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by

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PREPRINT SERIES IN MATHEMATICS
No.21/1976

BUCUREȘTI

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November 1976

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Abstract

Using some recent results concerning the time-optimal feedback control ([2]-[4], [11], [12]) it is proved that for some linear control systems (for example, for scalar control systems of dimension less than four or whose matrix has only real eigenvalues, etc.) a time-optimal feedback control using a state estimator can be constructed.

1. The statement of the problem

We consider the "input-output" control system defined by:

(1.1) $\frac{dx}{dt} = Ax + Bu \quad , x \in R^n, \quad u \in U \subset R^p$

(1.2) $y = C^* x \quad , y \in R^m$

where the matrices A, B, C and the control space, U, satisfy the following conditions:

(1.3) $U \subset R^p$ is a convex, compact polyhedron which contains the origin, $0 \in R^p$, in its interior;

(1.4) (A,B) is completely controllable (i.e. rank$(B, AB, \ldots A^{n-1}B) = n$

(1.5) (A, C^*) is completely observable (i.e. (A^*, C) is completely controllable; C^* denotes the transpose of C.

Since in many cases the state variable, x, of the system (1.1) cannot be "observed" (and therefore, "measured") and since the information about the state of the system is given by the output, y, the system (1.1)-(1.2) is controlled by using a "state estimator" ([8])

$$(1.6)\qquad \begin{cases} \frac{dx}{dt} = Ax + Bu \\ \frac{dz}{dt} = Az + K(y - KC^* z) + Bu \\ y = C^* x \end{cases}$$

where the matrix $K \in L(R^m, R^n)$ is chosen such that $A-KC^* \in L(R^n, R^n)$ is stable (i.e. it has only eigenvalues with negative real part).

The state estimator (1.6) works as follows: for any (measurable) control, $u(.) : [0,\infty) \longrightarrow U$, the output $y(t)$ is measured at each moment $t \geq 0$ and, if we denote by $(\varphi_u(.;x,0), \psi_u(.;x,0))$ the solution through $(x,0)$ of the system:

$$(1.7)\qquad \begin{cases} \frac{dx}{dt} = Ax + Bu(t) \\ \frac{dz}{dt} = Az + KC^*(x-z) + Bu(t) \end{cases}$$

then the component $\psi_u(t;x,0)$ is an "estimate" of the state $\varphi_u(t;x,0)$, of the system at the moment t. Due to the fact that the matrix $A-KC^*$ is stable, the estimate $\psi_u(t;x,0)$ (which can be computed from (1.7)) is as "close" as it is desired to the state $\varphi_u(t;x,0)$ after a sufficiently large interval.

A state estimator (1.7) is indispensable for the control of the system when instead of an open loop control, $u(.) : [0,\infty) \rightarrow U$, a feedback control, $v(.) : R^n \longrightarrow U$, is used. Since the state x of the system is not available, one cannot compute the input, $v(x)$, and therefore, using the estimate, z, of the state x, one can use instead of $v(x)$ the input $v(z)$.

The process of controlling the system by using an observer (state estimator) is therefore described by the differential system:

$$(1.8)\quad \begin{cases} \dfrac{dx}{dt} = Ax + Bv(z) \\ \dfrac{dz}{dt} = Az + KC^*(x - z) + Bv(z) \end{cases}$$

The aim of such a regulator is to steer every state of the system "close" to the origin - the equilibruium state of the system - as fast as possible.

In the classical engeneering control theory the following "linear regulator" is used ([8]):

$$(1.9)\quad \begin{cases} \dfrac{dx}{dt} = Ax - BLz \\ \dfrac{dz}{dt} = KC^*x + (A - KC^* - BL)z \end{cases}$$

corresponding to a linear feedback control:

$$(1.10)\qquad v(z) = - Lz$$

where the matrix $L \in L(R^n, R^p)$ is chosen such that $A - BL$ is a stable matrix.

Though very simple and easy to handle, the linear regulator (1.9) has the disadvantage that for initial states far away from the origin, the constraint $v(z) \in U$ will not be satisfied any more.

