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STABILITY OF CONCORDANCES AND THE SUSPENSION HOMEOMORPHISM

by

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Stability of Concordances and the Suspension Homeomorphism

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The canonical map $\varphi: S^{n-1} \to \Omega S^n$ induces the suspension homeomorphism $\pi_i(S^{n-1}) \to \pi_{i+1}(S^n)$. This map may be viewed as a map $\varphi: O_n/O_{n-1} \to \Omega(O_{n+1}/O_n)$, where O_n is the orthogonal group in R^n. As we will show, φ may be described as follows: For $0 \le t \le 1$, let $\rho_t \in O_2$ be rotation by the angle $t\pi$, and let $R_t \in O_{n+1}$ be the transformation $\begin{pmatrix} I_{n-1} & 0 \\ 0 & \rho_t \end{pmatrix}$. Define $\psi: I \times O_n \to O_{n+1}$ by $\psi(t,h) = R_t \circ h \times 1 \circ R_{-t}$, where O_n is considered as acting on the first n-coordinates of $R^{n+1} = R^n \times R$. Then ψ induces a map $\psi: I \times O_n/O_{n-1} \to O_{n+1}/O_n$ and in fact a map $\psi: \Sigma(O_n/O_{n-1}) \to O_{n+1}/O_n$ which we will show is the usual identification $\Sigma S^{n-1} \to S^n$. φ is the adjoint of ψ.

Now for other groups (or H-spaces) acting on R^n and containing O_n, such as Top_n or G_n, the group of homeomorphisms of R^n and the space of proper homotopy equivalences of R^n respectively; ψ also defines maps $\psi: \Sigma(Top_n/Top_{n-1}) \to Top_{n+1}/Top_n$ and $\psi: \Sigma(G_n/G_{n-1}) \to G_{n+1}/G_n$. (This is an abuse of notation since strictly speaking the quotient spaces G_n/G_{n-1}, for example, are not defined). The adjoint φ then induces homeomorphisms $\varphi_*: \pi_i(Top_n, Top_{n-1}) \to \pi_{i+1}(Top_{n+1}, Top_n)$ and similarly for G_n. Since

$\pi_i(O_n/O_{n-1}) \to \pi_i(O_{n+1}/O_n)$ is an isomorphism for $i < 2n-3$, it is natural to conjecture that the same holds for the other spaces. In fact, since the inclusion of O_n in G_n induces isomorphisms $\pi_i(O_n, O_{n-1}) \to \pi_i(G_n, G_{n-1})$ for $i < 2n-3$ (see [9]), the result holds for G_n.

In order to prove a similar result for Top_n it is natural to consider the homotopy theoretic fibre F_{n-1} of the map $Top_{n-1}/O_{n-1} \to Top_n/O_n$, or equivalently the map $O_n/O_{n-1} \to Top_n/Top_{n-1}$. Note that φ defines a map $\varphi: F_{n-1} \to \Omega F_n$. Now in [6] it is shown that this fibre is $C^t(S^{n-1})$, the group of topological concordances or pseudo-isotopies of S^{n-1}. Further it can be shown that φ corresponds to a composition of maps: $C^t(S^{n-1}) \xrightarrow{\emptyset} C^t(S^{n-1} \times I) \xrightarrow{\lambda} \Omega C^t(S^n)$, where $\emptyset$ is $f \to f \times id_I$ and λ is $2n-3$ connected.

Now Hatcher [2] has shown for any compact PL-manifold M^n that $\emptyset: C^t(M^n) \to C^t(M^n \times I)$ is $\gamma(n)$ connected, where $\gamma(n+1) \geq \gamma(n)$ and $\lim_n \gamma(n) = \infty$. Thus if $\gamma(n)$ were sufficiently large the result for Top_n would follow. Unfortunately, the best result for $\gamma(n)$ so far is roughly $n/3$. Explicitly

<u>Theorem</u> (A. Hatcher): $\emptyset: C^t(M^n) \to C^t(M^n \times I)$ is at least $(n-10)/3$ connected.

(Hatcher's result is for $C^{PL}(M^n)$, but for $n \geq 5$, $C^{PL}(M) \simeq C^t(M)$, see [1].)

It turns out that by using Morlet's comparison theorem [1] to translate from topological to smooth concordances we can not only overcome this problem but also prove a stability theorem for smooth concordances.

The sections of this paper are as follows

§1 The suspension map: It is proved that $\psi: \Sigma O_n/O_{n-1} \to O_{n+1}/O_n$ is the usual homeomorphism.

§2 Representation of $C^t(M)/C^d(M)$ as a space of sections (of a fibration over M and fiber F_n). The fibre is F_n and, in particular, it is shown that if M^n has a trivial tangent bundle then $C^t(M)/C^d(M)$ is homotopy equivalent to the space $F_n^{M/\partial M}$ of base point preserving maps of $M/\partial M$ into F_n. Similarly, it is shown that $C^t(M\times I)/C^d(M\times I) \simeq (\Omega F_{n+1})^{M/\partial M}$.

§3 The stability map: It is shown that $\phi: C^t(M)/C^d(M) \to C^t(M\times I)/C^d(M\times I)$ corresponds to a map of sections. Theorem A.

§4 Proof of Theorems B through E: Outline of the argument: If $M^n \subset \text{Int}\, D^n$, then $C(M) \simeq C(D^n)\times C_0(M)$, $C_0(M) = C(M)/C(D^n_\varepsilon)$, $D^n_\varepsilon \subset \text{Int}\, M^n$. Then $C_0^t(M)/C_0^d(M) \simeq F_n^{M^0/\partial M}$, $M^0 = M - \text{Int}\, D^n_\varepsilon$.

In particular, for $M = S^n \times D^k$, we have

(4.6)
$$\begin{array}{ccc} C_0^t(S^n\times D^k)/C_0^d(S^n\times D^k) & \xrightarrow{\ \phi\ } & C_0^t(S^n\times D^{k+1})/C_0^d(S^n\times D^{k+1}) \\ \downarrow \approx & & \downarrow \approx \\ \Omega^k F_{n+k} & \xrightarrow{\ \varphi\ } & \Omega^{k+1} F_{n+k+1} \end{array}$$

commutes. Further by [2], $\pi_i C_0^d(S^n \times D^k) = 0$ for $i \leq 2n-4$ Since $C^t(D^n)$ is trivial, $C^t(S^n \times D^k) \to C_0^t(S^n \times D^k)/C_0^d(S^n \times D^k)$ is $2n-3$ connected. Thus (4.6) and Hatcher's result shows the suspension map $\varphi: Top_n/Top_{n-1} \to \Omega\, Top_{n+1}/Top_n$ is roughly $n+\gamma(n)/2$ or $n+n/6$ connected (Theorem B). Theorem B and the result of §3 give the stability for $C^d(M)$ (Theorem C).

Theorems D and E give spectral results for k-connected manifolds.

§5 Invariance under k-equivalence. If $M_1^{n_1}$ and $M_2^{n_2}$ are smooth compact k-equivalent manifolds, then $\pi_i C^d(M_1) \simeq \pi_i C^d(M_2)$ for $i \leq k-2$, $k << (n_1, n_2)$.

§6 Concordances as homology theory: Let K be a finite complex and $N_n(K)$ be a regular neighborhood of K in R^n. Let $C^t(K) = \operatorname{Lim}_n C^t(N_n(K))$; then $\operatorname{Lim}_j \pi_{p+\gamma}\, C^t(S^j \wedge K) \simeq \mathcal{F}_p(K)$, where $\mathcal{F}$ is the spectrum $\{F_n\}$.

§7 Algebraic consequences of the suspension isomorphism. As was first noted by Iqusa, Theorem B implies $\hat{\pi}_i Top_n \simeq \hat{\pi}_i Top \oplus \pi_i^a Top_n$, where $\hat{\pi}_i(x) = \pi_i(x) \otimes Z(1/2)$, in a range roughly $i \leq n/6 + n$. Further, under inclusion in Top_{n+1}, the first factor goes isomorphically and the second trivially. Finally $\pi_i^a Top_n \simeq \pi_{i+2}^a Top_{n+2}$. (Here $\pi_i^a Top_n$ is the antisymmetric part of $\hat{\pi}_i(Top_n)$ under conjugation by $\begin{pmatrix} I_{n-1} & 0 \\ 0 & -1 \end{pmatrix}$.)

1. The suspension map $\psi: \Sigma O_n / O_{n-1} \to O_{n+1}/O_n$

Write $x = (x_1 \ldots, x_n)$ for $x \in S^{n-1} \subset R^n$. Let $e_n = (0, \ldots, 0, 1) \in R^n$.

Following Steenrod [12], define $\lambda : S^{n-1} - (-e_n) \to O_n$ by letting $\lambda(x)$ be the rotation which is the identity on points orthogonal to both x and e_n, and rotates the great circle through x and e_n so as to carry e_n into x. In coordinates,

$$\lambda(x) = \left| \begin{array}{c|c} \delta^\alpha_\beta \quad \dfrac{x_\alpha x_\beta}{1+x_n} & \begin{array}{c} x_1 \\ \vdots \\ x_{n-1} \end{array} \\ \hline -x_1 \ \ldots \ -x_{n-1} & x_n \end{array} \right| , \quad 1 \leq \alpha, \beta \leq n-1$$

Then $\lambda(x)e_n = x$. An easy computation gives

$$\psi(t, \lambda(x))e_{n+1} = (x_1 \sin t\pi, \ldots, x_{n-1} \sin t\pi, (1-x_n)\sin t\pi \cos t\pi, x_n \sin^2 t\pi + \cos^2 t\pi).$$

Now ψ extends to $-e_n$ and defines a homeomorphism from the (reduced) suspension of S^{n-1} to S^n.

§ 8. Applications to immersions : Haefliger-Millet have shown for the topological Stiefel manifold $V^t_{n,p} = \mathrm{Emb}^t(R^p, R^n)$ that there is a $2n - p - 3$ connected map : $V^t_{n,p} \longrightarrow G_n/G_{n-p}$. We show for a somewhat larger range that

$$\hat{\pi}_i(V^t_{n,p}) \simeq \hat{\pi}_i G_n/G_{n-p} \oplus \pi^d_i(V^t_{n,p}, V^d_{n,p})$$

and $\pi^d_i(V^t_{n,p}, V^d_{n,p}) = \pi^d_{i+2}(V^t_{n+2, p+2}, V^d_{n+2, p+2})$.

§ 9. Applications to automorphisms, concordances and nilpotencies : We show that in stable range (which means "for the Postnicov k-th term with $k \leq \tau(n)$, $\tau(n)$ being the stabilisation function in the appropiate category) and localised away 2, $A(M^{n-1} \times I)$ and $A(M^{n-1} \times S^1)$ splits up as product of $A^a(\ldots) \times {}^s(\ldots)$, where $A^a(\ldots)$ is homotopy equivalent to $\tilde{A}(\ldots)$ and $A^s(M \times I) \times A^s(M \times I^2)$ is homotopy equivalent (in stable range and localised away 2) to $C(M \times I)$. Moreover, $A^s(M \times I) \sim A^s(M \times I^3)$. We also show that $\mathcal{N}^C(M) \sim \mathcal{N}(M) \times \mathcal{N}(M \times I)$, $\mathcal{N}(M) \sim \mathcal{N}(M \times I^2)$ (in stable range and localised away 2). We mention that $\tilde{A}(\ldots)$ can be theoretically computed and carries the topological properties of M, while $C(M \times I)$ is only a homotopy theoretical invariant.

Let $A^t(M)$ be the group of homeomorphisms of M with the compact open topology and $A^d(M)$ the group of diffeomorphisms of M with the C^∞-topology, fixed on ∂M (or more generally on a compact smooth submanifold $N^{n-1} \subset \partial M$). Then Morlet's comparison theorem as elaborated in [1] states ($\approx$ is homotopy equivalence)

Theorem (Morlet): $A^t(M)/A^d(M) \approx \Gamma(B_n(M))$, where Γ denotes the space of sections $s: M \to B_n(M)$, with C-O topology such that $s|\partial M = s_0|\partial M$ (or $s|N = s_0|N$ and $s|\partial M: \partial M \to B_{n-1}(\partial M) \to B_n(M)$), and s represents a smoothing diffeomorphic to M. The homotopy equivalence is induced by the differential.

Remark. Strictly speaking, $A^d(M)$ is not a topological subgroup of $A^t(M)$, and $A^t(M)/A^d(M)$ should be thought of as the homotopy theoretic fibre of the map $BA^d(M) \to BA^t(M)$ of the universal base spaces. Actually in the proof of the above theorem one treats the groups semi-simplicially so that $A^d(M)$ is a subgroup of $A^t(M)$ and the s.s. space $A^t(M)/A^d(M)$ is defined.

Now let $C^t(M)$ be the group of topological concordances of M and $C^d(M)$ the group of C^∞ concordances, fixed on ∂M. I.e., $C(M)$ = the group of automorphisms of $M \times I$ fixed in $M \times 0 \cup \partial M \times I$.

2. Representation of $C^{top}(M)/C^{diff}(M)$ as a space of sections.

Let M^n be a smooth compact manifold, $n \geq 5$. Let $P(M)$ be the principal Top_n bundle associated to the tangent bundle of M. Let $B_n = Top_n/O_n$ and let $B_n(M) = P(M) \times_{Top_n} B_n$, the associated bundle with fibre Top_n/O_n.

By smoothing theory [7], the isotopy classes of smoothings of (the topological manifold) M are in one-to-one correspondence with the homotopy classes of sections of $B_n(M)$. Let $s_0 : M \to B_n(M)$ be a section in the homotopy class defined by the given smoothing of M. If $\partial M \neq \emptyset$, then the given smoothing of ∂M corresponds to a class of sections $\partial M \to B_{n-1}(\partial M)$ and we can assume that $s_0 | \partial M : \partial M \to B_n(M) | \partial M$ is of the form $\partial M \xrightarrow{t_0} B_{n-1}(\partial M) \xrightarrow{j} B_n(M) | \partial M$, where t_0 is in this latter class, and j is the natural map induced by the obvious map $Top_{n-1}/O_{n-1} \to Top_n/O_n$.

