$.

Let us denote by :

$$\mathcal{H}_n = \{ h \in \mathcal{H} ; h = \sum_{k \in n} A_k f_k , A_k \in \mathcal{L}(\mathcal{E}), f_k \in \{f_n\}_{-\infty}^{+\infty} \}$$
$$\mathcal{K}_n = \text{c.l.m}\{ k \in \mathcal{K} ; k = \gamma_{(a,h)} , a \in \mathcal{E}, h \in \mathcal{H}_n \}$$
$$\mathcal{K}_n^{(a)} = \text{c.l.m}\{ k \in \mathcal{K} ; k = \gamma_{(a,h)} , h \in \mathcal{H}_n \}$$
$$\mathcal{K}_{-\infty} = \bigcap_{-\infty}^{+\infty} \mathcal{K}_n , \quad \mathcal{K}_{\infty} = \bigvee_{-\infty}^{+\infty} \mathcal{K}_n , \quad \mathcal{K}_{-\infty}^{(a)} = \bigcap_{-\infty}^{+\infty} \mathcal{K}_n^{(a)} , \quad \mathcal{K}_{\infty}^{(a)} = \bigvee_{-\infty}^{+\infty} \mathcal{K}_n^{(a)} .$$

It is easy to see that $\mathcal{K}_n \subset \mathcal{K}_{n+1}$ and

(2.2) $$\mathcal{K}_{\infty} = \mathcal{K}_{-\infty} \oplus \bigoplus_{-\infty}^{+\infty} (\mathcal{K}_{n+1} \ominus \mathcal{K}_n)$$

When we consider two Γ-stationary processes $\{f_n\}_{-\infty}^{+\infty}$ and $\{g_n\}_{-\infty}^{+\infty}$, for avoid any confusion, we denote these spaces respectively : $\mathcal{H}_n^{\{f\}}$, $\mathcal{H}_n^{\{g\}}$, $\mathcal{K}_n^{\{f\}}$, $\mathcal{K}_n^{\{g\}}$, $\mathcal{K}_{\infty}^{\{f\}}$, etc.

THEOREM 1. <u>There exists an unitary operator U on $\mathcal{K}_{\infty}$, such that for any</u> $a \in \mathcal{E}$, $A \in \mathcal{L}(\mathcal{E})$ <u>and</u> $n \in \mathbb{Z}$ <u>we have</u>

(2.3) $$U \gamma_{(a, Af_n)} = \gamma_{(a, Af_{n+1})} .$$

<u>Proof.</u> Let us define U on a densly subspace in $\mathcal{K}$ as follows :

$$U \sum_i c_i \gamma_{(a_i, \sum_k A_{i_k} f_{i_k})} = \sum_i c_i \gamma_{(a_i, \sum_k A_{i_k} f_{i_k+1})} .$$

We have :

$$\left\langle \sum_i c_i \gamma_{(a_i, \sum_k A_{i_k} f_{i_k+1})}, \sum_j d_j \gamma_{(b_j, \sum_p A_{j_p} f_{j_p+1})} \right\rangle_{\gamma} =$$

$$= \sum_{i,j} c_i \bar{d}_j \left(\Gamma\left[\sum_p A_{j_p} f_{j_p+1}, \sum_k A_{i_k} f_{i_k+1} \right] a_i, b_j \right)_{\mathcal{E}} =$$

$$= \sum_{i,j} c_i \bar{d}_j \sum_{k,p} \left(A_{j_p}^* \Gamma[f_{j_p+1}, f_{i_k+1}] A_{i_k} a_i, b_j \right)_{\mathcal{E}} =$$

$$= \sum_{i,j} c_i \bar{d}_j \sum_{k,p} \left(A_{j_p}^* \Gamma[f_{j_p}, f_{i_k}] A_{i_k} a_i, b_j \right)_{\mathcal{E}} = \text{---} =$$

$$= \left\langle \sum_i c_i \gamma_{(a_i, \sum_k A_{i_k} f_{i_k})}, \sum_j d_j \gamma_{(b_j, \sum_p A_{j_p} f_{j_p})} \right\rangle_{\gamma} .$$

It is clear then that U can be defined on $\mathcal{K}_\infty$ as a unitary operator with required properties.

We shall call U the shift operator of the Γ-stationary process $\{f_n\}_{-\infty}^{+\infty}$.

REMARK 1. For any $a \in \mathcal{E}$, $\mathcal{K}_\infty^{(a)}$ is a reducing subspace for the shift operator U.

REMARK 2. Clearly $U\mathcal{K}_n = \mathcal{K}_{n+1}$, consequently $U(\mathcal{K}_n \ominus \mathcal{K}_{n-1}) = \mathcal{K}_{n+1} \ominus \mathcal{K}_n$, and $\mathcal{K}_{-\infty}$ reduces U. If we denote

$$\mathcal{F} = \mathcal{K}_0 \ominus \mathcal{K}_{-1} \tag{2.4.}$$

then we have

$$\mathcal{K}_\infty = \mathcal{K}_{-\infty} \oplus \bigoplus_{-\infty}^{+\infty} U^{*n} \mathcal{F} . \tag{2.5.}$$

(Wold decomposition for the shift operator of the process).

REMARK 3. Let us consider $V : \mathcal{E} \to \mathcal{K}_\infty$ defined by

$$Va = \gamma_{(a, f_0)} \tag{2.6.}$$

For any $a \in \mathcal{E}$ we have

$$\|Va\|^2 = \langle \gamma_{(a,f_0)}, \gamma_{(a,f_0)} \rangle = (\Gamma[f_0, f_0] a, a)_{\mathcal{E}} \leq \|\Gamma(0)\| \cdot \|a\|^2,$$

i.e.

$$\|Va\|^2 \leq \|\Gamma(0)\| \cdot \|a\|^2 \tag{2.7}$$

and

$$V^* V = \Gamma(0) . \tag{2.8}$$

Moreover

$$(V^* U^n Va, a)_{\mathcal{E}} = \langle U^n Va, Va \rangle_{\gamma} = \langle U^n \gamma_{(a,f_0)}, \gamma_{(a,f_0)} \rangle_{\gamma} =$$

$$= \langle \gamma_{(a,f_n)}, \gamma_{(a,f_0)} \rangle = (\Gamma[f_0, f_n]a, a)_{\mathcal{E}} = (\Gamma(n)a, a)_{\mathcal{E}} .$$

Therefore

$$\Gamma(n) = V^* U^n V \qquad n \in \mathbb{Z} . \tag{2.9}$$

Because $\gamma_{(a, Ah)} = \gamma_{(Aa, h)}$ we have

$$\mathcal{K}_\infty = \bigvee_{-\infty}^{+\infty} U^n V \mathcal{E} . \tag{2.10}$$

It follows that $[\mathcal{K}_\infty, V, U]$ is the <u>minimal unitary dilation</u> of the positive definite function $n \longrightarrow \Gamma(n)$ from $\mathbb{Z}$ into $\mathcal{L}(\mathcal{E})$.

