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ON INTERTWINING DILATIONS

by

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ON INTERTWINING DILATIONS

by ZOIA CEAUSESCU in Bucharest

Introduction. Let T, T′ by two contractions on the Hilbert space H and H′, and U, U′ be their isometric dilations on K and K′, respectively. For an operator $A \in L(H'; H)$ (the space of all bounded operators from H′ into H) intertwining T and T′ (i.e. TA=AT′) let us call an <u>intertwining dilation</u> of A any operator $B \in L(K'; K)$ satisfying: $P_H B|H' = A$, $UB = BU'$ and $B(K' \ominus H') \subset K \ominus H$. If, moreover, B satisfies $\|B\| = \|A\|$ it will be called an <u>exact</u> intertwining dilation of A. It is known that for any operator A intertwining T and T′ there exists at least one exact intertwining dilation (see Th.2.3 of [5]).

In the present paper we are concerned with the problem of uniqueness of such an exact intertwining dilation. We reduce this problem to the similar problem for the Hahn-Banach extensions of continuous functionals on some adequate quotient spaces of projective tensor products.[1)]

Our main results are contained in Sections 2 and 3. Thus we give a case in which only scalar multiple of isometric or coisometric intertwining operators have exact intertwining dilations (see Th. 2.1) and show that if an operator intertwining two contractions has a unique exact intertwining dilation, then all the operators which are "dominated" (in the sense of Definition 3.1) by it have the same property (see Th. 3.2). As an illustrative example, in the last section, an application of the above theorem to Hankel operators is given.

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1. Let K and G be two Hilbert spaces. We shall denote by $K^*\otimes G$ the subspace of $L(K;G)$ consisting of operators τ which admit a representation of the form

(1) $\tau = \sum_{j=1}^{n} k_j^*\otimes g_j$, where $k_j \in K$, $g_j \in G$, $1 \leq j \leq n$,

that is,

(2) $$\tau(k) = \sum_{h=1}^{n} (k, k_j)\, g_j \qquad (k \in K) .$$

We shall use the notation $\|\cdot\|_\pi$ for the nuclear norm on $K^*\otimes G$:

(3) $$\|\tau\|_\pi = \inf \left\{ \sum_{j=1}^{n} \|k_j\| \|g_j\| : \tau = \sum_{j=1}^{n} k_j^* \otimes g_j \right\}$$

Also, the space $K^*\otimes G$ endowed with this norm will be denoted by $K^*\underset{\pi}{\otimes} G$.

An immediate result is expressed by the following

Lemma 1.1. <u>For a subspace</u> H <u>of</u> K <u>the space</u> $H^*\underset{\pi}{\otimes} G$ <u>can be identified with the subspace</u> L <u>of</u> $K^*\underset{\pi}{\otimes} G$ <u>consisting of those</u> $\tau \in K^*\underset{\pi}{\otimes} G$ <u>for which</u>

(4) $$\tau \,|\, K \ominus H = 0$$

On account of Lemma 1.1 we may and will identify $H^*\underset{\pi}{\otimes} G$ with the subspace L defined by (4), of $K^*\underset{\pi}{\otimes} G$.

We shall denote by $K^*\hat{\otimes}_\pi G$ and $H^*\hat{\otimes}_\pi G$ the completions of $K^*\otimes_\pi G$ and $H^*\otimes_\pi G$, respectively.

Let us recall some well known properties (see [7]) of the completion of projective tensor product.

(i) Every element τ of $K^*\hat{\otimes}_\pi G$ is the sum of an absolutely convergent series:

$$\tau = \sum_{n=0}^{\infty} k_n^* \otimes g_n \ , \text{ and } \|\tau\|_\pi = \inf\left\{ \sum_{n=0}^{\infty} \|k_n\| \, \|g_n\| : \tau = \sum_{n=0}^{\infty} k_n^* \otimes g_n \right\}$$

(ii) The dual of $K^*\hat{\otimes}_\pi G$ is realized as the space $L(G;K)$.

Also, we shall consider operators U on K, T on H and Z on G and assume that H is a subspace of K invariant for U^*, and $U^*|H = T^*$.

We denote by $[Z, U]$ the operator on $L(K;G)$, defined by

(5) $$[Z, U]\, V = ZV - VU \quad \text{for } V \in L(K;G)$$

Note that $K^*\otimes G$ and $H^*\otimes G$ are invariant for $[Z, U]$, and in virtue of the condition $T^* = U^*|H$ we have

$$[Z, U]\,|\,H^*\otimes G = [Z, T]\,|\,H^*\otimes G$$

(where $[Z, T]$ is defined on $L(H;G)$ in the same way as $[Z, U]$ is on $L(K;G)$. The operators $[Z, T]$ and $[Z, U]$ can be extended continuously to $H^*\hat{\otimes}_\pi G$ and $K^*\hat{\otimes}_\pi G$, respectively.

