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FACTORIZATION OF SEMI–SPECTRAL MEASURES

by

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1. Introduction

The problem of factorization of a positive operator valued function by means of an operator valued analytic function is a crucial point in solving prediction problems for stationary processes. At the begining, Kolmogorov [9], [10] gave his elegant solution for prediction problems for stationary univariate processes (in the discret case) using the result of Szegö [17] about representation $f(t) = |q(e^{it})|^2$ of a scalar valued function $f(t) \geq 0$ by means of a scalar valued analytic function $q(\lambda)$.

In a natural attempt to build the prediction theory for the multivariate stationary processes along the lines of Kolmogorov's development, factorization theorems for operator valued functions are needed. For the matrix valued functions a complete theory was built in succesively papers, begining with Zasuhin [24] in 1941 and strongly continued by Wiener [20], Doob [2], Wiener and Masani [21], [22], Helson and Lowdenslager [6], [7].

In the prediction theory of a stationary processes with infinitely many components, factorization theorems for operator valued functions are necessary. Such theorems appeared and we mention here the results of Devinatz [1] and Lowdenslager [11]. In this case new difficulties related to the boundary values of an analytic function in the unit disc appear.

In the case of bounded analytic functions B.Sz.-Nagy and C.Foiaş [18], [19] avoid these difficulties by using the bounded convergence principle in order to construct the boundary function in the Fatou theorem about non-tangentially a.e. convergence (in strong sense) and consequentely, they obtained totaly satisfactory factorization theorems in the bounded case. In fact, using the boundedness condition, they made clear the proof of the factorization theorem which appear in the work [11] of Lowdenslager.

But, as Lowdenslager points out in the same work, in prediction theory any restriction of boundedness is unnatural, thus the efforts to made clear the proof of Lowdenslager's theorem in its full generality are justified.

H.Helson [5] proved a variant of Lowdenslager's theorem, but under supplementary conditions which also are unnatural conditions for prediction theory. R.G.Douglas [3] gave an example which shows that a result of Lowdenslager contained in [11] is not valid, but, as we shall see in section 5, his ''correct Lowdenslager theorem '' is also not well suited in prediction theory.

The aim of this paper is to make clear the Lowdenslager's theorem in its full generality in order for to be useful in the prediction theory. The main idea of the paper is to use Naymark dilation theorem [15] in order to construct not the boundary function for the L^2 - bounded analytic functions but the semi-spectral measure whose the Poisson integral it is. The difficulties in construction of the boundary function are thus of the same nature as in construction of the derivatives (a.e. in strong sense) for the bounded variation positive operator valued functions.

In section 3 we shall prove a Fatou theorem of such a type. In section 4 this theorem is used in the proof of factorization theorem for semi-spectral measure. In section 5 we shall use the factorization theorem to evaluate the prediction error operator for a stationary process whose covariance function is an operator valued pozitive defined function on the group of integers.

There are not but technical complications in the case of continuous stationary processes on the group of reals. But we hope that the method presented here can be used in more general cases of dinamical systems considered for example in [13].

Finally, we want to point out that in paralel to Szegő's analytic results another classical result, of geometric type, namely the Wold decomposition [23] played an important role in the development of prediction theory. Begining with Kolmogorov's works to the Lowdenslager's, the idea of Wold decomposition was intimately related and furnished effective contributions in prediction theorems.

2. Preliminaries

In this section we establish the notations and terminology which we shall use in that follows.

Let $\mathcal{E}$ be a separable Hilbert space and $L(\mathcal{E})$ the space of all linear bounded operators on $\mathcal{E}$. We shall adopt the terminology and notation from [19] ch.V, sec.1, concerning the Hilbert spaces $L^2(\mathcal{E})$, $L^2_+(\mathcal{E})$, $H^2(\mathcal{E})$, etc.

Recall that $L^2(\mathcal{E})$ is the Hilbert space of all measurable functions v defined on the unit circle $\mathbb{T}$ in the complex plane $\mathbb{C}$ with values in $\mathcal{E}$ such that

$$\|v\|^2_{L^2(\mathcal{E})} = \frac{1}{2\pi}\int_0^{2\pi} \|v(e^{it})\|^2_{\mathcal{E}}\, dt$$

There exists a one - to - one correspondence between the elements v in $L^2(\mathcal{E})$ and the sequences $\{a_k\}_{-\infty}^{+\infty}$, $a_k \in \mathcal{E}$ with $\sum_{-\infty}^{+\infty}\|a_k\|^2 < \infty$ such that

$$v(e^{it}) = \sum_{-\infty}^{+\infty} e^{ikt} a_k ,$$

the convergence of the serie been in the $L^2(\mathcal{E})$-norm.

We have also

$$\|v\|^2_{L^2(\mathcal{E})} = \sum_{-\infty}^{+\infty}\|a_k\|^2 ,$$

The space $L^2_+(\mathcal{E})$ is the subspace of $L^2(\mathcal{E})$ containing the functions v in $L^2(\mathcal{E})$ for which, in the above correspondence, $a_k = 0$ for $k < 0$. The space $L^2_+(\mathcal{E})$ is isometric isomorph to $H^2(\mathcal{E})$ - the Hilbert space of all analytic functions u defined in the open unit disc $\mathbb{D} = \{\lambda \in \mathbb{C} \,;\, |\lambda| < 1\}$ in the complex plane, with values in $\mathcal{E}$ for which

$$\|u\|^2_{H^2(\mathcal{E})} = \sup_{0\leq r<1} \frac{1}{2\pi}\int_0^{2\pi} \|u(re^{it})\|^2 \, dt < \infty$$

via the correspondence

$$u(\lambda) = \sum_{k=o}^{\infty} \lambda^k a_k \longrightarrow v(e^{it}) = \sum_{k=o}^{\infty} e^{ikt} a_k ,$$

$$\|u\|^2_{H^2(\mathcal{E})} = \|v\|^2_{L^2(\mathcal{E})} = \sum_0^{\infty} \|a_k\|^2$$

The functions $u(\lambda)$ and $v(e^{it})$ are also connected by Poisson's formula :

$$u(re^{it}) = \frac{1}{2\pi}\int_0^{2\pi} P_r(t-s)\, v(e^{is})\, ds$$

where

$$P_r(t) = \frac{1-r^2}{1-2r\cos t+r^2}$$

We have also the generalized Fatou theorem : $u(\lambda)$ tends a.e. to $v(e^{it})$ strongly (in $\mathcal{E}$) as λ tends to e^{it} non-tangentially with respect to the unit circle.

An $L(\mathcal{E})$ - valued semi-spectral measure on $\mathbb{T}$ is a map F defined on the family of Borel sets σ of $\mathbb{T}$ with values bounded operators on $\mathcal{E}$ such that for any $a \in \mathcal{E}$, $\sigma \longrightarrow (F(\sigma)a,a)$ is a positive Borel measure on $\mathbb{T}$. The semi-spectral measure E is spectral if $E(\sigma_1 \cap \sigma_2) = E(\sigma_1)E(\sigma_2)$ for any Borel sets σ_1, σ_2.