The triplet $[\mathcal{K}_\infty, V, U]$ will be called <u>the geometrical model</u> of the Γ - stationary process $\{f_n\}_{-\infty}^{+\infty}$.

REMARK 4. If we denote

$$\mathcal{K}_+ = \bigvee_0^\infty U^{*n} V \mathcal{E} \tag{2.11}$$

then $\mathcal{K}_+ = \mathcal{K}_0$, $\mathcal{K}_n = U^n K_+$, and if we denote by

$$U_+ = U^* | \mathcal{K}_+$$

then the Wold decomposition of U_+ is

$$\mathcal{K}_+ = \mathcal{K}_{-\infty} \oplus \bigoplus_0^\infty U_+^n \mathcal{F} \tag{2.12}$$

i.e. $\mathcal{F} = \mathcal{K}_+ \ominus U_+ \mathcal{K}_+$ and $\mathcal{K}_{-\infty} = \bigcap_0^\infty U_+^n \mathcal{K}_+$.

Hence all the geometrical elements of the prediction theory for the process $\{f_n\}_{-\infty}^{+\infty}$ can be obtained from its geometrical model $[\mathcal{K}_\infty, V, U]$.

Let Q be the orthogonal projection of $\mathcal{K}_\infty$ onto $K_{-\infty}$, P_n the orthogonal projection onto $\mathcal{K}_n$, and $P = I - Q$. Then we have :

$$\begin{aligned} P_n Q &= QP_n = Q \\ Q(I-P_n) &= 0 \\ (I-P_n)P &= P(I-P_n) = I-P_n \\ PP_nP &= P_nP = PP_n \end{aligned} \tag{2.13}$$

In what follows we shall use the notation and the therminology from [3]. The main result of [3] (see Theorem 2) says that if F is an $\mathcal{L}(\mathcal{E})$- valued semi-spectral measure on $\mathbb{T}$ and $[\mathcal{K}, V, E]$ its minimal dilation, then there exists an unique L^2- bounded outer analytic function $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ such that

(i) $F_{\Theta} \leq F$

(ii) For any other L^2-bounded analytic function $\{\mathcal{E}, \mathcal{G}, \Omega(\lambda)\}$ for which $F_{\Omega} \leq F$, we have also $F_{\Omega} \leq F_{\Theta}$.

In order that $F_{\Theta} = F$ it is necessary and sufficient that

$$\bigcap_{n=0}^{\infty} U^n \mathcal{K}_+ = \{0\}$$

where $\mathcal{K}_+ = \bigvee_0^{\infty} U^n V \mathcal{E}$, and U is the unitary operator corresponding to the spectral measure E.

The unique L^2 -bounded outer analytic function $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ will be called the <u>maximal outer function</u> attached to the $\mathcal{L}(\mathcal{E})$ -valued semi-spectral measure F.

If $[\mathcal{K}_{\infty}, V, U]$ is the geometrical model of the Γ -stationary process $\{f_n\}_{-\infty}^{+\infty}$, then we can attach a semi-spectral measure F as follows. Consider E the $\mathcal{L}(\mathcal{K}_{\infty})$ - valued spectral measure on $\mathbf{T}$ corresponding to the unitary operator U^*. Then we have

$$U = \int_0^{2\pi} e^{-it} d\, E(t)$$

and if we put

$$(2.14.) \qquad F = V^* E V$$

then we obtain an $\mathcal{L}(\mathcal{E})$ -valued semi-spectral measure F on $\mathbf{T}$. From (2.14.) it results

$$(2.15.) \qquad \Gamma(n) = \int_0^{2\pi} e^{-int} d\, F(t) .$$

The $\mathcal{L}(\mathcal{E})$-valued semi-spectral measure F is well defined by (2.15.) and it is called the <u>spectral distribution</u> of the process $\{f_n\}_{-\infty}^{+\infty}$.

The maximal outer function $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ attached to the spectral distribution F is also called the maximal outer function attached to the process $\{f_n\}_{-\infty}^{+\infty}$.

We remark that F_{Θ} has the spectral dilation $[L^2(\mathcal{F}), V_{\Theta}, E^{\times}]$, where $E^{\times}$ is the spectral measure of the multiplication by e^{it} on $L^2(\mathcal{F})$, $\mathcal{F}$ is given by (2.4.), $V_{\Theta} = \Phi^{\mathcal{F}} P V$, $\Phi^{\mathcal{F}}$ being the canonical isomorphism between $\bigoplus_{-\infty}^{+\infty} U^{*n} \mathcal{F}$ and $L^2(\mathcal{F})$, and P the orthogonal projection of $\mathcal{K}_{\infty}$ onto $\bigoplus_{-\infty}^{+\infty} U^{*n} \mathcal{F}$.

Also we remark that $F = F_{\Theta}$ if and only if

$$(2.16.) \qquad \mathcal{K}_{-\infty} = \{0\} .$$

If we use (2.13.), we obtain for $a \in \mathcal{E}$

$$(\Theta(0)^*\Theta(0)a,a) = \|\Theta(0)a\|^2 = \|(V_\Theta a)(0)\|^2_{\mathfrak{F}} = \inf_{\substack{v_0\in H^2(\mathfrak{F})\\ v_0(0)=0}} \|V_\Theta a - v_0\|^2_{H^2(\mathfrak{F})} =$$

$$= \inf_{\sum_1^m} \|V_\Theta a - \sum_1^m e^{ikt} V_\Theta a_k\|^2_{H^2(\mathfrak{F})} = \inf_{\sum_1^n} \|\Phi^{\mathfrak{F}} PVa - \sum_1^n \Phi^{\mathfrak{F}} PVa_k\|^2_{H^2(\mathfrak{F})} =$$

$$= \inf \|\Phi^{\mathfrak{F}} PVa - \Phi^{\mathfrak{F}} \sum_1^n U^{*k} PVa_k\|^2_{H^2(\mathfrak{F})} = \inf \|PVa - P\sum_1^n U^{*k} Va_k\|^2_{\mathcal{K}_\infty} =$$

$$= \|PVa - PP_{-1}PVa\|^2 = \|P(I-P_{-1})Va\|^2 = \|(I-P_{-1})Va\|^2 .$$

Hence we have

$$\|(I-P_{-1})Va\|^2 = (\Theta(0)^*\Theta(0)a, a) . \tag{2.17}$$

REMARK 5. For $a \in \mathcal{E}$ and $A \in \mathcal{L}(\mathcal{E})$ we can interpret $\|VAa\|^2_\gamma$ as the mean square value of the parameter a in the present state of the process when we act with A on the system.