Now, denoting

(6) $$R_U = \left([Z, U]\,(K^*\hat{\otimes}_\pi G)\right)^{-}, \qquad R_T = \left([Z, T]\,(H^*\hat{\otimes}_\pi G)\right)^{-}$$

where the closures are taken in the spaces $K^*\hat{\otimes}_\pi G$ and $H^*\hat{\otimes}_\pi G$, respectively. We shall consider the quotients modulo R_U and R_T of the nuclear norms on $K^*\hat{\otimes}_\pi G$ and $H^*\hat{\otimes}_\pi G$, respectively; thus, if ψ and φ denote the canonical epimorphism

$$\psi : K^* \hat{\otimes}_\pi G \longrightarrow (K^* \hat{\otimes}_\pi G)/R_U \,,\quad \varphi : H^* \hat{\otimes}_\pi G \longrightarrow (H^* \hat{\otimes}_\pi G)/R_T$$

then

$$\|\psi(\tau)\| = \inf_{\tau_1 \in R_U} \|\tau + \tau_1\|_\pi \qquad (\tau \in K^* \hat{\otimes}_\pi G) \text{ and}$$

$$\|\varphi(\tau)\| = \inf_{\tau_1 \in R_T} \|\tau + \tau_1\|_\pi \qquad (\tau \in H^* \hat{\otimes}_\pi G)$$

Since, $R_U \supset R_T$, we infer that

(7) $$\|\psi(\tau)\| \leq \|\varphi(\tau)\| \qquad \text{for } \tau \in H^* \hat{\otimes}_\pi G$$

Lemma 1.2. (i) The dual of the Banach space $(K^* \hat{\otimes}_\pi G)/R_U$ is isometric-isomorphic to the subspace

$$\{B \in L(G;H) : \ UB = BZ\} \quad \text{of} \quad L(G;K)$$

(ii) The dual of the Banach space $(H^* \hat{\otimes}_\pi G) / R_T$ is isometric-isomorphic to the subspace

$$\{A \in L(G;H) : \ TA = AZ\} \quad \text{of } L(G;H)$$

Proof. Ad (i): Firstly, let us observe that $\{B \in L(G;K): \ UB=BZ\}$ is isometric-isomorphic to $R_U^\perp$, where we denote by $R_U^\perp$ the orthogonal of R_U i.e.

$$R_U^\perp = \{f \in (K^* \hat{\otimes}_\pi G)' : \ f \,|\, R_U = 0\} .$$

Indeed, since $L(G;K)$ is isometric-isomorphic to $(K^* \hat{\otimes}_\pi G)'$, for any $B \in L(G;K)$ with the property $UB=BZ$ there is a unique f from $(K^* \hat{\otimes}_\pi G)'$ with the properties

(a) $\quad f(k^* \otimes g) = (Bg, k) \quad (k \in K, \ g \in G)$ and (b) $\|f\| = \|B\|$.

But, for this f and for any $k \in K$, $g \in G$, we also have:

$$f([Z, U] (k^* \otimes g)) = (BZg, k) - (UBg, k) = 0$$

Since the set $\{[Z, U](k^* \otimes g) : k \in K, g \in G\}$ spans R_U, it results readily $f \mid R_U = 0$

Conversely, for any $f \in (K^* \hat{\otimes}_\pi G)'$ with $f \mid R_U = 0$, there exists (since $L(G;K) \cong (K^* \hat{\otimes}_\pi G)'$) a unique $B \in L(G;K)$ satisfying conditions (a), (b) above; moreover, we have

$$((UB-BZ)g, k) = f([Z, U](k^* \otimes g)) = 0 \quad \text{for any } k \in K, g \in G.$$

Thus, the operator B has also the property UB=BZ.

Now, statement (i) of the Lemma results from the following general fact: If X is a Banach space and Y is a subspace of X, then the orthogonal $Y^\perp$ of Y is isometric-isomorphic to the dual of the quotient space X/Y.

Ad (ii): The proof is analogous to that of (i), due to the similar definition for the space $H^* \hat{\otimes}_\pi G$, and thus for $(H^* \hat{\otimes}_\pi G)/R_T$ too.

Lemma 1.3. The following two statements are equivalent :

(P_1) For any $A \in L(G;H)$ satisfing the condition $TA=AZ$, there exists at least one exact intertwining dilation $B \in L(G;K)$ of A.

(P_2) For any $\tau \in H^* \hat{\otimes}_\pi G$, we have $\|\psi(\tau)\| = \|\varphi(\tau)\|$.

Proof. First, we notice that, on account of Lemma 1.2, (P_1) is equivalent to the following:

(P_1') For any $f \in ((H^* \hat{\otimes}_\pi G)/R_T)'$ there exists an "extension" $\tilde{f} \in ((K^* \hat{\otimes}_\pi G)/R_U)'$ of f (i.e. $\tilde{f}\psi(\tau) = f\varphi(\tau)$ for all $\tau \in H^* \hat{\otimes}_\pi G$) such that:

$\|\tilde{f}\| = \|f\|$ (or equivalently, $\|\tilde{f}\psi\| = \|f\varphi\|$).