A celebrated dilation theorem of M.A. Naymark says that for any $L(\mathcal{E})$ - valued semi-spectral measure F on $\mathbb{T}$ there exist a Hilbert space $\mathcal{K}$, a bounded operator V from $\mathcal{E}$ into $\mathcal{K}$, and a $L(\mathcal{K})$ - valued spectral measure E on $\mathbb{T}$ such that for any Borel set σ we have

$$F(\sigma) = V^* E(\sigma) V .$$

We shall call such a triplet $[\mathcal{K}, V, E]$ a spectral dilation of F.

To any spectral measure E on $\mathbb{T}$ we can attach the spectral scale E(t), $0 \leq t \leq 2\pi$, such that the integral with respect to E of continuous scalar valued functions on $\mathbb{T}$ is the Lebesque Stieltjes integral with respect to the positive (projection valued) function E(t). If we denote $F(t) = V^{*}E(t)V$ then F(t) is a positive $L(\mathcal{E})$ - valued function on $\mathbb{T}$ and we can integrate continuous scalar valued functions in Lebesque-Stieltjes sense with respect to F(t). We have

$$\int_{0}^{2\pi} f(t)dF(t) = V^{*} \int_{0}^{2\pi} f(t)dE(t)\,V$$

Recall that there exists a one-to-one correspondence between the set of unitary operators on K and $L(\mathcal{E})$ - valued spectral measures on $\mathbb{T}$ given by

$$U^{n} = \int e^{int} dE(t), \qquad (n \in \mathbb{Z}),$$

and between $L(\mathcal{E})$ - valued semi-spectral measure on $\mathbb{T}$ and $L(\mathcal{E})$ - valued positive maps on the group $\mathbb{Z}$ of integers given by

$$R(n) = \int_{0}^{2\pi} e^{int} dF(t), \qquad (n \in \mathbb{Z}).$$

If $[\mathcal{K},V,E]$ is a spectral dilation of F then $[\mathcal{K},V,U]$ is a unitary dilation of R in the sense that

$$R(n) = V^{*} U^{n} V, \qquad (n \in \mathbb{Z}).$$

Under certain conditions of minimality like

$$\mathcal{K} = \bigvee_{-\infty}^{+\infty} U^{n} V \mathcal{E}$$

the spectral (or unitary) dilation is uniquely determined up to a unitarity which conserve the operator V. In this paper all considered dilation will be supposed minimal.

An important special case is the semi-spectral measure attached to a contraction T on $\mathcal{E}$. A very known B.Sz.-Nagy theorem says that the map R defined as

$$R(n) = \begin{cases} T^n & n > 0 \\ I & n = 0 \\ T^{*|n|} & n < 0 \end{cases}$$

is positive definite on Z. Since here R(o) = I, we have $V^*V = I$ for any unitary dilation $[\mathcal{K}, V, U]$ of R. Thus V is an isometric embeding of $\mathcal{E}$ in $\mathcal{K}$. If P is the orthogonal projection of $\mathcal{K}$ onto $\mathcal{E}$ (considered as a subspace of $\mathcal{K}$ with the isometric embeding V) then we have

$$F(\sigma) = P E(\sigma)|_{\mathcal{E}} \qquad (\sigma \subset \mathbb{T}, \text{ Borel set})$$
$$F(t) = P E(t)|_{\mathcal{E}} \qquad (o \leq t \leq 2\pi)$$
$$R(n) = T^n = PU^n|_{\mathcal{E}} \qquad (n \in \mathbb{Z})$$

In order for an $L(\mathcal{E})$ - valued semi-spectral measure on $\mathbb{T}$ to be the semi-spectral measure attached to a contraction T on $\mathcal{E}$ it is necessary and sufficient that it has a spectral dilation $[\mathcal{K}, V, U]$ with $V^*V = I$ and $\mathcal{E}$ (as $V\mathcal{E}$) to be semi-invariant for U, i.e.

$$V^*U^n V = (V^*UV)^n, \qquad (n \in \mathbb{Z})$$

This special case and some others results conected with Sz.-Nagy—Foiaş characteristic function will be discussed in a separate paper.

3. Fatou theorem for L^2 - bounded analytic functions

Let now $\mathcal{E}$, $\mathcal{F}$, be two separable Hilbert spaces. Consider a function $\Theta(\lambda)$ defined on $\mathbb{D}$ whose values are bounded linear operators from $\mathcal{E}$ to $\mathcal{F}$ given by

$$(3.1) \qquad \Theta(\lambda) = \sum_{k=o}^{\infty} \lambda^k \Theta_k$$

Θ_k being bounded linear operators from $\mathcal{E}$ to $\mathcal{F}$. The series is supposed to be convergent weakly, strongly, or in norm, which amounts to the same for power series.

Let us suppose, moreover, that for any $a \in \mathcal{E}$ we have

$$\sup_{0 \leq r < 1} \frac{1}{2\pi} \int_0^{2\pi} \|\Theta(re^{it})a\|_{\mathcal{F}}^2 \, dt \leq M^2 \|a\|_{\mathcal{E}}^2 , \tag{3.2}$$

or equivalently

$$\sum_{k=0}^{\infty} \|\Theta_k a\|_{\mathcal{F}}^2 \leq M^2 \|a\|_{\mathcal{E}}^2 \tag{3.3}$$

where M is a constant independent of a.

Such a function will be called L^2 - bounded analytic function and will be denoted by the triplet $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$.

Let us remark that if $\{\Theta_k\}_0^\infty$ is a sequence of bounded operators from $\mathcal{E}$ to $\mathcal{F}$ which verifies (3.3) then (3.1) defines an L^2 - bounded analytic function $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$. Indeed, for any $a \in \mathcal{E}$ we have

$$\Big\|\sum_m^n \lambda^k \Theta_k a\Big\|_{\mathcal{F}} \leq \sum_m^n |\lambda|^k \|\Theta_k a\|_{\mathcal{F}} \leq (1-|\lambda|^2)^{-1/2} \Big[\sum_m^n \|\Theta_k a\|_{\mathcal{F}}^2\Big]^{1/2} \longrightarrow 0$$

for $n > m \longrightarrow \infty$.

THEOREM 1. (Fatou type theorem). Let $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ be an L^2 - bounded analytic function on $\mathbb{D}$.

(a). There exists an $L(\mathcal{E})$ - valued semi-spectral measure F_Θ on $\mathbb{T}$ such that:

(i) F_Θ has a spectral dilation of the form $[L^2(\mathcal{F}), V_\Theta, E^x]$ where V_Θ is a bounded operator from $\mathcal{E}$ into $L^2(\mathcal{F})$ verifying $V_\Theta \mathcal{E} \subset L^2_+(\mathcal{F})$, and E^x is the spectral measure of the multiplication by e^{it} in $L^2(\mathcal{F})$.

(ii) For any $a \in \mathcal{E}$, $\Theta(\lambda)a$ tends a.e. to $(V_\Theta a)(e^{it})$ when λ tends to e^{it} non-tangentially with respect to the unit circle and

$$\Theta(re^{it})a = \frac{1}{2\pi} \int_0^{2\pi} P_r(t-s)(V_\Theta a)(s)\, ds \tag{3.4}$$

(b) If F is an $L(\mathcal{E})$ - valued semi-spectral measure on $\mathbb{T}$ which admits a spectral dilation of the form $[L^2(\mathcal{F}), V, E^x]$ where V is a bounded operator from $\mathcal{E}$ into $L^2(\mathcal{F})$ verifying $V\mathcal{E} \subset L^2_+(\mathcal{F})$ and E^x is the spectral measure of the multiplication by e^{it} on $L^2(\mathcal{F})$ then the Poisson integral

$$\Theta(re^{it})a = \frac{1}{2\pi}\int_0^{2\pi} P_r(t-s)(Va)(e^{is})ds \tag{3.5}$$

defines an L^2 - bounded analytic function $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ such that if F_Θ and V_Θ are as in the point (a), then $F = F_\Theta$, $V = V_\Theta$.