In what follows we study the possibility to use a Bang-Bang regulator of the type (1.8) where, instead of the linear feedback (1.10) one uses the time-optimal feedback control $v(.): G \subset R^n \longrightarrow U$ of the system (1.1). Such a regulator will allways comply to the control constraint $v(z) \in U$ and, moreover, as it is proved in the next section, it destroys the perturbations in a time close to the minimal one. However, the strong discontinuity of the time-optimal feedback control makes this possible only for a certain class of control systems.

The problem we are considering in what follows may be stated as follows: find the conditions under which the time-optimal observer-feedback control (1.8) stabilizes the system (1.1)-(1.2) in a time close to the minimal one.

The main result of this paper states that the time-optimal observer-feedback control described above stabilizes the system when the optimal feedback control has the property that the optimal trajectories coincide with the Filippov solutions.

2. The time-optimal feedback control for linear systems and Filippov solutions

We say that a measurable map, $u(.) : [0, t_1] \longrightarrow U$ is an admissible control with respect to the initial state $x \in R^n$ if the solution $\varphi_u(.;x)$ through x of the system:

$$\frac{dx}{dt} = Ax + Bu(t) \tag{2.1}$$

verifies the conditions: $\varphi_u(t;x) = 0$ for $t \in [0, t_1)$ and $\varphi_u(t_1;x) = 0$; we say also that $u(.)$ steers x to the origin $0 \in R^n$ in the time t_1.

For each $t \geqslant 0$ we denote by $\mathcal{R}(t)$ the reachability set at the moment t, that is, the set of all states $x \in R^n$ which can be steered to the origin in a time $t_1 \leqslant t$. We denote also by $G = \bigcup_{t \geqslant 0} \mathcal{R}(t)$ the controllability set of the system (1.1).

The following facts are well known from the now classical textbooks in Control Theory [1] , [9] etc. :

1. For any $t > 0$, the reachability set $\mathcal{R}(t)$ is a compact convex neighborhood of the origin and the map $t \longmapsto \mathcal{R}(t)$ is strictly increasing with respect to the set inclusion and continuous in the Pompeiu-Hausdorff topology of the compact subsets of R^n;

2. The controllability set, $G \subset R^n$, of the system (1.1) is a convex, open neighborhood of the origin; $G=R^n$ if A is stable;
3. For any $x \in G$ there exists a unique optimal control, $u_x(.):[0,T(x)] \longrightarrow U$ which steers x to the origin in the minimal time $T(x) \geqslant 0$; moreover, $u_x(.)$ is a piecewise constant map, takes values only in the set V of the vertices of the polyhedron U and satisfies the Maximum Principle;
4. The minimal-time function, $T(.): G \longrightarrow R_+$, is continuous;
5. There exists a uniquely defined map $v(.): G \longrightarrow U$ such that for any $x \in G$, the unique optimal trajectory $\tilde{\varphi}_u(.;x):[0,T(x)] \longrightarrow G$ corresponding to x is a classical (Carathéodory) solution of the differential system:

$$(2.2) \qquad \frac{dx}{dt} = Ax + Bv(x)$$

Moreover, $v(0) = 0 \in U$, and for any $x \in G \setminus \{0\}$, $v(x) \in V$.

The map $v(.)$ is called <u>the time-optimal feedback control</u> of the system (1.1).

The fact that $v(.)$ is discontinuous makes the optimal trajectories to be classical solutions of the <u>right-hand side discontinuous differential system (2.2)</u> whereas for such systems the natural concept of solution is that given by Filippov in [5]:

<u>Definition 2.1</u> ([5])

<u>Let $f : I \times G \subset R \times R^n \longrightarrow R^n$ be a measurable, bounded map that defines the differential system:</u>

$$(2.3) \qquad \frac{dx}{dt} = f(t,x)$$

<u>A map $\Psi(.):[t_0,t_1] \subset I \longrightarrow G$ is called a Filippov solution of the system (2.3) if it is absolutely continuous and:</u>

$$(2.4) \qquad \frac{d\Psi}{dt}(t) \in F(t, \Psi(t)) \qquad \text{a. e. on } [t_0,t_1]$$

where:

$$F(t,x) = \bigcap_{\delta>0} \bigcap_{\mu(E)=0} \overline{co}\, f(t, B_\delta(x) \setminus E) \tag{2.5}$$

$\overline{co}(M)$ being the closed convex hull of the set M, $\mu(E)$ the Lebesgue measure of the set E and $B_\delta(x)$ the ball of radius δ centered at x.