Indeed, if (P_1) holds then, in virtue of Lemma 1.2, for $f \in ((H^* \hat{\otimes}_\pi G)/R_T)'$ there is $\tilde{f} \in ((K^* \hat{\otimes}_\pi G)/R_U)'$ such that $\|\tilde{f}\| = \|f\|$ and $\tilde{f}\psi(h^* \otimes g) = f\varphi(h^* \otimes g)$ for all $h \in H$ and $g \in G$. Since, for $\tau \in H^* \hat{\otimes}_\pi G$ there are the representations

$\tau = \sum_{n \in N} h_n^* \otimes g_n$ where the series $\sum_{n \in N} h_n^* \otimes g_n$ is absolutely convergent, and since f, $\tilde{f}$, φ, ψ, are continuous, we also have

$$f\varphi(\tau) = \tilde{f}\psi(\tau) \quad \text{for all } \tau \in H^* \hat{\otimes}_\pi G.$$

The converse implication $(P_1') \Rightarrow (P_1)$ is, by Lemma 1.2 even more abvious.

Now, we assume that (P_1') holds. Let us take $\tau_0 \in H^* \hat{\otimes}_\pi G$ with $\varphi(\tau_0) \neq 0$. There exists $f \in ((H^* \hat{\otimes}_\pi G)/R_T)'$ with the properties:

$$\|f\| = \|f\varphi\| = 1, \qquad f\varphi(\tau_0) = \|\varphi(\tau_0)\|.$$

For this f there exists, according to (P_1'), $\tilde{f} \in ((K^* \hat{\otimes}_\pi G)/R_U)'$ such that $\|\tilde{f}\| = \|f\| = 1$ and $\tilde{f}\psi(\tau) = f\varphi(\tau) \quad (\tau \in H^* \hat{\otimes}_\pi G)$.

Thus, by (7),

$$\|\varphi(\tau_0)\| = \tilde{f}\psi(\tau_0) \leq \|\tilde{f}\| \, \|\psi(\tau_0)\| = \|\psi(\tau_0)\| \leq \|\varphi(\tau_0)\|.$$

If $\varphi(\tau_0) = 0$ then, by (7), $0 \leq \|\psi(\tau_0)\| \leq \|\varphi(\tau_0)\| = 0$

Consequently, we obtain $\|\varphi(\tau)\| = \|\psi(\tau)\|$ for all $\tau \in H^* \hat{\otimes}_\pi G$.

Let us now assume that $\|\varphi(\tau)\| = \|\psi(\tau)\|$ for all $\tau \in H^* \hat{\otimes}_\pi G$.

This means that the continuous canonical epimorphism

$$\varphi(H^* \hat{\otimes}_\pi G) = (H^* \hat{\otimes}_\pi G)/R_T \longrightarrow (H^* \hat{\otimes}_\pi G)/R_U = \psi(H^* \hat{\otimes}_\pi G)$$

is an isometry. Therefore, we can identify $(H^* \hat{\otimes}_\pi G)/R_T$ with the subspace $(H^* \hat{\otimes}_\pi G)/R_U$ of $(K^* \hat{\otimes}_\pi G)/R_U$. Now, the implication $(P_2) \Rightarrow (P_1')$ follows from the Hahn-Banach Theorem.

It is known that if T is a contraction on H, U an isometric dilation of T on K and Z an isometry on G, then the assertion (P_1) of Lemma 1.3 is true (cf. [5] Prop. II 2.2.). Thus we have

Theorem 1.1. _Let T be a contraction on H, U an isometric dilation of T, and Z an isometry on G. Then,_

$$(H^*\hat{\otimes}_\pi G)/([Z,T](H^*\hat{\otimes}_\pi G))^-$$

is linear canonically isometric to the image of $H^*\hat{\otimes}_\pi G$ in

$$(K^*\hat{\otimes}_\pi G)/([Z,U](K^*\hat{\otimes}_\pi G))^-$$

2. In the sequel we shall only treat the case considered in Theorem 1.1; that is, T a contraction on H, U an isometric dilation of T on K, and Z an isometry on G.

Remark 2.1. Let $A\in L(G;H)$ satisfy $TA = AZ$. In order that A should have a unique intertwining dilation $B\in L(G;K)$ with $\|B\| = \|A\|$ it is necessary and sufficient that the functional $f\in((H^*\hat{\otimes}_\pi G)/R_U)'$ (where $(H^*\hat{\otimes}_\pi G)/R_U$ is identified with $(H^*\hat{\otimes}_\pi G)/R_T$, in virtue of Theorem 1.1), corresponding to A by : $f\psi(h^*\otimes g) = (Ag, h)$, have a unique norm-preserving extension to the space $(K^*\hat{\otimes}_\pi G)/R_U$. On the other hand, a well-known consequence of the classical proof of the Hahn-Banach Theorem is that a functional $f\in((H^*\hat{\otimes}_\pi G)/R_U)'$ of norm 1 has a unique norm-preserving extension to $(K^*\hat{\otimes}_\pi G)/R_U$ if an only if for any $\mathfrak{z}\notin H^*\hat{\otimes}_\pi G$,

$$\sup\{\operatorname{Re} f(\dot{\mathfrak{z}}_1)-\|\dot{\mathfrak{z}}_1-\dot{\mathfrak{z}}\| : \dot{\mathfrak{z}}_1\in(H^*\hat{\otimes}_\pi G)/R_U\}=\inf\{\|\dot{\mathfrak{z}}_2+\dot{\mathfrak{z}}\|-\operatorname{Re} f(\dot{\mathfrak{z}}_2) : \dot{\mathfrak{z}}_2\in(H^*\hat{\otimes}_\pi G)/R_U\}.$$

