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ON BEST APPROXIMATION IN VECTOR - VALUED NORMS

by

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On best approximation in vector-valued norms

by

A. Bacopoulos, G. Godini and I. Singer

1. It is well known that in a (real or complex) normed linear space E the distance from an element $x \in E$ to a subset G of E is defined by the formula

(1) $$\operatorname{dist}(x,G) = \inf \{ \|x-g\| \mid g \in G \}$$

and the elements of best approximation of x, by means of the elements of G, are, by definition, those $g_0 \in G$ (if any), for which this inf is attained, i.e. for which

(2) $$\|x-g_0\| = \inf \{ \|x-g\| \mid g \in G \} = \operatorname{dist}(x,G);$$

the set of all such elements g_0 is denoted by $\mathcal{P}_G(x)$ (see e.g. [9] or [10]).

In many situations, there appears the necessity to approximate an element x of a linear space E, by means of the elements of a subset G of E, simultaneously in two (or more) given norms, say $\|\cdot\|_1$ and $\|\cdot\|_2$, on E. One possible way of doing this, is to consider a third norm $\|\cdot\|$ on E, which is a suitable combination of the norms $\|\cdot\|_1$ and $\|\cdot\|_2$, for example,

(3) $$\|x\| = \max(\|x\|_1, \|x\|_2) \qquad (x \in E),$$

or

(4) $$\|x\| = \|x\|_1 + \|x\|_2 \qquad (x \in E),$$

and then to consider (1) and (2) for this new norm; for results and literature in this direction, see e.g. [4] and the references therein.

In a large number of cases, however, another approach, suggested by some problems of mathematical economics(see e.g. [8]), appears more naturally. Namely, following [1], one can consider on E the "norm" with values in the plane R^2 (with its natural partial ordering), defined by

$$(5) \qquad [\![x]\!] = (\|x\|_1, \|x\|_2) \qquad (x \in E)$$

(for the theory of norms with values in partially ordered linear spaces see e.g. [7]) and to call an element $g_o \in G$ an <u>element of best vectorial approximation</u> of x, by means of the elements of G, if there is no other $g \in G$ which gives a "strictly better" approximation to x in the sense of the natural partial ordering of R^2 (we recall that $(\alpha_1,\alpha_2) \leq (\beta_1,\beta_2)$ if and only if both $\alpha_1 \leq \alpha_2$ and $\beta_1 \leq \beta_2$; one writes $(\alpha_1,\alpha_2) < (\beta_1,\beta_2)$, if $(\alpha_1,\alpha_2) \leq (\beta_1,\beta_2)$ and $(\alpha_1,\alpha_2) \neq \neq (\beta_1,\beta_2)$). We shall denote the set of all such $g_o \in G$ by $\mathcal{V}_G(x)$; thus, $g_o \in \mathcal{V}_G(x)$ if and only if there exists no element $g \in G$ such that

$$(6) \qquad [\![x-g]\!] = (\|x-g\|_1, \|x-g\|_2) < (\|x-g_o\|_1, \|x-g_o\|_2) = [\![x-g_o]\!].$$

Let us observe that instead of two scalar-valued norms $\|\cdot\|_1$ and $\|\cdot\|_2$ (or, equivalently, instead of the norm (5) with values in the plane R^2, with its natural order) one can give a similar definition of $\mathcal{V}_G(x)$ for the case of any (finite or infinite) number of scalar-valued norms (or, more generally, for the case of norms with values in any partially ordered linear space); in particular, a usual (scalar-valued) norm is a norm with values in the real line R and then $\mathcal{V}_G(x)$ reduces to the set $\mathcal{P}_G(x)$ defined by (2) (note that $\mathcal{V}_G(x) = \mathcal{P}_G(x)$ may also happen for the vector-valued norm (5), namely, in the particular case when $\|\cdot\|_1 = \|\cdot\|_2 = \|\cdot\|$).

