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# ON CROSSED PRODUCTS

by

S. STRATILA, D. VOICULESCU, L. ZSIDO

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# ON CROSSED PRODUCTS

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# INTRODUCTION

The cross-product construction for von Neumann algebras has recently received special attention, partly due to A.Connes([6]) discovery of the role it plays in the structure theory of type III factors and the successful generalization of this situation by M.Takesaki's duality theorem for crossed products of von Neumann algebras by locally compact abelian automorphism groups ([60]) . M.Landstad ( [38]) has obtained an important characterization of those von Neumann algebras which are crossed products by locally compact automorphism groups. Moreover this result of M.Landstad strongly indicates that the action of the dual group on the crossed product in the abelian case, should be replaced in general by a certain comodule structure.

On the other hand the development of the duality theory for locally compact groups (see 0.2.21 for references) has led to consider groups as objects of a certain category of Hopf - von Neumann algebras (Katz algebras).

Thus it became clear that one possible way of extending the theorems of M.Takesaki ([60]) and M.Landstad ( [38]) and related results about weights on crossed products due to T.Digerres ([8], [9]) , J.-L.Sauvageot ([50]) , and U.Haagerup ([25],[26]) was in the frame-work of comodules over Katz algebras. Another way has been taken by J.E.Roberts ([47]) , considering a different dual object for a locally compact group , motivated by superselection structure in particle physics (see [42],[43],[14]).

The aim of this paper is to present the duality theory for crossed products of von Neumann algebras by groups and group duals considered as Katz algebras. We have chosen a parallel presentation of both cases, as announced in ([53]), as a first step towards a unified treatement as crossed products by Katz algebras ([54]) .

The paper has four chapters, from 0 to III .

Chapter 0 , contains mainly the basic definitions for the rest of the work. Especially there is a careful discussion of the invariance of a weight with respect to an action (see Theorem 0.2.12) which leads to a brief introduction to the duality theory for Katz algebras and will also serve in § III.1.2 as an important technical tool.

Chapter I , deals with actions of groups. Theorem I.2.1 extends M.Takesaki's duality theorem by removing the commutativity condition on the group. Using M.Landstad's theorem ([38]) we give (I.2.3) a short proof of the commutation theorem for crossed products ([60], [9]).

Chapter II , is a parallel to Chapter I for actions of group duals. To deal with this case we had to prove an analogue (Theorem II.3.3) of M.Landstad's theorem ([38]) and to develop as well analogs of the preliminaries which were well known in the case of group actions. As a by-product we mention Proposition II.1.3 which is a "(ultra)-weak" result in connection with the conjecture (A) of P.Eymard ([21]).

In § III.1 and § III.2 dual weights on crossed products are introduced and in the case of group actions we recover the known results ([60], [8], [9], [50], [25], [26]) . Theorem III.1.2 extends a result of U.Haagerup ([25]) from the commutative case to the general case and was the main reason for the search of an appropriate definition of the invariance of a weight with respect to an action. In § III.3 the twisted Plancherel theorem of M.Takesaki ([60]) and T.Digernes ([9]) is extended again by removing the commutativity condition ; our proof seems to be somewhat simpler even in the commutative case.

We were led to these questions by the study of the works of A.Connes and M.Takesaki ([6],[7],[60]). On the technical side we owe very much to the knowledge of the preprints of the works of U.Haagerup ([24]) and M.Landstad ([38]) , the methods of which are intensively used in the present paper.

In the final stage of our work on crossed products by groups and group duals we have received the preprints ([25],[26]) of U.Haagerup which already contained a (slightly different) construction of the conditional expectation $P_G$ (I.3.1) and the computation (by the same method) of the dual weight on crossed products by groups. For the sake of completeness we have included in the present paper (I.3.1.,III.1.1) the proofs we have found independently.

After submitting the announcement ([53]) of our work, we have received the preprints of M.Landstad ([40]) and Y.Nakagami ([43]) which contain the same main results as those of Chapters I,II.

We gratefully acknowledge the receipt of a letter of M.Landstad ([39]) containing a beautiful proof that any action of a group dual is saturated (see II.1.1), while we knew this fact only for amenable groups. Consequently, using this proof of M.Landstad, we were able to remove the saturation condition on the action in some statements (compare with [53]) and to simplify some proofs.

Thanks are also due to professor H.Leptin who kindly informed us about the present state of research in connection with conjecture (H) of P.Eymard and communicated us a copy of [41].

# CONTENTS

# BASIC DEFINITIONS AND AUXILLIARY RESULTS.

## § 1. Auxilliary results

0.1.1. Let M be a von Neumann algebra and $\varphi$ be a normal weight on M (i.e. on $M^+$). From the works of F.Combes ([2],[4]), U.Haagerup ([22]), G.K.Pedersen and M.Takesaki ([46]) we know that the set $F_\varphi = \{f \in M_*^+ ; f \leq \varphi\}$ is $\varepsilon$-filtered, i.e. : $(\forall)$ $f_1, f_2 \in F_\varphi$, $(\forall)$ $\varepsilon > 0$, $(\exists)$ $f \in F_\varphi$ such that $f_1 \leq (1+\varepsilon)f$ , $f_2 \leq (1+\varepsilon)f$ , and :

(1) $$\langle x, \varphi \rangle = \sup_{f \in F_\varphi} \langle x, f \rangle \quad , \quad x \in M^+,$$

and moreover, there is a family $\{f_\iota\}_{\iota \in I} \subset M_*^+$ such that

(2) $$\langle x, \varphi \rangle = \sum_{\iota \in I} \langle x, f \rangle \quad , \quad x \in M^+.$$

In what follows we shall replace the subscript $f \in F_\varphi$ by $f \leq \varphi$.

We shall freely use the modular theory of n.s.f.(normal,semifinite, faithful) weights ([4],[6],[9],[23],[50],[51],[55]). The objects associated to $\varphi$ in this theory will be denoted, as usually, by $\mathfrak{N}_\varphi = \{x \in M ; \varphi(x^*x) < +\infty\}$, $\mathfrak{M}_\varphi = \mathfrak{N}_\varphi^* \mathfrak{N}_\varphi$, $\mathfrak{A}_\varphi = \mathfrak{N}_\varphi^* \cap \mathfrak{N}_\varphi$, $H_\varphi$ , $S_\varphi$ , $J_\varphi$ , $\Delta_\varphi$, $\sigma_t^\varphi$ . The (maximal) Tomita algebra will be denoted by $\mathfrak{T}_\varphi$. If $x \in \mathfrak{N}$, then $x_\varphi \in \mathfrak{N}_\varphi \subset H_\varphi$ stands for the canonical image of x in the Hilbert space $H_\varphi$ with scalar product $(x_\varphi | y_\varphi)_\varphi = \varphi(y^*x)$ , $x,y \in \mathfrak{N}_\varphi$. When the normal weight is not semifinite or faithful, the modular theory refers to its restriction to the n.s.f. part (see[46]). The support projection of $\varphi$ will be denoted as $s(\varphi)$ .

0.1.2.LEMMA. <u>Let M, A be von Neumann algebras, let $\varphi$ be a normal weight on M and $k \in A_*^+$. Then</u>

(3) $$\langle X, \varphi \otimes k \rangle = \sup_{f \in F_\varphi} \langle X, f \otimes k \rangle \quad , \quad X \in (M \bar{\otimes} A)^+$$

Proof. Since the set $\{f\otimes k \; ; \; f\in F_\varphi\}$ is $\varepsilon$-filtered, the right hand side of (3) defines a normal weight $\Phi$ on $M\otimes A$ and, for $X = x\otimes a$, $x\in \mathfrak{M}_\varphi^+$, $a\in A$, we have (by (1)) :

(4) $$< X , \varphi\otimes k > \; = \; < X , \Phi > .$$

Therefore we may assume that $\varphi$ is a n.s.f. weight and then (4) shows that $\varphi\otimes k$ and $\Phi$ agree on the w-dense, $\sigma_t^{\varphi\otimes k}$-invariant ($t\in\mathbb{R}$) $*$-subalgebra $\mathfrak{M}_\varphi\otimes A$ of $M\bar{\otimes}A$.

On the other hand, $f\in F_\varphi \Rightarrow f\circ\sigma_t^\varphi \leq \varphi\circ\sigma_t^\varphi = \varphi$, so that $f\in F_\varphi \Leftrightarrow f\circ\sigma_t^\varphi\in F_\varphi$. Therefore, for any $X\in (M\bar{\otimes}A)^+$ we have

$$\begin{aligned} < X , \Phi\circ\sigma_t^{\varphi\otimes k} > &= \sup_{f\in F_\varphi} < X , (f\otimes k)\circ(\sigma_t^\varphi\otimes\sigma_t^k) > \\ &= \sup_{f\in F_\varphi} < X , (f\circ\sigma_t^\varphi)\otimes(k\circ\sigma_t^k) > \\ &= \sup_{f\in F_\varphi} < X , f\otimes k > \\ &= < X , \Phi > , \end{aligned}$$

which means that $\Phi$ is $\sigma_t^{\varphi\otimes k}$-invariant.

By the theorem of G.K.Pedersen and M.Takesaki ([46], Prop. 5.9), it follows that $\Phi = \varphi\otimes k$. ∎

As an obvious consequence of 0.1.2 we note that if $N$ is another von Neumann algebra and if $\pi : N \longrightarrow M$ is a normal $*$-homomorphism then, for any normal weight $\varphi$ on $M$ and any $k\in A_*^+$, we have

(5) $$(\varphi\otimes k)\circ(\pi\otimes i_A) \; = \; (\varphi\circ\pi)\otimes k$$

as (normal) weights on $N\otimes A$.

Actually, owing to the work of G.K.Pedersen and M.Takesaki ([46]), it can be shown that for any normal weight $\varphi$ on $M$ there is a family $\{f_\iota\}_{\iota\in I}\subset M_*^+$ such that, for any normal weight $\psi$ on any $N$, we have

$$\varphi\otimes\psi = \sum_{\iota\in I}(f_\iota\otimes\psi) .$$

A similar result is well known for strictly semifinite weights ([5]).

0.1.3. LEMMA. Let $\varphi$ be a normal weight on $M$ and $x \in \mathfrak{N}_\varphi$, $x \geq 0$. If the linear mapping

$$f_x : \mathfrak{N}_\varphi^* \ni y \longmapsto \varphi(yx)$$

has a linear w-continuous extension to $M$, then $x \in \mathfrak{M}_\varphi^+$ and $\varphi(x) = f_x(s(x))$.

Proof. There is a sequence $\{e_n\}$ of projections commuting with $x$ such that

$$x e_n \geq n^{-1} e_n \quad , \quad e_n \uparrow s(x)$$

Since $e_n \leq n x e_n$, we have $e_n \leq n^2 x^* e_n x \in \mathfrak{M}_\varphi$, thus $e_n \in \mathfrak{N}_\varphi^*$ and, by the w-continuity of $f_x$,

$$\varphi(e_n x) = f_x(e_n) \longrightarrow f_x(s(x)) .$$

On the other hand, $e_n x = x^{1/2} e_n x^{1/2} \uparrow x$, so that, by the normality of $\varphi$,

$$\varphi(x) = \sup_n \varphi(e_n x) = f_x(s(x)) < +\infty . \quad \blacksquare$$

In particular, if $x \in \mathfrak{N}_\varphi$, $x \geq 0$ and $\varphi(yx) = 0$ for all $y \in \mathfrak{N}_\varphi^*$, then $\varphi(x) = 0$.

0.1.4. LEMMA.([16],[35]). Let $\varphi$ be a normal weight on $M$. For each $x \in \mathfrak{T}_\varphi^2$, the linear mapping

$$f_x : \mathfrak{N}_\varphi^* \ni y \longmapsto \varphi(yx)$$

has a linear w-continuous extension to $M$.

Proof. We may suppose that $\varphi$ is a n.s.f.weight. Moreover, we may restrict to the case $x = ab$, $a, b \in \mathfrak{T}_\varphi$. For any $y \in \mathfrak{N}_\varphi^*$ we then have

$$\begin{aligned}\varphi(yx) = (x_\varphi \mid y^*_\varphi)_\varphi &= (a_\varphi b_\varphi \mid y^*_\varphi)_\varphi = (b_\varphi \mid (a^* y^*)_\varphi)_\varphi = \\ &= (b_\varphi \mid S_\varphi (ya)_\varphi)_\varphi = ((ya)_\varphi \mid S^*_\varphi b_\varphi)_\varphi = \\ &= (y\, a_\varphi \mid S^*_\varphi b_\varphi)_\varphi .\end{aligned}$$

Thus $f_x$ is extended by the vector functional $\omega_{a_\varphi, S^*_\varphi b_\varphi}$. $\blacksquare$

0.1.5. Let $M$, $N$ be von Neumann algebras. We refer to the work of U.Haagerup ([24]) for the notion of extended positive part $\overline{N^+}$ of a von Neumann algebra. Also following U.Haagerup ([24]), but alterating slightly his terminology, an operator valued weight on $M^+$ with values in $\overline{N^+}$ will be an additive positive-homogeneous mapping $T : M^+ \longrightarrow \overline{N^+}$. If $N$ is a von Neumann subalgebra of $M$, a conditional expectation of $M$ on $N$ will be an operator valued weight on $M^+$ with values in $\overline{N^+}$ with the additional property

$$T(a^*xa) = a^*T(x)a \quad ; \quad a \in N\ ,\ x \in M^+ .$$

As in ([24]), $\mathfrak{N}_T = \{x \in M\ ;\ T(x^*x) \in N^+\}$, $\mathfrak{M}_T = \mathfrak{N}_T^* \mathfrak{N}_T$, but we denote also by $T$ the canonical extension to $\mathfrak{M}_T$.

The following outstanding result of U.Haagerup ([24], Cor.4.2) is the main technical tool in dealing with weights on crossed products

THEOREM (U.Haagerup, [24]). Let $N \subset M$ be von Neumann algebras and let $T$ be a n.s.f. conditional expectation of $M$ on $N$. For any n.s.f. weights $\varphi, \psi$ on $N$, $\varphi \circ T$, $\psi \circ T$ are n.s.f. weights on $M$ and

(i) $\sigma_t^{\varphi \circ T}(x) = \sigma_t^{\varphi}(x)$ , $x \in N$ , $t \in \mathbb{R}$ ,

(ii) $[D(\psi \circ T) : D(\varphi \circ T)]_t = [D\psi : D\varphi]_t$ , $t \in \mathbb{R}$ .

Recall that $\{[D\psi : D\varphi]_t\}_{t \in \mathbb{R}}$ denotes the A.Connes'cocycle([ ]).

0.1.6. DEFINITION([38],[64]). Let $M$, $A$ be von Neumann algebras and let $\omega$ be a n.s.f. weight on $A$. Then the formula

$$< E_M^{\omega}(X)\ ,\ f > \ = \ < X\ ,\ f \otimes \omega > , \quad X \in (M \overline{\otimes} A)^+ \ ,\ f \in M_*^+ ,$$

defines a n.s.f. operator valued weight $E_M^{\omega}$ on $(M \overline{\otimes} A)^+$ with values in $\overline{M^+}$ having the additional property

$$E_M^{\omega}((a^* \otimes 1_A)\ X\ (a \otimes 1_A)) = a^* E_M(X)\ a \quad , \quad X \in (M \overline{\otimes} A)^+, a \in M,$$

(that is, $X \longmapsto E_M^{\omega}(X) \otimes 1_A$ is a conditional expectation).

For any $k \in A_*^+$, $E_M^k : M \overline{\otimes} A \longrightarrow M$ is a w-continuous linear mapping and, using (0.1.2) we get

$$E_M^{\omega}(X) = \sup_{k \leq \omega} E_M^k(X) \quad , \quad X \in (M \overline{\otimes} A)^+ .$$

This procedure of "partial integration" ( or "slice mappings") was introduced by J.Tomiyama (see [64]) and is particularly useful in dealing with tensor products and crossed products.

Using this method, more precisely imitating the proof of ([64], Theorem 2.1 ), we obtain the following

0.1.7.LEMMA.(see also [61]). Let $M$ , $N$ , $A$ be von Neumann algebras, $N \subset M$ and $X \in M \overline{\otimes} A$ . If

$$\langle X \, , \varphi \otimes k \rangle = 0$$

for any $\varphi \in M_*$ which annihilates $N$ and for any $k \in A_*$, then

$$X \in N \overline{\otimes} A$$

Proof. By assumption we have $\langle E_M^k(X) \, , \varphi \rangle = 0$ for any $\varphi \in M_*$ which annihilates $N$ and any $k \in A_*$, therefore $E_M^k(X) \in N$ for all $k \in A_*$.

Supose $M$ acts on the Hilbert space $H$ , $M \subset B(H)$ , and consider $y' \in N' \subset B(H)$. For any $\varphi \in B(H)_*$ and any $k \in A_*$ we have :

$$\begin{aligned} \langle X(y' \otimes 1_A) \, , \varphi \otimes k \rangle &= \langle X \, , \varphi(\cdot y') \otimes k \rangle = \langle E_M^k(X) \, , \varphi(\cdot y') \rangle = \\ &= \langle E_M^k(X) \, y' \, , \varphi \rangle = \langle y' \, E_M^k(X) \, , \varphi \rangle = \\ &= \langle E_M^k(X) \, , \varphi(y' \cdot) \rangle = \langle X \, , \varphi(y' \cdot) \otimes k \rangle = \\ &= \langle (y' \otimes 1_A)X \, , \varphi \otimes k \rangle . \end{aligned}$$

Since $X(y' \otimes 1_A) - (y' \otimes 1_A)X \in B(H) \overline{\otimes} A$, it follows that $X$ commutes with $(y' \otimes 1_A)$ . Since $y' \in N'$ was arbitrary, it is clear that $X \in N \overline{\otimes} A$ . ∎

We shall need the following result , which is a direct consequence of ([24], Theorem 3.10 ).

0.1.8. LEMMA. Let $\varphi, \psi$ be n.s.f. weights on $M$, let $u \in M$ be unitary and let $\lambda \in \mathbb{R}$, $\lambda > 0$. Then

$$\sigma_t^{\varphi,\psi}(u) = \lambda^{it} u \iff \psi(a) = \lambda^{-1} \varphi(uau^*), \ (\forall)\ a \in M^+ .$$

Proof. Indeed,

$$\sigma_t^{\varphi,\psi}(u) = \lambda^{it} u \iff \sigma_{-i}^{\varphi,\psi}(u) = \lambda u$$

$$\iff (u, \lambda u) \in \mathrm{Graph}(\sigma_{-i}^{\varphi,\psi})$$

$$\text{(by [47], Theorem 3.10)} \iff u\mathfrak{N}_\psi^* \subset \mathfrak{N}_\varphi^*,\ \mathfrak{N}_\varphi u \subset \mathfrak{N}_\psi \text{ and } \varphi(ux) = \lambda \psi(xu), \ (\forall)\ x \in \mathfrak{N}_\psi^* \mathfrak{N}_\varphi$$

$$\iff \psi(a) = \lambda^{-1} \varphi(uau^*), \ (\forall)\ a \in M^+ .$$

The last equivalence is elementary. ∎

For another proof see ([26], Lemma 1.3).

0.1.9. Let $M$ be a von Neumann algebra. Then any norm bounded w-continuous $M$-valued function $F$ defined on a locally compact group $G$ defines a unique element $x_F \in M \overline{\otimes} L^\infty(G)$

$$< x_F, \varphi \otimes k > = \int < F(t), \varphi > k(t)\, dt, \ \varphi \in M_*, \ k \in L^1(G).$$

0.1.10. If $\Omega$ is locally compact space, $\mu$ is a positive Radon measure on $\Omega$ and

$$F : \Omega \ni t \longmapsto F(t) \in B(H)$$

is an operator valued function, then we shall say that $F$ is w-measurable (respectively s-measurable or $s^*$-measurable) with respect to $\mu$ if for any compact set $C \subset \Omega$ and for any $\varepsilon > 0$ there exists a compact set $K \subset C$ such that $\mu(C \setminus K) \leq \varepsilon$ and the restriction of $F$ to $K$ is w-continuous (respectively s-continuous or $s^*$-continuous).

By Lusin's theorem, the function $\Omega \ni t \longmapsto \|F(t)\| \in \mathbb{R}^+$ is then measurable with respect to $\mu$.

0.1.11. Notations and terminology. Usually we follow the current terminology and notations in operator algebras. However, by the term injection we abreviate the notion of faithful normal unit-preserving $*$-homomorphism between von Neumann algebras. The terms w-topology, s-topology mean respectively the ultra-weak and the ultra-strong topology.

The symbol $1_M$ stands for the unit of the von Neumann algebra $M$ and also for the scalar subalgebra of $M$, while the symbol $i_M$ denotes the identity automorphism of $M$. In some situations we replace the subscript $M$ by a simpler one recalling $M$, or by a number indicating a position in tensor products.

$B(H)$ is the algebra of all bounded linear operators on a Hilbert space $H$.

If $u : G \longrightarrow B(H)$ is a (continuous) unitary representation of the locally compact group $G$, then $\mathrm{Ad}\, u : G \longrightarrow \mathrm{Aut}(B(H))$ denotes the corresponding (continuous) $*$-automorphic representation of $G$ on $B(H)$:

$$\mathrm{Ad}(u(t))(x) = u(t)\, x\, u(t)^* , \quad t \in G , x \in B(H) .$$

Moreover, for any $f \in L^1(G)$ we denote

$$u(f) = \int f(t)\, u(t)\, dt \in B(H) .$$

By $\mathcal{R}\{X\}$ we denote the von Neumann algebra generated by $X$ and by $\mathrm{c.l.m.}\{X\}$ we denote the w-closed linear span of $X$.

Given two von Neumann algebras $M$, $N$ there is a unique $*$-isomorphism of $M \overline{\otimes} N$ onto $N \overline{\otimes} M$ which sends $x \otimes y$ in $y \otimes x$, $x \in M$, $y \in N$. This symmetry isomorphism will be denoted by $\tilde{\cdot}$. If $M \subset B(H)$, $N \subset B(K)$, then we denote by $\sim$ the unitary operator of $H \overline{\otimes} K$ onto $K \overline{\otimes} H$ which sends $\xi \otimes \eta$ in $\eta \otimes \xi$, $\xi \in H$, $\eta \in K$. For $X \in M \overline{\otimes} N$ we then have $\tilde{X} = \sim \circ X \circ \sim$.

Some other notations will appear on the way.

## § 2. Actions, invariant weights and crossed products.

0.2.1. Consider a triple $\{M, A, \delta\}$ where

$$\begin{cases} M, A \text{ are von Neumann algebras}, \\ \delta : M \to M \bar{\otimes} A \text{ is a normal } *\text{-homomorphism}, \ \delta(1_M) = 1_M \otimes 1_A . \end{cases}$$

For any $k \in A_*$ and any $x \in M$ denote $k \cdot x = F_M^k(\delta(x))$. Then

$$\langle k \cdot x, f \rangle = \langle \delta(x), f \otimes k \rangle, \quad f \in M_*,$$

the map $E_M^k \circ \delta : M \ni x \mapsto k \cdot x \in M$ is linear, w-continuous and of norm $\leq \|k\|$. Moreover

$$(k \cdot x)^* = k^* \cdot x^* \quad \text{for any } x \in M, \ k \in A_*,$$
$$k \cdot x \geq 0 \quad \text{for any } x \in M^+, \ k \in A_*^+.$$

Also, we shall denote

$$M^\delta = \{x \in M ; \delta(x) = x \otimes 1_A\}.$$

In a more precise situation, $\delta$ will be called an action of $A$ on $M$ and $M^\delta$ the centralizer of this action.

0.2.2. DEFINITION. Let $\{M, A, \delta\}$ be as in 0.2.1. A normal semifinite weight $\varphi$ on $M$ will be called $\delta$-invariant if

$$\langle \delta(x), \varphi \otimes k \rangle = \langle x \otimes 1_A, \varphi \otimes k \rangle, \quad x \in M^+, \ k \in A_*^+.$$

0.2.3. LEMMA. Let $\{M, A, \delta\}$ be as in 0.2.1. and let $\varphi$ be a $\delta$-invariant normal semifinite weight on $M$. Then

$$\langle (k \cdot x)^*(k \cdot x), \varphi \rangle^{1/2} \leq \|k\| \langle x^* x, \varphi \rangle^{1/2}, \quad x \in M, \ k \in A_*.$$

Proof. If $x \notin \mathfrak{N}_\varphi$, then the statement is obvious.

Let $x \in \mathfrak{N}_\varphi$, $k \in A_*$ and $k = |k|(\cdot v)$ be the polar decomposition of $k$. Then, for any $f \in M_*^+$, $f \leq \varphi$, we have

$$\langle (k \cdot x)^*(k \cdot x), f \rangle =$$
$$= \langle ((k \cdot x)^* \otimes 1_A) \delta(x), f \otimes k \rangle =$$
$$= \langle ((k \cdot x)^* \otimes v) \delta(x), f \otimes |k^*| \rangle \leq$$

$$\leq <(k\cdot x)^*(k\cdot x)\ ,\ f>^{1/2} \|k\|^{1/2} <\delta(x^*x)\ ,\ f\otimes |k^*|>^{1/2} \leq$$
$$\leq <(k\cdot x)^*(k\cdot x)\ ,\ f>^{1/2} \|k\|^{1/2} <\delta(x^*x)\ ,\ \varphi\otimes |k^*|>^{1/2} =$$
$$= <(k\cdot x)^*(k\cdot x)\ ,\ f>^{1/2} \|k\|^{1/2} < x^*x\ ,\ \varphi >^{1/2} \|k\|^{1/2} ,$$

hence

$$<(k\cdot x)^*(k\cdot x)\ ,\ f>^{1/2} \leq \|k\| < x^*x\ ,\ \varphi >^{1/2} .$$

Since $f \in M_*^+$ , $f \leq \varphi$ , was arbitrary, we conclude

$$<(k\cdot x)^*(k\cdot x)\ ,\ \varphi >^{1/2} \leq \|k\| < x^*x\ ,\ \varphi >^{1/2} . \blacksquare$$

It follows that, if $\varphi$ is $\delta$-invariant, then

(1) $\quad x \in \mathfrak{N}_\varphi \Longrightarrow k\cdot x \in \mathfrak{N}_\varphi$ and $\|(k\cdot x)_\varphi\|_\varphi \leq \|k\| \|x_\varphi\|_\varphi$, $\quad k \in A_*$,

thus the map $x \longmapsto k\cdot x$ defines a bounded linear operator $\pi_\varphi^\delta(k)$ on $H_\varphi$ :

(2) $$\pi_\varphi^\delta(k)\, x_\varphi = (k\cdot x)_\varphi \quad , \quad x \in \mathfrak{N}_\varphi ,$$

and $\|\pi_\varphi^\delta(k)\| \leq \|k\|$ . Hence we have a contractive (linear) representation $\pi_\varphi^\delta : A_* \longrightarrow B(H_\varphi)$ of the Banach space $A_*$ .