(Here, as in the sequel, we set $\dot{\mathfrak{z}}=\psi(\mathfrak{z})$ for $\mathfrak{z}\in K^*\hat{\otimes}_\pi G$). Hence, we easily infer the following sufficient and necessary condition for that an $A\in L(G;H)$, $\|A\| = 1$, satisfying $TA = AZ$ have a unique exact intertwining dilation :

For any $\varepsilon > 0$ and for any $\mathfrak{z}\in(K^*\hat{\otimes}_\pi G)\setminus(H^*\hat{\otimes}_\pi G)$ there exist $\mathfrak{z}_1,\mathfrak{z}_2\in H^*\hat{\otimes}_\pi G$ satisfying

$$\|\dot{\mathfrak{z}}_1+\dot{\mathfrak{z}}_2\|\le\|\dot{\mathfrak{z}}_1-\dot{\mathfrak{z}}\|+\|\dot{\mathfrak{z}}_2+\dot{\mathfrak{z}}\|<\operatorname{Re} f(\dot{\mathfrak{z}}_1+\dot{\mathfrak{z}}_2)+\varepsilon. \tag{8}$$

Remark 2.2. Let us also note that, since a linear bounded functional of norm a on a closed proper subspace has several linear continuous extensions on the whole space of norm $a' > a$, it follows easily:

For any $A \in L(G;H)$ intertwining T and Z there are several non exact intertwining dilations.

By this remark we shall establish the following

Theorem 2.1. Let T be a contraction on a Hilbert space H and X an operator on H with $\|X\| = 1$, double-commuting with T. Then, X has a unique exact commuting dilation if and only if X is an isometry or a coisometry.

Proof. First, assume that X is isometry or coisometry. Then X or X^* has a unique exact dilation relating to T or T^*, respectively (see [3], Prop.10.8). But it is easy to show that an operator X commuting with a contraction T has a unique exact dilation if and only if so has X^* (naturally in relation to T^*).

Now, assume that X has a unique exact dilation . Let $X = VR$ be the polar decomposition of X, where R is its absolute value and $V \mid ImR$ is an isometry. Since X double-commutes with T, the self-adjoint operator R commutes with T, too. Denote $H_0 = KerR = KerX$, $H_1 = \overline{ImR}$, $T_i = T \mid H_i$, $R_i = R \mid H_i$ $(i = 0,1)$. Obviously H_i is invariant for T_i and R_i $(i = 0,1)$. Let us consider for R_1 a spectral representation

$$R_1 = \int_{[0,1]} \lambda \, dE_\lambda .$$

Denoting for a fixed $\lambda \in]0,1[$, $H_{1,1} = E_\lambda H_1$ and $H_{1,2} = (I - E_\lambda)H_1$ we obtain the decompositions

$$H_1 = H_{1,1} \oplus H_{1,2}, \quad R_1 = R_{1,1} \oplus R_{1,2}, \quad T_1 = T_{1,1} \oplus T_{1,2}$$

where $R_{1,i} = P_{H_{1,i}} R_1 \mid H_{1,i}$ and $T_{1,i} = P_{H_{1,i}} T_1 \mid H_{1,i}$ $(i = 1,2)$. Then, the isometric dilation of T_1 on $K_1 = \bigvee_{n\geq 0} U^n H_1$ will be of the form $U_1 = U_{1,1} \oplus U_{1,2}$, where $U_{1,i}$ are the isometric dilations of T_i on $K_{1,i} = \bigvee_{n\geq 0} U^n_{1,i} H_{1,i}$. Also, since $\|R_{1,1}\| < 1$, there exist (cf. Remark 2.2) at least two distinct commuting dilations of $R_{1,1}$ on $K_{1,1}$, of norm 1. Therefore since the orthogonal sum of the dilations of $R_{1,1}$ and $R_{1,2}$ is a dilation of R_1, it results that R_1 has at least two distinct exact dilations on K_1.

Denote $V_1 = V \mid H_1$, $X_1 = X \mid H_1$. Since X and R commute with T, we have $V_1 T_1 = T V_1$. Let $\hat{V}_1: K_1 \longrightarrow K$ be an exact dilation of V_1 (where K is the space of isometric dilation U of T) and let $\hat{R}'_1$, $\hat{R}''_1$ be distinct exact dilations of R_1. Then, since $X_1 = V_1 R_1$, $\hat{V}_1 \hat{R}'_1$ and $\hat{V}_1 \hat{R}''_1$ are exact dilations of X_1 on K_1. But, by hypothesis X has a unique exact dilation and then so has X_1. Thus $\hat{V}_1 \hat{R}'_1 = \hat{V}_1 \hat{R}''_1$. From this, since $\hat{V}_1$ is an isometry (see [3], Prop. 10.8), we infer that $\hat{R}' = \hat{R}''$, which implies $H_1 = \{0\}$. Thus, we have $R_1 = I$ and consequently X is a partial - isometry.