Some results on the elements of best approximation by elements of convex sets in arbitrary linear spaces endowed with a vector-valued norm and, more generally, on convex vectorial optimization, have been given in [2],[5]. In the present paper we shall consider, generalizing (1), a new notion of "distance" for the case of norms with values in R^2. We shall derive this from a new notion of the "infimum" of any set $A \subset R^2$, which will no longer be a point, but a set in the plane R^2, denoted by INF A (see definition 1 below). Then our new "distance", generalizing (1), will be the set

$$(7) \qquad \text{DIST}(x,G) = \text{INF}\{[\![x-g]\!] \mid g \in G\} = \text{INF}\{(\|x-g\|_1, \|x-g\|_2) \mid g \in G\} \subset R^2$$

and we shall have $g_o \in \mathcal{V}_G(x)$ if and only if

$$(8) \qquad [\![x-g_o]\!] = (\|x-g_o\|_1, \|x-g_o\|_2) \in \text{DIST}(x,G);$$

of course, in general, DIST (x,G) may also contain elements of the form $\lim_{n\to\infty} [\![x-g_n]\!]$, where $\{g_n\} \subset G$.

In the present paper we shall give some results on the elements of INF A, for those sets $A \subset R^2$ which have a certain property of generalized convexity, which we shall call "property (C)". When G is a convex subset of E, the sets (7) need not be convex, but they will have property (C) and therefore we shall be able to derive from our theorems on INF A the results of [2] on best vectorial approximation by elements of convex sets G.

The proofs of the results on INF A and of their applications to best vectorial approximation by convex sets (and, more generally, to convex vectorial optimization), which are not given here, as well as some additional results on INF A, for sets $A \subset R^2$ which do not have property (C), will be given elsewhere [3].

2. We shall assume, without any special mention, that all sets $A \subset R^2$ occurring in the sequel are non-empty and bounded from below in the sense of the natural partial order of R^2 (i.e., there exists $q \in R^2$ such that $q \leq a$ for all $a \in A$). We shall denote by $\overline{A}$ the closure of A in the usual topology of R^2.

<u>Definition 1</u>. Let $A \subset R^2$ and let $p \in R^2$. We shall say that $p \in$ INF A if the following two conditions are satisfied:

1^o. There exists no $a \in A$ such that $a < p$.

2^o. $p \in \bar{A}$.

Let us observe that such a definition can be also given if we replace R^2 by any partially ordered topological space. One can also define a related notion of "infimum" of a set $A \subset R^2$ (or, more generally, of a set A in a partially ordered linear space with a "unit element" e), by conserving 1^o and modifying condition 2^o as follows:

$2'$. For each $\varepsilon > 0$ there exits an element $a \in A$ such that $a \leqslant p + \varepsilon e$ (where $e = (1,1)$ in R^2).

Obviously, in general this latter definition yields a larger set than the one obtained from 1^o and 2^o. The results which we shall give below for the sets INF A defined by 1^o and 2^o remain also valid, with easy modifications, if we use 1^o and $2'$. Other modifications of 2^o, $2'$ are also possible, but we shall not consider them here.

One can show that for any $A \subset R^2$ (of course, satisfying the assumptions mentioned before definition 1), INF A $\neq \emptyset$.

Dually, one can also define, in an obvious way, the set SUP $A \subset R^2$ and the results which we shall give below for INF A, admit dual results for SUP A. We believe that these concepts and results may be also of interest for other applications of the theory of partially ordered spaces.

Let us observe that our definition of INF A is different from that of inf A occurring in the theory of partially ordered spaces. Namely, we recall that inf A is the unique element $m \in R^2$ with the following two properties: i) $m \leqslant a$ for all $a \in A$; ii) for each $p \in R^2$ such that $p \leqslant a$ for all $a \in A$, we have $p \leqslant m$. Thus, if $A \subset R^2$, then $m = (m_1, m_2)$, where

$$(9) \qquad m_1 = \inf\{a_1 \mid (a_1, a_2) \in A\}, \quad m_2 = \inf\{a_2 \mid (a_1, a_2) \in A\}.$$

In the sequel we shall also use m_1, m_2 and the constants γ_1, γ_2 defined by

(10) $$\gamma_1 = \inf\{\bar{a}_1 \mid (\bar{a}_1, m_2) \in \bar{A}\}, \quad \gamma_2 = \inf\{\bar{a}_2 \mid (m_1, \bar{a}_2) \in \bar{A}\};$$

here (as well as in the sequel) we adopt the convention $\inf\{b \mid b \in \emptyset\} = +\infty$. Note that, clearly, $m_1 \leq \gamma_1 \leq \inf\{a_1 \mid (a_1, m_2) \in A\}$ (the latter can be also $+\infty$) and, dually, $m_2 \leq \gamma_2 \leq \inf\{a_2 \mid (m_1, a_2) \in A\}$.