Proof. (a) Let

$$\Theta(\lambda) = \sum_0^\infty \lambda^k \Theta_k$$

be the Taylor series of the L^2 - bounded analytic functions $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$. Since for any $a \in \mathcal{E}$ we have

$$\sum_0^\infty \|\Theta_k a\|^2_{\mathcal{F}} \leq M^2 \|a\|^2_{\mathcal{E}},$$

we can define the function v_a in $L^2_+(\mathcal{F})$ by

$$v_a(e^{it}) = \sum_0^\infty e^{ikt}\Theta_k a$$

If we put $V_\Theta a = v_a$ we have

$$\|V_\Theta a\|^2_{L^2(\mathcal{F})} = \|v_a\|^2_{L^2(\mathcal{F})} = \sum_0^\infty \|\Theta_k a\|^2_{\mathcal{F}} \leq M^2 \|a\|^2_{\mathcal{E}}.$$

Thus V_Θ is a bounded operator from $\mathcal{E}$ into $L^2(\mathcal{F})$ and clearly $V_\Theta\mathcal{E} \subset L^2_+(\mathcal{F})$. Let E^x be the spectral measure of the multiplication by e^{it} in $L^2(\mathcal{F})$. Then if for any borelian set σ in $\mathbb{T}$ we put

$$F_\Theta(\sigma) = V^*_\Theta E(\sigma) V_\Theta$$

we obtain an $L(\mathcal{E})$ - valued semi-spectral measure F on $\mathbb{T}$ such that $[L^2(\mathcal{F}), V_\Theta, E^\times]$ is a spectral dilation for F_Θ. Since for any $a \in \mathcal{E}$ the function $u_a(\lambda) = \Theta(\lambda)a$ from $H^2(\mathcal{F})$ has v_a as a boundary limit, clearly all assertions in (a) hold.

(b) Let F be an $L(\mathcal{E})$ - valued semi-spectral measure on $\mathbb{T}$ as in (b). For $a \in \mathcal{E}$ we have :

$$\sup_{0 \leq r < 1} \left\| \frac{1}{2\pi} \int_0^{2\pi} P_r(t-s)(V_a)(s)\,ds \right\|^2_{L^2(\mathcal{F})} = \|V_a\|^2_{L^2(\mathcal{F})} \leq \|V\|^2 \|a\|^2$$

Then is clear that the Poisson integral (3.5) defines an L^2 - bounded analytic function $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ such that if F_Θ, V_Θ are as in point (a), we have $F = F_\Theta$, $V = V_\Theta$.

The proof of the theorem is complete.

REMARK 1. If there exists a 0 - Lebesque measure set $\sigma \subset \mathbb{T}$ such that for any $t \notin \sigma$ and any $a \in \mathcal{E}$, $(V_\Theta a)(e^{it})$ exists as a radial limit of $\Theta(re^{it})a$, then $\Theta(\lambda)$ tends strongly to a bounded operator $\Theta(e^{it})$ from $\mathcal{E}$ into $\mathcal{F}$ as λ tends non-tangentially to e^{it}. This hapens for example when $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ is a bounded analytic function (cf. [19] , chap.V, sec.2).
In such a case we have

$$(V_\Theta a)(e^{it}) = \Theta(e^{it})\, a \qquad \text{a.e.}$$

and

$$\frac{d\,F_\Theta(t)}{dt} = \Theta(e^{it})^* \,\Theta(e^{it}) \qquad \text{strongly a.e.}$$

REMARK 2. There exist L^2 - bounded analytic functions which have no (strongly a.e.) radial limit. Indeed, let $\{H^2, \mathbb{C}, \Theta(\lambda)\}$ be the L^2 - bounded analytic function defined as

$$\Theta(\lambda) f = f(\lambda) \qquad (\lambda \in \mathbb{D}) .$$

./.

If $\Theta(re^{it})$ has strongly radial limit a.e. then there exists a 0 - Lebesgue measure set $\omega \subset [0,2\pi]$ such that for any $t \notin \omega$, $f(re^{it})$ have radial limit for every f in H^2, which is impossible.

We remark that for this $\Theta(\lambda)$, the semi-spectral measure attached F_Θ is the semi-spectral measure of the shift operator on H^2, and V_Θ is the usual embeding of H^2 into L^2.

Let us summarise some known facts about bounded analytic functions (cf. [19], ch.V.) as follows :

An L^2 - bounded analytic function $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ is bounded if and only if there exists a bounded operator Θ_+ from $L^2_+(\mathcal{E})$ into $L^2_+(\mathcal{F})$ such that $e^{it}\Theta_+ v = \Theta_+ e^{it} v$, for $v \in L^2_+(\mathcal{E})$, and $\Theta_+|_{\mathcal{E}} = V_\Theta$.

If Q is a bounded operator from $L^2_+(\mathcal{E})$ into $L^2_+(\mathcal{F})$ such that $e^{int}Q = Qe^{int}$, then there exists a bounded analytic function $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ such that $\Theta_+ = Q$.

The L^2 - bounded analytic function $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ will be called inner if it is bounded and the corresponding Θ_+ is an isometry. The L^2 - bounded analytic function $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ will be called L^2 - bounded outer function if :

$$\bigvee_0^\infty e^{int} V_\Theta \mathcal{E} = L^2_+(\mathcal{F}) .$$

As in the bounded case we can show that :

(i) For every L^2 - bounded outer function $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ we have $\Theta(\lambda)\mathcal{E} = \mathcal{F}$ for all $\lambda \in \mathbb{D}$.

(ii) Any L^2 - bounded analytic function which is simultaneously inner and outer, is a unitary constant function.

We say that the L^2 - bounded analytic function $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ has a scalar multiple if there exists a scalar function $\delta(\lambda) \not\equiv 0$ in the Hardy class H^2 and a contractive analytic function $\{\mathcal{F}, \mathcal{E}, \Omega(\lambda)\}$ such that

$$\Omega(\lambda)\Theta(\lambda) = \delta(\lambda) 1_{\mathcal{E}} , \quad \Theta(\lambda)\Omega(\lambda) = \delta(\lambda) 1_{\mathcal{F}} .$$

PROPOSITION 1. If $\dim \mathcal{E} = \dim \mathcal{F} = n < \infty$, then every L^2 - bounded analytic function $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ such that $\Theta(\lambda)$ is invertible for at least one λ in $\mathbb{D}$, has a scalar multiple.