Explicitly if $P^d(M)$ is the associated principal O_n bundle to the tangent vector bundle, then we can take $P(M) = P^d(M) \times_{O_n} Top_n$. Then $B_n(M) = P(M)/O_n = P^d(M) \times_{O_n} Top_n/O_n$ has the natural section $s_0 : M = P^d(M) \times_{O_n} O_n/O_n \to P^d(M) \times_{O_n} Top_n/O_n$.

(2.3) $F_n \to E_n(M) \xrightarrow{p} B_n(M)$, $p[x, \lambda] = [x, \lambda(0)]$, with section $\sigma: B_n(M) \to E_n(M)$, $\sigma[x, b] = [x, \sigma(b)]$. We may then describe $C^t(M)/C^d(M)$ as sections $s: M \to E_n(M)$ such that $p \circ s = s_0$ and $s | \partial M = \sigma s_0 | \partial M$.

Equivalently, taking the induced fibration over M by s_0:

$$\begin{array}{ccc} F_n & = & F_n \\ \downarrow & & \downarrow \\ s_0^* E_n(M) & \longrightarrow & E_n(M) \\ \downarrow & & p \downarrow \uparrow \sigma \\ M & \xrightarrow[s_0]{} & B_n(M) \end{array}$$

(2.4) $C^t(M)/C^d(M) \approx \Gamma(s_0^* E_n(M))$, sections $s: M \to s_0^* E_n(M)$ such that $s | \partial M = \overline{s}_0 | \partial M$, where $\overline{s}_0 : M \to s_0^* E_n(M)$ is the induced section from σs_0.

Remark 1. If the tangent vector bundle of M is trivial, then $C^t(M)/C^d(M) \approx F_n^{M/\partial M}$, where we write X^A for the space of a base pointed maps of A in X with the C-O topology.
(If $\partial M = \emptyset$, $M/\partial M = M \cup \text{pt.}$).

2. F_n is the homotopy theoretic fibre of the map

$$Top_n/O_n \to Top_{n+1}/O_{n+1} .$$

3. Naturality: Let $N^n \subset \text{Int } M^n$ be a compact smooth submanifold of codimension zero. Since $h \in C(N)$ is fixed in $\partial N \times I$ it extends by the identity to $\hat{h} \in C(M)$. Thus $C(N) \subset C(M)$. Also

Then by Morlet's theorem we have, since $B_{n+1}(M\times I) = B_{n+1}(M)\times I$ ($B_{n+1}(M) = P(M) \underset{Top_n}{\times} B_{n+1}$)

(2.1) $C^t(M)/C^d(M) \approx \Gamma(B_{n+1}(M)\times I)$

sections $s: M\times I \to B_{n+1}(M)\times I$, with $s|M\times 0 \cup \partial M\times I = s_0\times id$ and $s|M\times 1: M\times 1 \to B_n(M)\times 1 \to B_{n+1}(M)\times 1$; $s_0: M \to B_n(M) \to B_{n+1}(M)$ representing the given smoothing.

To put this another way, s is a path of sections of $B_{n+1}(M)$ beginning at s_0 and ending in $B_n(M) \subset B_{n+1}(M)$. To exploit this we make some definitions: If $X, Y \subset Z$ we let $\Lambda(Z; X, Y)$ be paths in Z beginning in X and ending in Y (with the C-O topology). Then we let:

$E_n = \Lambda(B_{n+1}; B_n, B_n)$

$F_n = \Lambda(B_{n+1}; *, B_n)$

We have the fibration

(2.2) $F_n \xrightarrow{i} E_n \xrightarrow{p} B_n$, $p(\lambda) = \lambda(0)$.

Let $\sigma: B_n \to E_n$ be the cross-section $\sigma(b): I \to b$, the constant path. Now Top_n acts on B_n, B_{n+1} and hence on E_n by $(g\lambda)(t) = g(\lambda(t))$. Further p and σ commute with this action. Thus if we let $E_n(M) = P(M) \underset{Top_n}{\times} E_n$, we have the fibration is in X for $r \leq 0$ and in Y for $t = 0, 1$. Let

$$G_n = \Lambda^2(B_{n+2}; B_{n+1}, B_{n+1})$$

$$H_n = \Lambda^2(B_{n+2}; *, B_{n+1}).$$

We have the fibration

(2.6) $H_n \to G_n \xrightarrow{p} B^I_{n+1}$, $p(\mu) = \mu | r = 0$.

Note that H_n may be identified with ΩF_{n+1}

Take a fixed retract $\rho : I^2 \to$ points with $r = 0$. Then we have a cross-section $\sigma_2 : B^I_{n+1} \to G_n$, $\sigma_2(\omega) = \omega \circ \rho$. Top_n acts on G_n and B^I_{n+1} by $(g\mu)(t, r) = g(\mu(t, r))$ and $g(\omega)(t) = g(\omega(t))$. Further p and σ_2 commute with this action. Thus if we let $G_n(M) = P(M) \times_{Top_n} G_n$ and $B^I_{n+1}(M) = P(M \times_{Top_n} B^I_{n+1})$, we have the fibration

(2.7) $H_n \to G_n(M) \underset{\sigma_2}{\overset{p}{\rightleftarrows}} B^I_{n+1}(M)$

We may then describe $C^t(M \times I)/C^d(M \times I)$ as sections $s: M \to G_n(M)$ such that $ps = \sigma_1 s_0$ and $s | \partial M = \sigma_2 \sigma_1 s_0 | \partial M$, where $M \xrightarrow{s_0} B_{n+1}(M) \xrightarrow{\sigma_1} B^I_{n+1}(M)$, σ_1 induced by $\sigma_1 : B_{n+1} \to B^I_{n+1}$, $\sigma_1(b) : I \to b$ the constant path.

(2.8) Again we may describe $C^t(M \times I)/C^d(M \times I)$ as sections of the pullback $s_0^* \sigma_1^*(G_n(M)) \to M$ with fibre $H_n = \Omega F_{n+1}$, $s | \partial M = \bar{s}_0 | \partial M$, $\bar{s}_0$ the section induced from $\sigma_2 \sigma_1 s_0$.

$P(N) \subset P(M)$ and $s_0^* E_n(N) \subset s_0^* E_n(M)$ are bundle restrictions. Further, $s \in \Gamma(s_0^* E_n(N))$ extends to $\hat{s} \in \Gamma(s_0^* E_n(M))$ by setting $s = \bar{s}_0$ outside N. Then

$$\begin{array}{ccc} C^t(N)/C^d(N) & \approx & \Gamma(s_0^* E_n(N)) \\ \downarrow & & \downarrow \\ C^t(M)/C^d(M) & \approx & \Gamma(s_0^* E_n(M)) \end{array} \quad \text{commutes.}$$

Next we want a similar representation of $C^t(M \times I)/C^d(M \times I)$ as a space of sections. Since $C(M \times I)$ = automorphisms of $M \times I^2$ fixed on $M \times I \times 0 \cup \partial(M \times I) \times I$ and $\overset{\circ}{B}_{n+2}(M \times I^2) = B_{n+2}(M) \times I^2$ we have:

(2.5) $\quad C^t(M \times I)/C^d(M \times I) \approx \Gamma(\overset{\circ}{B}_{n+2}(M) \times I^2)$,

sections $s: M \times I^2 \to B_{n+2}(M) \times I^2$ with $s | M \times I \times 0 \cup \partial(M \times I) \times I = s_0 \times id$ and $s | M \times I \times 1 : M \times I \times 1 \to B_{n+1}(M) \times I \times 1 \to B_{n+2}(M) \times I \times 1$; $s_0 : M \to B_n(M) \to B_{n+2}(M)$ representing the given smoothing.

It will be convenient to use polar coordinates in I^2 so let $I^2 = [-1, 1] \times [0, 1]$ and take coordinates (r, t), $0 \leq t \leq 1$, $r \leq 1$, where $(x, y) \in I^2$ is of the form $((r-1)\cos t\pi, 1+(r-1)\sin t\pi)$. Note that $C(M \times I)$ has as deformation retract the automorphisms of $M \times I^2$ fixed for $r \leq 0$.

Now s is a 2-disc of sections of $B_{n+2}(M)$ with $s = s_0$ for $r \leq 0$ and with s in $B_{n+1}(M)$ for $t = 0, 1$. For $X \subseteq Y \subseteq Z$, let $\Lambda^2(Z; X, Y)$ be the space of maps μ of I^2 into Z such that the image

3. The Stability Map

Define $\varphi: E_n \to G_n$ by $\varphi(\lambda)(r,t) = \psi(t, \lambda(r))$. Then

a) $\varphi | F_n : F_n \to H_n = \Omega F_{n+1}$ is the map defined in the introduction.

b) $p\varphi = \sigma_1 \circ p$

$$\sigma_1 \circ p(\lambda) = \sigma_1(\lambda(0)) = \text{constant path at } \lambda(0) \in B_n \subset B_{n+1}$$

$$p\varphi(\lambda)(t) = \psi(t, \lambda(0)) = \text{constant path at } \lambda(0)$$

c) φ commutes with the action to Top_n .

d) $\varphi \circ \sigma = \sigma_2 \circ \sigma_1$.

(3.1) Here φ induces $\varphi: E_n(M) \to G_n(M)$ with

$$\begin{array}{ccc} F_n & \xrightarrow{\varphi} & H_n \\ \downarrow & & \downarrow \\ E_n(M) & \xrightarrow{\varphi} & G_n(M) \\ p\downarrow & & \downarrow p \\ B_n(M) & \xrightarrow{\sigma_1} & B^I_{n+1}(M) \end{array} \qquad \text{commuting}$$

Further, given $s: M \to E_n(M)$ with $ps = s_0$ and $s|\partial M = \sigma s_0 |\partial M$,

$$\begin{array}{ccccc} & \overset{s}{\nearrow} & E_n(M) & \xrightarrow{\varphi} & G_n(M) \\ & & p\downarrow\uparrow\sigma & & p\downarrow\uparrow\sigma_2 \\ M & \xrightarrow{s_0} & B_n(M) & \xrightarrow{\sigma_1} & B^I_{n+1}(M) \end{array} .$$

we get $\varphi s: M \to G_n(M)$ with $p\varphi s = \sigma_1 \circ p \circ s = \sigma_1 \circ s_0$ and $\varphi s|\partial M = \varphi\sigma s_0 = \sigma_2\sigma_1 s_0$.

Remark 4. If the tangent vector bundle of M is trivial,

$$C^t(M\times I)/C^d(M\times I) \approx (\Omega F_{n+1})^{M/\partial M} .$$

On the other hand, applying Remark 1 to $M\times I$ instead of M, we have $C^t(M\times I)/C^d(M\times I) \approx F_{n+1}^{M\times I/\partial(M\times I)} = F_{n+1}^{\Sigma(M/\partial M)}$. It is easy to check from our definitions that these two representations correspond under the adjoint identification $F_{n+1}^{\Sigma(M/\partial M)} = \Omega F_{n+1}^{M/\partial M}$.

5. $C^t(M\times I)/C^d(M\times I) \approx \Gamma(s_0^* \sigma_1^* G_n(M))$ is natural with respect to $N^n \subset M^n$ (see Remark 3).

I.e. we conjugate $h\times 1$ with $id_M \times \gamma$, where $\gamma: [0,1]\times[0,1] \to [-1,1]\times[0,1]$ is defined by $\gamma(r,t) = ((r-1)\cos t\pi, 1 + (r-1)\sin t\pi)$.

<u>Definition.</u> $\emptyset(h) = id_M \times \gamma \circ h \times 1 \circ id_M \times \gamma^{-1}$.

<u>Remarks.</u> Since h = identity for $r = 0$, $\emptyset(h)$ extends to all of $M\times I\times I$ by making $\emptyset(h)$ = identity outside the half disc.

Since h is a product near $r = 0, 1$, $\emptyset(h)$ is smooth if h is smooth.

We now compute the differential $D\emptyset(h)$ in terms of Dh. First we need the differential of γ:

$$D\gamma = \begin{pmatrix} \cos t\pi & -\pi(r-1)\sin t\pi \\ \sin t\pi & \pi(r-1)\cos t\pi \end{pmatrix} = \begin{pmatrix} \cos t\pi & -\sin t\pi \\ \sin t\pi & \cos t\pi \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \pi(r-1) \end{pmatrix};$$

Then $D\emptyset(h) = R_t \begin{pmatrix} I & 0 \\ 0 & \pi(r-1) \end{pmatrix} \circ Dh\times 1 \circ \begin{pmatrix} I & 0 \\ 0 & \pi(r-1) \end{pmatrix}^{-1} R_t^{-1}$. Since $\begin{pmatrix} I & 0 \\ 0 & \pi(r-1) \end{pmatrix} = id \times (\pi(r-1))$ commutes with $Dh\times 1$, we have:

(3.3) $$D\emptyset(h) = R_t \circ Dh\times 1 \circ R_t^{-1}, \text{ and}$$

<u>Theorem A.</u> $\emptyset: C^t(M)/C^d(M) \to C^t(M\times I)/C^d(M\times I)$ corresponds to the map of sections $\varphi: \Gamma(s_0^*(E_n(M)) \to \Gamma(s_0^*\sigma_1^*G_n(M))$ of (3.2).