Definition 2. We shall say that a set $A \subset R^2$ has property (C), if for each $a', a'' \in A$ and each λ with $0 < \lambda < 1$ there exists an element $a \in A$ such that

(11) $$a \leq \lambda a' + (1-\lambda) a''.$$

For example, every convex set $A \subset R^2$ has property (C) and the closure $\bar{A}$ of any set A with property (C) also has this property. Furthermore, as we have already mentioned in § 1, for any convex subset G of E, the (not necessarily convex) set (7) has property (C).

In our first theorem we describe INF A, for $A \subset R^2$ having property (C), by determining its intersections with the vertical and horizontal lines in the plane R^2, i.e., the sets $D'_{p_1} \cap \text{INF } A$ and $D''_{p_2} \cap \text{INF } A$, where

(12) $$D'_{p_1} = \{(p_1, p_2) \in R^2 \mid -\infty < p_2 < +\infty\} \quad (-\infty < p_1 < +\infty),$$

(13) $$D''_{p_2} = \{(p_1, p_2) \in R^2 \mid -\infty < p_1 < +\infty\} \quad (-\infty < p_2 < +\infty).$$

Theorem 1. *Let $A \subset R^2$ be a set with property (C) and let $-\infty < p_1 < +\infty$. Then*

(14) $$D'_{p_1} \cap \text{INF } A = \begin{cases} \emptyset \ \text{if} \ -\infty < p_1 < m_1 \\ \bar{A} \cap \{(m_1, p_2) \mid \gamma_2 \leq p_2 \leq \inf\{a_2 \mid (m_1, a_2) \in A\}\} \ \text{if} \ p_1 = m_1 \\ (p_1, \inf\{a_2 \mid (a_1, a_2) \in A, \ a_1 \leq p_1\}) = \\ \quad = (p_1, \inf\{\bar{a}_2 \mid (p_1, \bar{a}_2) \in \bar{A}\}) < (p_1, +\infty) \ \text{if} \ m_1 < p_1 \leq \gamma_1 \\ (p_1, m_2) \ \text{if} \ \gamma_1 < p_1 \leq \inf\{a_1 \mid (a_1, m_2) \in A\} \ \text{and} \ (p_1, m_2) \in \bar{A} \\ \emptyset \quad \text{if} \ \gamma_1 < p_1 \leq \inf\{a_1 \mid (a_1, m_2) \in A\} \ \text{and} \ (p_1, m_2) \notin \bar{A} \\ \emptyset \quad \text{if} \ \inf\{a_1 \mid (a_1, m_2) \in A\} < p_1 < +\infty, \end{cases}$$

where $(p_1,p_2)<(p_1,+\infty)$ means that $p_2<+\infty$.

For $D''_{p_2} \cap \mathrm{INF}\ A$ the dual formulae hold (i.e., in which the indices 1 and 2 and the first and second coordinates are interchanged).

Note that if $\gamma_1 = +\infty$ or $\inf\{a_1 \mid (a_1,m_2)\in A\} = +\infty$, then (14) admits some obvious simplifications. Let us also observe that it is only the part $m_1<p_1\leq\gamma_1$ where the assumption that A has property (C) cannot be omitted.

Clearly, from theorem 1 it follows that for $A\subset R^2$ with property (C), INF A is closed.

It may also happen that $m_1 = \gamma_1$ (or $m_2 = \gamma_2$), but then, necessarily, $m_2 = \gamma_2$ (respectively, $m_1=\gamma_1$) and $\mathrm{INF}\ A = I_1\cup I_2$, where

(15) $$I_1 = \{(m_1,p_2)\in \bar{A} \mid \gamma_2 \leq p_2 \leq \inf\{a_2 \mid (m_1,a_2)\in A\}\},$$

(16) $$I_2 = \{(p_1,m_2)\in \bar{A} \mid \gamma_1 \leq p_1 \leq \inf\{a_1 \mid (a_1,m_2)\in A\}\};$$

clearly, in this case $I_1\cap I_2$ is the point (m_1,m_2). In the sequel we shall exclude this simple case, i.e., we shall assume that $m_1<\gamma_1$ (and $m_2<\gamma_2$).