Proof. : Let $\{e_i\}$ and $\{f_i\}$ ($i = 1, \ldots n$) be the orthonormal bases in $\mathcal{E}$ and in $\mathcal{F}$. The corresponding matrix $\Theta(\lambda) = [\theta_{ij}(\lambda)]$ ($i, j = 1, \ldots n$) of $\Theta(\lambda)$ is defined by

$$\Theta(\lambda)e_j = \sum_{i=1}^{n} \theta_{ij}(\lambda)f_i \qquad (j = 1, \ldots n),$$

i.e.

$$\theta_{ij}(\lambda) = (\Theta(\lambda)e_j, f_i) .$$

Let $\omega(\lambda) = [\omega_{ij}(\lambda)]$ be the algebraic adjoint of $\Theta(\lambda)$. Since $\Theta(\lambda)$ is invertible for at least one λ in $\mathbb{D}$, we have $d(\lambda) = \det[\theta_{ij}(\lambda)] \not\equiv 0$, and

$$\omega(\lambda)\,\Theta(\lambda) = \Theta(\lambda)\,\omega(\lambda) = d(\lambda)\, I_n, \tag{3.6}$$

where I_n is the unit matrix of order n.

Obviously $\theta_{ij}(\lambda)$ are functions in H^2, $\omega_{ij}(\lambda)$ are functions in $H^{2/n-1}$, and $d(\lambda)$ is in $H^{2/n}$. As H^p, $p>0$, is in Nevanlinna class, using a well known theorem of F. and R. Nevanlinna (Cf. [4], pag.16), we have $\omega_{ij}(\lambda) = u_{ij}(\lambda)/v_{ij}(\lambda)$, $d(\lambda) = u(\lambda)/v(\lambda)$, where u_{ij}, v_{ij}, u, v are functions in H^∞.

Let us consider $\gamma = v \prod_{i,j=1}^{n} v_{ij}$, and $\gamma_{ij} = \gamma/v_{ij}$, $\gamma_o = \gamma/v$.

We have

$$\omega_{ij} = \frac{\gamma_{ij}\, u_{ij}}{\gamma} \quad \text{and} \quad d = \frac{\gamma_o\, u_{ij}}{\gamma} .$$

If we denote by $\delta'_{ij} = \gamma_{ij} u_{ij}$ and $\delta' = \gamma_o u_{ij}$, we obtain from (3.6.) that

$$[\delta'_{ij}(\lambda)]\,[\theta_{ij}(\lambda)] = [\theta_{ij}(\lambda)]\,[\delta'_{ij}(\lambda)] = \delta'(\lambda)\, I_n . \tag{3.7}$$

If $\rho = \sup_{\lambda\in\mathbb{D}} \| \delta'_{ij}(\lambda) \|$, and $\delta_{ij}(\lambda) = \frac{1}{\rho}\delta'_{ij}(\lambda)$, $\delta(\lambda) = \frac{1}{\rho}\delta'(\lambda)$

it results that

$$\left[\delta_{ij}(\lambda)\right]\left[\Theta_{ij}(\lambda)\right] = \left[\Theta_{ij}(\lambda)\right]\left[\delta_{ij}(\lambda)\right] = \delta(\lambda) I_n,$$

and that the analytic operator valued function $\{\mathfrak{F}, \mathcal{E}, \Delta(\lambda)\}$ defined by

$$\Delta(\lambda) f_j = \sum_{i=1}^{n} \delta_{ij}(\lambda) e_i \qquad (j = 1, \dots n)$$

is a contractive analitic function.

Hence $\delta(\lambda)$ is a scalar multiple for the L^2 - bounded function $\{\mathcal{E}, \mathfrak{F}, \Theta(\lambda)\}$.

4. Factorizations

In this section we shall prove the variant needed in prediction theory of the Lowdenslager factorization theorem. As in the bounded case [19], we shall begin with the following

PROPOSITION 2. *Let $\{\mathcal{E}, \mathfrak{F}, \Theta(\lambda)\}$ and $\{\mathcal{E}, \mathfrak{F}_1, \Theta_1(\lambda)\}$ be two L^2 - bounded analytic functions, the second one being outer, and F_Θ, F_{Θ_1} the corresponding semi-spectral measure. Suppose that*

(4.1) $\qquad F_\Theta \le F_{\Theta_1}$.

Then there exists a contractive analytic function $\{\mathfrak{F}_1, \mathfrak{F}, \Theta_2(\lambda)\}$ such that

$$\Theta(\lambda) = \Theta_2(\lambda)\Theta_1(\lambda) \qquad (\lambda \in \mathbb{D}) .$$

If in (4.1) the equality sign holds, then $\Theta_2(\lambda)$ is inner. If, moreover, $\Theta(\lambda)$ is outer, then $\Theta_2(\lambda)$ is a unitary constant function.

Proof. Define X from $L^2_+(\mathfrak{F}_1)$ into $L^2_+(\mathfrak{F})$ as follows : put firstly for any analytic polinomial p and $a \in \mathcal{E}$

$$X p V_{\Theta_1} a = p V_{\Theta} a .$$

We have

$$\|p V_{\Theta} a\|^2_{L^2(\mathfrak{F})} = (V^*_{\Theta} |p|^2 V_{\Theta} a, a)_{\mathcal{E}} =$$

$$= \int_0^{2\pi} |p(e^{it})|^2 d(F_{\Theta}(t)a, a)_{\mathcal{E}} \leq \int_0^{2\pi} |p(e^{it})|^2 d(F_{\Theta_1}(t)a, a)_{\mathcal{E}} =$$

$$= (V^*_{\Theta_1} |p|^2 V_{\Theta_1} a, a)_{\mathcal{E}} = \|p V_{\Theta_1} a\|^2_{L^2(\mathfrak{F}_1)}$$

Since $\Theta_1(\lambda)$ is outer, it is clear that X can be defined as a contraction from $L^2_+(\mathfrak{F}_1)$ into $L^2_+(\mathfrak{F})$ and clearly

$$X e^{it} = e^{it} X .$$

The proof works further exactly as in the bounded case (cf. [19], chap.V, sec.4, prop.4.1).

THEOREM 2. (Lowdenslager, Sz.-Nagy, Foiaş factorization theorem). Let F be an $L(\mathcal{E})$ - valued semi-spectral measure on $\mathbb{T}$ and $[\mathcal{K}, V, E]$ its minimal spectral dilation. There exists an L^2 - bounded outer function $\{\mathcal{E}, \mathfrak{F}_1, \Theta_1(\lambda)\}$ with the following properties :

(i) $F \geqslant F_{\Theta_1}$.

(ii) For every other L^2 - bounded analytic functions $\{\mathcal{E}, \mathfrak{F}, \Theta(\lambda)\}$ for which $F \geqslant F_{\Theta}$ we have also $F_{\Theta_1} \geqslant F_{\Theta}$.

The properties (i) and (ii) determine the outer function $\Theta_1(\lambda)$ up to a constant unitary factor from the left. In order that equality holds in (i) it is necessary and sufficient that the condition

$$\bigcap_{n=0}^{\infty} U^n \mathcal{K}_+ = 0$$

be satisfied, where $\mathcal{K}_+ = \bigvee_{n=0}^{\infty} U^n V\mathcal{E}$ and U is the unitary operator on $\mathcal{K}$ corresponding to the spectral measure E.