0.2.4. LEMMA. <u>Let</u> $\{M, A, \delta\}$ <u>be as in</u> 0.2.1. <u>and let</u> $\varphi$ <u>be a</u> $\delta$ <u>-invariant normal semifinite weight on</u> M . <u>Then</u>

$$s(\varphi) \in M^\delta .$$

<u>Proof</u>. Clearly $(1_M - s(\varphi))_\varphi = 0_\varphi$ and, by (1), this implies $(k\cdot(1_M - s(\varphi)))_\varphi = 0_\varphi$ , $k \in A_*$, therefore $(k\cdot(1_M - s(\varphi)))\, s(\varphi) = 0$ , $k \in A_*$. For any $f \in M_*$ , $k \in A_*$ we have

$$<\delta(1_M - s(\varphi))\ (s(\varphi)\otimes 1_A)\ ,\ f\otimes k > = <(k\cdot(1_M - s(\varphi)))\ s(\varphi)\ ,\ f>=0$$

so that $\delta(1_M - s(\varphi))\ (s(\varphi)\otimes 1_A) = 0$ . This proves that

$$\delta(s(\varphi)) \leq s(\varphi) \otimes 1_A .$$

On the other hand, put $e = (s(\varphi)\otimes 1_A) - \delta(s(\varphi))$ . Since

$$<\delta(1_M - s(\varphi))\ ,\ \varphi\otimes k > = <(1_M - s(\varphi))\otimes 1_A\ ,\ \varphi\otimes k > = 0\ ,\ k\in A_*^+$$

we have $< e\ ,\ \varphi\otimes k > = 0$ , $k \in A_*^+$, so that $e(s(\varphi)\otimes s(k)) = 0$, $k \in A_*^+$, and hence

$$e = e(s(\varphi)\otimes 1_A)e = \bigvee_{k\in A_*^+} e(s(\varphi)\otimes s(k))e = 0 . \blacksquare$$

0.2.5. LEMMA. Let $\{M, A, \delta\}$ be as in 0.2.1. and let $\varphi$ be a $\delta$-invariant normal semifinite weight on $M$. For any $x, y \in \mathfrak{N}_\varphi$ and any $k \in A_*^+$, we have

$$(y^* \otimes 1_A)\delta(x) \in \mathfrak{M}_{\varphi \otimes k}, \quad y^*(k \cdot x) \in \mathfrak{M}_\varphi \quad \text{and}$$

(3) $$\langle (y^* \otimes 1_A)\delta(x), \varphi \otimes k \rangle = \langle y^*(k \cdot x), \varphi \rangle .$$

Proof. By the $\delta$-invariance of $\varphi$ we have

$$\langle \delta(x)^* \delta(x), \varphi \otimes k \rangle = \langle x^* x, \varphi \rangle \langle 1_A, k \rangle < +\infty,$$

so that $\delta(x) \in \mathfrak{N}_{\varphi \otimes k}$ and consequently $(y^* \otimes 1_A)\delta(x) \in \mathfrak{M}_{\varphi \otimes k}$.

Write $(y^* \otimes 1_A)\delta(x) = a_1 - a_2 + ia_3 - ia_4$ with $a_j \in \mathfrak{M}^+_{\varphi \otimes k}$, $j = 1,2,3,4$. For any $f \in M_*^+$ we have

$$\langle a_j, f \otimes k \rangle = \langle E^k_M(a_j), f \rangle, \quad j = 1,2,3,4,$$

thus, using Lemma 0.1.2, we get

(4) $$\langle a_j, \varphi \otimes k \rangle = \langle E^k_M(a_j), \varphi \rangle, \quad j = 1,2,3,4.$$

In particular, all $E^k_M(a_j)$ belong to $\mathfrak{M}^+_\varphi$, so that

$$y^*(k \cdot x) = E^k_M((y^* \otimes 1_A)(x)) =$$

$$= E^k_M(a_1) - E^k_M(a_2) + iE^k_M(a_3) - iE^k_M(a_4),$$

and using again (4) we obtain (3). ∎

0.2.6. Consider now a quadruple $\{M, A, \delta, j\}$, where

$$\begin{cases} \{M, A, \delta\} \text{ are as in 0.2.1,} \\ j : A \longrightarrow A \text{ is an involutive } (j \circ j = i_A) \text{ *-antiautomorphism.} \end{cases}$$

For any $k \in A_*$ denote

$$k^o = k^* \circ j .$$

Then $A_* \ni k \longmapsto k^o \in A_*$ is an isometric involution on the Banach space $A_*$. Note that

$$(k^o)^* = (k^*)^o, \quad k \in A_* .$$

We shall introduce a stronger notion of invariance :

0.2.7. DEFINITION. Let $\{M, A, \delta, j\}$ be as in 0.2.6. A normal semifinite weight $\varphi$ on $M$ will be called $(\delta, j)$-invariant if it is $\delta$-invariant and

(5) $\langle (y^* \otimes 1_A)\delta(x), \varphi \otimes k \rangle = \langle \delta(y^*)(x \otimes 1_A), \varphi \otimes (k \circ j) \rangle$

for any $x, y \in \mathfrak{N}_\varphi$, and any $k \in A_*^+$.

Using Lemma 0.2.5 we see that both sides of (5) are well defined and

$$\langle (y^* \otimes 1_A)\delta(x), \varphi \otimes k \rangle = \langle y^*(k\,x), \varphi \rangle ,$$

$$\langle \delta(y^*)(x \otimes 1_A), \varphi \otimes (k \circ j) \rangle = \langle (k^0 \cdot y)^* x, \varphi \rangle .$$

It follows that, if $\varphi$ is $(\delta, j)$-invariant, then (4) holds and

$$((k \cdot x)_\varphi \mid y_\varphi)_\varphi = (x_\varphi \mid (k^0 \cdot y)_\varphi)_\varphi \quad , \quad x, y \in \mathfrak{N}_\varphi , \; k \in A_* .$$

which in turn means that

$$\pi_\varphi^\delta(k)^* = \pi_\varphi^\delta(k^0) \quad , \quad k \in A_* .$$

Therefore

0.2.8. LEMMA. *If $\varphi$ is $(\delta, j)$-invariant, then $\pi_\varphi^\delta : A_* \longrightarrow B(H_\varphi)$ is a contractive representation of the involutive Banach space $(A_*, {}^0)$.*

We shall see that this property characterizes the $(\delta, j)$-invariance. To be more explicit, we introduce the following

0.2.9. DEFINITION. Let $\{M, A, \delta, j\}$ be as in 0.2.6 and $\varphi$ be a normal semifinite weight on $M$. We say that $\pi_\varphi^\delta$ is a *bounded representation of the involutive Banach space* $(A_*, {}^0)$ *if, for some* $\lambda > 0$,

(i) $x \in \mathfrak{N}_\varphi \Longrightarrow k \cdot x \in \mathfrak{N}_\varphi$ and $\|(k \cdot x)_\varphi\|_\varphi \le \lambda \|k\| \|x_\varphi\|_\varphi$ , $k \in A_*$,

(ii) $x, y \in \mathfrak{N}_\varphi \Longrightarrow ((k \cdot x)_\varphi \mid y_\varphi)_\varphi = (x_\varphi \mid (k^0 \cdot y)_\varphi)_\varphi$ , $k \in A_*$.

By condition (i), formula (2) defines a bounded linear operator $\pi_\varphi^\delta(k) \in B(H_\varphi)$, $\|\pi_\varphi^\delta(k)\| \le \lambda \|k\|$ and by (ii), $\pi_\varphi^\delta(k)^* = \pi_\varphi^\delta(k^0)$.

Note also that, if $\pi_\varphi^\delta$ is a bounded representation of the involutive Banach space $(A_*, {}^0)$, then, with the same argument as in the proof of Lemma 0.2.4, we obtain $\delta(s(\varphi)) \leq s(\varphi) \otimes 1_A$. It follows that

(6) $$\delta(M_{s(\varphi)}) \subset M_{s(\varphi)} \overline{\otimes} A \;.$$

where $M_{s(\varphi)} = s(\varphi)\, M\, s(\varphi)$.

0.2.10. LEMMA. Let $\{M, A, \delta, j\}$ be as in 0.2.6 and $\varphi$ be a normal semifinite weight on $M$. If $\pi_\varphi^\delta$ is a bounded representation of the involutive Banach space $(A_*, {}^0)$, then $\delta$ commutes with the modular automorphism group $\{\sigma_t^\varphi\}_{t\in\mathbb{R}}$ of $\varphi$:

$$\delta(\sigma_t^\varphi(x)) = (\sigma_t^\varphi \otimes 1_A)(\delta(x)), \quad x \in M_{s(\varphi)} \; ; \; t \in \mathbb{R}.$$

Proof. Whitout restriction of the generality, but rennouncing to the assumption $\delta(1_M) = 1_M \otimes 1_A$, we may suppose that $\varphi$ is a n.s.f. weight (see (6)).

Consider the left Hilbert algebra $\mathfrak{A}_\varphi = \mathfrak{N}_\varphi^* \cap \mathfrak{N}_\varphi \subset H_\varphi$ associated to $\varphi$, the closed antilinear operator $S_\varphi$ in $H_\varphi$ defined by

$$S_\varphi x_\varphi = (x^*)_\varphi \quad , \quad x_\varphi \in \mathfrak{A}_\varphi \subset H_\varphi \; ;$$

and the modular operator $\Delta_\varphi = S_\varphi^* S_\varphi$. Then

(7) $$\pi_\varphi^\delta(k)\mathfrak{A}_\varphi \subset \mathfrak{A}_\varphi \;,\quad \pi_\varphi^\delta(k)^* = \pi_\varphi^\delta(k^0) \;,\quad k \in A_* \;,$$

and for any $x_\varphi \in \mathfrak{A}_\varphi$, $k \in A_*$, we have

$$\begin{aligned} S_\varphi \pi_\varphi^\delta(k)\, x_\varphi &= S_\varphi\,(k\cdot x)_\varphi = ((k\cdot x)^*)_\varphi = \\ &= (k^*\cdot x^*)_\varphi = \pi_\varphi^\delta(k^*)\,(x^*)_\varphi = \pi_\varphi^\delta(k^*)\, S_\varphi\, x_\varphi \;. \end{aligned}$$

Since $S_\varphi$ is the closure of its restriction to $\mathfrak{A}_\varphi$ and since $\pi_\varphi^\delta(k)$, $\pi_\varphi^\delta(k^*)$ are bounded, we infer that

(8) $$\xi \in \mathrm{Dom}(S_\varphi) \Longrightarrow \pi_\varphi^\delta(k)\xi \in \mathrm{Dom}(S_\varphi) \;,\; S_\varphi \pi_\varphi^\delta(k)\xi = \pi_\varphi^\delta(k^*) S_\varphi \xi \;.$$

Consider now $\eta \in \mathrm{Dom}(\Delta_\varphi)$ and $\xi \in \mathrm{Dom}(S_\varphi)$. Then $\eta \in \mathrm{Dom}(S_\varphi)$, $S_\varphi \eta \in \mathrm{Dom}(S_\varphi^*)$ and, using (7) and (8), we obtain

$$\begin{aligned} (S_\varphi \pi_\varphi^\delta(k)\eta \,|\, S_\varphi \xi)_\varphi &= (\pi_\varphi^\delta(k^*) S_\varphi \eta \,|\, S_\varphi \xi)_\varphi \\ &= (S_\varphi \eta \,|\, \pi_\varphi^\delta(k^{*0}) S_\varphi \xi)_\varphi \\ &= (S_\varphi \eta \,|\, S_\varphi \pi_\varphi^\delta(k^0)\xi)_\varphi \\ &= (\pi_\varphi^\delta(k^0)\xi \,|\, S_\varphi^* S_\varphi \eta)_\varphi \\ &= (\xi \,|\, \pi_\varphi^\delta(k) S_\varphi^* S_\varphi \eta)_\varphi \;. \end{aligned}$$

Therefore, for any $\eta \in \mathrm{Dom}(\Delta_\varphi)$ and any $k \in A_*$,

$$S_\varphi \pi_\varphi^\delta(k)\eta \in \mathrm{Dom}(S_\varphi^*) \quad \text{and} \quad S_\varphi^* S_\varphi \pi_\varphi^\delta(k)\eta = \pi_\varphi^\delta(k) S_\varphi^* S_\varphi \eta ,$$

or, equivalently,

$$\eta \in \mathrm{Dom}(\Delta_\varphi) \Longrightarrow \pi_\varphi^\delta(k)\eta \in \mathrm{Dom}(\Delta_\varphi) , \quad \Delta_\varphi \pi_\varphi^\delta(k)\eta = \pi_\varphi^\delta(k)\Delta_\varphi \eta .$$

Thus $\pi_\varphi^\delta(k)$ commutes with $\Delta_\varphi$, so that

$$\pi_\varphi^\delta(k)\, \Delta_\varphi^{it} = \Delta_\varphi^{it}\, \pi_\varphi^\delta(k) \quad , \quad t \in \mathbb{R} , \ k \in A_* .$$

Consequently,

$$(\sigma_t^\varphi(k\cdot x))_\varphi = \Delta_\varphi^{it}\, (k\cdot x)_\varphi = \Delta_\varphi^{it}\, \pi_\varphi^\delta(k)\, x_\varphi =$$
$$= \pi_\varphi^\delta(k)\, \Delta_\varphi^{it}\, x_\varphi = \pi_\varphi^\delta(k)\, (\sigma_t^\varphi(x))_\varphi = (k\cdot\sigma_t^\varphi(x))_\varphi ,$$

for any $x \in \mathfrak{N}_\varphi$ , $k \in A_*$, that is

$$\sigma_t^\varphi(k\cdot x) = k\cdot\sigma_t^\varphi(x) \quad , \quad x \in M , \ k \in A_* , \ t \in \mathbb{R} ,$$

which in turn gives

$$\begin{aligned} \langle \delta(\sigma_t^\varphi(x)) , f \otimes k \rangle &= \langle k\cdot\sigma_t^\varphi(x) , f \rangle \\ &= \langle \sigma_t^\varphi(k\cdot x) , f \rangle \\ &= \langle k\cdot x , f\circ\sigma_t^\varphi \rangle \\ &= \langle \delta(x) , (f\circ\sigma_t^\varphi)\otimes k \rangle \\ &= \langle (\sigma_t^\varphi \otimes i_A)(\delta(x)) , f \otimes k \rangle , \end{aligned}$$

for all $f \in M_*$. This proves the Lemma. ∎

0.2.11. LEMMA. <u>Let</u> $\{M, A, \delta, j\}$ <u>be as in</u> 0.2.6 <u>and</u> $\varphi$ <u>be a normal semifinite weight on</u> $M$ . <u>If</u> $\pi_\varphi^\delta$ <u>is a bounded representation of the involutive Banach space</u> $(A_*, {}^\circ)$, <u>then</u> $\varphi$ <u>is</u> $\delta$-<u>invariant</u>.

<u>Proof</u>. We have to show that, for any $k \in A_*^+$,

$$(9) \qquad (\varphi \otimes k)\circ\delta = \langle 1_A , k \rangle \varphi$$

as (normal) weights on $M$ .

Since $(k\cdot(1_M - s(\varphi)))\, s(\varphi) = 0$ , using Lemma 0.1.2. we obtain

$$\langle \delta(1_M - s(\varphi)) , \varphi \otimes k \rangle = \langle k\cdot(1_M - s(\varphi)) , \varphi \rangle = 0 ,$$

that is, $s((\varphi\otimes k)\circ\delta) \le s(\varphi)$ , hence $s((\varphi\otimes k)\circ\delta) = s((\varphi\otimes k)\circ\delta|_{M_{s(\varphi)}})$.
On the other hand, clearly, $s(\langle 1_A, k\rangle\varphi) = s(\langle 1_A, k\rangle\varphi|_{M_{s(\varphi)}})$.

Therefore we may assume that $\varphi$ is a n.s.f. weight.

Then the square $B = \mathcal{T}_\varphi^2$ of the (maximal) Tomita algebra is a w-dense, $\sigma_t^\varphi$-invariant $*$-subalgebra of $M$. Remark that $B$ is the linear span of its positive part $B^+$.

If $x \in B^+$ and $k \in A_*^+$, then $k\cdot x \in \mathfrak{M}_\varphi^+$. Indeed, since $\pi_\varphi^\delta(k)^* = \pi_\varphi^\delta(k^0)$, for any $y \in \mathfrak{N}_\varphi$ we have

$$\langle y^*(k\cdot x), \varphi\rangle = \langle (k^0\cdot y)^* x, \varphi\rangle$$

and the assertion follows using Lemma 0.1.3. and Lemma 0.1.4..

Thus, given $x \in B^+$ and $k \in A_*^+$, we may write

$$x = a^2 \quad , \quad k\cdot x = b^2 \quad \text{with} \quad a, b \in \mathfrak{N}_\varphi^+ .$$

By the proof of ([46], Lemma 5.2 ), there is a net $\{y_\iota\} \subset \mathfrak{N}_\varphi^+$ of analytic elements such that

$$\sigma_\alpha^\varphi(y_\iota) \xrightarrow{\text{s-topology}} 1_M \quad , \quad \alpha \in \mathbb{C} .$$

We have

$$\begin{aligned}
\langle x, (\varphi\otimes k)\circ\delta\rangle &= \langle \delta(x), \varphi\otimes k\rangle \\
&= \langle k\cdot x, \varphi\rangle \\
&= \langle b^2, \varphi\rangle \\
&= (b_\varphi \mid b_\varphi)_\varphi \\
&= (b_\varphi \mid S_\varphi b_\varphi)_\varphi \\
&= (b_\varphi \mid J_\varphi \Delta_\varphi^{1/2} b_\varphi)_\varphi \\
&= \lim_\iota (b_\varphi \mid J_\varphi \sigma_{-i/2}^\varphi(y_\iota)\Delta_\varphi^{1/2} b_\varphi)_\varphi \\
&= \lim_\iota (b_\varphi \mid J_\varphi \Delta_\varphi^{1/2} y_\iota b_\varphi)_\varphi \\
&= \lim_\iota (b_\varphi \mid S_\varphi (y_\iota b)_\varphi)_\varphi \\
&= \lim_\iota (b_\varphi \mid (b y_\iota)_\varphi)_\varphi \\
&= \lim_\iota ((k\cdot x)_\varphi \mid (y_\iota)_\varphi)_\varphi \\
&= \lim_\iota (x_\varphi \mid (k^0\cdot y_\iota)_\varphi)_\varphi
\end{aligned}$$

$$= \lim_{\iota} \ (a_\varphi \mid (a(k^0 \cdot y_\iota))_\varphi)_\varphi$$

$$= \lim_{\iota} \ (a_\varphi \mid S_\varphi((k^0 \cdot y_\iota)a)_\varphi)_\varphi$$

$$= \lim_{\iota} \ (a_\varphi \mid J_\varphi \Delta_\varphi^{1/2} (k^0 \cdot y_\iota)\, a_\varphi)_\varphi$$

$$= \lim_{\iota} \ (a_\varphi \mid J_\varphi \sigma^\varphi_{-i/2}(k^0 \cdot y_\iota)\, \Delta_\varphi^{1/2} a_\varphi)_\varphi$$

(by Lemma 0.2.10) $$= \lim_{\iota} \ (a_\varphi \mid J_\varphi(k^0 \cdot \sigma^\varphi_{-i/2}(y_\iota))\, \Delta_\varphi^{1/2} a_\varphi)_\varphi$$

$$= (a_\varphi \mid J_\varphi(k^0 \cdot 1_M)\, \Delta_\varphi^{1/2} a_\varphi)_\varphi$$

$$= (a_\varphi \mid S_\varphi a_\varphi)_\varphi \langle 1_A, k\rangle$$

$$= (a_\varphi \mid a_\varphi)_\varphi \langle 1_A, k\rangle$$

$$= \langle a^2, \varphi\rangle \langle 1_A, k\rangle$$

$$= \langle x, \langle 1_A, k\rangle \varphi\rangle .$$

Therefore the two weights appearing in (9) agree on $B \subset \mathcal{M}_{\langle 1_A, k\rangle\varphi}$. In particular $(\varphi \otimes k)\circ\delta$ is semifinite.

Using again Lemma 0.2.10, we see that $(\varphi \otimes k)\circ\delta$ is invariant with respect to the modular automorphism group $\{\sigma_t^\varphi\}$ of $\langle 1_A, k\rangle\varphi$.

Thus the equality (9) follows by ([46], Proposition 5.9) and this proves the Lemma. ∎

0.2.12. THEOREM. Let $M, A$ be von Neumann algebras, $\delta : M \longrightarrow M \bar{\otimes} A$ be a normal $*$-homomorphism such that $\delta(1_M) = 1_M \otimes 1_A$ and $j : A \longrightarrow A$ be an involutive $*$-antiautomorphism.

Let $\varphi$ be a normal semifinite weight on $M$.

Then the following assertions are equivalent :

(i) $\varphi$ is $(\delta, j)$-invariant ;

(ii) $\pi_\varphi^\delta$ is a contractive representation of the involutive Banach space $(A_*, {}^0)$ ;

(iii) $\pi_\varphi^\delta$ is a bounded representation of the involutive Banach space $(A_*, {}^0)$ ;

(iv) the following holds :

a) $s(\varphi) \in M^{\delta}$ ;

b) $\delta(\sigma_t^{\varphi}(x)) = (\sigma_t^{\varphi} \otimes i_A)(\delta(x))$ , $x \in M_{s(\varphi)}$ , $t \in \mathbb{R}$ ;

c) <u>there is a</u> w-<u>dense</u> $\sigma_t^{\varphi}$-<u>invariant</u> $*$-<u>subalgebra</u> B <u>of</u> $\mathfrak{M}_{\varphi} \subset M_{s(\varphi)}$ <u>such that</u>

$$\langle \delta(x), \varphi \otimes k \rangle = \langle x \otimes 1_A, \varphi \otimes k \rangle \quad , x \in B^+, \; k \in A_*^+ ;$$

d) <u>there is a</u> $\|\cdot\|_{\varphi}$-<u>dense subset</u> D <u>of</u> $\mathfrak{N}_{\varphi}$ <u>and</u> <u>there is a norm-dense subset</u> F <u>of</u> $\{k \in A_*^+ \; ; \; k \circ j = k\}$ <u>such that</u>

$$x, y \in D \; , \; k \in F \implies \delta(x), \delta(y) \in \mathfrak{N}_{\varphi \otimes k} \quad \text{and}$$

$$\langle (y^* \otimes 1_A)\delta(x), \varphi \otimes k \rangle = \langle \delta(y^*)(x \otimes 1_A), \varphi \otimes k \rangle \quad .$$

<u>Proof</u>. (i) $\implies$ (ii) by Lemma 0.2.8;

(ii) $\implies$ (iii) obvious;

(iii) $\implies$ (i) by Lemma 0.2.11, and an easy argument using Lemma 0.2.5.

If the equivalent conditions (i),(ii),(iii) are satisfied, then (iv) is verified as follows :

a) by Lemma 0.2.4;

b) by Lemma 0.2.10;

c) obvious with $B = \mathfrak{M}_{\varphi}$ ;

d) obvious with $D = \mathfrak{N}_{\varphi}$ , $F = \{k \in A_*^+ \; ; \; k \circ j = k\}$.

(iv) $\implies$ (ii) By a) we may assume that $\varphi$ is a n.s.f. weight. Using b),c) and ([46],Proposition 5.9) we obtain the equality $(\varphi \otimes k)\circ\delta = \langle 1_A, k\rangle \varphi$ , $k \in A_*^+$, so that $\varphi$ is $\delta$-invariant.

Then $\pi_{\varphi}^{\delta}$ is defined and contractive (see (1),(2)) and using d) we get the equality $\pi_{\varphi}^{\delta}(k)^* = \pi_{\varphi}^{\delta}(k^0)$, first for $k \in F$, then by density for any $k \in A_*^+$, $k \circ j = k$ , and finally for any $k \in A_*$, since $A_*$ is the linear span of $\{k \in A_*^+ \; ; \; k \circ j = k\}$ . ∎

0.2.13. COROLLARY. Let $\{M, A, \delta, j\}$ be as in 0.2.6 and let $\varphi, \psi$ be $(\delta, j)$-invariant n.s.f. weights on $M$. Then

$$[D\psi : D\varphi]_t \in M^{\delta}, \quad t \in \mathbb{R}.$$

Proof. Denote by $Mat_2(M)$ the von Neumann algebra of $2 \times 2$ matrices over $M$. Then $Mat_2(M) = M \otimes Mat_2(\mathbb{C})$ and $\delta_2 = \delta \otimes i_{\mathbb{C}}$ is a normal $*$-homomorphism of $Mat_2(M)$ into $Mat_2(M) \bar{\otimes} A$ which preserves the units:

$$\delta_2\left(\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}\right) = \begin{pmatrix} \delta(x_{11}) & \delta(x_{12}) \\ \delta(x_{21}) & \delta(x_{22}) \end{pmatrix}.$$

Consider also the balanced weight $\theta = \theta(\varphi, \psi)$ on $Mat_2(M)$ ([6]):

$$\theta\left(\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}\right) = \varphi(x_{11}) + \psi(x_{22})$$

Since $\varphi, \psi$ are $(\delta, j)$-invariant, a straightforward computation shows that $\theta$ is $(\delta_2, j)$-invariant. By Theorem 0.2.12 it follows that $\delta_2$ commutes with $\sigma_t^{\theta}$.

Denote $u_t = [D\psi : D\varphi]_t$, $t \in \mathbb{R}$. Then ([6]):

$$\begin{pmatrix} 0 & 0 \\ \delta(u_t) & 0 \end{pmatrix} = \delta_2\left(\begin{pmatrix} 0 & 0 \\ u_t & 0 \end{pmatrix}\right) = \delta_2\left(\sigma_t^{\theta}\left(\begin{pmatrix} 0 & 0 \\ 1_M & 0 \end{pmatrix}\right)\right) =$$

$$= (\sigma_t^{\theta} \otimes i_A)\left(\delta_2\left(\begin{pmatrix} 0 & 0 \\ 1_M & 0 \end{pmatrix}\right)\right)$$

$$= (\sigma_t^{\theta} \otimes i_A)\left(\begin{pmatrix} 0 & 0 \\ 1_M & 0 \end{pmatrix} \otimes 1_A\right)$$

$$= \begin{pmatrix} 0 & 0 \\ u_t \otimes 1_A & 0 \end{pmatrix}.$$

Therefore $\delta(u_t) = u_t \otimes 1_A$, i.e. $u_t \in M^{\delta}$. ■

0.2.14. Some statements in the sequel will refer to the following situation:

$$\begin{cases} A \text{ is a von Neumann algebra} \\ \delta : A \longrightarrow A \overline{\otimes} A \text{ is a normal } *\text{-homomorphism}, \ \delta(1_A) = 1_A \otimes 1_A \\ j : A \longrightarrow A \text{ is an involutive } *\text{-antiautomorphism} \\ \text{and } \delta, \ j \text{ are related by} \\ \qquad (j \overline{\otimes} j) \circ \delta = \tilde{\cdot} \circ \delta \circ j \ . \end{cases}$$

that is we particularize the situation in 0.2.6 to the case $M = A$ and we impose a compatibility condition for $\delta$ and $j$.

