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HOLOMORPHICALLY INDUCED REPRESENTATIONS OF SOLVABLE LIE GROUPS

by

HENRI MOSCOVICI and ANDREI VERONA

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Henri MOSCOVICI [*] and Andrei VERONA [*]

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[*] Institute of Mathematics, str.Academiei,14 , Bucharest,1

ABSTRACT

The holomorphically induced representations of a simply connected solvable Lie group,arising by Kostant's quantization procedure from Hamiltonian G-spaces,are studied.Our main result gives necessary and sufficient conditions for such a representation to be primary,primary of type I,respectively irreducible.It is shown further that the irreducible unitary representations constructed by this method are sufficiently many to separate the elements of G.

INTRODUCTION

In this paper we are going to investigate some important properties of the holomorphically induced representations of a simply connected solvable Lie group which arise by Kostant's quantization procedure.For details concerning the quantization theory,the reader is referred to [4] and [12].

Let G be a connected and simply connected Lie group and let (X,ω) be a Hamiltonian G-space.If (L,α) is a complex line bundle with connection and invariant Hermitian structure over X, we denote by $\ell=[L,\alpha]$ its equivalence class. $\Lambda(X,\omega)$ stands for the set of all equivalence classes $\ell=[L,\alpha]$ whose curvature form is ω. The quantization procedure associates to a pair (ℓ,F) consisting of an element $\ell\in\Lambda(X,\omega)$ and a G-invariant polarization F of the symplectic manifold (X,ω) an equivalence class of unitary representations $\rho(\ell,F)$ of G. One of the fundamental questions one can raise about this construction is to find out first to what extent it is independent of the choice of a polarization and then to determine the structure of the representations so obtained.It is the purpose of the present paper to answer these questions in the special case of a solvable Lie group,where powerful results due to Auslander-Kostant [1] and Pukanszky ([9], [10], [11]) are available.

Our main result establishes the independence of $\rho(\ell,F)$ of the choice of a "good" polarization F and, which is more important,gives necessary and sufficient conditions in order that $\rho(\ell,F)$ be primary,primary of type I, respectively irreducible.In particular,this led us to a natural extension of the Auslander-Kostant procedure for obtaining irreducible unitary representations, by which we may associate to every (not necessarily integral) orbit under the coadjoint action of G a set of irreducible representations.Although our construction can not aim at a complete parametrization of all equivalence classes of irreducible unitary representations (excepting the case of a type I group, when it adds nothing to the original construction of Auslander and Kostant), it yields a family of irreducible unitary representations $\Lambda(G)$ satisfying the following completeness property: each primitive ideal of the group C^*-algebra $C^*(G)$ of G is the kernel of an irreducible representation in $\Lambda(G)$, when it is viewed as a representation of $C^*(G)$. In particular, $\Lambda(G)$ constitutes a separating family of unitary irreducible representations for G. We show also that every equivalence class of irreducible normal representations is a member of $\Lambda(G)$.

1. PRELIMINARIES

Throughout this paper G will denote a connected and simply connected solvable Lie group, $\mathfrak{g}$ its Lie algebra and $\mathfrak{g}^*$ the dual vector space of $\mathfrak{g}$, on which G acts by the coadjoint representation.

1.1. Given $g \in \mathfrak{g}^*$, $G(g)$ denotes the isotropy subgroup of G at g with respect to the coadjoint action, $\mathfrak{g}(g)$ its Lie algebra, $X_g = G \cdot g$ the orbit of g under G and ω_g the G-invariant 2-form on X_g induced by the 2-cocycle $-dg$ on $\mathfrak{g}$. Since the identity component $G(g)_o$ of $G(g)$ is simply connected, there exists a unique character χ_g on $G(g)_o$

such that $d\chi_g = 2\pi i \cdot g|\mathfrak{g}(g)$; its kernel Q_g is a normal subgroup of $G(g)$.

We shall denote by $\mathcal{S}(g)$ the set of all subgroups Γ of $G(g)$ containing $G(g)_o$ .For $\Gamma \in \mathcal{S}(g)$ we put $\Gamma^c = \{\alpha \in G(g)$; $\chi_g(\alpha\gamma\alpha^{-1}\gamma^{-1}) = 1$ for any $\gamma \in \Gamma$. This definition makes sense since, $G(g)/G(g)_o$ being abelian (cf. [8], p.492), the commutator subgroup $[G(g),G(g)]$ of $G(g)$ is contained in $G(g)_o$. Note also that Γ^c is the inverse image in $G(g)$ of the centralizer of Γ/Q_g in $G(g)/Q_g$.

Let $\mathcal{A}(g)$ denote the set of all $\Gamma \in \mathcal{S}(g)$ such that Γ/Q_g is abelian, and $\mathcal{A}_{max}(g)$ the subset of those $\Gamma \in \mathcal{A}(g)$ with the property that Γ/Q_g is a maximal abelian subgroup of $G(g)/Q_g$ .It is easily seen that $\mathcal{A}(g) = \{\Gamma \in \mathcal{S}(g); \Gamma \subset \Gamma^c\}$ and $\mathcal{A}_{max}(g) = \{\Gamma \in \mathcal{S}(g) ; \Gamma = \Gamma^c\}$.