For this concept of solution, A.F. Filippov has proved in [5] the analogous of all fundamental results in the Theory of Ordinary Differential Equations: existence, uniqueness, continuous dependence on parameters and initial data, etc..Very simple examples show that the classical solution is ill behaved for discontinuous right-hand side differential equations. For example, if we know that a system (2.3) has a classical solution through each point and if we perturb the system then we cannot state even the existence of (classical) solutions for the new system.

H. Hermes, in [7], was the first to remark that in order that the optimal trajectories of a control system to have a "good" behavior with respect to some sort of perturbations of the feedback system (2.2) it is necessary that they were also Filippov solutions of (2.2). In [7] he defined the concept of "stability to measurements" for a differential system of the form (2.3) and proved that if the system admits a classical solution which is not a Filippov solution then the system is not stable to measurement.

Later on, in [2], P. Brunovský found a necessary and sufficient condition for the optimal trajectories to coincide with the Filippov solutions of the system (2.2) in the case $n = p = 2$; in [3] he proved that the systems (2.2) that have this property are stable in a certain sense to small perturbations of the right-hand side.

In [4] it is proved that if the time-optimal feedback control for the systems in which $p = 1$ defines a regular synthesis in the sense of Boltyanskii ([1], [10]) with an additional transversality condition then the optimal trajectories coincide with the Filippov solutions of (2.2). In [11] and [12] it is proved that this happens if $n \leq 3$ or if the matrix A has only real eigenvalues.

3. The time-optimal observer-feedback control

Definition 3.1

We say that a system of the form (1.7) is a time-optimal observer-feedback control for (1.1) if $v(.): G \longrightarrow U$ is the time-optimal feedback control of (1.1) and $K \in L(R^m, R^n)$ is chosen such that $A - KC^*$ is a stable matrix.

The possibility of construction of a time-optimal or linear or any other kind of observer is given by the existence of some "pole assignement" algoritms. In what follows we need the following special result:

Lemma 3.2

For any completely controllable pair (A^*, C) there exists the real constants $c, k > 0$ such that for any $\alpha > 0$ there exists a matrix $K \in L(R^m, R^n)$ such that:

$$\| \exp(A-KC^*)t \| \leq k \alpha^{n-1} \exp(-\alpha t) \quad , \ t \geq 0 \tag{3.1}$$

$$\| A-KC^* \| \leq c \alpha^n \tag{3.2}$$

Proof:

We consider the positive integer r such that $\text{rank}(C, A^* C, (A^*)^2 C, \ldots (A^*)^{r-1} C) = n$, then we define $\lambda_1, \ldots \lambda_r$ by $\lambda_j = -j\alpha$, $j = 1, 2, \ldots r$; it is easy to see now that $\text{rank}((\lambda_1 I - A^*)^{-1} C, (\lambda_2 I - A^*)^{-1} C, \ldots (\lambda_r I - A^*)^{-1} C) = n$.

hence we may define the invertible matrix $X = col(x_1, x_2, \dots x_n)$ where $x_j = (\lambda_{i_j} I - A^*)^{-1} C_{\ell_j}$, $j = 1,2,\dots n$, C_{ℓ_j} being columns of the matrix C. We define now the following matrices: $Y = X^{-1}$, Z such that $col(C_{\ell_1}, C_{\ell_2}, \dots C_{\ell_n}) = CZ^*$ and $K = -Y^* Z$, $L = diag(\lambda_{i_1}, \lambda_{i_2}, \dots \lambda_{i_n})$.