0.2.15. PROPOSITION. *Let $\{A, \delta, j\}$ be as in 0.2.14 and let $\omega$ be a non-zero $\delta$-invariant normal semifinite weight on* A. *Then $\omega$ is faithful and*

$$A^{\delta} = \mathbb{C} \cdot 1_A \ .$$

*Proof*. Let $e$ be an arbitrary projection in $A^{\delta}$. Then

$$\delta(j(e)) = \tilde{\cdot}(j \otimes j)(\delta(e)) = 1_A \otimes j(e) \ .$$

Thus, for any $x \in \mathfrak{M}_{\omega}^{+}$ and any $k \in A_{*}^{+}$ we have

$$\begin{aligned} \langle j(e) x j(e), \omega \rangle \langle 1_A, k \rangle &= \langle \delta(j(e) x j(e)), \omega \otimes k \rangle \\ (10) \qquad &= \langle (1_A \otimes j(e)) \delta(x) (1_A \otimes j(e)), \omega \otimes k \rangle \\ &= \langle \delta(x), \omega \otimes k(j(e) \cdot j(e)) \rangle \\ &= \langle x, \omega \rangle \langle j(e), k \rangle \ . \end{aligned}$$

Since $\omega$ is non-zero, $s(\omega) \neq 0$, so that there exists $k \in A_{*}^{+}$, $k \neq 0$, with $\langle j(1_A - s(\omega)), k \rangle = 0$. By Lemma 0.2.4 we have $s(\omega) \in A^{\delta}$, thus, replacing $e$ by $1_A - s(\omega)$ in (10), we get

$$\langle j(1_A - s(\omega)) x j(1_A - s(\omega)), \omega \rangle = 0 \quad , \quad x \in \mathfrak{M}_{\omega}^{+} \ .$$

It follows that $j(1_A - s(\omega)) s(\omega) = 0$, i.e.:

$$(11) \qquad s(\omega) = j(s(\omega)) s(\omega) \ .$$

Now supose that $s(\omega) \neq 1_A$. Then there exists $k \in A^{+}$, $k \neq 0$, with $\langle j(s(\varphi)), k \rangle = 0$; and using again (10), we get

$$< j(s(\omega))\, x\, j(s(\omega)) \,, \omega > = 0 \quad , \quad x \in \mathfrak{M}_{\omega}^{+} \,,$$

thus

(12) $$j(s(\omega))\, s(\omega) = 0$$

and using (11),(12) we obtain $s(\omega) = 0$ , a contradiction.

Therefore $\omega$ is faithful.

If $e \in A^{\delta}$ , $e \neq 1_A$ , then there exists $k \in A^{+}$ , $k \neq 0$ , with $< j(e) , k > = 0$ and (10) yields

$$< j(e)\, x\, j(e) \,, \omega > = 0 \quad , \quad x \in \mathfrak{M}_{\omega}^{+} \,,$$

which means that

$$j(e) = j(e)\, s(\omega) = 0 \,.$$

Therefore the only projections in $A$ are the trivial ones.

This proves the proposition. ∎

0.2.16. COROLLARY. <u>Let $\{A, \delta, j\}$ be as in 0.2.14 and let $\omega, \tau$ be two non-zero $(\delta, j)$-invariant normal semifinite weights on $A$ . Then $\omega, \tau$ are faithful and proportional, i.e. there is $\lambda \in \mathbb{R}$ , $\lambda > 0$ such that</u>

$$\tau = \lambda \omega \,.$$

<u>Proof</u>. By Proposition 0.2.15 , $\omega$ and $\tau$ are faithful , by Coroll - ry 0.2.13 we have

$$[D\tau : D\omega]_t \in A^{\delta} \quad , \quad t \in \mathbb{R} \,,$$

and using again Proposition 0.2.15, we get

$$[D\tau : D\omega]_t \in \mathbb{C} \quad , \quad t \in \mathbb{R} \,.$$

It follows that $[D\tau : D\omega]_t = \lambda^{it}$ , $t \in \mathbb{R}$ , for some $\lambda > 0$ , and hence $\tau = \lambda \omega$ . ∎

0.2.17. In § III.1. a similar argument will be used for a characterization of dual weights on crossed products. In fact, our aim was to extend the result of U.Haagerup ([25], Theorem 3.7) from the commutative case to the general case and for this purpose we had to find a definition for the invariance of a weight with respect to an "action" such that the following requirements be satisfied:

- it reduces to the usual notion of invariant weight under group actions, when $A$ corresponds to a group (see § 0.3);
- it entails the commutation of the modular automorphism group of the weight with the action $\delta$;
- it is preserved under passage to balanced weights;
- any dual weight on a crossed product is invariant under the dual action (see § III.1).

In the first version of ([53]) we called $\varphi$ "invariant" if

$$<(x^*\otimes 1)\delta(a)(x\otimes 1),\varphi\otimes k> = <(i_M\otimes j)(\delta(x^*))\ (a\otimes 1)\ (i_M\otimes j)(\delta(x)),\varphi\otimes k>$$

for any $a\in M^+$, $x\in M$, $k\in A_*^+$.

Although this definition works (at the level of the associated Hilbert spaces), we rennounced to it in the present approach because $(i_M\otimes j)$ has not an obvious sense in $M\overline{\otimes} A$.

In what follows we specialize our objects $\{M,A,\delta,j\}$ to more usual ones.

0.2.18. DEFINITIONS. A <u>Hopf - von Neumann algebra</u> ([48]) $(A,\delta_A)$ is a von Neumann algebra $A$ together with an injection $\delta_A : A \longrightarrow A\overline{\otimes} A$ called co-multiplication, which is co-associative:

$$(i_A\otimes\delta_A)\circ\delta_A = (\delta_A\otimes i_A)\circ\delta_A .$$

By an <u>action</u> of $(A,\delta_A)$ on a von Neumann algebra $M$ we mean an injection $\delta : M \longrightarrow M\overline{\otimes} A$ such that

$$(i_M\otimes\delta_A)\circ\delta = (\delta\otimes i_A)\circ\delta \ ;$$

alternatively, $M$ is called an $A$ - comodule.

The <u>centralizer</u> of this action is $M^{\delta} = \{x\in M\ ;\ \delta(x) = x\otimes 1_A\}$.

An automorphism $\sigma$ of $M$ <u>commutes</u> with $\delta$ if it is an $A$-comodule map, i.e. $\delta\circ\sigma = (\sigma\otimes i_A)\delta$.

Two actions $\delta_1, \delta_2$ of $A_1, A_2$ respectively, on the same $M$ are said to commute if $(\delta_1 \otimes i_2)\delta_2 = (i_M \otimes \tilde{\cdot})(\delta_2 \otimes i_1)\delta_1$ .

If $(A, \delta_A)$ is a Hopf - von Neumann algebra, then the predual space $A_*$ of $A$ becomes a Banach algebra ([57]) with multiplication defined by

$$\langle x , h\cdot k \rangle = \langle \delta_A(x) , h \otimes k \rangle , \quad x \in A , \quad h,k \in A_*.$$

If $A$ is co-commutative, i.e. if $\delta_A$ is symmetric $(\tilde{\delta}_A = \delta_A)$ , then $A_*$ is commutative.

If there is an action $\delta : M \longrightarrow M \bar{\otimes} A$ of $A$ on $M$ , then the map $A_* \times M \ni (k,x) \longmapsto k\cdot x \in M$ (see 0.2.1) defines a left $A_*$-module structure on $M$ :

$$h\cdot(k\cdot x) = (h\cdot k)\cdot x \quad , \quad x \in M \ , \quad h,k \in A_* \ .$$

A co-involutive Hopf - von Neumann algebra ([57]) $(A, \delta_A, j_A)$ is a Hopf - von Neumann algebra $(A, \delta_A)$ together with an involutive $*$-anti automorphism $j_A$ of $A$ such that $(j_A \otimes j_A)\delta_A = \tilde{\cdot}\,\delta_A j_A$ .

In this case, the Banach algebra $A_*$ with the involution $k \longmapsto k^o$ (see 0.2.6) becomes an involutive Banach algebra.

0.2.19. DEFINITION. Let $(A, \delta, j)$ be a co-involutive Hopf - von Neumann algebra. A non-zero normal semifinite weight $\omega$ on $A$ is called left Haar weight if it is $(\delta, j)$ - invariant.

By Corollary 0.2.16 , the left Haar weight is faithful and unique up to multiplication with strictly positive constants.

We can also define the right Haar weight as a non-zero normal semifinite weight $\tau$ on $A$ which is $(\tilde{\delta}, j)$ - invariant $(\tilde{\delta} = \tilde{\cdot} \circ \delta)$ .

It is easy to see that $\tau = \omega \circ j$ .

The left Haar weight $\omega$ commutes ([46]) with the right Haar weight $\omega \circ j$ .

This follows from Lemma 0.2.10 and the following more general result :

0.2.20. LEMMA. Let $\{A,\delta,j\}$ be as in 0.2.14 and let $\omega$ be a $\delta$-invariant normal semifinite weight on $A$. Suppose there are given :

$\alpha : A \longrightarrow A$, a positive linear mapping ,

$\beta : A \longrightarrow A$, a normal completely positive linear mapping, $\beta(1_A)=1_A$,

such that

$$(\beta \otimes i_A)\circ\delta = \delta\circ\alpha .$$

Then

$$\omega\circ j\circ\alpha = \omega\circ j .$$

Proof. Fix an arbitrary $k \in A^+$, $\langle 1_A , k\rangle = 1$. Then for any $a \in A^+$ we have

$$\begin{aligned} \langle\alpha(a) , \omega\circ j\rangle &= \langle j(\alpha(a))\otimes 1_A , \omega\otimes k\rangle \\ &= \langle\delta(j(\alpha(a))) , \omega\otimes k\rangle \\ &= \langle\tilde{\ }(j\otimes j)(\beta\otimes i_A)\delta(a) , \omega\otimes k\rangle \\ &= \langle(i_A\otimes(j\beta j))\tilde{\ }(j\otimes j)\delta(a) , \omega\otimes k\rangle \\ &= \langle\delta(j(a)) , \omega\otimes(k\circ(j\beta j))\rangle \\ &= \langle j(a)\otimes 1_A , \omega\otimes(k\circ(j\beta j))\rangle \\ &= \langle a , \omega\circ j\rangle . \blacksquare \end{aligned}$$

Following the terminology introduced in ([16]) we put

0.2.21. DEFINITION. A Katz algebra $(A,\delta,j,\omega)$ is a co-involutive Hopf - von Neumann algebra $(A,\delta,j)$ which has a left Haar weight satisfying the (stronger) commutation relation

$$\sigma_t^{\omega\circ j} = \sigma_t^{\omega} , \qquad t \in \mathbb{R} ,$$

that is

$$j\circ\sigma_{-t}^{\omega}\circ j = \sigma_t^{\omega} , \qquad t \in \mathbb{R} .$$

We remark that the last condition is automatically satisfied in the following two cases :

1) if A is commutative ( then $\sigma_t^\omega = i_A$ , $t \in \mathbb{R}$ ) ;

2) if A is co-commutative ( by Corollary 0.2.16, $\omega \circ j$ is then proportional to $\omega$ ) .

There are many ways to reformulate this definition, more precisely the invariance condition . The aim of all these definitions was to obtain a category $\mathcal{A}$ and a duality equivalence functor on $\mathcal{A}$ such that :

- $\mathcal{A}$ contains all locally compact groups among its objects ;
- the duality functor reduces to the Pontrjagin duality for commutative locally compact groups ;
- the duality functor reduces to the Tannaka duality for compact groups .

As a result of several mathematical efforts ([1],[18],[19],[20],[21],[28],[29],[30],[31],[32],[33],[36],[37],[48],[51],[52],[57],[58],[60],[61],[62],[63],[65],[66]; we apologize for possible omissions ), such a theory is now available in full generality in two paralel works ([34],[35];[16],[17]).

0.2.22. Following the ideas of the above cited works, we now sketch the construction of the dual Katz algebra $(\hat{A},\hat{\delta},\hat{j},\hat{\omega})$ of a given Katz algebra $(A,\delta,j,\omega)$.

The von Neumann algebra $\hat{A}$ is defined by

$$\hat{A} = \mathcal{R}\{\pi_\omega^\delta(A_*)\} \subset B(H_\omega) \quad ,$$

By the $\delta$-invariance of $\omega$ , the map $x \otimes y \longmapsto \delta(x)(1 \otimes y)$ defines a linear isometry $V$ on $H_\omega \otimes H_\omega$ and it can be shown that $W = V^*$ is also an isometry, so that $W$ is unitary. Moreover $W \in \hat{A} \overline{\otimes} A$ and

$$\delta(x) = W^*(x \otimes 1)W \quad ; \quad x \in A .$$

Consider the unitary operator $\hat{W} = \tilde{W}^* = \sim W^* \sim$. Then the formula

$$\hat{\delta}(x) = \hat{W}^*(x \otimes 1)\hat{W} \quad , \quad x \in \hat{A} \quad ,$$

defines a co-associative co-multiplication $\hat{\delta}$ on $\hat{A}$.

On the other hand, the formula

$$\hat{j}(x) = J_\omega x^* J_\omega \quad , \quad x \in \hat{A} \quad ,$$

defines an involutive $*$-antiautomorphism $\hat{j}$ of $\hat{A}$ compatible with $\hat{\delta}$.

Thus $\{\hat{A}, \hat{\delta}, \hat{j}\}$ becomes a co-involutive Hopf - von Neumann algebra.

It can be shown that the correspondence

$$x_\omega \longmapsto (j(x^*))_\omega$$

gives rise to a closed antilinear operator $\hat{S}$ in $H_\omega$.

Also, there is a $*$- subalgebra D of A contained in $\mathcal{T}^2_\omega$ which, regarded as a subset of $A_*$ (see Lemma 0.1.4), is also an involutive subalgebra of the involutive algebra $A_*$ and moreover,

$$\overline{S_\omega|_D} = S_\omega \quad , \quad \overline{\hat{S}|_D} = \hat{S} \quad .$$

With the algebraic operations inherited from $A_*$ and the scalar product inherited from $H_\omega$, D becomes a left Hilbert algebra and the left von Neumann algebra associated to it identifies with $\hat{A}$.

The natural weight associated to this left Hilbert algebra defines a n.s.f. weight $\hat{\omega}$ on $\hat{A}$ which is $(\hat{\delta}, \hat{j})$ - invariant.

One thus obtain the dual Katz algebra $(\hat{A}, \hat{\delta}, \hat{j}, \hat{\omega})$ of $(A, \delta, j, \omega)$.

Note also that, with the algebraic operations inherited from A and the same scalar product, D is again a left Hilbert algebra and its associated left von Neumann algebra identifies with A, $\omega$ being the associated natural weight.

This "Tomita bi-algebra" structure on D is the nerve of the duality theory and it makes almost obvious <u>the duality theorem</u>, namely that the second dual Katz algebra identifies with the original one :

$$(\hat{\hat{A}}, \hat{\hat{\delta}}, \hat{\hat{j}}, \hat{\hat{\omega}}) = (A, \delta, j, \omega)$$

For the details see ([29],[30],[58],[34],[35],[46],[17]).

0.2.23. DEFINITION. Let $\delta : M \longrightarrow M \bar{\otimes} A$ be an action of the Katz algebra $(A, \delta_A, j_A, \omega_A)$ on the von Neumann algebra $M$.

The von Neumann algebra

$$M \times_\delta A$$

generated in $M \bar{\otimes} B(H_A)$ by

$$\delta(M) \quad \text{and} \quad 1_M \otimes \hat{A}$$

will be called the crossed product of $M$ by the action $\delta$.

In this definition $H_A$ is the Hilbert space corresponding to the standard representation of $(A, \omega_A)$ and $\hat{A} \subset B(H_A)$ is the dual Katz algebra.

The category of Katz algebras with appropriate morphisms will be denoted by $\mathcal{A}$.

In the next Section we shall apply these definitions to the objects of $\mathcal{A}$ which correspond to locally compact groups and to their dual objects.

0.2.24. Notes. The definition of $\delta$-invariant weights (0.2.2) is the same as the usual definition of "integrals" on Hopf algebras ([56]) and the $\delta$-invariance is the most natural extension of the invariance under group actions.

In the context considered in this Section, in particular in the context of Katz algebras (or "ring groups", [35]), there are some special situations for which the $\delta$-invariance is sufficient. This happens, for instance, in finite-dimensional cases ([33],[44]) or in the case of group actions (see 0.3.8).

While the construction of the dual Katz algebras given in 0.2.22 is almost a reproduction of the ideas of the original works cited there, our definition of $(\delta, j)$-invariance, as presented in Theorem 0.2.12, has some advantages in comparison with the axioms concerning the invariant weights, as presented in ([35]) and/or ([46]), namely :

- it applies equally to weights on co-modules ( that is, not only to weights on the Hopf algebra itself) ;

- it does not contain redundant requirements ;

- it does not make explicitely use of the tensor product of the identity automorphism with a $*$-antiautomorphism , which has not an obvious sense for von Neumann algebras .

The uniqueness of the "invariant weight" on a "ring group" (compare with Corollary 0.2.16) was asserted without proof by G.I Katz and L.I. Vainerman ([35], page 203).

The assertion "$\omega$ is faithful" in Proposition 0.2.15 strenghtnes and generalizes the result of M. Walter ([66], Remark on page 158). Also, another result of M. Walter ([66], Prop. 3., page 157) can be extended, namely any two $\delta$-invariant n.s.(f). weights which commute are proportional.

## § 3. Actions of groups and actions of group duals.

Let $G$ be a locally compact group with neutral element $e$, left Haar measure $dg$ and modular function $\Delta_G$.

Denote by $C_{oo}(G)$ the set of continuous functions with compact support on $G$.

0.3.1. The set $C_{oo}(G)$ endowed with the scalar product from $L^2(G)$ and with the operations of multiplication and conjugation,

$$(\xi\eta)(s) = \xi(s)\eta(s) ,$$
$$\xi^{\flat}(s) = \overline{\xi(s)} ,$$

becomes a commutative Hilbert algebra.

The associated modular operator and antiunitary involution are

$$\hat{\Delta}_G \xi = \xi ,$$
$$\hat{J}_G \xi = \overline{\xi} .$$

The corresponding (left and right) von Neumann algebra is $L^\infty(G)$ acting on $L^2(G)$ by multiplication,

$$(f\xi)(s) = f(s)\xi(s) .$$

In particular, $L^\infty(G)$ is maximal abelian in $B(L^2(G))$,

$$L^\infty(G)' = L^\infty(G) .$$

The natural weight corresponding to this Hilbert algebra is denoted by $\mu_G$ and called the left Haar weight on $L^\infty(G)$. Clearly,

$$\mu_G(f) = \int f(s)\, ds \quad , \quad f \in L^\infty(G)^+ .$$

0.3.2. The set $C_{oo}(G)$ endowed with the scalar product from $L^2(G)$ and with the operations of convolution and involution,

$$(\xi * \eta)(s) = \int \xi(t)\eta(t^{-1}s)\, dt$$
$$\xi^*(s) = \Delta_G(s)^{-1}\overline{\xi(s^{-1})}$$

becomes a left Hilbert algebra.

The associated modular operator and antiunitary involution are

$$(\Delta_G \xi)(s) = \Delta_G(s)\,\xi(s) \quad ,$$
$$(J_G \xi)(s) = \Delta_G(s)^{-1/2}\,\overline{\xi(s^{-1})} \quad .$$

The corresponding left von Neumann algebra is generated by the left regular representation $\lambda : G \ni g \longmapsto \lambda(g) \in B(L^2(G))$ ,

$$(\lambda(g)\xi)(s) = \xi(g^{-1}s) \quad , \quad \xi \in L^2(G) \quad ,$$

and the corresponding right von Neumann algebra is generated by the right regular representation $\rho : G \ni g \longmapsto \rho(g) \in B(L^2(G))$ ,

$$(\rho(g)\xi)(s) = \Delta_G(g)^{1/2}\,\xi(sg) \quad , \quad \xi \in L^2(G) \quad .$$

In particular , putting

$$L(G) = \mathcal{R}\{\lambda(g)\ ;\ g \in G\} \subset B(L^2(G)) \quad ,$$
$$R(G) = \mathcal{R}\{\rho(g)\ ;\ g \in G\} \subset B(L^2(G)) \quad ,$$

we have the commutation theorem for the regular representations,

$$L(G)' = R(G) \quad .$$

The natural weight corresponding to this left Hilbert algebra is denoted by $\omega_G$ and called the Plancherel weight on $L(G)$ .

For any $f \in L^1(G)$ we have $\lambda(f) \in L(G)$ and

$$\lambda(f_1 * f_2) = \lambda(f_1)\,\lambda(f_2) \quad , \quad \lambda(f^*) = \lambda(f)^* \quad .$$

For any continuous $f \in L^1(G)$ and of positive type , that is $\lambda(f) \in L(G)^+$ , we have

$$\omega_G(\lambda(f)) = f(e) \quad ,$$

and for any $f \in L^1(G)$ we have $\lambda(f^* * f) \in L(G)^+$ and

$$\omega_G(\lambda(f^* * f)) = \int |f(s)|^2\, ds = \text{"}(f^* * f)(e)\text{"} \quad .$$

For any $g \in G$ we have

$$\omega_G \circ \mathrm{Ad}(\lambda(g)) = \Delta_G(g)\,\omega_G \quad .$$

The proof of these facts are direct applications of the definitions (see [26], [51], [52], [66]).

0.3.3. It is clear that

$$B(L^2(G)) = \mathcal{R}\{L^\infty(G), L(G)\} = \mathcal{R}\{L^\infty(G), R(G)\}.$$

The formula

$$(W_G \zeta)(s,t) = \zeta(s,st) \quad , \quad \zeta \in L^2(G\times G) \ , \ s,t \in G ,$$

defines a unitary operator $W_G$ on $L^2(G) \overline{\otimes} L^2(G)$ . Consider also the unitary operator $\hat{W}_G = \tilde{W}_G^* = \sim W_G^* \sim$ on $L^2(G) \overline{\otimes} L^2(G)$ . Then

$$W_G \in L^\infty(G) \overline{\otimes} L(G) \quad , \quad \hat{W}_G \in L(G) \overline{\otimes} L^\infty(G) .$$

0.3.4. The object of the category $\mathcal{K}$ (see 0.2.23) corresponding to the locally compact group $G$ is

$$\underline{G} = \{L^\infty(G) , \iota_G , k_G , \mu_G\} ,$$

where the co-multiplication $\iota_G : L^\infty(G) \longrightarrow L^\infty(G) \overline{\otimes} L^\infty(G)$ is defined by

$$\iota_G(f) = \hat{W}_G^*(f \otimes 1_G) \hat{W}_G \quad , \quad f \in L^\infty(G) ,$$

that is

$$(\iota_G(f))(s,t) = f(ts) ,$$

and the involutive $*$-(anti)automorphism $k_G : L^\infty(G) \longrightarrow L^\infty(G)$ is defined by

$$k_G(f) = J_G \bar{f} J_G \quad , \quad f \in L^\infty(G) ,$$

that is

$$(k_G(f))(s) = \check{f}(s) = f(s^{-1}) .$$

The $(\iota_G, k_G)$-invariance of $\mu_G$ follows by the left invariance of the Haar measure $dg$ .

Note that $\underline{G}$ is a commutative Katz algebra . Any commutative Katz algebra corresponds to a unique locally compact group ([57]) .

0.3.5. The dual object of $\underline{G}$ in $\mathcal{K}$ is

$$\hat{\underline{G}} = \{L(G) , \delta_G , j_G , \omega_G\} ,$$

where the co-multiplication $\delta_G : L(G) \longrightarrow L(G)\overline{\otimes}L(G)$ is defined by

$$\delta_G(x) = W_G^*(x\otimes 1_G)W_G, \quad x \in L(G),$$

in particular

$$\delta_G(\lambda(g)) = \lambda(g)\otimes\lambda(g),$$

and the involutive $*$-antiautomorphism $j_G : L(G) \longrightarrow L(G)$ is defined by

$$j_G(x) = \hat{J}_G x^* \hat{J}_G, \quad x \in L(G),$$

in particular

$$j_G(\lambda(g)) = \lambda(g^{-1}).$$

The $(\delta_G, j_G)$-invariance of $\omega_G$ is easily verified using Theorem 0.2.12.

Note that $\hat{\underline{G}}$ is a co-commutative Katz algebra, in particular we have $\omega_G \circ j_G = \omega_G$. Any co-commutative Katz algebra is the dual object of a unique locally compact group ([L65]).

0.3.6. The predual space $L^1(G)$ of the co-involutive Hopf - von Neumann algebra $(L^\infty(G), \iota_G, k_G)$ has an involutive algebra structure (see 0.2.18), namely

$$h\cdot k = k * h, \quad h,k \in L^1(G),$$
$$k^{\cdot} = k^{\star}, \quad k \in L^1(G).$$

It is easy to see that

$$\pi^{\iota_G}_{\mu_G}(k) = \lambda(\overline{k^{\star}}), \quad k \in L^1(G),$$

thus the involutive algebra structure on $L^1(G) = L^\infty(G)_*$ is inherited from $L(G)$ via the identification of $k \in L^1(G)$ with $\lambda(\overline{k^{\star}}) \in L(G)$.

0.3.7. The predual space of the co-involutive Hopf - von Neumann algebra $(L(G), \delta_G, j_G)$, with the corresponding involutive algebra structure (see 0.2.18) is denoted by $A(G)$ and called <u>the Eymard algebra</u> of $G$.

For $h,k \in A(G)$ and $\xi \in C_{oo}(G)$ we have

$$\begin{aligned}
\langle k\cdot\lambda(\xi)\ ,\ h\rangle &= \langle \delta_G(\lambda(\xi))\ ,\ h\otimes k\rangle \\
&= \langle \int \xi(s)\,\lambda(s)\otimes\lambda(s)\,ds\ ,\ h\otimes k\rangle \\
&= \langle \int \xi(s)\langle\lambda(s)\ ,\ k\rangle\lambda(s)\,ds\ ,\ h\rangle \\
&= \langle \lambda(k(\ )\xi)\ ,\ h\rangle
\end{aligned}$$

where we have denoted

$$k(s) = \langle\lambda(s)\ ,\ k\rangle\ ,\qquad s\in G\ .$$

Then $k(\cdot)$ is a continuous $L^\infty$-function, $\|k(\cdot)\|_\infty \leq \|k\|_{A(G)}$ and we have

$$\begin{aligned}
\pi^{\delta_G}_{\omega_G}(k)\xi &= \pi^{\delta_G}_{\omega_G}(k)\,(\lambda(\xi))_{\omega_G} = (k\cdot\lambda(\xi))_{\omega_G} = (\lambda(k(\cdot)\xi))_{\omega_G} = \\
&= k(\cdot)\xi\ ,
\end{aligned}$$

therefore

$$\pi^{\delta_G}_{\omega_G}(k) = k(\cdot)\in L^\infty(G)\ .$$

We shall identify $k\in A(G)$ with the corresponding function $k(\cdot)\in L^\infty(G)$. Then the involutive algebra structure on $A(G)$ coincides with that inherited from $L^\infty(G)$ :

$$h\cdot k = hk\ ,\qquad h,k\in A(G)\ ,$$

$$k^o = \overline{k}\ ,\qquad k\in A(G)\ .$$

Indeed,

$$\begin{aligned}
(h\cdot k)(s) &= \langle\lambda(s)\ ,\ h\cdot k\rangle = \langle\delta_G(\lambda(s))\ ,\ h\otimes k\rangle = \\
&= \langle\lambda(s)\otimes\lambda(s)\ ,\ h\otimes k\rangle = h(s)\,k(s)\ ,\\
k^o(s) &= \langle\lambda(s)\ ,\ k^o\rangle = \overline{\langle j_G(\lambda(s))\ ,\ k^*\rangle} = \\
&= \overline{\langle(\lambda(s^{-1}))^*,\ k\rangle} = \overline{\langle\lambda(s)\ ,\ k\rangle} = \overline{k(s)}\ .
\end{aligned}$$

For the properties of the Eymard algebra we refer to ([21]) .