Denote $H'_0 = \operatorname{Ker} X^*$, $H'_1 = X H_1$ and $T'_i = T \mid H'_i$ $(i = 0,1)$. Since X double commutes with T, it results that the spaces H'_0, H'_1 are invariant for T. Now, let us consider the following decompositions of H and T, respectively:

$$H = H_0 \oplus H_1 = H'_0 \oplus H'_1, \text{ and } T = T_0 \oplus T_1 = T'_0 \oplus T'_1$$

Then, the isometric dilation of T on K is of the form $U = U_0 \oplus U_1 = U'_0 \oplus U'_1$, where U_i, U'_i are isometric dilations of T_i, T'_i on the spaces $K_i = \bigvee_{n\geq 0} U^n H_i$, $K'_i = \bigvee_{n\geq 0} U^n H'_i$ $(i = 0,1)$, respectively.

Note that $X_0 = X \mid H_0$ $(: H_0 \longrightarrow H'_0) = 0$, $X_1 = X \mid H_1$ $(: H_1 \longrightarrow H'_1)$ is an unitary operator and $X_i T_i = T'_i X_i$ $(i = 0,1)$. Let $\hat{X}_i : K_i \to K'_i$ $(i = 0,1)$ be intertwining dilations of X_i, of norm 1 $(i = 0,1)$.

Then, $\hat{X}_0 \oplus \hat{X}_1$ [2] is an exact dilation of X on K. By Remark 2.2 for X_0 there exist at least two distinct intertwining dilations of norm 1. But, since X has a unique exact dilation, we finally infer that $H_0 = \{0\}$ (i.e. X is an isometry) or $H_0' = \{0\}$ (i.e. X is a coisometry).

3. We introduce the following definition for contractions on Hilbert spaces:

Definition 3.1. Let $A_1, A_2 \in L(H_1; H_2)$ be two contractions. We say that A_1 Harnack-dominates A_2 if for some positive constants C, C′ we have:

(9) $\qquad \|D_{A_2} h\| \leq C \|D_{A_1} h\| \quad$ and $\quad \|(A_2 - A_1)h\| \leq C' \|D_{A_1} h\|$

for all $h \in H_1$. Here D_{A_1}, D_{A_2} are the defect operators of A_1, A_2, i.e. $D_{A_i} = (1 - A_i^* A_i)^{\frac{1}{2}}$ $\quad (i = 1,2)$.

Remark 3.1. Let us introduce, for the contractions $A_1, A_2 \in L(H_1, H_2)$, the following isometries:

$$\hat{A}_i = \begin{pmatrix} A_i \\ D_{A_i} \end{pmatrix} : H_1 \to \begin{matrix} H_2 \\ \oplus \\ \mathcal{D}_{A_i} \end{matrix} \qquad (i = 1,2),$$

where $\mathcal{D}_{A_i} = \overline{D_{A_i} H_1}$ $(i = 1,2)$ Then, conditions (9) of Definition 3.1 are plainly equivalent to the following:

The exists a bounded operator

$$K : \begin{matrix} H_2 \\ \oplus \\ \mathcal{D}_{A_1} \end{matrix} \to \begin{matrix} H_2 \\ \oplus \\ \mathcal{D}_{A_2} \end{matrix}$$

such that

(10) $\qquad K\begin{pmatrix} h_2 \\ 0 \end{pmatrix} = \begin{pmatrix} h_2 \\ 0 \end{pmatrix}$ $\quad$ for all $h_2 \in H_2$, and $\hat{A}_2 = K\hat{A}_1$.

Remark 3.2. We note that, if H_1 and H_2 coincide, then the equivalence relation for contractions on H, defined by:

$$A_1 \text{ Harnack-dominates } A_2 \text{, and } A_2 \text{ Harnack-dominates } A_1$$

coincides with the Harnack-equivalence as defined in 4 , p.362.

For two operators $A_1, A_2 \in L(G;H)$, intertwining T and Z, denote by f_{A_1}, f_{A_2} the functionals $\in ((H^* \hat{\otimes}_\pi G)/R_U)'$, corresponding to A_1 and A_2, respectively, and by F_{A_1}, F_{A_2} the functionals $\in (H^* \hat{\otimes}_\pi G)'$, satisfying $F_{A_1} | R_U = F_{A_2} | R_U = 0$, which correspond to f_{A_1}, f_{A_2} by virtue of the isometric - isomorphism

$$((H^* \hat{\otimes}_\pi G)/R_U)' \cong R_U^{\perp}.$$

Lemma 3.1. *Let $A_1, A_2 \in L(G;H)$ be two operators intertwining T and Z, $\|A_1\| = \|A_2\| = 1$, and such that A_1 Harnack-dominates A_2. Then,*

$$\|\tau\|_\pi - \operatorname{Re} F_{A_1}(\tau) \leq \varepsilon \quad \text{(for some } \varepsilon > 0 \text{ and } \tau \in H^* \hat{\otimes}_\pi G)$$

implies

$$\operatorname{Re} F_{A_1}(\tau) \leq \operatorname{Re} F_{A_2}(\tau) + 2\varepsilon(\|K\|^2 - 1)$$

(K is the bounded operator satisfying (10), which exists by Remark 3.1.)