By theorem 1, for each p_1 with $m_1<p_1<\gamma_1$ the set $D'_{p_1}\cap \mathrm{INF}\ A$ is the single point $(p_1,\ \inf\{a_2 \mid (a_1,a_2)\in A,\ a_1\leq p_1\})\in R^2$ and therefore, in the next theorem we shall consider the real-valued function f defined by

(17) $$f(p_1) = \inf\{a_2 \mid (a_1,a_2)\in A,\ a_1\leq p_1\} \qquad (m_1<p_1<\gamma_1).$$

In terms of the function f, theorem 1 yields that $\mathrm{INF}\ A = \Gamma(f)\cup I_1\cup I_2$, where $\Gamma(f)\subset R^2$ is the graph of f and I_1, I_2 are the sets (15), (16); if $m_1<\gamma_1$, these three sets are pairwise disjoint.

Theorem 2. Let $A\subset R^2$ be a set with property (C) and with $m_1<\gamma_1$. Then the function f defined by (17) is convex (hence continuous) and non-increasing.

We note that if $\gamma_2<+\infty$ (respectively, $\gamma_1<+\infty$), then $(m_1,\gamma_2)\in \mathrm{INF}\ A$ (respectively, $(\gamma_1,m_2)\in \mathrm{INF}\ A$) and therefore, in these cases it is natural to extend the definition of f by putting $f(m_1) = \gamma_2$ (respectively, $f(\gamma_1) = m_2$). Then the function f extended in this way will still remain convex, continuous and non-increasing. From theorem 1 and from this remark it follows, in particular, that <u>if</u> $\gamma_2 = \inf \{a_2 \mid (m_1, a_2) \in A\}$ <u>and</u> $\gamma_1 = \inf \{a_1 \mid (a_1, m_2) \in A\}$ <u>and if both</u> $\gamma_1, \gamma_2 < +\infty$, <u>then</u> INF A <u>is either an arc or a point</u>.

Theorem 1 above permits to reduce the computation of INF A $\cap$ $\cap \{(p_1,p_2) \in R^2 \mid m_1 < p_1 < \gamma_1\}$, to the computation of the infimum of a certain set of scalars (or, alternatively, of a functional on a subset of A, since a_2 may be also regarded as a functional on $A \cap \{(a_1,a_2) \in R^2 \mid a_1 \leq p_1\}$). The following theorem gives another "scalarization", of "Kuhn-Tucker type" (see [2]), for the problem of finding INF $A \cap \{(p_1,p_2) \in R^2 \mid m_1 < p_1 < \gamma_1\}$:

<u>Theorem 3</u>. <u>Let</u> $A \subset R^2$ <u>be a set with property</u> (C) <u>and with</u> $m_1 < \gamma_1$. <u>Then for every</u> $(p_1,p_2) \in$ INF A <u>with</u> $m_1 < p_1 < \gamma_1$ <u>there exists a number</u> λ <u>with</u> $0 \leq \lambda \leq 1$ (<u>depending on</u> (p_1,p_2)) <u>such that</u>

$$\lambda p_1 + (1-\lambda)p_2 = \inf \{\lambda a_1 + (1-\lambda)a_2 \mid (a_1,a_2) \in A\}. \tag{18}$$

The following partial converse is immediate: If $A \subset R^2$ (not necessarily with property (C)) and $(p_1,p_2) \in \bar{A}$ and if there exists λ with $0 < \lambda < 1$ such that we have (18), then $(p_1,p_2) \in$ INF A; as shown by simple examples, here the assumption $(p_1,p_2) \in \bar{A}$ cannot be omitted

3. Now we shall show how the above results can be applied to obtain again the results of [2] on best vectorial approximation.