3. Prediction-error operator

The prediction problems for the stationary process consist in obtaining informations about the process up to the moment p+r, $r > 0$, from the "knowledge" of the process up to the moment p, (the prediction of lag r). We can obtain informations about the past and the present of the process acting on it with some specific experiences. The results of the experiences are measured in a "measuring system" intimately related to the nature of the experiences. In our case the experiences are contained in the correlated action $\{\mathcal{E}, \mathcal{H}, \Gamma\}$ and the measuring system is given by the metric of the attached Aranszajn space $\mathcal{K}$.

Let $\{f_n\}_{-\infty}^{+\infty}$ be a Γ-stationary process and $\mathcal{H}_p$, $\mathcal{K}_p$, $\mathcal{K}_\infty$, etc., as in section 2. Denote P_p the orthogonal projection of $\mathcal{K}_\infty$ on $\mathcal{K}_p$. When we say that we know the process up to the moment p we mean that we know the subspace $\mathcal{K}_p$; more precisely that we can measure the "mean value" $\|\gamma_{(a,g)}\|$ of the parameter a for any element g in H_p obtained by succesive actions on the process up to the moment p. We can obtain the best information on the process at the moment p+1 if we can find the elements of the best approximation in $\mathcal{K}_p$ for the elements in $\mathcal{K}_{p+1}$.

More precisely, we can formulate the prediction problems (of lag 1) as follows :

For any $a \in \mathcal{E}$ find :

(1) a sequence $(a_1, \dots, a_k)_m$ of finite systems in $\mathcal{E}$, a sequence

$(A_{1_j}, \dots\dots, A_{k_j})_m$ of finite systems in $\mathcal{L}(\mathcal{E})$, a sequence $(c_1, \dots\dots, c_k)_m$ of finite systems of complex numbers and a sequence of finite systems $(n_{1_j}, \dots\dots, n_{k_j})_m$ of integers each of them less or equal to p, such that :

$$\lim_{m\to\infty} \| \gamma_{(a, f_{p+1})} - \sum_k c_k \gamma_{(a_k, \sum_j A_{k_j} f_{n_{k_j}})} \| = \| P_p \gamma_{(a, f_{p+1})} \|$$

(2) the mean value of the prediction error of lag 1

$$G(a) = \| (I - P_p) \gamma_{(a, f_{p+1})} \|^2 .$$

Similarly, we can formulate the prediction problems for lag $r > 1$. In this paper we obtain some results concerning the problem (2). We begin with the following :

THEOREM 2. Let $\{f_n\}_{-\infty}^{+\infty}$ be a discrete Γ-stationary process. There exists a positive operator G in $\mathcal{L}(\mathcal{E})$ such that for any integer p we have :

(1^o) G is the infimum in the set of positive operators in $\mathcal{L}(\mathcal{E})$ of the family of positive operators $\left\{ \Gamma\left[f_{p+1} - g,\ f_{p+1} - g\right] ;\ g \in \mathcal{H}_p \right\}$, i.e.

(i) $G \leq \Gamma\left[f_{p+1} - g,\ f_{p+1} - g\right] \qquad g \in \mathcal{H}_p .$

(ii) ~~Any~~ Any positive A in $\mathcal{L}(\mathcal{E})$ which verifies

(3.0) $$A \leq \Gamma\left[f_{p+1} - g,\ f_{p+1} - g\right] \qquad g \in \mathcal{H}_p$$

also verifies

$$A \leq G .$$

(2^o) For any $a \in \mathcal{E}$ and $A \in \mathcal{L}(\mathcal{E})$ we have

(3.1) $$(GAa,\ Aa) = \| (I - P_p)\ \gamma_{(a, Af_{p+1})} \|^2$$

(3^o) For any $a \in \mathcal{E}$ we have

(3.2) $$(Ga,\ a) = \inf \left(\sum_{i,j=0}^{m} \Gamma[g_j, g_i]\, a_i, a_j \right)$$

where the infimum is taken over all finite systems

$g_o = f_{p+1};\ g_1, \ldots\ldots, g_m \in \mathcal{H}_p; \quad a_o = a,\ a_1, \ldots\ldots, a_m \in \mathcal{E}$.

(4^o) If $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ is the maximal outer function attached to the Γ-stationary process $\{f_n\}_{-\infty}^{+\infty}$, then

$$G = \Theta(0)^* \Theta(0) \quad . \tag{3.3.}$$

Proof. We define G by

$$G = V^* U^{*p+1} (I - P_p)\, U^{p+1}\, V \tag{3.4.}$$

Obviously G is a positive operator in $\mathcal{L}(\mathcal{E})$. For any $a \in \mathcal{E}$ and $A \in \mathcal{L}(\mathcal{E})$ we have

$$(GAa, Aa) = (V^* U^{*\,p+1} (I-P_p)\, U^{p+1}\, V A a, A a) = \| (I-P_p)\, U^{p+1}\, V A a \|^2 =$$
$$= \| (I-P_p)\, U^{p+1}\, \gamma(Aa, f_o) \|^2 = \| (I-P_p)\, \gamma(Aa, f_{p+1}) \|^2 =$$
$$= \| (I-P_p)\, \gamma(a, Af_{p+1}) \|^2$$

and the assertion (2^o) is proved.

The assertion (4^o) is a consequence of (3.1.) and (2.17).

To prove (3^o) we have :

$$(Ga, a) = \| (I-P_p)\, \gamma(a, f_{p+1}) \|^2 = \inf_{k \in \mathcal{K}_p} \| \gamma(a, f_{p+1}) - k \|^2 =$$

$$= \inf_{\substack{g_1, \ldots\ldots, g_m \in \mathcal{H}_p \\ a_1, \ldots\ldots, a_m \in \mathcal{E}}} \| \gamma(a, f_{p+1}) - \sum_{k=1}^{m} \gamma(a_k, g_k) \|^2 =$$

$$= \inf_{\substack{g_1, \ldots\ldots, g_m \in \mathcal{H}_p \\ a_1, \ldots\ldots, a_m \in \mathcal{E}}} \left(\sum_{i,j=0}^{m} \Gamma[g_j, g_i]\, a_i, a_j \right) ,$$

where we have denoted $g_o = f_{p+1}$ and $a_o = a$.