Thus

(3.2) $\quad \varphi$ defines a map of fibrations; i.e. a commutative diagram

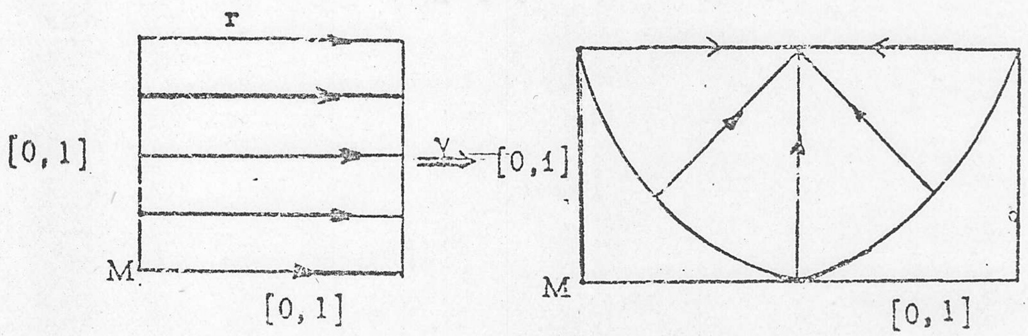
$$\begin{array}{ccc} F_n & \xrightarrow{\varphi} & H_n = \Omega F_{n+1} \\ \downarrow & & \downarrow \\ s_0^* E_n(M) & \xrightarrow{\varphi} & s_0^* \sigma_1^* G_n(M) \\ \downarrow & & \downarrow \\ M & = & M \end{array}$$

Further $\varphi: \Gamma(s_0^* E_n(M)) \to \Gamma(s_0^* \sigma_1^* G_n(M))$, the sections s with $s|\partial M = \overline{s}_0|\partial M$

Next we show that this map φ of sections corresponds to the map $\emptyset: C(M) \to C(M\times I)$ under the differential.

First we may assume our concordances $h: M\times I \to M\times I$, $h|M\times 0 \cup \partial M\times I$ = identity, also satisfy $h|M\times[0,\varepsilon]$ = identity and $h|M\times[1-\varepsilon, 1] = h_1 \times \mathrm{id}$, $h_1 \in A(M)$. I.e. the subspace of such concordances is a deformation retract.

The map $\emptyset$ is essentially $h \to h\times 1: (M\times I)\times I \to (M\times I)\times I$, except that $h\times 1$ is not the identity on $\partial(M\times I)\times I$. In order to correct this and also to make $\emptyset(h)$ smooth when h is smooth we view $M\times I\times I$ as in the picture below (see [4])



Thus it is natural to use the notation $C_0^t(M)/C_0^d(M)$ for $C^t(M) \underset{C^d(M)}{\times} C^d(D^n)$.

Now from the map of bundles

$$
\begin{array}{ccccc}
C^d(D^n) & \to & C^t(M) \underset{C^d(M)}{\times} C^d(D^n) & \to & C^t(M)/C^d(M) \\
\| & & \downarrow & & \downarrow \\
C^d(D^n) & \to & C^t(D^n) \underset{C^d(D^n)}{\times} C^d(D^n) & \to & C^t(D^n)/C^d(D^n)
\end{array}
$$

we see that we have a homotopy fibration

(4.2) $$C_0^t(M)/C_0^d(M) \to C^t(M)/C^d(M) \to C^t(D^n)/C^d(D^n),$$

since $C^t(D^n) = C^t(D^n) \underset{C^d(D^n)}{\times} C^d(D^n)$ is contractible.

Since $D_\varepsilon^n/\partial D_\varepsilon^n \rightleftarrows M/\partial M \rightleftarrows D^n/\partial D^n$ is a homotopy equivalence we see that $M/\partial M \approx (M^0/\partial M) \vee D^n/\partial D^n$, where $M^0 = M - \text{Int } D_\varepsilon^n$. Thus

$$F_n^{M/\partial M} \simeq F_n^{D_\varepsilon^n/\partial D_\varepsilon^n} \times F_n^{M^0/\partial M}.$$

Hence $C_0^t(M)/C_0^d(M) \to C^t(M)/C^d(M) \xrightarrow{d} F_n^{M/\partial M} \to F_n^{M^0/\partial M}$ is a homotopy equivalence. Thus the differential induces

(4.3) $$C_0^t(M)/C_0^d(M) \overset{d}{\approx} F_n^{M^0/\partial M}.$$

Given $N^n \subset \text{Int } M^n$, if we take $D_\varepsilon^n \subset N^n \subset M^n \subset D^n$, then (4.1), (4.2), (4.3) are natural w.r.t the inclusion $C(N) \subset C(M)$. Similarly, we have a natural homotopy equivalence

4. Proof of Theorems B through E.

Assume that the tangent vector bundle of M is trivial and let $D_\varepsilon^n \subset \mathrm{Int}\, M^n$. Then by naturality $C^t(D_\varepsilon^n)/C^d(D_\varepsilon^n) \to C^t(M)/C^d(M)$

$F_n^{D_\varepsilon^n/\partial D_\varepsilon^n} \to F_n^{M/\partial M}$ under the map $M/\partial M \to D_\varepsilon^n/\partial D_\varepsilon^n$ which collapses $M^0 = M - \mathrm{Int}\, D_\varepsilon^n$ to the base point (see Remarks 1, 2, 3).

If further, $M^n \subset \mathrm{Int}\, D^n$, then $C(D_\varepsilon^n) \subset C(M^n) \subset C(D^n)$. Now $C(D_\varepsilon^n)$ is a deformation retract of $C(D^n)$, by uniqueness of collars. Let $C_0(M) = C(M)/C(D_\varepsilon^n)$ and let $g: C(M) \to C_0(M)$ be the quotient map. Then $C(M) \xrightarrow{d} C(M) \times C(M) \xrightarrow{q \times i} C_0(M) \times C(D^n)$ is a homotopy equivalence. Since $C^t(D^n)$ is contractible by the Alexander trick (see [2]), $C^t(M) \to C_0^t(M)$ is a homotopy equivalence.

Taking the associated bundle to $C^d(M) \xrightarrow{i \times i \circ d} C^t(M) \times C^d(D^n) \to C^t(M) \times_{C^d(M)} C^d(D^n)$ with fibre $C_0^d(M) = C^d(M)/C^d(D_\varepsilon^n))$,

$$\begin{array}{ccc} C^d(M)/C^d(D^n) & \to C^t(M) \times_{C^d(D_\varepsilon^n)} C^d(D^n) \to & C^t(M) \times_{C^d(M)} C^d(D^n) \\ \| & \approx \downarrow & \\ C_0^d(M) & \longrightarrow C_0^t(M) & \end{array}$$

we see from the commutativity of the above that we have a homotopy fibration

(4.1) $$C_0^d(M) \to C_0^t(M) \to C^t(M) \times_{C^d(M)} C^d(D^n) .$$

(4.8)
$$\begin{array}{ccc} \pi_i C^t(S^n \times D^k) & \xrightarrow{\ \phi_*\ } & \pi_i C^t(S^n \times D^{k+1}) \\ d_* \downarrow & & \downarrow d_* \\ \pi_i \Omega^k F_{n+k} & \xrightarrow{\ \varphi_*\ } & \pi_i \Omega^{k+1} F_{n+k+1} \end{array}$$

commutes; and for $i \leq 2n-4$, d_* is an isomorphism and for $i \leq 2n-3$, d_* is surjective.

Remark. The above argument shows that (4.8) holds for $k > 0$. However, as remarked in the introduction $C^t(S^n) \to F_n$ is a homotopy equivalence and it is not difficult to show the diagram commutes for $k = 0$.

Thus, since ϕ is $\gamma(n+k)$ connected, we get that φ is $\inf(\gamma(n+k), 2n-3)$ connected, $\varphi : \Omega^k F_{n+k} \to \Omega^{k+1} F_{n+k+1}$. Taking k sufficiently large so that $\gamma(n+k) \geq 2n-3$, and letting $r = n+k$, we get that $\varphi : \Omega^k F_r \to \Omega^{k+1} F_{r+1}$ is $2n-3$ connected, and hence that $\varphi : F_r \to \Omega F_{r+1}$ is $2n-3+k = r+n-3$ connected. Letting $s = n-2$ we have:

(4.9) $\quad \varphi : F_r \to \Omega F_{r+1}$ is $r+s-1$ connected when $2s+1 \leq \gamma(r)$.

From the map of exact sequences

$$\begin{array}{ccccc} \to \pi_i(O_{r+1}/O_r) & \to & \pi_i Top_{r+1}/Top_r & \to & \pi_i(Top_{r+1}/Top_r, O_{r+1}/O_r) \xrightarrow{\gamma} \\ \downarrow \varphi & & \downarrow \varphi & & \downarrow \varphi \\ \pi_{i+1}(O_{r+2}/O_{r+1}) & \to & \pi_{i+1}(Top_{r+2}/Top_{r+2}) & \to & \pi_{i+1}(Top_{r+2}/Top_{r+1}, O_{r+2}/O_{r+1}) \xrightarrow{\gamma} \end{array}$$

using that $\varphi : O_{r+1}/O_r \to (O_{r+2}/O_{r+1})$ is $2r-1$ connected and that $\pi_i(Top_{r+1}/Top_r, O_{r+1}/O_r) \simeq \pi_{i-1}(F_r)$, we get that

(4.4) $$C_0^t(M\times I)/C_0^d(M\times I) \overset{d}{\approx} (\Omega F_{n+1})^{M^0/\partial M} .$$

Further, from the commutativity of the diagram

$$\begin{array}{ccccccc}
C^t(M) \underset{C^d(M)}{\times} C^d(D^n) & \to & C^t(M)/C^d(M) & \xrightarrow{d} & F_n^{M/\partial M} & \to & F_n^{M^0/\partial M} \\
\Big\downarrow \phi & & \Big\downarrow \phi & & \Big\downarrow \varphi & & \Big\downarrow \varphi \\
C^t(M\times I) \underset{C^d(M\times I)}{\times} C^d(D^n\times I) & \to & C^t(M\times I)/C^d(M\times I) & \xrightarrow{d} & (\Omega F_{n+1})^{M/\partial M} & \to & (\Omega F_{n+1})^{M^0/\partial M}
\end{array}$$

we get

(4.5)
$$\begin{array}{ccc}
C_0^t(M)/C_0^d(M) & \overset{d}{\approx} & F_n^{M^0/\partial M} \\
\Big\downarrow \phi & & \Big\downarrow \varphi \\
C_0^t(M\times I)/C_0^d(M\times I) & \overset{d}{\approx} & (\Omega F_{n+1})^{M^0/\partial M}
\end{array}$$
commutes.

In particular, take $M = S^n \times D^k$, then $M/\partial M \simeq S^k \vee S^{n+k}$ and $M^0/\partial M = S^k$. Thus from (4.5) we have:

$$\begin{array}{ccc}
C_0^t(S^n\times D^k)/C_0^d(S^n\times D^k) & \xrightarrow{\phi} & C_0^t(S^n\times D^{k+1})/C_0^d(S^n\times D^{k+1}) \\
\Big\downarrow \approx & & \Big\downarrow \approx \\
\Omega^k F_{n+1} & \xrightarrow{\varphi} & \Omega^{k+1} F_{n+k+1}
\end{array}$$

Now by [2], $\pi_i C_0^d(S^n\times D^k) = 0$ for $i \leq 2n-4$. Also since $C^t(D^{n+k})$ is contractible, $\pi_i C^t(S^n\times D^k) \simeq \pi_i C_0^t(S^n\times D^k)$, all i. Hence

(4.7) $\pi_i C^t(S^n\times D^k) \to \pi_i(C_0^t(S^n\times D^k)/C_0^d(S^n\times D^k))$ is an isomorphism for $i \leq 2n-4$ and surjective for $i \leq 2n-3$

Theorem C. a) $\emptyset: C^t(M)/C^d(M) \to C^t(M\times I)/C^d(M\times I)$ is s-1 connected, $2s+1 \leq \gamma(n)$.

b) $\emptyset: C^d(M) \to C^d(M\times I)$ is s-2 connected, $2s+1 \leq \gamma(n)$.

If M^n is an arbitrary k-1 connected compact smooth manifold, $\pi_i(C_0^d(M)) = 0$ for $i \leq 2k-3$, and $k \leq n-4$ if ∂M is not 1-connected, by Theorem 3.1' of [2].

Theorem D. Let M^n be an k-connected compact smooth manifold. Then

a) $\pi_i(C^t(M)) \simeq \pi_i(C_0^t(M), C_0^d(M))$ for $i \leq 2k-3$, and $k \leq n-4$ if ∂M not 1-connected.

b) $\pi_i(C^t(M)) \simeq \pi_i(\Gamma(s_0^* E_n(M)), F_n^{D^n/\partial D^n})$, $i \leq 2k-3$, and $k \leq n-4$ if ∂M not 1-connected.

c) If further $M^n \subset \operatorname{Int} D^n$, $\pi_i(C^t(M)) \simeq \pi_i(C_0^t(M)/C_0^d(M)) \simeq \pi_i(F_n^{M^0/\partial M})$, $i \leq 2k-3$, and $k \leq n-4$ if ∂M not 1-connected.

Proof. $\pi_i(C^t(M)) \simeq \pi_i(C_0^t(M)) \simeq \pi_i(C_0^t(M), C_0^d(M))$

$$\simeq \pi_i(C^t(M)/C^d(M), C^t(D^n)/C^d(D^n)), \quad i \leq 2k-3.$$

Thus a) and b) follow using (2.4) and Remark 1. c) follows from (4.3).

If $M^n \subset \operatorname{Int} D^n$ is a k-connected compact manifold, note that $\dim(M^0, \partial M) \leq n-k-1$, and so from (4.5) and Theorem D:

$\varphi: Top_{r+1}/Top_r \to \Omega\, Top_{r+2}/Top_{r+1}$ is $\inf(r+s, 2r-1)$ connected, $2s+1 \leq \gamma(r)$.