Let G be a convex set in a linear space E endowed with two norms, $\|\cdot\|_1$ and $\|\cdot\|_2$ and let $x \in E$. We shall denote by dist_1 and dist_2 the corresponding distances and by $\mathcal{P}_G^1(x)$ and $\mathcal{P}_G^2(x)$ the set of all elements of best approximation of x by means of the elements of G, in the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ respectively. Let

(19) $A = \{(\|x-g\|_1, \|x-g\|_2) \mid g \in G\} \subset R^2.$

Then, by (9) and (10),

(20) $m_1 = \inf\{\|x-g\|_1 \mid g \in G\} = \text{dist}_1(x,G), \ m_2 = \text{dist}_2(x,G),$

(21) $\gamma_1 = \inf\{\underline{\lim}\, \|x-g_n\|_1 \mid \{g_n\} \subset G, \ \lim_{n\to\infty} \|x-g_n\|_2 = \text{dist}_2(x,G)\},$

(22) $\gamma_2 = \inf\{\underline{\lim}\, \|x-g_n\|_2 \mid \{g_n\} \subset G, \ \lim_{n\to\infty} \|x-g_n\|_1 = \text{dist}_1(x,G)\}.$

Furthermore, the other expressions occurring in (14) are

(23) $\inf\{a_2 \mid (m_1,a_2) \in A\} = \inf\{\|x-g\|_2 \mid g \in \mathcal{P}_G^1(x)\} = \text{dist}_2(x,\mathcal{P}_G^1(x)),$

(24) $\inf\{a_2 \mid (a_1,a_2) \in A, \ a_1 \leqslant p_1\} = \inf\{\|x-g\|_2 \mid g \in G, \|x-g\|_1 \leqslant p_1\},$

(25) $\inf\{a_1 \mid (a_1,m_2) \in A\} = \inf\{\|x-g\|_1 \mid g \in \mathcal{P}_G^2(x)\} = \text{dist}_1(x,\mathcal{P}_G^2(x)),$

and the obvious dual to (24).

We recall the main result of [2] on best vectorial approximation ([2], theorem 2.1): <u>For every</u> $c \in R = (-\infty, +\infty)$ <u>satisfying</u>

(26) $\text{dist}_1(x,G) \leqslant c \leqslant \text{dist}_1(x,\mathcal{P}_G^2(x)),$

<u>we have</u>

(27) $\mathcal{V}_G(x) \cap \{y \in E \mid \|x-y\|_1 = c\} = \mathcal{P}^2_{G\cap\{y \in E \mid \|x-y\|_1 \leqslant c\}}(x).$

In order to deduce this theorem from theorem 1 above, the following obvious observation is used: we have $g_0 \in \mathcal{V}_G(x)$ if and only if $(\|x-g_0\|_1, \|x-g_0\|_2) \in D'_{\|x-g_0\|_1} \cap \text{INF } A$. Now, the inclusion $\subset$ in (27) follows immediately by considering the various cases occurring in (14) and taking into account (20)-(25). To prove the opposite inclusion, let $g_0 \in \mathcal{P}^2_{G\cap\{y\in E \mid \|x-y\|_1 \leqslant c\}}(x)$, i.e., $g_0 \in G$, $\|x-g_0\|_1 \leqslant c$ and $\|x-g_0\|_2 = \inf\{\|x-g\|_2 \mid g \in G, \|x-g\|_1 \leqslant c\}$.

<u>Case 1°</u>. If $\|x-g_0\|_2 > \text{dist}_2(x,G)$, then, using (21), it follows that $c \leqslant \gamma_1$. Now, if $\text{dist}_1(x,G) = c$, then $\|x-g_0\|_1 = c$ and $\|x-g_0\|_2 = \text{dist}_2(x,\mathcal{P}_G^1(x))$, whence, by (14) and (20), (23) we get $(\|x-g_0\|_1, \|x-g_0\|_2) \in \text{INF } A$. On the other hand, if $m_1 = \text{dist}_1(x,G) < c \leqslant \gamma_1$, then, by (14) and (24), $(c, \|x-g_0\|_2) \in \text{INF } A$. Hence, since $(\|x-g_0\|_1, \|x-g_0\|_2) \leqslant$

$\leqslant (c, \|x-g_o\|_2)$, where $(\|x-g_o\|_1, \|x-g_o\|_2) \in A$, we obtain $\|x-g_o\|_1 = c$ and $(\|x-g_o\|_1, \|x-g_o\|_2) \in \mathrm{INF}\ A$.