0.3.8. Let M be a von Neumann algebre.

Consider a w-continuous representation $\alpha : G \longrightarrow \mathrm{Aut}(M)$ by $*$-automorphisms of M. Then $\alpha$ is automatically $s^*$-continuous.

For any $x \in M$ the function $g \longmapsto \alpha_g^{-1}(x)$ defines (0.1.9) an element $\iota_\alpha(x) \in M \bar{\otimes} L^\infty(G)$. The injection $\iota_\alpha : M \longrightarrow M \bar{\otimes} L^\infty(G)$ is then an action (0.2.18) of $\underline{G}$ on M.

Conversely, consider an action $\iota : M \longrightarrow M \bar{\otimes} L^\infty(G)$ of $\underline{G}$ on M. A standard approximate unit argument ([11],[67]) based on the equality

$$h\cdot(k\cdot x) = (k * h)\cdot x \quad , \quad x \in M , \quad h,k \in L^1(G) ,$$

shows that there is a unique w-continuous $*$-automorphic representation $\alpha : G \longrightarrow \mathrm{Aut}(M)$ such that $\iota = \iota_\alpha$. For details see ([43]).

Consequently, we shall identify the actions of $\underline{G}$ on M with the w-continuous $*$-automorphic representations of G on M.

Given an action $\alpha : G \longrightarrow \mathrm{Aut}(M)$ of G on M we shall also need the injection $\iota_{\alpha^{-1}} : M \longrightarrow M \bar{\otimes} L^\infty(G)$ defined by $(\iota_{\alpha^{-1}}(x))(g) = \alpha_g(x)$, $g \in G$, $x \in M$.

For an action $\alpha : G \longrightarrow \mathrm{Aut}(M)$ of $\underline{G}$ on M and a n.s.f. weight $\varphi$ on M, it is easy to see that

$$\varphi \text{ is } \iota_\alpha\text{-invariant} \Longrightarrow \varphi \circ \alpha_g = \varphi \quad , \quad g \in G$$
$$\Longrightarrow \varphi \text{ is } (\iota_\alpha, k_G)\text{-invariant}$$
$$\Longrightarrow \varphi \text{ is } \iota_\alpha\text{-invariant} \quad ,$$

thus the $(\iota_\alpha, k_G)$-invariance of $\varphi$ is just the usual $\iota_\alpha$-invariance.

Also, given an action $\alpha : G \longrightarrow \mathrm{Aut}(M)$ of $\underline{G}$ on M, the crossed product $M \times_{\iota_\alpha} G$ (0.2.23) is equal to the usual crossed product $M \times_\alpha G$ ([60]), being the von Neumann algebra generated in $M \bar{\otimes} B(L^2(G))$ by $\iota_\alpha(M)$ and $1_M \otimes L(G)$.

The formula

$$\iota_G(x) = \hat{W}_G^* (x \otimes 1) \hat{W}_G \quad , \quad x \in B(L^2(G)) \quad ,$$

defines an action $\iota_G$ of $\underline{G}$ on $B(L^2(G))$ and we have

$$\iota_G = \iota_{Ad\lambda} \quad .$$

Using the commutation theorem $L(G)' = R(G)$ , we get

$$B(L^2(G))^{\iota_G} = R(G) \quad .$$

0.3.9. According to the general definition of an action (0.2.18), an action of $\hat{\underline{G}}$ on a von Neumann algebra $M$ is an injection $\delta : M \longrightarrow M \bar{\otimes} L(G)$ such that $(i_M \otimes \delta_G)\delta = (\delta \otimes i_G)\delta$ and the crossed product (0.2.23) $M \times_\delta \hat{G}$ is the von Neumann algebra generated in $M \bar{\otimes} B(L^2(G))$ by $\delta(M)$ and $1_M \otimes L^\infty(G)$ .

The formulas

$$\delta_G(x) = W_G^* (x \otimes 1) W_G \quad , \quad x \in B(L^2(G)) \quad ,$$

$$\delta_G^*(x) = W_G (x \otimes 1) W_G^* \quad , \quad x \in B(L^2(G)) \quad ,$$

define the actions $\delta_G$ , $\delta_G^*$ of $\hat{\underline{G}}$ on $B(L^2(G))$ .

Using the commutation theorem $L^\infty(G)' = L^\infty(G)$ , it is easy to see that

$$B(L^2(G))^{\delta_G} = L^\infty(G) = B(L^2(G))^{\delta_G^*} \quad .$$

Indeed, if $x \in B(L^2(G))$ and $\delta_G(x) = x \otimes 1$ , then for any vectors $\xi, \eta, \varphi, \psi \in L^2(G)$ we have

$$(W_G(\xi \otimes \varphi) | (x \otimes 1)(\eta \otimes \psi)) = ((x \otimes 1)(\xi \otimes \varphi) | W_G^*(\eta \otimes \psi)) \quad ,$$

which yields

$$(xf\xi | \eta) = (fx\xi | \eta) \quad , \quad \xi, \eta \in L^2(G) \quad ,$$

where

$$f(s) = \int \varphi(st)\overline{\psi(t)}\, dt = \int \varphi(t)\overline{\psi(s^{-1}t)}\, dt \quad , \quad \varphi, \psi \in L^2(G) \quad .$$

0.3.10. If $G$ is commutative and $\hat{G}$ is the Pontrjagin dual group then the Fourier - Plancherel isomorphism identifies the objects

$$(L(G)\,,\,\delta_G\,,\,j_G\,,\omega_G) \text{ and } (L^\infty(\hat{G})\,,\,\iota_{\hat{G}}\,,\,k_{\hat{G}}\,,\,\mu_{\hat{G}})\,,$$

the corresponding actions and the corresponding crossed products.

# I. THE FIRST DUALITY THEOREM FOR CROSSED PRODUCTS.

## § 1. Preliminaries.

Let $M \subset B(H)$ be a von Neumann algebra, $G$ be a locally compact group and $\alpha : G \longrightarrow \mathrm{Aut}(M)$ be an action of $G$ on $M$, which yields the injection $\iota_\alpha : M \longrightarrow M \overline{\otimes} L^\infty(G)$ defined by:

$$(\iota_\alpha(x))(t) = \alpha_{t^{-1}}(x) \quad , \quad t \in G\ , x \in M .$$

Suppose there is a unitary representation $G \ni t \longmapsto u(t) \in B(H)$ which implements the action $\alpha$, i.e. $\alpha_t(x) = u(t)\, x\, u(t)^*$ . $t \in G$, $x \in M$. Denote by $U$ the unitary operator on $H \overline{\otimes} L^2(G)$ defined by :

$$(U\xi)(t) = u(t)\, \xi(t) \quad ; \quad t \in G\ , \ \xi \in L^2(G,H) .$$

Then $U \in B(H) \overline{\otimes} L^\infty(G)$ and putting

$$\theta_\alpha(X) = U^* X U \quad , \quad X \in M \overline{\otimes} L^\infty(G) ,$$

it is easy to verify that $\theta_\alpha$ is a $*$-automorphism of $M \overline{\otimes} L^\infty(G)$ and:

$$\theta_\alpha(x \otimes 1_G) = \iota_\alpha(x)\ , \ x \in M \quad ; \quad \theta_\alpha(1_M \otimes f) = 1_M \otimes f\ , \ f \in L^\infty(G) .$$

As a consequence of this remark we get

I.1.1. PROPOSITION. For any action $\alpha : G \longrightarrow \mathrm{Aut}(M)$ of $G$ on $M$ we have

$$M \overline{\otimes} L^\infty(G) = \mathcal{R}\{\iota_\alpha(M)\ , \ 1_M \otimes L^\infty(G)\} .$$

In particular

$$M \overline{\otimes} B(L^2(G)) = \mathcal{R}\{\iota_\alpha(M)\ , \ 1_M \otimes B(L^2(G))\}.$$

Proof. Since the statement does not depend on the Hilbert space on which $M$ acts, we may assume $M \subset B(H)$ such that the action $\alpha$ is implemented by a unitary representation of $G$ on $H$ as above(see e.g. § I.2.(6) below). Then clearly

$$M \overline{\otimes} L^\infty(G) = \theta_\alpha(M \overline{\otimes} L^\infty(G)) = \mathcal{R}\{\iota_\alpha(M)\ , \ 1_M \otimes L^\infty(G)\} . \blacksquare$$

Also , it is easy to verify that

$$(\alpha_t \otimes \mathrm{Ad}(\rho(t))) \circ \theta_\alpha = \theta_\alpha \circ (i_M \otimes \mathrm{Ad}(\rho(t))) \quad , \quad t \in G ,$$

on $M \overline{\otimes} L^\infty(G)$ and using this fact we get

I.1.2. PROPOSITION. <u>For any action</u> $\alpha : G \longrightarrow \mathrm{Aut}(M)$ <u>of</u> $G$ <u>on</u> $M$ <u>we have</u> :

$$(M \overline{\otimes} L^\infty(G))^{\alpha \otimes \mathrm{Ad}\rho} = \iota_\alpha(M) .$$

<u>Proof</u>. Indeed, for $X \in M \overline{\otimes} L^\infty(G)$ we have :

$$\begin{aligned} X \in (M \overline{\otimes} L^\infty(G))^{\alpha \otimes \mathrm{Ad}\rho} &\Longleftrightarrow (\alpha_t \otimes \mathrm{Ad}(\rho(t)))\, X = X \\ &\Longleftrightarrow (i_M \otimes \mathrm{Ad}(\rho(t)))\, \theta_\alpha^{-1}(X) = \theta_\alpha^{-1}(X) \\ &\Longleftrightarrow \theta_\alpha^{-1}(X) \in M \otimes 1_G \\ &\Longleftrightarrow X \in \theta_\alpha(M \otimes 1_G) = \iota_\alpha(M) , \end{aligned}$$

since clearly $(M \overline{\otimes} L^\infty(G))^{i_M \otimes \mathrm{Ad}\rho} = M \otimes 1_G$. ∎

Let us record two more results which are essentially the same as I.1.2. :

(a) <u>For</u> $X \in M \overline{\otimes} L^\infty(G)$ <u>we have</u> :

$$X \in \iota_\alpha(M) \Longleftrightarrow (\iota_\alpha \otimes i_G)(X) = (i_M \otimes \iota_G)(X) .$$

(b) <u>Let</u> $N$ <u>be a von Neumann subalgebra of</u> $M$ <u>and</u> $x \in M$ . <u>Then</u> :

$$\iota_\alpha(x) \in N \overline{\otimes} L^\infty(G) \Longrightarrow x \in N .$$

We shall not use these results in the sequel , so we omit their (easy) proofs .

§ 2. The duality theorem.

For an action $\alpha : G \longrightarrow \mathrm{Aut}(M)$ of $G$ on $M \subset B(H)$ we consider the crossed product

$$N = M \times_\alpha G$$

which is the von Neumann algebra generated in

$$P = M \mathbin{\overline{\otimes}} B(L^2(G))$$

by $\iota_\alpha(M) \subset M \mathbin{\overline{\otimes}} L^\infty(G)$ and $1_M \otimes L(G)$.

There is an action

$$\delta = i_M \otimes \delta_G : P \longrightarrow P \mathbin{\overline{\otimes}} L(G)$$

of $\hat{G}$ on $P$ defined by

$$\delta(X) = (1_M \otimes W_G^*)(X \otimes 1_G)(1_M \otimes W_G) \quad , \quad X \in P .$$

For $X \in P$ we have $X \in P^\delta$ if and only if

$$\begin{aligned} X \otimes 1_G &\in (M \mathbin{\overline{\otimes}} B(L^2(G)) \otimes 1_G) \cap \{1_M \otimes W_G\}' \\ &= (M \mathbin{\overline{\otimes}} B(L^2(G)) \otimes 1_G) \cap (B(H) \mathbin{\overline{\otimes}} \{W_G\}') \\ &= M \mathbin{\overline{\otimes}} ((B(L^2(G)) \otimes 1_G) \cap \{W_G\}') \\ &= M \mathbin{\overline{\otimes}} (B(L^2(G)))^{\delta_G} \otimes 1_G \\ &= M \mathbin{\overline{\otimes}} L^\infty(G) \otimes 1_G . \end{aligned}$$

Thus

$$P^\delta = M \mathbin{\overline{\otimes}} L^\infty(G) \tag{1}$$

and in particular

$$\iota_\alpha(M) \subset P^\delta \cap N . \tag{2}$$

Therefore $\delta(\iota_\alpha(M)) = \iota_\alpha(M) \otimes 1_G$ and obviously $\delta(1_M \mathbin{\overline{\otimes}} L(G)) \subset (1_M \otimes L(G)) \mathbin{\overline{\otimes}} L(G)$, so that $\delta(N) \subset N \mathbin{\overline{\otimes}} L(G)$ and hence $\delta$ restricts to an action

$$\hat{\alpha} = \delta|_N : N \longrightarrow N \mathbin{\overline{\otimes}} L(G)$$

of $\hat{G}$ on $N = M \times_\alpha G$ which we call <u>the dual action</u> of $\alpha$ .

Consider also the action

$$\beta = \alpha \otimes \mathrm{Ad}\varsigma \text{ of } G \text{ on } P = M \overline{\otimes} B(L^2(G)) .$$

By I.1.2. we have $\iota_\alpha(M) \subset P^\beta$ and clearly $1_M \otimes L(G) \subset P^\beta$ , thus

(3) $$N \subset P^\beta .$$

There is an injection $v : L^\infty(G) \ni f \longmapsto 1_M \otimes f \in P$ of $L^\infty(G)$ in $P$ and we have

(4) $$\beta_t(v(f)) = v(\mathrm{Ad}(\varsigma(t))(f)) \quad , \quad t \in G , f \in L^\infty(G)$$

All the above results are well known ([25],[38]).

The following theorem is our first extension of M.Takesaki's duality theorem ([60], Theorem 4.5 ) to the case of arbitrary locally compact groups.

I.2.1. THEOREM. <u>Let</u> $\alpha : G \longrightarrow \mathrm{Aut}(M)$ <u>be an action of</u> $G$ <u>on</u> $M$ <u>and let</u> $\hat{\alpha}$ <u>be its dual action</u> . <u>Then</u>

$$(M \times_\alpha G) \times_{\hat{\alpha}} \hat{G} \text{ \underline{is} } *\text{-\underline{isomorphic to} } M \overline{\otimes} B(L^2(G)) .$$

<u>Proof</u>. With the above introduced notations we define an injection

$$\pi : P \longrightarrow P \overline{\otimes} B(L^2(G))$$

by

$$\pi(X) = (1_M \otimes W_G^*) \iota_{\beta^{-1}}(X) (1_M \otimes W_G) \quad , \quad X \in P .$$

If $X \in N$ , then $X \in P^\beta$ by (3) , thus $\iota_{\beta^{-1}}(X) = X \otimes 1_G$ and

$$\pi(X) = \hat{\alpha}(X) .$$

If $f \in L^\infty(G)$ , then it is easy (see e.g. § II.3.1. below) to verify that

$$\pi(1_M \otimes f) = 1_P \otimes f .$$

Therefore , using I.1.2 and the definition of $N \times_\delta \hat{G}$ , we get

$$\pi(P) = \pi(\mathcal{R}\{N , 1_M \otimes L^\infty(G)\}) = \mathcal{R}\{\hat{\alpha}(N) , 1_P \otimes L^\infty(G)\} = N \times_{\hat{\alpha}} \hat{G}$$

and this proves the theorem . ∎

Moreover, it is easy to verify that $\mathcal{U}^{-1}$ transforms the second dual action $\hat{\hat{\alpha}}$ $(= i \otimes \mathrm{Ad}\rho)$ of $G$ on $(M \times_\alpha G) \times_{\hat{\alpha}} \hat{G}$, i.e. the dual action of $\hat{\alpha}$ (see § II.2), into the action $\alpha \otimes \mathrm{Ad}\rho$ of $G$ on $M \bar{\otimes} B(L^2(G))$.

We remark that Theorem I.2.1. is in fact a particular case of Theorem II.3.4 below. However, the "generation part", which is the hard part in II.3.4, is rather trivial here.

With the same notations as before, we now continue the discussion interrupted by Theorem I.2.1., in order to obtain simple proofs for two known results which we need in the sequel.

Since

$$\delta_G(\mathrm{Ad}(\rho(t)(x)) = (\mathrm{Ad}(\rho(t)) \otimes i_G)(\delta_G(x)) \quad , \quad t \in G ,$$

for $x \in L^\infty(G)$ and for $x \in L(G)$, hence for all $x \in B(L^2(G))$, it follows that $\delta$ commutes with $\beta$, i.e. $\delta \circ \beta_t = (\beta_t \otimes i_G) \circ \delta$ on $P$, $t \in G$ (see also § II.3.1 below). We infer that

$$X \in P^\delta \Longrightarrow \beta_t(X) \in P^\delta,\ t \in G \quad ; \quad X \in P^\beta \Longrightarrow \delta(X) \in P^\beta \otimes L(G) .$$

Thus, by restriction we obtain

an action $\beta : G \longrightarrow \mathrm{Aut}(P^\delta)$ of $G$ on $P^\delta$,

an action $\delta : P^\beta \longrightarrow P^\beta \bar{\otimes} L(G)$ of $\hat{G}$ on $P^\beta$.

Using (1), (2), (3) and I.1.2. we get

$$\iota_\alpha(M) \subset N^\delta \subset (P^\beta)^\delta = (P^\delta)^\beta = (M \bar{\otimes} L^\infty(G))^{\alpha \otimes \mathrm{Ad}\rho} = \iota_\alpha(M)$$

and therefore

$$(5) \qquad \iota_\alpha(M) = N^\delta = (P^\beta)^\delta .$$

The first equality is a result due to M.Landstad which we state separately for further reference :

I.2.2. PROPOSITION (M.Landstad, [38]). <u>For any action</u> $\alpha : G \longrightarrow \mathrm{Aut}(M)$ <u>of</u> $G$ <u>on</u> $M$ <u>we have</u>

$$\iota_\alpha(M) = (M \times_\alpha G)^{\hat{\alpha}}$$

<u>where</u> $\hat{\alpha}$ <u>is the dual action of</u> $\alpha$.

There is a natural unitary representation $u : G \longrightarrow N$, namely

$$u(t) = 1_M \otimes \lambda(t) \ , \ t \in G \ ,$$

and it is easy to see that

(6) $$\iota_\alpha \circ \alpha_t \circ \iota_\alpha^{-1} \Big|_{\iota_\alpha(M)} = \mathrm{Ad}(u(t)) \Big|_{\iota_\alpha(M)} \ , \quad t \in G \ ,$$

which means that the action $\alpha$ of $G$ on $M$ is implemented via $u$ in the crossed product $M \times_\alpha G$. On the other hand it is clear that :

(7) $$\delta(u(t)) = u(t) \otimes \lambda(t) \quad , \quad t \in G \ .$$

This anticommutation relation is the main idea of the outstanding theorem of M.Landstad ([38], Theorem 1 ; stated below as Theorems I.3.3. and I.3.4.) , which we now use in order to obtain a simple proof of the following result due to T.Digernes :

I.2.3. THEOREM (T.Digernes ,[9]). For any action $\alpha : G \longrightarrow \mathrm{Aut}(M)$ of $G$ on $M$ we have

$$M \times_\alpha G = (M \overline{\otimes} B(L^2(G)))^{\alpha \otimes \mathrm{Ad}\rho} \ .$$

Proof. With the same notations as before, we have seen that there is an action $\delta : P^\beta \longrightarrow P^\beta \overline{\otimes} L(G)$ of $\hat{G}$ on $P^\beta$ and a unitary representation $u : G \longrightarrow N \subset P^\beta$ (by (3)) , which are related by (7). By M.Landstad's theorem ([38], Theorem 1 ; see I.3.7 below) it follows that $P^\beta$ is generated by $(P^\beta)^\delta = \iota_\alpha(M)$ (see (5)) and $u(G) = \{1_M \otimes \lambda(g) \ ; \ g \in G\}$ , hence $P^\beta = N$ . ∎

It is clear that this result is equivalent to the general commutation theorem for crossed products due to M.Takesaki ([60]) and to T.Digernes ([9]).

## § 3. The conditional expectation $P_G$ and the characterization of crossed products by actions of G.

Let N be a von Neumann algebra. The Plancherel weight $\omega_G$ on L(G) yields a n.s.f. operator valued weight $E_N^{\omega_G}$ on $(N \bar{\otimes} L(G))^+$ with values in $\overline{N^+}$, namely (0.1.6) :

$$\langle E_N^{\omega_G}(X), \varphi \rangle = \langle X, \varphi \otimes \omega_G \rangle \quad , \quad X \in (N \bar{\otimes} L(G))^+ \ , \ \varphi \in N_*^+ \ .$$

I.3.1. PROPOSITION. (see also [26], [38]). Consider an action $\delta : N \longrightarrow N \bar{\otimes} L(G)$ of $\hat{G}$ on N and assume there is a unitary representation $u : G \longrightarrow N$ such that

$$(1) \qquad \delta(u(t)) = u(t) \otimes \lambda(t) \quad , \quad t \in G \ .$$

Then the formula

$$(2) \qquad P_G^N(x) = P_G(x) = E_N^{\omega_G}(\delta(x)) \quad , \quad x \in N^+ \ ,$$

defines a n.s.f. conditional expectation $P_G$ of N on $N^\delta$ and

$$(3) \qquad P_G(u(f)) = f(e)\, 1_N$$

for any continuous function with compact support f on G such that $u(f) \geq 0$.

Proof. It is clear that $P_G$ is normal, faithful, additive and positive-homogeneous. Also, for $x \in N^+$ and $a \in N^\delta$ we have

$$\begin{aligned} P_G(a^* x a) &= E_N^{\omega_G}(\delta(a^* x a)) \\ &= E_N^{\omega_G}((a^* \otimes 1)\, \delta(x)\, (a \otimes 1)) \\ &= a^* E_N^{\omega_G}(\delta(x))\, a \\ &= a^* P_G(x)\, a \ . \end{aligned}$$

For any continuous function f with compact support and of positive type and for any $\varphi \in N_*^+$, we have

$$\langle P_G(u(f)), \psi \rangle = \langle \delta(u(f)), \psi \otimes \omega_G \rangle$$

$$= \sup_{k \le \omega_G} \langle \int f(t)\,(u(t) \otimes \lambda(t))\,dt, \psi \otimes k \rangle$$

$$= \sup_{k \le \omega_G} \langle \int f(t)\,\psi(u(t))\,\lambda(t)\,dt, k \rangle$$

$$= \langle \int f(t)\,\psi(u(t))\,\lambda(t)\,dt, \omega_G \rangle$$

$$= f(e)\,\psi(u(e))$$

$$= \langle f(e)\,1_N, \psi \rangle .$$

This proves (3) and the semifiniteness of $P_G$.

We shall now prove, in a number of steps, that $P_G(x) \in \overline{(N^\delta)}^+$ for all $x \in N^+$.

