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On a Theorem of M.D.Choi and E.G.Effros

by Dan Voiculescu

Recently, M.D.Choi and E.G.Effros have obtained an important new existence theorem for completely positive liftings [5]. One of the reasons for interest in completely positive liftings is due to the connection discovered by W.B.Arveson [2] with the problem whether the Brown-Douglas-Fillmore Ext of a C^*-algebra is a group [4].

In this paper we give a new proof of the Choi-Effros theorem. While the original proof of M.D.Choi and E.G.Effros used tensor products and convexity, the present proof is based on the existence of quasicentral approximate units proved by C.A.Akeman and G.K.Pedersen in [1] and by W.B.Arveson in [3].

§ 1.

The first step in our proof is to obtain the existence of completely positive liftings for the completely positive mappings of the C^*-algebra of $n \times n$ matrices into a quotient C^*-algebra.

Thus let $\mathcal{M}_n$ denote the C^*-algebra of $n \times n$ matrices with the system of matrix units $(e_{ij})_{1\leq i,j\leq n}$

$$\left(\sum e_{ii} = 1,\ e_{ij} = e_{ji}^*,\ e_{ij}\cdot e_{kl} = \delta_{jk} e_{il}\right)$$

Let further A be a C^*-algebra.

Lemma 1.1.*)

A map $\Phi : \mathcal{M}_n \longrightarrow A$ is completely positive if and only if the element $\sum_{1\leq i,j\leq n} \Phi(e_{ij}) \otimes e_{ij}$ of $A \otimes \mathcal{M}_n$ is positive.

Proof.

If Φ is completely positive then $\sum_{1\leq i,j\leq n} \Phi(e_{ij}) \otimes e_{ij}$

*) This is a particular case of Lemma 4.1 in [6]

is necessarily positive since $\sum\limits_{1 \le i,j \le n} e_{ij} \otimes e_{ij} = (\sum\limits_{i} e_{1i} \otimes e_{1i})^* \cdot (\sum\limits_{j} e_{1j} \otimes e_{1j})$ is a positive element of $\mathcal{M}_n \otimes \mathcal{M}_n$.

Conversely, suppose $\sum \Phi(e_{ij}) \otimes e_{ij}$ is positive, then we can find elements $x_{ij} \in A$ such that

$$\sum \Phi(e_{ij}) \otimes e_{ij} = (\sum x_{ij} \otimes e_{ij})^* (\sum x_{ij} \otimes e_{ij}) =$$

$= \sum\limits_{i} (\sum\limits_{j,k} x_{ij}^* x_{ik} \otimes e_{jk})$. Then defining $\Phi_i : \mathcal{M}_n \longrightarrow A$ by $\Phi_i(e_{jk}) = x_{ij}^* \; x_{ik}$ it is immediate that Φ_i is completely positive and $\Phi = \sum\limits_{i} \Phi_i$.

Q.E.D.

Consider now $\mathcal{J}$ a closed two-sided ideal of A and $\pi : A \longrightarrow A/\mathcal{J}$ the canonical map.

<u>Proposition 1.2.</u>

Let $\varphi : \mathcal{M}_n \longrightarrow A/\mathcal{J}$ be completely positive. Then we can find $\psi : \mathcal{M}_n \longrightarrow A$ a completely positive mapping such that $\pi \circ \psi = \varphi$. Moreover if φ is contractive (resp. A is unital and φ is unit-preserving) then ψ can be chosen contractive (resp. unit-preserving).

<u>Proof:</u>

Since by Lemma 1.1, $\sum\limits_{i,j} \varphi(e_{ij}) \otimes e_{ij}$ is a positive element of $A/\mathcal{J} \otimes \mathcal{M}_n$ there are elements $x_{ij} \in A/\mathcal{J}$ such that

$$\sum_{i,j} \varphi(e_{ij}) \otimes e_{ij} = (\sum_{i,j} x_{ij} \otimes e_{ij})^* (\sum_{i,j} x_{ij} \otimes e_{ij}),$$

Let $y_{ij} \in A$ be such that $\pi(y_{ij}) = x_{ij}$, then defining $\psi : \mathcal{M}_n \longrightarrow A$ by the relation

$$\sum_{i,j} \psi(e_{ij}) \otimes e_{ij} = (\sum_{i,j} y_{ij} \otimes e_{ij})^*(\sum_{i,j} y_{ij} \otimes e_{ij})$$

it is clear that $\pi \circ \psi = \varphi$ and because of Lemma 1.1., ψ is completely positive.

In case φ is contractive, let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function

$$f(t) = \begin{cases} 1 & \text{for } t \leq 1 \\ t^{-1/2} & \text{for } t > 1 \end{cases}$$

and consider $f(\psi(1)) = C$ in the C^*-algebra obtained from A by adjoining a unit .