Proof. Let $\tau \in H^* \hat{\otimes}_\pi G$ be such that:

$$\|\tau\|_\pi - \operatorname{Re} F_{A_1}(\tau) \leq \varepsilon$$

for some $\varepsilon > 0$. There exists a representation of τ , say

$$\tau = \sum_{n \in N} h_n^* \otimes g_n ,$$

with

$\| g_n \| = 1$, $\sum_{n \in N} \|h_n\| < \infty$, and $\|\tau\|_\pi \leq \sum_{n \in N} \|h_n\| < \|\tau\|_\pi + \varepsilon$.

Since $F_{A_i}(h_n^* \otimes g_n) = (A_i g_n, h_n)$ $(i = 1,2)$, and since F_{A_i} are continuous it results that the series $\sum_{n \in N} (A_i g_n, h_n)$ $(i = 1,2)$ are absolutely convergent, and

$$F_{A_i}(\tau) = \sum_{n \in N} (A_i g_n, h_n) \quad .$$

Consequently,

$$\sum_{n \in N} \|h_n\| - \sum_{n \in N} \operatorname{Re}(A_1 g_n, h_n) \leq 2\varepsilon$$

Now let us notice that

$$1 - \operatorname{Re}(A_i g_n, f_n) = \frac{1}{2} \left\| \begin{bmatrix} A_i g_n - f_n \\ D_{A_i} g_n \end{bmatrix} \right\|^2 = \frac{1}{2} \left\| \hat{A}_i g_n - \hat{f}_n \right\|^2$$

where $f_n = \dfrac{h_n}{\|h_n\|}$ and $\hat{f}_n = \begin{pmatrix} f_n \\ 0 \end{pmatrix}$ $(n \in N)$. Since A_1 Harnack-dominates A_2 in virtue of Remark 3.1 we also have

$$\|\hat{A}_2 g_n - \hat{f}_n\|^2 = \|K(\hat{A}_1 g_n - \hat{f}_n)\|^2 \leq \|K\|^2 \|\hat{A}_1 g_n - \hat{f}_n\|^2$$

Therefore,

$$\operatorname{Re}(A_1 g_n, h_n) - \operatorname{Re}(A_2 g_n, h_n) \leq \frac{1}{2}(\|K\|^2 - 1) \|\hat{A}_1 g_n - \hat{f}_n\|^2 \|h_n\|, (n \in N)$$

Whence,

$$\operatorname{Re} F_{A_1}(\tau) - \operatorname{Re} F_{A_2}(\tau) \leq (\|K\|^2 - 1) \sum_{n \in N} \frac{1}{2} \|\hat{A}_1 g_n - \hat{f}_n\|^2 \|h_n\| =$$

$$= (\|K\|^2 - 1) \sum_{n \in N} \left[\|h_n\| - \operatorname{Re}(A_1 g_n, h_n) \right] < 2\varepsilon (\|K\|^2 - 1).$$

We may now state and prove our main theorem concerning the uniqueness of exact intertwining dilation.

Theorem 3.1. Let A_1, $A_2 \in L(G;H)$ be operators with the properties:

$$TA_1 = A_1 Z, \quad TA_2 = A_2 Z, \quad \|A_1\| = \|A_2\| = 1, \quad A_1 \text{ Harnack-dominates } A_2 .$$

Then, if A_1 has a unique exact intertwining dilation so has A_2.

Proof. By Remark 2.1, we must show that if the functional $f_{A_1} \in ((H^* \hat{\otimes}_\pi G)/R_U)'$ defined by A satisfies condition (8), then the functional $f_{A_2} \in ((H \times G)/R_U)'$ defined by A_2, also satisfies it.

Assume that for $\varepsilon > 0$ and $\tau \in (K^* \hat{\otimes}_\pi G) \setminus (H^* \hat{\otimes}_\pi G)$ we have

(11) $$\|\dot{\tau}_1 + \dot{\tau}_2\| \leq \|\dot{\tau}_1 - \dot{\tau}\| + \|\dot{\tau}_2 + \dot{\tau}\| < \operatorname{Re} f_{A_1}(\dot{\tau}_1 + \dot{\tau}_2) + \varepsilon$$

for some $\tau_1, \tau_2 \in H^* \hat{\otimes}_\pi G$. Since $\|\dot{\tau}\| = \|f(\tau)\| = \|\psi(\tau)\|$ for all $\tau \in H^* \hat{\otimes}_\pi G$, there exists $\tau' \in R_T$ such that

$$\|\tau_1 + \tau_2 + \tau'\|_\pi < \|\varphi(\tau_1 + \tau_2)\| + \varepsilon = \|\dot{\tau}_1 + \dot{\tau}_2\| + \varepsilon .$$

Denote $\tau_2' = \tau_2 + \tau'$ and note that

$$\|\dot{\tau}_1 + \dot{\tau}_2'\| = \|\dot{\tau}_1 + \dot{\tau}_2\| \quad \text{and} \quad f_{A_i}(\dot{\tau}_1 + \dot{\tau}_2') = f_{A_i}(\dot{\tau}_1 + \dot{\tau}_2) .$$