To prove (1^o), let us consider $g \in \mathcal{H}_p$ and $a \in \mathcal{E}$. If we denote $g_o = f_{p+1}$, $g_1 = -g$, $a_o = a$, $a_1 = a$, we have :

$$\left(\sum_{i,j=0}^{1} \Gamma\left[g_j, g_i\right] a, a \right) = \left(\Gamma\left[f_{p+1}, f_{p+1}\right] a, a \right) -$$

$$- \left(\Gamma\left[f_{p+1}, g\right] a, a \right) - \left(\Gamma\left[g, f_{p+1}\right] a, a \right) + \left(\Gamma\left[g, g\right] a, a \right) =$$

$$= \left(\Gamma\left[f_{p+1} - g, f_{p+1} - g\right] a, a \right)$$

and from (3.2) it results that

$$(Ga, a) \leq \left(\Gamma\left[f_{p+1} - g, f_{p+1} - g\right] a, a \right)$$

i.e. the assertion (i) in (1^o) is proved.

Let A be a positive operator in $\mathcal{L}(\mathcal{E})$ which verifies (3.0). For any $a_o = a, a_1, \ldots, a_m \in \mathcal{E}$ and $g_o = -f_{p+1}, g_1, \ldots, g_m \in \mathcal{H}_p$, if we consider $A_k \in \mathcal{L}(\mathcal{E})$ such that $A_k a = a_k$ then :

$$\sum_{i,j=0}^{m} \left(\Gamma[g_j, g_i] a_i, a_j \right) = \left(\Gamma[f_{p+1}, f_{p+1}] a, a \right) - \sum_{j=1}^{m} \left(\Gamma[g_j, f_{p+1}] a, A_j a \right) -$$

$$- \sum_{i=1}^{m} \left(\Gamma[f_{p+1}, g_i] A_i a, a \right) + \sum_{i,j=1}^{m} \left(\Gamma[g_j, g_i] A_i a, A_j a \right) =$$

$$= \left(\Gamma[f_{p+1}, f_{p+1}] a, a \right) - \sum_{j=1}^{m} \left(\Gamma[A_j g_j, f_{p+1}] a, a \right) -$$

$$- \sum_{i=1}^{m} \left(\Gamma[f_{p+1}, A_i g_i] a, a \right) + \sum_{i,j=1}^{m} \left(\Gamma[A_j g_j, A_i g_i] a, a \right) =$$

$$= \left(\Gamma\left[f_{p+1} - \sum_{j=1}^{m} A_j g_j, f_{p+1} - \sum_{i=1}^{m} A_i g_i\right] a, a \right) \geq (Aa, a).$$

From (3^o) it follows that $A \leq G$, and the proof of (1^o) is finished.

It remains only to show that G does not depend on p. For $1 \leq k \leq m$ let us denote $g_k = \sum_{j=1}^{m_k} A_{jk} f_{n_{jk}} \in \mathcal{H}_p$, and $g_o = f_{p+1} = \sum_{j=0}^{1} A_{jo} f_{n_{jo}}$, where $f_{n_{oo}} = f_{p+1}$, $A_{jo} = \delta_{jo} \cdot I_{\mathcal{E}}$, $a_o = a, a_1, \ldots, a_m \in \mathcal{E}$.

Also denote $g'_k = \sum_{j=1}^{m_k} A_{jk} f_{n_{jk}+1}$ and $g'_o = f_{p+2}$. Then we have :

$$\sum_{\ell,k=0}^{m} (\Gamma[g_k, g_\ell] a_\ell, a_k) = \sum_{\ell,k=0}^{m} (\Gamma[\sum_{j=0}^{m_k} A_{jk} f_{n_{jk}}, \sum_{i=0}^{m_\ell} A_{i\ell} f_{n_{i\ell}}] a_\ell, a_k)$$

$$= \sum_{\ell,k=0}^{m} \sum_{i,j=0}^{m_\ell, m_k} (A_{jk}^* \Gamma[f_{n_{jk}}, f_{n_{i\ell}}] A_{i\ell} a_\ell, a_k) =$$

$$= \sum_{\ell,k=0}^{m} \sum_{i,j=0}^{m_\ell, m_k} (A_{jk}^* \Gamma[f_{n_{jk}+1}, f_{n_{i\ell}+1}] A_{i\ell} a_\ell, a_k) =$$

$$= - - - = \sum_{\ell,k=0}^{m} (\Gamma[g'_k, g'_\ell] a_\ell, a_k).$$

It is clear that we can obtain any system $\{g_k\}$ from $\{g'_k\}$ in the same way, and follows that in (3°) the infimum is taken on the same set even at the moment p, also at p+1. It results that G does not depend on p.

The proof of the theorem is finished.

The operator G will be called <u>prediction-error operator</u> (of lag 1) of the Γ-stationary process $\{f_n\}_{-\infty}^{+\infty}$. From point ($2^\circ$) of the Theorem 2 it results that $\| Ga \|$ is the minimum prediction-error for the parameter a. The point (1°) tell us that we can obtain simultaneously these errors by a minimizing procedure in the set of the positive operators in $\mathcal{L}(\mathcal{E})$. The expression (3.2) is an intrinsec computation formula for the error in terms of actions and correlations, while (3.3) permit us to obtain the prediction-error operator of the process $\{f_n\}_{-\infty}^{+\infty}$ using the maximal outer function attached to the spectral distribution.

Let us remark that in general

$$(Ga, a) \leq \inf_{g \in \mathcal{H}_p} (\Gamma[f_{p+1} - g, f_{p+1} - g] a, a).$$

Hence the estimation of the parameters given by the prediction in $\mathcal{K}_\infty$ (multivariate prediction) is better as the estimation given by the prediction in $\mathcal{K}_\infty^{(a)}$ (univariate prediction).

4. Deterministic, white noise, and moving average processes

If we take in (3.2) $g_1 = g_2 = \ldots = g_m = 0$, then we obtain

(4.1) $$0 \leq G \leq \Gamma(0).$$

The Γ-stationary process $\{f_n\}_{-\infty}^{+\infty}$ is called :

(i) deterministic if $G = 0$,

(ii) white noise if $G = \Gamma(0)$.

REMARK 6. By Theorem 2, point (1^{o}), it results that <u>the process $\{f_n\}_{-\infty}^{+\infty}$ is deterministic if and only if</u>

$$\mathcal{K}_{-\infty} = \mathcal{K}_p = \mathcal{K}_{+\infty} .$$

PROPOSITION 1. <u>The process</u> $\{f_n\}_{-\infty}^{+\infty}$ <u>is of white noise if and only if</u>

$$\Gamma[f_n, f_m] = \Gamma(0) \cdot \delta_{n,m} \tag{4.2}$$

<u>Proof</u>. If $\Gamma[f_n, f_m] = \Gamma(0) \cdot \delta_{n,m}$ then the positive operator $\Gamma(0)$ satisfies (3.0) because

$$\Gamma(0) \leq 2\, \Gamma(0) = \Gamma[f_{p+1} - g, f_{p+1} - g] , \quad g \in \mathcal{H}_p,$$

therefore $\Gamma(0) \leq G$. From (4.1) it follows then $G = \Gamma(0)$.