Theorem B.

a) $\varphi: \pi_i(Top_{r+1}/O_{r+1}, Top_r/O_r) \to \pi_{i+1}(Top_{r+2}/O_{r+2}, Top_{r+1}/O_{r+1})$ is surjective for $i \leq r+s$ and injective for $i < r+s$, $2s+1 \leq \gamma(r)$.

b) $\varphi_*: \pi_i(Top_{r+1}/Top_r) \to \pi_{i+1}(Top_{r+2}/Top_{r+1})$ is surjective for $i \leq r+s$ and injective for $i < r+s$, $2s+1 \leq \inf(\gamma(r), 2r-1)$.

Next we use Theorem A together with Theorem B to get results in the stability of $C^d(M)$.

Lemma 4.9. Let $p_i: E_i \to B$ be fibrations with fiber F_i, $i = 1, 2$ and let $\varphi: E_1 \to E_2$ be a map of fibrations over the identity. Let $s_0: B \to E_1$ be a section. Let (B, A) be a simplicial pair of dim n and $\Gamma(E_i)$ the space of sections $s: B \to E_i$ such that $s|A = s_0|A$ for $i = 1$ and $s|A = \varphi \circ s_0$ for $i = 2$. Then if $\varphi: F_1 \to F_2$ is $n+k$ connected, $\Gamma(\varphi): \Gamma(E_1) \to \Gamma(E_2)$ is k-connected.

Proof. We need to consider homotopy classes of sections s of $id \times p_i: S^q \times E_i \to S^q \times B$ such that $s| * \times B \cup S^q \times A = id \times s_0$ (or $id \times \varphi \circ s_0$). Since the fibre of $id \times \varphi: S^q \times E_1 \to S^q \times E_2$ is the same as the fibre of $F_1 \to F_2$, a section $s: S^q \times B \to S^q \times E_2$ lifts to a section of $S^q \times E_1$ if $q+n \leq k+n$. The lift is unique up to homotopy if $q+n < k+n$.

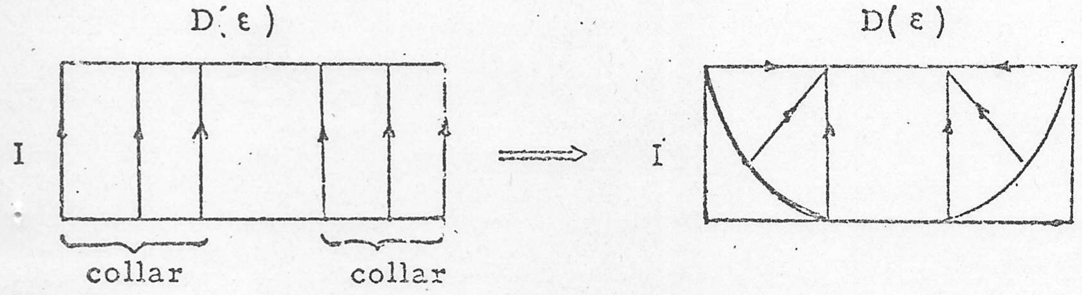
Thus we have:

5. Invariance under k-equivalence

In [3], Hatcher proves that if $M_1^{n_1}$ and $M_2^{n_2}$ are compact k-equivalent PL manifolds, then $\pi_i C^{p\ell}(M_1) \simeq \pi_i C^{p\ell}(M_2)$ for $i << k$. In this section we demonstrate a similar result for the smooth category.

Let M^n be a smooth compact manifold and let ξ be a smooth vector bundle over M, and $D = D(\xi)$ the associated disc bundle. Now $C^d(M) \subset C^\infty(M\times I, M\times I; \mathrm{mod}\ M\times 0 \cup \partial M \times I)$, and the function space is contractible to the identity map. Hence we have a map from $C^d(M)$ to smooth bundle maps of $D(\xi)\times I$ the identity over $M\times 0 \cup \partial M\times I$. This last maps into $A^d(D\times I;\ \mathrm{mod}\ D\times 0 \cup (D|\partial M)\times I)$.

Now there is a map $\overline{\gamma}: A^d(D\times I\,;\ \mathrm{mod} D\times 0 \cup D|\partial M\times I) \to C^d(D)$; i.e. by using a collar neighborhood of the sphere bundle $S = S(\xi)$ in $D(\xi)$, we may identify $D\times I$ with itself in such a way that $S\times I \cup D\times 1$ is identified with $D\times 1$ and the collar of $S\times 0$ in $D\times 0$ is identified with $S\times I$. In fact, we may assume $h \in A^d(D\times I;\ \mathrm{mod}\ D\times 0 \cup D|\partial M\times I)$ satisfies $h|(\text{collar } S)\times I = (h|S\times I)\times$ identity on the collar parameter. Thus as in section 3, conjugate $h|S\times I\times I$, where the last factor is the collar parameter, using essentially $\gamma^{-1}|[-1,0]\times[0,1]$, as in the diagram below.



Theorem E. Let $M^n \subset \operatorname{Int} D^n$. Then if M is k-connected

a) $\emptyset: C_0^t(M)/C_0^d(M) \to C_0^t(M\times I)/C_0^d(M\times I)$ is $k+s$ connected, $2s+1 \leq \gamma(n)$.

b) $\emptyset: C^t(M) \to C^t(M\times I)$ is $\inf(k+s, 2k-3)$ connected, $2s+1 \leq \gamma(n)$, and

$k \leq n-4$ if ∂M not 1-connected.

Theorem F. Let M^n be a smooth compact manifold and $\mathcal{E}$ a smooth vector bundle over M, then $\rho_{\mathcal{E}*}: \pi_i C(M) \to \pi_i C(D(\mathcal{E}))$ is an isomorphism for $i \leq s-3$, $2s+1 \leq \gamma(n)$.

Now suppose $M_1^{n_1}$ and $M_2^{n_2}$ are two k-equivalent smooth compact manifolds. Let N_i, $i = 1, 2$ be the regular neighborhoods of the k-skeletons of M_i. Then $\pi_i(C(N_i)) \to \pi_i(C(M_i))$ is an isomorphism for $i \leq k-2$, $k \leq n-4$ by Theorem 3.1' of [2].

Now embed N_2 in R^{n_2+s}, s sufficiently large. Let ν be the normal bundle. Then if s is sufficiently large, the homotopy equivalence $N_1 \to N_2$ is homotopic to an embedding of N_1 in Int $D(\nu_2)$. If ν_1 is the normal bundle of N_1 in Int $D(\nu_2)$, we have $D(\nu_1) \subset \text{Int}\, D(\nu_2)$. Then again by Theorem 3.1' of [2], $\pi_i C(D(\nu_1)) \simeq \pi_i C(D(\nu_2))$ for $i \leq k-2$, $k \leq n_2 + s-4$. Thus using Theorem F, we have:

Theorem G. Let $M_1^{n_1}$, $M_2^{n_2}$ be k-equivalent smooth compact manifolds. Then $\pi_i C^d(M_1) \simeq \pi_i C^d(M_2)$ for $i \leq \inf(k-2, s-3)$, $k \leq n_j - 4$, $2s+1 \leq \gamma(n_j)$, $j = 1, 2$.

Note that if ξ is the trivial line bundle, i.e. $D(\xi) = M \times I$, $\rho: C(M) \to C(M\times I)$ is homotopic to the stabilization map $\emptyset$. In fact, the map $C(M) \to$ smooth bundle maps of $D(\xi)\times I$, the identity over $M\times 0 \cup \partial M\times I$, is the product map $g \to g\times id_I$. But in this case $\rho = \emptyset$, up to an obvious deformation.

Next observe that if ξ and η are two vector bundles over M, then pulling η back over $D(\xi)$,

$$\begin{array}{ccc} & & C(D(\varepsilon)) \\ & \overset{\rho_\varepsilon}{\nearrow} & \big\downarrow \rho_\eta \\ C(M) & \underset{\rho_{(\varepsilon\oplus\eta)}}{\searrow} & C(D(\varepsilon + \eta)) \end{array}$$

commutes up to homotopy.

Thus if η^s is an inverse of ξ^r, then $\xi \oplus \eta = 1_{r+s}$ and $\rho_{\varepsilon\oplus s}$ is homotopic to the stabilization map $\emptyset: C(M) \to C(M\times D^{r+s})$. Since $\emptyset_*$ is an isomorphism on $\pi_i C(M)$, $i << n$, $\rho_{\varepsilon *}$ is injective and ρ_η is epi if $i << n$. Likewise $C(D(\varepsilon)) \xrightarrow{\rho_\eta} C(D(\varepsilon\oplus\eta)) \xrightarrow{\rho_\varepsilon} C(D(\varepsilon\oplus\eta\oplus\varepsilon)) = C(D(\varepsilon)\times D^{r+s})$ induces an isomorphism on π_i, $i << n+r$, and hence ρ_η is injective and hence an isomorphism for $i << n$. Hence $\rho_\varepsilon: \pi_i C(M) \subset \pi_i C(D(\varepsilon))$ is an isomorphism for $i << n$. Thus we have proved

In fact, $\pi_p \zeta_0^{t/d}(K) = \operatorname{Lim}_n [S^p \wedge K_n^*, F_n] \simeq \operatorname{Lim}_n \pi_{p+n}(F_n \wedge K) = \mathcal{F}_p(K)$.

Since $\mathcal{F}_p(K)$ is an homology theory, it follows from Theorem H that $\pi_{p+j} \zeta_0^{t/d}(S^j \wedge K) \simeq \pi_p \zeta_0^{t/d}(K)$. Now $\zeta_0^{t/d}(S^j \wedge K) = \operatorname{Lim}_n C_0^t(N_n(S^j \wedge K))/C_0^d(N_n(S^j \wedge K))$. But by Theorem D, $\pi_i C^t(N_n(S^j \wedge K)) \simeq \pi_i C_0^t(N_n(S^j \wedge K))/C_0^d(N_n(S^j \wedge K))$ for $i << 2j-5$, for n large. Hence if we let $\zeta^t(K) = \operatorname{Lim}_n C^t(N_n(K))$, we have:

<u>Theorem I.</u> $\operatorname{Lim}_j \pi_{p+j} \zeta^t(S^j \wedge K) \simeq \mathcal{F}_p(K)$.

<u>Remark.</u> Since $F_n \simeq C^t(S^n)$, $\mathcal{F}_p(S^0) = \operatorname{Lim}_n \pi_{p+n}(F_n) = \operatorname{Lim}_n \pi_{p+n} C^t(S^n) = \operatorname{Lim}_n \pi_p \Omega^n C^t(S^n)$.

6. Concordances as a homology theory

Let K be a finite complex. Then for n sufficiently large, K embeds in S^n. Let $N_n = N_n(K)$ be a regular neighborhood of K in S^n. Again for n sufficiently large $N_n(K)$ is independent of the embedding. Now by (4.3) $C_0^t(N_n)/C_0^d(N_n) \simeq F_n^{N^0/\partial N}$. But $N_n^0/\partial N \simeq \Sigma K_{n-1}^*$, K_{n-1}^* the Spanier-Whitehead dual of K in S^n; i.e. $K_{n-1}^* = S^n - \text{Int}\, N_n(K)$. In fact, if we let $D^n = S^n - \text{Int}\, D_\varepsilon^n$, $D_\varepsilon^n \subset \text{Int}\, N \subset S^n$, then $K_{n-1}^* \subset \text{Int}\, D^n$ and $N^0/\partial N = D^n/K_{n-1}^* \simeq \Sigma K_{n-1}^*$. Further, it is standard that $\Sigma K_{n-1}^* \simeq K_n^*$ and $N_{n+1}(K) = N_n(K) \times I$. Thus we have:

$$(6.1) \qquad C_0^t(N_n)/C_0^d(N_n) \simeq F_n^{K_n^*},$$

and by (4.5) the commutative diagram

$$(6.2) \qquad \begin{array}{ccc} C_0^t(N_n)/C_0^d(N_n) & \simeq & F_n^{K_n^*} \\ \downarrow \phi & & \downarrow \varphi \\ C_0^t(N_{n+1})/C_0^d(N_{n+1}) & \simeq & (\Omega F_{n+1})^{K_n^*} \simeq F_{n+1}^{K_{n+1}^*} \end{array}$$

Let $\mathcal{F}$ be the spectrum $\{F_n\}$. Write $\mathcal{K}$ for the suspension spectrum of K and $\mathcal{K}^*$ for $\{K_n^*\}$. Let $\mathcal{C}_0^{t/d}(K) = \underset{n}{\text{Lim}}\, C_0^t(N_n(K))/C_0^d(N_n(K))$. Then by Spanier-Whitehead Duality, [11], $\mathcal{K}^*$ is the dual spectrum of $\mathcal{K}$ and

<u>Theorem H.</u> $\pi_p \mathcal{C}_0^{t/d}(K) \simeq \mathcal{F}_p(K)$.

If (H_n, H'_n) is a pair of the above groups or H-spaces such that $H'_n \subset H_n$, we let $\lambda_n(H/H')$ be such that $\varphi: (H_n/H_{n-1}, H'_n/H'_{n-1}) \to (\Omega(H_n/H_{n-1}), \Omega(H'_n/H'_{n-1}))$ is $n + \lambda_n(H/H')$ connected.

Note that

(7.1) a) $\lambda_n(H/H') \geq \inf(\lambda_n(H), \lambda_n(H') + 1)$

b) $\lambda_n(H) \geq \inf(\lambda_n(H'), \lambda_n(H/H'))$.

From Theorem B and our introductory remarks we have for example

$$\lambda_n(H) = n-3 \text{ for } O_n \text{ or } G_n$$

$$\lambda_n(H/H') \geq (\gamma(n-1)-3)/2 \text{ for } PL_n/O_n \text{ or } Top_n/O_n.$$

Thus λ_n for the other groups or pairs can be determined by (7.1).