Then, from (11) we readily infer that

$$\|\tau_1 + \tau_2'\|_\pi < \operatorname{Re} f_{A_1}(\dot{\tau}_1 + \dot{\tau}_2') + 2\varepsilon = \operatorname{Re} F_{A_1}(\tau_1 + \tau_2') + 2\varepsilon .$$

Consequently, in virtue of Lemma 3.1, it follows

$$\operatorname{Re} F_{A_1}(\tau_1 + \tau_2') \leq \operatorname{Re} F_{A_2}(\tau_1 + \tau_2') + 2\varepsilon(\|K\|^2 - 1)$$

or, equivalently,

$$\operatorname{Re} f_{A_1}(\dot{\tau}_1 + \dot{\tau}_2) \leq \operatorname{Re} f_{A_2}(\dot{\tau}_1 + \dot{\tau}_2) + 2\mathcal{E}(\|K\|^2 - 1).$$

Whence it results that f_{A_2} satisfies the condition

$$\|\dot{\tau}_1 - \dot{\tau}\| + \|\dot{\tau}_2 + \dot{\tau}\| < \operatorname{Re} f_{A_2}(\dot{\tau}_1 + \dot{\tau}_2) + 2\mathcal{E}(\|K\|^2 - 1).$$

Thus, we can conclude that f_{A_2} satisfies (8) too.

As a corollary of the previous theorem we have the following more general result:

Theorem 3.2. Let T, T' be two contractions on the Hilbert spaces H and H', respectively. Moreover let $A_1, A_2 \in L(H';H)$ satisfy the conditions:

$$TA_1 = A_1T', \quad TA_2 = A_2T', \quad \|A_1\| = \|A_2\| = 1, \quad A_1 \text{ Harnack-dominates } A_2$$

Then, if A_1 has a unique exact intertwining dilations so has A_2.

Indeed, denoting by Z the isometric dilation of T' it is known (see [5], Th.2.3) that any exact intertwining dilation of A_i $(i = 1,2)$ is obtained as exact intertwining dilation of the operators $B_i = A_iP_{H'}$ $(i = 1,2)$ intertwining T and Z.

4. Let T, T' be two contractions on the Hilbert space H and H', and let U, U' be their minimal isometric dilations on the spaces K and K', respectively.

Theorem 4.1. Let $B_1, B_2 \in L(K';K)$ have the properties:

$$\|B_1\| = \|B_2\| = 1, \quad UB_i = B_iU', \quad PB_i(I-P') = 0 \quad (i = 1,2) \quad \text{where } P = P_H, \; P' = P_{H'}$$

$$B_1 \text{ Harnack-dominates } B_2,$$

and let $A_1, A_2 \in L(H';H)$ be the operators $A_i = PB_i \mid H'$ $(i = 1,2)$.

Then, if B_1 is an exact intertwining dilation of A_1, B_2 is an exact intertwining dilation of A_2; moreover, if B_1 is the unique exact intertwining dilation for A_1, so is B_2 for A_2.

Proof. First, by hypothesis we observe that $PB_i = A_iP'$ and A_i is intertwining T and T'. Thus, B_i is an intertwining dilation of A_i (i = 1,2).

Now, in order to prove that B_2 is an exact intertwining dilation for A_2 if B_1 is so for A_1, it suffices to show that $\|A_2\| = 1$.

Clearly, we have (by definition of A_2) $\|A_2\| \leq 1$.

For the converse inequality we observe that, since B_1 Harnack-dominates B_2 i.e. $\|D_{B_2}k'\| \leq C\|D_{B_1}k'\|$ and $\|(B_2-B_1)k'\| \leq C'\|D_{B_1}k'\|$ with C, $C' > 0$, we have for $h' \in H'$

$$\|(I-P)B_2h'\| \leq \|(I-P)B_1h'\| + \|(I-P)(B_2-B_1)h'\| \leq \|D_{A_1}h'\| + \|(B_2-B_1)h'\|$$

$$\leq \|D_{A_1}h'\| + C'\|D_{B_1}h'\| \leq (1+C')\|D_{A_1}h'\|$$

and therefore,

$$\|D_{A_2}h'\|^2 = \|D_{B_2}h'\|^2 + \|(I-P)B_2h'\|^2 \leq (C^2 + (1+C')^2)\|D_{A_1}h'\|^2 =$$

$$= C''\|D_{A_1}h'\|^2, \text{ for any } h' \in H'.$$

Since $\|A_1\| = 1$, we infer from this inequality that $\|A_2\| = 1$ too, thus B_2 is an exact intertwining dilation of A_2.

The above relation with the following one:

$$\|(A_2-A_1)h'\| \leq \|(B_2-B_1)h'\| \leq C'\|D_{B_1}h'\| \leq C'\|D_{A_1}h'\| \quad (h' \in H')$$

mean that A_1 Harnack-dominates A_2. Now the second statement of this theorem can be obtained by referring to Theorem 3.2.