Conversely, if $G = \Gamma(0)$ then for $n \neq m$ we have $\Gamma[f_n, f_m] = 0$. For this we can suppose $n < m$. From (3.0) it results

$$(\Gamma(0)\, a, a) = (G\, a, a) \leq (\Gamma[f_m \pm \varepsilon f_n, f_m \pm \varepsilon f_n]\, a, a) =$$

$$= (\Gamma(0)\, a, a) + \varepsilon^2 (\Gamma(0)\, a, a) \pm 2\, \varepsilon \cdot \mathrm{Re}\, (\Gamma[f_m, f_n]\, a, a) .$$

Hence

$$\varepsilon (\Gamma(0)\, a, a) \geq \pm 2\, \mathrm{Re}\, (\Gamma[f_m, f_n]\, a, a)$$

for any $\varepsilon \geq 0$, and

$$\mathrm{Re}\, (\Gamma[f_m, f_n]\, a, a) = 0.$$

Analogously we obtain

$$\mathrm{Im}\, (\Gamma[f_m, f_n]\, a, a) = 0$$

and follows that

$$\Gamma[f_m, f_n] = 0 \qquad \text{for } m \neq n.$$

The proposition is proved.

REMARK 7. The process $\{f_n\}_{-\infty}^{+\infty}$ is a white noise process if and only if $dF = \Gamma(0)\, dt$. In this case $K_\infty = L^2(\mathcal{F})$, where $\mathcal{F} = \overline{\Gamma(0)\mathcal{E}}$, U is the operator of multiplication by e^{it} on $L^2(\mathcal{F})$, and $V = \sqrt{\Gamma(0)}$. The function $\Theta(\lambda)$ is, in this case, the constant function $\Theta(\lambda)a = \sqrt{\Gamma(0)}\, a$.

We say that the process $\{f_n\}_{-\infty}^{+\infty}$ contains the white noise $\{g_n\}_{-\infty}^{+\infty}$ if

1) $g_p \in \mathcal{H}_p^{\{f\}}$

2) $\operatorname{Re}\Gamma[f_p - g_p, g_p] \geqslant 0$

3) $\Gamma[f_p, g_k] = 0$ for $k > p$.

PROPOSITION 2. _If the process_ $\{f_n\}_{-\infty}^{+\infty}$ _contains the white noise_ $\{g_n\}_{-\infty}^{+\infty}$, _then_

$$G^{\{g\}} = \Gamma[g_o, g_o] \leqslant G^{\{f\}} \tag{4.3}$$

Proof. From Theorem 2, point (1^o), it follows that it is sufficient to show that

$$\Gamma[g_o, g_o] \leqslant \Gamma[f_o - g, f_o - g], \qquad (g \in \mathcal{H}_{-1}^{\{f\}}).$$

Since $\Gamma[f_n, g_o] = 0$ for $n < 0$, it results that $\Gamma[g, g_o] = 0$ for any $g \in \mathcal{H}_{-1}^{\{f\}}$. We have

$$\Gamma[f_o - g, f_o - g] = \Gamma[f_o - (g+g_o) + g_o, f_o - (g+g_o) + g_o] =$$
$$= \Gamma[f_o - (g+g_o), f_o - (g+g_o)] + \Gamma[g_o, g_o] + 2\operatorname{Re}\Gamma[f_o - (g+g_o), g_o].$$

But

$$\operatorname{Re}\Gamma[f_o - (g+g_o), g_o] = \operatorname{Re}\Gamma[f_o - g_o, g_o] \geqslant 0$$

therefore

$$\Gamma[g_o, g_o] \leqslant \Gamma[f_o - g, f_o - g]$$

$$G^{\{g\}} = \Gamma[g_o, g_o] \leqslant G^{\{f\}}$$

COROLLARY 1. _If the process_ $\{f_n\}_{-\infty}^{+\infty}$ _is deterministic, then it does not contain any white noise._

PROPOSITION 3. _If_ $\{f_n\}_{-\infty}^{+\infty}$ _contains the white noise process_ $\{g_n\}_{-\infty}^{+\infty}$ _then for any_ $a \in \mathcal{E}$ _we have_

$$(4.4) \qquad \gamma_{(a,g_n)} \in \mathcal{K}_n^{\{f\}} \ominus \mathcal{K}_{n-1}^{\{f\}}$$

and

$$(4.5) \qquad \| \gamma_{(a,g_n)} \| \leq \| (I-P_{n-1}) \gamma_{(a,f_n)} \| .$$

Proof. For any $h \in \mathcal{H}_{n-1}^{\{f\}}$ and $a, b \in \mathcal{E}$ we have

$$\langle \gamma_{(a,g_n)}, \gamma_{(b,h)} \rangle = (\Gamma [h, g_n] a, b) = (\Gamma [\sum_{k\leq n-1} A_k f_k, g_n] a, b) =$$

$$= \sum_{k\leq n-1} (A_k^* \Gamma [f_k, g_n] a, b) = 0.$$

From (4.3) and (3.1) we obtain

$$\| \gamma_{(a,g_n)} \|^2 = (\Gamma [g_n, g_n] a, a) = (\Gamma [g_o, g_o] a, a) \leq (G^{\{f\}} a, a) \leq$$

$$\leq \| (I-P_{n-1}) \gamma_{(a,f_n)} \|^2 .$$

The proposition is proved.

The Γ-stationary process $\{f_n\}_{-\infty}^{+\infty}$ is called moving average process if $\mathcal{K}_{\infty}^{\{f\}}$ is spanned by $\mathcal{K}_{\infty}^{\{g\}}$ when $\{g_n\}_{-\infty}^{+\infty}$ runs over all white noises contained in $\{f_n\}_{-\infty}^{+\infty}$.

PROPOSITION 4. If $\{f_n\}_{-\infty}^{+\infty}$ is a moving average process then

$$\mathcal{K}_{-\infty}^{\{f\}} = \{0\}$$

and consequently by (2.16) we have $F = F_{\Theta}$.