Proof. Let

$$\mathcal{K}_+ = \bigoplus_0^{\infty} U^n \mathfrak{F}'_1 \oplus \bigcap_{n=0}^{\infty} U^n \mathcal{K}_+$$

be the Wold decomposition of the isometry $U_+ = U|\mathcal{K}_+$ and denote by P the orthogonal projection of $\mathcal{K}_+$ onto $\bigoplus_{0}^{\infty} U^n \mathfrak{F}_1$. Let V_1 be the bounded operator from $\mathcal{E}$ into $L^2(\mathfrak{F}_1)$ obtained by composing the operator PV with the usual isomorfism between $\bigoplus_{-\infty}^{+\infty} U^n \mathfrak{F}_1$ and $L^2(\mathfrak{F}_1)$. Clearly $V_1 \mathcal{E} \subset L^2_+(\mathfrak{F}_1)$. Let us put for any Borel set $\sigma \subset \mathbb{T}$

$$F_1(\sigma) = V_1^{\times} E^{\times}(\sigma) V_1$$

where $E^{\times}$ is the spectral measure of the multiplication by e^{it} on $L^2(\mathfrak{F}_1)$. Then F_1 is an $L(\mathcal{E})$ - valued semi-spectral measure on $\mathbb{T}$ which has $[L^2(\mathfrak{F}_1), V_1, E^{\times}]$ as a spectral dilation. From Theorem 1, point (b) it results that there exists an L^2 - bounded analytic function $\{\mathcal{E}, \mathfrak{F}_1, \Theta_1(\lambda)\}$ such that $F_1 = F_{\Theta_1}$, $V_1 = V_{\Theta_1}$. Since PU = UP it results easily that

$$\bigvee_{n=0}^{\infty} e^{int} V_{\Theta_1} \mathcal{E} = L^2_+(\mathfrak{F}_1)$$

i.e. $\Theta_1(\lambda)$ is outer.

For any analytic polynominal p we have :

$$\int |p(e^{it})|^2 d(F(t)a, a)_{\mathcal{E}} = \|p(U)Va\|^2_{\mathcal{K}} \geq \|Pp(U)Va\|^2 =$$

$$= \|p(U)PVa\|^2_{\mathcal{K}} = \|X_{\mathfrak{F}_1} p(U)PVa\|^2_{L^2(\mathfrak{F}_1)} = \|pX_{\mathfrak{F}_1} PVa\|^2_{L^2(\mathfrak{F}_1)} =$$

$$= \|pV_1 a\|^2_{L^2(\mathfrak{F}_1)} = \int |p(e^{it})|^2 d(F_1(t)a, a)_{\mathcal{E}} ,$$

where we denoted by $X_{\mathfrak{F}_1}$ the Fourier representation of $\bigoplus_{-\infty}^{+\infty} U^n \mathfrak{F}_1$ onto $L^2(\mathfrak{F}_1)$. Clearly then

$$\int |p(e^{it})|^2 d(F(t)a, a)_{\mathcal{E}} \geq \int |p(e^{it})|^2 d(F_{\Theta_1}(t)a, a)_{\mathcal{E}}$$

holds for any trigonometric polinomial p. Thus $F \geq F_{\Theta_1}$.

Let now $\{\mathcal{E}, \mathfrak{F}, \Theta(\lambda)\}$ be another L^2 - bounded analytic function such that $F \geq F_{\Theta}$. For any polinomial p, and $a \in \mathcal{E}$ we have :

$$\|pV_{\Theta}a\|^2_{L^2(\mathfrak{F})} = (pV_{\Theta}a, pV_{\Theta}a)_{L^2(\mathfrak{F})} = (V^*_{\Theta}|p|^2 V_{\Theta}a, a)_{\mathcal{E}} =$$

$$= \int |p(e^{it})|^2 \, d(F_{\Theta}(t)a, a) \leq \int |p(e^{it})|^2 d(F(t)a, a) = \|p(U)Va\|^2 .$$

Thus we can define the contraction Y from $\mathcal{K}_+$ into $L^2_+(\mathfrak{F})$ by

$$Y(p(U)Va) = pV_{\Theta}a$$

for any analytic polinomial p, and $a \in \mathcal{E}$.

Clearly $YU = e^{it}Y$ and

$$Y(\bigcap_{n=o}^{\infty} U^n\mathcal{K}_+) \subset \bigcap_{n=o}^{\infty} YU^n\mathcal{K}_+ = \bigcap_{n=o}^{\infty} e^{int}Y\mathcal{K}_+ \subset \bigcap_{n=o}^{\infty} e^{int}L^2_+(\mathfrak{F}) = \{0\}$$

It results that $Y = YP$. Then for any analytic polinomial p we have

$$\int |p(e^{it})|^2 d(F_{\Theta}(t)a, a)_{\mathcal{E}} = \|pV_{\Theta}a\|^2_{L^2(\mathfrak{F})} =$$

$$= \|Yp(U)Va\|^2_{\mathcal{K}} = \|YPp(U)Va\|^2_{\mathcal{K}} \leq \|Pp(U)Va\|^2_{\mathcal{K}} =$$

$$= \|p(U)PVa\|^2 = \|pX_{\mathfrak{F}_1}PV_1\|^2_{L^2(\mathfrak{F}_1)} = \|pV_{\Theta_1}a\|^2_{L^2(\mathfrak{F}_1)} =$$

$$= \int |p(e^{it})|^2 \, d(F_{\Theta_1}(t)a, a) .$$

Clearly then

$$\int |p(e^{it})|^2 \, d(F_{\Theta}(t)a, a)_{\mathcal{E}} \leq \int |p(e^{it})|^2 \, d(F_{\Theta_1}(t)a, a)_{\mathcal{E}}$$

for any trigonometric polynomial p and $a \in \mathcal{E}$, hence $F_{\Theta} \leq F_{\Theta_1}$.

Let now $\{\mathcal{E}, \mathfrak{F}_1', \Theta_1'(\lambda)\}$ be an L^2 - bounded outer function which satisfies (i) and (ii). Then $\mathfrak{F}_{\Theta_1'} = \mathfrak{F}_{\Theta_1}$ and preceding proposition shows that $\Theta_1'(\lambda) = Z\Theta_1(\lambda)$ with a unitary operator Z from $\mathfrak{F}_1'$ to $\mathfrak{F}_1$.

It is clear that $F = F_{\Theta_1}$ if and only if $PV = V$, thus if and only if $\bigcap_0^\infty U^n \mathcal{K}_+ = \{0\}$.

The proof of the theorem is complete.

COROLARRY. Any L^2 - bounded analytic function $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ can be uniquely factorized into the form

$$\Theta(\lambda) = \Theta_i(\lambda)\Theta_e(\lambda)$$

Where $\{\mathcal{E}, \mathcal{F}_1, \Theta_e(\lambda)\}$ is an L^2 - bounded outer function, and $\{\mathcal{F}_1, \mathcal{F}, \Theta_i(\lambda)\}$ is an inner function.

Proof: If F_Θ is the $L(\mathcal{E})$ - valued semi-spectral measure on $\mathbb{T}$ attached to $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ then from Theorem 2 there exists an L^2 - bounded outer function $\{\mathcal{E}, \mathcal{F}_1, \Theta_e(\lambda)\}$ with properties (i) and (ii). Hence $F_\Theta = F_{\Theta_1}$. Thus, by virtue of Proposition 2 there exists an inner function $\{\mathcal{F}_1, \mathcal{F}, \Theta_i(\lambda)\}$ such that

$$\Theta(\lambda) = \Theta_i(\lambda)\Theta_e(\lambda) \qquad (\lambda \in \mathbb{D})$$

This factorization is unique in the sense that if

$$\Theta(\lambda) = \Theta'_i(\lambda)\Theta'_e(\lambda) \qquad (\lambda \in \mathbb{D})$$

is any factorization with some outer function $\Theta'_e(\lambda)$ and inner function $\Theta'_i(\lambda)$, and with some intermediary space $\mathcal{F}'_1$, then there exists a unitary operator Z from $\mathcal{F}_1$ to $\mathcal{F}'_1$ such that

$$\Theta'_e(\lambda) = Z\Theta_e(\lambda) \quad \text{and} \quad \Theta'_i(\lambda) = \Theta_i(\lambda)Z^{-1}, \qquad (\lambda \in \mathbb{D}).$$

This follows from Proposition 1.