Also note that $\lambda_{n+1}(H) \geq \lambda_n(H)$ and $\lambda_{n+1}(H/H') \geq \lambda_n(H/H')$.

We consider the linear transfrmations of R^n:

$$T_k = \begin{pmatrix} I_{n-k} & 0 \\ 0 & -I_k \end{pmatrix}.$$

Let τ_k be the involution on H_n given by conjugation with T_k. Note that τ_2 on H_{n+1}, i.e., conjugation by $\begin{pmatrix} I_{n-1} & & O \\ & -1 & \\ O & & -1 \end{pmatrix}$ leaves H_n invariant and $\tau_2|H_n = \tau_1$ on H_n.

From the fibration $H_n \subset H_{n+1} \to H_{n+1}/H_n$ we get the following two exact sequences as soon as we notice that τ_2 induces the identity on $\pi_*(H_{n+1})$:

7. Algebraic consequences of suspension isomorphism

The main result of this section was first noted by Igusa [5]. He used our Theorem C but we will use Theorem B directly.

<u>Properties of</u> $Z(1/2)$ <u>modules:</u> Let (M, τ) be a commutative $Z(1/2)$ module M with an involution $\tau: M \to M$; i.e. τ is a $Z(1/2)$ morphism with $\tau^2 =$ identity. Then

a) $M = M^s \oplus M^a$, where $M^s = \{x \in M \mid \tau(x) = x\}$ and $M^a = \{x \in M \mid \tau(x) = -x\}$.

b) Let $f: (M_1, \tau_1) \to (M_2, \tau_2)$ be a morphism of $Z(1/2)$ modules with involution; i.e. $f: M_1 \to M_2$ is a $Z(1/2)$ morphism and $\tau_2 f = f\tau_1$. Then $f(M_1^s) \subset M_2^s$, $f(M_1^a) \subset M_2^a$ and $f = f^s \oplus f^a$.

c) Let $\to M_i \xrightarrow{f_i} M_{i+1} \xrightarrow{f_{i+1}} M_{i+2} \to \cdots$ be an exact sequence of $Z(1/2)$-modules such that each M_i has an involution and f_i is a morphism of $Z(1/2)$-modules with involution. Then

$$\to M_i^s \xrightarrow{f_i^s} M_{i+1}^s \xrightarrow{f_{i+1}^s} M_{i+2}^s \to \cdots$$

$$\to M_i^a \xrightarrow{f_i^a} M_{i+1}^a \xrightarrow{f_{i+1}^a} M_{i+2}^a \to \cdots$$

are exact.

<u>Notation.</u> Let $\hat{\pi}_i(X) = \pi_i(X) \otimes Z(1/2)$

Now let H_n be any one of the groups or H-spaces O_n, PL_n, Top_n, G_n. Let $\lambda_n(H)$ be such that $\varphi: H_n/H_{n-1} \to \Omega(H_{n+1}/H_n)$ is $n + \lambda_n(H)$ connected.

Since T_1, T_2 and hence τ_1, τ_2 commute, $\tau_1 \tau_2$ is an involution. Further $\tau_1\tau_2$ on H_{n+1} satisfies $(\tau_1\tau_2)_* = \tau_{1*}$ and $\tau_1\tau_2|H_n = \tau_2\tau_1|H_n = \tau_2|H_n = \tau_1$ on H_n. Hence we have the exact sequence

$$(7.7) \qquad \to \pi_i^{s_1}(H_n) \to \pi_i^{s_1}(H_{n+1}) \to \pi_i^{s_{12}}(H_{n+1}/H_n) \to \pi_{i-1}^{s_1}(H_n) \to$$

where $\pi_i^{s_{12}}$ is the symmetric part under $\tau_1\tau_2$. From (7.3) and (7.5) we must have for the anti-symmetric part

$$(7.8) \qquad \pi_i^{a_{12}}(H_{n+1}/H_n) \simeq \pi_i^{a_1}(H_{n+1}) \oplus \pi_{i-1}^{a_1}(H_n).$$

Also from (7.7), (7.4) and (7.3) we get

$$(7.9) \qquad \pi_i^{s_1}(H_{n+1}/H_n) \simeq \pi_i^{s_{12}}(H_{n+1}/H_n) \oplus \pi_i^{a_2}(H_{n+1}/H_n),$$

and from (7.7), (7.2) and (7.5) we get

$$(7.10) \qquad \pi_i^{s_2}(H_{n+1}/H_n) \simeq \pi_i^{s_{12}}(H_{n+1}/H_n) \oplus \pi_i^{a_1}(H_{n+1}/H_n) .$$

Proposition 7.11. $\partial_* \varphi_* : \hat{\pi}_i(H_{n+1}/H_n) \to \pi_i^{a_1}(H_{n+1})$.

Proof. $\varphi : H_{n+1}/H_n \to \Omega(H_{n+2}/H_{n+1})$ is given by:

$$\varphi[h](t) = [R_t \circ h \times 1 \circ R_t^{-1}] = [R_t \circ h \times 1 \circ R_t^{-1} \circ h^{-1} \times 1] .$$

Now φ may be factored as:

$$\varphi : H_{n+1}/H_n \xrightarrow{\tilde{\varphi}} \Lambda(H_{n+2}, *, H_{n+1}) \xrightarrow{q} \Omega(H_{n+2}/H_{n+1}),$$

where q is the homotopy equivalence $q(\lambda)(t) = [\lambda(t)]$, and $\tilde{\varphi}[h](t) = R_t \circ h \times 1 \circ R_t^{-1} \circ h^{-1} \times 1$. Then $\partial_* \varphi_* = p_* \tilde{\varphi}_*$:

$p : \Lambda(H_{n+2}, *, H_{n+1}) \to H_{n+1}$, $p(\lambda(t)) = \lambda(1)$. But

$$(7.2)\qquad \to \pi_i^{s_1}(H_n) \to \hat{\pi}_i(H_{n+1}) \to \pi_i^{s_2}(H_{n+1}/H_n) \xrightarrow{\delta} \pi_{i-1}^{s_1}(H_n) \to$$

$$(7.3)\qquad 0 \to \pi_i^{a_2}(H_{n+1}/H_n) \xrightarrow{\approx} \pi_{i-1}^{a_1}(H_n) \to 0\,,$$

where s_i, a_i, $i = 1, 2$ are the symmetric and anti-symmetric parts of $\hat{\pi}_i$ with respect to τ_i.

Since τ_1 on H_{n+1} leaves H_n fixed, we get

$$(7.4)\qquad \to \hat{\pi}_i(H_n) \to \pi_i^{s_1}(H_{n+1}) \to \pi_i^{s_1}(H_{n+1}/H_n) \to \hat{\pi}_{i-1}(H_n) \to$$

$$(7.5)\qquad 0 \to \pi_i^{a_1}(H_{n+1}) \xrightarrow{\approx} \pi_i^{a_1}(H_{n+1}/H_n) \to 0\,.$$

<u>Proposition 7.6.</u> $\varphi_* : \hat{\pi}_i(H_{n+1}/H_n) \to \hat{\pi}_{i+1}(H_{n+2}/H_{n+1})$ satisfies

a) $\varphi_* \tau_{1*} = \tau_{2*} \varphi_*$, b) $\varphi_* \tau_{2*} = \tau_{1*} \varphi_*$.

<u>Proof.</u>

a) For $h \in H_{n+1}$,
$$R_t \circ \tau_1(h) \times 1 \circ R_t^{-1} = R_t \circ T_2 \circ h \times 1 \circ T_2^{-1} \circ R_t^{-1}$$
$$= T_2 \circ R_t \circ h \times 1 \circ R_t^{-1} \circ T_2^{-1}\,,$$

and the result follows from the definition of φ.

b) For $h \in H_{n+1}$,
$$R_t \circ \tau_2(h) \times 1 \circ R_t^{-1} = R_t \circ T_3 \circ h \times 1 \circ T_3^{-1} \circ R_t^{-1}$$
$$= T_3 \circ R_t \circ h \times 1 \circ R_t^{-1} \circ T_3^{-1}\,.$$

But

$$T_3 = \begin{pmatrix} I_{n-1} & & & O \\ & -1 & & \\ & & -1 & \\ O & & & 1 \end{pmatrix} \circ T_1$$

is homotopic to T_1 by a homotopy leaving H_{n+1} invariant.

<u>Lemma 7.16</u>: If for some $i \leq \inf(n+1+\lambda_n(H), n+\lambda_{n+1}(H))$, $\pi_i^{s_{12}}(H_{n+1}/H_n) \neq 0$, then for all $j \geq i$, $\hat{\pi}_j(H) \neq 0$.

<u>Proof.</u> By (7.12), $\pi_i^{s_{12}}(H_{n+1}/H_n) \neq 0$ implies $\pi_i^{s_1}(H_{n+1}) \neq 0$. By (7.13), $\pi_i^{s_1}(H_{n+1}) \to \hat{\pi}_i(H)$ is a monomorphism, and $\pi_i(H) \neq 0$. On the other hand, $\pi_i^{s_{12}}(H_{n+1}/H_n) \simeq \pi_{i+1}^{s_{12}}(H_{n+2}/H_{n+1})$ by (7.14), and $\hat{\pi}_{i+1}(H) \neq 0$, etc.

Also we note

(7.17) All the results (7.2) through (7.16) hold for pairs (H_n, H'_n).

Since $\hat{\pi}_i(O) = 0$ for $i \not\equiv 3.7 \bmod 8$ we have:

<u>Proposition 7.18.</u>

a) $\pi_i^{s_{12}}(O_{n+1}/O_n) \simeq \pi_i^{s_{12}}(G_{n+1}/G_n) = 0$ for $i \leq 2n-2$

b) $\pi_i^{s_1}(O_n) \simeq \hat{\pi}_i(O)$ and $\pi_i^{s_1}(G_n) \simeq \hat{\pi}_i(G)$ for $i \leq 2n-3$.

<u>Proof.</u>

a) follows from (7.16) and the fact that $\lambda_n(O) = \lambda_n(G) = n-3$.

b) follows from (7.13).

Since $\hat{\pi}_i(G/Top) = \hat{\pi}_i(G/PL) = 0$ for $i \not\equiv 0 \bmod 4$ we have:

<u>Proposition 7.19.</u>

a) $\pi_i^{s_{12}}(G_{n+1}/G_n, Top_{n+1}/Top_n) = \pi_i^{s_{12}}(G_{n+1}/G_n, PL_{n+1}/PL_n) = 0$ for $i \leq \inf(2n-2, n+1+\lambda_n(Top))$.

b) $\pi_i^{s_{12}}(Top_{n+1}/Top_n) \simeq \pi_i^{s_{12}}(PL_{n+1}/PL_n) = 0$ for $i \leq \inf(n+\lambda_n(Top), 2n-3)$.

$$p\breve{\varphi}[h] = R_1 \circ h \times 1 \circ R_t^{-1} \circ h^{-1} \times 1 = T_2 \circ h \times 1 \circ T_2^{-1} \circ h^{-1} \times 1$$
$$= \tau_1(h) \times 1 \circ h^{-1} \times 1 = \tau_1(h) \circ h^{-1} .$$

Hence $\tau_1(p\breve{\varphi}[h]) = h \circ \tau_1(h)^{-1} = (p\breve{\varphi}[h])^{-1}$. So $\tau_{1*} \circ \partial_* \varphi_* = -\partial_* \varphi_*$, proving the proposition.

<u>Corollary 7.12.</u> For $i \leq n+1+\lambda_n(h)$, $\partial_* : \pi_i^{s_{12}}(H_{n+1}/H_n) \to \hat{\pi}_{i-1}(H_n)$ is trivial.

<u>Proof.</u> By (7.7), $\partial_* : \pi_i^{s_{12}}(H_{n+1}/H_n) \to \pi_{i-1}^{s_1}(H_n)$. But if φ_* is surjective onto $\pi_i^{s_{12}}$, Image $\partial_* \subset \pi_{i-1}^{a_1}(H_n)$ by (7.11).

<u>Corollary 7.13.</u> For $i \leq n+\lambda_n(H)$ we have the exact sequence

$$0 \to \pi_i^{s_1}(H_n) \to \pi_i^{s_1}(H_{n+1}) \to \pi_i^{s_{12}}(H_{n+1}/H_n) \to 0.$$

<u>Lemma 7.14</u> $\varphi_* : \pi_i^{s_{12}}(H_n/H_{n-1}) \to \pi_{i+1}^{s_{12}}(H_{n+1}/H_n)$,

$$\varphi_* : \pi_i^{a_1}(H_n/H_{n-1}) \to \pi_{i+1}^{a_2}(H_{n+1}/H_n), \text{ and}$$
$$\varphi_* : \pi_i^{a_2}(H_n/H_{n-1}) \to \pi_{i+1}^{a_1}(H_{n+1}/H_n)$$

are surjective for $i \leq n+\lambda_n(H)$ and injective for $i \leq n-1+\lambda_n(H)$.

<u>Proof.</u> From (7.6), φ_* sends $\pi_i^{a_1} \to \pi_{i+1}^{a_2}$, $\pi_i^{a_2} \to \pi_{i+1}^{a_1}$ and $\pi_i^{s_{12}} \to \pi_{i+1}^{s_{12}}$. The result then follows using (7.9) and (7.10).

<u>Corollary 7.15.</u> $\pi_i^{a_1}(H_n) \simeq \pi_{i+1}^{a_2}(H_{n+1}/H_n) \xrightarrow{(\varphi_*)^2} \pi_{i+3}^{a_2}(H_{n+3}/H_{n+2}) \simeq \pi_{i+2}^{a_1}(H_{n+2})$ is surjective for $i \leq n+\lambda_{n+1}(H)$ and injective for $i \leq n-1+\lambda_{n+1}(H)$.