Proof. For any white noise $\{g_n\}_{-\infty}^{+\infty}$ contained in $\{f_n\}_{-\infty}^{+\infty}$ let $[\mathcal{K}_{\infty}^{\{g\}}, V_{\{g\}}, U_{\{g\}}]$ be its geometrical model. Using (2.10) and (4.4) it results that

$$(4.6) \qquad \mathcal{K}_{\infty}^{\{g\}} = \bigvee_{-\infty}^{+\infty} U_{\{g\}}^n V_{\{g\}} \mathcal{E} \subset \bigoplus_{-\infty}^{+\infty} \mathcal{K}_n^{\{f\}} \ominus \mathcal{K}_{n-1}^{\{f\}} .$$

Hence

$$\mathcal{K}_{\infty}^{\{f\}} = \bigvee_{\{g_n\}} \mathcal{K}_{\infty}^{\{g\}} \subset \bigoplus_{-\infty}^{+\infty} \mathcal{K}_n^{\{f\}} \ominus \mathcal{K}_{n-1}^{\{f\}} \subset \mathcal{K}_{\infty}^{\{f\}} .$$

From (2.2) it follows that

$$\mathcal{K}_{-\infty}^{\{f\}} = \{0\} .$$

In general it is possible that the process $\{f_n\}_{-\infty}^{+\infty}$ contains no white noises although it is nondeterministic. Also we can have $\mathcal{K}_{-\infty}^{\{f\}} = \{0\}$ without $\{f_n\}_{-\infty}^{+\infty}$ be a moving average of its (interior) white noises. This is natural because the process $\{f_n\}_{-\infty}^{+\infty}$ can be influenced by some white noises whose relations which $\{f_n\}_{-\infty}^{+\infty}$ cannot be controled with our correlated action $\{\mathcal{E}, \mathcal{H}, \Gamma\}$.

In a next paper, where we shall introduce the notion of complete correlated action, we shall show that there exists an "maximal" white noise whose moving average determine the prediction of the process. Hence we shall be able to proof the Wold decomposition of the process in time domain.

5. Evaluation functions. Reduction of parameters.

An outer L^2-bounded analytic function $\{\mathcal{E}, \mathcal{E}_1, \Delta(\lambda)\}$ is called an evaluation function (of $\mathcal{E}$ in $\mathcal{E}_1$) if the operator V_Δ from $\mathcal{E}$ in $L^2_+(\mathcal{E}_1)$, (or $H^2(\mathcal{E}_1)$) is isometric. This name is justified because V_Δ embeds $\mathcal{E}$ in $H^2(\mathcal{E}_1)$ such that

$$\Delta(\lambda)a = (V_\Delta a)(\lambda) \qquad a \in \mathcal{E}$$

or simpler

$$\Delta(\lambda)a = a(\lambda) \qquad a \in \mathcal{E} \subset H^2(\mathcal{E}_1).$$

We remark that

PROPOSITION 5. The L^2-bounded analytic function $\{\mathcal{E}, \mathcal{E}_1, \Delta(\lambda)\}$ is an evaluation function if and only if the Hilbert space $\mathcal{E}$ is isomorphic to a cyclic subspace for multiplication by λ on $H^2(\mathcal{E}_1)$.

In this case $\dim \mathcal{E}_1 \leq \dim \mathcal{E}$.

Proof. If $\{\mathcal{E}, \mathcal{E}_1, \Delta(\lambda)\}$ is an evaluation function then $V_\Delta \mathcal{E}$ is isomorphic to $\mathcal{E}$, and (because $\Delta(\lambda)$ is outer) $V_\Delta \mathcal{E}$ is a cyclic subspace of $H^2(\mathcal{E}_1)$.

Conversely, if $\mathcal{E}$ is a cyclic (or isomorphic to a cyclic) subspace in $H^2(\mathcal{E}_1)$, then if we define

$$\Delta(\lambda)a = a(\lambda) \qquad a \in \mathcal{E} \subset H^2(\mathcal{E}_1)$$

then we obtain an L^2-bounded analytic function $\{\mathcal{E}, \mathcal{E}_1, \Delta(\lambda)\}$ which is an evaluation function.

Let $P_{\mathcal{E}_1}$ be the projection from $H^2(\mathcal{E}_1)$ on $\mathcal{E}_1$. If there exist $a_1 \in \mathcal{E}_1$ such that

$$(a_1, P_{\mathcal{E}_1} a) = 0 \qquad a \in \mathcal{E}$$

then $(a_1, \lambda^n a) = 0$ for $n = 0, 1, 2, \ldots$, and from cyclicity it follows that $a_1 = 0$, and

./.

$\overline{P_{\mathcal{E}_1}\mathcal{E}} = \mathcal{E}_1$. Therefore $\dim \mathcal{E}_1 \leq \dim \mathcal{E}$.

THEOREM 3. For any L^2-contractive analytic function $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ there exist an evaluation function $\{\mathcal{E}, \mathcal{E}_1, \Delta(\lambda)\}$ and a contractive analytic function $\{\mathcal{E}_1, \mathcal{F}, M(\lambda)\}$ such that

(5.1) $$\Theta(\lambda) = M(\lambda)\Delta(\lambda) \qquad \lambda \in \mathbb{D}.$$

Proof. Let us denote $D_\Theta = (I - V_\Theta^* V_\Theta)^{1/2}$, $\mathcal{D}_\Theta = \overline{D_\Theta \mathcal{E}}$ and $\mathcal{G} = \mathcal{F} \oplus \mathcal{D}_\Theta$. If we consider

$$F_\Omega = F_\Theta + D_\Theta^2 \, dt$$

then obviously F_Ω is an $\mathcal{L}(\mathcal{E})$-valued semi-spectral measure on $\mathbb{T}$, which has $[L^2(\mathcal{G}), V_\Theta \oplus D_\Theta, E^x]$ as a spectral dilations such that $(V_\Theta \oplus D_\Theta)\mathcal{E} \subset L^2_+(\mathcal{G})$. Thus F_Ω is the semi-spectral measure attached to an L^2-contractive function. By [3], Theorem 2, there exists an L^2-contractive outer function $\{\mathcal{E}, \mathcal{E}_1, \Delta(\lambda)\}$ such that $F_\Delta = F_\Omega$. Since

$$F_\Delta(\mathbb{T}) = F_\Theta(\mathbb{T}) + D_\Theta^2 = V_\Theta^* V_\Theta + I - V_\Theta^* V_\Theta = I$$

it results that $V_\Delta^* V_\Delta = I$, hence $\{\mathcal{E}, \mathcal{E}_1, \Delta(\lambda)\}$ is an evaluation function.

Because $F_\Theta \leq F_\Delta$, Proposition 2 (see [3]) implies that there exists a contractive analytic function $\{\mathcal{E}_1, \mathcal{F}, M(\lambda)\}$ such that

$$\Theta(\lambda) = M(\lambda)\Delta(\lambda) \qquad \lambda \in \mathbb{D}.$$

The proof of the theorem is finished.