It is easy to see that $XL = A^* X + CZ^*$ and hence $XLX^{-1} = A^* + CZ^* Y = A^* - CK^*$ and therefore $A-KC^*$ has the eigenvalues $\lambda_{i_1}, \lambda_{i_2}, \dots \lambda_{i_n}$, being a matrix in the diagonal canonical form. The estimates (3.1) and (3.2) follow directly from the construction.

Concerning the performance of a time optimal observer-feedback control, we prove the following result:

Theorem 3.3

Let A, B, C, U that define the input-output control system (1.1)-(1.2) satisfy the conditions (1.3)-(1.4) and let us suppose that the Filippov solutions of (2.2) coincide with the time-optimal trajectories of (2.1).

Then, for any $\varepsilon, \tau > 0$ and any compact subset $G_0 \subset G$ there exists a matrix $K \in L(R^m, R^n)$ satisfying (3.1) and (3.2) such that the time optimal observer (1.8) has the following property: for any $x \in G_0$, any Filippov solution $(\varphi(.;x,0), \psi(.;x,0))$ through $(x,0)$ of (1.8) verifies:

(3.3) $$\| \varphi(t;x,0) \| \leq \varepsilon \quad \text{for} \quad t \geq T(x) + \tau .$$

Proof:

Since the minimal-time function, $T(.):G \longrightarrow R_+$, is continuous, for $\tau > 0$ and the compact subset $G_0 \subset G$ there exists $\theta > 0$ such that for any $x, y \in G_0$ satisfying $\| x-y \| < \theta$

we have:

(3.4) $\quad T(y) < T(x) + \tau$

Obviously, we can take $\theta < \varepsilon$ and we denote:

(3.5) $\quad \tau_1 = \tau + \max\{T(x) \mid x \in G_0\}$

According to theorem 1 in [3], for $\tau_1, \varepsilon > 0$ there exists $\delta > 0$ such that for any $z_0 \in \mathcal{R}(\tau_1)$ and any measurable map, $\eta(.) : [0, \infty) \longrightarrow R^n$ that satisfies:

(3.6) $\quad \| \eta(t) \| \leq \delta$ a.e. on $[0, \infty)$,

any Filippov solution, $\psi_\eta(.;z_0)$, through z_0 of the system:

(3.7) $\quad \dfrac{dz}{dt} = Az + Bv(z) + \eta(t)$

verifies:

(3.8) $\quad \| \psi_u(t;z_0) \| \leq \varepsilon/2 \qquad$ for $t \geq T(z_0)$

We denote $a = \| A \|$, $L = \max\{\| Bu \| \mid u \in U\}$.

For continuity reasons, it is easy to see that for $\theta > 0$ and the compact subset $G_0 \subset G$ there exists $\tau_0 > 0$ such that:

(3.9) $\quad (\| x \| + L/a)(\exp(at) - 1) < \theta/2$ for $t \in [0, \tau_0]$, $x \in G_0$. We take $\tau_0 < \tau$.

Let $c, k > 0$ be the constants associated to the completely controllable pair (A^*, C) by lemma 3.2.

Since $\alpha^p \exp(-\alpha t) \longrightarrow 0$ as $\alpha \to \infty$ for any positive integer p and any $t > 0$ it follows that for $c, k\, \tau, \theta > 0$ and the compact subset $G_0 \subset G$ there exists $\alpha > 0$ such that:

(3.10) $\quad (c\alpha^n + a)k\alpha^{n-1}\exp(-\alpha t)\| x \| < \delta$ for $t \geq \tau_0$ and $x \in G_0$

(3.11) $\quad k\alpha^{n-1}\exp(-\alpha t)\| x \| < \theta/2 \qquad$ for $t \geq \tau_0$ and $x \in G_0$

Let $K \in L(R^m, R^n)$ be the matrix corresponding to (A^*, C) and $\alpha > 0$ that satisfies (3.10) and (3.11) according to lemma 3.2 .