Given $\Gamma \in \mathcal{S}(g)$ we denote by $\hat{\Gamma}$ the set of all unitary characters χ on Γ which extends χ_g .

1.2. LEMMA. With the previous notations we have:

(i) $\mathcal{A}(g) = \{\Gamma \in \mathcal{S}(g) ; \hat{\Gamma} \neq \phi\}$.

(ii) Let $\Gamma, \Gamma' \in \mathcal{A}(g)$ and suppose that $\Gamma \subset \Gamma'$.Then every $\chi \in \hat{\Gamma}$ can be extended to a character $\chi' \in \hat{\Gamma'}$.

Proof. Let $\Gamma \in \mathcal{S}(g)$ .Assume first that $\hat{\Gamma} \neq \phi$ and choose $\chi \in \hat{\Gamma}$. If $\alpha \in \Gamma$,then $\chi_g(\alpha\gamma\alpha^{-1}\gamma^{-1}) = \chi(\alpha\gamma\alpha^{-1}\gamma^{-1}) = 1$ for all $\gamma \in \Gamma$, hence $\alpha \in \Gamma^c$ .Therefore $\Gamma \subset \Gamma^c$, that is $\Gamma \in \mathcal{A}(g)$.

Conversely, let us suppose $\Gamma \in \mathcal{A}(g)$. Consider the set $\mathcal{F}(\Gamma)$ of all pairs (Γ',χ') with $\Gamma' \in \mathcal{S}(g)$, $\Gamma' \subset \Gamma$ and $\chi' \in \hat{\Gamma'}$,endowed with the following ordering: $(\Gamma',\chi') \prec (\Gamma'',\chi'')$ if $\Gamma' \subset \Gamma''$ and $\chi''|\Gamma' = \chi'$. Obviously, $\mathcal{F}(\Gamma)$ is totally ordered and, since the pair $(G(g)_o , \chi_g)$ belongs to it, nonvoid. There exists then a maximal element (Γ_m, χ_m) in $\mathcal{F}(\Gamma)$. We assert that $\Gamma_m = \Gamma$, which would be sufficient for concluding the proof of the first statement in our lemma. Indeed, assuming the contrary, one can find an $\alpha \in \Gamma$ such that $\alpha \notin \Gamma_m$. Then, the subgroup Γ' of Γ generated by Γ_m and α is strictly larger than Γ_m. Now let p denote the order of the image of α in the quotient group $G(g)/\Gamma_m$ and choose $t \in \mathbb{C}$ to be a p-root of the complex number (of modulus 1) $\chi_m(\alpha^p)$ if p is finite, or $t = 1$ otherwise. Then, using the fact that $\chi_g(\alpha\gamma\alpha^{-1}\gamma^{-1}) = 1$ for any $\gamma \in \Gamma_m$, it is easily seen that the formula

$$\chi'(\gamma\alpha^n) = \chi_m(\gamma)t^n \quad , \gamma \in \Gamma_m \, , \, n \in \mathbb{Z} \, ,$$

makes sense and defines a character χ' on Γ' which extends χ. This contradicts the maximality of (Γ_m, χ_m).

A similar argument can be used to prove the assertion (ii). Another way is to pick an $\eta \in \hat{\Gamma'}$, which surely exists by (i), and to observe that $\chi\cdot(\bar{\eta}|\Gamma)$ is trivial on $G(g)_o$, hence it can be viewed as a character of $\Gamma/G(g)_o$. But this is a subgroup of the free abelian group $\Gamma'/G(g)_o$, hence we may extend the above character to a character of $\Gamma'/G(g)_o$, which lifts back to Γ' and gives a character ν on Γ'. Finally, $\chi' = \nu\cdot\eta$ is the desired extension of χ.

1.3. Let (X, ω) be a Hamiltonian G-space. It covers a well-determined G-orbit (X_g, ω_g) in $\mathfrak{g}^*$ and there exists a $\Gamma \in \mathcal{S}(g)$ such that (X, ω) is isomorphic to $(X_\Gamma, \omega_\Gamma)$, where $X_\Gamma = G/\Gamma$ and ω_Γ is the G-invariant 2-form on X_Γ induced by the 2-cocycle $-dg$ on $\mathfrak{g}$. Further, we recall that $\Lambda(X, \omega) \neq \phi$ if and only if $\hat{\Gamma} \neq \phi$ ([4], Theorem

5.7.1) and if this is the case each $\ell \in \Lambda(X, \omega)$ arises from a character $\chi \in \hat{\Gamma}$. Finally, every G-invariant polarization F on (X, ω) comes from a polarization $\mathfrak{h}$ at g (for this latter notion, see [1], Definition I.4.1).