(i) Let $\delta_N, \delta_M$ be actions of $\hat{G}$ on $N$, $M$ respectively, and assume there is an injection $\pi : N \longrightarrow M$ such that

$$\delta_M \circ \pi = (\pi \otimes i_G) \circ \delta_N .$$

Then, for any $x \in N^+$ and any $\varphi \in M_*^+$ we have

$$\langle P_G^M(\pi(x)), \varphi \rangle = \langle \delta_M(\pi(x)), \varphi \otimes \omega_G \rangle$$

$$= \langle (\pi \otimes i_G)\delta_N(x), \varphi \otimes \omega_G \rangle$$

$$= \langle \delta_N(x), (\varphi \circ \pi) \otimes \omega_G \rangle$$

$$= \langle P_G^N(x), \varphi \circ \pi \rangle$$

$$= \langle \pi(P_G^N(x)), \varphi \rangle ,$$

so that

(4) $$P_G^M(\pi(x)) = \pi(P_G^N(x)) , \quad x \in N^+ .$$

On the other hand, for any $x \in N$ we have

$$x \in N^{\delta_N} \iff x \in N \text{ and } \delta_N(x) = x \otimes 1_G$$

$$\iff \pi(x) \in \pi(N) \text{ and } (\pi \otimes i_G)\delta_N(x) = \pi(x) \otimes 1_G$$

$$\iff \pi(x) \in \pi(N) \text{ and } \delta_M(\pi(x)) = \pi(x) \otimes 1_G$$

$$\Longleftrightarrow \pi(x) \in \pi(N) \cap M^{\delta_M}$$

therefore

$$\pi(N^{\delta_N}) = \pi(N) \cap M^{\delta_M}$$

and, using the uniqueness of the spectral resolution of extended positive elements ([24], 1.5.), it follows that

$$(5) \qquad \pi(\overline{(N^{\delta_N})^+}) = \pi(\overline{N^+}) \cap \overline{(M^{\delta_M})^+}$$

Suppose now that $P_G^M$ has the desired property, i.e. $P_G^M$ takes its values in $\overline{(M^{\delta_M})^+}$. Then the same is true for $P_G^N$. Indeed, using (4) and (5), for any $x \in N^+$ we then get :

$$\pi(P_G^N(x)) = P_G^M(\pi(x)) \in \pi(\overline{N^+}) \cap \overline{(M^{\delta_M})^+} = \pi(\overline{(N^{\delta_N})^+})$$

and hence

$$P_G^N(x) \in \overline{(N^{\delta_N})^+} .$$

(ii) Let $\delta : N \longrightarrow N \overline{\otimes} L(G)$ be an action of $\hat{G}$ on $N$ and assume that $N$ acts on the Hilbert space $K$, $N \subset B(K)$. By the definition of an action of $\hat{G}$, we then have the following commutative diagram :

$$\begin{array}{ccc} N & \xrightarrow{\quad\delta\quad} & N \overline{\otimes} L(G) \\ \delta \downarrow & & \downarrow \delta \otimes i_G \\ B(K) \overline{\otimes} B(L^2(G)) & \xrightarrow[\quad i_K \otimes \delta_G \quad]{} & B(K) \overline{\otimes} B(L^2(G)) \overline{\otimes} L(G) \end{array}$$

Thus, the discussion in (i) shows that it is sufficient to prove that $P_G$ takes its values in $\overline{(N^{\delta})^+}$ only for

$$N = B(K) \overline{\otimes} B(L^2(G)) \quad \text{and} \quad \delta = i_K \otimes \delta_G .$$

(iii) Any $s^*$-measurable function (see 0.1.10)

$$G \times G \ni (s,r) \longmapsto X(s,r) \in B(K)$$

such that

$$\iint \| X(s,r) \|^2 \, dr ds < +\infty$$

defines an operator $X \in B(K) \overline{\otimes} B(L^2(G)$ by

$$(X\xi|\eta) = \iint (X(s,r)\xi(r)|\eta(s))\, drds \quad , \quad \xi,\eta \in L^2(G,K).$$

We shall compute $(i_K \otimes \delta_G)(X) \in B(K)\overline{\otimes} B(L^2(G))\overline{\otimes} L(G)$. For any $\zeta \in L^2(G \times G, K)$ we have :

$$((i_K \otimes \delta_G)(X)\zeta \mid \zeta) =$$

$$= ((1_K \otimes W_G^*)(X \otimes 1_G)(1_K \otimes W_G)\zeta \mid \zeta)$$

$$= ((X \otimes 1_G)(1_K \otimes W_G)\zeta \mid (1_K \otimes W_G)\zeta)$$

$$= \iint (X(s,r)((1_K \otimes W_G)\zeta)(r,\cdot) \mid ((1_K \otimes W_G)\zeta)(s,\cdot))\, drds$$

$$(6) \qquad = \iiint (X(s,r)((1_K \otimes W_G)\zeta)(r,t) \mid ((1_K \otimes W_G)\zeta)(s,t))\, drdsdt$$

$$= \iiint (X(s,r)\zeta(r,rt) \mid \zeta(s,st))\, drdsdt$$

$$= \iiint (X(s,r)\zeta(r,rs^{-1}t) \mid \zeta(s,t))\, drdsdt$$

$$= \iiint (X(s,r^{-1}s)\zeta(r^{-1}s,r^{-1}t) \mid \zeta(s,t))\, \Delta_G(r^{-1}s)\, drdsdt$$

Now, for any $k \in L(G)_*^+ \subset A(G)$ there is a vector $\theta \in L^2(G)$ such that $k = \omega_\theta$ , that is

$$(7) \qquad k(r) = \langle \lambda(r), k \rangle = (\lambda(r)\theta \mid \theta) = \int \theta(r^{-1}t)\overline{\theta(t)}\, dt \quad .$$

If $\eta \in L^2(G,K)$ , then $\zeta = \eta \otimes \theta \in L^2(G \times G,K)$ , $\zeta(s,t) = \theta(t)\eta(s)$, $s,t \in G$, and $\omega_\zeta = \omega_\eta \otimes k$. By (6) and (7) we get

$$\langle (i_K \otimes \delta_G)(X), \omega_\eta \otimes k \rangle =$$

$$= \iint (X(s,r^{-1}s)\eta(r^{-1}s) \mid \eta(s))\, k(r)\, \Delta_G(r^{-1}s)\, drds$$

$$(8) \qquad = \int k(r)\, \left(\int \Delta_G(r^{-1}s)\, (X(s,r^{-1}s)\eta(r^{-1}s) \mid \eta(s))\, ds\right) dr$$

$$= \langle \lambda(f), k \rangle \quad ,$$

where $f \in L^1(G)$ is given by

(9) $$f(r) = \int \Delta_G(r^{-1}s)\,(X(s,r^{-1}s)\eta(r^{-1}s)\,|\,\eta(s))\,ds \quad , \quad r \in G.$$

If $X \in (B(K)\,\overline{\otimes}\,B(L^2(G)))^+$ , then (8) shows that $\lambda(f) \geq 0$ and, using again (8), we get

$$\begin{aligned} < P_G(X)\,,\,\omega_\eta > &= <(i_K \otimes \delta_G)(X)\,,\,\omega_\eta \otimes \omega_G > \\ &= \sup_{k \leq \omega_G} <(i_K \otimes \delta_G)(X)\,,\,\omega_\eta \otimes k > \\ &= \sup_{k \leq \omega_G} <\lambda(f)\,,\,k > \\ &= <\lambda(f)\,,\,\omega_G > . \end{aligned} \qquad (10)$$

We take now a particular $X$ , namely, for an arbitrary fixed $\xi \in L^2(G,K)$, $\|\xi\|_2 = 1$ , we define

(11) $$X(s,r) = (\cdot\,|\,\xi(r))\,\xi(s) \in B(K) \quad , \quad s,r \in G .$$

It is clear that the resulting $X \in B(K)\,\overline{\otimes}\,B(L^2(G)) = B(L^2(G,K))$ is the one-dimensional projection onto the space spanned by $\xi$.

In this case we have

$$f(r) = \int \Delta_G(r^{-1}s)\,(\xi(s)\,|\,\eta(s))\,\overline{(\xi(r^{-1}s)\,|\,\eta(r^{-1}s))}\,ds = (h * h^{\star})(r),$$

thus $f = h * h^{\star}$ , where $h \in L^1(G)$ , $h(s) = (\xi(s)\,|\,\eta(s))$ , $s \in G$ , and $^{\star}$ is the involution in $L^1(G)$ , $h^{\star}(s) = \Delta_G(s^{-1})\,\overline{h(s^{-1})}$ , $s \in G$ .

But in this case it is known (0.3.2) that $<\lambda(f)\,,\,\omega_G>$ is equal to the "formal value" of $f$ in $e \in G$ . Thus , from (10) we infer that, for $X$ given by (11),

(12) $$< P_G(X)\,,\,\omega_\eta > = f(e) = \int \Delta_G(s)\,(X(s,s)\eta(s)\,|\,\eta(s))\,ds \quad .$$

On the other hand, any $\delta^*$-measurable positive function

$$G \ni s \longmapsto F(s) \in B(K)^+$$

defines an extended positive element $F \in (\overline{B(K)\,\overline{\otimes}\,L^\infty(G)})^+$ by

(13) $$< F\,,\,\omega_\eta > = \int (F(s)\eta(s)\,|\,\eta(s))\,ds \quad , \quad \eta \in L^2(G,K) .$$

Thus , for any $X$ given by (11) ,

(14) $\quad P_G(X) = F \quad \text{where} \quad F(s) = \Delta_G(s)\, X(s,s) \quad , \quad s \in G \quad ,$

so that

(15) $\quad P_G(X) \in \overline{(B(K)\overline{\otimes} L^\infty(G))}^{+} = \overline{((B(K) \quad B(L^2(G)))^{i_K \otimes \delta_G}}{}^{+}$

for any positive finite rank operator $X \in B(L^2(G,K))$.

Since any positive operator in $B(L^2(G,K))$ can be obtained as the supremum of an increasing net of positive finite rank operators, the normality of $P_G$ entails that (15) holds for any $X \in (B(L^2(G,K))^+$

This completes the proof of the proposition. ■

I.3.2. The only trouble in writting (12) for other X's is to establish that, for the function $f$ given by (9), the equality

(16) $$\langle \lambda(f) , \omega_G \rangle = f(e)$$

still holds. This is the case, for instance, if $f$ is a continuous $L^1$-function and $\lambda(f) \geqslant 0$ (0.3.2).

In particular, if $M \subset B(K)$ is a von Neumann algebra and

$$G \times G \ni (s,r) \longmapsto X(s,r) \in M$$

is a $s^*$-continuous function with compact support in both variables, then the corresponding operator $X$ belongs to $M \overline{\otimes} B(L^2(G)$ and the corresponding function $f$ is continuous with compact support. Therefore, if $X \geqslant 0$ we have $\lambda(f) \geqslant 0$ by (8), the equality (16) holds and we get

(17) $$(P_G(X))(s) = \Delta_G(s)\, X(s,s) \quad , \quad s \in G \, .$$

We shall use this remark in § III.3.3.

The last part of the proof of I.3.1 is equivalent to the proof of the following equality

(18) $$(\varphi \otimes \omega_G) \circ (i_M \otimes \delta_G) = (\varphi \circ a_M) \otimes \omega_G \quad , \quad \varphi \in (M \overline{\otimes} L(G))^+_*$$

where $a_M$ stands for the ampliation of $M$ into $M \overline{\otimes} L(G)$ .

In particular this yields the following "weak" invariance property of $\omega_G$

$$(19) \qquad (k \otimes \omega_G) \circ \delta_G = k(e)\,\omega_G \quad , \quad k \in L(G)_*^+ \subset A(G)$$

which can be easily verified using ([46], 5.9).

The conditional expectation $P_G$ is the main technical tool used in the proof of the following theorem of M.Landstad :

I.3.3. THEOREM (M.Landstad,[38]). Let N be a von Neumann algebra and assume there is an action $\delta : N \longrightarrow N \bar{\otimes} L(G)$ of $\hat{G}$ on N and a unitary representation $u : G \longrightarrow N$ such that

$$\delta(u(t)) = u(t) \otimes \lambda(t) \quad , \quad t \in G .$$

Then N is generated by $N^\delta$ and $u(G)$ :

$$N = \mathcal{R}\{N^\delta , u(G)\} .$$

We shall not reprove here this Theorem. We just mention that the main idea of the proof ([38]) is a rigorous version of the following formula

$$x = \int P_G(a\,u(t))\,u(t^{-1})\,dt \quad , \quad a \in \mathfrak{M}_{P_G}^+ .$$

The next theorem, also due to M.Landstad, characterizes the von Neumann algebras arising as crossed products by actions of G .

I.3.4. THEOREM (M.Landstad,[38]). A von Neumann algebra N is $*$-isomorphic with the crossed product of another von Neumann algebra by an action of G , if and only if there is an action $\delta : N \to N \bar{\otimes} L(G)$ of $\hat{G}$ on N and a unitary representation $u : G \longrightarrow N$ such that

$$(20) \qquad \delta(u(t)) = u(t) \otimes \lambda(t) \quad , \quad t \in G .$$

In this case , N is $*$-isomorphic with $M \times_\alpha G$ , where $M = N^\delta$ and $\alpha$ is implemented via u .

Proof.([38]). If $N = M \times_\alpha G$ , then the desired conclusions follow from § I.2.(5), I.2.(6), I.2.(7).

Conversely , the action Ad u of G on N restricts to an action $\alpha$ of G on $M = N^{\delta}$ (by (20)) and the injection

$$\pi : N \longrightarrow N \overline{\otimes} B(L^2(G))$$

defined by

$$\pi(x) = U^* \delta(x) U \quad , \quad x \in N ,$$

where $U \in N \overline{\otimes} L^{\infty}(G)$ is the unitary operator defined by the function $t \longmapsto u(t)$ , has the properties

$$\pi(x) = \iota_{\alpha}(x) \quad , \quad x \in M \quad ; \quad \pi(u(t)) = 1_M \overline{\otimes} \lambda(t) \quad , \quad t \in G ,$$

therefore, taking into account Theorem I.3.3. and the definition of $M \times_{\alpha} G$ , it follows that

$$\pi(N) = M \times_{\alpha} G$$

and this proves the theorem. ∎

We have presented the proof of this Theorem for the paralell with the other parts of the present paper, especially because it was our source in similar situations (see I.2.1., II.2.1., II.3.4.).

I.3.5. Since there is an action $\delta_G$ of $\hat{G}$ on $B(L^2(G))$ such that $B(L^2(G))^{\delta_G} = L^{\infty}(G)$ and a unitary representation $\lambda : G \longrightarrow B(L^2(G))$ such that (20) holds, it follows that $B(L^2(G))$ is $*$-isomorphic to the crossed product $L^{\infty}(G) \times_{Ad\lambda} G$ and there is a n.s.f. conditional expectation $P_G$ of $B(L^2(G))$ on $L^{\infty}(G)$.

# II. THE SECOND DUALITY THEOREM FOR CROSSED PRODUCTS.

## § 1. Preliminaries.

Let $M \subset B(H)$ be a von Neumann algebra, $G$ be a locally compact group and $\delta : M \longrightarrow M \overline{\otimes} L(G)$ be an action of $\hat{G}$ on $M$.

II.1.1. PROPOSITION. For any action $\delta : M \longrightarrow M \overline{\otimes} L(G)$ of $\hat{G}$ on $M$ and any $X \in M \overline{\otimes} L(G)$ we have

$$X \in \delta(M) \iff (i_M \otimes \delta_G)(X) = (\delta \otimes i_G)(X).$$

Proof. (M. Landstad, [39]). Since $(i_M \otimes \delta_G)\delta = (\delta \otimes i_G)\delta$ the part "$\Longrightarrow$" is trivial. In order to prove the part "$\Longleftarrow$", consider the actions $i_H \otimes \delta_G$ and $i_H \otimes \delta_G^*$ of $\hat{G}$ on $B(H) \overline{\otimes} B(L^2(G))$ and recall that, for $X \in B(H) \overline{\otimes} B(L^2(G))$,

$$(i_H \otimes \delta_G)(X) = (1_H \otimes W_G^*)(X \otimes 1_G)(1_H \otimes W_G),$$
$$(i_H \otimes \delta_G)(X) = (1_H \otimes W_G)(X \otimes 1_G)(1_H \otimes W_G^*).$$

If $X \in \delta(M)' \subset B(H) \overline{\otimes} B(L^2(G))$, then $(i\ \otimes \delta_G^*)(X) \in \delta(M)' \overline{\otimes} L(G)$ since, for any $x \in M$,

$$\begin{aligned}
((i_H \otimes \delta_G^*)(X))\ (\delta(x) \otimes 1_G) &= \\
&= (1_M \otimes W_G)(X \otimes 1_G)(1_M \otimes W_G^*)(\delta(x) \otimes 1_G)(1_M \otimes W_G)(1_M \otimes W_G^*) \\
&= (1_M \otimes W_G)(X \otimes 1_G)((i_M \otimes \delta_G)\delta(x))(1_M \otimes W_G^*) \\
&= (1_M \otimes W_G)(X \otimes 1_G)((\delta \otimes i_G)\delta(x))(1_M \otimes W_G^*) \\
&= (1_M \otimes W_G)((\delta \otimes i_G)\delta(x))(X \otimes 1_G)(1_M \otimes W_G^*) \\
&= (1_M \otimes W_G)((i_M \otimes \delta_G)\delta(x))(X \otimes 1_G)(1_M \otimes W_G^*) \\
&= (1_M \otimes W_G)(1_M \otimes W_G^*)(\delta(x) \otimes 1_G)(1_M \otimes W_G)(X \otimes 1_G)(1_M \otimes W_G^*) \\
&= (\delta(x) \otimes 1_G)\ ((i_H \otimes \delta_G^*)(X)).
\end{aligned}$$

Therefore $i_H \otimes \delta_G$ restricts to an action $\delta' : \delta(M)' \longrightarrow \delta(M)' \overline{\otimes} L(G)$ of $\hat{G}$ on $\delta(M)'$. On the other hand, $u'(t) = 1_M \otimes \rho(t) \in \delta(M)'$ and $\delta'(u'(t)) = u'(t) \otimes \lambda(t)$, $t \in G$. By M. Landstad's theorem (see I.3.5) it follows that $\delta(M)' = \mathcal{R}\{(\delta(M)')^{\delta'}, u'(G)\}$.

Now consider $X \in M \overline{\otimes} L(G)$, $(i_M \otimes \delta_G)(X) = (\delta \otimes i_G)(X)$. It is clear that $X$ commutes with $u'(G)$. If $Y \in (\delta(M)')^{\delta'}$ then $Y \in \delta(M)'$ and

$$(1_M \otimes W_G)(Y \otimes 1_G)(1_M \otimes W_G^*) = Y \otimes 1_G ,$$

thus

$$\begin{aligned} XY \otimes 1_G &= (X \otimes 1_G)(Y \otimes 1_G) \\ &= (1_M \otimes W_G)(1_M \otimes W_G^*)(X \otimes 1_G)(1_M \otimes W_G)(Y \otimes 1_G)(1_M \otimes W_G^*) \\ &= (1_M \otimes W_G)((i_M \otimes \delta_G)(X))(Y \otimes 1_G)(1_M \otimes W_G^*) \\ &= (1_M \otimes W_G)((\delta \otimes i_G)(X))(Y \otimes 1_G)(1_M \otimes W_G^*) \\ &= (1_M \otimes W_G)(Y \otimes 1_G)((\delta \otimes i_G)(X))(1_M \otimes W_G^*) \\ &= (1_M \otimes W_G)(Y \otimes 1_G)((i_M \otimes \delta_G)(X))(1_M \otimes W_G^*) \\ &= (1_M \otimes W_G)(Y \otimes 1_G)(1_M \otimes W_G^*)(X \otimes 1_G)(1_M \otimes W_G)(1_M \otimes W_G^*) \\ &= (Y \otimes 1_G)(X \otimes 1_G) \\ &= YX \otimes 1_G , \end{aligned}$$

so that $XY = YX$. Since $\delta(M)' = \mathcal{R}\{(\delta(M)')^{\delta'}, u'(G)\}$, it follows that $X \in (\delta(M)')' = \delta(M)$. ∎

In a first version of this paper (cf.[53]) we have obtained the result of Proposition II.1.1. only for amenable $G$, using Lemma II.1.4. below and the result of H.Leptin ([42]). We are very grateful to M.Landstad for communicating us in his letter ([39]) the above presented proof for the general case.

II.1.2. COROLLARY. <u>Let</u> $N$ <u>be a von Neumann subalgebra of</u> $M$ <u>such that</u> $\delta(N) \subset N \overline{\otimes} L(G)$ <u>and</u> $x \in M$. <u>Then</u>

$$x \in N \iff \delta(x) \in N \overline{\otimes} L(G) .$$

<u>Proof</u>. By assumption, $\delta$ restricts to an action $\delta_N$ of $\hat{G}$ on $N$. If $\delta(x) \in N \overline{\otimes} L(G)$, then $(i \otimes \delta_G)\delta(x) = (\delta \otimes i)\delta(x) = (\delta_N \otimes i)\delta(x)$ and Proposition II.1.1. yields $\delta(x) \in \delta_N(N) = \delta(N)$, thus $x \in N$. ∎

In the following proposition we use the notations introduced in (0.2.18., 0.3.7.) for the action $\delta$ of the Hopf-von Neumann algebra $\hat{G} = L(G)$ on $M$ and the predual space $A(G)$ of $L(G)$.

II.1.3. PROPOSITION. <u>For any action</u> $\delta : M \longrightarrow M \bar{\otimes} L(G)$ <u>of</u> $\hat{G}$ <u>on</u> $M$ <u>and any</u> $x \in M$ <u>we have</u>

$$x \in \mathcal{R}\{k \cdot x \; ; \; k \in A(G)\} .$$

<u>Proof</u>. Denote $N = \mathcal{R}\{A(G) \cdot x\}$. By II.1.2. it is sufficient to show that $\delta(N) \subset N \bar{\otimes} L(G)$ and that $\delta(x) \in N \bar{\otimes} L(G)$.

For any $\varphi \in M_*$ which annihilates $N$ and any $h, k \in A(G)$ we have

$$\langle \delta(x) \, , \varphi \otimes h \rangle = \langle h \cdot x \, , \varphi \rangle = 0 \, ,$$
$$\langle \delta(k \cdot x) \, , \varphi \otimes h \rangle = \langle (hk) \cdot x \, , \varphi \rangle = 0 \, ,$$

thus $\delta(x) \in N \bar{\otimes} L(G)$ and $\delta(k \cdot x) \in N \bar{\otimes} L(G)$, by the result of J.Tomiyama (0.1.7).

In the case $M = L(G)$, $\delta = \delta_G$, the <u>conjecture</u> (H) of P.Eymard ([21],4.15) says that

(H) $\qquad x \in \overline{A(G) \cdot x}^{w}$ <u>for any</u> $x \in L(G)$

which is stronger than II.1.3. While the conjecture (H) is true for amenable groups (i.e. A(G) has an approximate unit ,[42]), we do not know if it is true or not for arbitrary groups.*) We mention that Corollary 4.5 of Y.Nakagami ([43]) asserts that conjecture (H) is true in the general case but, unfortunately, we have not understood its proof.

II.1.4. LEMMA. <u>Let</u> $\delta : M \longrightarrow M \bar{\otimes} L(G)$ <u>be an action of</u> $\hat{G}$ <u>on</u> $M$. <u>For any</u> $x \in M$ <u>and any</u> $f, k \in A(G)$ , <u>both functions with compact support, we have</u> :

$$(1) \qquad \int (1_M \otimes \lambda(g)) \delta({}_g f \cdot k \cdot x) \Delta_G(g)^{-1} \, dg = (k \cdot x) \otimes \lambda(f \Delta_G^{-1})$$

<u>where</u> ${}_g f(t) = f(gt)$ <u>and the integral is understood in the</u> w-<u>topology</u>.

<u>Proof</u>. The function $g \longmapsto (1 \otimes \lambda(g)) \delta({}_g f \cdot k \cdot x) \Delta_G(g)^{-1}$ is w-continuous and has compact support since ${}_g f \cdot k = 0$ outside of the set $(\mathrm{supp}\, f)(\mathrm{supp}\, k)^{-1}$. Therefore the integral appearing in (1) is logi-

*) see the discussion in [41].

time and represents an element of $M \overline{\otimes} L(G)$.

Then, for any $\varphi \in M_*$ and any $h \in A(G)$ we have :

$$< \int (1 \otimes \lambda(g)) \delta({}_g f \cdot k \cdot x) \Delta_G(g)^{-1} \, dg \; , \varphi \otimes h > =$$

$$= \int < (1 \otimes \lambda(g)) \delta({}_g f \cdot k \cdot x) \; , \varphi \otimes h > \Delta_G(g)^{-1} \, dg =$$

$$= \int < \delta({}_g f \cdot k \cdot x) \; , \varphi \otimes {}_g h > \Delta_G(g)^{-1} \, dg =$$

$$= \int < {}_g h \cdot {}_g f \cdot k \cdot x \; , \varphi > \Delta_G(g)^{-1} \, dg =$$

$$= \int < \delta(x) \; , \varphi \otimes {}_g(hf) \cdot k > \Delta_G(g)^{-1} \, dg =$$

$$= < \delta(x) \; , \varphi \otimes \int {}_g(hf) \cdot k \, \Delta_G(g)^{-1} \, dg = ./.$$

The new integral that appeared is legitime in $A(G)$ with its $\sigma(A(G), L(G))$-topology and represents an element, say, $b \in A(G)$. We compute it as follows : for $t \in G$ we have

$$b(t) = < \lambda(t) \; , b > \; = < \lambda(t) \; , \int {}_g(hf) \cdot k \, \Delta_G(g)^{-1} \, dg >$$

$$= \int < \lambda(t) \; , {}_g(hf) \cdot k > \Delta_G(g)^{-1} \, dg$$

$$= \int h(gt) \, f(gt) \, k(t) \, \Delta_G(g)^{-1} \, dg$$

$$= \left( \int h(g) \, f(g) \, \Delta_G(g)^{-1} \, dg \right) k(t)$$

$$= < \lambda(f \Delta_G^{-1}) \; , h > k(t) \; ,$$

therefore $b = < \lambda(f \Delta_G^{-1}) \; , h > k$ and we may continue our first chain of equalities with

$$./. = < \delta(x) \; , \varphi \otimes k > < \lambda(f \Delta_G^{-1}) \; , h > =$$

$$= < k \cdot x \; , \varphi > < \lambda(f \Delta_G^{-1}) \; , h > =$$

$$= < (k \cdot x) \otimes \lambda(f \Delta_G^{-1}) \; , \varphi \otimes h >$$

and this proves the lemma. ∎

II.1.5. COROLLARY. *Let* $\delta : M \longrightarrow M \overline{\otimes} L(G)$ *be an action of* $\hat{G}$ *on* $M$. *For any* $x \in M$ *and any* $k \in A(G)$ *we have*

(2) $(k\cdot x) \otimes 1_G \in \text{c.l.m.}\{(1_M \otimes \lambda(g))\delta(h\cdot k\cdot x)\ ;\ g \in G\ ,\ h \in A(G)\}$.

*Proof*. Indeed, Lemma II.1.4. shows that

(3) $(k\cdot x)\otimes\lambda(f\Delta_G^{-1}) \in \text{c.l.m.}\{(1_M\otimes\lambda(g))\delta(h\cdot k\cdot x)\ ;\ g \in G\ ,\ h \in A(G)\}$

for any $x \in M$ and any $k,f \in A(G)$, both functions with compact support. By the regularity properties of $A(G)$ ([21], 3.2), we see that there is an approximate unit $\{f_\iota \Delta_G^{-1}\}_{\iota\in I}$ for $L^1(G)$ with $f_\iota \in A(G)$ functions with compact support. Then the net $\{\lambda(f_\iota \Delta_G^{-1})\}_{\iota\in I}$ is w-convergent to $1_G$, thus using (3), we get (2) for $k \in A(G)$ with compact support. Since the functions $k \in A(G)$ with compact support are (norm) dense in $A(G)$ ([21], 3.26), we get (2) for all $k \in A(G)$. ∎

II.1.6. PROPOSITION. *Let* $\delta : M \longrightarrow M \overline{\otimes} L(G)$ *be an action of* $\hat{G}$ *on* $M$. *For any* $x \in M$ *we have*

(4) $$x \otimes 1_G \in \mathcal{R}\{\delta(A(G)\cdot x)\ ,\ 1_M \otimes L(G)\}.$$

*In particular*

(5) $$M \overline{\otimes} L(G) = \mathcal{R}\{\delta(M)\ ,\ 1_M \overline{\otimes} L(G)\},$$

(6) $$M \overline{\otimes} B(L^2(G)) = \mathcal{R}\{\delta(M)\ ,\ 1_M \overline{\otimes} B(L^2(G))\}.$$

*Proof*. The relation (4) follows from II.1.5. and II.1.3., and then (4) $\Rightarrow$ (5) $\Rightarrow$ (6). ∎

The next result improves Corollary II.1.2.

II.1.7. PROPOSITION. *Let* $\delta : M \longrightarrow M \overline{\otimes} L(G)$ *be an action of* $\hat{G}$ *on* $M$, *let* $N$ *be any von Neumann subalgebra of* $M$ *and* $x \in M$. *Then*

$$\delta(x) \in N \overline{\otimes} L(G) \implies x \in N.$$

*Proof*. If $\delta(x) \in N \overline{\otimes} L(G)$, then for any $h,k \in A(G)$ and for any $\varphi \in M_*$ which annihilates $N$ we have

$$<\delta(k\cdot x), \varphi \otimes h> = <\delta(x), \varphi \otimes hk> = 0 ,$$

thus $\delta(k\cdot x) \in N \overline{\otimes} L(G)$ by the result of J.Tomiyama (0.1.7.). Then, using II.1.5. we get $(k\cdot x) \otimes 1_G \in N \overline{\otimes} L(G)$ , that is $k\cdot x \in N$ , and using II.1.3. we conclude $x \in N$ . ∎

## § 2. The duality theorem.

For an action $\delta : M \longrightarrow M \overline{\otimes} L(G)$ of $\hat{G}$ on $M$ we consider the crossed product

$$N = M \times_{\delta} \hat{G}$$

which is the von Neumann algebra generated in

$$P = M \overline{\otimes} B(L^2(G))$$

by $\delta(M) \subset M \overline{\otimes} L(G)$ and $1_M \overline{\otimes} L^{\infty}(G)$.