It can happen that in Theorem 2 $\mathcal{F}_1 = \{0\}$. We admit the null function as an outer function of the from $\{\mathcal{E}, \{0\}, \Theta(\lambda) \equiv 0\}$. It results from the theorem that, in this case, if $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ is an L^2 - bounded analytic function such that $F \geqslant F_\Theta$, then Θ is the null function. In prediction theory it is important to decide in terms of the semi-spectral measure F when $\mathcal{F}_1$ is or not $\{0\}$. The fact that there are no L^2 - bounded analytic function "under" F is, however, a characterization of $\mathcal{F}_1 = \{0\}$, but a too vague one.

Similarly as in the bounded case we can give a sufficient condition for $\mathcal{F}_1 \neq \{0\}$ which becomes necessary too in the scalar case and in certain finite dimensional cases.

Define the Borel measure on $\mathbb{T}$

$$\mu_a(\sigma) = (F(\sigma)a, a) \qquad (a \in \mathcal{E}).$$

Let $d\mu_a = \frac{1}{2\pi} h_a dt + d\tau_a$ be the Lebesgue decomposition of $d\mu_a$ with respect to Lebesgue measure dt.

THEOREM 3. The following assertions are equivalent :

(i) There exists $h \in L^1(dt)$, $h \geq 0$ such that for any $a \in \mathcal{E}$, $\|a\| = 1$, $h \leq h_a$ a.e. and

$$\int \log h(t)dt > -\infty \qquad (4.2.)$$

(ii) The L^2 - bounded outer function $\{\mathcal{E}, \mathcal{F}_1, \Theta_1(\lambda)\}$ attached to F as in Theorem 2 admits scalar multiple.

(iii) There exists a non-zero L^2 - bounded analytic function $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ which admits scalar multiple such that

$$F_{\Theta} \leq F$$

If one of the equivalent assertions (i) - (iii) holds then $\dim \mathcal{E} = \dim \mathcal{F}_1$.

Proof : (i) $\Rightarrow$(ii). From (4.2) it results that there exists a scalar outer function $\delta_1 \in H^2$ such that $|\delta_1|^2 = h$ a.e. For any analytic polynominal p and $a \in \mathcal{E}$ we have :

$$\|\delta_1 pa\|^2_{L^2(\mathcal{E})} = \frac{1}{2\pi} \int_0^{2\pi} |\delta_1(e^{it})|^2 |p(e^{it})|^2 \|a\|^2 dt = \frac{1}{2\pi} \int_0^{2\pi} |p(e^{it})|^2 h(t)\|a\| dt$$

$$\leq \frac{1}{2\pi} \int_0^{2\pi} |p(e^{it})|^2 h_a(t)\, dt \leq \int_0^{2\pi} |p(e^{it})|^2 (dF(t)a, a) =$$

$$= \int_0^{2\pi} |p(e^{it})|^2 d(E(t)Va, Va) = \|p(U)Va\|^2 .$$

Thus we can define the contraction Δ_1 from $\mathcal{K}_+$ into $L^2_+(\mathcal{E})$ by

$$\Delta_1 p(U)V_a = p\delta_1 a$$

Clearly $\Delta_1 U = e^{it}\Delta_1$, thus

$$\Delta_1 \bigcap_{n\geq 0} U^n\mathcal{K}_+ \subseteq \bigcap_{n\geq 0} e^{int}\Delta_1\mathcal{K}_+ \subseteq \bigcap_{n\geq 0} e^{int} L^2_+(\mathcal{E}) = \{0\}$$

But $\mathcal{K}_+ = L^2_+(\mathcal{F}_1) \oplus \bigcap_{n\geq 0} U^n\mathcal{K}_+$, hence $\Delta_1 = \Delta_1 P$, where P is the projection from $\mathcal{K}_+$ onto $L^2_+(\mathcal{F}_1)$. Let $\{\mathcal{F}_1, \mathcal{E}, \Delta_1(\lambda)\}$ be the contractive analytic function corresponding to Δ_1.

Denoting by $w_a(\lambda) = \Delta_1(\lambda)\Theta_1(\lambda)\, a$ and $u_a(\lambda) = \Theta_1(\lambda)a$, and considering $H^2(\mathcal{E})$, $H^2(\mathcal{F}_1)$ identified with $L^2_+(\mathcal{E})$, $L^2_+(\mathcal{F}_1)$, respectively, we have :

$$w_a = \Delta_1 u_a = \Delta_1 PV_a = \Delta_1 V_a = \delta_1 a$$

i.e. $\Delta_1(\lambda)\Theta_1(\lambda)\, a = \delta_1(\lambda)a, \qquad a \in \mathcal{E}$. Hence

$$\Delta_1(\lambda)\Theta_1(\lambda) = \delta_1(\lambda)I_{\mathcal{E}}$$

Multiplyng by $\Theta_1(\lambda)$ from the left yields

$$(\delta_1(\lambda)I_{\mathcal{F}} - \Theta_1(\lambda)\,\Delta_1(\lambda))\,\Theta_1(\lambda) = 0, \qquad (\lambda \in \mathbb{D}).$$

Since $\Theta_1(\lambda)$ is outer we have $\overline{\Theta_1(\lambda)\mathcal{E}} = \mathcal{F}$, so we conclude that

$$\Theta_1(\lambda)\,\Delta_1(\lambda) = \delta_1(\lambda)I_{\mathcal{F}} \qquad (\lambda \in \mathbb{D})$$

Thus the L^2 - bounded outer function $\{\mathcal{E}, \mathcal{F}_1, \Theta_1(\lambda)\}$ admits a scalar multiple.

(ii) $\Longrightarrow$ (iii) is obvious.