Fix now an $\ell \in \Lambda(X_\Gamma, \omega_\Gamma)$ which arises from a character $\chi \in \hat{\Gamma}$ and a polarization F on $(X_\Gamma, \omega_\Gamma)$ which comes from a polarization $\mathfrak{h}$ at g. Assume in addition that $\mathfrak{h}$ is positive and satisfies the Pukanszky condition (see [1], Definitions I.4.4 and I.5.1). Then the unitary representation $\rho(\ell, F)$ associated to these data by Kostant's procedure (cf. [12]) is equivalent to the holomorphically induced representation $\rho(g, \chi, \mathfrak{h})$ which is defined as follows.

Let $\partial = \mathfrak{h} \cap \mathfrak{g}$ and let D_0 be the analytic subgroup corresponding to ∂. We form the group $D_\Gamma = D_0 \cdot \Gamma$ and note that it is closed (as follows from [1], Proposition I.5.1) and there exists a unique character η on D_Γ which extends χ and satisfies $d\eta = 2\pi i \cdot g|\partial$; this last assertion can be easily checked by arguments similar to those in the proof of Proposition I.5.10 in [1]. Denote by $\Delta(D_\Gamma)$, $\Delta(G)$ the modular functions of D_Γ and G respectively, and let $\delta_\Gamma = \Delta(D_\Gamma)/\Delta(G)$. Consider further the space $\mathcal{K}(G;D_\Gamma)$ of all continuous functions ψ on G, with compact support modulo D_Γ, satisfying $\psi(ad) = \delta_\Gamma(d)\varphi(a)$ for $a \in G$ and $d \in D_\Gamma$. There exists a positive G-invariant linear functional on $\mathcal{K}(G;D\ )$, unique up to a multiplicative constant, which we denote here $\psi \longmapsto \oint_{G/D_\Gamma} \psi(a)d\dot{a}$. Consider now all C^∞-functions φ on G, with compact support modulo D_Γ, which verify:

(i) $\varphi(ad) = \eta(d)^{-1}\delta_\Gamma(d)^{1/2}\varphi(a), \quad a \in G, \quad d \in D_\Gamma$;

(ii) $x \cdot \varphi = (-2\pi i \langle g, x \rangle + 1/2\, \vartheta(x))\varphi, \quad x \in \mathfrak{h}$,

where $(x \cdot \varphi)(a) = \frac{d}{dt}\varphi(a \cdot \exp tx)\Big|_{t=0}$ for $x \in \mathfrak{g}$ and one extends it by linearity for $x \in \mathfrak{g}_{\mathbb{C}}$, and $\vartheta(x)$ stands for the trace of the operator induced by ad x on $\mathfrak{g}_{\mathbb{C}}/\mathfrak{h} + \bar{\mathfrak{h}}$. This space of functions has a norm given by

(iii) $\|\varphi\|^2 = \oint_{G/D_\Gamma} |\varphi(a)|^2 \, d\dot{a}$

and we let $\mathcal{H}(g, \chi, \mathfrak{h})$ denote its completion. Finally we define $\rho(g, \chi, \mathfrak{h})$ as being the representation of G on this Hilbert space by left translations.

Note that $\rho(g, \chi, \mathfrak{h})$ is a subrepresentation of the representation of G induced by the character η on D_Γ . The latter will be denoted by ind(η,G).

1.4. There is another way for obtaining representations of G starting with a functional $g \in \mathfrak{g}^*$, due to Auslander and Kostant, which involves crucially the Mackey little group method. We shall briefly review it, following closely Vergne's expository paper [13] .

Let $N = [G,G]$, $\mathfrak{n} = [\mathfrak{g},\mathfrak{g}]$, $g \in \mathfrak{g}^*$, $f = g \,|\, \mathfrak{n}$; let $M = G(f)$ be the isotropy subgroup of G at f , G acting on $\mathfrak{n}^*$, $\mathfrak{m} = \mathfrak{g}(f)$ be the Lie algebra of M and $m = g \,|\, \mathfrak{m}$. Denote by χ_f the unique character of the simply connected group N(f) whose differential is $2\pi i \cdot f \,|\, \mathfrak{n}(f)$ and by Q_f the identity component of Ker χ_f . Then Q_f is normal in M, in particular in the identity component M_o of M , and M_o/Q_f is simply connected and nilpotent. The functional $m \in \mathfrak{m}^*$ vanishes on $\mathfrak{q}_f = \ker(f \,|\, \mathfrak{n}(f))$ and becomes a functional on the Lie algebra $\mathfrak{m}/\mathfrak{q}_f$ of M_o/Q_f , giving rise by the Kirillov construction to an irreducible unitary representation of M_o/Q_f which, when regarded as a representation of M_o will be denoted $\rho_o(m)$.