Let $\{\mathcal{E}, \mathcal{H}, \Gamma\}$ be a correlated action and $\{f_n\}_{-\infty}^{+\infty}$ be a Γ-stationary process in $\mathcal{H}$. We recall that we have denoted by $\Gamma(n)$ the correlation function, F the spectral distribution, and $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ the maximal outer function attached to the process $\{f_n\}_{-\infty}^{+\infty}$.

PROPOSITION 6. If $\{f_n\}_{-\infty}^{+\infty}$ is a moving average and $\Gamma(0) = I$, then its maximal outer function $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ is an evaluation function, and

(5.2) $$(Ga, a)_{\mathcal{E}} = \| a(0) \|^2_{\mathcal{F}}$$

where the function $a(\lambda)$ is the image of a by the embeding of $\mathcal{E}$ in $H^2(\mathcal{F})$.

Proof. By Proposition 4 we have $F = F_\Theta$, and because

$$V_\Theta^* V_\Theta = F_\Theta(\mathbb{T}) = F(\mathbb{T}) = \Gamma(0) = I$$

it follows that $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ is an evaluation function, and using (3.3).

$$(Ga, a)_{\mathcal{E}} = (\Theta(0)^* \Theta(0) a, a)_{\mathcal{E}} = \| \Theta(0) a \|^2_{\mathcal{F}} = \| a(0) \|^2_{\mathcal{F}}.$$

PROPOSITION 7. Let $\{f_n\}_{-\infty}^{+\infty}$ and $\{g_n\}_{-\infty}^{+\infty}$ be Γ-stationary processes in $\mathcal{H}$, $G^{\{f\}}$, $G^{\{g\}}$ be the prediction-error operators, $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ and $\{\mathcal{E}, \mathcal{F}', \Theta'(\lambda)\}$ be their corresponding maximal outer functions. If there exists a contractive analytic function $\{\mathcal{F}', \mathcal{F}, M(\lambda)\}$ such that

$$\Theta(\lambda) = M(\lambda)\,\Theta'(\lambda) \tag{5.3.}$$

then $G^{\{f\}} \leq G^{\{g\}}$.

Proof. For any $a \in \mathcal{E}$ we have

$$(G^{\{f\}}a, a) = (\Theta(0)^{*}\Theta(0)a, a) = \|\Theta(0)a\|^2 = \|M(0)\Theta'(0)a\|^2 \leq$$

$$\leq \|\Theta'(0)a\|^2 = (G^{\{g\}}a, a)$$

hence $G^{\{f\}} \leq G^{\{g\}}$.

In this way we say that if we can determine either contractive factors or multiplies for $\Theta(\lambda)$, then we can obtain either increases or decreases for the prediction-error operator G. In particular we have the following corollary which may be useful in testing.

COROLLARY 2. If there exists an outer scalar function $\delta(\lambda)$ in H^2 such that either

$$\delta(\lambda)V_1 = M_1(\lambda)\Theta(\lambda) \qquad \text{or} \qquad \Theta(\lambda) = M_2(\lambda)\delta(\lambda)V_2 \tag{5.4.}$$

where $\{\mathcal{F}, \mathcal{E}_1, M_1(\lambda)\}$ and $\{\mathcal{E}_2, \mathcal{F}, M_2(\lambda)\}$ are contractive analytic functions, $V_1 : \mathcal{E} \to \mathcal{E}_1$ and $V_2 : \mathcal{E} \to \mathcal{E}_2$ are isometric operators, then either

$$G \geq \exp\left[\frac{1}{2\pi}\int_0^{2\pi} \log|\delta|\,dt\right] \tag{5.5.}$$

or

$$G \leq \exp\left[\frac{1}{2\pi}\int_0^{2\pi} \log|\delta|\,dt\right]. \tag{5.6.}$$

If the process $\{f_n\}_{-\infty}^{+\infty}$ is a moving average, and $\Gamma(0) = I$ then we have seen that in the functional model given by the embeding V_Θ of $\mathcal{E}$ in $H^2(\mathcal{F})$ the error is obtained by $\|a(0)\|^2_{\mathcal{F}}$. In the next we shall show how the Theorem 3 permit us an eventual reduction of parameters in the error calculation.

Let us consider the factorization of the maximal outer function $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ attached to the Γ-stationary process $\{f_n\}_{-\infty}^{+\infty}$ given by Theorem 3

$$\Theta(\lambda) = M(\lambda)\,\Delta(\lambda)$$

where $\{\mathcal{E}, \mathcal{E}_1, \Delta(\lambda)\}$ is the evaluation function, and $\{\mathcal{E}_1, \mathcal{F}, M(\lambda)\}$ is a contractive analytic function. We additionaly suppose that $\mathcal{E}_1 \subset V_\Delta\mathcal{E}$. In this case $\{\mathcal{E}_1, \mathcal{F}, M(\lambda)\}$ is a outer contractive function. Indeed let M be the contraction from $L^2(\mathcal{E}_1)$ to $L^2(\mathcal{F})$, (see [4], chap V, sec. 2). Then because

$$V_\Delta a = \sum_0^\infty e^{int} a_n \qquad a_n \in \mathcal{E}_1$$

we have for any $a \in \mathcal{E}$

$$V_{\circledast} a = M V_{\Delta} a = \sum_{0}^{\infty} e^{int} M a_n = \sum_{0}^{\infty} e^{int} V_M a_n \in \bigvee_{0}^{\infty} e^{int} V_M \mathcal{E}_1 .$$

It follows that

$$L^2_+(\mathfrak{F}) = \bigvee_{0}^{\infty} e^{int} V_{\circledast} \mathcal{E} \subset \bigvee_{0}^{\infty} e^{int} V_M \mathcal{E}_1 \subset L^2_+(\mathfrak{F})$$

therefore $\{\mathcal{E}_1, \mathfrak{F}, M(\lambda)\}$ is an outer contractive function.

We consider the correlated action $\{\mathcal{E}_1, \mathcal{H}, \Gamma\}$ as follows : for $A_1 \in \mathcal{L}(\mathcal{E}_1)$ and $f \in \mathcal{H}$ we put

(5.7) $$A_1 f = (A_1 P_{\mathcal{E}_1}) f$$

and

(5.8) $$\Gamma_1 [f, g] = P_{\mathcal{E}_1} \Gamma[f, g] \big| \mathcal{E}_1 .$$

Clearly that $\{\mathcal{E}_1, \mathcal{H}, \Gamma_1\}$ is a correlated action of $\mathcal{L}(\mathcal{E}_1)$ onto $\mathcal{H}$ and the Γ-stationary process $\{f_n\}_{-\infty}^{+\infty}$ is Γ_1-stationary process too.