(iii) $\Longrightarrow$ (i). Let $\{\mathcal{F}, \mathcal{E}, \Delta(\lambda)\}$ be a contractive analytic function and $\delta(\lambda)$ in H^2 such that

$$\Delta(\lambda)\Theta(\lambda) = \delta(\lambda)I_{\mathcal{E}}$$
$$\Theta(\lambda)\Delta(\lambda) = \delta(\lambda)I_{\mathcal{F}} .$$

For any analytic polynomial p and $a \in \mathcal{E}$ we have

$$\int_0^{2\pi} |p|^2 \, d(F(t)a, a) \geqslant \int_0^{2\pi} |p|^2 \, d(F_{\Theta}(t)a, a) =$$

$$= \int_0^{2\pi} |p(e^{it})|^2 \, d(E^X(t)V_{\Theta}a, V_{\Theta}a) = \|pV_{\Theta}a\|^2_{L^2(\mathcal{F})} = \|pV_{\Theta}a\|^2_{H^2(\mathcal{F})} =$$

$$= \|p(\lambda)\Theta(\lambda)a\|^2_{H^2} = \sup_{0\leq r<1} \frac{1}{2\pi}\int_0^{2\pi} \|p(re^{it})\Theta(re^{it})a\|^2 \, dt \geqslant$$

$$\geqslant \sup_{0\leq r<1} \frac{1}{2\pi}\int_0^{2\pi} \|p(re^{it})\Delta(re^{it})\Theta(re^{it})a\|^2 \, dt =$$

$$= \sup_{0\leq r<1} \frac{1}{2\pi}\int_0^{2\pi} |p(re^{it})\delta(re^{it})|^2 \|a\|^2 dt = \frac{1}{2\pi}\int_0^{2\pi} |p(e^{it})|^2 |\delta(e^{it})|^2 \cdot \|a\|^2 dt .$$

It results that for any trigonometric polynomial p we have

$$\int |p|^2 \, d(F(t)a, a) \geqslant \int |p|^2 |\delta|^2 \, dt \cdot \|a\|^2 .$$

If we put $h = |\delta|^2$ then we have $h \in L^1(dt)$, $0 \leq h \leq h_a$ a.e. and

$$\int \log h(t)dt > -\infty .$$

In case $\dim \mathcal{E} < \infty$ it is known that a.e. there exists $F'(t) = \frac{dF(t)}{dt}$. In this case the assertion (i) is equivalent to

(i') $\int \log \det \left[F'(t)\right] > -\infty .$

Thus we have the following results which in prediction theory was firstly proved by Wiener and Masani in [21].

<u>COROLARRY</u>. <u>Let F be an $L(\mathcal{E})$ - valued semi-spectral measure on $\mathbb{T}$, where $\dim \mathcal{E} < \infty$. Then the following are equivalent</u> :

(i) $\int \log \det \left[F'(t) \right] dt > -\infty$.

(ii) There exists an L^2 - bounded outer function $\{\mathcal{E}, \mathcal{F}_1, \Theta_1(\lambda)\}$ such that $\dim \mathcal{E} = \dim \mathcal{F}_1$, and $F \geqslant F_{\Theta}$.

Proof. It results from Theorem 3 and Proposition 1.

5. Prediction error operator formula

It is known that the prediction theory for stationary processes consists essentially in the study of theirs covariance or correlation functions, which are scalar valued in the case of simple processes, matrix valued in the case of finite multivariate processes, and (why not ?) operator valued in the case of infinite variate processes. In all the casses, the stationarity conditions imply that these functions are positive definite maps on the base group (in general of time, but can be and others).

We suppose here that the base group is the group $\mathbb{Z}$ of integers and construct a geometrical model for prediction based on a positive definite function R on $\mathbb{Z}$ with values bounded operators on a separable Hilbert space $\mathcal{E}$.

Recall that an $L(\mathcal{E})$ - valued positive definite function R on $\mathbb{Z}$ is a map $k \longrightarrow R(k)$ from $\mathbb{Z}$ into $L(\mathcal{E})$ such that for any finite sistem $\{k_1, k_2, \ldots k_n\}$ in $\mathbb{Z}$ and any $a_1, \ldots a_n$ in $\mathcal{E}$ we have

$$\sum_{i,j=1}^{n} (R(k_i - k_j)a_i, a_j) \geqslant 0 \quad .$$

Using Naymark dilation theorem we deduce that there exists the triplet $[\mathcal{K}, V, U]$, $\mathcal{K}$ - a Hilbert space, V - a bounded operator from $\mathcal{E}$ into $\mathcal{K}$, and U a unitary operator on $\mathcal{K}$, such that

(5.1.)
$$R(n) = V^* U^n V \qquad (n \in \mathbb{Z})$$
$$\mathcal{K} = \bigvee_{-\infty}^{+\infty} U^n V \mathcal{E}$$

If E is the ($L(\mathcal{K})$ - valued) spectral measure of U^* then $F = V^* E V$

is an $L(\mathcal{E})$ - valued semi-spectral measure on $\mathbb{T}$ and we have

$$R(n) = \int_0^{2\pi} e^{-int} \, dF(t)$$

From the unicity part of the Naymark dilation theorem it results that R, U, E, F are in an one-to-one correspondence.

We shall call a <u>geometrical model</u> for the prediction theory of a (possible) stationary processes with covariance function R, any triplet $[\mathcal{K}, V, U]$ which is related with R by (5.1). We shall not formulate in details the complete problem of prediction in this context (this will be do in a separate paper), but we shall construct the prediction error operator and apply factorization theorem for to evaluate it.

Let us put

$$\mathcal{M}_n = \bigvee_{-\infty}^{n} U^k V\mathcal{E} \quad , \mathcal{M}_{-\infty} = \bigcap_{-\infty}^{\infty} \mathcal{M}_n .$$

We have $\mathcal{M}_n \subset \mathcal{M}_{n+1}$ and if we denote by P_n the orthogonal projection of $\mathcal{K}$ onto $\mathcal{M}_n$ then

$$U^n P_k U^j = P_{k+n} U^{j+n}$$

(5.2.) $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (j,k,n \in \mathbb{Z})$

$$U^n \mathcal{M}_k = \mathcal{M}_{k+n}$$

For any integer n and $f, g \in \mathcal{E}$ we have :

$$(U^n Vf - P_{n-1} U^n Vf, U^n Vg - P_{n-1} U^n Vg) =$$

$$= (U^n Vf, U^n Vg) - (U^n Vf, P_{n-1} U^n Vg) - (P_{n-1} U^n Vf, U^n Vg) +$$

$$+ (P_{n-1} U^n Vf, P_{n-1} U^n Vg) = (U^n Vf, U^n Vg) - (U^n Vf, P_{n-1} U^n Vg) =$$

$$= (U^{n-k} Vf, U^{n-k} Vg) - (U^{n-k} Vf, P_{(n-k)-1} U^{n-k} Vg) =$$

$$= \ldots = (U^{n-k} Vf - P_{(n-k)-1} U^{n-k} Vf, U^{n-k} Vg - P_{(n-k)-1} U^{n-k} Vg).$$

Thus we can define the operator G on $\mathcal{E}$ by

$$(5.3) \qquad (Gf, g) = (U^n Vf - P_{n-1} U^n Vf, U^n Vg - P_{n-1} U^n Vg)$$

and G does not depend of n.

We have

$$(5.4) \qquad (Gf, f) = \| U^n Vf - P_{n-1} U^n Vf \|^2 = \| Vf - P_{-1} Vf \|^2 .$$

Thus G is a positive operator on $\mathcal{E}$, so called prediction error operator (with lag 1, see [21]), attached to the stationary process which has R as a covariance function.

From (5.4) it results that

$$(5.5) \qquad (Gf, f) = \inf_{p \in \mathcal{P}_o} \| Vf - p(U^*) Vf \|^2$$

where $\mathcal{P}_o$ is the set of all analytic polinomials p which verifies $p(o) = 0$.

From (5.3) it results that G = 0 if and only if

$$\mathcal{M}_{-\infty} = \dots = \mathcal{M}_{-1} = \mathcal{M}_o = \mathcal{M}_1 = \dots = \mathcal{K},$$

thus if and only if R is the covariance function of a deterministic process.