Now let us denote by $\mathcal{R}(g)$ the set of all unitary representations σ of $G(g)$ such that $\sigma | G(g)_0$ is a multiple of χ_g. Starting with a representation $\sigma \in \mathcal{R}(g)$ one can obtain a representation of G in the following manner. One form first the representation $\sigma \otimes \rho_0(m)$ of the direct product $G(g) \times M_0$ and one observe that it factorizes to a representation $(\sigma \otimes \rho_0(m))^\wedge$ of the group $M_g = G(g) \cdot M_0$, which is just the stabilizer of $\rho_0(m) \in \hat{M}_0$ in M, and that $(\sigma \otimes \rho_0(m))^\wedge | M_0$ is a multiple of $\rho_0(m)$. One considers then the representation $\tau(\sigma) = \mathrm{ind}((\sigma \otimes \rho_0(m))^\wedge, M)$. It lifts to a representation $\tau(\sigma)^\vee$ of the semi-direct product $M \times_s N$. Before proceeding we recall that, according to [1] (Proposition III.2.2 and Theorem III.3.1), the irreducible representation $\rho(f)$ of N, associated by the Kirillov procedure to $f \in \mathfrak{n}^*$, has a canonical extension $\nu(f)$ to $M \times_s N$. Now, by forming the tensor product $\tau(\sigma)^\vee \otimes \nu(f)$ we get a representation of $M \times_s N$ which is trivial on the kernel of the canonical map of $M \times_s N$ onto $K = M \cdot N$ and hence drops down to a representation of K; we call it $(\tau(\sigma)^\vee \otimes \nu(f))^\wedge$. Note that K is the stabilizer of $\rho(f) \in \hat{N}$ in G and the restriction of $(\tau(\sigma)^\vee \otimes \nu(f))^\wedge$ to N is a multiple of $\rho(f)$. The final step of this construction is to take $\pi(\sigma) = \mathrm{ind}((\tau(\sigma)^\vee \otimes \nu(f)), G)$, obtaining thus a unitary representation of G.

The crucial fact about $\pi(\sigma)$, which follows by a careful inspection of its construction in the light of the Mackey theory [8], is that the commuting rings of the representations σ and $\pi(\sigma)$ are algebraically isomorphic, so that $\pi(\sigma)$

is primary if and only if σ is primary and when both are primary they have the same type and one is irreducible if and only if the other is also.

2. THE MAIN RESULT

As it was explained 1.3, the unitary representations of G arising by the quantization procedure from Hamiltonian G-spaces can also be viewed as arising by the device of holomorphic induction. In these terms, the answer to the problem of determining the structure of such a representation is settled by the following result.

2.1. THEOREM. *Let $g \in \mathfrak{g}^*$, $\Gamma \in \mathcal{A}(g)$ and $\chi \in \hat{\Gamma}$.*

(1) The equivalence class of the representation $\rho(g,\chi,\mathfrak{h})$ does not depend on the choice of the positive strongly admissible polarization $\mathfrak{h}$ at g ; accordingly, it will be denoted in the sequel $\rho(g,\chi)$.

(2) $\rho(g,\chi)$ is primary if and only if $\Gamma^{cc} = \Gamma$; when this is so, $\rho(g,\chi)$ is of type I if and only if Γ is of finite index in Γ^{c}.

(3) $\rho(g,\chi)$ is irreducible if and only if $\Gamma \in \mathcal{A}_{\max}(g)$, or equivalently $\Gamma^{c} = \Gamma$; when this is so, $\rho(g,\chi)$ is normal if and only if X_g is locally closed in $\mathfrak{g}^*$ and the cohomology class $[\omega_g] \in H^2(X_g,\mathbb{R})$ is rational.

(4) Let $\Gamma' \in \mathcal{A}(g)$ be such that $\Gamma \subset \Gamma'$ and let be an extension of χ . Then

$$\rho(g,\chi) \simeq \int^{\oplus}_{(\Gamma'/\Gamma)^{\wedge}} \rho(g, \chi'\cdot\check{\nu})\,d\nu ,$$

where $\check{\nu}$ stands for the pull back of the character ν of the abelian group Γ'/Γ , and $d\nu$ is the Haar measure on the *character group $(\Gamma'/\Gamma)^\wedge$.*

We are going to break the proof of the theorem into several lemmas. Before stating them, let us make a few preliminary comments. First we denote $\sigma(g,\chi) = \mathrm{ind}(\chi, G(g))$ and notice that an immediate application of the Mackey subgroup theorem [5] ensures us that $\sigma(g,\chi) \in \mathcal{R}(g)$. Thus, starting with $\sigma = \sigma(g,\chi)$ we may form as in 1.4 the representations $\tau(\sigma)$ and $\pi(\sigma)$ of M and G respectively. They will be denoted in what follows by $\tau(g,\chi)$, respectively $\pi(g,\chi)$.