Then $\psi'(X) = C\,\psi(X)\,C$ is a completely positive map taking values in A, $\pi \circ \psi' = \varphi$ and $\|\psi'\| = \|\psi'(1)\| \leq 1$.

In case A is unital and φ is unit-preserving let ψ be a contractive completely positive lifting of φ and define $\psi'(X) = f(X)(1 - \psi(1)) + \psi(X)$, where f is any state of $\mathcal{M}_n$. Then ψ' will do.

Q.E.D.

§ 2.

In this section we prove the Choi-Effros theorem.

There will be no essential loss of generality, to suppose <u>throughout this section all C^*-algebras separable.</u>

We recall that a completely positive contractive map $\varphi : A \longrightarrow B$ (A, B - C^*-algebras) is called <u>nuclear</u> ([5] section 3) if there are completely positive contractive maps σ_j, τ_j :

$$A \xrightarrow{\sigma_j} \mathcal{M}_{n(j)} \xrightarrow{\tau_j} B$$

such that $\tau_j \circ \sigma_j$ converge to φ in the point-norm topology.

We shall also mention the following fact ([5] section 3, p.13): if A, B, φ are unital and φ is nuclear, then σ_j, τ_j can be chosen unital.

The next thing to recall is the Akeman-Arveson-Pedersen result on <u>quasi-central approximate units</u>. Under our separability assumptions it is : for B a C^*-algebra and $J \subset B$ a two-sided ideal there is a sequence $0 \leq u_1 \leq u_2 \leq \ldots$, $\|u_j\| \leq 1$ of elements of J such that for any $x \in J$ we have $\lim_{n \to \infty} \|u_n x - x\| = 0$ and for any $y \in B$ we have $\lim_{n \to \infty} \|[u_n, y]\| = 0$.

We shall also need the following simple lemma from [3] p.6:

Let $\varepsilon > 0$ and f be a continuous function on $[0,1]$ satisfying $f(0) = 0$. Then there is $\delta > 0$ such that for any pir of elements a, b in the unit ball of a C^*-algebra A, $a \geq 0$, we have

$$\|[a,b]\| \leq \delta \Rightarrow \|[f(a), b]\| \leq \varepsilon$$

In the proof of the theorem below $\delta(\varepsilon)$ will denote the greatest δ corresponding to ε in the above lemma for $f(x) = \sqrt{x}$.

<u>Theorem 2.1. (Choi-Effros)</u>

Let A, B be separable C^*-algebras, J a closed two-sided ideal of B and $\pi : B \longrightarrow B/J$ the canonical map. Let also $\varphi : A \longrightarrow B/J$ be a nuclear completely positive contraction. Then there is a nuclear completely positive contraction $\psi : A \longrightarrow B$ such that $\pi \circ \psi = \varphi$. Moreover if A, B, are unital, then ψ can be chosen to be also unital.

<u>Proof.</u>

Let $\{x_j\}_{j=1}^{\infty}$ be a total sequence in A and let $y_j \in B$, be such that $\pi(y_j) = \varphi(x_j)$. suppose moreover

$\|x_j\| < 1/2$, $\|y_j\| < 1/2$

We can also suppose that the approximate factorizations of φ

$$A \xrightarrow{\sigma_j} \mathcal{M}_{n(j)} \xrightarrow{\tau_j} B/J$$

are such that

$$\|(\tau_j \circ \sigma_j)(x_i) - \varphi(x_i)\| < 1/j^2$$

for $1 \leq i \leq j$.

In view of Proposition 1.2. , we can find completely positive contractions $\tau'_j : \mathcal{M}_{n(j)} \longrightarrow B$ such that $\pi \circ \tau'_j = \tau_j$, and consider $\chi_j = \tau'_j \circ \sigma_j$. Let $0 = u_0 \leq u_1 \leq u_2 \leq \dots$ be a quasicentral approximate unit of J with respect to B .

Replacing $\{u_j\}_1^\infty$ by some subsequence we can suppose that the following three conditions are satisfied :

(i) $\|[u_n, \ \chi_j(x_i) - y_i]\| < 1/2\, \delta(1/n^2)$, $(1 \leq i,j \leq n)$

(ii) $\|[u_n, y_i]\| < 1/2\, \delta(1/n^2)$ $(1 \leq i \leq n)$

(iii) $\|(1 - u_n)(\chi_j(x_i) - y_i)\| < 1/j^2$ $(1 \leq i \leq j \leq n)$

(Condition (iii) can be satisfied since

$\|\pi(\chi_j(x_i) - y_i)\| = \|(\tau_j \circ \sigma_j)(x_i) - \varphi(x_i)\| < 1/j^2$ for $1 \leq i \leq j$.