Lemma 4.1. *Let* B_1, B_2 $L(K';K)$, $\|B_1\| = \|B_2\| = 1$ *be of the form* $B_i = B_0 \oplus S_i$ *where* S_i *are strict contractions* (i = 1,2). *Then* B_1, B_2 *Harnack-dominate each other.*

Proof. Consider the decomposition $K' = K_0' \oplus K_1'$ for which

$$B_1 P_{K_0'} = B_2 P_{K_0'} = B_0 \quad \text{and} \quad S_i = B_i P_{K_1'} = B_i(1 - P_{K_0'})$$

and note that

$$\| D_{B_i} k' \|^2 = (\| k_0' \|^2 - \| B_0 k_0' \|^2) + (\| k_1' \|^2 - \| S_i k_1' \|^2)$$

$$\geqslant \| k_1' \|^2 - \| S_i k_1' \|^2 \geqslant (1 - \| S_i \|^2) \| k_1' \|^2 .$$

(Here $k_0' = P_{K_0'} k'$, $k_1' = P_{K_1'} k'$.)

Whence, by taking $C = \max \left\{ (1 - \| S_1 \|^2)^{-\frac{1}{2}} , \ (1 - \| S_2 \|^2)^{-\frac{1}{2}} \right\}$ it follows

$$\| P_{K_1'} k' \| \leqslant C \| D_{B_i} k' \| \quad \text{for all } k' \in K' .$$

Therefore, we have $\| (B_2 - B_1) k' \| \leqslant \| S_2 - S_1 \| \ \| k_1' \| \leqslant C' \| D_{B_i} k' \|$ and also

$$\| D_{B_2} k' \|^2 = \| k' \|^2 - \| B_0 k_0' \|^2 - \| S_2 k_1' \|^2 =$$

$$= \| D_{B_1} k' \|^2 + (\| S_1 k_1' \| - \| S_2 k_1' \|) \ (\| S_1 k_1' \| + \| S_2 k_1' \|)$$

$$\leqslant \| D_{B_1} k' \|^2 + \| S_1 - S_2 \| (\| S_1 \| + \| S_2 \|) \ \| k_1' \|^2 ,$$

hence $\| D_{B_2} k' \| \leqslant C'' \| D_{B_1} k' \|$ for all $k' \in K$ where, C', C'' are constants.

Thus B_1 Harnack-dominates B_2.

By symmetry B_2 also Harnack-dominates B_1.

Theorem 4.1 and Lemma 4.1 have the following

Corollary 4.1. <u>Let</u> B_1, $B_2 \in L(K';K)$ <u>be two operators as in Lema</u> 4.1, intertwining U and U' and such that : $B_i(K' \ominus H') \subset K \ominus H$ ($i = 1,2$). Then, B_1 is an exact intertwining dilation of $A_1 = P_H B_1 | H'$, if and only if B_2 is an exact intertwining dilation of $A_2 = P_H B_2 | H'$; moreover, B_1 is the unique intertwining dilation for A_1 if and only if B_2 is so for A_2.

In virtue of Theorems 2 and 5 of [2], we also have the following corollary of Theorem 4.1, concerning the Hankel operators [3)].

Corollary 4.2. Let $F_1, F_2 \in L^\infty(\mathcal{E},\mathcal{F})$ ($\mathcal{E},\mathcal{F}$ - separable Hilbert spaces) have the properties :

$$\|F_1\| = \|F_2\| = 1$$

$F_1(t) = F_2(t)$ whenever $\max\{\|F_1(t)\|, \|F_2(t)\|\} > 1 - \theta$ for some fixed θ, $0 < \theta < 1$.

Then, if one of these functions is a minifunction for its Hankel operator, then so is the other. Moreover, if one of them is the unique minifunction of its Hankel operator so is the other.

Proof. Let us set

$$\mathcal{L}_0 = \chi\{t : \max\{\|F_1(t)\|, \|F_2(t)\|\} > 1-\theta\} L^2(\mathcal{E}), \text{ and}$$

$$\mathcal{L}_1 = \chi\{t : \max\{\|F_1(t)\|, \|F_2(t)\|\} \le 1-\theta\} L^2(\mathcal{E})$$

where χ is the characteristic function. Then $L^2(\mathcal{E})$ is the orthogonal sum $L^2(\mathcal{E}) = \mathcal{L}_0 \oplus \mathcal{L}_1$ and the spaces $\mathcal{L}_0$, $\mathcal{L}_1$ are invariant for the operator $U^x(e^{it}) = e^{it}$.

Also, notice that the operators $B_i f = F_i f$ ($f \in L^2(\mathcal{E})$, $i = 1,2$) can be written $B_i = B_0 \oplus S_i$, where $B_0 = B_i P_{\mathcal{L}_0}$ and $S_i = B_i(I - P_{\mathcal{L}_0})$, ($i = 1,2$), with $\|S_i\| < 1$.

Now, Corollary 4.2. follows at once by Corollary 4.1.

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Foot - notes

1) This reduction already was done in some more or less particular cases (see for instance [6]).

2) $\hat{X}_o \oplus \hat{X}_1$ denotes the operator from $K = K_o \oplus K_1$ to $K = K'_o \oplus K'_1$ given by the matrix $\begin{pmatrix} \hat{X}_o & O \\ O & \hat{X}_1 \end{pmatrix}$.

3) This corollary can be also obtained as a consequence of Theorems 1.3 and 3.1 of [1].