2.2. LEMMA. *For any positive strongly admissible polarization $\mathfrak{h}$ at g, $\rho(g,\chi,\mathfrak{h})$ is equivalent to $\pi(g,\chi)$.*

Proof. The proof can be achieved along the lines of the technique developped by Auslander and Kostant in [1], Chapter III, by looking closely at the construction of both these representations and using essentially the independence of polarization in the nilpotent case, to which the general case can be reduced. For the convenience of the reader we shall sketch below the arguments in more detail.

The notation being as in 1.4, let $\mathfrak{k} = \mathfrak{m} + \mathfrak{n}$ be the Lie algebra of $K = M \cdot N$, $k = g|\mathfrak{k}$, $M_\Gamma = M_o \cdot \Gamma$ and $K_\Gamma = M_\Gamma \cdot N$. It is clear that $M_o \subset M_\Gamma \subset M_g \subset M$ and $K_o = M_o \cdot N \subset K_\Gamma \subset K_g = M_g \cdot N \subset K$. Using the fact that $M_o \cap G(g) = G(g)_o$ (see [13], 2.2.3), the results in [1], II.1 and also similar arguments, one proves that $K_\Gamma(k) = \Gamma \cdot N(f)$ and there exists a unique character χ_k on $K_\Gamma(k)$ such that $\chi_k|\Gamma = \chi$ and $\chi_k|N(f) = \chi_f$. Note also that $\mathfrak{h} \subset \mathfrak{k}$ (cf. [1], Theorem II. 2.1) and $D_o \cdot K_\Gamma(k = D_o \cdot \Gamma = D_\Gamma$. We may form therefore, following the scheme given in 1.3, the holomorphically induced representation $\rho(k, \chi_k, \mathfrak{k})$ of K_Γ. By an "induction in stages" argument (see [2], Proposition 4.3.5) one sees that $\rho(g, \chi, \mathfrak{k}) \simeq \mathrm{ind}(\rho(k, \chi_k, \mathfrak{k}), G) \simeq \mathrm{ind}(\mathrm{ind}(\rho(k, \chi_k, \mathfrak{k}), K), G)$. The first reduction step is thus achieved and we have to prove now that $\rho_K = \mathrm{ind}(\rho(k, \chi_k, \mathfrak{k}), K)$ is equivalent to $(\tau(g, \chi)^\vee \otimes \nu(f))^\wedge$. In turn, this amounts to show that the pull back $\rho_K^\vee$ of ρ_K to $M \times_s N$ and $\tau(g, \chi)^\vee \otimes \nu(f)$ are equivalent representations.

We shall now proceed by looking more closely at $\rho = \rho(k, \chi_k, \mathfrak{k})$, trying to describe it in a more convenient way. We notice first that $M_\Gamma(m) = \Gamma \cdot N(f)$ and then we put $\chi_m = \chi_k$. As in [1], II.2, let $\mathfrak{k}_1 = \mathfrak{k} \cap \mathfrak{n}_{\mathbb{C}}$, $\mathfrak{k}_2 = \mathfrak{k} \cap \mathfrak{m}_{\mathbb{C}}$, $\partial_1 = \mathfrak{k}_1 \cap \mathfrak{n}$, $\partial_2 = \mathfrak{k}_2 \cap \mathfrak{m}$ and D_0^1, D_0^2 be the analytic subgroups corresponding to ∂_1, ∂_2 respectively. Furthermore, let $D_\Gamma^2 = D_0^2 \cdot M_\Gamma(m) = D_0^2 \cdot \Gamma$ and remark that $D^1 = D_0^1 \cdot N(f) = D_0^1$ since $N(f)$ is connected. Now χ_m extends to a well determined character η_m on D_Γ^2 satisfying $d\eta_m = 2\pi i \cdot m | \partial_2$ and also χ_f extends to the character η_f on (the simply connected group) D^1 given by $d\eta_f = 2\pi i \cdot f | \partial_1$. These being settled, we may consider the holomorphically induced representations $\rho_1 = \rho(f, \chi_f, \mathfrak{k}_1)$ of N and $\rho_2 = \rho(m, \chi_m, \mathfrak{k}_2)$ of M_Γ. Now ρ_2 lifts to a representation $\rho_2^\vee$ of $M_\Gamma \times_s N$ and ρ_1, which is just $\rho(f)$, extends to the representation $\nu_\Gamma(f) = \nu(f) | M_\Gamma \times_s N$. Arguing as in [1], III.4.1 (see also [2], Chap. VIII, §4) one can see that the pull back $\rho^\vee$ of ρ to $M_\Gamma \times_s N$ is equivalent to $\rho_2^\vee \otimes \nu_\Gamma(f)$.