The prediction error operator G depends only of the positive definite function R and not of the particular choosen geometrical model. More precisely, for R and R′ two $L(\mathcal{E})$ respectively $L(\mathcal{E}')$ - valued positive function on $\mathbb{Z}$, and G respectively G′ the corresponding prediction error operators, if there exists a unitary operator X from $\mathcal{E}$ onto $\mathcal{E}'$ such that $XR(n) = R'(n)X$, $n \in \mathbb{Z}$, then $XG = G'X$.

In fact G depends only of the so called innovation part of the process. Indeed let Q denotes the orthogonal projection of $\mathcal{K}$ onto $\mathcal{M}_{-\infty}$ and $P = I - Q$. Since $\mathcal{M}_{-\infty}$ reduces U we have

$$P_n Q = QP_n = Q$$
$$Q(I - P_n) = 0$$

$$(I - P_n)P = P(I - P_n) = I - P_n$$

$$PP_nP = P - P(I - P_n)P = P - (I - P_n)P = P_nP$$

Then we have

$$\|(I - P_{-1})Vf\|^2 = \|P(I - P_{-1})Vf\|^2 = \|(I - P_{-1})PVf\|^2 = \tag{5.6}$$

$$= \| PVf - P_{-1}PVf \|^2 .$$

Denote $\mathcal{K}' = P\mathcal{K}$, $V' = PV$, $U' = U_{|\mathcal{K}'}$. Then the triplet $[\mathcal{K}', V', U']$ is a geometrical model for prediction of a certain process with the covariance function $R'(n) = V'^{*}U'^{n}V'$ - the innovation process of the initial one.

We have

$$\mathcal{M}'_n = \bigvee_{-\infty}^{n} U'^{n}\mathcal{K}' = \bigvee_{-\infty}^{n} U^{n}P\mathcal{K} = P \bigvee_{-\infty}^{n} U^{n}\mathcal{K} = P\mathcal{M}_n .$$

Thus

$$P'_nP\mathcal{K} = PP_n\mathcal{K} ,$$

i.e.

$$P'_nP = PP_nP = P_nP . \tag{5.7}$$

Using (5.6) and (5.7) we obtain

$$(G'f, f) = \|V'f - P'_{-1}V'f\|^2 = \|Vf - P_{-1}Vf\|^2 = (Gf, f) . \tag{5.8}$$

Thus the prediction error operator is the same for both processes.

Let F, F' be the $L(\mathcal{E})$ - valued semi-spectral measure of R, R', respectively, and $\{\mathcal{E}, \mathcal{F}_1, \Theta_1(\lambda)\}$, $\{\mathcal{E}, \mathcal{F}'_1, \Theta'_1(\lambda)\}$ be the L^2 - bounded outer functions attached to F respectively F', as in Theorem 2. Since

$$\bigcap_{n=0}^{\infty} U'^{*n}\mathcal{K}'_{+} \subset \bigcap_{n=0}^{\infty} U^{*n}\mathcal{K}_{+} \subset \mathcal{M}_{-\infty} ,$$

it results that

$$\bigcap_{n=0}^{\infty} U'^{*n} \mathcal{K}'_{+} = \{0\} \; .$$

Thus $F' = F_{\Theta'_1}$. Since clearly $F' \leq F$ it results, again from Theorem 2, that

$F_{\Theta'_1} \leq F_{\Theta_1}$, i.e.

$$(5.9) \qquad F' \leq F_{\Theta_1} \leq F .$$

From (5.5), (5.8) and (5.9) it results :

$$(Gf, f) = \inf_{p\in\mathcal{P}_o} \| Vf - p(U^*)Vf\|^2_{\mathcal{K}} = \inf_{p\in\mathcal{P}_o} \int |1-p(e^{it})|^2 \, d(F(t)f, f)_{\mathcal{E}} \geq$$

$$\geq \inf_{p\in\mathcal{P}_o} \int_o^{2\pi} |1 - p(e^{it})|^2 \, d\,(F_{\Theta_1}(t)f, f)_{\mathcal{E}} =$$

$$= \inf_{p\in\mathcal{P}_o} \frac{1}{2\pi} \int_o^{2\pi} |1-p(e^{it})|^2 \, \|(V_{\Theta_1} f)(e^{it})\|^2_{\mathcal{F}_1} dt =$$

$$= \inf \left\{ \frac{1}{2\pi} \int_o^{2\pi} \|(V_{\Theta_1} f)^2(e^{it}) - v_o(e^{it})\|_{\mathcal{F}_1} dt \; ; \; v_o \in L^2_+(\mathcal{F}'_1), \; v_o(o) = o \right\} =$$

$$= \|(V_{\Theta_1} f)(o)\|^2_{\mathcal{F}_1} = (\Theta_1(o)^* \Theta_1(o)f, f)_{\mathcal{E}}$$

and

$$(Gf, f) = \inf_{p\in\mathcal{P}_o} \| V'f - p(U'^*)V'f \|^2 =$$

$$= \inf_{p\in\mathcal{P}_o} \int |1 - p(e^{it})|^2 \, d(F'(t)f, f) \leq$$

$$\leq \inf_{p\in\mathcal{P}_o} \int |1 - p(e^{it})|^2 \, d\,(F_{\Theta_1}(t)f, f) = (\Theta_1(o)^* \Theta_1(o)f, f) \; .$$

Hence

$$G = \Theta_1(o)^* \Theta_1(o) \; .$$

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Summing up we have

THEOREM 4. Let R be an $L(\mathcal{E})$ - valued positive definite function on $\mathbb{Z}$, G the prediction error operator of R and F the semi-spectral measure of R. If $\{\mathcal{E}, \mathcal{F}_1, \Theta_1(\lambda)\}$ is the L^2- bounded outer function attached to F as in Theorem 2, then we have

$$G = \Theta_1(o)^* \Theta_1(o) \quad .$$

In prediction theory it is important to decide in terms of R or of its semi-spectral measure F, when G is or is not equal to 0 and in case $G \neq 0$ to compute it. Theorem 4 gives a solution to this problem via the factorization Theorem 2. We remark that from the prediction theory point of view, the assumption $F = F_{\Theta_1}$ in Theorem 2 is not essential, the only important thing being to decide when $\Theta_1(\lambda)$ is or is not the null function. Let us remark in this context, that the example presented by R.G.Douglas in [3] shows only that, in infinite dimensional case, the closeness of the operator function F to 0 is not relevant to the fact that in Theorem 2 we have $F = F_{\Theta_1}$. The closeness to 0 remains relevant to the factorability of F if we mean by " factorability of F " the fact that the function $\Theta_1(\lambda)$ which appear in Theorem 2 is not the null function. As we already remarked, only this kind of factorability is relevant in prediction theory. With this kind of factorability, Lowdenslager's initial result [11] remains valid in his full generality.

Suppose now dim $\mathcal{E} < \infty$. Using Theorem 4 and Corolarry to the Theorem 3, we can prove that if F is an $L(\mathcal{E})$ - valued semi-spectral measure on $\mathbb{T}$, and G its prediction error operator, then G is invertible (full rank process) if and only if $\log \det \left[\frac{dF(t)}{dt}\right] \in L^1(dt)$

and we have

$$\det [G] = \exp \frac{1}{2\pi} \int_0^{2\pi} \log \det \left[\frac{dF(t)}{dt}\right] dt \quad ,$$

which is the result of Wiener and Masani [21] mentioned in the last part of section 4.

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