There is an action

$$\alpha = i_M \otimes \mathrm{Ad} \quad : G \longrightarrow \mathrm{Aut}(P)$$

of $G$ on $P$ and we clearly have

(1) $$P^{\alpha} = M \overline{\otimes} L(G) \quad ,$$

in particular

(2) $$\delta(M) \subset P^{\alpha} \cap N \quad .$$

Therefore $\alpha_t(\delta(M)) = \delta(M)$ and , obviously , $\alpha_t(1_M \otimes L^{\infty}(G)) = 1_M \otimes (G)$ , so that $\alpha_t(N) = N$ , $t \in G$ , and hence $\alpha$ restricts to an action

$$\hat{\delta} = \alpha|_N \; : G \longrightarrow \mathrm{Aut}(N)$$

of $G$ on $N = M \times_{\delta} \hat{G}$ which we call <u>the dual action</u> of $\delta$ .

We now define a mapping $\nabla : P \longrightarrow P \overline{\otimes} L(G)$ by

$$\nabla(X) = (1_M \otimes W_G)(1_M \otimes \sim)((\delta \otimes i_G)(X))(1_M \otimes \sim)(1_M \otimes W_G^*) \quad , \quad X \in P .$$

It may be useful to read the definition of $\nabla$ on the diagram below :

$$\begin{array}{ccc}
M \overline{\otimes} B(L^2(G)) & \xrightarrow{\;\delta \otimes i_G\;} & M \overline{\otimes} L(G) \overline{\otimes} B(L^2(G)) \\
 & & \downarrow \; i_M \otimes (\sim \cdot \sim) \\
 & \searrow^{\nabla} & M \overline{\otimes} B(L^2(G)) \overline{\otimes} L(G) \\
 & & \downarrow \; i_M \otimes (W_G \cdot W_G^*) \\
 & & M \overline{\otimes} B(L^2(G)) \overline{\otimes} L(G)
\end{array}$$

There is also a unitary representation of $G$ in $P$ :

$$u : G \ni t \longmapsto u(t) = 1_M \otimes \rho(t) \in P \quad .$$

II.2.0. LEMMA. $\nabla : P \longrightarrow P \overline{\otimes} L(G)$ <u>is an action of</u> $\hat{G}$ <u>on</u> $P$ <u>and we have</u> :

(3) $$N \subset P^{\nabla} \quad ,$$

(4) $$\nabla(u(t)) = u(t) \otimes \lambda(t) \quad , \quad t \in G \; .$$

<u>Proof</u>. It is clear that $\nabla$ is an injection.

For $x \in M$ , $\delta(x) \in \delta(M) \subset P$ and we have

$$\begin{aligned} \nabla(\delta(x)) &= (1_M \otimes W_G)(1_M \otimes \sim)((\delta \otimes i_G)\delta(x))(1_M \otimes \sim)(1_M \otimes W_G^*) \\ &= (1_M \otimes W_G)(1_M \otimes \sim)((i_M \otimes \delta_G)\delta(x))(1_M \otimes \sim)(1_M \otimes W_G^*) \\ &= (1_M \otimes W_G \sim W_G^*)(\delta(x) \otimes 1_G)(1_M \otimes W_G \sim W_G^*) \\ &= \delta(x) \otimes 1_G \end{aligned}$$

since $\delta(x) \in M \overline{\otimes} L(G)$ and for any $y \in M$ , $t \in G$ , we have

$$\begin{aligned} &(1_M \overline{\otimes} W_G \sim W_G^*)(y \otimes \lambda(t) \otimes 1_G)(1_M \otimes W_G \sim W_G^*) \\ &\quad = y \otimes (W_G \sim W_G^*(\lambda(t) \otimes 1_G)W_G \sim W_G^*) \\ &\quad = y \otimes (W_G \sim (\lambda(t) \otimes \lambda(t)) \sim W_G^*) \\ &\quad = y \otimes (W_G(\lambda(t) \otimes \lambda(t))W_G^*) \\ &\quad = y \otimes \lambda(t) \otimes 1_G \; . \end{aligned}$$

For $f \in L^\infty(G)$ , $1_M \otimes f \in 1_M \otimes L^\infty(G) \subset P$ and we have

$$\begin{aligned} \nabla(1_M \otimes f) &= (1_M \otimes W_G)(1_M \otimes \sim)((\delta \otimes i_G)(1_M \otimes f))(1_M \otimes \sim)(1_M \otimes W_G^*) \\ &= (1_M \otimes W_G)(1_M \otimes \sim)(1_M \otimes 1_G \otimes f)(1_M \otimes \sim)(1_M \otimes W_G^*) \\ &= (1_M \otimes W_G)(1_M \otimes f \otimes 1_G)(1_M \otimes W_G^*) \\ &= 1_M \otimes f \otimes 1_G \quad , \end{aligned}$$

since $W_G^*(f \otimes 1_G)W_G = \delta_G(f) = f \otimes 1_G$ .

The above computations prove (3) . We prove (4) :

$$\begin{aligned} \nabla(1_M \otimes \rho(t)) &= (1_M \otimes W_G)(1_M \otimes \sim)((\delta \otimes i_G)(1_M \otimes \rho(t)))(1_M \otimes \sim)(1_M \otimes W_G^*) \\ &= (1_M \otimes W_G)(1_M \otimes \sim)(1_M \otimes 1_G \otimes \rho(t))(1_M \otimes \sim)(1_M \otimes W_G^*) \\ &= (1_M \otimes W_G)(1_M \otimes \rho(t) \otimes 1_G)(1_M \otimes W_G^*) \end{aligned}$$

$$= 1_M \otimes \rho(t) \otimes \lambda(t)$$

since $W_G(\rho(t) \otimes 1_G)W_G^* = \delta_G^*(\rho(t)) = \rho(t) \otimes \lambda(t)$.

It is now an easy matter to verify the condition

$$(i_P \otimes \delta_G) \circ \nabla = (\nabla \otimes i_G) \circ \nabla$$

on $\delta(M)$, $1_M \otimes L^\infty(G)$ and $1_M \otimes R(G)$. Using II.1.6. we see that this condition is satisfied on the whole $P$, thus $\nabla$ is indeed an action of $\hat{G}$ on $P$. ∎

Due to its structure, the action $\nabla$ of $\hat{G}$ on $P = M \overline{\otimes} B(L^2(G))$ could be denoted as $\delta * \delta_G$ and called the convolution of the actions $\delta$ and $\delta_G$ of $\hat{G}$ on $M$ and $B(L^2(G))$, respectively. It is the analogue of the tensor product of two actions of $G$.

The following theorem is our second extension of M.Takesaki's duality theorem ([60], 4.5) to the case of arbitrary locally compact groups.

II.2.1. THEOREM. <u>Let</u> $\delta : M \longrightarrow M \overline{\otimes} L(G)$ <u>be an action of</u> $\hat{G}$ <u>on</u> $M$ <u>and</u> $\hat{\delta}$ <u>be its dual action. Then</u>

$(M \times_\delta \hat{G}) \times_{\hat{\delta}} G$ <u>is $*$-isomorphic to</u> $M \overline{\otimes} B(L^2(G))$.

<u>Proof</u>. We use the above introduced notations. As in the proof of I.3.4., we consider the unitary operator $U \in P \overline{\otimes} L^\infty(G)$ defined by the function $t \longmapsto u(t)$, we define an injection

$$\pi : P \longrightarrow P \overline{\otimes} B(L^2(G))$$

by

$$\pi(X) = U^* \nabla(X) U, \quad X \in P,$$

and we verify that $\pi(X) = \iota_\alpha(X)$ for $X \in N \subset P^\nabla$ (by(3)) and that $\pi(u(t)) = 1_P \otimes \lambda(t)$ for $t \in G$.

Thus, using also II.1.6, we get

$$\pi(P) = \pi(\mathcal{R}\{N, 1_M \otimes R(G)\}) = \mathcal{R}\{\iota_\alpha(N), 1_N \otimes L(G)\} = N \times_\alpha G$$

and this proves the theorem. ∎

Moreover, it is easy to verify that $\mathcal{M}^{-1}$ transforms the <u>second dual action</u> $\hat{\hat{\delta}}$ $(= i \otimes \delta_G)$ of $\hat{G}$ on $(M \times_\delta \hat{G}) \times_{\hat{\delta}} G$, i.e. the dual action of $\hat{\delta}$ (see § I.2.), into the action $\nabla$ of $\hat{G}$ on $M \overline{\otimes} B(L^2(G))$.

We remark that we could first prove that $P^\nabla = N$ (see II.2.3 below) and then Theorem II.2.1. would follow at once from I.3.4 and II.2.0. On this way it is also possible to avoid the use of II.1.6., since there is another (rather tedious) proof that $\nabla$ is an action of $\hat{G}$ on $P$.

We now pursue the paralell with § I.2, by discussing the relations between $\alpha$ and $\nabla$. It is no more true that they commute, since

$$\nabla(\alpha_g(u(t))) = 1_M \otimes \rho(gtg^{-1}) \otimes \lambda(gtg^{-1}),$$

$$(\alpha_g \otimes i_G)(\nabla(u(t)) = 1_M \otimes \rho(gtg^{-1}) \otimes \lambda(t) \quad .$$

However, using (4) and the fact that $\alpha_g$ is implemented by $u(g)$, it is easy to see that

$$X \in P^\nabla \Longrightarrow \alpha_g(X) \in P^\nabla \quad , \quad g \in G.$$

Thus, by restriction we obtain

an action $\alpha : G \longrightarrow \mathrm{Aut}(P^\nabla)$ of $G$ on $P^\nabla$.

Using (1), (2), (3) we get

$$\delta(M) \subset N^\alpha \subset (P^\nabla)^\sim = P^\nabla \cap (M \overline{\otimes} L(G)).$$

Consider $X \in M \overline{\otimes} L(G)$ with $\nabla(X) = X \otimes 1_G$. Then

$$(1_M \otimes W_G)(1_M \otimes \sim)((\delta \otimes i_G)(X))(1_M \otimes \sim)(1_M \otimes W_G^*) = X \otimes 1_G$$

or

$$\begin{aligned}(\delta \otimes i_G)(X) &= (1_M \otimes \sim)(1_M \otimes W_G^*)(X \otimes 1_G)(1_M \otimes W_G)(1_M \otimes \sim) \\ &= (1_M \otimes \sim)((i_M \otimes \delta_G)(X))(1_M \otimes \sim) \\ &= (i_M \otimes \delta_G)(X)\end{aligned}$$

since $\delta_G$ is symmetric. By II.1.1. it follows that $X \in \delta(M)$. Therefore

(5) $$\delta(M) = N^\alpha = (P^\nabla)^\alpha \quad .$$

We state separately the first equality :

II.2.2. PROPOSITION. <u>For any action</u> $\delta : M \longrightarrow M \bar{\otimes} L(G)$ <u>of</u> $\hat{G}$ <u>on</u> $M$ <u>we have</u>

$$\delta(M) = (M \times_\delta \hat{G})^{\hat{\delta}}$$

<u>where</u> $\hat{\delta}$ <u>is the dual action of</u> $\delta$.

In order to still pursue the paralell with § I.2, it is now necessary to introduce the analogue of the unitary implementation of a group of $*$-automorphisms and to prove a dual version of Landstad's theorem (I.3.3). However, it seems preferable to discuss these topics separately in § II.3 and to continue here the last part of the analogy with § I.2, assuming some familiarity with § II.3.

With the notations introduced in the present Section, there is a natural injection $v : L^\infty(G) \longrightarrow N$, namely $v(f) = 1_M \otimes f$, $f \in L^\infty(G)$. It is easy to see that $(v \otimes i_G)(W_G) = 1_M \otimes W_G$, thus $v$ implements the action $(i_M \otimes \delta_G)|_N : N \longrightarrow N \bar{\otimes} L(G)$ of $\hat{G}$ on $N$. Since $(i_M \otimes \delta_G)\delta = (\delta \otimes i_G)\delta$, we have

(6) $\quad (\delta \otimes i_G)\delta(x) = (v \otimes i_G)(W_G)^* (\delta(x) \otimes 1_G)(v \otimes i_G)(W_G) \quad , \quad x \in M .$

Therefore we may say that the action $\delta$ of $G$ on $M$ is implemented <u>via</u> $v$ in the crossed product $M \times_\delta \hat{G}$. On the other hand it is clear that

(7) $\quad \alpha_t(v(f)) = v(\mathrm{Ad}(\rho(t))f) \quad , \quad t \in G \, , \, f \in L^\infty(G) .$

II.2.3. THEOREM. <u>For any action</u> $\delta : M \longrightarrow M \bar{\otimes} L(G)$ <u>of</u> $\hat{G}$ <u>on</u> $M$ <u>we have</u>

$$M \times_\delta \hat{G} = (M \bar{\otimes} B(L^2(G)))^{\delta * \delta_G} .$$

<u>Proof</u>. With the same notations as before, we have seen that there is an action $\alpha : G \longrightarrow \mathrm{Aut}(P^\nabla)$ of $G$ on $P^\nabla$ and an injection $v : L^\infty(G) \longrightarrow N \subset P^\nabla$ (by (3)) which are related by (7). By Theorem II.3.3. below, it follows that $P^\nabla$ is generated by $(P^\nabla)^\alpha = \delta(M)$ (see (5)) and $v(L^\infty(G)) = 1_M \otimes L^\infty(G)$, hence $P^\nabla = N$. ∎

As Theorem I.2.3. corresponds to the commutation theorem for crossed products by $G$ , the above Theorem II.2.3. should correspond to a commutation theorem for crossed products by $\hat{G}$ . This is indeed the case and the corresponding commutation theorem was found independently by M.Landstad ([40], Theorem 5).

## § 3. The conditional expectation $Q_G$ and the characterization of crossed products by actions of $\hat{G}$.

Consider a von Neumann algebra $N$ and an injection

$$v : L^{\infty}(G) \longrightarrow N$$

of $L^{\infty}(G)$ into $N$. Then

$$v \otimes i_G : L^{\infty}(G) \bar{\otimes} L(G) \longrightarrow N \bar{\otimes} L(G)$$

is also an injection and, since $W_G \in L^{\infty}(G) \otimes L(G)$, we may consider the unitary operator

$$W_v = (v \otimes i_G)(W_G) \in N \otimes L(G)$$

and we may define an injection

$$\delta_v : N \longrightarrow N \bar{\otimes} L(G)$$

by

$$\delta_v(x) = W_v^*(x \otimes 1_G) W_v \quad , \quad x \in N .$$

The property $(i_1 \otimes \delta_G)\delta_G = (\delta_G \otimes i_2)\delta_G$ of $\delta_G$ means that

$$(W_G \otimes 1_3)(1_1 \otimes \sim)(W_G \otimes 1_3)(1_1 \otimes \sim)(1_1 \otimes W_G^*)(W_G^* \otimes 1_3) \in 1_1 \otimes B(L^2(G \times G)).$$

Translating this relation by $v \otimes i_2 \otimes i_3$, we get

$$(W_v \otimes 1_3)(1_N \otimes \sim)(W_v \otimes 1_3)(1_N \otimes \sim)(1_N \otimes W_G^*)(W_v^* \otimes 1_3) \in 1_N \otimes B(L^2(G \times G)),$$

which in turn means that

$$(i_N \otimes \delta_G)\delta_v = (\delta_v \otimes i_G)\delta_v .$$

Thus, any injection $v : L^{\infty}(G) \longrightarrow N$ defines an action $\delta_v$ of $\hat{G}$ on $N$. We then say that $\delta_v$ is implemented via $v$.

For instance, $\delta_G$ is implemented via the injection

$$i_G : L^{\infty}(G) \ni f \longrightarrow f \in B(L^2(G)) ,$$

$\delta_G^*$ is implemented via the injection

$$k_G : L^{\infty}(G) \ni f \longmapsto \check{f} \in B(L^2(G))$$

and any action $\delta$ of $\hat{G}$ on $M$ is implemented in the crossed product $M \times_{\delta} \hat{G}$ (see § II.2.(6)).

II.3.1. LEMMA. Consider an action $\alpha : G \longrightarrow \mathrm{Aut}(N)$ of $G$ on $N$ and an injection $v : L^\infty(G) \longrightarrow N$ such that

(1) $$\alpha_t(v(f)) = v(\mathrm{Ad}(\rho(t))f) \quad , \quad t \in G \; , \; f \in L^\infty(G) \; .$$

Then

(i) $\delta_v \circ \alpha_t = (\alpha_t \otimes i_G) \circ \delta_v$ , $t \in G$ , i.e. $\delta_v$ commutes with $\alpha$ ;

(ii) $W_v^* \iota_{\alpha^{-1}}(v(f)) W_v = 1_N \otimes f$ , $f \in L^\infty(G)$.

Proof. (i) It is easy to verify that

$$\delta_G(\mathrm{Ad}(\rho(t))(x)) = (\mathrm{Ad}(\rho(t)) \otimes i_G)(\delta_G(x)) \quad , \quad t \in G \; ,$$

for $x \in L^\infty(G)$ and for $x \in L(G)$ , therefore the same is true for all $x \in B(L^2(G))$. This means that $W_G(\rho(t) \otimes 1_G) W_G^*(\rho(t)^* \otimes 1_G)$ commutes with $x \otimes 1_G$ for any $x \in B(L^2(G))$ , that is

$$W_G \cdot (\mathrm{Ad}(\rho(t)) \otimes i_G)(W_G^*) \in 1_G \otimes L(G) \; .$$

Using the assumption and applying here $v \otimes i_G$ , we get

$$W_v \cdot (\alpha_t \otimes i_G)(W_v^*) \in 1_N \otimes L(G) \; .$$

This in turn entails that $W_v \cdot (\alpha_t \otimes i_G)(W_v^*)$ commutes with $\alpha_t(x) \otimes 1_G$ for any $x \in N$ , which yields

$$\delta_v(\alpha_t(x)) = (\alpha_t \otimes i_G)(\delta_v(x)) \quad , \quad x \in N \; .$$

(ii) For $f \in L^\infty(G)$ , denote by $F \in L^\infty(G \times G) \subset L^\infty(G) \otimes L^\infty(G)$ the function

$$F(s,t) = f(st) \quad , \quad s,t \in G \; .$$

With $\xi \in L^2(G \times G)$ we have

$$(W_G^* F W_G \xi)(s,t) = f(t)\xi(s,t) = ((1_G \otimes f)\xi)(s,t) \; ,$$

so that

$$W_G^* \, F \, W_G = 1_G \otimes f \; .$$

Applying here $v \otimes i_G$ we get

$$W_v^* \, ((v \otimes i_G)(F)) \, W_v = 1_N \otimes f \; .$$

Since $F$ is defined by the function

$$G \ni t \longmapsto \mathrm{Ad}(\rho(t))f \in L^\infty(G)$$

and $\iota_{\alpha^{-1}}(v(f))$ is defined by the function

$$G \ni t \longmapsto \alpha_t(v(f)) = v(Ad(\rho(t))f) \in N ,$$

we see that

$$\iota_{\alpha^{-1}}(v(f)) = (v \otimes i_G)(F)$$

which entails the desired conclusion. ∎

II.3.2. PROPOSITION. Consider an action $\alpha : G \longrightarrow Aut(N)$ of $G$ on $N$ and assume there is an injection $v : L^\infty(G) \longrightarrow N$ such that

(1) $$\alpha_t(v(f)) = v(Ad(\rho(t))f) , \quad t \in G , f \in L^\infty(G) .$$

Then the formula

(2) $$Q_G(x) = \int \alpha_t(x) \, dt , \quad x \in N^+ ,$$

defines a n.s.f. conditional expectation $Q_G$ of $N$ on $N^\alpha$ and

(3) $$Q_G(v(f)) = (\int f(t) \, dt) 1_N , \quad f \in L^\infty(G)^+ .$$

Proof.([7],[26],[38]). It is clear that $Q_G$ is normal, faithful, additive, positive-homogeneous and

$$Q_G(a^* x a) = a^* Q_G(x) a , \quad x \in N^+ , \quad a \in N^\alpha .$$

For any $f \in L^\infty(G)^+$ and for any $k \in L^\infty(G)^+_* \subset L^1(G)$, we have

$$\int \langle Ad(\rho(t))f , k \rangle dt = \iint f(st) k(s) \, ds dt = (\int f(t) \, dt) \langle 1_G , k \rangle$$

thus, for any $\psi \in N^+_*$ we get

$$\int \langle \alpha_t(v(f)) , \psi \rangle dt = \int \langle v(Ad(\rho(t)f) , \psi \rangle dt$$

$$= \int \langle Ad(\rho(t)f , \psi \circ v \rangle dt$$

$$= (\int f(t) \, dt) \langle 1_N , \psi \rangle .$$

This proves (3) and the semifinitness of $Q_G$.

Due to the left invariance of Haar measure, it is clear that $\alpha_g(Q_G(x)) = Q_G(x)$ , $x \in N^+$, $g \in G$. Using the uniqueness of the spectral resolution of extended positive elements ([24], 1.5 ), we infer that all spectral projections of $Q_G(x)$ are $\alpha_g$-invariant , so that

$Q_G(x) \in \overline{(N^{\alpha})^+}$ . ∎

We can derive the conditional expectation $Q_G$ by the same general method which was use in § I.3 for $P_G$ . Indeed , the Haar weight $\mu_G$ on $L^{\infty}(G)$ yields an operator valued weight $E_N^{\mu_G}$ on $(N \bar{\otimes} L^{\infty}(G))^+$ with values in $\overline{N^+}$ , namely (0.1.6) :

$$< E_N^{\mu_G}(X) , \varphi > = < X , \varphi \otimes \mu_G > , \quad X \in (N \bar{\otimes} L^{\infty}(G))^+ , \varphi \in N_*^+ ,$$

and we reobtain $Q_G$ by

$$Q_G(x) = E_N^{\mu_G}(\iota_{\alpha^{-1}}(x)) , \quad x \in N^+ .$$

II.3.3. THEOREM. Let $N$ be a von Neumann algebra and assume there is an action $\alpha : G \longrightarrow \mathrm{Aut}(N)$ of $G$ on $N$ and an injection $v : L^{\infty}(G) \longrightarrow N$ such that

$$\alpha_t(v(f)) = v(\mathrm{Ad}(\rho(t))f) \quad ; \quad t \in G , f \in L^{\infty}(G) .$$

Then $N$ is generated by $N^{\alpha}$ and $v(L^{\infty}(G))$ :

$$N = \mathcal{R}\{ N^{\alpha} , v(L^{\infty}(G)) \} .$$

Proof. Consider a fundamental system $\{V_{\iota}\}_{\iota \in I}$ of compact neighborhoods of $e \in G$ and choose the continuous functions $f_{\iota}$ on $G$ such that

$$f_{\iota} \geq 0 \quad , \quad \mathrm{supp}\, f_{\iota} \subset V_{\iota} \quad , \int f_{\iota}(t)\, dt = 1 \quad ; \iota \in I .$$

For any $x \in \mathfrak{M}_{Q_G}$ and for any continuous function with compact support $h$ on $G$ , we shall show that :

$$(4) \qquad x\, v(h) = \lim_{\iota \in I} \int Q_G(x\, v(h \cdot \mathrm{Ad}(\lambda(g))f_{\iota}))\, v(\mathrm{Ad}(\lambda(g))\check{f}_{\iota})\, dg$$

in the sense of w-topology.

The integral appearing in (4) is legitime since the function we integrate is w-continuous and has compact support , namely $h\, \mathrm{Ad}(\lambda(g))f_{\iota} = 0$ for $g \notin (\mathrm{supp}\, h) \cdot (\mathrm{supp}\, f_{\iota})^{-1}$.

Thus, the right hand side in (4) is an element of $\mathcal{R}\{ N^{\alpha} , v(L^{\infty}(G)) \}$

and (4) shows that $x\cdot v(h) \in \mathcal{R}\{N^{\alpha}, v(L^{\infty}(G))\}$. Since the continuous functions with compact support are w-dense in $L^{\infty}(G)$ and since $\mathfrak{M}_{Q_G}$ is w-dense in $N$, the theorem follows from (4).

In order to prove (4) put $a = x\cdot v(h)$ and remark that :

$$\int \mathrm{Ad}(\lambda(g))\check{f}_{\iota}\, dg = 1_G \text{ in } L^{\infty}(G) \text{ and } Q_G(v(\mathrm{Ad}(\lambda(g))f_{\iota}) = 1_N \text{ in } N .$$

Thus

$$a = \int a\, Q_G(v(\mathrm{Ad}(\lambda(g))f_{\iota}))\, v(\mathrm{Ad}(\lambda(g))\check{f}_{\iota})\, dg$$

and, for any $\psi \in N_*$, we have

$$< \int Q_G(a\cdot v(\mathrm{Ad}(\lambda(g))f_{\iota}))\, v(\mathrm{Ad}(\lambda(g))\check{f}_{\iota})\, dg \; - \; a \;, \psi > =$$

$$= < \int (Q_G(a\, v(\mathrm{Ad}(\lambda(g))f_{\iota})) - a\, Q_G(v(\mathrm{Ad}(\lambda(g))f_{\iota})))\, v(\mathrm{Ad}(\lambda(g))f_{\iota}) \;, \psi > dg =$$

$$= \int < (\int (\alpha_s(a) - a)\, \alpha_s(v(\mathrm{Ad}(\lambda(g))f_{\iota}))\, ds)\, v(\mathrm{Ad}(\lambda(g))\check{f}_{\iota}) \;, \psi > dg =$$

$$= \iint < (\alpha_s(a) - a)\, v(\mathrm{Ad}(\rho(s))\mathrm{Ad}(\rho(g))f_{\iota}\cdot \mathrm{Ad}(\lambda(g))\check{f}_{\iota}) \;, \psi > ds dg =$$

$$= \int_{V_{\iota}^2} < (\alpha_s(a) - a)\, v(\int \mathrm{Ad}(\lambda(g))\, (\mathrm{Ad}(\rho(s))f_{\iota}\cdot \check{f}_{\iota})\, dg) \;, \psi > ds$$

since $\mathrm{Ad}(\rho(s))f_{\iota}\cdot \check{f}_{\iota} = 0$ for $s \notin V_{\iota}^2$.