With this description of $\rho^\vee$ at hand, let us return to the representation $\rho_K^\vee$. It is clearly equivalent to the representation $\mathrm{ind}(\rho^\vee, M \times_s N)$. A direct computation which involves writing down explicitely the corresponding Hilbert spaces and finding out the appropriate intertwining operator , shows that ind($\rho_2^\vee \otimes \vartheta_\Gamma(f)$, $M \times_s N$) is equivalent to $\rho_M^\vee \otimes \vartheta(f)$, where $\rho_M^\vee$ stands for the pull back of $\rho_M =$ ind(ρ_2, M) to $M \times_s N$. Consequently , $\rho_K^\vee$ is equivalent to $\rho_M^\vee \otimes \vartheta(f)$.

Recalling that our aim was to establish the equivalence between $\rho_K^\vee$ and $\tau(g,\chi)^\vee \otimes \vartheta(f)$ we are left to verify that ρ_M is equivalent to $\tau(g,\chi)$. To this end it suffices to prove that ind(ρ_2 ,M_g) is equivalent to $(\sigma(g,\chi) \otimes \rho_0(m))^\wedge$, which will be our concern in the sequel.

Using the fact that M_Γ/D_Γ^2 is isomorphic to M_0/D_0 , which is easily seen , one can check that ρ_2 when restricted to M_0 is equivalent to $\rho_0(m)$. Furthermore , since $[M_0,\Gamma] \subset [M_0,M_g(m)] \subset Q_f$ (cf. [2],Chap.VIII , Proposition 4.3), it is not difficult to see that $\rho_2|\Gamma$ is a multiple of χ . These two remarks ensure us that ρ_2 when lifted to $\Gamma \times_s M_0$ can be viewed as a representation of the direct product $\Gamma \times M_0$, in which case it is equivalent to $\chi \otimes \rho_0(m)$. At this moment , let us consider the representation ind($\chi \otimes \rho_0(m)$,$G(g) \times M_0$ It is clearly equivalent to $\sigma(g,\chi) \otimes \rho_0(m)$, hence it is a representation of the semi-direct product $G(g) \times_s M_0$ too and , in that case , it is equivalent to ind($\chi \otimes \rho_0(m)$,$G(g) \times_s M_0$). Finally , this last representation is in turn equivalent to the pull back of ind(ρ_2 , M_g) to $G(g) \times_s M_0$, while $\sigma(g,\chi) \otimes \rho_0(m)$ is the pull back of $(\sigma(g,\chi) \otimes \rho_0(m))^\wedge$ to the same group. This completes the proof of our lemma.

2.3. LEMMA. *The representation $\sigma(g,\chi)$ is primary if and only if $\Gamma^{cc}=\Gamma$. When this is the case, $\sigma(g,\chi)$ is of type I if and only if Γ is of finite index in Γ^{c}.*

Proof. Let us denote $\lambda=\mathrm{ind}(\chi,\Gamma^{c})$ and $\sigma=\sigma(g,\chi)=$ $=\mathrm{ind}(\chi,G(g))$. We notice first that χ, λ and σ are trivial on Q_g, hence they drop down and define the representations $\hat{\chi}$, $\hat{\lambda}$ and $\hat{\sigma}$ of the groups Γ/Q_g, Γ^{c}/Q_g, and $G(g)/Q_g$ respectively. Now Γ^{c}/Q_g is just the stabilizer of $\hat{\chi}$ in $G(g)/Q_g$, Γ/Q_g is abelian (hence of type I) and $\hat{\lambda}$ when restricted to Γ/Q_g, which is central in Γ^{c}/Q_g, is a multiple of $\hat{\chi}$. These facts entitle us to use Theorem 8.1 in [8] and to infer from this (more exactly from its proof) that the commuting rings of λ and $\sigma\simeq\mathrm{ind}(\lambda,G(g))$ are algebraically isomorphic. Therefore, λ and σ are primary in the same time and, if so, they are of the same type.

The key step of the present proof is to observe now that, under the assumption $\Gamma^{cc}=\Gamma$, Γ/Q_g is precisely the center of Γ^{c}/Q_g; hence we are in a position to apply Proposition 1.1 in [9] and to infer from this that λ is primary, and that it is of type I if and only if Γ is of finite index in Γ^{c}.

To complete the proof of the lemma it now suffices to show that λ is not primary if $\Gamma^{cc}\neq\Gamma$. Assume therefore that Γ is strictly contained in Γ^{cc}. Further, note that $\Gamma^{cc}\subset\Gamma^{c}$ and that Γ^{cc}/Q_g is abelian, hence by Lemma 1.2 $\widehat{\Gamma^{cc}}\neq\phi$. Moreover, the same lemma ensures us that $\chi\in\hat{\Gamma}$ extends to a character $\eta\in\widehat{\Gamma^{cc}}$. Then it is merely a matter of routine to prove that

$$\operatorname{ind}(\chi, \Gamma^{cc}) \simeq \int^{\oplus}_{(\Gamma^{cc}/\Gamma)^{\wedge}} \eta\cdot\check{\nu}\, d\nu ,$$

where $\check{\nu}$ stands for the pull back to Γ^{cc} of the character $\nu \in (\Gamma^{cc}/\Gamma)^{\wedge}$, and $d\nu$ is the Haar measure on the character group $(\Gamma^{cc}/\Gamma)^{\wedge}$. It follows further that

$$\lambda \simeq \operatorname{ind}(\operatorname{ind}(\chi, \Gamma^{cc}), \Gamma^{c}) \simeq \int_{(\Gamma^{cc}/\Gamma)^{\wedge}} \operatorname{ind}(\eta\cdot\check{\nu}, \Gamma^{c})\, d\nu .$$