Also 1 is the unit of the C^*-algebra obtained from B by adjoining a unit).

From conditions (i) - (iii) we infere, δ being an increasing function :

$$\|[u_{n+1} - u_n, \chi_n(x_i) - y_i]\| < \delta(1/n^2) \qquad (1 \leq i \leq n)$$

$$\|[u_{n+1} - u_n , y_i]\| < \delta(1/n^2) \qquad (1 \leq i \leq n)$$

$$\|(u_{n+1} - u_n)(\chi_n(x_i) - y_i)\| < 2/n^2 \qquad (1 \leq i \leq n)$$

$$\|[(1 - u_n), (\chi_n(x_i) - y_i)]\| < \delta(1/n^2) \qquad (1 \leq i \leq n) .$$

Hence for $R_n = (u_{n+1} - u_n)^{1/2}$ $(n \geq 0)$ we shall have :

(1) $\| [R_n, \chi_n(x_i) - y_i] \| < 1/n^2$ $(1 \leq i \leq n)$

(2) $\| [R_n, y_i] \| < 1/n^2$ $(1 \leq i \leq n)$

(3) $\| R_n^2 (\chi_n(x_i) - y_i) \| < 2/n^2$ $(1 \leq i \leq n)$

(4) $\| [(1 - u_n)^{1/2}, \chi_n(x_i) - y_i] \| < 1/n^2$ $(1 \leq i \leq n)$

Consider now $\psi_n : A \longrightarrow B$

$$\psi_n(x) = \sum_{j=0}^{n} R_j \chi_j(x) R_j + (1-u_{n+1})^{1/2} \chi_{n+1}(x)(1-u_{n+1})^{1/2}$$

Then ψ_n is a completely positive contraction which factors exactly through a finite-dimensional C^*-algebra.

We have :

$$\psi_n(x_i) = \sum_{j=0}^{n} R_j(\chi_j(x_i) - y_i)R_j + \sum_{j=0}^{n} [R_j, y_i] R_j + y_i \sum_{i=0}^{n} R_j^2 +$$

$$+ (1 - u_{n+1})^{1/2} [\chi_{n+1}(x_i) - y_i , (1-u_{n+1})^{1/2}] +$$

$$+ (1 - u_{n+1})(\chi_{n+1}(x_i) - y_i) +$$

$$+ [(1 - u_{n+1})^{1/2}, y_i] (1 - u_{n+1})^{1/2} + y_i(1 - u_{n+1}) =$$

$$= y_i + \sum_{j=0}^{n} R_j(\chi_j(x_i) - y_i)R_j +$$

$$+ \sum_{j=0}^{n} [R_j, y_i] R_j + (1 - u_{n+1})^{1/2} [\chi_{n+1}(x_i) - y_i, (1-u_{n+1})^{1/2}] +$$

$$+ (1 - u_{n+1})(\chi_{n+1}(x_i) - y_i) + [(1 - u_{n+1})^{1/2}, y_i] (1 - u_{n+1})^{1/2}.$$

The last three terms converge to zero for $n \longrightarrow \infty$ because of (4) respectively (iii) and respectively the lemma mentioned before the theorem . On the other hand

$$\| R_j(\chi_j(x_i) - y_i)R_j \| \leq$$

$$\leq \| R_j^2 (\chi_j(x_i) - y_i) \| + \| [R_j, \chi_j(x_i) - y_i] \| < 3/j^2$$ for $j \geq i$ because of (1) and (3) .

Also $\sum_{j=0}^{\infty} [R_j, y_i] R_j$ is absolutely norm-convergent because of (2) . Thus we have

$$\lim_{n \to \infty} \psi_n(x_i) = y_i + \sum_{j=0}^{\infty} R_j(\chi_j(x_i) - y_i)R_j + \sum_{j=0}^{\infty} [R_j, y_i] R_j$$

The maps ψ_n being all contractive it follows that they are point-norm convergent to a completely positive contraction $\psi : A \longrightarrow B$, which being the limit of the ψ_n is nuclear.

The infinite sums in the expresion of $\psi(x_i)$ being absolutely convergent it follows that $\pi(\psi(x_i) - y_i) = 0$ and $\{x_i\}_{i=1}^{\infty}$ being total in A we infere $\pi \circ \psi = \varphi$. So ψ satisfies all our requirements.

In the unital case, the same argument used at the end of the proof of Proposition 1.2 gives us the possibility to make ψ unital without affecting its nuclearity.

Q.E.D.

Remarks :

1) In case A is separable and B arbitrary, the theorem still holds since B can be replaced by some separable subalgebra

2) In fact the preceding proof consists in the construction of a localization map in the sense of [3] (with some additional properties) which is used to glue togerther the χ_j .

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