<u>Case 2^o</u>. If $\|x-g_o\|_2 = \mathrm{dist}_2(x,G)$, then $g_o \in \mathcal{P}^2_G(x)$, whence, using also (26), $\|x-g_o\|_1 \leqslant c \leqslant \inf\{\|x-g\|_1 \mid g \in \mathcal{P}^2_G(x)\} \leqslant \|x-g_o\|_1$, so $\|x-g_o\|_1 = c$. Also, from (21), $g_o \in \mathcal{P}^2_G(x)$ and (26), we get $\gamma_1 \leqslant c \leqslant \mathrm{dist}_1(x, \mathcal{P}^2_G(x))$. Hence, by (14) and (25), $(\|x-g_o\|_1, \|x-g_o\|_2) \in \mathrm{INF}\ A$ whenever $\gamma_1 < c$, while if $\gamma_1 = c$ $(< +\infty)$, then $(\|x-g_o\|_1, \|x-g_o\|_2) = (\gamma_1, m_2) \in \mathrm{INF}\ A$, which completes the proof.

From (25) and the observation made after (10) it follows that $\gamma_1 \leqslant \mathrm{dist}_1(x, \mathcal{P}^2_G(x))$. We note that if for every c such that $\gamma_1 < c < \mathrm{dist}_1(x, \mathcal{P}^2_G(x))$, the right hand side of (27) is non-empty, then already $\gamma_1 = \mathrm{dist}_1(x, \mathcal{P}^2_G(x))$. (Indeed, if $\gamma_1 < c < c' < \mathrm{dist}_1(x, \mathcal{P}^2_G(x))$, and $g_o \in \mathcal{P}^2_{G \cap \{y \in E \mid \|x-y\|_1 \leqslant c\}}(x)$, $g'_o \in \mathcal{P}^2_{G \cap \{y \in E \mid \|x-y\|_1 \leqslant c'\}}(x)$, then, by (27), $g_o, g'_o \in \mathcal{V}_G(x)$ and $\|x-g_o\|_1 = c$, $\|x-g'_o\|_1 = c'$. But then, by (14), $\|x-g_o\|_2 = \|x-g'_o\|_2 = m_2$. Since $c < c'$, this contradicts the definition of $\mathcal{V}_G(x)$).

4. There appears the problem, how to extend in a natural way some well known properties (see e.g. [9]) of the numbers (1) to the case of vector-valued norms, i.e., to the sets (7). For example, if we want to extend the implications

$$(28) \qquad G_1 \subset G_2 \Rightarrow \mathrm{dist}(x, G_1) \geqslant \mathrm{dist}(x, G_2),$$

$$(29) \qquad G_1 \subset G_2 \subset \ldots, \overline{\bigcup_{n=1}^{\infty} G_n} = E \Rightarrow \lim_{n \to \infty} \mathrm{dist}(x, G_n) = 0,$$

then it is necessary to define first a suitable partial order relation $\geqslant$, respectively a suitable topology, at least for the collection of all sets in R^2 of the form (7). Or, one can ask, how to extend to vector-valued norms the inequality

$$(30) \qquad \mathrm{dist}(x_1 + x_2, G) \leqslant \mathrm{dist}(x_1, G) + \mathrm{dist}(x_2, G),$$

where G is a linear subspace. Some results in this direction will be given in [6].

<u>Note</u>. After this paper has been completed, there appeared the

paper of L.Cesari and M.B.Suryanarayana, "Existence theorems for Pareto optimization in Banach spaces" (Bull.Amer.Math.Soc.82(1976), 306-308), in which the authors introduce, for a different problem, the notions of weak and strong Pareto extremum of a set A in a Banach space Z with a closed convex cone. For $Z = R^2$, with the natural positive cone, the set of all these Pareto extrema coincides with our INF A (since in R^2 the weak and norm topologies coincide), but there is no other overlapping of that paper with our present paper.

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