Since obviously $\operatorname{ind}(\eta\cdot\check{\nu}, \Gamma^{c})$ restricted to Γ^{cc} is equivalent to $I_n \otimes \eta\cdot\check{\nu}$, where n is the index of Γ^{cc} in Γ^{c} and I_n denotes the identity operator in the standard n-dimensional Hilbert space, we obtain that λ, when restricted to Γ^{cc}, is equivalent to $\eta\cdot I_n \otimes \check{R}$, where R is the regular representation of the abelian group Γ^{cc}/Γ and $\check{R}$ stands for its pull back to Γ^{cc}. Bearing in mind the fact that Γ^{cc}/Γ is not trivial, it is now fairly clear that λ can be split into disjoint parts.

2.4. LEMMA. The representation $\sigma(g,\chi)$ is irreducible if and only if $\Gamma = \Gamma^{c}$.

Proof. The arguments in the preceding proof show us that $\sigma = \sigma(g, \chi)$ is irreducible if and only if λ is so. Applying to $\lambda = \operatorname{ind}(\chi, \Gamma^{c})$ a criterion for irreducibility of Mackey ([6], Theorem 6') we get that λ is irreducible when and only when $\chi^{\alpha} \neq \chi$ for every $\alpha \in \Gamma^{c} \setminus \Gamma$, where χ^{α} is the character of Γ given by $\chi^{\alpha}(\gamma) = \chi(\alpha^{-1}\gamma\alpha)$, $\gamma \in \Gamma$. By the very definition of Γ^{c}, this happens only in the extreme situation when $\Gamma = \Gamma^{c}$.

2.5. LEMMA. Let $\Gamma' \in \mathcal{A}(g)$ be such that $\Gamma \subset \Gamma'$ and let $\chi' \in \widehat{\Gamma'}$ be an extension of χ . Then

$$\sigma(g,\chi) \simeq \int^{\oplus}_{(\Gamma'/\Gamma)^{\wedge}} \sigma(g,\chi'\cdot\check{\nu})\, d\nu \ .$$

Proof. As we have already seen in a similar situation ,

$$\mathrm{ind}(\chi,\Gamma') \simeq \int^{\oplus}_{(\Gamma'/\Gamma)^{\wedge}} \chi'\cdot\check{\nu}\, d\nu \quad ,$$

hence , by inducing to G(g) , we get the desired formula.

2.6. These preparatives enable us to write down in a few lines the proof of the theorem. The central rôle in the proof is played by Lemma 2.2 which not only gives us at once the first claim of the theorem , but also entitle us to replace in the remaining assertions $\rho(g,\chi)$ by $\pi(g,\chi)$.

As we have already noticed at the end of section 1.4 , the commuting rings of $\pi(g,\chi)$ and $\sigma(g,\chi)$ are isomorphic. Combining this with Lemma 2.3 , we get the second claim of the theorem and also the first assertion in (3). To complete the proof of (3) , assume that $\rho(g,\chi)$ is irreducible. Then , owing to [3], Proposition ,p.5 , $\rho(g,\chi)$ is normal if and only if X_g is locally closed in $\mathfrak{g}^*$ and $G(g)^c$ has finite index in $G(g$ which , as observed by Pukanszky ([9],p.465) , can be rephrased by requiring that the cohomology class $[\omega_g] \in H^2(X_g,\mathbb{R})$ is rational.

Finally , it is only a matter of routine to verify that the decomposition given by Lemma 2.5 passes over all the steps which are involved in the construction described in 1.4 and yields ultimately the last assertion of the theorem.

3. CONSEQUENCES

We shall turn now our attention to the irreducible unitary representations which arise by the method discussed above. It would be interesting to characterize collectively these irreducible representations for an arbitrary solvable Lie group, but so far we have not succeeded in obtaining such a result. Instead, we shall collect here some results which may give an idea about the size of this family of representations.