For each $k \in L^{\infty}(G)_*^+ \subset L^1(G)$ we have

$$< \int \mathrm{Ad}(\lambda(g))\, (\mathrm{Ad}(\rho(s)f_{\iota}\cdot \check{f}_{\iota})\, dg \;, \; k > =$$

$$= \int < \mathrm{Ad}(\lambda(g))\, (\mathrm{Ad}(\rho(s)f_{\iota}\cdot \check{f}_{\iota}) \;, \; k > dg \; =$$

$$= \iint f_{\iota}(g^{-1}ts)\, f_{\iota}(t^{-1}g)\, k(t)\, dt dg \; =$$

$$= \iint f_{\iota}(g^{-1}s)\, f_{\iota}(g)\, k(t)\, dg dt \qquad = (f_{\iota} * f_{\iota})(s)\, \|k\|_1$$

so that

$$\| v(\int \mathrm{Ad}(\lambda(g))\, (\mathrm{Ad}(\rho(s))f_{\iota}\cdot \check{f}_{\iota})\, dg) \| \leq (f_{\iota} * f_{\iota})(s) \, .$$

On the other hand, given $\varepsilon > 0$, for "large" $\iota \in I$ we have

$$s \in V_{\iota}^2 \implies < (\alpha_s(a) - a)(\alpha_s(a) - a)^* \;, \psi > \leq \varepsilon^2 / \|\psi\| \, .$$

Using the Schwartz inequality :

$$|<XY, \Psi>| \leq \Psi(XX^*)^{1/2}\Psi(Y^*Y)^{1/2} \leq \Psi(XX^*)^{1/2} \|\Psi\|^{1/2} \|Y\| \quad ,$$

we get

$$|<\int Q_G(a\cdot v(\mathrm{Ad}(\lambda(g))f_L))\, v(\mathrm{Ad}(\lambda(g)\check{f}_L)\, dg \; - \; a \;, \psi>| \leq$$

$$\leq \varepsilon \int (f_L * f_L)(s)\, ds \;\leq \varepsilon \|f_L\|_1^2 \;= \varepsilon \;.$$

This proves (4) and the theorem. ∎

The following theorem characterizes the von Neumann algebras arising as crossed products by actions of $\hat{G}$ .

II.3.4. THEOREM. <u>A von Neumann algebra</u> $N$ <u>is</u> $*$<u>-isomorphic with the crossed product of another von Neumann algebra by an action of</u> $\hat{G}$ <u>if and only if there is an action</u> $\alpha : G \longrightarrow \mathrm{Aut}(N)$ <u>of</u> $G$ <u>on</u> $N$ <u>and an injection</u> $v : L^\infty(G) \longrightarrow N$ <u>such that</u> :

(1) $$\alpha_t(v(f)) = v(\mathrm{Ad}(\rho(t))f) \quad , \quad t \in G \quad , \quad f \in L^\infty(G).$$

<u>In this case</u> , $N$ <u>is</u> $*$<u>-isomorphic with</u> $M \times_\delta \hat{G}$ , <u>where</u> $M = N^\alpha$ <u>and</u> $\delta$ <u>is implemented via</u> $v$ .

<u>Proof</u>. If $N = M \times_\delta \hat{G}$, then the desired conclusions follow from § II.2.(5), II.2.(6), II.2.(7).

Conversely, we define $M = N^\alpha$ and we consider the action $\delta = \delta_v$ of $\hat{G}$ on $N$ . By II.3.1.(i) , $\delta$ commutes with $\alpha$ , so that $\delta(M) \subset M \bar{\otimes} L(G)$ and $\delta$ restricts to an action, still denoted by $\delta$ , of $\hat{G}$ on $M$ .

To show that $N$ is $*$-isomorphic to $M \times_\delta \hat{G}$ , we define an injection

$$\pi : N \longrightarrow N \bar{\otimes} B(L^2(G))$$

by

$$\pi(x) = W_v^* \iota_{\alpha^{-1}}(x) W_v \quad , \quad x \in N \, ,$$

where, we recall, $W_v = (v \otimes i_G)(W_G) \in N \bar{\otimes} L(G)$ .

For $x \in M = N^{\alpha}$ we have

$$\pi(x) = W_V^* \iota_{\alpha^{-1}}(x) W_V = W_V^*(x \otimes 1_G) W_V = \delta(x) \quad ,$$

while, by II.3.1.(ii), for $f \in L^{\infty}(G)$ we have

$$\pi(v(f)) = W_V^* \iota_{\alpha^{-1}}(v(f)) W_V = 1_N \otimes f \quad .$$

Therefore, taking into account Theorem II.3.3 and the definition of $M \times_{\delta} \hat{G}$ , it follows that

$$\pi(N) = \pi(\mathcal{R}\{N^{\alpha} , v(L^{\infty}(G))\}) = \mathcal{R}\{\delta(M) , 1_M \otimes L^{\infty}(G)\} = M \times_{\delta} \hat{G}$$

and this proves the theorem. ∎

II.3.5. Since there is an action $Ad_{\rho}$ of $G$ on $B(L^2(G))$ such that $B(L^2(G))^{Ad_{\rho}} = L(G)$ and an injection $i_G : L^{\infty}(G) \to B(L^2(G))$ such that (1) holds, it follows that $B(L^2(G))$ is $*$-isomorphic to the crossed product $L(G) \times_{\delta_G} \hat{G}$ and there is a n.s.f. conditional expectation $Q_G$ of $B(L^2(G))$ on $L(G)$ .

# III. DUALITY FOR WEIGHTS ON CROSSED PRODUCTS.

## § 1. Dual weights on crossed products by actions of G .

Let $\alpha : G \to \mathrm{Aut}(M)$ be an action of a locally compact group G on a von Neumann algebra M and $\varphi$ be a n.s.f. weight on M .

The dual weight $\tilde{\varphi}$ on $M \times_\alpha G$ was constructed by M.Takesaki ([60]) T.Digernes ([ 8 ],[ 9 ]) and J.-L.Sauvageot ([50]) by means of Hilbert algebras. Then U.Haagerup ([24]) has showed that there is a unique n.s.f. operator valued weight P of $M \times_\alpha G$ on M such that $\tilde{\varphi} = \varphi \circ P$ for any n.s.f. weight on M and M.Landstad ([38]) has explicitely constructed the "bounded part" of $P_G$ .

We have constructed "the whole" $P_G$ ( § I.3.1 ; and also U.Haagerup [26]) and we shall define $\tilde{\varphi}$ by $\tilde{\varphi} = \varphi \circ \iota_\alpha^{-1} \circ P_G$ and study the values of $\tilde{\varphi}$ on some particular elements as well as its modular automorphism group, which in turn entails that $\tilde{\varphi}$ is indeed the dual weight of $\varphi$ as originally defined in ([60],[ 8 ],[50]).

The same result had already been obtained by U.Haagerup ([26]). For the sake of completeness we shall include the proofs which we found independently.

On the other hand, we shall prove that the dual weights on $M \times_\alpha G$ are exactly the n.s.f. weights on $M \times_\alpha G$ which are invariant - in an appropriate sense - with respect to the dual action $\hat{\alpha}$ of $\hat{G}$ . This extends for arbitrary locally compact groups the result of U.Haagerup ([25], Theorem 3.7.) in the commutative case, which in fact suggested us the generalization.

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* *

Let $\alpha : G \to \mathrm{Aut}(M)$ be an action of G on M , $\delta = \hat{\alpha}$ be the dual action of $\hat{G}$ on $M \times_\alpha G$ and put $u(g) = 1_M \otimes \lambda(g) \in M \times_\alpha G$ ,

$g \in G$ . Recall that ( § I.2.(7) ) :

$$\delta(u(g)) = u(g) \otimes \lambda(g) \quad , \quad g \in G .$$

Consider a n.s.f. weight $\varphi$ on $M$ . By § I.3.1. and § I.2.2. there is a n.s.f. conditional expectation $P_G$ of $M \times_\alpha G$ on $\iota_\alpha(M)$, so that ([24]) we may define a n.s.f. weight $\tilde{\varphi}$ on $M \times_\alpha G$ as follows

$$\tilde{\varphi}(X) = \varphi(\iota_\alpha^{-1}(P_G(X))) \quad , \quad X \in (M \times_\alpha G)^+ .$$

The weight $\tilde{\varphi}$ is called <u>the dual weight</u> of $\varphi$.

If $f : G \longrightarrow M$ is a w-continuous function with compact support, then we may define an element $\Lambda_f \in M \bar{\otimes} L(G)$ by

(1) $$\Lambda_f = \int (f(g) \otimes \lambda(g))\, dg$$

and , if $\Lambda_f \geq 0$ , we have

(2) $$E_M^{\omega_G}(\Lambda_f) = f(e) .$$

Indeed , for any $\varphi \in M_*^+$ we have

$$\begin{aligned} < E_M^{\omega_G}(\Lambda_f), \varphi > &= < \Lambda_f , \varphi \otimes \omega_G > \\ &= \sup_{k \leq \omega_G} < \int (f(g) \otimes \lambda(g))\, dg \;, \varphi \otimes k > \\ &= \sup_{k \leq \omega_G} < \int \varphi(f(g)) \lambda(g)\, dg, \; k > \\ &= < \lambda(\varphi \circ f) \;, \omega_G > \\ &= (\varphi \circ f)(e) \\ &= < f(e) , \varphi > . \end{aligned}$$

On the other hand , for the same function $f$ we define an element $T_f \in M \times_\alpha G$ by

(3) $$T_f = \int \iota_\alpha(f(g))\, u(g)\, dg$$

(recall that $u(g) = 1_M \otimes \lambda(g)$) and if $T_f \geq 0$ , we have

(4) $$P_G(T_f) = \iota_\alpha(f(e)) \quad .$$

Indeed, we have

$$\delta(T_f) = \int \delta(\iota_\alpha(f(g))\, u(g))\, dg$$

$$= \int (\iota_\alpha(f(g))\, u(g)) \otimes \lambda(g)\, dg$$

so that, using (2) we get

$$P_G(T_f) = E^{\omega_G}_{(M\times_\alpha G)}(\delta(T_f)) = \iota_\alpha(f(e))\, u(e) = \iota_\alpha(f(e)).$$

It follows that

(5) $$\tilde{\varphi}(T_f) = \varphi(f(e))$$

for any w-continuous function $f : G \longrightarrow M$ with compact support and such that $T_f \geqslant 0$.

In what follows we shall identify $M$ with $\iota_\alpha(M) = (M \times_\alpha G)^{\delta}$. In addition to previous notations we shall freely use the usual notations of the relative modular theory ([6],[9]).

Thus $P_G$ is a n.s.f. conditional expectation of $M \times_\alpha G$ on $M$, $\varphi$ is a n.s.f. weight on $M$ and $\tilde{\varphi} = \varphi \circ P_G$. It follows from the work of U.Haagerup ([24], Corollary 4.2 ) that :

(6) $$\sigma_t^{\tilde{\varphi}}(x) = \sigma_t^{\varphi}(x), \qquad x \in M,\ t \in \mathbb{R},$$

(7) $$[D(\widetilde{\varphi\circ\alpha_g}) : D\tilde{\varphi}]_t = [D(\varphi\circ\alpha_g) : D\varphi]_t, \qquad g \in G,\ t \in \mathbb{R}.$$

On the other hand, by the relative modular theory ([9], §2) ,

$$\sigma_t^{\tilde{\varphi}}(u(g)) = \sigma_t^{\tilde{\varphi},\widetilde{\varphi\circ\alpha_g}}(u(g))[D(\widetilde{\varphi\circ\alpha_g}) : D\tilde{\varphi}]_t$$

(8)

$$= \sigma_t^{\tilde{\varphi},\widetilde{\varphi\circ\alpha_g}}(u(g))[D(\varphi\circ\alpha_g) : D\varphi]_t$$

and we shall show that

(9) $$\sigma_t^{\tilde{\varphi},\widetilde{\varphi\circ\alpha_g}}(u(g)) = \Delta_G(g)^{it}\, u(g), \qquad g \in G,\ t \in \mathbb{R}.$$

Then (8) and (9) will entail

(10) $$\sigma_t^{\tilde{\varphi}}(u(g)) = \Delta_G(g)^{it}\, u(g)\ \ D(\varphi\circ\alpha_g) : D\varphi]_t \quad .$$

Concerning (9), we remark that, by (0.1.8), it is equivalent to

(11) $$(\widetilde{\varphi\circ\alpha_g})(X) = \Delta_G(g)^{-1}\tilde{\varphi}(u(g)\,X\,u(g)^*)\ ,\ X\in(M\times_\alpha G)^+,\ g\in G.$$

But

$$(\widetilde{\varphi\circ\alpha_g})(X) = (\varphi\circ\alpha_g)(P_G(X)) = \varphi(u(g)\,P_G(X)\,u(g)^*)\ ,$$

$$\Delta_G(g)^{-1}\tilde{\varphi}(u(g)\,X\,u(g)^*) = \Delta_G(g)^{-1}\varphi(P_G(u(g)\,X\,u(g)^*))\ ,$$

so that (11) follows once we prove that

(12) $$P_G(u(g)\,X\,u(g)^*) = \Delta_G(g)\,u(g)\,P_G(X)\,u(g)^*\ ,\ X\in(M\times_\alpha G)^+,\ g\in G.$$

This property of $P_G$, which corresponds to the property (0.3.2) of $\omega_G$, is proved by the following computation with $\varphi\in M_*^+$ :

$$\begin{aligned}
&< P_G(u(g)\,X\,u(g)^*)\ ,\ \varphi >\\
&= <\delta(u(g)\,X\,u(g)^*)\ ,\ \varphi\otimes\omega_G>\\
&= <(u(g)\otimes\lambda(g))\,\delta(X)\,(u(g)^*\otimes\lambda(g)^*)\ ,\ \varphi\otimes\omega_G>\\
&= <\delta(X)\ ,\ (\varphi\circ Ad(u(g)))\otimes(\omega_G\circ Ad(\lambda(g)))>\\
&= <\delta(X)\ ,\ (\varphi\circ Ad(u(g)))\otimes\omega_G>\Delta_G(g)\\
&= <P_G(X)\ ,\ \varphi\circ Ad(u(g))>\ \Delta_G(g)\\
&= <\Delta_G(g)\,u(g)\,P_G(X)\,u(g)^*\ ,\ \varphi>.
\end{aligned}$$

III.1.1. THEOREM ([60],[8],[50],[26]). Let $\alpha : G \longrightarrow Aut(M)$ be an action of $G$ on $M$ and $\varphi$ be a n.s.f. weight on $M$. Then the dual weight

$$\tilde{\varphi} = \varphi\circ\iota_\alpha^{-1}\circ P_G$$

is the unique n.s.f. weight on $M\times_\alpha G$ with the properties :

(i) for any w-continuous function $f : G \longrightarrow M$ with compact support and such that $T_f \geq 0$,

(5) $$\tilde{\varphi}(T_f) = \varphi(f(e))$$

(ii) <u>for any</u> $x \in M$ , $g \in G$ , $t \in \mathbb{R}$ ,

(6) $$\sigma_t^{\tilde{\varphi}}(\iota_\alpha(x)) = \iota_\alpha(\sigma_t^{\varphi}(x))$$

(10) $$\sigma_t^{\tilde{\varphi}}(1_M \otimes \lambda(g)) = \Delta_G(g)^{it}(1_M \otimes \lambda(g))\, \iota_\alpha([D(\varphi\circ\alpha_g) : D\varphi]_t).$$

<u>Proof</u>. By the above , $\tilde{\varphi}$ has the required properties.

Since $M \times_\alpha G$ is generated by $\iota_\alpha(M)$ and $1_M \otimes L(G)$ , (6) and (10) completely determine the modular automorphism group of $\tilde{\varphi}$ , while (5) determines the values of $\tilde{\varphi}$ on a w-dense $\sigma_t^{\tilde{\varphi}}$-invariant $*$-subalgebra of $M \times_\alpha G$ , so that the uniqueness follows by the theorem of G.K. Pedersen and M.Takesaki ([46], Proposition 5.9 ). ∎

We shall say that a n.s.f. weight on $M \times_\alpha G$ is <u>invariant with respect to the dual action</u> $\hat{\alpha}$ of $\hat{G}$ on $M \times_\alpha G$ if it is $(\hat{\alpha}, j_G)$ - invariant (see 0.2.7 , 0.2.12).

The following theorem extends the theorem of U.Haagerup ([25], Theorem 3.7) from the commutative case to the general case.

III.1.2. THEOREM. <u>Let</u> $\alpha : G \longrightarrow Aut(M)$ <u>be an action of</u> $G$ <u>on</u> $M$. <u>Then the dual weights on</u> $M \times_\alpha G$ <u>are exactly the n.s.f. weights on</u> $M \times_\alpha G$ <u>which are invariant under the dual action</u> $\hat{\alpha}$ <u>of</u> $\hat{G}$ <u>on</u> $M \times_\alpha G$.

<u>Proof</u>. For any $s^*$-continuous function with compact support $f : G \longrightarrow M$ and any $k \in A(G)$ we have :

(13) $$(T_f)^* = T_{f^\star} ,$$

(14) $$k \cdot T_f = T_{kf} ,$$

where $f^\star(g) = \Delta_G(g)^{-1} f(g^{-1})^*$ , $(kf)(g) = k(g)\, f(g)$ , $g \in G$ .

Also, for any two $s^*$-continuous functions with compact support $f_1, f_2 : G \longrightarrow M$ , we have

(15) $$T_{f_1} T_{f_2} = T_{f_1 * f_2} ,$$

where $(f_1 * f_2)(g) = \int f_1(s)\, f_2(s^{-1}g)\, ds \quad , \quad g \in G$ .

These formulas follow directly from definitions.

Using - in an obvious way - Theorem 0.2.12 , Theorem III.1.1 and formulas (13) , (14) , (15) , it is straightforward to verify that any dual weight on $M \times_\alpha G$ is $(\hat{\alpha}, j_G)$ - invariant.

Conversely, let $\Psi$ be a n.s.f. weight on $M \times_\alpha G$ , invariant with respect to the dual action $\hat{\alpha}$ of $\hat{G}$ on $M \times_\alpha G$ and consider an arbitrary dual (hence invariant) weight $\tilde{\varphi}$ on $M \times_\alpha G$. By Corollary 0.2.13 we have

$$[D\Psi : D\tilde{\varphi}]_t \in (M \times_\alpha G)^{\hat{\alpha}} = \iota_\alpha(M) \quad \text{(by § I.2.2).}$$

By the work of A.Connes ([6], Théorème 1.2.4), there is a unique n.s.f. weight $\psi$ on M such that

$$[D\psi : D\varphi]_t = \iota_\alpha^{-1}([D\Psi : D\tilde{\varphi}]_t) \quad , \quad t \in \mathbb{R} \ ,$$

and using the result of U.Haagerup ([24], Corollary 4.2), we get

$$[D\Psi : D\tilde{\varphi}]_t = [D\tilde{\psi} : D\tilde{\varphi}]_t \quad , \quad t \in \mathbb{R} \ ,$$

so that $\Psi = \tilde{\psi}$ .

This proves the Theorem. ∎

## § 2. Dual weights on crossed products by actions of $\hat{G}$.

Let $\delta : M \longrightarrow M \bar{\otimes} L(G)$ be an action of $\hat{G}$ on $M$ and let $\alpha = \hat{\delta}$ be the dual action of $G$ on $M \times_\delta \hat{G}$.

Consider a n.s.f. weight $\varphi$ on $M$. By § II.3.2. and § II.2.2., there is a n.s.f. conditional expectation $Q_G$ of $M \times_\delta \hat{G}$ on $\delta(M)$, so that (by [24]) we may define a n.s.f. weight $\tilde{\varphi}$ on $M \times_\delta \hat{G}$ as follows :

$$\tilde{\varphi}(X) = \varphi(\delta^{-1}(Q_G(X))) \quad , \quad X \in (M \times_\delta \hat{G})^+ .$$

The weight $\tilde{\varphi}$ is called <u>the dual weight</u> of $\varphi$.

We have not an analogue of Theorem III.1.1. for this situation. We just remark that, by the general result of U.Haagerup ([24], 4.2 ), we still have

(1) $$\sigma_t^{\tilde{\varphi}}(\delta(x)) = \delta(\sigma_t^{\varphi}(x)) \quad , \ t \in \mathbb{R} \ , \ x \in M ,$$

(2) $$[D\tilde{\psi} : D\tilde{\varphi}]_t = \delta([D\psi : D\varphi]_t) \quad , \ t \in \mathbb{R} ,$$

for any n.s.f. weights $\varphi, \psi$ on $M$. We also mention that, if $\varphi$ is $\delta$-invariant , then $\sigma_t^{\tilde{\varphi}}$ acts as an identity on $1_M \otimes L^\infty(G)$.

On the other hand, the conditional expectation $Q_G$ being clearly invariant with respect to the dual action $\alpha$, it follows that any dual weight is invariant with respect to the dual action.

If $\Psi$ is any $\alpha$-invariant n.s.f. weight on $M \times_\delta \hat{G}$ and if $\tilde{\varphi}$ is any dual weight (automatically $\alpha$-invariant) on $M \times_\delta \hat{G}$, then it is easy to verify that $[D\Psi : D\tilde{\varphi}]_t$ is also $\alpha$-invariant, thus

$$[D\Psi : D\tilde{\varphi}]_t \in (M \times_\delta \hat{G})^\alpha = \delta(M) \quad \text{(by § II.2.2 )} .$$

By the work of A.Connes ([6], Théorème 1.2.4 ) , there is a unique n.s.f. weight $\psi$ on $M$ such that

$$[D\psi : D\varphi]_t = \delta^{-1}([D\Psi : D\tilde{\varphi}]_t) \quad , \quad t \in \mathbb{R} ,$$

and using (2) we get

$$[D\Psi : D\tilde{\varphi}]_t = [D\tilde{\varphi} : D\tilde{\varphi}]_t \quad , \quad t \in \mathbb{R} ,$$

so that $\Psi = \tilde{\varphi}$ .

Thus, the same proof as in the commutative case ([25], Theorem 3.7) yields the following

III.2.1. THEOREM. Let $\delta : M \longrightarrow M \bar{\otimes} L(G)$ be an action of $\hat{G}$ on $M$ . Then the dual weights on $M \times_\delta \hat{G}$ are exactly the n.s.f weights on $M \times_\delta \hat{G}$ which are invariant under the dual action $\hat{\delta}$ of $G$ on $M \times_\delta \hat{G}$ .

## § 3. The duality theorem.

Let $\alpha : G \longrightarrow \mathrm{Aut}(M)$ be an action of $G$ on $M$, $\delta = \hat{\alpha}$ be the dual action of $\hat{G}$ on $M \times_\alpha G$ and $\beta = \hat{\delta} = \hat{\hat{\alpha}}$ be the second dual action of $G$ on $(M \times_\alpha G) \times_{\hat{\alpha}} \hat{G}$ (see § I.2 , § II.2).

We identify $(M \times_\alpha G) \times_{\hat{\alpha}} \hat{G}$ with $M \overline{\otimes} B(L^2(G))$ as in Theorem I.2.1. Then, as we have already noted, the second dual action $\beta$ is given by:

$$\beta_g = \alpha_g \otimes \mathrm{Ad}(\rho(g)) \quad , \quad g \in G .$$

Consider a n.s.f. weight $\varphi$ on $M$. Then $\tilde{\varphi}$ is a n.s.f. weight on $M \times_\varphi G$ and its dual weight $\tilde{\tilde{\varphi}}$ is a n.s.f. weight on $M \overline{\otimes} B(L^2(G))$.

On the other hand, let $\mathrm{Tr}$ denote the canonical trace on $B(L^2(G))$. Since $\Delta_G$ is a positive self-adjoint operator on $L^2(G)$, we may consider the derived weight $\mathrm{Tr}_{\Delta_G}$ on $B(L^2(G))$ ([46],§4) and then the tensor product weight $\varphi \otimes \mathrm{Tr}_{\Delta_G}$ on $M \overline{\otimes} B(L^2(G))$. Furthermore, consider the unitary element $U_t \in M \overline{\otimes} L(G)$, $t \in \mathbb{R}$, defined by the function

$$G \ni g \longmapsto U_t(g) = [D(\varphi \circ \alpha_g) : D\varphi]_t \in M .$$

Then it is easy to verify (see [9]) that

$$U : t \longmapsto U_t$$

is a unitary cocycle with respect to $\varphi \otimes \mathrm{Tr}_{\Delta_G}$ in the sense of ([6], 1.2.4) and we denote by $(\varphi \otimes \mathrm{Tr}_{\Delta_G})_U$ the corresponding derived weight ([6], 1.2.4.), which is a n.s.f. weight on $M \overline{\otimes} B(L^2(G))$.

We shall show that

(1) $$\tilde{\tilde{\varphi}} = (\varphi \otimes \mathrm{Tr}_{\Delta_G})_U ,$$

or equivalently, we shall prove :

III.3.1. THEOREM. <u>For any action</u> $\alpha : G \longrightarrow \mathrm{Aut}(M)$ <u>of a locally compact group</u> $G$ <u>on a von Neumann algebra</u> $M$, <u>and any n.s.f. weight</u> $\varphi$ <u>on</u> $M$, <u>we have</u> :

(2) $$[D\tilde{\tilde{\varphi}} : D(\varphi \otimes \mathrm{Tr})]_t = \Delta_G^{it} U_t \quad , \quad t \in \mathbb{R} .$$

In the case of commutative groups $G$, this result was obtained by M.Takesaki ([60],Theorem 6.7) and T.Digernes ([9],Theorem 4.2). Our proof of Theorem III.3.1. avoids the use of Hilbert algebras and, even in the commutative case, seems to be somewhat simpler than the proofs in [60],[9].

The idea of the proof is to compute both sides of equality (1) for elements $X \in (M \overline{\otimes} B(L^2(G)))^+$ defined by $s^*$-continuous $M$-valued kernels with compact support, to establish that the two weights in (1) commute and then to apply the theorem of G.K.Pedersen and M.Takesaki ([46],Proposition 5.9).

*

* *

Consider a $s^*$-continuous function

$$G \times G \ni (s,r) \longmapsto X(s,r) \in M$$

with compact support and such that the corresponding element $X \in M \overline{\otimes} B(L^2(G))$ (see § I.3) be positive. It is then easy to see that $X(g,g) \in M^+$ for all $g \in G$.

III.3.2. LEMMA. <u>In the above conditions we have</u> :

$$\tilde{\tilde{\varphi}}(X) = \int \varphi(\alpha_g(X(g,g)))\Delta_G(g)\, dg \quad .$$

<u>Proof</u>. We have

$$\tilde{\tilde{\varphi}} = \tilde{\varphi} \circ Q_G$$

where (see § II.3.2) $Q_G$ is the n.s.f. conditional expectation of $M \overline{\otimes} B(L^2(G))$ on $M \times_\alpha G$ defined by :

$$Q_G(Y) = \int (\alpha_g \otimes \mathrm{Ad}(\rho(g)))\,(Y)\, dg \quad , \quad Y \in (M \overline{\otimes} B(L^2(G)))^+ .$$

We may assume that $M \subset B(H)$ and that there is a unitary representation $u : G \longrightarrow B(H)$ such that $\alpha_g = \mathrm{Ad}(u(g))$, $g \in G$.