Let $x_o \in G_o$ and $(\varphi(.;x_o,0), \psi(.;x_o,0))$ be a Filippov solution through $(x_o,0)$ of the system (1.8). It follows that $\varphi(.;x_o,0)$ is a Filippov solution through x_o of the system:

$$\frac{dx}{dt} = Ax + Bv(\psi(t;x_o,0)) \tag{3.12}$$

According to lemma 4 in [5], there exists an integrable map, $\chi(.):[0,\infty) \longrightarrow R^n$, such that $\varphi(t;x_o,0) = x_o + \int_0^t \chi(s)ds$ and $\chi(s) \in A\varphi(s;x_o,0) + BU$ for $s \in [0,\infty)$ and therefore there exists a measurable map, $w(.):[0,\infty) \longrightarrow BU$ ($w(s) = \chi(s) - A\varphi(s;x_o,0)$) such that:

$$\varphi(t;x_o,0) = x_o + \int_0^t [A\varphi(s;x_o,0) + w(s)]ds \tag{3.13}$$

hence $\varphi(.;x_o,0)$ is a classical solution of the system: $\frac{dx}{dt} = Ax + w(t)$. From the variaton of the constants formula it follows that : $\varphi(t;x_o,0) = \exp(tA)x_o + \int_0^t \exp((t-s)A)w(s)ds$

and therefore, since $w(s) \in BU$ for $s \in [0,\infty)$, we have:

$$\| \varphi(t;x_o,0) - x_o \| \leqslant (\|x_o\| + L/a)(\exp(at)-1), t \geqslant 0 \tag{3.14}$$

On the other hand, from (1.8) it follows that the map $\xi(.;x_o):[0,\infty) \longrightarrow R^n$, given by:

$$\xi(t;x_o) = \psi(t;x_o,0) - \varphi(t;x_o,0) \tag{3.15}$$

is the solution through $-x_o$ of the system:

$$\frac{dx}{dt} = (A - KC^*)x \tag{3.16}$$

and therefore , from (3.2) it follows that:

$$\| \xi(t;x_o) \| \leqslant k\alpha^{n-1}\exp(-\alpha t)\|x_o\| \text{ , for } t \geqslant 0,\ x_o \in G_o. \tag{3.17}$$

From (3.14) and (3.17) it follows that for $\tau_o > 0$ satisfying (3.9) we have:

(3.18) $\| \psi(\tau_0;x_0,0)-x_0 \| \leq k\alpha^{n-1}\exp(-\alpha\tau_0)\|x_0\| + (\|x_0\| + L/a)(\exp(a\tau_0) - 1).$

If we take $\alpha > 0$ satisfying (3.10) and (3.11), we have:

(3.19) $\| \psi(\tau_0;x_0,0) - x_0 \| < \theta$

and, using (3.4), it follows:

(3.20) $T(\psi(\tau_0;x_0,0)) < T(x_0) + \tau$

Further on, since $\psi(\cdot;x_0,0)$ is a Filippov solution through $0 \in R^n$ of the system:

(3.21) $\frac{dz}{dt} = Az + Bv(z) - KC^{*}\xi(t;x_0)$

and since $\|KC^{*}\| \leq \|A - KC^{*}\| + \|A\|$, from (3.1) and (3.16) it follows:

(3.22) $\| KC^{*}\xi(t;x_0) \| \leq (c\alpha^{n} + a)k\alpha^{n-1}\exp(-\alpha t)\|x_0\|$ for $t \geq 0, x \in G_0$

and if $\alpha > 0$ satisfies (3.10) then $\eta(t) = -KC\xi(t;x_0)$ satisfies:

(3.23) $\| \eta(t) \| \leq \delta$ for $t \geq \tau_0$

According to theorem 1 in [3], for any $x_0 \in G_0$, any Filippov solution, $\psi(\cdot;x_0,0)$, of (3.21) verifies:

(3.24) $\| \psi(t;x_0,0) \| \leq \varepsilon/2$ for $t \geq T(x_0) + \tau$

(according to (3.20), $T(x_0) + \tau > T(\psi(\tau_0;x_0,0))$.