3.1. First of all we shall settle the question of determining when two such representations are equivalent.

Let $\mathcal{L}(\mathfrak{g}) = \{(g, \chi);\ g \in \mathfrak{g}^*$ and $\chi \in \hat{\Gamma}$ for some $\Gamma \in \mathcal{A}_{\max}(g)\}$ and let $\Lambda(G)$ be the subset of the unitary dual $\hat{G}$ of G, consisting of all representations of the form $\rho(g, \chi)$. Standard arguments ensure us that, for $g_i \in \mathfrak{g}^*$, $\Gamma_i \in \mathcal{A}_{\max}(g_i)$ and $\chi_i \in \hat{\Gamma}_i$, $i = 1,2$, one has $\rho(g_1, \chi_1) = \rho(g_2, \chi_2)$ if and only if $g_2 = a^{-1}\cdot g_1$ and $\sigma(g_1, \chi_1) \simeq \sigma(g_1, \chi_2^a)$ for some $a \in G$. In turn, according to Theorem 7' in [6], the last condition can be rephrased as follows: $\Gamma_1 \cap a\Gamma_2 a^{-1}$ is of finite index in both Γ_1 and $a\Gamma_2 a^{-1}$ and there exists $\alpha \in G(g_1)$ such that $\chi_1 | \Gamma_1 \cap a\Gamma_2 a^{-1} = \chi_2^{a\alpha} | \Gamma_1 \cap a\Gamma_2 a^{-1}$.

We are thus led to introduce the following equivalence relation on $\mathcal{L}(\mathfrak{g})$: $(g_1, \chi_1) \sim (g_2, \chi_2)$ if there exists $a \in G$ such that $g_2 = a^{-1}\cdot g_1$, $\Gamma_1 \cap a\Gamma_2 a^{-1}$ is of finite index in both Γ_1 and $a\Gamma_2 a^{-1}$, and $\chi_1 | \Gamma_1 \cap a\Gamma_2 a^{-1} = \chi_2^a | \Gamma_1 \cap a\Gamma_2 a^{-1}$.

It is now clear that the map which associates to any

$(g,\chi) \in \mathcal{L}(\mathfrak{g})$ the representation $\rho(g,\chi)$ gives rise to a bijection of the quotient set $\Lambda(\mathfrak{g}) = \mathcal{L}(\mathfrak{g})/\!\sim$ onto $\Lambda(G)$.

3.2. PROPOSITION. *Let $\hat{G}_{\mathrm{norm}}$ denote the set of all equivalence classes of irreducible normal representations of G. Then $\hat{G}_{\mathrm{norm}} \subset \Lambda(G)$.*

Proof. According to Proposition 2, §1 in [3], if π is an equivalence class of irreducible normal representations of G, it is quasi-equivalent to one of the form $\rho(g,\eta)$ with $\eta \in \widehat{G(g)}^c$, X_g locally closed in $\mathfrak{g}^*$ and $[\omega_g]$ rational. Choose now an arbitrary $\Gamma \in \mathcal{A}_{\max}(g)$. In view of Lemma 1.2, we can find a character $\chi \in \hat{\Gamma}$ such that $\chi\,|\,G(g)^c = \eta$. Then, taking into account that $\Gamma/G(g)^c$ is finite and applying 2.1.(4) we get

$$\rho(g,\eta) = \sum\nolimits^{\oplus}_{\nu\in(\Gamma/G(g)^c)^\wedge} \rho(g,\chi\cdot\check{\nu}) .$$

Now Theorem 1 in [9] tells us that, in the case at hand, $\rho(g,\eta)$ is primary of type I, while 2.1.(3) ensures us that the components in the above direct sum decomposition are irreducible. It follows that $\rho(g,\eta)$ is a multiple of $\rho(g,\chi)$ and thus $\pi = \rho(g,\chi)$.

3.3. PROPOSITION. *The map which associates to any $\pi \in \Lambda(G)$ the kernel $\mathrm{Ker}_{C^*(G)}\pi$ of the corresponding representation of $C^*(G)$ is a surjection of $\Lambda(G)$ onto the set $\mathrm{Prim}\, C^*(G)$ of all primitive ideals of $C^*(G)$.*

Proof. Let J be a primitive ideal in $C^*(G)$. In view of Proposition 4 in [10], there exist $g \in \mathfrak{g}^*$ and $\eta \in \widehat{G(g)}^c$ such that $J = \mathrm{Ker}_{C^*(G)}\,\rho(g,\eta)$. Now pick a $\Gamma \in \mathcal{A}_{\max}(g)$ and then a character $\chi \in \hat{\Gamma}$ which extends η . By 2.1.(4) we have

$$\rho(g,\eta) = \int^{\oplus}_{(\Gamma/G(g)^c)^\wedge} \rho(g,\chi\cdot\check{\nu})\,d\nu \ .$$

But $\rho(g,\eta)$ is primary, hence homogeneous. Then, according to Lemma 1.9 in [5] , $J = \mathrm{Ker}_{C^*(G)}\,\rho(g,\chi\cdot\check{\nu})$ for almost all $\nu \in (\Gamma/G(g)^c)^\wedge$.

3.4. COROLLARY. If $a \in G$ and $a \neq 1$, there exists $\pi \in \Lambda(G)$ such that $\pi(a) \neq I$.

This can be inferred from the preceding proposition by imitating the way in which Pukanszky deduces in [11] the corollary on p. 120 from Theorem 1.

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