2. For $i \leq \inf(2n-3, n + \lambda_n(\mathrm{Top}))$,

a) $\hat{\pi}_i \mathrm{Top}_n \simeq \hat{\pi}_i \mathrm{Top} \oplus \pi_i^{a_1} \mathrm{Top}_n$, where under the inclusion in Top_{n+1}, the first factor goes isomorphically and the second trivially. Further, $\pi_{i-1}^{a_1} \mathrm{Top}_n \simeq \pi_{i+1}^{a_1} \mathrm{Top}_{n+2}$.

b) $\hat{\pi}_i \mathrm{Top}_n/O_n \simeq \hat{\pi}_i \mathrm{Top}/O \oplus \pi_i^{a_1} \mathrm{Top}_n/O_n$, where under the inclusion in $\mathrm{Top}_{n+1}/O_{n+1}$, the first factor goes isomorphically and the second trivially. Further $\pi_{i-1}^{a_1}(\mathrm{Top}_n/O_n) \simeq \pi_{i+1}^{a_1}(\mathrm{Top}_{n+2}/O_{n+2})$.

c) $\pi_i^{a_1} \mathrm{Top}_n \simeq \pi_i^{a_1} O_n \oplus \pi_i^{a_1} \mathrm{Top}_n/O_n$, and

$\hat{\pi}_{i-1} C^{\mathrm{Top}}(S^{n-1}) \simeq \pi_i^{a_1} \mathrm{Top}_n/O_n \oplus \pi_{i-1}^{a_1} \mathrm{Top}_{n-1}/O_{n-1}$.

The same results hold for PL_n.

<u>Proof.</u>

1. $\hat{\pi}_i O_n = \pi_i^{s_1} O_n \oplus \pi_i^{a_1} O_n$, and $\pi_i^{s_1} O_n \simeq \pi_i O$ by (7.18b).

$\pi_i^{a_1} O_n \to \pi_i^{a_1} O_{n+1}$ is trivial by (7.8),

$\pi_i^{a_1} O_n \simeq \pi_i^{a_1} O_{n+2}$ by (7.15).

Now by (7.18a), $\hat{\pi}_i S^n \simeq \hat{\pi}_i O_{n+1}/O_n \simeq \pi_i^{a_{12}} O_{n+1}/O_n \simeq \pi_i^{a_1} O_{n+1} \oplus \pi_{i-1}^{a_1} O_n$, for $i \leq 2n-2$. For n odd, $\hat{\pi}_i O_{n+1} \to \hat{\pi}_i S^n$ is split surjective; i.e., there is a map $\rho: S^n \to O_{n+1}$ such that $S^n \xrightarrow{\rho} O_{n+1} \to S^n$ is of degree 2, (see [12]). Hence $\pi_{i-1}^{a_1} O_n = 0$ for $i \leq 2n-2$ or $\pi_i^{a_1} O_n = 0$ for $i \leq 2n-3$.

For n even $\hat{\pi}_i O_n \simeq \hat{\pi}_{i+2} O_{n+2} \simeq \hat{\pi}_{i+2} S^{n+1} \simeq \hat{\pi}_i S^{n-1}$ for $i \leq 2n-3$.

c) $\pi_i^{s_1}(G_n/Top_n) \simeq \hat{\pi}_i(G/Top)$ for $i \leq \inf(2n-3, n+\lambda_n(Top))$

$\pi_i^{s_1}(G_n/PL_n) \simeq \hat{\pi}_i(G/PL)$ for $i \leq \inf(2n-3, n+\lambda_n(PL)$

d) $\pi_i^{s_1}(Top_n) \simeq \hat{\pi}_i(Top)$ for $i \leq \inf(n+\lambda_n(Top), 2n-3)$

$\pi_i^{s_1}(PL_n) \simeq \hat{\pi}_i(PL)$ for $i \leq \inf(n+\lambda_n(PL), 2n-3)$

Proof.

a) follows from (7.16) and b) follows from a) and (7.18a).

c) and d) follow from a) and b) and (7.13).

Proposition 7.20.

a) $\pi_i^{s_{12}}(Top_{n+1}/Top_n, O_{n+1}/O_n) \simeq \pi_i^{s_{12}}(PL_{n+1}/PL_n, O_{n+1}/O_n) = 0$ for $i \leq \inf(2n-3, n+\lambda_n(top))$.

b) $\pi_i^{s_1}(Top_n/O_n) \simeq \hat{\pi}_i(Top/O)$ for $i \leq \inf(2n-3, n+\lambda_n(top))$.

$\pi_i^{s_1}(Pl_n/O_n) \simeq \hat{\pi}_i(PL/O)$ for $i \leq \inf(2n-3, n+\lambda_n(PL))$.

Proof. By (7.18a), $\pi_i^{s_{12}}(G_{n+1}/G_n, O_{n+1}/O_n) = 0$ for $i \leq 2n-2$. From this and (7.19a) we get a). b) follows from a) and (7.13).

Theorem J. 1. For $i \leq 2n-3$, $\hat{\pi}_i O_n \simeq \hat{\pi}_i O \oplus \pi_i^{a_1} O_n$, where under the inclusion in O_{n+1}, the first factor goes isomorphically and the second trivially. Further, $\pi_i^{a_1} O_n \simeq \pi_{i+2}^{a_1} O_{n+2}$ and $\pi_i^{a_1} O_n \simeq \begin{cases} 0 & \text{for } n \text{ odd} \\ \hat{\pi}_i S^{n-1} & \text{for } n \text{ even} \end{cases}$.

The same result holds for G_n.

i.e. multiplication by 2 is an isomorphism. If X is homotopy commutative, so is $X_{(2)}$ and $[K, X_{(2)}]$ is abelian.

Now suppose $\tau : X \to X$ is an H-map such that $\tau^2 \simeq$ identity. Then τ_* induces an automorphism of $\pi_i(X)$ of order 2. Hence

$\hat{\pi}_i(X) = \pi_i(X)_{(2)} = \pi_i(X_{(2)}) \simeq \pi_i^s(X) \oplus \pi_i^a(X)$.

Proposition 7.21. Let X be a homotopy commutative CW weak group, and $\tau : X \to X$ an H-map with $\tau^2 \simeq 1_X$. Then there exists CW complexes X^s and X^a and a homotopy equivalence $h : X^s \times X^a \to X_{(2)}$ such that $h_* = h_*^s \oplus h_*^a$, $h_*^s : \pi_i(X^s) \simeq \pi_i^s(X)$, $h_*^a : \pi_i(X^a) \simeq \pi_i^a(X)$.

Proof. Consider $[K, X_{(2)}]_s$, resp. $[K, X_{(2)}]_a$, the homotopy classes $\alpha \in [K, X_{(2)}]$ such that $\tau\alpha = \alpha$, resp. $\tau\alpha = -\alpha$. Note that $[K, X_{(2)}] = [K, X_{(2)}]_s \oplus [K, X_{(2)}]_a$. We show that $[\ , X_{(2)}]_s$ and $[\ , X_{(2)}]_a$ are representable functors. Now the wedge axiom is obvious. So consider maps $f : A \to K$, $g : A \to L$ of base pointed connected CW complexes, and let $K \cup_A L$ be the double mapping cylinder. We need to show that if in the commutative diagram:

$$\begin{array}{ccc}
[A, X_{(2)}]_s & \xleftarrow{\ f^*\ } & [K, X_{(2)}]_s \\
\uparrow{\scriptstyle g^*} & & \downarrow{\scriptstyle \iota_K^*} \\
[L, X_{(2)}]_s & \xleftarrow[\ \iota_K^*\ ]{} & [K \cup_A L, X_{(2)}]_s
\end{array}$$

2. a) and b) follow from (7.19d), (7.20b), (7.8) and (7.15).

c) follows from the fact that $Top_n/Top_{n-1} \simeq S^{n-1} \times BC(S^{n-1})$, where $O_n/O_{n-1} \to Top_n/Top_{n-1} \to S^{n-1}$ is homotopic to the identity (see [6]).

Now we may apply the same analysis to the block groups $\widetilde{PL}_n$ or $\widetilde{Top}_n$, using the fact that $\widetilde{PL}_{n+1}/\widetilde{PL}_n \simeq \widetilde{Top}_{n+1}/\widetilde{Top}_n \simeq G_{n+1}/G_n$ for $n \geq 5$, to get:

<u>Theorem K.</u> a) For $i \leq 2n-3$, $\hat{\pi}_i \widetilde{Top}_n \cong \hat{\pi}_i Top \oplus \pi_i^{a_1} \widetilde{Top}_n$, where

$$\pi_i^{a_1} \widetilde{Top}_n = \begin{cases} 0 & \text{for } n \text{ odd} \\ \hat{\pi}_i S^{n-1}, & n \text{ even} \end{cases}$$

b) $\hat{\pi}_i Top_n \to \hat{\pi}_i \widetilde{Top}_n$ is split surjective with kernel $\pi_i^{a_1} \widetilde{Top}_n/O_n$ for $i \leq \inf(2n-3, n+\lambda_n(Top))$

The same results hold for PL_n.

Finally we note that all the above algebraic splittings correspond to geometric splittings at the loop space level. This will follow from general considerations on involutions of topological groups and H-spaces:.

Let X be a weak group (i.e. satisfying the group axioms up to homotopy). Assume also that X is a connected CW complex. Let $X_{(2)}$ be X localized away from 2. Then $X_{(2)}$ is also a weak group. Further any connected base pointed CW complex K, $[K, X_{(2)}]$ is a local 2-group;

4. For any CW complex Y with involution τ_1, $\Omega^2\tau_1 : \Omega^2 Y \to \Omega^2 Y$, and $(\Omega_0^2 Y)_{(2)}$ splits.

Thus for example we have:

(7.10) $(\Omega_0 \mathrm{Top}_n)_{(2)} \simeq X^s \times X^a$, $\pi_i X^s \simeq \pi^s_{i+1} \mathrm{Top}_n$, $\pi_i X^a \simeq \pi^a_{i+1} \mathrm{Top}_n$.

Further, $X^s \to (\Omega_0 \mathrm{Top}_n)_{(2)} \to (\Omega_0 \mathrm{Top})_{(2)}$ is $\inf(2n-4, n+\lambda_n(\mathrm{Top})-1$ connected.

(7.11) $(\Omega_0^2 \mathrm{Top}_n/O_n)_{(2)} \simeq X^s \times X^a$, $\pi_i X^s \simeq \pi^s_{i+2} \mathrm{Top}_n/O_n$, $\pi_i X^a \simeq \pi^a_{i+2} \mathrm{Top}_n$.

Further, $X^s \to (\Omega_0^2 \mathrm{Top}_n/O_n)_{(2)} \to (\Omega_0^2 \mathrm{Top}/O)_{(2)}$ is $\inf(2n-5, n+\lambda_n(\mathrm{Top})-2)$ connected.

we are given $\alpha \in [K, X_{(2)}]_s$ and $\beta \in [L, X_{(2)}]_s$ such that $f^*\alpha = g^*\beta$, then there exists $\gamma \in [K \underset{A}{\cup} L, X_{(2)}]_s$ with $\iota_K^*\gamma = \alpha$ and $\iota_L^*\gamma = \beta$. Similarly for $[\ , X_{(2)}]_a$.

Now obviously there exists $\gamma_1 \in [K \underset{A}{\cup} L, X_{(2)}]$ with $\iota_K^*\gamma_1 = \alpha$, $\iota_L^*\gamma_2 = \beta$. Let $\gamma = \frac{1}{2}(\gamma_1 + \tau\gamma_1)$. (Recall that multiplication by 2 is an isomorphism.) Then $\tau\gamma = \gamma$ and $\iota_K^*\gamma = \iota_K^*\gamma_{1/2} + \iota_K^*\tau\gamma_{1/2} = \iota_K^*\gamma_1 = \alpha$. Likewise $\iota_L^*\gamma = \beta$. A similar argument using $\gamma = \frac{1}{2}(\gamma_1 - \tau\gamma_1)$ works for $[\ , X_{(2)}]_a$.

Thus there exist X^s and X^a such that $[\ , X_{(2)}]_s = [\ , X^s]$ and $[\ , X_{(2)}]_a = [\ , X^a]$. Hence $[\ , X_{(2)}] = [\ , X_{(2)}]_s \oplus [\ , X_{(2)}]_a = [\ , X^s] \oplus [\ , X^a] = [\ , X^a \times X^s]$; and there exists $h: X^s \times X^a \simeq X_{(2)}$ such that $h: [K, X^s \times X^a] \to [K, X_{(2)}]$ satisfies $h_* = h_*^s \oplus h_*^a$.

<u>Remarks</u>. 1. X^s and X^a are unique up to homotopy equivalence.

2. τ is homotopic to the identity in $X_{(2)}$ if and only if X^a is trivial. Hence τ is homotopic to the identity on $X_{(2)}$ if and only if τ is the identity on $\pi_i(X)$, all i.

3. If Y is an H-space and $\tau_1: Y \to Y$ is an H-map with $\tau_1^2 \simeq 1_Y$, then τ_1 induces $\Omega\tau_1$ on ΩY. If Y is the homotopy type of a CW complex then the component $\Omega_0 Y$ of the base point in ΩY is the homotopy type of a CW complex X satisfying the hypothesis of (7.19). Hence $h: X^s \times X^a \simeq (\Omega_0 Y)_{(2)}$ exists and $h_* = h_*^s \oplus h_*^a$.

we see that $\pi_i CE^d(D^p, D^n) \to \pi_i CE^d(D^{p+1}, D^{n+1})$ is an isomorphism for $i < s-2$, $2s+1 \le \gamma(n)$, and surjective for $i = s-2$. Hence

(8.2) $\qquad CIm^t(D^p, D^n) \to CIm^t(D^{p+1}, D^{n+1})$ is $\inf(2r-p-3, s-2)$ connected, $2s+1 \le \gamma(n)$.