On the other hand, since $\tau_0 \leq \tau$ and $\theta < \varepsilon$ from (3.11) and (3.17) it follows that $\| \xi(t;x_0) \| < \theta/2 < \varepsilon/2$ for $t \geq T(x_0) + \tau \geq \tau > \tau_0$ and therefore: $\| \varphi(t;x_0,0) \| \leq$
$\| \varphi(t;x_0,0) - \psi(t;x_0,0) \| + \| \psi(t;x_0,0) \| \leq \| \xi(t;x_0) \| + \| \psi(t;x_0,0) \| < \varepsilon$ for $t \geq T(x_0) + \tau$ and the theorem is completely proved.

Remark 3.4

The proof of the theorem does not suggest, unfortunately, methods to estimate the constant α (and therefore the matrix K which defines the observer) when the "admi$sible error", $\varepsilon > 0$, the "delay", $\tau > 0$, and the compact subset $G_0 \subset G$ are given.

In practical applications one choses normally the best possible state estimator (1.8) with respect to some other criteria than the mathematical motivations in theorem 3.3 such as some technological constraints, the computing facilities at hand, etc. . The theorem above suggets the fact that such a regulator will stabilize the system for the initial states in some neighborhood of the origin when an admiisible error, $\varepsilon > 0$ and an admissible delay, $\tau > 0$, are given.

A more suitable result from this point of view is the following:

Theorem 3.5

Let the hypotheses of theorem 3.3 be satisfied. Then, for any $\varepsilon, \tau, \alpha > 0$ there exists a matrix $K \in L(R^m, R^n)$ that satisfies (3.1) and (3.2) and a compact neighborhood of the origin, $G_0 \subset G$, such that for any $x_0 \in G_0$, any Filippov solution through $(x_0, 0)$, $(\varphi(.;x_0,0), \psi(.;x_0,0))$, of (1.8) satisfies the estimation (3.3).

Proof:

For every θ , $r > 0$ we define the real number:

$$(3.25) \qquad \omega(\theta, r) = \sup\{|T(y) - T(x)| \ ,\ x,y \in \bar{B}_r(0),\ y \in \bar{B}_\theta(x)\}$$

where $\bar{B}_r(0)$ and $\bar{B}_\theta(x)$ denote the closed balls of radius r and θ centered at 0 and x respectively.

Since the minimal-time function, $T(.): G \longrightarrow R_+$, is continuous, for each $r > 0$, the map $\theta \longmapsto \omega(\theta, r)$ is non-decreasing and $\lim \omega(\theta, r) = 0$ as $\theta \searrow 0$. From (3.25) it is obvious that for any $\theta > 0$, the map $r \longmapsto \omega(\theta, r)$ is non-decreasing too.

For any $r > 0$ we denote $s(r) = \max\{T(x) \mid x \in \bar{B}_r(0)\}$. According to theorem 1 in [3] , for ε , $s(r) > 0$ there exists $\delta(\varepsilon, r) > 0$ such that for any measurable map $\eta(\cdot): [0, \infty) \longrightarrow R^n$ that satisfies (3.6) , for any $z_0 \in \mathcal{R}(s(r))$, any Filippov solution, $\Psi_\eta(\cdot; z_0)$, through z_0 of the system (3.7) satisfies the estimation (3.8).

Obviously, for any fixed $\varepsilon > 0$, the map $r \longrightarrow \delta(\varepsilon, r)$ is non-increasing and for any fixed $r > 0$, $\lim \delta(\varepsilon, r) = 0$ as $\varepsilon \searrow 0$.