Then, for any $g,s \in G$ and any $\xi \in L^2(G,H)$, we have :

$$(((\alpha_g \otimes \mathrm{Ad}(\rho(g)))(X))\xi)(s) =$$

$$= ((u(g)\otimes\rho(g))\, X\, (u(g)^* \otimes \rho(g)^*)\xi)(s)$$

$$= u(g)\Delta_G(g)^{1/2}\, (X(u(g)^*\otimes\rho(g)^*)\xi)(sg)$$

$$= u(g)\Delta_G(g)^{1/2}\int X(sg,r)((u(g)^*\otimes\rho(g)^*)\xi)(r)\, dr$$

$$= u(g)\int X(sg,r)\, u(g)^*\xi(rg^{-1})\, dr$$

$$= \Delta_G(g)\int u(g)\, X(sg,rg)\, u(g)^*\xi(r)\, dr$$

$$= \Delta_G(g)\int \alpha_g(X(sg,rg))\,\xi(r)\, dr \quad ,$$

thus

$$(Q_G(X)\xi\,|\,\xi) = \iiint \Delta_G(g)\, (\alpha_g(X(sg,rg))\xi(r)\,|\,\xi(s))\, drdsdg \; .$$

Consider the function $f : G \longrightarrow M$ defined by :

$$f(t) = \int \alpha_{tg}(X(tg,g))\Delta_G(g)\, dg \quad , \quad t\in G \; .$$

It is a w-continuous function with compact support and, with the notation $T_f$ as in § III.1.(3), we have :

$$(T_f\xi\,|\,\xi) = \iiint \Delta_G(g)\, (\alpha_g(X(sg,rg))\xi(r)\,|\,\xi(s))\, drdsdg \; .$$

Therefore

$$(Q_G(X)\xi\,|\,\xi) = (T_f\xi\,|\,\xi) \quad , \quad \xi\in L^2(G,H) \; ,$$

which means that $T_f \geq 0$ and

$$Q_G(X) = T_f \quad \text{in } (M\times_\alpha G)^+ \; .$$

Finally, using III.1.(5) and ([46],3.1), we get :

$$\tilde{\tilde{\varphi}}(X) = \tilde{\varphi}(Q_G(X))$$

$$= \tilde{\varphi}(T_f)$$

$$= \varphi(f(e))$$

$$= \varphi\left(\int \Delta_G(g)\,\alpha_g(X(g,g))\, dg\right)$$

$$= \int \varphi(\alpha_g(X(g,g)))\,\Delta_G(g)\, dg \; . \blacksquare$$

The computation of $(\varphi \otimes \mathrm{Tr}_{\Delta_G})_U(X)$ is a little more difficult.

There is a n.s.f. conditional expectation $P_G$ of $B(L^2(G))$ on $L^\infty(G)$ (§ I.3.5 ) and we have the (left) Haar weight $\mu_G$ on $L^\infty(G)$. Therefore ([24]) we get a n.s.f. weight $\mu_G \circ P_G$ on $B(L^2(G))$.

III.3.3. LEMMA. We have

$$\mathrm{Tr}_{\Delta_G} = \mu_G \circ P_G$$

as weights on $B(L^2(G))$.

Proof. Consider $X \in B(L^2(G))^+$ defined by a continuous kernel

$$G \times G \ni (s,r) \longmapsto X(s,r) \in \mathbb{C}$$

with compact support. By § I.3.2 it follows that $P_G(X) \in L^\infty(G)^+$ and

$$P_G(X)(g) = \Delta_G(g)\, X(g,g) \quad , \quad g \in G \ ,$$

so that

$$(\mu_G \circ P_G)(X) = \int X(g,g)\, \Delta_G(g)\, dg \ .$$

On the other hand, $\Delta_G X$ is defined by the kernel

$$(\Delta_G X)(s,r) = \Delta_G(s)\, X(s,r) \quad , \quad s,r \in G$$

and, using ([15], Ch. IX , § 8 , Ex. 49(c) ) , we get

$$\mathrm{Tr}_{\Delta_G}(X) = \int X(g,g)\, \Delta_G(g)\, dg \ .$$

Therefore, $\mu_G \circ P_G$ and $\mathrm{Tr}_{\Delta_G}$ agree on the linear span $A$ of the elements $X \in B(L^2(G))^+$ which are defined by continuous kernels with compact support. It is clear that $A$ is a w-dense $*$-subalgebra of $B(L^2(G))$. Since

$$\sigma_t^{\mathrm{Tr}_{\Delta_G}}(X) = \Delta_G^{it}\, X\, \Delta_G^{-it} \quad , \quad t \in \mathbb{R} \ ,$$

it follows that $A$ is $\sigma_t^{\mathrm{Tr}_{\Delta_G}}$ - invariant.

Moreover, since $\Delta_G^{it} \in L^\infty(G)$ and $P_G$ is a conditional expectation of $B(L^2(G))$ on $L^\infty(G)$ , which is commutative, we have :

$$P_G(\Delta_G^{it} X \Delta_G^{-it}) = \Delta_G^{it} P_G(X) \Delta_G^{-it} = P_G(X) \quad , \quad X \in B(L^2(G))^+ .$$

Therefore $\mu_G \circ P_G$ is invariant with respect to the modular automorphism group of $Tr_{\Delta_G}$.

Using the theorem of G.K.Pedersen and M.Takesaki ([46], 5.9), we get the desired equality. ∎

There is also a n.s.f. conditional expectation $P_G = P_G^{M \bar{\otimes} B(L^2(G))}$ of $M \bar{\otimes} B(L^2(G))$ on $M \bar{\otimes} L^\infty(G)$, obtained via the action $\bar{\delta}_G = i_M \otimes \delta_G$ of $\hat{G}$ on $M \bar{\otimes} B(L^2(G))$ as in § I.3.1.(iii).

III.3.4. LEMMA. We have

$$(\varphi \otimes \mu_G) \circ P_G = \varphi \otimes Tr_{\Delta_G}$$

as weights on $M \bar{\otimes} B(L^2(G))$.

Proof. By Lemma III.3.3. the two weights agree on $\mathfrak{M}_\varphi \otimes \mathfrak{M}_{Tr_{\Delta_G}}$. The modular automorphism group $\sigma_t = \sigma_t^\varphi \otimes (\Delta_G^{it} \cdot \Delta_G^{-it})$ of $\varphi \otimes Tr_{\Delta_G}$ commutes with $\bar{\delta}_G$, i.e. $\bar{\delta}_G \circ \sigma_t = (\sigma_t \otimes i_G) \circ \bar{\delta}_G$. Using § I.3.(4) we see that $\sigma_t$ commutes with $P_G$. Finally, $\varphi \otimes \mu_G$ is clearly $\sigma_t$-invariant and therefore $(\varphi \otimes \mu_G) \circ P_G$ is $\sigma_t$-invariant.

Thus the Lemma follows using again ([46],5.9). ∎

We remark that the above proof shows that

$$P_G^{M \bar{\otimes} B(L^2(G))} = i_M \otimes P_G^{B(L^2(G))}$$

in the sense defined by U.Haagerup ([27],Theorem 5.5).

Now, $U = \{U_t\}_{t \in \mathbb{R}} \subset M \bar{\otimes} L^\infty(G)$ is also a unitary cocycle with respect to the n.s.f. weight $\varphi \otimes \mu_G$ on $M \bar{\otimes} L^\infty(G)$, so that we may consider the derived weight $(\varphi \otimes \mu_G)_U$ on $M \bar{\otimes} L^\infty(G)$ ([6], 1.2.4).

III.3.5. LEMMA. *We have*

$$(\varphi \otimes \mathrm{Tr}_{\Delta_G})_U = (\varphi \otimes \mu_G)_U \circ P_G$$

*as weights on* $M \bar{\otimes} B(L^2(G))$ .

*Proof.* This follows from Lemma III.3.4 , using the result of U.Haagerup ([24],4.2) and the uniqueness of the derived weight corresponding to a given cocycle ([6],1.2.4 ). ∎

The action $\alpha : G \longrightarrow \mathrm{Aut}(M)$ defines a $*$-automorphism $\tau_\alpha = \theta_\alpha^{-1}$ of $M \bar{\otimes} L^\infty(G)$ , as was pointed out in § I.1. If we regard an element $x \in M \bar{\otimes} L^\infty(G)$ as an M-valued $L^\infty$-function $g \longmapsto x(g)$ , then

$$(\tau_\alpha(x))(g) = \alpha_g(x(g)) \quad , \quad g \in G .$$

We recall that

(3) $$U_t(g) = [D(\varphi \circ \alpha_g) : D\varphi]_t \quad , \quad g \in G \quad , \quad t \in \mathbb{R} .$$

III.3.6. LEMMA. *We have*

$$(\varphi \otimes \mu_G)_U = (\varphi \otimes \mu_G) \circ \tau_\alpha$$

*as weights on* $M \bar{\otimes} L^\infty(G)$.

*Proof.* Indeed, using the original construction of A.Connes' cocycles ([6], § 1.2 ) or the KMS-boundary conditions ([9],2.2 ), it is easy to verify that

$$[D((\varphi \otimes \mu_G) \circ \tau_\alpha) : D(\varphi \otimes \mu_G)]_t = U_t \quad , \quad t \in \mathbb{R} . \ ∎$$

From Lemmas III.3.5. and III.3.6. we infer that

(4) $$(\varphi \otimes \mathrm{Tr}_{\Delta_G})_U = ((\varphi \otimes \mu_G) \circ \tau_\alpha) \circ P_G$$

as weights on $M \bar{\otimes} B(L^2(G))$.

We are now able to compute $(\varphi \otimes \mathrm{Tr}_{\Delta_G})_U (X)$ :

III.3.7. LEMMA. <u>For any</u> $X \in (M \overline{\otimes} B(L^2(G)))^+$ <u>defined by a</u> $s^*$<u>-continuous compactly supported M-valued kernel</u>

$$G \times G \ni (s,r) \longmapsto X(s,r) \in M$$

<u>we have</u>

$$(\varphi \otimes \mathrm{Tr}_{\Delta_G})_U (X) = \int \varphi(\alpha_g(X(g,g)))\Delta_G(g)\, dg \ .$$

Proof. Indeed, using (4) and § I.3.2. we get :

$$\begin{aligned}(\varphi \otimes \mathrm{Tr}_{\Delta_G})_U (X) &= (\varphi \otimes \mu_G)(\tau_\alpha(P_G(X))) \\ &= \int \varphi((\tau_\alpha(P_G(X)))(g))\, dg \\ &= \int \varphi(\alpha_g(P_G(X)(g)))\, dg \\ &= \int \varphi(\alpha_g(\Delta_G(g)X(g,g)))\, dg \\ &= \int \varphi(\alpha_g(X(g,g)))\Delta_G(g)\, dg \ . \blacksquare\end{aligned}$$

Our next objective is to show that $\tilde{\tilde{\varphi}}$ commutes with $(\varphi \otimes \mathrm{Tr}_{\Delta_G})_U$ , i.e. that $\tilde{\tilde{\varphi}}$ is invariant with respect to the modular automorphism group of $(\varphi \otimes \mathrm{Tr}_{\Delta_G})_U$ .

In order to simplify the notations we shall denote :

$$\sigma_t = \sigma_t^{(\varphi \otimes \mathrm{Tr}_{\Delta_G})_U} \in \mathrm{Aut}(M \overline{\otimes} B(L^2(G))) \quad , \quad t \in \mathbb{R} ,$$

$$V_t(g) = \Delta_G^{it}(g)\, U_t(g) \in M \quad , \quad g \in G ,\ t \in \mathbb{R} ,$$

We recall that

$$\beta_g = \alpha_g \otimes \mathrm{Ad}(\rho(g)) \in \mathrm{Aut}(M \overline{\otimes} B(L^2(G))) \quad , \quad g \in G .$$

Also, there is a n.s.f. conditional expectation $Q_G$ of $M \overline{\otimes} B(L^2(G))$ on $M \times_\alpha G$ defined by (see § II.3.2.) :

$$Q_G(X) = \int \beta_g(X)\, dg \quad , \quad X \in (M \overline{\otimes} B(L^2(G)))^+$$

and

$$\tilde{\tilde{\varphi}} = \tilde{\varphi} \circ Q_G \ .$$

III.3.8. LEMMA. With the above notations, we have :

(5) $$\sigma_t |_{M \times_\alpha G} = \sigma_t^{\tilde{\varphi}} \quad , \quad t \in \mathbb{R} ,$$

(6) $$Q_G \circ \sigma_t = \sigma_t \circ Q_G \quad , \quad t \in \mathbb{R} .$$

III.3.9. COROLLARY. $\tilde{\tilde{\varphi}}$ commutes with $(\varphi \otimes \mathrm{Tr}_{\Delta_G})_U$ .

Proof of Corollary III.3.9. Using (5) and (6) we get

$$Q_G \circ \sigma_t = \sigma_t^{\tilde{\varphi}} \circ Q_G \quad , \quad t \in \mathbb{R} ,$$

and therefore

$$\tilde{\tilde{\varphi}} \circ \sigma_t = \tilde{\varphi} \circ Q_G \circ \sigma_t = (\tilde{\varphi} \circ \sigma_t^{\tilde{\varphi}}) \circ Q_G = \tilde{\varphi} \circ Q_G = \tilde{\tilde{\varphi}} \quad , \quad t \in \mathbb{R} ,$$

since $\tilde{\varphi}$ is clearly $\sigma_t^{\tilde{\varphi}}$ -invariant. ∎

Proof of Lemma III.3.8. We first remark that

(7) $$\sigma_t(X) = V_t \, ((\sigma_t^{\varphi} \otimes i_G)(X)) \, V_t^* \quad , \quad X \in M \bar{\otimes} B(L^2(G)) .$$

For $f \in L^\infty(G)$ , $1_M \otimes f \in 1_M \otimes L^\infty(G) \subset M \bar{\otimes} B(L^2(G))$ and we have

(8) $$\sigma_t(1_M \otimes f) = V_t((\sigma_t^{\varphi} \otimes i_G)(1_M \otimes f))V_t^* = V_t(1_M \otimes f)V_t^* = 1_M \otimes f ;$$

since $V_t \in M \bar{\otimes} L^\infty(G)$ and $L^\infty(G)$ is commutative.

For $x \in M$ , $\iota_\alpha(x) \in \iota_\alpha(M) \subset M \bar{\otimes} B(L^2(G))$ and , owing to (3) and to § III.1.1.(6) , we get

$$\begin{aligned}
(\sigma_t(\iota_\alpha(x)))(g) &= (V_t \, ((\sigma_t^{\varphi} \otimes i_G)(\iota_\alpha(x))) \, V_t^*)(g) \\
&= V_t(g) \, \sigma_t^{\varphi}((\iota_\alpha(x))(g)) \, V_t(g)^* \\
&= U_t(g) \, \sigma_t^{\varphi}(\alpha_{g^{-1}}(x)) \, U_t(g)^* \\
&= \sigma_t^{\varphi \circ \alpha_g}(\alpha_{g^{-1}}(x)) \\
&= \alpha_g^{-1} \circ \sigma_t^{\varphi} \circ \alpha_g \, (\alpha_g^{-1}(x)) \\
&= \alpha_{g^{-1}}(\sigma_t^{\varphi}(x)) \\
&= (\iota_\alpha(\sigma_t^{\varphi}(x)))(g)
\end{aligned}$$

$$= (\sigma_t^{\tilde{\varphi}}(\iota_\alpha(x)))(g) \qquad , \quad g \in G ,$$

thus

(9) $$\sigma_t(\iota_\alpha(x)) = \sigma_t^{\tilde{\varphi}}(\iota_\alpha(x)) .$$

For $g \in G$ , $1_M \otimes \lambda(g) \in 1_M \otimes L(G) \subset M \bar{\otimes} B(L^2(G))$ and we have

$$\sigma_t(1_M \otimes \lambda(g)) = V_t \, ((\sigma_t^{\varphi} \otimes i_G)(1_M \otimes \lambda(g))) \, V_t^*$$

(10) $$= V_t \, (1_M \otimes \lambda(g)) \, V_t^*$$

$$= (1_M \otimes \lambda(g)) \, (\mathrm{Ad}(1_M \otimes \lambda(g)^* )(V_t)) \, V_t^* .$$

On the other hand, for each $r \in G$ ,

$$((\mathrm{Ad}(1_M \otimes \lambda(g)^* )(V_t) \, V_t^*)(r) = V_t(gr) \, V_t(r)^*$$

(11) $$= \Delta_G(g)^{it} \, U_t(gr) \, U_t(r)^*$$

$$= \Delta_G(g)^{it} (\iota_\alpha([D(\varphi \circ \alpha_g) : D\varphi]_t))(r) .$$

The last equality is justified as follows. By the "chain rule" ([6], 1.2.3 ) , we get

$$[D(\varphi \circ \alpha_{gr}) : D(\varphi \circ \alpha_r)]_t \, [D(\varphi \circ \alpha_r) : D\varphi]_t = [D(\varphi \circ \alpha_{gr}) : D\varphi]_t$$

thus

$$U_t(gr) \, U_t(r)^* = [D(\varphi \circ \alpha_{gr}) : D(\varphi \circ \alpha_r)]_t$$

and , using ([ ], 2.3 ) ,

$$U_t(gr) \, U_t(r)^* = [D((\varphi \circ \alpha_g) \circ \alpha_r) : D(\varphi \circ \alpha_r)]_t$$

$$= \alpha_r^{-1}([D(\varphi \circ \alpha_g) : D\varphi]_t)$$

$$= (\iota_\alpha([D(\varphi \circ \alpha_g) : D\varphi]_t))(r) .$$

Combining (10) and (11) we infer

$$\sigma_t(1_M \otimes \lambda(g)) = \Delta_G(g)^{it} (1_M \otimes \lambda(g)) \, \iota_\alpha([D(\varphi \circ \alpha_g) : D\varphi]_t)$$

and finally , owing to § III.1.1.(10) , we obtain

(12) $$\sigma_t(1_M \otimes \lambda(g)) = \sigma_t^{\tilde{\varphi}}(1_M \otimes \lambda(g))$$

Since $M \times_\alpha G = \mathcal{R}\{L_\alpha(M)\ ,\ 1_M \otimes L(G)\}$, the relations (9) and (12) entail the first conclusion (5) of the Lemma.

Since $M \times_\alpha G \subset (M \overline{\otimes} B(L^2(G)))^\beta$, the same relations (9) and (12) show that

(13) $$(\beta_g \circ \sigma_t)(X) = (\sigma_t \circ \beta_g)(X) \quad \text{for} \quad X \in M \times_\alpha G \ .$$

Using (8) we see that (13) also holds for $X \in 1_M \otimes L^\infty(G)$. But(§I.1.1)

$$M \overline{\otimes} B(L^2(G)) = \mathcal{R}\{M \times_\alpha G\ ,\ 1_M \otimes L^\infty(G))\}$$

and therefore (13) holds for any $X \in M \overline{\otimes} B(L^2(G))$. This in turn entails the other conclusion (6) of the Lemma. ∎

Proof of Theorem III.3.1. We shall construct a w-dense $\sigma_t$-invariant $*$-subalgebra $\mathcal{B}$,

$$\mathcal{B} \subset \mathfrak{M}_{(\varphi \otimes Tr_{\Delta_G})_U} \subset M \overline{\otimes} B(L^2(G)) \quad ,$$

on which $(\varphi \otimes Tr_{\Delta_G})_U$ and $\tilde{\tilde{\varphi}}$ are equal.

To this end we consider :

$\mathcal{X} = \{ X \in M \overline{\otimes} B(L^2(G))$ ; $X$ is defined by a $s^*$-continuous compactly supported M-valued kernel

$$G \times G \ni (s,r) \longmapsto X(s,r) \in M \},$$

$\mathcal{S} = \{ S \in B(L^2(G))$ ; $S$ is defined by a continuous compactly supported scalar valued kernel

$$G \times G \ni (s,r) \longmapsto S(s,r) \in \mathbb{C} \},$$

$\mathcal{W}$ = the set of $s^*$-continuous functions $W : G \times G \longmapsto M$ such that $W(s,r)$ is unitary for all $s,r \in G$ .

For each $S \in \mathcal{S}$, $W \in \mathcal{W}$, $x \in \mathfrak{N}_\varphi \subset M$, we define an operator

$$Y = Y(S,W,x) \in \mathcal{X} \subset M \overline{\otimes} B(L^2(G))$$

by the kernel

$$Y(s,r) = S(s,r)\, W(s,r)\, \alpha_r^{-1}(x) \quad , \quad s,r \in G \ .$$

Then $Y^*$ is defined by the kernel

$$Y^*(s,r) = Y(r,s)^* = \overline{S(r,s)}\,\alpha_s^{-1}(x^*)\,W(r,s)^* , \quad r,s \in G .$$

Thus $X = Y^*Y$ is defined by the kernel

$$X(s,r) = \int Y^*(s,h)\,Y(h,r)\,dh$$

$$= \int \overline{S(h,s)}\,S(h,r)\,\alpha_s^{-1}(x^*)\,W(h,s)^*\,W(h,r)\,\alpha_r^{-1}(x)\,dh ,$$

$$r,s \in G .$$

In particular

$$X(g,g) = \left(\int |S(h,g)|^2\,dh\right)\alpha_g^{-1}(x^*x) , \quad g \in G ,$$

and, using Lemma III.3.7, we get

$$(\varphi \otimes \mathrm{Tr}_{\Delta_G})_U(X) = \int \varphi(\alpha_g(X(g,g)))\,\Delta_G(g)\,dg$$

$$= \varphi(x^*x)\iint |S(h,g)|^2\,dh\,dg < +\infty .$$

Therefore, for any $S \in \mathcal{S}$, $W \in \mathcal{W}$, $x \in \mathcal{N}_\varphi$, we have

$$Y^*(S,W,x)\,Y(S,W,x) \in \mathcal{M}^+_{(\varphi \otimes \mathrm{Tr}_{\Delta_G})_U} .$$

Now let $\mathcal{F}$ be the face ([3],[45]) <u>of</u> $\mathfrak{X}^+$ (<u>not</u> <u>of</u> $(M \overline{\otimes} B(L^2(G)))^+$) generated by the set

$$\{ Y^*(S,W,x)\,Y(S,W,x) \; ; \; S \in \mathcal{S} , W \in \mathcal{W} , x \in \mathcal{N}_\varphi \} .$$

Since $\mathcal{M}^+_{(\varphi \otimes \mathrm{Tr}_{\Delta_G})_U}$ is a face of $(M \overline{\otimes} B(L^2(G)))^+$ containing this set, it follows that $\mathcal{F} \subset \mathcal{M}^+_{(\varphi \otimes \mathrm{Tr}_{\Delta_G})_U}$.

The linear span of $\mathcal{F}$ is a facial $*$- subalgebra $\mathcal{B}$ of $\mathfrak{X}$ (see [3], Proposition 1.3).

Thus $\mathcal{B}$ <u>is a</u> $*$-<u>subalgebra of</u> $\mathcal{M}_{(\varphi \otimes \mathrm{Tr}_{\Delta_G})_U}$ .

Since $\mathcal{B} \subset \mathfrak{X}$ , and $\mathcal{B}$ is the linear span of its positive part, Lemma III.3.2 and Lemma III.3.7 show that $(\varphi \otimes Tr_{\Delta_G})_U$ <u>and</u> $\tilde{\varphi}$ <u>are equal on</u> $\mathcal{B}$ .

If $W_o \in \mathcal{W}$ is defined by $W_o(s,r) = 1_M$ for any $s,r \in G$ , then

$$Y(S,W_o,x) = (1_M \otimes S)\, \iota_\alpha(x) , \quad S \in \mathcal{S} , x \in \mathfrak{N}_\varphi .$$

Owing to Proposition I.1.1, we infer that the $*$-subalgebra $\mathcal{B}$ <u>is w-dense in</u> $M \bar{\otimes} B(L^2(G))$ .

Recall that the modular automorphism group $\{\sigma_t\}_{t\in\mathbb{R}}$ of $(\varphi\otimes Tr_{\Delta_G})_U$ is given by

$$\sigma_t(X) = V_t\,((\sigma_t^\varphi \otimes i_G)(X))\, V_t^* , \quad t \in \mathbb{R} ,$$

where $V_t \in M \bar{\otimes} L^\infty(G)$ and

$$V_t(g) = \Delta_G^{it}(g)\,[D(\varphi\circ\alpha_g) : D\varphi]_t , \quad g \in G , t \in \mathbb{R} .$$

For any $S \in \mathcal{S}$ , $W \in \mathcal{W}$ , $x \in \mathfrak{N}_\varphi$, the operator $Y = Y(S,W,x)$ is defined by the kernel

$$Y(s,r) = S(s,r)\, W(s,r)\, \alpha_r^{-1}(x) , \quad s,r \in G ,$$

and, for any $t \in \mathbb{R}$ , the operator $\sigma_t(Y)$ also belongs to $\mathcal{X}$ , being defined by the kernel

$$(\sigma_t(Y))(s,r) = S(s,r)\, V_t(s)\, \sigma_t^\varphi(W(s,r))\, \sigma_t^\varphi(\alpha_r^{-1}(x))\, V_t(r)^*$$
$$= S(s,r)\, V_t(s)\, \sigma_t^\varphi(W(s,r))\, V_t(r)^*\, V_t(r)\, \sigma_t^\varphi(\alpha_r^{-1}(x))\, V_t(r)^*$$
$$= S_t^\varphi(s,r)\, W_t^\varphi(s,r)\, \alpha_r^{-1}(\sigma_t^\varphi(x))$$

where $S_t^\varphi \in \mathcal{S}$ , $W_t^\varphi \in \mathcal{W}$ are defined by

$$S_t^\varphi(s,r) = S(s,r) , \quad s,r\in G , t \in \mathbb{R} ,$$
$$W_t^\varphi(s,r) = V_t(s)\, \sigma_t^\varphi(W(s,r))\, V_t(r)^* , \quad s,r \in G , t\in\mathbb{R} .$$

Thus ,

$$\sigma_t(Y(S,W,x)) = Y(S_t^\varphi, W_t^\varphi, \sigma_t^\varphi(x)) , \quad t \in \mathbb{R} .$$

It follows that the $*$-subalgebra $\mathcal{B}$ <u>is</u> $\sigma_t$-<u>invariant</u> .

Therefore $(\varphi \otimes \mathrm{Tr}_{\Delta_G})_U$ and $\tilde{\tilde{\varphi}}$ agree on the w-dense $\sigma_t$-invariant $*$-subalgebra $\mathcal{B}$ of $M \overline{\otimes} B(L^2(G))$ and, since they commute by Corollary III.3.9, the theorem of G.K.Pedersen and M.Takesaki ([46], Proposition 5.9) shows that they are equal as n.s.f. weights on $M \overline{\otimes} B(L^2(G))$ .

This proves the Theorem. ∎

Recall that there is a n.s.f. conditional expectation $Q_G$ of $B(L^2(G))$ on $L(G)$ (see § II.3.5) and we have the Plancherel weight $\omega_G$ on $L(G)$ . A particular case of Theorem III.3.1 is the following

III.3.10. COROLLARY. *We have*

$$\mathrm{Tr}_{\Delta_G} = \omega_G \circ Q_G$$

*as weights on* $B(L^2(G))$.

The results III.3.3 and III.3.10 are dual to each other. They could be regarded as "Plancherel theorems" for $G$ .

Lemma III.3.3 is a particular case of a dual version of Theorem III.3.1. Concerning such a theorem we just mention, without proof, that if we start with an action $\delta$ of $\hat{G}$ on $M$ and with a $(\delta, j_G)$-invariant n.s.f. weight $\varphi$ on $M$ , then

$$\tilde{\tilde{\varphi}} = \varphi \otimes \mathrm{Tr}_{\Delta_G}$$

as weights on $M \overline{\otimes} B(L^2(G)) \cong (M \times_{\delta} \hat{G}) \times_{\hat{\delta}} G$ .

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