Now $\pi_i CIm^t(D^p, D^n) = \pi_{i+p}(V^t_{n+1,p+1}, V^t_{n,p})$. Hence we have:

(8.3) $\qquad \pi_j(V^t_{n+1,p+1}, V^t_{n,p}) \to \pi_{j+1}(V^t_{n+2,p+2}, V^t_{n+1,p+1})$ is $\inf(2n-3, s+p-2)$ connected, $2s+1 \le \gamma(n)$

On the other hand, by Haefliger-Millet [10] it is known that $V^t_{n,p} \to G_n/G_{n-p}$ is $2n-p-3$ connected. Hence the above gives information when $s+p-2 \ge 2n-p-3$ or $s \ge 2n-2p-5$, or $\frac{\gamma(n)-1}{2} \ge 2n-2p-5$ or $\gamma(n) \ge 4(n-p) - 9$.

Now applying essentially the same argument as in section 7, we get:

(8.4) $$\hat{\pi}_i(V^t_{n,p}) \simeq \hat{\pi}_i G/G_{n-p} \oplus \pi_i^{a_1}(V^t_{n,p})$$

for $i \le \inf(2n-3, s+p-2)$, $2s+1 \le \gamma(n)$

Further, $\pi_i^{a_1}(V^t_{n,p}) \simeq \pi_{i+2}^{a_1}(V^t_{n+2,p+2})$.

<u>Remark.</u> $\hat{\pi}_i G_n/G_{n-p} \simeq \hat{\pi}_i G/G_{n-p} \oplus \pi_i^{a_1}(G_n)$ for $i \le 2n-3$. Hence for $i < 2n-p-3$,

$$\pi_i^{a_1}(V^t_{n,p}) = \pi_i^{a_1}(G_n) = \begin{cases} 0 & , \text{ n odd} \\ \pi_i S^{n-1} & , \text{ n even} \end{cases}$$

More generally, one has

8. Application to Immersions

Given $M \subset N$, let $CE(M, N)$ be the space of proper embeddings of $M \times I$ in $N \times I$, the inclusion in $M \times 0 \cup \partial M \times I$. Similarly, let $CIm(M, N)$ be the space of proper immersions of $M \times I$ in $N \times I$, the incusion on $M + 0 \cup \partial M \times I$. Then in [8], we have shown that:

(8.1) $CE^d(D^p, D^n) \to CIm^d(D^p, D^n) \to CIm^t(D^p, D^n)$ is a homotopy fibration.

Using the same construction as in section 3, we have the commutative diagram

$$\begin{array}{ccccc} CE^d(D^p, D^n) & \to & CIm^d(D^p, D^n) & \to & CIm^t(D^p, D^n) \\ \downarrow & & \downarrow & & \downarrow \\ CE^d(D^{p+1}, D^{n+1}) & \to & CIm^d(D^{p+1}, D^{n+1}) & \to & CIm^t(D^{p+1}, D^{n+1}) \end{array}$$

Now $\pi_i CIm^d(D^p, D^n) = \pi_{i+p}(O_{n+1}/O_{n-p}, O_n/O_{n-p}) = \pi_{i+p} O_{n+1}/O_n$ and $\pi_i CIm^d(D^{p+1}, D^{n+1}) = \pi_{i+p+1}(O_{n+2}/O_{n+1})$. Hence $\pi_i CIm^d(D^p, D^n) \to \pi_i CIm^d(D^{p+1}, D^{n+1})$ is an isomorphism for $i \leq 2n-p-3$.

From the fibrations

$$\begin{array}{ccccc} C^d(S^{n-p-1} \times D^{p+1}) & \to & C^d(D^n) & \to & CE^d(D^p, D^n) \\ \downarrow & & \downarrow & & \downarrow \\ C^d(S^{n-p-1} \times D^{p+2}) & \to & C^d(D^{n+1}) & \to & CE^d(D^{p+1}, D^{n+1}) \end{array}$$

§§ 9.

All spaces we work with in this paragraph are assumed to be <u>localised away 2</u>, i.e. their homotopy groups are $Z(\frac{1}{2})$ modules. We also denote by X_k the k-th Postnicov term in the Postnicov decomposition of X.

If X is an homotopy commutative H-space and $\tau: X \to X$ an involution we have seen in §§ 7 that X decomposes as $X \sim X^a \times X^s$ the antisymmetric resp. symmetric factors.

For a manifold M which belongs to one of the geometric categories $\mathcal{D}iff$, $\mathcal{P}L$, $\mathcal{T}op$ we denote by $\Gamma(M) =$ {the bigest integer k so that $C(M) \xrightarrow{\phi} C(M \times I)$ is a k-homotopy equivalence} and by $\delta(M) =$ {the bigest integer k so that $C(M \times D^s) \xrightarrow{\phi} C(M \times D^s \times I)$ is a $(k - s)$ - homotopy equivalence for $s = 0, 1, \ldots k$ }. Actually $\Gamma(M) \geq \delta(M) \geq (n - 10)/3$ for $\mathcal{P}l$ and $\mathcal{T}op$ and $\Gamma(M) \geq \delta(M) \geq n{-}25/6$ for $\mathcal{D}iff$.

Let τ be the involution on $A(M \times I)$ resp. $\tilde{A}(M \times I)$ defined by conjugation with $id_M \times \Gamma/_I$ $\Gamma: [-1,1] = I \to [-1,1]$ $\Gamma(t) = -t$.

<u>Lemma 9.1</u> : a) The inclusion $A^s(M \times I) \subset A(M \times I) \subset C(M)$ has a natural (with respect to inclusions $N^n \subset M^n$ homotopy invers $p^s: C(M) \to A^s(M \times I)$

b) $\tilde{A}^s(M \times I) \simeq *$ and $\tilde{A}^a(M \times I) \simeq A(M \times I)$

<u>Proof</u> : a) Let $\sigma: \Omega A(M) \to A(M \times I)$ be the natural map defined by $\sigma(h): M \times I \to M \times I$ $\sigma(h)(m,t) = (h_t(m), t)$

$$\pi_i O_n/O_{n-1} = \pi_i(V^d_{n,p}, V^d_{n-1,p-1}) \to \pi_i(V^t_{n,p}, V^t_{n-1,p-1}) \to \pi_i(V^t_{n,1}, *)$$

$$\approx \pi_i(V^t_{n,1}) = \pi_i(\mathrm{Emb}(R^1, R^n)) \to \pi_i S^{n-1}$$

is an isomorphism. Thus $\pi_i^{a_1} O_n/O_{n-p} \simeq \pi_i^{a_1} O_n \simeq \pi_i^{a_1} G_n$ is a direct summand of $\pi_i^{a_1} V^t_{n,p}$, and

(8.5) $$\pi_i^{a_1} V^t_{n,p} \simeq \pi_i^{a_1} G_n \oplus \pi_i^{a_1}(V^t_{n,p}, V^d_{n,p}).$$

Hence we have:

<u>Theorem L.</u> $\hat{\pi}_i(V^t_{n,p}) \simeq \pi_i G_n/G_{n-p} \oplus \pi_i^{a_1}(V^t_{n,p}, V^d_{n,p})$ for $i \leq \inf(2n-3, s+p-2)$, $2s+1 \leq \gamma(n)$. Further

$$\pi_i^{a_1}(V^t_{n,p}, V^d_{n,p}) \simeq \pi_{i+2}^{a_1}(V^t_{n+2,p+2}, V^d_{n+2,p+2}).$$

defined by conjugation with $id_M \times \Upsilon \times id_I$ (clearly τ' is homotopic to τ defined taking $M \times I$ instead M).

Theorem 9.2 : 1) There is a natural (with respect to inclusions $(N^n \subset M^n)$) homotopy equivalence :

$$C(M \times I)_k \xrightarrow{\simeq} A^s(M \times I)_k \times A^s(M \times I^2)_k$$

for $k \leq \Upsilon(M)$.

2) $A^a(M \times I)_k \xrightarrow{\simeq} (BA^a(M \times I)^2)_k$ is a natural (with respect to inclusions $N^n \subset M^n$) homotopy equivalence for $k \leq \Upsilon(M)$ hence $A^a(M \times I)_k \xrightarrow{\sim} \tilde{A}(M \times I)_k$ is for $k \quad (M)$.

Proof: Consider the principal fibration $A(M \times I^2) \to C(M \times I) \to A(M \times I)$ with involution τ' on $A(M \times I^2)$ and $C(M \times I)$ and $A(M \times I)$. The maps are clearly equivariant, consequentely this principal fibration decomposes as product of the following principal fibrations :

$$A^s(M \times I^2) \xrightarrow{i^s} C^s(M \times I) \longrightarrow A^s(M \times I)$$

$$A^a(M \times I^2) \xrightarrow{i^a} C^a(M \times I) \longrightarrow A^a(M \times I)$$

By Lemma 9.1 we know $A^s(M \times I^2) \to C^s(M \times I) \subset C(M \times I)$ has a natural homotopy inverse hence i^s has , consequentely $C^s(M \times I) \xrightarrow{\sim} A^s(M \times I^2) \times A^s(M \times I)$ is a natural homotopy equivalence.

Let t be the trivial involution on $C(M)$; since $\phi : C(M) \to C(M \times I)$ is equivariant, $\phi : C^s(M)_k \simeq$ $\simeq C(M)_k \xrightarrow{\sim} C^s(M \times I)_k \simeq C(M \times I)_k$ and $\phi : C^a(M) \simeq * \to C^a(M \times I)_k \simeq *$

and let $\bar{\tau}$ be the involution on $\Omega A(M)$ defined by $\bar{\tau}(h_t)=\omega_t$ $\omega_t=h_{-t}$. As σ is equivariant, we have also an involution induced on the homotopy theoretical fiber F_σ and consequentely the fibration $F_\sigma \to \Omega A(M) \to A(M \times I)$ factor as product of the principal fibrations :

$$F^s_\sigma \to (\Omega A(M))^s \xrightarrow{\simeq} A^s(M \times I)$$

and

$$F^a \to (\Omega A(M))^a \to A^a(M \times I)$$

Notice that $(\Omega A(M))^s \simeq *$, since the involution induced by $\bar{\tau}$ on $[K, \Omega A(M)]$ is given by $\bar{\tau}_*(x) = - x$ and $[K, \Omega A(M)]$ is $Z(\frac{1}{2})$ module ; consequentely , $A^s(M \times I) \to BF^s_\sigma$ is a homotopy equivalence. As the composite $\pi^s.i^s$ is identity, with i^s and π^s the inclusion from the symmetric part respectively the projection on the symmetric part which are group-homorphisms, we obtain the commutative diagram

$$\begin{array}{ccccc} BF^s & \xrightarrow{Bi^s} & BF & \xrightarrow{B\pi^s} & BF^s \\ \uparrow & & \uparrow & & \uparrow \\ A^s(M \times I) & \to & A(M \times I) & \to & A^s(M \times I) \end{array}$$

which concludes our statement as soon as we identify $A(M\times I) \to BF_\sigma$ to $A(M \times I) \hookrightarrow C(M)$.

b), follows straightforward simply observing that $\Omega A(M) \to \tilde{A}(M \times I)$ is a homotopy equivalence (actually an s.s isomorphism if we use an appropiate definition of Ω for s.s complexes) and again observing that $(\Omega A(M))^s \simeq *$; hence $A(M \times I)^s \simeq *$ hence $\tilde{A}(M \times I) \simeq \tilde{A}^a(M \times I)$.

Let τ' be the involution on $A(M \times I^2)$ and $C(M \times I)$

with the up-line exact sequence. Since

$$\pi_0^a(\bar{C}(MxI^k))=0 \quad \text{then} \quad \pi_0^a(A(MxI^k)) \longrightarrow \pi_0(\tilde{A}(MxI^k))$$

is an isomorphism and Proposition 9.2 is proved.

<u>Proposition 9.3</u> : If $k \leq \Upsilon(MxI)$, $A^s(MxI)_k \simeq A^s(MxI^3)_k$

<u>Proof.</u> Consider the diagram

$$\begin{array}{ccccc} A^s_k(M \times I^2) & \xrightarrow{i_1} & C(M \times I)_k & \xrightarrow{p_1} & A^s(M \times I)_k \\ & & \downarrow \phi & & \\ A^s_k(M \times I^3) & \xrightarrow{i_2} & C(M \times I^2)_k & \xrightarrow{p_2} & A^s(M \times I^2)_k \subset A(M \times I^2)_k \end{array}$$

First notice that $p_2 \cdot \phi \cdot i_1$ induces for homotopy the group-homomorphism $x \to x + \tau_*(x)$ where τ_* is the involution induced by τ. As $x \in \pi^s_*(A(M \times I^2)) = \pi_*(A^s(M \times I^2))$, we have $\tau_*(x) = x$, hence $x \to 2x$. Since the homotopy groups are $Z(\frac{1}{2})$ modules $p_2 \cdot \phi \cdot i_1$ induces isomorphism for all homotopy groups hence it is a homotopy equivalence.

Next we show the same is true for $p_1 \cdot \phi^{-1} \cdot i_2 = \omega$.
Actually is immediate that ω_* is injective using the injectivity of $(p_2 \cdot \phi \cdot i_1)$. To show that ω_* is surjective choose $x \in \pi_*(A^s(M \times I))$ and $y \in \pi_*(C(M \times I^2)_k)$ with $(p_1 \cdot \phi^{-1})_*(y)=x$. If $(p_2)_*(y) = 0$ we find $z \in \pi_*(A^s_k(M \times I^3))$ with $i_{2*}(z)=y$, hence $\omega_*(z) = x$. If not, let $u = (p_2)_*(y)$ and let $y' = (\phi \cdot i_1)_*(\frac{u}{2})$; clearly $p_{2*}(y-y') = 0$ and $(p_1 \cdot \phi^{-1})_*(y-y')=x$, hence we reduced to the previous case.