Let τ , $\varepsilon > 0$ be fixed; for any $r > 0$ we define now $\theta(r) \in (0, \infty]$ by

(3.27) $\theta(r) = \sup\{\theta \mid \omega(\theta, r) \leq \tau\}$

and $r_0 \in (0, \infty]$ by:

(3.28) $r_0 = \sup\{r > 0 \mid \theta(r) \leq \varepsilon\}$

From the properties above of the function $\omega(.,.)$ it is obvious that $r_0 > 0$. We denote $\theta_0 = \theta(r_0) \leq \varepsilon$ and $\delta_0 = \delta(\varepsilon, r_0)$

From continuity reasons and the properties of the functions $\omega(.,.)$ and $\delta(.,.)$ it follows that for the constants a, L, k, c > 0, defined in the proof of theorem 3.3 there exist $r_1 \leq r_0$ and $\tau_1 \leq \tau$ such that :

(3.28) $(r_1 + L/a)(\exp(a\tau_1) - 1) < \theta_0/2$

(3.29) $(c\alpha^n + a)k\alpha^{n-1}\exp(-\alpha\tau_1)r_1 < \delta_0 < \delta(\varepsilon, r_1)$

(3.30) $k\alpha^{n-1}\exp(-\alpha\tau_1)r_1 < \theta_0/2$

It follows immediately now that for any $x \in G_0 = \overline{B}_{r_1}(0)$, the statement in theprem 3.3 holds and therefore the theorem 3.5 is completely proved.

Corollary 3.6

Let us consider the system (1.1)-(1.2) in the case $p = 1$, $U = [-1,+1]$, $n \leq 3$ or, if $n > 3$ then the mat.. ..x A which defines (1.1) has only real eigenvalues.

Then the statements in the theorems 3.3 and 3.5 hold.

Proof:

According to the theorems proved in [11] and [12] , the time-optimal feedback control for the systems described above defines a regular synthesis in the sense of Boltyanskii ([1] , [10]) On the other hand, the theorems proved in [4] state that for such a system the Filippov solutions of (2.2) coincide with the optimal trajectories and therefore the hypothesis of theorem 3.3 is satisfied.

Corollary 3.7

Let us consider the system (1.1)-(1.2) in the case $n = p = 2$ and the polyhedron $U \subset R^2$ and the matrix A are such that for any vertex v of U the polar cone corresponding to v ,

$\Lambda_v = \{ \lambda \in R^2 \mid < \lambda , Bv > \geqslant < \lambda , Bv' > \text{ for any vertex } v' \text{ of } U \}$,

intersects only that eigenspace of $-A^*$ which corresponds to its largest eigenvalue. Then the statements in the theorems 3.3 and 3.5 hold.

Proof:

According to the theorem proved in [2] , for such systems the Filippov solutions of (2.2) coincide with the optimal trajectories and therefore the hypothesis oh the theorem 3.3 is satisfied.

Remark 3.8

From the point of view of the theory of Ordinary Differential Equations, the property in the theorem 3.3 may be interpreted as follows: the first component, $\varphi(.;x_0,0)$, of the solution through $(x_0,0)$ of (1.18) is a Filippov solution through x_0 of the differential system:

$$(3.31) \qquad \frac{dx}{dt} = Ax + Bv(x+\xi(t;x_0))$$

where $\xi(.;x_0)$ is defined by (3.15), which is obtained from (2.2) by adding to the argument of $v(.)$ the "perturbation" $\xi(.;x_0)$. This perturbation is produced by the fact that one cannot observe the state x and only the output y is available.

Thus, the properties in theorems 3.3 and 3.5 may be interpreted as "stability to observations" of the differential system (2.2).

The property called "stability to measurements" introduced by H. Hermes in [7] means, for the same system, that the classical solutions (when they exist) of the perturbed systems:

$$(3.32) \qquad \frac{dx}{dt} = A(x+\xi(t)) + Bv(x+\xi(t))$$

are "close" to the optimal trajectories.

Hence, the stability to measurements assumes the possibility of observing and therefore of measuring the state, in contrast to the problem studied in this paper where the state cannot be measured.

On the other hand, the property of stability to measurements does not say anything about the Filippov solutions of the perturbed system (3.31) (or (3.32)) which may differ from the classical ones, although, for the unperturbed system, (2.2) these solutions coincide.

From this point of view the "stability" studied in [3] and in this paper is, apparently, more natural. It would be, though, interesting, to find a deeper relationship between the two kinds of stability of the solutions of the differential system (2.2) at the perturbations of the right-hand side.

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