If we define $V_1 = V \big| \mathcal{E}_1$ and consider

$$\mathcal{K}'_\infty = \bigvee_{-\infty}^{+\infty} U^n V_1 \mathcal{E}_1 \subset \mathcal{K}_\infty$$

then denoting $U_1 = U|\mathcal{K}'_\infty$ we obtain that the triplet $\{\mathcal{K}'_\infty, V_1, U_1\}$ is the geometrical model of the Γ_1-stationary process $\{f_n\}_{-\infty}^{+\infty}$. Let G' be the prediction-error operator, $\mathbf{F}_1$ be the spectral distribution, and $\{\mathcal{E}_1, \mathfrak{F}_1, \Theta_1(\lambda)\}$ be the maximal outer function attached to the Γ_1-stationary process $\{f_n\}_{-\infty}^{+\infty}$.

THEOREM 4. The function $\{\mathcal{E}_1, \mathbf{F}_1, \Theta_1(\lambda)\}$ coincides with the outer contractive analytic function $\{\mathcal{E}_1, \mathfrak{F}, M(\lambda)\}$ given by Theorem 3. Moreover, for any $a \in \mathcal{E}$ we have

$$(Ga, a) = (G' a(0), a(0)) .$$

Proof. From [3], Theorem 2, it is sufficient to show that $F_{\Theta_1} = F_M$ or equivalent

(5.9) $$dF_{\Theta_1} = dF_M = \frac{1}{2\pi} M(e^{it})^* M(e^{it}) dt .$$

For any analytic polynomial p and $a_1 \in \mathcal{E}_1$ we have

$$\int_0^{2\pi} |p(e^{it})|^2 d(F_M(t) a_1, a_1)_{\mathcal{E}_1} = \frac{1}{2\pi} \int_0^{2\pi} |p(e^{it})|^2 \| M(e^{it}) a_1 \|^2_{\mathfrak{F}} dt =$$

$$= \| p M a_1 \|^2_{L^2_+(\mathfrak{F})} = \| p M V_\Delta a_1 \|^2_{L^2_+(\mathfrak{F})} = \| p V_{\circledast} a_1 \|^2_{L^2_+(\mathfrak{F})} =$$

$$= \| p \Phi^{\mathfrak{F}} P V a_1 \|^2_{L^2_+(\mathfrak{F})} = \| p(U) P V a_1 \|^2_{\mathcal{K}_\infty} = \| P p(U) V a_1 \|^2_{\mathcal{K}_\infty} \leq$$

$$\leq \| p V a_1 \|^2_{\mathcal{K}_\infty} = \| p V_1 a_1 \|^2_{\mathcal{K}'_\infty} = \int_0^{2\pi} |p(e^{it})|^2 d(F_1(t) a_1, a_1)_{\mathcal{E}_1} .$$

It follows that $F_M \leq F_1$, and, because F_M is the semi-spectral measure of an analytic function, it results from the factorization theorem that

$$(5.10) \qquad F_M \leq F_{\Theta_1} .$$

Conversely, for any analytic polynomial p, and $a_1 \in \mathcal{E}_1$ we have

$$\int_0^{2\pi} |p(e^{it})|^2 d(F_{\Theta_1}(t) a_1, a_1)_{\mathcal{E}_1} \leq \int_0^{2\pi} |p(e^{it})|^2 d(F_1(t) a_1, a_1)_{\mathcal{E}_1} =$$

$$= \| p(U_1) a_1 \|^2_{\mathcal{K}'_\infty} = \| p(U) a_1 \|^2_{\mathcal{K}_\infty} = \int_0^{2\pi} |p(e^{it})|^2 d(F(t) a_1, a_1)_{\mathcal{E}_1} .$$

But F_Θ is the maximal semi-spectral measure such that $F_\Theta \leq F$, hence $F_{\Theta_1} \leq F_\Theta$. Now we have

$$\int_0^{2\pi} |p(e^{it})|^2 d(F_{\Theta_1}(t) a_1, a_1)_{\mathcal{E}_1} \leq \int_0^{2\pi} |p(e^{it})|^2 d(F_\Theta(t) a_1, a_1) =$$

$$= \| p V_\Theta a_1 \|^2_{L^2_+(\mathcal{F})} = \| p M V_\Delta a_1 \|^2_{L^2_+(\mathcal{F})} = \| p M a_1 \|^2_{L^2_+(\mathcal{F})} =$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \| p(e^{it}) M(e^{it}) a_1 \|^2_{\mathcal{F}} dt = \frac{1}{2\pi} \int_0^{2\pi} |p(e^{it})|^2 (M(e^{it})^* M(e^{it}) a_1, a_1)_{\mathcal{E}_1} dt =$$

$$= \int_0^{2\pi} |p(e^{it})|^2 d(F_M(t) a_1, a_1)_{\mathcal{E}_1} .$$

Therefore $F_{\Theta_1} \leq F_M$ and by (5.10) we have (5.9).

Using (3.3) we obtain for any $a \in \mathcal{E}$

$$(G a, a)_{\mathcal{E}} = \| \Theta(0) a \|^2_{\mathcal{F}} = \| M(0) \Delta(0) a \|^2_{\mathcal{F}} = \| \Theta_1(0) \Delta(0) a \|^2_{\mathcal{F}} =$$

$$= \| \Theta_1(0) a(0) \|^2_{\mathcal{F}} = (G' a(0), a(0)) .$$

Such a way, the factorization theorem given by (5.1) permits us the reduction of the parameters in prediction up to $\mathcal{E}_1$.

Following this way, on can construct a minimal subspace $\mathcal{E}_0$ in $\mathcal{E}$ which gives us the prediction, hence we can exhibit the set of essential parameters for the prediction of the process $\{ f_n \}_{-\infty}^{+\infty}$.

REFERENCES

[1]. Aronszajn , N., *Theory of reproducing kernels,* Trans. Amer. Soc. 68 (1950), 337 - 404.

[2]. Masani, P., *An explicit treatment of dilation theory,* (preprint)

[3]. Suciu, I. and Valuşescu I., *Factorization of semi-spectral measures,* Rev. Roum. Math. pures et appl. 6 (1976), 773 - 793.

[4]. Sz. - Nagy, B. and Foiaş C., *Harmonic Analysis of Operators on Hilbert space,* Acad. Kiadó Budapest - North Holland Company Amsterdam - London 1970.

[5]. Wiener, N. and Masani P. *The prediction theory of multivariate stochastic processes I,* Acta Math. 98 (1957), 111-150.

[6]. Wiener, N. and Masani P. *The prediction theory of multivariate stochastic process II,* Acta Math. 99 (1958), 93 - 139 .