Putting together all these results we obtain:

are homotopy equivalences.

Since $C^a(M \times I)_k \sim *$ and $C^s(M \times I)_k \sim C(M \times I)_k$ the natural maps $A^a(M \times I)_k \longrightarrow BA^a(M \times I^2)_k$ and $C(M \times I)_k \longrightarrow A^s(M \times I^2)_k \times A^s(M \times I)_k$ are homotopy equivalences.

To finish the proof of 2), consider the commutative diagram:

$$\begin{array}{ccc} A^a(M \times I) & \xrightarrow{l_1} & BA^a(M \times I^2) \\ \downarrow l_2 & & \downarrow l_3 \\ A^a(M \times I) & \xrightarrow{\sim} & BA^a(M \times I^2) \end{array}$$

Since l_1 is a k-homotopy equivalence, in order to show the same about l_3, it suffices to check that "$A^a(M \times I^2) \longrightarrow \tilde{A}^a(M \times I^2)$ is a $(k-1)$ - homotopy equivalence " which is equivalent to "$A^a(M \times I^3) \longrightarrow \tilde{A}^a(M \times I^3)$ is a $(k-2)$-homotopy equivalence", ... which is equivalent to "$A^a(M \times I^k) \to \tilde{A}^a(M \times I^k)$ induces an isomorphism for π_o ".

Consider the diagram

$$\begin{array}{ccccc} \Omega\, A(M \times I^k)/A(M \times I^k) & \longrightarrow & A(M \times I^k) & \longrightarrow & \tilde{A}(M \times I^k) \\ & \nwarrow & & \nearrow & \\ & & C(M \times I^k) & & \end{array}$$

whose up-line is a fibration and all arrows equivariant with respect to the involutions induced by τ.

We obtain the following diagram of groups

$$\begin{array}{ccccc} \pi_1^a(\tilde{A}(M \times I^k)/A(M \times I^k)) & \longrightarrow & \pi_o^a(A(M \times I^k)) & \to & \pi_o(\tilde{A}(M \times I^k)) \to 0 \\ & \nwarrow & & \nearrow & \\ & & \pi_o^a(C(M \times I^k)) & & \end{array}$$

It is not hard to see using the same kind of arguments as presented above that

1) The surjectivity in the homotopy category of

$$A(M)_k \longrightarrow \tilde{A}(M)_k$$

2) The nulity in the homotopy category of

$$A^a(M \times I)_k \rightarrow C(M)_k$$

3) The homotopy equivalence

$$A(M)_k \simeq \tilde{A}(M)_k \times (C(M)/A^s(M \times I))_k$$

are equivalent.

With the notations of Burghelea-Lashof-Rothenberg 2 we consider the nilpotencies $\mathcal{N}(M)$'s resp. $\mathcal{N}^C(M)$'s which accour in the natural decompositions

$$*)\begin{cases} A(M \times S^1) \simeq A(M \times I) \times BA(M \times I) \times \mathcal{N}(M) & \text{if } \partial M \neq \emptyset \\ A(M \times S^1) \simeq A(M \times I) \times BA(M \times I) \times \mathcal{N}(M) \times S^1 & \text{if } \partial M = \emptyset \\ C(M \times S^1) \simeq C(M \times I) \times BC(M \times I) \times \mathcal{N}^C(M) & \end{cases}$$

Using the same arguments as before and the naturality of *) we derive the following proposition about the structure of $\mathcal{N}^C(M)$.

<u>Proposition 9.6</u> : 1) $\mathcal{N}^C(M^{n-1} \times I)_k \simeq \mathcal{N}^s(\mathring{M}^{n-1} \times I)_k \times \mathcal{N}^s(M \times I^2)_k^{+)}$

+) The involutions τ's on $A(M \times I \times S^1)$, $A(M \times I \times I)$ and $\mathcal{E}mb^A(M \times I, M \times I \times V)$ induced by conjugation with $id_M \times \Upsilon \times id_{S^1}$, $id_M \times \Upsilon \times id_I$ and, by right composition with $id_M \times \Upsilon$ and left composition with $id_M \times \Upsilon \times id_V$, gives rise to the involution τ on $\mathcal{N}(M \times I)$

<u>Theorem 9.4</u> : 1) $A^{s}(M \times I)_k \simeq \tilde{A}(M \times I)_k$ for $k \leq \delta(M)$ which says that the canonical inclusion $A(M \times I) \hookrightarrow \tilde{A}(M \times I)$ has a homotopy right invers in stable ranges.

2) $A^{s}(M \times I)_k$ sits as a direct factor in $C(M \times I)$ with other factor $A^{s}(M \times I^2)_k$, $k \leq \gamma(M)$, and

$$A^{s}(M \times I)_k \simeq A^{s}(M \times I^3)_k \text{ if } k \leq \gamma(M \times I)$$

<u>Corollary 9.5</u> : 1) If $k \leq \delta(M)$

$$A(M \times I)_k \times A(M \times I^2)_k \simeq C(M \times I)_k \times \tilde{A}(M \times I)_k \times \tilde{A}(M \times I^2)_k$$

$$2) \ \tilde{A}(M \times I)/A(M \times I) \times \tilde{A}(M \times I^2)/A(M \times I^2) \simeq (B\, C(M \times I))_k$$

Theorem 9.4 and Corollary 9.5 give the structure (in stable range) of $A(M \times I)$. For general M , the answer is not as precised as above. What we can say is only what can be deduced from Lemma 9.1, namely $\Omega\, \tilde{A}(M)/A(M)$ has the same k-homotopy type as $C(M)/A^{s}(M \times I)$, and $C(M) \sim C(M)/A^{s}(M \times I) \times A^{s}(M \times I)$.

$$\begin{array}{ccccc} \widetilde{A}(M \times I^2) & \to & \widetilde{A}(M \times I \times S^1) & \to & \mathcal{E}mb^A(M \times I,\ M \times I \times S^1) \\ \uparrow & & \uparrow & & \uparrow \\ A^a(M \times I^2) & \to & A^a(M \times I \times S^1) & \to & \mathcal{E}mb^{A\ a}(M\times I,\ M \times I \times S^1) \end{array}$$

Since $A^a(M \times I^2) \to \widetilde{A}(M \times I^2)$ and $A^a(M \times I \times S^1) \to \widetilde{A}(M \times I \times S^1)$ are $\rho(\dim M + 2)$ - homotopy equivalences and the both lines are trivial fibrations[++], $\mathcal{E}mb^{A\ a}(M \times I,\ M \times I \times S^1) \longrightarrow \mathcal{E}mb^A(M \times I,\ M \times I \times S^1)$ is a $\rho(\dim M + 2)$ - homotopy equivalence. Before localisation "away 2" $\widetilde{A}(M \times I) \to \mathcal{E}mb^A(M \times I, M \times I \times S^1)$ has as homotopy theoretic fiber a space whose homotopy groups are the Rotenberg-groups of $\pi_1(M)$ according to [13] Theorem 2.7, hence 2-primary groups. Consequentely after localisation "away 2" the obvious inclusion $\widetilde{A}(M \times I) \to \widetilde{\mathcal{E}mb}^A(M \times I,\ M \times I \times S^1)$ is a homotopy equivalence, hence $A^a(M \times I \times S^1) \simeq \widetilde{A}(M \times I) \times \widetilde{A}(M)$. Even more, the natural commutative diagram

$$\begin{array}{ccccc} B\widetilde{A}(M \times I^2) \simeq A(M \times I) & \xrightarrow{\widetilde{\delta}} & \widetilde{\mathcal{E}mb}^A(M \times I,\ M \times I \times S^1) & \to & * \\ \uparrow & & \uparrow & & \uparrow \\ BA^a(M \times I \times I) & \xrightarrow{\delta} & \mathcal{E}mb^{A\ a}(M\times I,\ M \times I \times S^1) & \to & \mathcal{N}^a(M \times I) \end{array}$$

whose horisontal lines are fibrations implies $\mathcal{N}^a(M \times I)$ is $\rho(\dim M + 2)$ - connected, hence $\mathcal{N}^a(M \times I) \to \mathcal{N}'(M \times I)$ is a $\rho(\dim M + 2)$- homotopy equivalence.

a) follows then from Proposition 9.6.

To prove b), let us observe that for any manifold M^n there exists a codimension one submanifold V^{n-1} with trivial

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++) since the first line is a trivial fibration

$$2)\quad \mathcal{N}^s(M^{n-1} \times I)_k \simeq \mathcal{N}^s(M^{n-1} \times I^3)_k$$

for $k \leq \rho(n+1)$, $\rho(n)$ being the stability function in $\mathcal{T}$op.

<u>Theorem 9.7</u> : a) If $M^n = N^{n-1} \times I$ then

$$1)\ \mathcal{N}^c(N \times I)_k \simeq \mathcal{N}(N \times I)_k \times \mathcal{N}(N \times I^2)_k$$

$$2)\ \mathcal{N}(N \times I)_k \simeq \mathcal{N}(N \times I^3)_k \quad \text{for} \quad k \leq \rho(n+1)$$

$$b)\ 1)\ \mathcal{N}^c(M)_{k'} \simeq \mathcal{N}(M)_{k'} \times \mathcal{N}(M \times I)_{k'}$$

$$2)\ \mathcal{N}(M)_{k'} \simeq \mathcal{N}(M \times I^2)_{k'}$$

$$\text{for } k' \leq \inf\left(\rho(n),\ \left[\tfrac{n-1}{2}\right] - 3\right)$$

<u>Proof of Theorem 9.7</u> : The commutative diagram of principal fibrations

$$\begin{array}{ccccc} \tilde{A}(M \times I^2) & \longrightarrow & \tilde{A}(M \times I \times S^1) & \longrightarrow & \tilde{\mathcal{E}}mb^A(M \times I,\ M \times I \times S^1) \\ \uparrow & & \uparrow & & \uparrow \\ A(M \times I^2) & \longrightarrow & A(M \times I \times S^1) & \longrightarrow & \mathcal{E}mb^A(M \times I,\ M \times I \times S^1) \end{array} \quad {}^{++)}$$

has all arrows equivariant with respect to the involutions τ's, hence decomposes as product of two diagrams, the symetric respectively antisymetric diagram. Since $\tilde{A}(P \times I) \simeq \tilde{A}^a(P \times I)$ which implies $\tilde{\mathcal{E}}mb^A(M \times I,\ M \times I \times S^1) \simeq \tilde{\mathcal{E}}mb^{A\ a}(M \times I,\ M \times I \times S^1)$ we obtain the following commutative diagram whose lines are principal fibrations.

++) The up-line is a principal fibration according to [13], Proposition 2.4

can be theoretically computed by surgery methods.

The other one, $A^S(M)_k$, is "almost" a homotopy theoretic invariant, in the sence that it is a distinguished factor of the homotopy theoretic invariant $C(M)_k$, $k < \varrho(\dim M)$, and satisfies the stability $A^S(M)_k \simeq A^S(M \times I^2)_k$.

normal bundle so that $\pi_i(M^n, V^{n-1}) = 0$ for $i \leq [\frac{n-1}{2}]$. To see this, we choose a Morse function $f : M \to [0, \infty)$ with critical values integer numbers and all critical points corresponding to i of index i. Take $V^{n-1} = f^{-1}([\frac{n-1}{2}] + \frac{1}{2})$, and identify $f^{-1}([\frac{n-1}{2}] + \frac{1}{3}, [\frac{n-1}{2}] + \frac{2}{3})$ to $V^{n-1} \times I$. Since $\pi_i(M, V^{n-1} \times I) = 0$ for $i \leq [\frac{n-1}{2}]$ by Theorem 3.1 of [13] we obtain $\pi_i(\mathcal{N}(M^n), \mathcal{N}(V^{n-1} \times I)) = 0$ for $i \leq [\frac{n-1}{2}] - 2$, hence $\mathcal{N}(V^{n-1} \times I)$ and $\mathcal{N}(M^n)$ have the same $([\frac{n-1}{2}] - 3)$ - homotopy type. Then a) implies b). As consequence we get.

<u>Theorem 9.8</u> : a) $A(M^n \times S^1)_k \simeq \{\overbrace{A(M) \times \Omega A(M)}^{I}\}_k \times \{\overbrace{A^s(M \times I) \times BA^s(M \times I) \times \mathcal{N}(M)}^{II}\}_k \times T$, $k \leq \rho(n+1)$ T = point if $\partial M \neq \phi$ and $T = S^1$ if $\partial M = \phi$.

b) $\{A(M^n \times S^1) \times A(M^n \times I \times S^1)\}_k \simeq \{\overbrace{\tilde{A}(M) \times (\Omega\tilde{A}(M))^2 \times \Omega^2 A(M))}^{I}\}_k \times \{\overbrace{(C(M))^2 \times \mathcal{N}^C(M)}^{II}\}_k \times T$ for $k \leq \inf(\rho(n+1), [\frac{n-1}{2}] - 3)$ T = point if $\partial M \neq \phi$ and $T = S^1$ if $\partial M = \phi$.

The proof follows straightforward from *), Theorems 9.4, 9.7 (and Proposition 2.8 [ ] which claims $\Omega\tilde{A}(M) \sim \tilde{A}(M \times I)$).

Remark that "I" is a geometric type invariant which depends deeply on the topology of M and the category A, while II is a homotopy theoretic invariant at least in case b).

Comments : The main goal of these considerations were to split up A(M) for $M = N \times I$ or $N \times S^1$ as product of two factors : One, the geometric factor (in stable range) is $\tilde{A}(M)$ and depends on the topology of M and the geometric category A ; it

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